Mini Project 01

Mini Project Group 16

Group Members

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Contribution: The three team members collaborated seamlessly, each putting in an equal amount of effort to complete the assigned mini project.

Q1(A)

According to the question the following variables are considered:

X_A = Lifetime of Block A

X_B = Lifetime of Block B

T = Lifetime of the satellite

It is given that XA and XB follow independent exponential distributions with mean 10.

Thus, E(T) = 1/λ = 10 and hence **λ = 0.1 years.**

We know that the lifetime of the satellite will be till at least one of them works.

Thus, the lifetime of the satellite T = max (XA, XB)

Since it is given that the lifetimes follow independent exponential distributions, we can conclude,

$$F_T(t) = F_{XA}(t) * F_{XB}(t)$$
$$= (1 - \exp^{(-0.1t)})^2$$
$$= 1 - 2\exp^{(-0.1t)} + \exp^{(-0.2t)} \qquad \text{............ (1)}$$

Now to find the pdf we need to differentiate the equation (1)

Thus, **fT (t) = F_T'(t)**

$$\boxed{f_T(t) = \begin{cases} 0.2\ exp^{(-0.1t)} - 0.2\ exp^{0.2t}, & 0 \leq t < \infty \\ 0. & otherwise \end{cases}}$$

Now the expected lifetime of the satellite is,

$$E(t) = \int t f T(t) = \int_0^\infty (0.2exp^{(-0.1t)} - 0.2exp^{(0.2t)} dt$$

Solving the integral we get,
E(T) = 0.2(100-25)

E(T) = 15 years

Now using the density function given by equation (1), we need to compute the probability that the lifetime of the satellite exceeds 15 years.

P(T>15) = 1- P (T ≤ 15)

So, let us the find the probability of P (X ≤ 15) first,

$$\int_0^{15} f(T)\, dt$$
$$= \int_0^{15} (0.2\exp(-0.1t) - 0.2\exp(-0.2t))\, dt$$
$$\left(Because \int exp(nx)dx = \left(\frac{1}{n}\right)\exp(nx) + c\right)$$
$$= [(0.2/-0.1) *\exp((-0.1)t) - (0.2/-0.2) *\exp((-0.2)t)]_0^{15}$$
$$= \left[(-2) * \exp\left((-0.1)t\right) + (1) * \exp\left((-0.2)t\right)\right]_0^{15}$$
$$= [(-2) * \exp((-0.1)t) + (1) * \exp((-0.2)t)]_0^{15}$$
$$= \left[\exp\left((-0.2)t\right) - 2 * \exp\left((-0.1)t\right)\right]_0^{15}$$
$$= \left(\exp\left(-0.2(15)\right) - 2 * \exp\left((-0.1) * 15\right)\right) - \left(\exp\left(-0.2(0)\right) - 2 * \exp\left((-0.1) * 0\right)\right)$$
$$= (\exp(-3) - 2 * \exp(-1.5)) - (\exp(0) - 2 * \exp(0))$$
$$= (\exp(-3) - 2 * \exp(-1.5)) - (\exp(0) - 2 * \exp(0))$$
$$= (-0.3964) - (1 - 2)$$
$$= (-0.3964) - (-1)$$
$$= (-0.3964) + 1$$
$$= 1 - 0.3964$$
$$= 0.6036$$

Thus, P (T ≤ 15) = 0.6036
So, P(T>15) = 1 – P (T ≤ 15)
= 1- 0.6036
= 0.3964
Hence,
The probability that the lifetime of the satellite exceeds 15 years

P (T > 15) = 0.3964

Q1(B)

Now we will use Monte Carlo approach to compute E(T) and P(T>15).

(I & II)

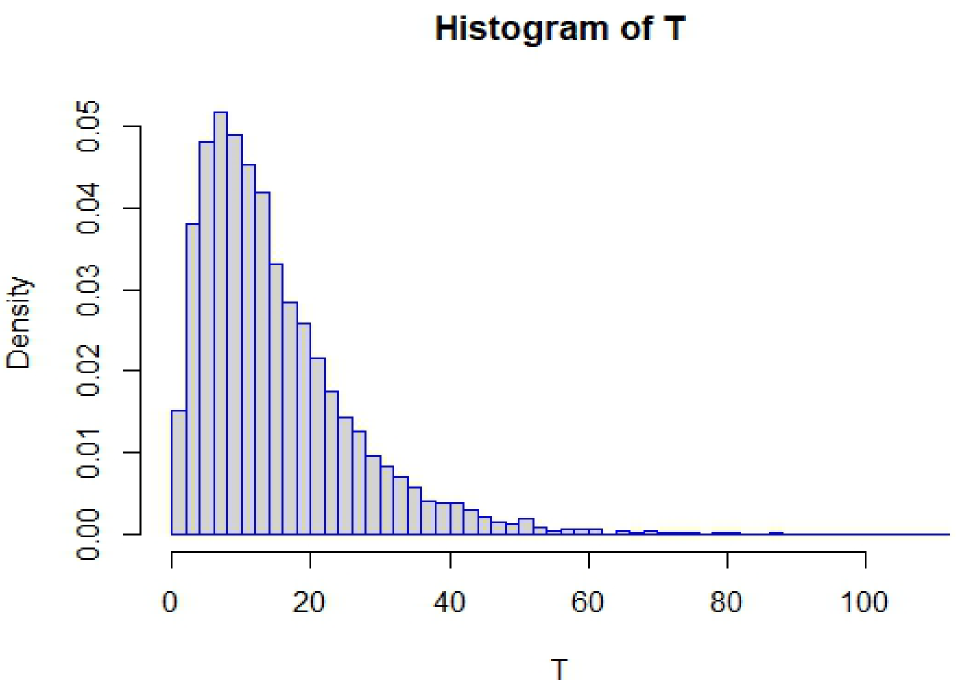
To simulate exponential distribution, we use the function rexp (n, rate) in R. As per the question, the rate is 0.1 (λ=1/10) and T is the max of the lifetimes of the two satellites. After that we need to repeat the above step 10,000 times; so, we use the replicate function.

```
> T = replicate (10000, max (rexp (1,0.1), rexp (1,0.1)))
```

(III)

Now to get the histogram drawn below of the draw T, we use the hist() function in R.

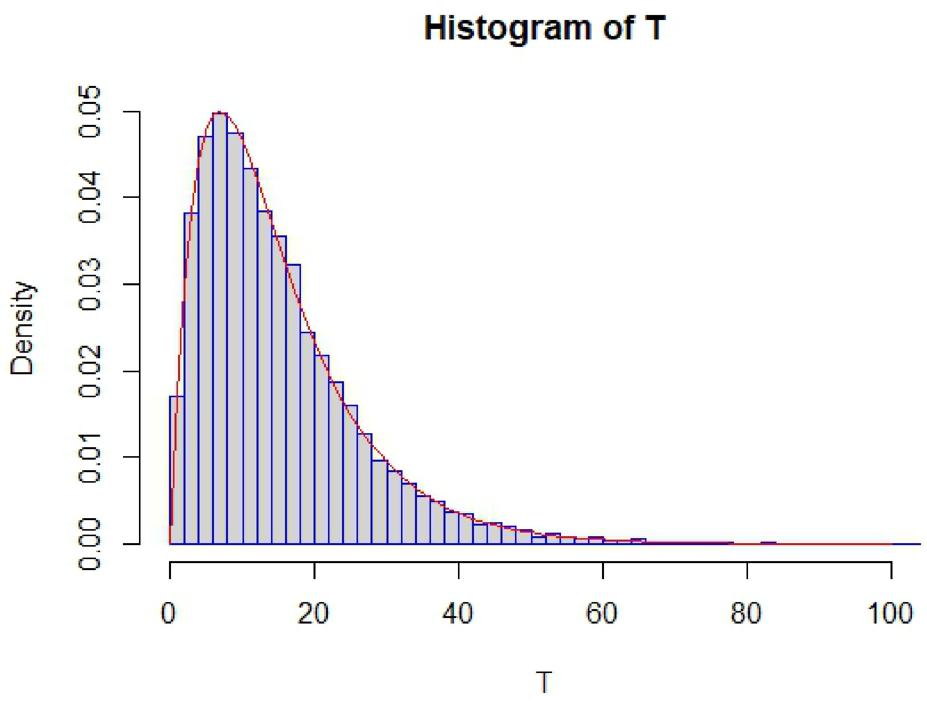
```
> hist (T, breaks=50, prob=TRUE, border="blue")
```



To superimpose the density function calculated in equation 1, we use the curve() function in R.

```
> curve (0.2*exp(-0.1*x)-0.2*exp(-0.2*x),
from=0, to=100, col ="red", add = TRUE)
```

Thus, the density function superimposed on the histogram will look like,



(IV)

Now to calculate the estimated value of T (i.e. E(T)), we will use the mean() function in R on the T variable. This will give the estimated lifetime of the satellite.

```
> mean(T)
[1] 14.96173
```

So, we get the estimated lifetime value as 14.96173, which is approximately equal to the value which we computed before i.e., 15.

(V)

Now we will calculate the probability of the lifetime of the satellite that exceeds 15 years. That is, we need to calculate P (T>15). To calculate this, we will have to count all the exponential random observations having values greater than 15 and then we will divide this by the sample size which is the length of the observations. To count the random observations we will use the sum() and for length of the observation we will use length() function in R.

```
> sum (T > 15)/ length(T)
[1] 0.3954
```

Thus, we can see that the probability of the lifetime of the satellite that exceeds 15 years is 0.3954 which is equal to the value we obtained theoretically.

(VI)

Now we are repeating the above process of getting the estimate of E(T) and the estimate of the probability P (T>15) four more times and we are creating a table which list down all these values.

Sr No	E(T)	P (T>15)
1	14.90355	0.3957
2	14.83966	0.3933
3	14.87017	0.3953
4	14.87768	0.3889

We can see that each time we are getting different values for our estimation. But these values differ basically by a small amount. They are approximately equal to the value which was calculated theoretically.

Q1(C) Now we repeat all the above steps five more times using 1,000 and 100,000 Monte Carlo replications instead of 10,000. Then we will present the observation onto a table.

1. Using 1,000 Monte Carlo replications

Sr No	E(T)	P (T>15)
1	14.72312	0.386
2	15.00746	0.388
3	14.34775	0.364
4	15.00815	0.391
5	14.67979	0.376

2. Using 100,000 Monte Carlo replications

Sr No	E(T)	P (T>15)
1	14.9763	0.39512
2	14.9986	0.39775
3	14.9910	0.39718
4	15.00960	0.39655
5	14.93427	0.39279

By looking at the table we can see that the number of replications does not change the mean of T in a significant manner.

R Code:

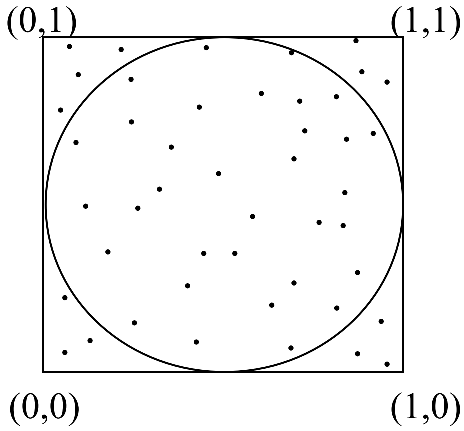
```
T = replicate (10000, max (rexp (n=1, rate=0.1), rexp (n=1, rate=0.1)))
hist (T, breaks=50, prob=TRUE, border="blue")
curve (0.2*exp(-0.1*x)-0.2*exp(-0.2*x), from=0, to=100, col ="red", add = TRUE)
mean(T)
sum(T>15)/length(T)


T = replicate (1000, max (rexp (n=1, rate=0.1), rexp (n=1, rate=0.1)))
mean(T)
sum(T>15)/length(T)


T = replicate (100000, max (rexp (n=1, rate=0.1), rexp (n=1, rate=0.1)))
mean(T)
sum(T>15)/length(T)
```

Q2

Given a circle with center (0.5, 0.5) and coordinates of a unit square – (0,0), (0,1), (1,0), and (1,1).



If we plot uniformly distributed random points on the unit square, we will observe that some of the points may lie inside the circle, while some of the points will lie out of the circle.

We know that the area of square is $4x^2$ where x is the value of side.
And Area of circle = πx^2 where x is the radius of the circle.

Hence, the probability that a randomly selected point in this unit square falls in the circle is:

$$P(X) = \frac{Area\ of\ the\ Circle}{Area\ of\ the\ Square} = \frac{\pi r^2}{a^2}$$

Because we need to find the proportion of the points falling inside the circle to the total number of the points inside the square.

a) Suppose X is the event that a randomly distributed number which is in the square falls inside the circle. Radius of the circle = 0.5; Length of the side of the square = 1

$$P(X) = \frac{\pi * (0.5)^2}{1^2}$$

$$P(X) = \frac{\pi}{4}$$

$$\boldsymbol{\pi = 4 * P(X)} \qquad \textbf{.... (1)}$$

Now, we know that the general equation for a circle is $(x - h)^2 + (y - k)^2 = r^2$, where (h, k) is the center and r is the radius. So, the equation for the above circle is:-
$(x - 0.5)^2 + (y - 0.5)^2 = 0.5^2$
Any point that lies within this circle will satisfy the Equation:-

$$\sqrt{(\mathbf{x} - \mathbf{0.5})^2 + (\mathbf{y} - \mathbf{0.5})^2} \leq \mathbf{0.5}^2 \text{ --- } \mathbf{(2)}$$

We can now use Monte Carlo approach to estimate the value of π by generating 10,000 random values for coordinates x and y that lie inside the square in R using runif(n,0,1) command.

```
> Xcord <- runif (10000)
> Ycord <- runif (10000)
```

We can substitute these coordinates in equation (2) to find all the points within the circle. P(X) is obtained by dividing the count of those coordinates by the total number of coordinates (10,000).

```
> P = sum(sqrt((Xcord-0.5)^2 + (Ycord-0.5)^2) <=
0.5)/10000
```

Then use equation (1) to get value of π.

```
> PI = P * 4
> PI
[1] 3.1404
```

As per the simulation, the value of **π** is 3.1404.
The correct value of π = 3.141593.
We will get a more precise value if the value of replications is large.

R Code:

```
Xcord <- runif (10000)

Ycord <- runif (10000)

p <- sum(sqrt((Xcord-0.5)^2 + (Ycord-0.5)^2) <=
0.5)/10000

PI <- 4*p
```