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Mathematical Aspects of Relativity – I

Special Relativity

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Contents

Disclaimer

I am grateful to my Special and General Relativity professors at Imperial College London for teaching the subject with depth and rigour. I am also grateful to the professors at University of Cambridge and Dr. Dexter Chua who have shared their precious notes on the web from which I have studied these subjects in ever more depth (I may not be allowed to share them here, however you can also find them for your personal usage and preparation using a simple Web Search).

I have tried my best to make the notes self contained and easily accessible to anyone with basic knowledge of first year undergraduate Mathematics and Physics. Many concepts which could have been included in the first two chapters but are not directly used in further are omitted, so the reader can focus on specific contents required.

Further, I have attempted to avoid any bias in my writing in regards to copying from what I have learnt at my MSc tenure as well as from the resources I have prepared from, yet the notes may reflect parts of language according to my learnings. Even then, all errors and mistakes are almost surely mine.

Recommended Resources and Books

The first set of resources are from YouTube for absolute beginners.

- Intro to Special Relativity Course by minutephysics
- Special relativity by Khan Academy Physics
- MIT 8.20 Introduction to Special Relativity, January IAP 2021 by MIT OpenCourseWare

The following two are lecture series by Prof Leonard Susskind on stanford YouTube channel,they are primarily recommended in accordance with my frequency of understanding the subject and my preparation. In my understanding, these two are better for students who have a decent grasp of undergraduate level mathematics. If you can smoothly sail through these, the notes mostly complement these lectures for beginners.

- Classical Mechanics
- Special Relativity

Finally, the formal set of recommendations - the Books. As this subject is a century old and extremely famous, there is a plethora of good and amazing books on it. The following is the combination of books suggested by my professors at ICL and Professor David Tong of University of Cambridge in his notes.

Following book is for the preparation or revision on the fundamentals of Newtonian mechanics.

- Douglas Gregory, Classical Mechanics

Next two books are for the students who want to go in further depth with the Lagrangian and Hamiltonian formalisms. Book by Landau and Lifshitz is really concise and discuses the overview, whereas, Goldstein is rather for the absolute pros.

- L. Landau and E. Lifshitz, Mechanics
- H. Goldstein, C. Poole and J. Safko, Classical Mechanics

The classical theory of field, suggested by my Special Relativity professor, is the second part of the series by Landau, and builds on the topics discussed in the first part - Mechanics towards Special Relativity (Electromagnetism and General Relativity if you wish to go further). The book by French is suggested by Prof Tong and I personally find it really beautifully written with all the diagrams, the figures and images of actual researchers and experiments.

- L. Landau, The Classical Theory of Fields
- A. French, Special Relativity

1. Mathematical Preliminaries

As you will observe throughout this course, we will heavily rely on different Mathematical objects in order to comprehend the physical ideas discussed. Why do we do that? Because, language can be vague and be subject to interpretation. Whereas, mathematical concepts are definitive and based on structured axioms, theorems and corresponding proofs. Hence, we will begin by learning a few fundamental concepts required to understand the ideas discussed in Special Relativity.

As said in the declaimer, these notes are meant to be self contained up to the level of first year undergraduate STEM degree. Hence, the following preliminary material is provided as these concepts are utilised to understand topics in chapters 3-7. However, if you already have a clear idea of what is being discussed, feel free to begin with chapter 2.

1.1 Review of Linear Algebra

1.1.1 Euclidean Vectors

Why Study Them?

Five kilogram of sugar, two litres of water, one second. What is common here? These quantities such as time, mass, volume, temperature, are all measured just with their values (called their magnitude). They are known as scalar. A scalar in simplest terms is an entity that you can talk about without invoking the question where it came from or where it is going (i.e., its direction). Five kilogram of sugar would be exactly the same amount whether it came from North or East or if it goes in any direction.
Whereas, there are certain quantity for which it becomes absolutely necessary to ask the question of direction. However, with the direction we still have to know its magnitude. That is we now have to know of two concepts related to that physical quantity, i.e., its direction and magnitude.

A professor walked 20 meter in the east. Here, 20 meter is the distance that professor walked (as distance has only some magnitude it is a scalar) and that distance when combined with direction (that is east here) becomes the displacement of that professor. As displacement has both direction and magnitude, it is a vector. A bike is going towards north with 20 km per hour speed. Here, 20 kmph is called the speed of that bike which is scalar. Speed combined with the direction makes the velocity, which is a vector. We threw a ball upwards with 0.2N force. Here we have magnitude of the force (0.2N) as well as the direction "upward", that makes Force a vector. Other examples include Electric and Magnetic fields, direction of the propagation of a wave. Hence, it becomes important to mathematically describe vectors so that we can perform operations on them and manipulate these quantities.

Definition (Vector)**.** *A quantity that is specified by a magnitude (value) and a direction in (2-D/3-D) space is called a vector.*

Remarks.

- Vectors are generally represented in bold, i.e., $\mathbf{F}$ for force vector.
- Magnitude of the vector is represented with normal font or in modules, i.e., F or $|\mathbf{F}|$ for the force.

Geometric Representation

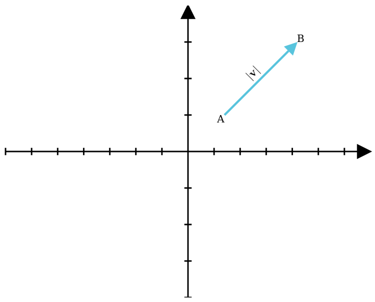

Figure 1: Vector $\mathbf{v}$ as line segment $\overrightarrow{AB}$

On Cartesian Coordinate Plane (2D) or Space (3D) a vector $\mathbf{v}$ is represented with line segment say $\overrightarrow{AB}$. Its direction is considered from one point to other, say from point A to point B just like the direction of a line segment. Whereas the magnitude $|\mathbf{v}|$ is represented by length of the line segment.

With a clear idea of what scalars and vectors are, let us now understand the corresponding fields.

Definition (Scalar Field)**.** *A scalar that is function of position is called a scalar field.*

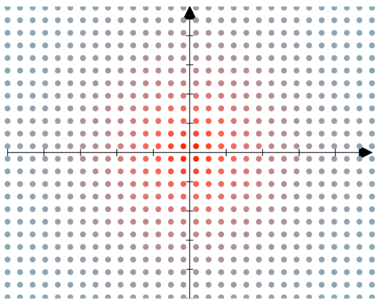

Figure 2: Temperature field as a function of distance from the boiler

In mathematical terms the scalar field is a function that is $f : \mathbb{R}^3 \to \mathbb{R}$. That is it takes vector valued entries and gives out a scalar value.
An example of such function is speed (not velocity) $s \equiv s(x, y, z)$ of water near a waterfall. As you move around the edge of the fall, the speed changes. But as speed just has a value and no direction, hence it is a scalar field.
Another example is temperature of a plate as it is boiled at the centre from below. The temperatures are higher near the boiler and they change the value with the change in distance. But there is no sense of direction.

Definition (Vector Field)**.** *A vector that is function of position is called a vector field.*

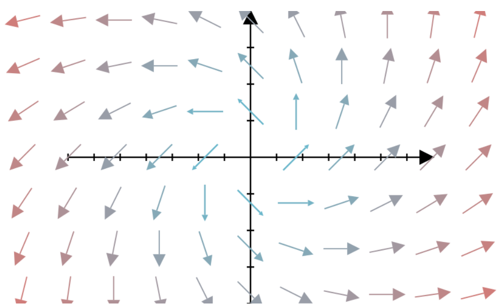

Figure 3: Magnetic Field Proportional to the Position

In mathematical terms the scalar field is a function that is $f : \mathbb{R}^3 \to \mathbb{R}^3$. That is it takes vector valued entries and gives out a vector outcome.

An example of vector field is the Magnetic field. The components of magnetic field are $(B_x, B_y) = (x \cdot \sin(45°))$

Properties(Addition and Multiplication by scalar)

1.1.2 Vector Spaces

Algebraic Definition Properties

1.1.3 Important Topics

Scalar Product
Orthogonal Bases
Vector Component Identities
Higher Dimensions
Suffix Notation
Scalar Product in suffix notation
Summation convention Kronecker delta
Matrices
Matrix vector multiplication
Multiplication of matrices

1.2 Review of Calculus in Euclidean Space

1.2.1 Derivatives and Integrals

1.2.2 Overview of Differential Equations

2. Review of Mechanics

2.1 Historical Development of Classical Dynamics

From Indian and Greek to medieval European, different civilisations. Kepler to Galileo to Newton - Physics development. Descartes - Mathematical development.

2.2 Newtonian Mechanics

2.2.1 Laws of Motion

2.3 Lagrangian

2.4 Hamiltonian (Brief Overview)

3. Historical Development and Motivation for Relativity

3.1 Galilean relativity

3.2 Maxwell's equations

3.3 Michelson-Morley experiment

3.4 Lorentz and Einstein

3.5 Minkowski spacetime

3.6 Need for a new theory: Motivation for special relativity

4. Postulates and Foundations

4.1 Postulates of Relativity

N-2 + Speed of Light

4.2 Derivations of Lorentz Transformation

5. Key Concepts

5.1 Length Contraction

5.2 Time Dilation

5.3 Addition of Velocities in SR

5.4 Relativistic Doppler Effect

6. Four-Vectors and Concepts of Spacetime

6.1 Introduction to Four-Vectors

6.1.1 Euclidean vectors vs. four-vectors

6.1.2 Transformation properties

6.2 Concept of Spacetime

6.3 Metric and Its Invariance

6.3.1 Three Types of Intervals

6.4 Worldlines and Light Cones

7. Advanced Topics

7.1 SR Lagrangian

7.2 SR Equation of Motion

7.3 Momentum in Special Relativity

7.4 $E = mc^2$ Mass-Energy Equivalence

7.5 Conservation of Energy and Momentum

7.6 4-Vectors and Examples

7.7 Calculus in Special Relativity