Estimating Varying Parameters for Stock Return and of CAPM

A Data Analyst and Econometrician's View

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AMS 520
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October 18, 2023

The Objective I

A close look at model specification, data structure, and estimation

- Estimating βs in Fama-French (1992, pp. 430 - 431)
- Estimating μs in Chitsiripanich et al. (2022, Table 1)

They don't have explicit model specifications. Some lecture notes have different model specification, which can cause confusion and misleading. In conclusion, for estimating Fama-French (1992)'s βs,

(1) **Brunnermeier (2015, Princeton Univ.)'s lecture note is a correct one**
(2) Different notations and conventions among Panel data econometric modeling, Statistical factor analysis, and Seemingly unrelated regression.

The Objective II

Modeling and estimating varying intercepts and/or slopes

- **In cross-sectional time-series or multi-level setting**
  - **Mixed model** allows random components
  - Andrew Gelman (Columbia University)'s work is famous
  - E.g. Kim and Montgomery (2012)'s work

- In the setting of either time-series only or cross-section only
  - Usually not estimable but possible with imposing more structure.
  - Bayesian: Hierarchical structure with hyper-parameters
  - State-space model: Hidden state

Literature Review

Estimating factor models is still an open question

- Connor, Hagmann, and Linton (2021): Semiparametric estimation of the Fama-French model
- Fama and French (2015): A five-factor asset pricing model
- Fama and French (2004): The capital asset pricing model: Theory and evidence
- Bai (2003): Inferential theory of factor models of large dimensions

Distinguish two different types of factors

- Observable factors
- Unobservable factors
  - **Statistical factor analysis** is to estimate unobservable factors.

Notations: A Review

For Stock Return (**r**)

- **Time-series data**: r_t, $t = 1, 2, \cdots, T$ (e.g. monthly).
- **Cross-sectional**: r_j, $j = 1, \cdots, J$. (e.g. portfolio)
- **Panel data**: $r_{j,t}$, $j = 1, \cdots, J$ and $t = 1, 2, \cdots, T$.
- **Multi-level**
  - $r_{j,i}$ Stock-level, Portfolio-level, and so on.
  - $r_{k,t}$ Daily, Monthly, and so on.
- **Multi-level cross-sectional time-series**: e.g. $r_{j,i,t}$

Estimating β_j in Fama-French (1992, pp. 430 - 431)

For each β,

- **Data**: A sample of 330 time series $(t = 1, 2, \cdots, 330)$.
- **Model**: $r_t = \alpha + \beta r_t^m + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma^2)$.

$$\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_T \end{bmatrix} = \begin{bmatrix} 1 & r_1^m \\ 1 & r_2^m \\ \vdots & \vdots \\ 1 & r_T^m \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_T \end{bmatrix}$$

  where r_t is excess return at time t.

- **Estimation**: OLS

Repeat it for all the portfolios $j = 1, 2, \cdots, J$. (And then use $\hat{\beta}_{j,OLS}$ as that of each of the stocks in the portfolio j for the next estimation procedure).

(1) Lecture note 1

Estimate the CAPM time-series regression:

$$R^e_{it} = \alpha_i + \beta_{iM} R^e_{Mt} + \varepsilon_{it}$$

(2) Lecture note 2

Stage 1: Use *time series* data to obtain estimates for each individual stock's β^j

$$R^j_t - R^f = \alpha + \beta^j (R^m_t - R^f_t) + \epsilon^j_t$$

Comparisons of Model Specifications and Data Structures

Data: Cross-sectional time-series data $t = 1, 2, \cdots, T, \quad j = 1, 2, \cdots, J$
Model Specifications:

- Model I: Same intercept and same slope

$$r_{j,t} = \alpha + \beta r_t^m + \varepsilon_{j,t}$$

- Model II: Different intercept and different slope

$$r_{j,t} = \alpha_j + \beta_j r_t^m + \varepsilon_{j,t}$$

Model I: Same intercept and slope model

Model: $r_{j,t} = \alpha + \beta r_t^m + \varepsilon_{j,t}, \quad t = 1, 2, \cdots, T, \quad j = 1, 2, \cdots, J.$

$$\begin{bmatrix} r_{1,1} \\ r_{1,2} \\ \vdots \\ r_{1,T} \\ r_{2,1} \\ r_{2,2} \\ \vdots \\ r_{2,T} \\ \vdots \\ r_{J,1} \\ r_{J,2} \\ \vdots \\ r_{J,T} \end{bmatrix} = \begin{bmatrix} 1 & r_1^m \\ 1 & r_2^m \\ \vdots & \vdots \\ 1 & r_T^m \\ 1 & r_1^m \\ 1 & r_2^m \\ \vdots & \vdots \\ 1 & r_T^m \\ \vdots & \vdots \\ 1 & r_1^m \\ 1 & r_2^m \\ \vdots & \vdots \\ 1 & r_T^m \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,1} \\ \varepsilon_{1,2} \\ \vdots \\ \varepsilon_{1,T} \\ \varepsilon_{2,1} \\ \varepsilon_{2,2} \\ \vdots \\ \varepsilon_{2,T} \\ \vdots \\ \varepsilon_{J,1} \\ \varepsilon_{J,2} \\ \vdots \\ \varepsilon_{J,T} \end{bmatrix}$$

Model II: A varying intercepts and slopes model

Model: $r_{j,t} = \alpha_j + \beta_j r_t^m + \varepsilon_{j,t}$

In matrix form, for a given j

$$\mathbf{r}_j = \alpha_j \iota_T + \beta_j \mathbf{r}^m + \varepsilon_j$$

where $\mathbf{r}_j = [r_{j,1}, r_{j,2}, \cdots, r_{j,T}]'$, $\iota_T = [1, 1, \cdots, 1]'$, $\mathbf{r}^m = [r_1^m, r_2^m, \cdots, r_T^m]'$.
For all the data,

$$\mathbf{r} = (I_J \otimes \iota_T)\alpha + (I_J \otimes r^m)\beta + \varepsilon$$

where $\alpha = [\alpha_1, \alpha_2, \cdots, \alpha_J]'$ and $\beta = [\beta_1, \beta_2, \cdots, \beta_J]'$. That is,

$$\begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_J \end{bmatrix} = \begin{bmatrix} \iota_T & 0 & \cdots & 0 & r^m & 0 & \cdots & 0 \\ 0 & \iota_T & \cdots & 0 & 0 & r^m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & \iota_T & 0 & 0 & \cdots & r^m \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_J \end{bmatrix}$$

Estimating μs in Chitsiripanich et al. (2022, Table 1, p.14)

For each sample mean μ (e.g. Excessive Return, alpha, $\cdots$),

- **Data**: $\{y_t\}$ from Jan. 1973 to Dec. 2020.
- **Model**: $y_t = \mu + \varepsilon_t, \;\; \varepsilon_t \sim N(0, \sigma^2)$.
- **Estimation**: OLS (Ordinary Least Square)

Repeat it for each of the portfolios $j = low, 2, 3, 4, high, high - low$.

To estimate portfolio-specific μ, my recommendation is to use the mixed model, $y_{j,t} = \mu + u_j + \eta_{j,t},\; u_j \sim N(0, \sigma_u^2),\; \eta_{j,t} \sim N(0, \sigma_\eta^2)$.

- Utilize the full sample, not losing the degree of freedom.
- Take into account correlation among cross-sectional units.
- Flexible modeling: For example, it can quantify the differences among portfolios with a set of dummy variables
- Estimation: GLS, Bayesian (Shrinkage happens).
- $\mathbf{y} = (\iota_T \otimes \iota_J)\mu + (\iota_T \otimes I_J)\mathbf{u} + \eta$

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