Understanding the differences between Newtonian and Relativistic Physics by investigating the Apsidal Precession of Mercury using Computational Methods

To what extent does Einstein's theory of General Relativity provide a better prediction for the orbit of Mercury compared to Newton's Universal Law of Gravitation?

Extended Essay in Physics

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Table of Contents

Introduction

At the beginning of the 20^{th} century, Albert Einstein revolutionized physics with his theories on the photoelectric effect, special theory of relativity and general theory of relativity. These ideas served as a launching point for all of modern physics, and also piqued my interest in physics when I was younger. As a child, reading about these complex and counterintuitive ideas captured my imagination because they were so difficult to wrap my head around, and this is why I aspire to be a physicist. This motivated me to learn as much physics and math as possible in the hopes of building up enough knowledge to eventually understand Einstein's theories of relativity.

Gaining an understanding of Special and General Relativity is extremely important for physicists because on the cutting edge of undiscovered physics is an effort to create a Grand Unified Theory of physics which unites the realm of Relativistic physics with the other main branch of modern physics, Quantum Mechanics. Attempts to come up with theories which bring Relativity together with Quantum Mechanics such as String Theory are inconclusive. String theory offers a possible way to unite Relativity and Quantum Mechanics, however the theory has no experimental evidence to support it. In order to understand and work towards forming a conclusion on whether String theory is viable or a flawed theory, physicists must have a solid understanding of Relativity. Another branch of cutting edge physics is Nuclear Physics. This includes the Standard Model and Quantum Field Theory. Many physicists consider the Standard Model of Physics as the crowning achievement of modern physics, since it is the closest we have come to a Grand Unified Theory. The Standard Model explains 3 out of the 4 fundamental forces, the electromagnetic force, strong force and weak force. Gravity (or General Relativity) is the only fundamental force which does not fit in with the Standard Model yet. In order to reconcile Gravity together with the Standard Model, it is possible that General Relativity is a flawed theory which needs to be adjusted, and thus an understanding of General Relativity is important.

Einstein's theories of relativity made adjustments to the framework of Newtonian ideas that all of Physics up until that point was built on. These adjustments were made in order to incorporate phenomena which Newtonian dynamics could not predict. On most everyday scales Newtonian physics serves as an accurate approximation for what is going on, however it begins to break down on very large scales, for example when objects have very large mass or are moving at speeds on the magnitude of the speed of light in a vacuum. One example of a phenomenon in which Newtonian dynamics and Relativistic dynamics differ are in its prediction of the orbit of Mercury.

While Newtonian Physics correctly predicts the elliptical shape of Mercury's orbit, the problem occurs in the apsidal precession of Mercury's orbit each time it orbits the sun. Apsidal Precession is the rotation of the line of apsides of the elliptical orbit, and it is measured in the angle between the line of apsides from one orbit to the next.

Figure 1.

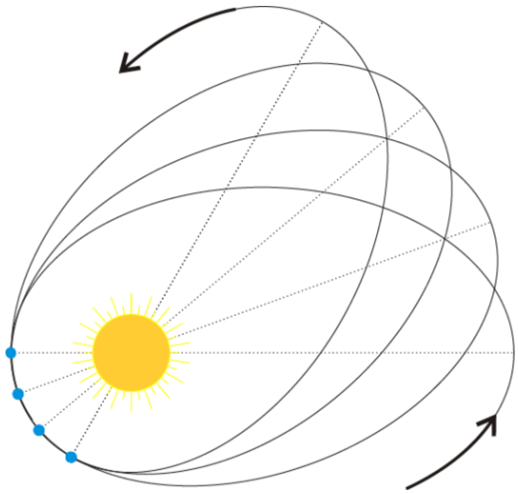


Diagram of Apsidal Precession, dotted lines represent lines of apsides in each orbit, Image Credit: https://aasnova.org/2017/07/12/wasp-12b-and-its-possible-fiery-demise/

While Newtonian Physics is able to predict 5.32” arcseconds of precession per year (an arcsecond is $\frac{1}{3600}$ degrees, indicated by symbol ”), the total observed precession per orbit is 5.75”, therefore there is a discrepancy of 0.43” (Fitzpatrick, 2012). This essay aims to investigate the phenomenon of apsidal precession of Mercury by exploring the differences in the physics of Newtonian and Relativistic theories through computational simulations. In order to explore this, the research question guiding the exploration is: **To what extent does Einstein’s theory of General Relativity provide a better prediction for the orbit of Mercury compared to Newton’s Universal Law of Gravitation?**

Spherical and Polar Coordinates

Since the orbits of planets are typically circular or elliptical, it is easier to describe their behaviors using polar/spherical coordinates rather than cartesian coordinates. Most physics problems use cartesian coordinates in which the vectors which describe the position and motion of the orbit of a planet are broken up into vertical and horizontal components. In polar coordinates, vectors are described in terms of angles and a radius, which makes problems easier to think about when everything is rotating around a fixed point such as in orbit.

Part A: Comparing Newtonian and Relativistic Physics

Newton’s Second Law

The foundation of Newtonian physics is Newton’s second law, which states:

$$F_{net} = ma \quad (3)$$

The goal of Newtonian physics is to analyse the forces acting on an object and then find the acceleration in order to describe the motion. The forces involved in the orbit of Mercury are all gravitational, and thus can be described using the Universal Law of Gravitation:

$$F_G = \frac{GMm}{r^2}\hat{\boldsymbol{r}} \ (4)$$

Note that in this equation $\hat{\boldsymbol{r}}$ represents the separation vector which points the force in the direction of the other object in the gravitational interaction.

Applying Newton's Second Law the equation for acceleration due to gravity can be found as:

$$a = \frac{GM}{r^2}\ \hat{\boldsymbol{r}} \ (5)$$

If we assume that the distance between Mercury and the Sun (r) stays constant, then the motion is simply uniform circular motion, since the acceleration along the radial direction will only change the linear/tangential velocity. In reality, the acceleration along the radial direction changes the distance between Mercury and the Sun, therefore r is a function of time. This means the orbit of Mercury cannot be described with uniform circular motion, and the analysis of motion is greatly complicated.

Typically, Newtonian physics problems with non-uniform acceleration can be solved using vector analysis. This is accomplished by breaking the acceleration vector into Cartesian $(\mathrm{x}, \mathrm{y}, \mathrm{z})$ components, and then casting the problem into differential equations (Taylor, 2004):

$$\overline{\boldsymbol{a}} = \left(a_x\,, a_y\,, a_z\right)$$

$$a_x = f(t), a_y = g(t), a_z = h(t)$$

$$\frac{d^2x}{dt^2} = f(t), \frac{d^2y}{dt^2} = g(t), \frac{d^2z}{dt^2} = h(t) \ \ (6)$$

This process is not as straightforward when using spherical/polar coordinates, however there is an alternate formulation of Newton's Second Law known as the Principle of Least Action.

The Principle of Least Action

The Principle of Least action formulates Newton's Laws based around analysing forces into the analysis of energy. It is also known as Lagrangian Mechanics.[1] Since energy is a scalar quantity compared to force which is a vector, this eliminates the vector analysis which is tricky for spherical coordinates. As an equation, the Principle of Least action states:

$$\delta\int_{t_0}^{t_f} \mathcal{L}(q, \dot{q}, t)\ dt = 0 \ (7)$$

While this equation looks intimidating, it can be broken down to show the actual physics it represents. Firstly, the most important quantity is $\mathcal{L}(q, \dot{q}, t)$. This quantity is known as the Lagrangian, and it simply represents the difference between kinetic energy and potential energy, $E_k - E_p$. Thus, the integration of the Lagrangian is simply summing together $E_k - E_p$ at every

[1] Discussion of Lagrangian Mechanics in this section comes from the textbook "Classical Mechanics" by John R. Taylor (Taylor, 2004)

point in time between t_0 & t_f, and this quantity is what physicists define action as. δ represents a special derivative known as a variational derivative, however for our purposes it is very similar to a normal derivative. Taking the derivative of some quantity and setting it equal to 0 finds the point where a function is at a minimum. Therefore what this equation is doing is finding when action is at a minimum. In conclusion, the physics this equation is telling us is the motion of objects always follows the path where there is the least amount of action possible.

When the special derivative δ is applied, the equation becomes:

$$\frac{d\mathcal{L}}{dq_i} = \frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{q}_i}\right) \text{ (8)}$$

While it may not seem apparent at first, Eq. 8 is a generalized version of Newton's Second Law. Firstly, the summation of q_i represents some generalized set of coordinates. For example, Cartesian coordinates can be represented as $(x, y, z) \rightarrow (q_1, q_2, q_3)$. Similarly when spherical coordinates are being used, (q_1, q_2, q_3) instead represent (r, θ, ϕ). When there is a dot on top of a quantity i.e. $\dot{q}_i$, it represents the derivative of that quantity with respect to time, A.K.A. the velocity. It is important to distinguish the fact that not all of these velocities are equivalent quantities, for example $\dot{x}, \dot{y}, \dot{z}$ & $\dot{r}$ are linear velocities, whereas $\dot{\theta}$ & $\dot{\phi}$ are angular velocities which do not behave in the same way.

Considering the simple example in one dimension of a mass attached to a spring will help to demonstrate the parallels between this equation and Newton's Second Law. The Lagrangian for this example is as follows:

$$\mathcal{L} = E_K - E_P \text{ (9)}$$

The potential energy of a mass attached to a spring is $\frac{1}{2}kx^2$ therefore:

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \text{(10)}$$

Firstly, taking the derivative of the Lagrangian with respect to the velocity $\dot{x}$:

$$\frac{d\mathcal{L}}{d\dot{x}} = m\dot{x} = mv \text{ (11)}$$

The derivative of the Lagrangian with respect to some velocity effectively yields a momentum, however if an angular velocity is used it yields an angular momentum. This is why the quantity $\frac{d\mathcal{L}}{d\dot{q}_i}$ is referred to as generalized momentum. This parallel's Newton's second law because Newton's second law can be written as:

$$F = \frac{d}{dt}(mv) \text{ (12)}$$

Similarly, Eq. 8 contains the time derivative of generalized momentum.

Next, we need to consider the left side of equation 8, $\frac{d\mathcal{L}}{dq_i}$. Usually the only quantity which is a function of position is the potential energy. By definition, force is the negative derivative of potential energy with respect to position:

$$F = -\frac{d}{dx}E_P \ (13)$$

Therefore the quantity $\frac{d\mathcal{L}}{dq_i}$ is known as generalized force. This differs from regular force because if, for example, the coordinate used is an angle, the generalized force is torque.

Returning to the main problem being explored, the Lagrangian for a planet orbiting the Sun such as Mercury using polar coordinates is:

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + \frac{GmM}{r} \ (14)$$

Note that this Lagrangian uses polar coordinates (r, ϕ) rather than spherical coordinates (r, θ, ϕ) because the symmetry of an orbit allows us to pick the coordinates in such a way that $\theta = \frac{\pi}{2}$ at all times by choosing the r & ϕ coordinates to be the same plane as the orbit. Therefore there is no motion in the θ coordinate, and this reduces the problem down to two dimensions.

Using Eq. 8, the equations for the acceleration are:

$$\ddot{r} = a_r = r\dot{\phi}^2 - \frac{GM}{r^2} \ (15)$$

$$\ddot{\phi} = a_\phi = -\frac{2}{r}\dot{r}\dot{\phi} \ (16)$$

Note that the double dot represents a second derivative with respect to time.

Eq. 15 and 16 are differential equations, which mean they show the relationship between unknown functions and their derivatives. The unknown functions are the position functions $r(t)$ & $\phi(t)$. Since the equations are in terms of the second derivative they are considered second order differential equations, and since there are terms containing r and ϕ in both equations, they are considered a system of coupled differential equations. A system of second order, coupled differential equations are difficult to solve, so I will return to this later on in the exploration.

Relativistic Mechanics

One of the main differences between Newtonian Physics and Relativistic Physics is that time is not absolute in relativity. In Newtonian Physics it is crucial that time is absolute because all motion is analysed as a function of time. Instead, in relativity objects move through time at different rates based on their velocity as well as whether they are in the presence of gravitational fields. This is the phenomena known as time dilation. Because time is variable for different objects, something is defined known as proper time, which describes how time is experienced through one individual object. Proper time is denoted with the Greek letter tau "τ", and all motion in Relativity is analysed as a function of proper time. Because of this, time "t" in relativity is treated the same way as a spatial coordinate, therefore the coordinate system becomes (t, r, θ, ϕ). However, this exploration is examining the spatial orbit of Mercury, not the time dilation effects Mercury experiences. Therefore there are only two main changes this discussion causes. Firstly, while previously a dot above a variable indicated differentiation with respect to time, but now it represents differentiation with respect to proper time. Secondly, an extra dimension is introduced to the problem, which means there will be one extra differential equation.

Newtonian physics focuses on analysing forces in order to find the motion they cause. In Einstein's theory of special relativity, the idea of analysing forces using Newton's Second Law is present, except it is slightly modified to fit with the relativistic ideas of what momentum is. However from Einstein's general theory of relativity we can see that gravity is not treated like a force. Rather, general relativity treats gravity as a theory of geometry. Objects with mass or energy change the geometry of the space and time around it, and other objects will move through space and time in a straight lines (called geodesics) in this curved geometry. This is the main difference between the Newtonian and Relativistic theories of gravity. Newtonian Physics describes gravitational interactions as the property of mass in an object exerting a force on another object, thus affecting its motion. Relativistic Physics on the other hand describes gravitational interactions as the property of mass changing the spatial and temporal geometry around it, which affects the motion of other objects around it.

Understanding the Geometry of General Relativity

The differential line element ds is how General Relativity[2] describes the geometry of a space. A differential line element represents the length of an infinitely small displacement in space. For example, 2D flat space using Cartesian coordinates can be described as:

$$ds^2 = dx^2 + dy^2$$

$$ds = \sqrt{dx^2 + dy^2} \ (17)$$

This line element follows from the Pythagorean theorem, as if you displace an object an infinitely small amounts in space dx and dy, the resulting length of this displacement is the line element ds.

Recalling Eq. 7 for the principle of least action, we can plug in the line element instead of action:

$$\delta \int_A^B ds = 0 \ (18)$$

Instead of minimizing action between two points, this equation finds the path of minimum length between point A and point B. Therefore, differential line elements are able to describe different geometrical spaces by telling us the path of the shortest distance between two points in a space.

[2] Discussion of General Relativity in this section sourced from "Spacetime and Geometry, An Introduction to General Relativity" by Sean M. Carroll (Carrol, 2004)

These paths of minimum length are called geodesics, and this is important because the orbit of Mercury is a geodesic path.

Einstein's Field Equations

$$R_{\mu\nu} + \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \; (19)$$

Eq. 19 may look like one equation, however it is actually a set of 16 equations since each variable with the μ and ν indices represents a 4x4 matrix. The right side of the equation essentially describes the mass of objects, as $T_{\mu\nu}$ represents the energy-momentum tensor. The left side of the equation describes the related changes in geometry caused by the mass. $R_{\mu\nu}$ represents the Ricci Curvature tensor, and $g_{\mu\nu}$ represents the metric tensor, which has to do with the differential line element discussed earlier.

In order to solve these equations, an energy-momentum tensor is plugged in describing a specific arrangement of masses, energies etc. and the solution can be put in the form of a differential line element that compactly describes the geometry caused by the given energy-momentum tensor.

The Schwarzschild Solution

The Schwarzschild Solution to Einstein's Field Equations is a solution for a non-charged, non-rotating spherical mass. While technically the sun has charge and does rotate about its axis, the effects of this are nearly negligible. The net charge on the sun is 77 coulombs (Neslušan, 2001), which compared to the massive size of the sun leads to its effects being basically none, and similarly the sun only rotates on its axis on average once every 27 days (Zell, 2013). In conclusion, the assumptions are safe to make in order to apply the Schwarzschild Solution to the sun.

The Schwarzschild Solution to Einstein's Field Equations is the differential line element:

$$-\left(1 - \frac{r_s}{r}\right) c^2 dt^2 + \left(\frac{1}{1 - \frac{r_s}{r}}\right) dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi \; (20)$$

The value r_s in this equation is the Schwarzschild radius. The Schwarzschild radius is defined as the following quantity:

$$r_s = \frac{2GM}{c^2} \; (21)$$

Physicists use a clever trick to simplify this expression by getting rid of the physical constants c and G, which are the speed of light and Newton's Gravitational constant respectively. They accomplish this by redefining the units of all physical quantities such that:

$$c = G = 1 \; (22)$$

Redefining units in this way is known as geometrized units, and the following table shows the way to convert from SI units to geometrized units:

Table 1.

Variable	SI Unit	Geom. Unit	Factor	Geometrized unit → SI unit
mass	kg	m	c^2G^{-1}	1 m $\rightarrow 1.3466 \times 10^{27}$ kg
length	m	m	1	1 m $\rightarrow$ 1 m
time	s	m	c^{-1}	1 m $\rightarrow 3.3356 \times 10^{-9}$ s
energy	kg m^2 s^{-2}	m	c^4G^{-1}	1 m $\rightarrow 1.2102 \times 10^{44}$ kg m^2 s^{-2}
momentum	kg m s^{-1}	m	c^3G^{-1}	1 m $\rightarrow 4.0370 \times 10^{35}$ kg m s^{-1}
velocity	m s^{-1}	dimensionless	c	1 $\rightarrow 2.9979 \times 10^{8}$ m s^{-1}
angular momentum	kg m^2 s^{-1}	m^2	c^3G^{-1}	1 m^2 $\rightarrow 4.037 \times 10^{35}$ kg m^2 s^{-1}
force	kg m s^{-2}	dimensionless	c^4G^{-1}	1 $\rightarrow 1.2102 \times 10^{44}$ kg m s^{-2}
acceleration	m s^{-2}	m^{-1}	c^2	1 m^{-1} $\rightarrow 8.9875 \times 10^{16}$ m s^{-2}
energy density	kg m^{-1} s^{-2}	m^{-2}	c^4G^{-1}	1 m^{-2} $\rightarrow 1.2102 \times 10^{44}$ kg m^{-1} s^{-2}

Table from "Natural System of Units in General Relativity" (Myers, 2016)

Using geometrized units, the Schwarzschild line element becomes:

$$-\left(1-\frac{2M}{r}\right)c^2dt^2+\left(\frac{1}{1-\frac{2M}{r}}\right)dr^2+r^2d\theta^2+r^2sin^2\theta d\phi \quad (23)$$

The derivation of the equations of the geodesics involves complicated maths outside of the scope of this essay, however based on the Schwarzschild line element, the following equations were obtained:

$$\ddot{t}=-\frac{2M}{r^2\left(1-\frac{2M}{r}\right)}\dot{r}\dot{t}$$

$$\ddot{r}=-\frac{M}{r^2}\left(1-\frac{2M}{r}\right)\dot{t}^2+\frac{M}{\left(1-\frac{2M}{r}\right)r^2}\dot{r}^2+\left(1-\frac{2M}{r}\right)(r\dot{\theta}^2+r\sin^2\theta\,\dot{\phi}^2)$$

$$\ddot{\theta}=-\frac{2}{r}\dot{r}\dot{\theta}+\sin\theta\cos\theta\dot{\phi}^2$$

$$\ddot{\varphi}=-\frac{2}{r}\dot{\phi}\dot{r}-2\frac{cos\theta}{sin\theta}\dot{\phi}\dot{\theta} \quad (24)$$

These equations can be simplified in the same way as the equations for the Newtonian orbit of Mercury by setting theta equal to the constant $\frac{\pi}{2}$ (note that from this the angular velocity $\dot{\theta}$ becomes 0) .

The equations then simplify to become:

$$\ddot{t} = a_t = -\frac{2M}{r^2\left(1-\frac{2M}{r}\right)}\dot{r}\dot{t}$$

$$\ddot{r} = a_r = -\frac{M}{r^2}\left(1-\frac{2M}{r}\right)\dot{t}^2 + \frac{M}{\left(1-\frac{2M}{r}\right)r^2}\dot{r}^2 + \left(1-\frac{2M}{r}\right)r\dot{\phi}^2$$

$$\ddot{\phi} = a_\phi = -\frac{2}{r}\dot{\phi}\dot{r} \quad (25)$$

These equations parallel the equations found which describe the acceleration in the case of Newtonian Physics. They are also a system of coupled second order differential equations, and as such are difficult to solve. Both of these systems can be solved computationally in order to interpret the physics and motion they describe.

Part B: Interpreting the orbit of Mercury with Computational Methods

In order to interpret the systems of differential equations found in Part A, a computational solver can be used. The differential equations obtained in Part A are a general case for any object orbiting a mass, however in order to apply them to the specific case of Mercury orbiting the sun we have to set constants which describe the specific problem.

Newtonian Case

The main constant which needs to be set is M as the mass of the sun. The mass of the sun is simply 1.989$\times 10^{30}$ kg. Additionally, in order to solve differential equations numerically, the initial positions and velocities at t=0 need to be specified $(r_0, \phi_0, \dot{r}_0, \dot{\phi}_0)$. The initial position can be picked at any point during Mercury's orbit, however where it is selected affects the selection of the other initial conditions. The initial position I am going to use is the perihelion of Mercury's orbit, which is the point during the orbit in which Mercury is closest to the sun. The initial radius is thus $r_0 = 4.6 \times 10^{10}\ m$ (Williams, 2021). The initial angle can be picked to be any angle so I will choose it to be $\phi_0 = 0$. The initial radial velocity will be $\dot{r}_0 = 0$, since Mercury does not start off with any velocity tending it to go towards or away from the sun, any radial velocity is caused later on by the force of gravity. The initial angular velocity $\dot{\phi}_0$ can be found based on another value cosmologists use called orbital velocity. The orbital velocity is the tangential/linear velocity of an orbiting body at some point, and according to NASA (Williams, 2021) the orbital velocity of Mercury at the perihelion is 58980 $\frac{m}{s}$. The angular velocity is related to the orbital velocity by:

$$\dot{\phi}_0 = \frac{v}{r} = \frac{58980}{4.6 \times 10^{10}} = 1.3 \times 10^{-6}\ \frac{rad}{s} \quad (26)$$

Specifying all of these constants, the system of differential equations from Eq. 15-16 was solved in MATLAB, and the resulting orbit was plotted as seen in Fig. 3.

Relativistic Case

The constants which need to be specified are all the same as those specified in the Newtonian case, however there are two differences. The first difference is that two additional constants need to be specified, the initial time and initial "time-velocity" t & $\dot{t}$. For obvious reasons, the initial time is 0, and $\dot{t}$ was set to be 1 however this does not affect the spatial solution therefore it is not important. The second difference is that the constants specified in the Newtonian Case need to be converted from SI units to geometrized units referring to Table 1 for conversion factors:

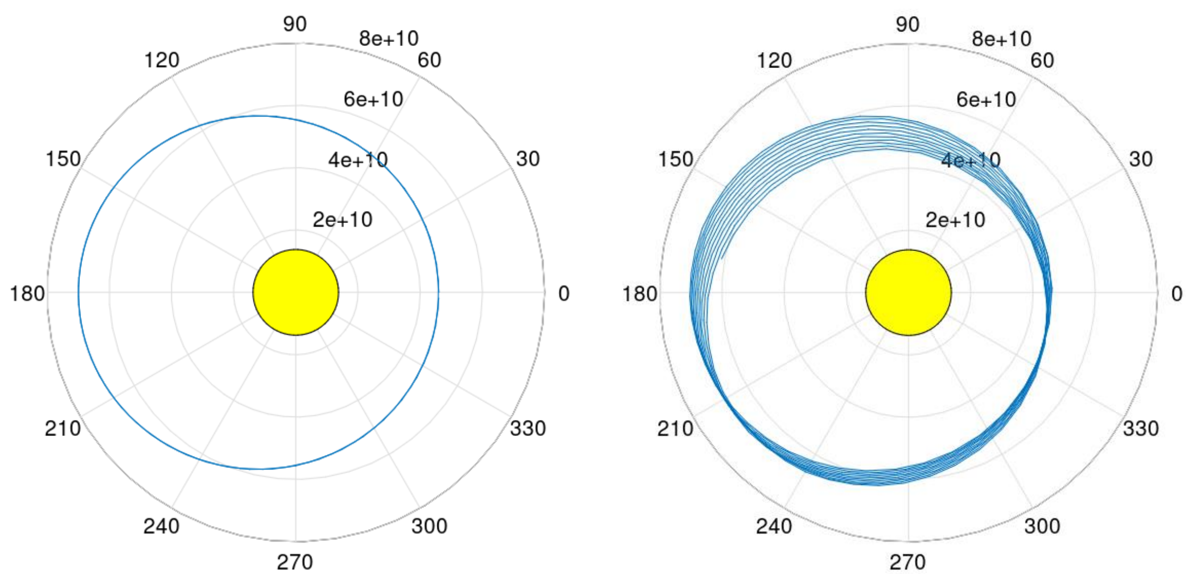
$$M = \frac{1.989 \times 10^{30}}{1.3466 \times 10^{27}} = 2.95 \times 10^{3}\ m\ (27)$$

The constants r_0, ϕ_0 & $\dot{r}_0$ do not need to be converted. This is because ϕ_0 & $\dot{r}_0$ are 0, and r_0 has units of length, which stay the same between SI and geometrized units. This leaves $\dot{\phi}_0$. The conversion for angular velocity is the reciprocal of the conversion factor for time, since the SI unit is radians per second, and radians do not need to be converted. Therefore the conversion is:

$$\dot{\phi}_0 = \frac{1.3 \times 10^{-6}}{3 \times 10^{8}} = 4.33 \times 10^{-15} \frac{rad}{m}\ (28)$$

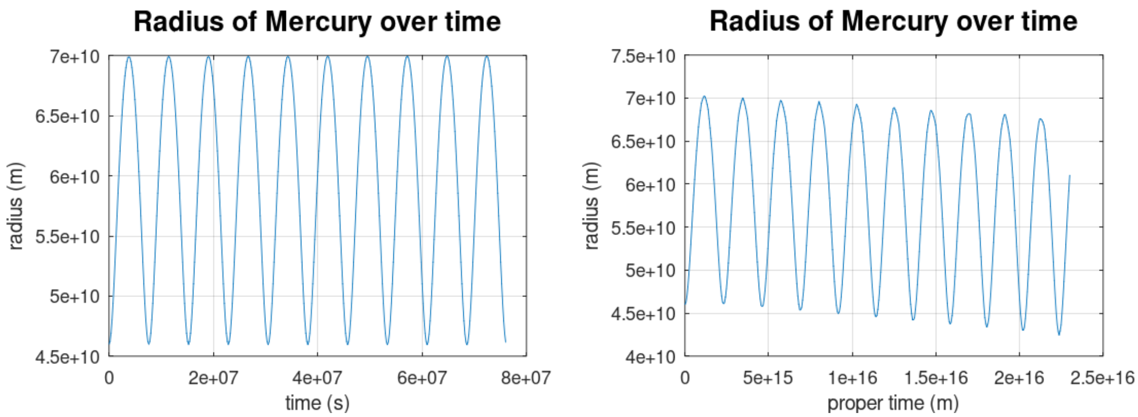
With these constants established, the system of differential equations from Eq. 25 can be solved computationally in MATLAB.

The resulting orbits from each of the solutions are shown in the figures below:
Figure 3.



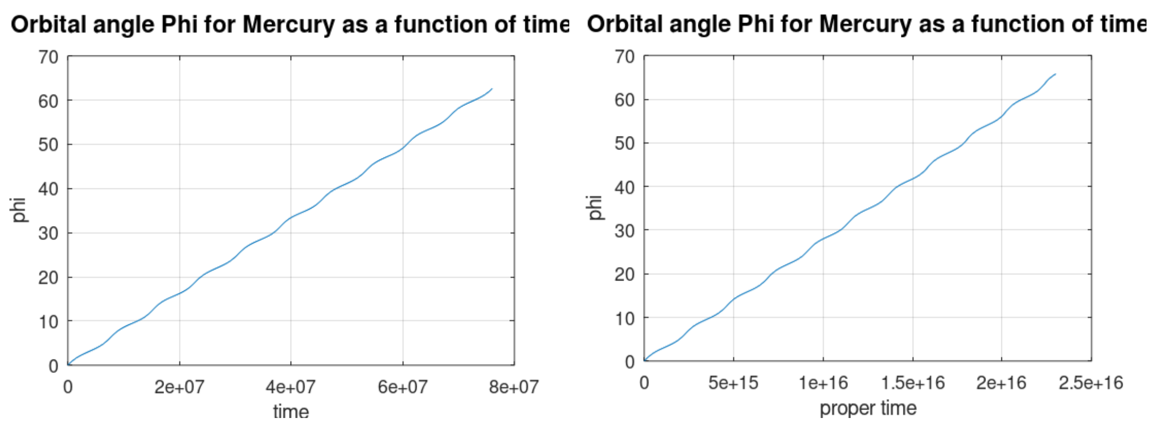
Newtonian orbit of mercury (left) and 10 orbits of Mercury in General Relativity (right)

Figure 4.



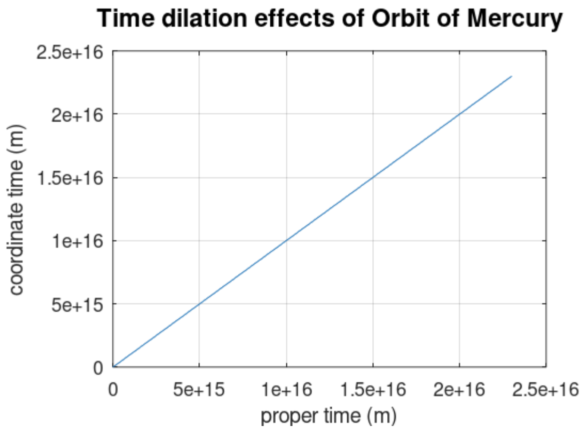
Graphs showing how the radius of Mercury changes over the course of 10 orbits, Newtonian (left) and Relativistic (right)

Figure 5.



Graphs showing how the orbital angle Phi changes over the course of 10 orbits, Newtonian (left) and Relativistic (right)

Figure 6.



This figure shows that there are no time dilation effects on Mercury because the proper time for Mercury is moving at the same rate as coordinate time, since the slope of the line is 1

Analysis of Computational Simulations and Conclusion

The results of the simulation are as expected, the orbit of Mercury is elliptical with a perihelion (point closest to the sun) of around 4.6×10^{10}m and aphelion (point furthest from the sun) of around 7×10^{10}m (Williams, 2021). Comparing the Newtonian and Relativistic orbits from Fig. 3, the apsidal precession can be seen in the Relativistic orbit, as the orbit shifts slightly each time it orbits around the sun. There is no apsidal precession at all in the Newtonian Case as it forms a closed elliptical orbit. This does not line up with the expected 532" of apsidal precession that Newtonian Physics predicts, however this can be attributed to the fact that the 532" of apsidal precession is caused by the effects of the forces of the other planets in the solar system on Mercury. The model used in this exploration only accounts for the Sun and Mercury, since accounting for the effects of other planets greatly complicates the analysis. Therefore it can be deduced that the apsidal precession seen in the Relativistic simulation is the remaining 43" that is not accounted for by the effects of other planets.

Ideally if I had more knowledge of differential equations I would solve the differential equations in this exploration mathematically, however since I am not able to, I can still analyse the results of the computational simulations to gain understanding. This is a valuable skill which physicists often use since differential equations are very often too difficult to solve. However, there are limitations towards computationally simulating physical systems, since the system could be chaotic, meaning small changes towards the initial conditions or the amount of time steps greatly impact the solution. Additionally there is a trade-off between accuracy in terms of how small each time-step is, and the computational power required.

The first thing I notice from the computational solutions is from Fig. 6 it can be seen that there are no time dilation effects. Therefore the equation for "time-acceleration" in Eq. 25 can be ignored. This also means for this problem it does not matter that everything in Relativity is a function of proper time, as it would have yielded the same results if everything was a function of normal time. Therefore, time dilation is not a difference between Newtonian and Relativistic theory that accounts for the precession of Mercury.

Comparing Eq. 15-16 with Eq. 25, the equations describing the angular acceleration are equivalent. This is not surprising as in both the Newtonian and Relativistic case if two objects satisfy the right conditions one will orbit periodically around the other, so the way the angle evolves should be the same. Fig. 5 confirms this as the way the orbital angle ϕ changes over time is the same in the Newtonian and Relativistic case.

Next, looking at the equations for the radial acceleration in Eq. 15-16 vs. Eq. 25 is where the significant difference shows. While there are some similar terms such as $r\dot{\phi}^2$ and $-\frac{M}{r^2}$, the Relativistic equation contains significantly more terms. The effect of this can be seen in Fig. 4. While both graphs look sinusoidal in nature, in the relativistic graph it appears the sine wave is superimposed with a negatively sloped line. Therefore it can be deduced that the difference in radial acceleration is what is causing apsidal precession in the relativistic case.

In conclusion, this exploration discovered many similarities and differences between Newtonian and Relativistic Physics. Newtonian physics describes gravity through the analysis of forces, whereas Relativistic physics describes gravity through the analysis of geometry. The path that the object takes in Newtonian gravity follows the principle of least action, while the path that the object takes in Relativistic gravity is the “principle of least distance” between two points in a curved geometry. Ultimately, these different approaches lead to describing very similar elliptical motion for the Orbit of Mercury. However, to answer the research question: General Relativity provides a better prediction for the Orbit of Mercury by approaching gravity through geometry, which leads to a different radial acceleration that describes the effect of apsidal precession.

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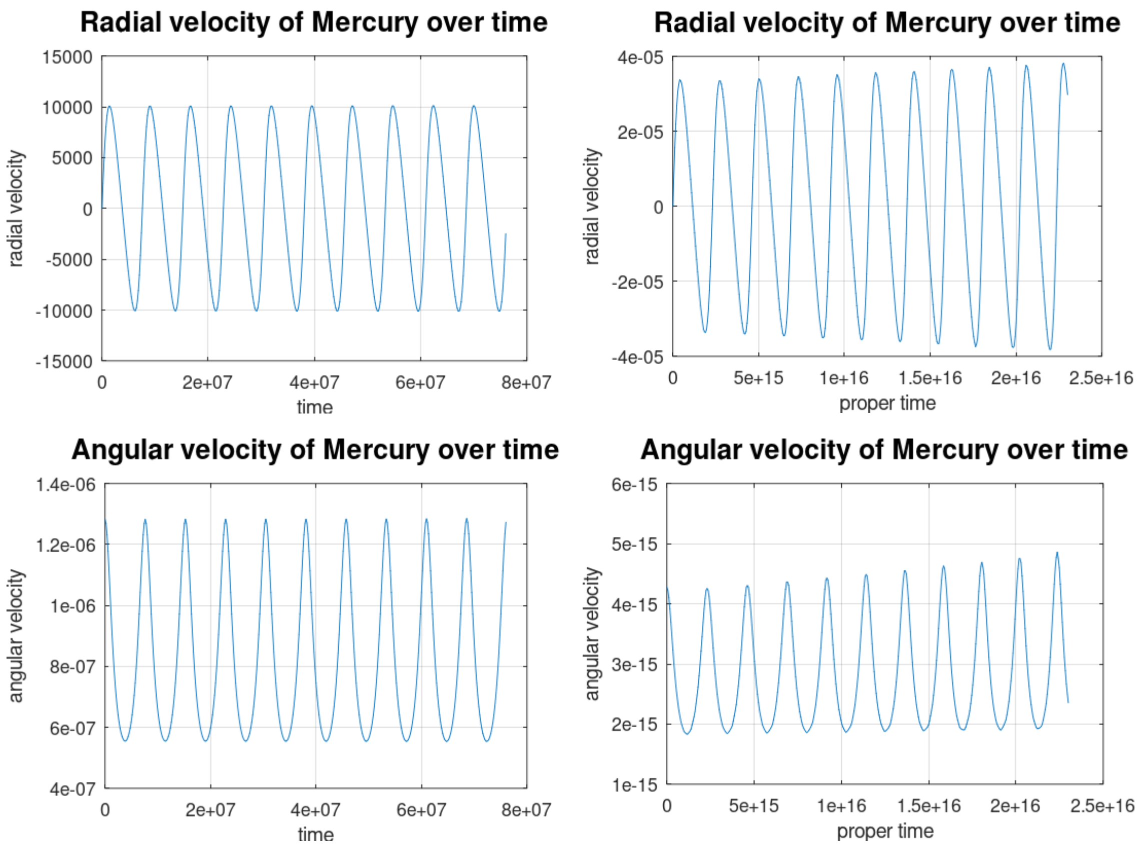
Appendix A:

Extra graphs of radial and angular velocity of orbit of Mercury, Newtonian (left), Relativistic (right)



Appendix B:

Code for Relativistic MATLAB Solver:

```
%Defines Differential Equation

function dy = Geodesic2(~,y)
 % y(1) = r
 %y(2) = theta -> t
 %y(3) = phi
 %y(4) = rdot
 %y(5) = thetadot -> tdot
 %y(6) = phidot

  rS = 2954.106638;
  dy = [y(4);
  y(5);
  y(6);
  (((-rS/(2*y(1)^2))*(1-(rS/y(1)))*y(5)^2)-((rS*(y(4)^2))/(2*(1-(rS/y(1)))*(y(1)^2)))-((rS-y(1))*(y(6)^2)));
  %-1;
  ((-rS*y(4)*y(5))/(y(1)^2*(1-(rS/y(1)))));
  (((-2/y(1))*y(6)*y(4)));
   ];


end
```

```
clc;
clear;
close all;

% assign f to differential equation
f = @(t,y) Geodesic2(t,y);

% tf = final time
tf = 10*2.3e15;

% Create 400 time-steps between 0 and final time
t_proper = linspace(0,tf,400);

% Define Initial Conditions
y0 = [46e9  0 0 0 1 4.2739e-15];

% implements differential equation solver
[t,y]=ode23s(f,t_proper,y0);

% retrieves the solutions to the differential equation
r = y(:,1);
tSol = y(:,2);
phi = y(:,3);
rdot = y(:,4);
tdot = y(:,5);
phidot = y(:,6);
```

```
% plots each solution variable over time
figure;
plot(t,r);
title('Radius of Mercury over time','FontSize',15);
xlabel('Proper time (m)');
ylabel('Radius (m)');
grid on;


figure;
plot(t,tSol);
title('Time dilation effects of Orbit of Mercury','FontSize',13);
xlabel('proper time (m)');
ylabel('coordinate time (m)');
grid on;

figure;

plot(t,phi);
title('Orbital angle Phi for Mercury as a function of time','FontSize',13);
xlabel('proper time (m)');
ylabel('phi (rad)');

grid on;

figure;
plot(t,rdot);
title('Radial velocity of Mercury over time','FontSize',15);
xlabel('proper time (m)');
ylabel('radial velocity');
grid on;


figure;
plot(t,phidot);
title('Angular velocity of Mercury over time','FontSize',15);
xlabel('proper time (m)');
ylabel('angular velocity');
grid on;
```

```
%plots orbit of Mercury

figure;

r = y(:,1);
phi= y(:,3);

polar(phi,r);
hold on
plot(0,0,'yo', 'markersize',40, 'markerfacecolor','y','markeredgecolor','k');

grid on;
```

Appendix C:

Code for Newtonian MATLAB Solver:

```
% Define differential equation

% y(1) = r
% y(2) = phi
% y(3) = rdot
% y(4) = phidot

function dy = NewtonGravity(~,y)
  M=1.9891e30;
  G = 6.67e-11;
  dy = [y(3);y(4);(y(1)*y(4)^2)-(G*M/(y(1)^2));-2*y(3)*y(4)/y(1)];
end
```

```
clc;
clear;
close all;
% assign f to differential equation
f = @(t,y) NewtonGravity(t,y);

% tf = final time
tf = 10*7.6e6;

% Create 100 time-steps between 0 and final time
t_span = linspace(0,tf,100);

% Define Initial Conditions
y0 = [46e9 0 0 0.00000128217];

% implements differential equation solver
[t,y]=ode23s(f,t_span,y0);


% retrieves the solutions to the differential equation
r = y(:,1);
phi = y(:,2);
rdot = y(:,3);
phidot = y(:,4);
```

```
% plots each solution variable over time
figure;
plot(t,phi);
xlabel('t');
ylabel('y(t)');
legend('r','phi','rdot','phidot');
grid on;

figure;
plot(t,r);
title('Radius of Mercury over time','FontSize',15);
xlabel('time (s)');
ylabel('radius (m)');
grid on;

figure;

plot(t,phi);
title('Orbital angle Phi for Mercury as a function of time','FontSize',13);
xlabel('time');
ylabel('phi');
grid on;

figure;
plot(t,rdot);
title('Radial velocity of Mercury over time','FontSize',15);
xlabel('time');
ylabel('radial velocity');
grid on;


figure;
plot(t,phidot);
title('Angular velocity of Mercury over time','FontSize',15);
xlabel('time');
ylabel('angular velocity');
grid on;
```

```
%plots orbit of Mercury

figure;


polar(phi,r);
hold on
plot(0,0,'yo', 'markersize',40, 'markerfacecolor','y','markeredgecolor','k');


grid on;
```