26th sept:

we had seen strong parito $\Rightarrow$ weak parito

eg: R:
$$\begin{matrix} a & a & c & d \\ b & c & b & c \\ c & b & a & b \\ d & d & d & a \end{matrix}$$
R':
$$\begin{matrix} d & c & b & b \\ a & a & c & a \\ b & b & a & d \\ c & d & d & c \end{matrix}$$

Relative position b/w a and b in R, R' are same for every player

so aggregate should also match pref order

Defn: $R_i, R_i' \in \mathcal{R}$ agree on $\{a,b\}$ if for agent i:
$a P_i b \Leftrightarrow a P_i' b$, $b P_i a \Leftrightarrow b P_i' a$, $a I_i b \Leftrightarrow a I_i' b$

we use $R_i|_{a,b} = R_i'|_{a,b}$ to denote for i
if $\forall i \in N \Rightarrow R|_{a,b} = R'|_{a,b}$

Defn: (Independence of irrelevant Alternative) An ASWF F satisfies IIA if for all $a,b \in A$ and for every pair $R|_{a,b} = R'|_{a,b}$ then $F(R)|_{a,b} = F(R')|_{a,b}$

one way to have F is to have scoring rule based mechanism

eg:
$$\begin{array}{cc} s^1 & 3 \\ s^2 & 2 \\ s^3 & 1 \\ s^4 & 0 \end{array}\begin{bmatrix} a & a & c & d \\ b & c & b & c \\ c & b & a & b \\ d & d & d & a \end{bmatrix}$$
$a = 3+3+1+0 = 7$ score

eg: election: top most -1, all else 0 i.e. plurality rule

Note: plurality is not satisfying IIA:

$a F_{plu}(R) b$ is there as $a \to 2$, $b \to 0$ for first case

$a F_{plu}(R') b$ is not there as $b \to 2$, $a \to 0$ so $b F_{plu}(R') a$

Defn: Dictatorship is $\exists$ predetermined agent d, then $F^d(R) = R_d$

Theorem: (Arrow's Impossibility result) For $|A| \geq 3$, if an ASWF F satisfies WP and IIA, then it must be dictatorial

Defn: (Decisiveness) let $F: \mathcal{R}^n \to \mathcal{R}$ be given, $G \subseteq N$, $G \neq \phi$ a is a group,
G is almost decisive over $\{a,b\}$ if $\forall R$,
$[a P_i b \ \forall i \in G, \ b P_j a \ \forall j \in N \setminus G] \Rightarrow [a \hat{F}(R) b]$
we write this as $\bar{D}_G(a,b)$: "G is almost decisive over $\{a,b\}$ wrt F"

G is decisive over $\{a,b\}$ if $\forall R$:
$[a P_i b, \forall i \in G] \Rightarrow [a \hat{F}(R) b]$ we call this $D_G(a,b)$

Note: as $[a P_i b, \forall i \in G, b P_j a, \forall j \in N \setminus G] \Rightarrow [a P_i b, \forall i \in G]$
so, $D_G(a,b) \Rightarrow \bar{D}_G(a,b)$

The theorem will be solved in two parts using two lemmas, both assuming WP & IIA

Part 1: Field expansion lemma: If a group is almost decisive over a pair of alternatives it is decisive over all pair of alternatives

Part 2: Group contraction lemma: If a group is decisive, then a strict non-empty subset of that group is also decisive

we can see if F is IIA and WP, then $\forall R$, $G = N$ is s.t
$[a P_i b, \forall i \in N] \Rightarrow [a \hat{F}(R) b]$ from WP

and so $\bar{D}_N(a,b)$ occurs, so we can use field expansion lemma to show that $\forall a,b \in R$, $D_N(a,b)$, and then we can use group contraction lemma to show that $\exists G \neq \phi$ s.t $G \subsetneq N$

$\vdots$

we repeat to have $G = \{i\}$ for some i s.t $D_{\{i\}}$ and so then arrows theorm follows

i.e $\forall a,b$, if $[a P_i b] \Rightarrow [a \hat{F}(R) b]$ $\forall R$ i.e F is ditorial
or $F^d(R) = R_i \Rightarrow i$th player

Lemma: (Field expansion lemma) Let F satisfy WP and IIA, then $\forall a,b,x,y, G \subseteq N$
$G \neq \phi$, $a \neq b$, $x \neq y$
$\bar{D}_G(a,b) \Rightarrow D_G(x,y)$

proof: cases to consider:

1. $\bar{D}_G(a,b) \Rightarrow D_G(a,y)$, $y \neq a,b$
2. $\bar{D}_G(a,b) \Rightarrow D_G(x,b)$, $x \neq a,b$
3. $\bar{D}_G(a,b) \Rightarrow D_G(x,y)$, $x \neq a,b$ $y \neq a,b$, $x \neq y$
4. $\bar{D}_G(a,b) \Rightarrow D_G(x,a)$, $x \neq a,b$
5. $\bar{D}_G(a,b) \Rightarrow D_G(b,y)$, $y \neq a,b$
6. $\bar{D}_G(a,b) \Rightarrow D_G(a,b)$
7. $\bar{D}_G(a,b) \Rightarrow D_G(b,a)$

case I: $\bar{D}_G(a,b) \Rightarrow D_G(a,y)$, $y \neq a,b$, as $\bar{R}_G(a,b)$, $[a P_i b, \forall i \in G, b P_i a, \forall i \in N \setminus G] \Rightarrow [a \hat{F}(R) b]$

pick any $R \in \mathcal{R}^n$, s.t $a P_i y$, $\forall i \in G$, we have to show $a \hat{F}(R) y$

now let R' be s.t

G		$N \setminus G$	
a	a	b	b
b	b	a	y
y	y	y	a

change this so b above y

ie positions of a and y in $N \setminus G$ s.t $R'|a,y = R|a,y$

we change this position to make $R'|a,y = R|a,y$

now, $\bar{D}_G(a,b) \Rightarrow a \hat{F}(R') b$, from WP over $b,y \Rightarrow b \hat{F}(R') y$

now as $a \hat{F}(R') b$ and $b \hat{F}(R') y$
$\Rightarrow a \hat{F}(R') y$

now, as $R'|a,y = R|a,y$ from construction
from IIA we get
$a \hat{F}(R) y$

so, $\forall R \in \mathcal{R}^n$ s.t $a P_i y$, $\forall i \in G$ we get
$a \hat{F}(R) y$ so, $D_G(a,y)$

case II: $\bar{D}_G(a,b) \Rightarrow D_G(x,b)$, $x \neq a,b$
any $R \in \mathcal{R}^n$ s.t $x P_i b$, $\forall i \in G$ then we have to show $x \hat{F}(R) b$
we construct R' s.t:

G		$N \setminus G$	
x	x	x	b
a	a	b	x
b	b	a	a

switch position to make $R'|x,b = R|x,b$, then
$\bar{D}_G(a,b) \Rightarrow a \hat{F}(R') b$
WP on $x,a \Rightarrow x \hat{F}(R') a$
and so $x \hat{F}(R') b \Rightarrow x \hat{F}(R) b$ from IIA and so $D_G(x,b)$

Case 3: $\bar{D}_G(a,b) \Rightarrow D_G(a,y)$ from ① when $y \neq a,b$
$\Rightarrow \bar{D}_G(a,y)$ from defn
$\Rightarrow D_G(x,y)$ from ② when $x \neq a,y$

Case 4: $\bar{D}_G(a,b) \Rightarrow D_G(x,b)$ $(x \neq a,b)$
$\Rightarrow \bar{D}_G(x,b)$ $(x \neq a,b)$
$\Rightarrow D_G(x,a)$ $(x \neq a,b)$ from ①

Case 5: $\bar{D}_G(a,b) \Rightarrow D_G(a,y)$ $(y \neq a,b)$ from ①
$\Rightarrow \bar{D}_G(a,y)$ $(y \neq a,b)$
$\Rightarrow D_G(b,y)$ $(y \neq a,b)$ from ②

Case 6: $\bar{D}_G(a,b) \Rightarrow D_G(x,b)$ $(x \neq a,b)$ from ①
$\Rightarrow \bar{D}_G(x,b)$
$\Rightarrow D_G(a,b)$ from ②

Case 7: $\bar{D}_G(a,b) \Rightarrow D_G(b,y)$ $(y \neq a,b)$ from ⑤
$\Rightarrow \bar{D}_G(b,y)$
$\Rightarrow D_G(b,a)$ from ①

Lemma: (group contraction lemma) let F satisfy WP and IIA, let $G \subseteq N$, $G \neq \emptyset$, $|G| \geq 2$, be decisive, then $\exists G' \subset G$, $G' \neq \emptyset$ which is also decisive

proof: G, $|G| \geq 2$ is given
let $G_1 \subset G$, $G_2 = G \setminus G_1$, $G_1, G_2 \neq \emptyset$ arbitrary
then let R be:

G_1	G_2	$N \setminus G$
a	c	b
b	a	c
c	b	a

as aP_ib $\forall i \in G$ and G is decisive
$\Rightarrow a\hat{F}(R)b$

Case I: $a\hat{F}(R)c$
then for G_1: as $a\hat{F}(R)b$, aP_ic $\forall i \in G_1$, cP_ia $\forall i \in N \setminus G_1$
every R' where above relation holds
then by IIA $a\hat{F}(R')c$
so, $\bar{D}_{G_1}(a,c)$ so by FEL D_{G_1}

Case II: $\neg(a\hat{F}(R)c) \Rightarrow cF(R)a$, as $a\hat{F}(R)b$ and $cF(R)a$ we get $c\hat{F}(R)b$
for G_2, cP_ib $\forall i \in G_2$, bP_ic $\forall i \in N \setminus G_2$
for all above such R' by IIA
$c\hat{F}(R')b$ so, $\bar{D}_{G_2}(c,b)$, by FEL D_{G_2}

1st oct:

The social choice setup:

we need a social order from pref profile, arrow's result says impossible
ways out ① consider a social choice setup
② put restrictions on agent preference

social choice fution (SCF)

$$f: \mathcal{P}^n \longrightarrow A$$

strictly linear

$A = \{a_1, a_2, \ldots a_m\}$ finite set of alternatives
$N = \{1, 2, \ldots n\}$ finite set of players
$\mathcal{P}$ set of all linear ordering

eg: voting:

P			
a	a	c	d
b	b	b	c
c	c	d	b
d	d	a	a

$\in$

$\xrightarrow{f} A = \{a, b, c, d\}$

SCF

alternative set

scoring rules:
$(s_1, s_2 \ldots s_m)$, $s_i \geq s_{i+1}$ $i = 1, 2 \ldots m-1$
dec order of scores is final order

plurality $(1, 0, \ldots 0)$
veto: $(1, 1, \ldots 1, 0)$
Borda: $(m-1, m-2, \ldots 1, 0)$
harmonic: $(1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{m})$
K-approval: $(\underbrace{1, 1, \ldots 1}_{K}, 0, 0 \ldots 0)$

Note: plurality with runoff is also called two round system (TRS) when first round - Regular plurality, second rand - only between top two

b is better choice

a	a	a	b	b	c	c
b	b	b	c	c	b	b

maxmin: maximises the minimum lead against other candidate

$$\text{score}(a) = \min_y |\{i : a \succ_i y\}|$$

P			
a	a	c	d
b	b	b	c
c	c	d	b
d	d	a	a

score$(a) = 2$ ← pick this
score$(d) = 1$

copeland: no of wins in pairwise election

Condorcet consistency:

Defn: A voting rule is condorcet consistency if it selects the condorcet winner when one exist

Note: condorcet winner is a candidate who defeates all other in pairwise election

P		
a	b	c
b	c	a
c	a	b

is condorcet paradox

can choose anyting

we want to know which rules are condorcet consistant

Eg:

30%	30%	40%
a	b	c
b	a	a
c	c	b

a – 30
b – 30
c – 40
} c is winner by plurality

a over b : 70%
a over c : 60%
so a is winning in pairwise against all, but above a not winner

Eg.

30%	30%	40%
a	b	c
b	a	a
c	c	b

copeland → yes
maximin → yes

Note: No other scoring rule is condorcet consistent

SCF: $f : P^n \to A$

Defn: (Pareto domination) an alternative a is pareto dominated by b if $\forall i \in N$, $b P_i a$ (a is called pareto dominated if such b exist)

Defn: (pareto efficiency) An SCF f is pareto efficient (PE) if $\forall P$ and $a \in A$, if a is pareto dominated, then $f(P) \neq a$

Defn: (unanimity) An SCF f is unanimous (UN) if $\forall P$ satisfying $P_1(1) = P_2(1) \cdots = P_n(1) = a$ then it holds $f(P) = a$

Pareto efficiency ⇒ unanimity

$$P_1 = \begin{bmatrix} a & \cdots & a \\ \vdots & \vdots & \vdots \end{bmatrix} \qquad P_2 = \begin{bmatrix} a & b & c & \cdots & a \\ d & a & a & \cdots & d \\ & d & a & & \end{bmatrix}$$

$f(P_1) = a$
$f(P_2) = d$
} UN but not PE

Defn: (onto) An SCF is onto (ONTO) if $\forall a \in A, \exists P^{(a)} \in P^n$ s.t. $f(P^{(a)}) = a$

UN ⇒ ONTO

Defn: (Manipulability) an SCF f is manipulable if $\exists i \in N$ and a profile P s.t. $f(P_i', P_{-i}) P_i f(P_i, P_{-i})$ for some P_i'

manipulable truthful

Just changing one vote will change outcome

Eg: plurality with fixed tie-breaking

$a \succ b \succ c$

4	4	1
a	b	c
b	a	b
c	c	a

⇒

4	4	1
a	b	b
b	a	c
c	c	a

winner a → winner b

as $b P_i a$ → this is manipulable

copeland with fixed tie-breaking

$a \succ b \succ c$

1	1	1
a	b	c
b	c	a
c	a	b

a – 3 b – 3
c – 3

⇒

1	1	1
a	c	c
b	b	a
c	a	b

c – 4

as $c P_i a$ this is manipulable

Defn: (strategy proof) An SCF is strategyproof (SP) if it is not manipulable by any agent at any profile

Defn: (monotonicity) An SCF is monotone (MONO) if $\forall$ two profiles P and P' and $f(P)=a$, $D(a,P_i) \subseteq D(a,P_i') \ \forall i \in N$, then $f(P')=a$

eg:

$$\begin{matrix} a & a & c & d \\ b & b & b & c \\ c & c & d & b \\ d & d & a & a \end{matrix} \Rightarrow \begin{matrix} c & a & c & d \\ b & c & b & c \\ a & b & d & b \\ d & d & a & a \end{matrix}$$

order of c goes up or stays same $\forall i \in N$

Dominate set of an alternative

$$D(a,P_i) := \{b \in A \mid a P_i b\}$$

a above b

Theorem: An SCF f is SP iff MONO

proof: $(\Rightarrow)$

P and P' and $f(P)=a$ and $D(a,P_i) \subseteq D(a,P_i') \ \forall i \in N$

let:

$$\underset{P=P(0)}{(P_1, P_2 \dots P_n)} \longrightarrow \underset{P(1)}{(P_1', P_2, \dots P_n)}$$

$\vdots$

$$P(n) = P' = (P_1', P_2', \dots P_n')$$

if we show $f(P^{k}) = a \ \forall k$ we are done,

if not $\exists P(k-1), P(k)$ s.t

$f(P(k-1)) = a$

$f(P(k)) = b \neq a$

this happens when:

case I: $a P_k b$, $a P_k' b$, here $f(P(k)) = b \neq a$, only when we misreport P_k'

case II: $b P_k a$, $b P_k' a$, here $f(P(k)) = b$ only when we misreport P_k

case III: $b P_k a$, $a P_k' b$ here $f(P(k)) = b$ only when we misreport both P_k, P_k'

so in all the cases we need misreporting, this is a contradiction to SP

$(\Leftarrow)$ we can show $\sim SP \Rightarrow \sim MONO$

now if $\sim SP$ but MONO then

$\sim SP \Rightarrow \exists i, P_i, P_i', P_{-i}$ s.t

$$\underbrace{f(P_i', P_{-i})}_{b \text{ (wlog)}} P_i \underbrace{f(P_i, P_{-i})}_{a \text{ (wlog)}} = b P_i a$$

let P'' be s.t $P''_{-i} = P_{-i}$ $\quad P_i''(1) = b \quad P_i''(2) = a$

$(P_i, P_{-i}) \rightarrow (P_i'', P_{-i})$

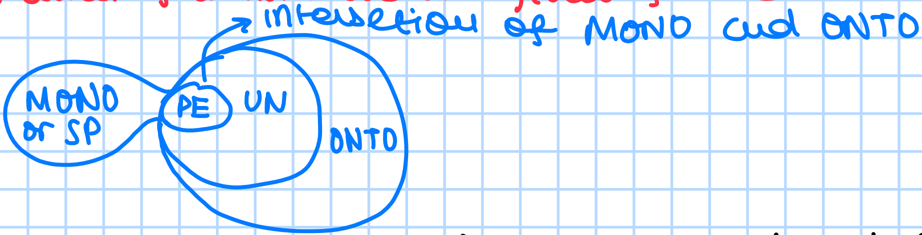
then $D(a, P_i) \subseteq D(a, P_i'')$ as $b P_i a$

$\Rightarrow f(P_i'', P_{-i}) = a$ ($\because$ MONO)

$(P_i', P_{-i}) \rightarrow (P_i'', P_{-i})$

$D(b, P_i') \subseteq D(b, P_i'')$ as $P_i''(1) = b \Rightarrow f(P_i'', P_{-i}) = b$ ($\because$ MONO) so, $f(P_i^{d'}, P_{-i}) = a$ and $f(P_i'', P_{-i}) = b$, this is a contradiction

Lemma: If an SCF f is MONO and ONTO, then f is PE



proof: suppose not, i.e f is MONO and ONTO but not PE

then $\exists a, b, P$ s.t $b P_i a \ \forall i \in N$ but (from ~PE) $f(P) = a$, now let P'' be s.t

$P_i''(1) = b$

$P_i''(2) = a \ \forall i \in N$

as f is ONTO, $\exists P'$ s.t $f(P') = b$

now, $D(b, P_i') \subseteq D(b, P_i'') \ \forall i \in N$ ($\because P_i''(1) = b$)

so, by MONO

$f(P'') = b$ ($\because f(P') = b$)

now, $D(a, P_i) \subseteq D(a, P_i'')$ ($\because b P_i a \ \forall i \in N$ and $P_i''(2) = a$)

from $f(P) = a$ and MONO

$\Rightarrow f(P'') = a$

so, $f(P'') = b$ and $f(P'') = a$ when $a \neq b$, this is a contradiction

Note: f is SP/MONO + ONTO $\Leftrightarrow$ f is SP + UN $\Leftrightarrow$ f is SP + PE

Theorem: (Gibbard-Satterthwaite theorem) suppose $|A| \geq 3$, f is ONTO and SP iff f is dictatorial

Note: above theorem works for f PE (or UN) and SP as all are equivalent
also this means that no reasonable voting rule is truthful

Note: in above theorem, we assume the preferences are unrestricted i.e all $m!$ pref profiles in domain of SCF f

3rd oct:

Note: For $|A|=2$, the GS does not hold as plurality with fixed tie is SP, ONTO and non-dictorial

lemma: $|A|\geq 3$, $N=\{1,2\}$ and f is ONTO and SP, then for every pref profile P, $f(P)\in\{P_1(1),P_2(1)\}$

proof:

if $P_1(1)=P_2(1)$ then from UN $f(P)=P_1(1)$ ($\because$ ONTO + SP $\Leftrightarrow$ UN + SP)

now, if $P_1(1)=a\neq b=P_2(1)$

let $f(P)=c\neq a,b$ ($\because |A|\geq 3$)

P_1	P_2
a	b
⋮	⋮

P_1	P_2'
a	b
⋮	a
	⋮

P_1'	P_2'
a	b
b	a
⋮	⋮

P_1'	P_2
a	b
b	⋮
⋮	

$f(P_1,P_2')\in\{a,b\}$ as all alternatives except b are pareto dominated by A ($\because$ PE)

if $f(P_1,P_2')=b$ then

$f(P_1,P_2')\ P_2\ f(P_1,P_2)=b\,P_2\,c$ which is true, so 2 manipulates P_2 to P_2'

$\Rightarrow f(P_1,P_2')=a$ ($\because$ SP)

similarly, $f(P_1',P_2)=b$

then by MONO: $P_1',P_2\rightarrow P_1',P_2'$

$\Rightarrow f(P_1',P_2')=b$

similarly $P_1,P_2'\longrightarrow P_1',P_2'$

$\Rightarrow f(P_1',P_2')=a$

as $f(P_1',P_2')=a$ and b where $a\neq b$

this is a contradiction

$\Rightarrow f(P)\in\{P_1(1),P_2(1)\}$

lemma: (two player version of GS theorem) suppose $|A|\geq 3$, $N=\{1,2\}$ and f is ONTO and SP

let $P: P_1(1)=a\neq b=P_2(1)$; $P_1'(1)=c$, $P_2'(1)=d$

then, $f(P)=a\Rightarrow f(P')=c$

$f(P)=b\Rightarrow f(P')=d$

proof: if $c=d$ then UN proves lemma, then for $c\neq d$:

cases:	c	d
1	a	b
2	$\neq a,b$	b
3	$\neq a,b$	$\neq b$
4	a	$\neq a,b$
5	b	$\neq a,b$
6	b	a

if we show that $f(P)=a\Rightarrow f(P')=c$, we are done as other side is symmetric

case 1: $c=a$, $d=b$:

P_1	P_2
a	b
⋮	⋮

P_1'	P_2'
a	b
⋮	⋮

$\hat{P}_1$	$\hat{P}_2$
a	b
b	a
⋮	⋮

we know from previous lemma $f(P')\in\{a,b\}$

if $f(P')=b$

$\underset{a}{P_1, P_2} \xrightarrow{\text{MONO}} \underset{a}{\hat{P}_1, \hat{P}_2}$ and $\underset{b}{P_1', P_2'} \xrightarrow{\text{MONO}} \underset{b}{\hat{P}_1, \hat{P}_2}$

$\Rightarrow f(\hat{P}_1, \hat{P}_2) = a$ $\quad\Rightarrow f(\hat{P}_1, \hat{P}_2) = b$

this is a contradiction

case 2: $c \neq a, b$ $\quad d = b$

$$\begin{array}{cc|cc|cc} P_1 & P_2 & P_1' & P_2' & \hat{P}_1 & P_2 \\ a & b & c & b & c & b \\ \vdots & \vdots & \vdots & \vdots & a & \vdots \\ & & & & \vdots & \end{array}$$

$f(P') \in \{c, b\}$

if $f(P') = b$ then as $f(P') = b$
from case I
$f(\hat{P}_1, P_2) = b$

and so, $f(P_1, P_2) \hat{P}_1 f(\hat{P}_1, P_2)$
$= a \hat{P}_1 b$ this is true so agent 1 misreports

$\Rightarrow f(P') = c$

case 3: $c \neq a, b$ $\quad d \neq b$

$$\begin{array}{cc|cc|cc} P_1 & P_2 & P_1' & P_2' & \hat{P}_1 & \hat{P}_2 \\ a & b & c & d & c & d \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

$f(P') = d$ then
$P' \rightarrow \hat{P}$ from case 2 $f(\hat{P}) = b$
$P \rightarrow \hat{P}$ from case 2 $f(\hat{P}) = c$

$\Rightarrow f(P') = c$

case 4: $c = a$, $d \neq b, a$

$$\begin{array}{cc|cc|cc} P_1 & P_2 & P_1' & P_2' & \hat{P}_1 & \hat{P}_2 \\ a & b & c=a & d & a & b \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

if $f(P') = d$
$P' \rightarrow \hat{P} \Rightarrow f(\hat{P}) = b$ from case 2
$P \rightarrow \hat{P} \Rightarrow f(\hat{P}) = a$ from case 1

$\Rightarrow f(P') = c$

case 5: $c = b$, $d \neq b, a$

$$\begin{array}{cc|cc|cc|cc} P_1 & P_2 & P_1' & P_2' & \hat{P}_1 & \hat{P}_2 & P_1 & \hat{P}_2 \\ a & b & c=b & d & c & d & a & d \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

$f(P') = d$
$P' \rightarrow \hat{P} \Rightarrow f(\hat{P}) = d$ case 1
$P \rightarrow (P_1, \hat{P}_2) \Rightarrow f(P_1, \hat{P}_2) = a$ case 4
$(P_1, \hat{P}_2) \rightarrow \hat{P} \Rightarrow f(\hat{P}) = a$ case 2

$\Rightarrow f(P') = c$

case 6: $c = b$, $d = a$

$$\begin{array}{cc|cc|cc} P_1 & P_2 & P_1' & P_2' & \hat{P}_1 & \hat{P}_2 \\ a & b & b & a & c & b \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

if $f(P') = a$ then $P \rightarrow \hat{P} \Rightarrow f(\hat{P}) = c$ from case 2
$P' \rightarrow \hat{P} \Rightarrow f(\hat{P}) = b$ from case 5

$\Rightarrow f(P') = b$

Note: $n \geq 3$ agent case uses induction

Domain restriction:

we know G-S theorm holds for unrestricted preferences

$$f : P^n \longrightarrow A$$

P contains all strict preference, one reason for restrictive result like G-S theorm is that the domain of SCF is large

A potential manipulator has many options to manipulate

Defn: (strategyproofness) An alternative defination is:

$$f(P_i, P_{-i}) \, P_i \, f(P_i', P_{-i}) \quad \text{or} \quad f(P_i, P_{-i}) = f(P_i', P_{-i})$$

$$\forall P_i, P_i' \in P, \forall i \in N, \forall P_{-i} \in P^{n-1}$$

we want to reduce the set of feasible preferences from P to $S \subset P$

if f SP on P then f SP on S

but potentially more f's can be SP on this restricted domain

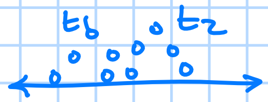
Domains:

① single peaked preferences
② Divisible goods allocation
③ Quasi-linear preferences

Each of this domain have intersting non-dictatorial SCFs that is SP

Single peaked preferences:

Temprature of a room, for every agent most comfortable temperature t_i^*, anything above or below are monotically less preferred



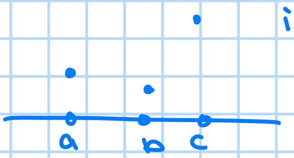
we have one common order over all alternatives, agent pref are single peaked

ordering via < (as in real numbers)

any relation is trasitive and antisymetric

assumption ① alternatives live on real line
② consider only one-dimensional single-peakedness

Eg:



i.e

P_i	P_j
c	a
a	c
b	b

not possible

Defn: (single peaked preferences) A pref ordering P_i (linear over A) of agent i is single peaked wrt common order < of alternatives if

$$\forall b, c \in A, \; b < c \leq P_i(1), \; c P_i b$$

$$\forall b, c \in A, \; P_i(1) \leq b < c, \; b P_i c$$

S be set of single peaked pref   SCF: $f: S^n \longrightarrow A$

Here this circumvents GS theorem as each player's pref has peak
if f picks leftmost peak, for agent having leftmost pick → no misrep
only way is left of leftmost for other agents → strictly worse than current outcome

f picking any k^{th} pick is SP, even median $k = [\frac{n}{2}]$

Median voter SCF:

An SCF $f: S^n \longrightarrow A$ is median voter SCF if $\exists B = \{y_1, \ldots y_{n+1}\}$ s.t
$f(P) = \text{median}(B, \text{peaks}(P))\ \forall P \in S^n$

Note: median is wrt $<$

B are called phantom voters, B is fixed for f and does not change with P

$f^{leftmost} \equiv (B_{left}, \text{peak}(P))$

$B_{left} = \{y_L, \ldots y_L\}$ i.e all phantom peaks on left

$f^{rightmost}(\cdot)$, same way

So phantom votes give us 'complete spectrum'

Theorem: (Moulin 1980) Every median voter SCF is strategyproof

proof: if $f(P) = a$ and player has peak $P_i(1)$ to be left of a
no benefit to misreport $P_i'(1) > a$, same for other case

Note: mean does not have this property

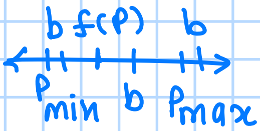
Claim: let P_{min} and P_{max} be leftmost and rightmost peaks of P according to $<$, then f is PE iff $f(P) \in [P_{min}, P_{max}]$

proof:

($\Rightarrow$) suppose $f(P) \notin [P_{min}, P_{max}]$ then wlog $f(P) < P_{min}$
then every agent prefers P_{min} over $f(P)$

i.e $f(P)$ is Pareto dominated
this is a contradiction

($\Leftarrow$) If $f(P) \in [P_{min}, P_{max}]$ then $b P_i f(P)\ \forall i \in N$ never occurs



as if $b < f(P) \leq P_{max}(1)$

$\Rightarrow f(P)\ P_{max}\ b$

$P_{min}(1) \leq f(P) < b$

$\Rightarrow f(P)\ P_{min}\ b$

so, $b P_i f(P)$ does not hold true $\forall i \in N$

$\Rightarrow f(P)$ is PE

Defn: (Monotonicity) An SCF is MONO if $\forall P, P'$ s.t $f(P)=a$ and $D(a,P_i) \subseteq D(a,P_i')$ $\forall i \in N$ holds then $f(P')=a$

Median Voter SCF and Monotonicity:

Note: Results are similar to unrestricted pref in this restricted domains of single peaked preferences, but the proofs differ since we cannot construct profiles as freely as before

Theorem: f is SP $\Rightarrow$ f is MONO

proof: proof similar to previous ones

Theorem: Let $f: S^n \to A$ is SP SCF, then f is onto $\Leftrightarrow$ f is UN $\Leftrightarrow$ f is PE

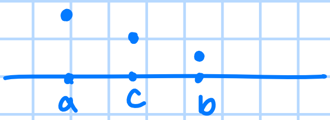
proof:

as PE $\Rightarrow$ UN $\Rightarrow$ ONTO

or PE + SP $\Rightarrow$ UN + SP $\Rightarrow$ ONTO + SP

if we show ONTO+SP $\Rightarrow$ PE+SP we are done

so, if f is SP and ONTO but not PE, then $\exists a,b \in A$ s.t $a P_i b$, $\forall i \in N$ but $f(P)=b$



as single peaked, $\exists$ alternative $c \in A$ s.t $c P_i b$ $\forall i \in N$ (c can be a also)

as ONTO $\Rightarrow \exists P'$ s.t $f(P')=c$

now P'' s.t $P_i''(1)=c$ $\forall i \in N$

$P_i''(2)=b$,

then $\underset{b}{P} \longrightarrow \underset{b}{P''}$ (MONO) $\Rightarrow f(P'')=b$

also, $\underset{c}{P'} \longrightarrow \underset{c}{P''}$ (MONO) $\Rightarrow f(P'')=c$

as $c \neq b$ this is a contradiction, or f is PE

Anonymity:

Anonymity is when outcome insensitive to agent identities

$\sigma: N \to N$ be a permutation, we apply σ to profile P to construct another profile as: pref of $i \to \sigma(i)$ in new profile

new profile: P^σ

Eg: $N=\{1,2,3\}$

$\sigma: \sigma(1)=2, \sigma(2)=3, \sigma(3)=1$

P_1	P_2	P_3		P_1^σ	P_2^σ	P_3^σ
a	b	b		b	a	b
b	a	c		c	b	a
c	c	a		a	c	c

Defn: (ANON) An SCF $f: S^n \to A$ is Anonymous if $\forall P$ and $\forall \sigma$, $f(P^\sigma)=f(P)$

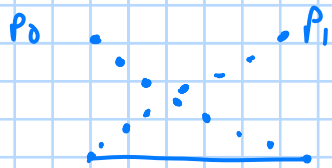
Note: Dictatorship is not anonymous

Theorm: (Median voter theorm) A SP SCF f is ONTO and ANON iff it is a median voter SCF

Proof: ($\Leftarrow$) Median voter SCF is SP (previous theorm)
it is also ANON as if we permute the agents with peak unchanged, the outcome does not change
it is also ONTO as we pick any arbitary alternative a, put peaks of all players at a, the outcome will be a irrespective of positions of phantom peaks as $(n-1)$

($\Rightarrow$) Now if $f: S^n \to A$ is SP + ANON + ONTO

let P_i^0: agent i's pref with peak at leftmost wrt $<$
P_i^1: agent i's pref with peak at rightmost wrt $>$



let $y_j = f(\underbrace{P_1^0, P_2^0, \ldots P_{n-j}^0}_{\substack{n-j \text{ peaks} \\ \text{leftmost}}}, \underbrace{P_{n-j+1}^1, \ldots P_n^1}_{\substack{j \text{ peaks} \\ \text{rightmost}}}) \quad j = 1, 2 \ldots n-1$

claim: $y_j \leq y_{j+1}$ $\quad j = 1, 2 \ldots n-2$ i.e peaks are non-decreasing
as $y_{j+1} = f(P_1^0, \ldots P_{n-j-1}^0, P_{n-j}^1, \ldots P_n^1)$ due to SP
$y_j P_{n-j}^0 y_{j+1} \Rightarrow y_j < y_{j+1}$
or $y_j = y_{j+1}$

now, consider $P = (P_1, P_2 \ldots P_n)$ s.t $P_i(1) = P_i$ (the peaks)

Claim: if f satisfies SP + ONTO + ANON then
$f(P) = \text{median}(P_1, P_2, \ldots P_n, y_1, \ldots y_{n-1})$
wlog $P_1 \leq P_2 \ldots \leq P_n$ ($\because$ ANON)
then if $a = \text{median}(P_1, P_2 \ldots P_n, y_1, \ldots y_{n-1})$

Case 1: a is a phantom peak i.e $\exists j \in \{1 \ldots n-1\}$ s.t $a = y_j$

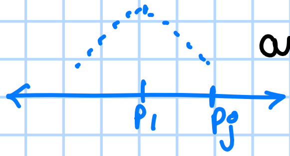
then, this is median of $2n-1$ points, $j-1$ phantom peaks on left and rest $n-j$ are agent peaks

$(j-1)$ phatom $\longleftarrow y_j \longrightarrow (n-1-j)$ phantom
$(n-j)$ agents $\qquad\qquad j$ agents

i.e $P_1 \leq \cdots \leq P_{n-j} \leq y_j = a \leq P_{n-j+1} \leq \ldots \leq P_n$

now as $f(P_1^0, \ldots P_{n-j}^0, P_{n-j+1}^1, \ldots P_n^1) = y_j$ (by defination)

if $f(P_1, P_2^0, \ldots P_{n-j}^0, P_{n-j+1}^1, \ldots P_n^1) = b$
then by SP, $y_j P_1^0 b \Rightarrow y_j \leq b$ $\quad$ ①
also by SP: $b P_1 y_j$ but as $P_1 \leq P_j$



as $bP_1 y_j$ and $P_1 \le P_j$
$\Rightarrow b \le P_j$ (single peaked) ②

from ①, ② $\Rightarrow b = y_j$

repeating this for first $n-j$:

$$f(P_1, P_2, \dots P_{n-j}, P'_{n-j+1}, \dots P'_n) = y_j$$

now if $f(P_1 \dots P_{n-j}, P'_{n-j+1}, \dots P_n) = b$ then
$b = y_j$ ($\because$ similar argument)

so, $f(P_1, P_2, \dots P_n) = y_j$

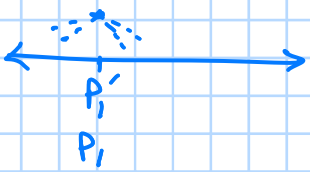
$\Rightarrow f(P_1, P_2, \dots P_n) = \text{median}(y_1, y_2 \dots y_{n-1}, P_1, P_2, \dots P_n)$
$\Rightarrow f(P) = \text{median}(P_1, P_2, \dots P_n, y_1, \dots y_{n-1})$

<u>case 2</u>: a is an agent peak
then if $|N| = 2$ ie for 2 players (general case repeats same argument)

claim: $N = \{1, 2\}$ let P and P' be s.t $P_i(1) = P'_i(1), \forall i \in N$, then $f(P) = f(P')$

let $a = P_1(1) = P'_1(1)$
$P_2(1) = P'_2(1) = b$
$f(P) = x$ and $f(P'_1, P_2) = y$

since f is SP, $x P_1 y$ and $y P'_1 x$



if x and y on same side of peak
then $x = y$

if x and y on differt sides and wlog $x < a < y$ and $a < b$
as f is SP + ONTO $\Leftrightarrow$ f is SP + PE
PE requires $f(P) \in [a, b]$ but $f(P) = x < a$
so x and y on same side this is a contradiction

$\Rightarrow f(P'_1, P_2) = x$, similarly $f(P'_1, P'_2) = x$

now, profile: $(P_1, P_2) = P$

$P_1(1) = a, \quad P_2(1) = b$

y_1 is phantom peak and median(a, b, y_1) is agent peak
wlog median $= a$
if $f(P) = c \ne a$ then by PE
as $f(P) \in [a, b] \Rightarrow c \in [a, b]$
or $[b, a]$ or $[b, a]$
so, we will have 2 subcases:
$b < a < y_1$
and $y_1 < a < b$

<u>case 2.1</u>: $b < a < y_1$:
then by PE as $c \in [b, a]$ and $c \ne a \Rightarrow c < a$
let P'_1 be s.t $P'_1(1) = a = P_1(1)$
and $y_1 P'_1 c$ (possible as differt sides of a)

then from earlier claim as $f(P) = c$
and $P_1'(1) = P_1(1) \Rightarrow f(P_1', P_2) = c$

also $(P_1^!, P_2)$ ($P_1^!$ has rightmost point)

$P_2(1) = b < y_1 \leq P_1^!(1) \Rightarrow$ median $(b, y_1, P_1^!(1)) = y_1$ which is phantom peak

so from case 1
$f(P_1^!, P_2) = y_1$

but $y_1 P_1' c$ and $f(P_1', P_2) = c$ so

we get agent 1 manipulates $P_1' \to P_1^!$
contradiction to f being SP

Case 2.2: $y_1 < a < b$ by PE as $c \in [a, b]$ and $c \neq a$
$\Rightarrow a < c$

now, P_1' s.t $P_1'(1) = a = P_1(1)$ and $y_1 P_1' c$ ($\because$ similar to above)

$f(P_1', P_2) = c$ (by claim)

and now
(P_1^0, P_2) : $P_1^0(1) \leq y_1 < b \Rightarrow f(P_1^0, P_2) = y_1$ but $y_1 P_1' c$ and
so agent 1 manipulates
this is a contradiction

Both cases contradiction, so

$f(P) = a =$ median (a, b, y_1)

8^{th} Oct:

we saw median voting rule and theorm

$f: S^n \longrightarrow A$ where S ← single peak property

we saw if f is SP, f is ONTO + ANON $\Leftrightarrow$ f is median

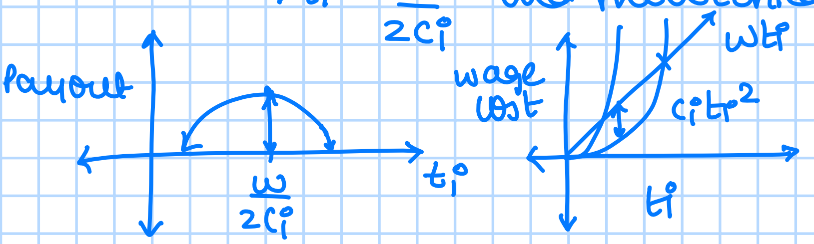
Task allocation domain:

unit amount of task to be shared among n agents
agent i gets $s_i \in [0,1]$
$$\sum_{i \in N} s_i = 1$$
Payoff: every agent has most preferred share of work

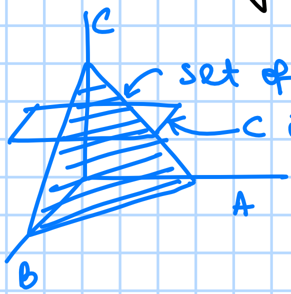
Eg: task has reward w per unit time
$i \to t_i$ time then wt_i reward
task has cost: $c_i t_i^2$

net payout $= wt_i - c_i t_i^2$ so we want to maximise

$\Rightarrow t_i^* = \dfrac{w}{2c_i}$ and monotonically dec on both sides



Note: this is single peaked over share of the task and not over alternatives



set of all task allocations
c is indefeent for all points here

let domain of task allocation T
a is allocation of task
$a = (a_i \in [0,1], i \in N)$

SCF: $f: T^n \longrightarrow A$
$P \in T^n$
$f(P) = (f_1(P), \ldots f_n(P))$
$f_i(P) \in [0,1] \;\; \forall i \in N$
$$\sum_{i \in N} f_i(P) = 1$$
Player i has peak P_i over the shares of task

Defn: (PE) An SCF f is PE if there does not exist any profile P where $\exists$ a task allocation $a \in A$ s.t it is weakly preferred over $f(P)$ by all agents and strictly prefered by atleast one

$\nexists P$ where $\exists\, a \in A$ s.t $\quad a R_i f(P) \;\; \forall i \in N$
$a P_j f(P), \; \exists j \in N$

Eg:

	A	B	C
peaks:	0.5	0.4	0.7
curr:	0.2	0.6	0.2
better:	0.4	0.4	0.2

as $\sum p_i > 1$

Note: if $\sum_{i \in N} p_i = 1$ then $\exists$ unique PE allocation

Note: if $\sum_{i \in N} p_i > 1$, $\exists k \in N$ s.t. $f_k(P) < p_k$

From PE

as $f_j(P) \leq p_j \; \forall j$ and $\exists j$ s.t. $f_j(P) < p_j$

as if $\exists j$ s.t. $f_j(P) > p_j$, then inc k, red j makes new better (Not PE)

Note: If $\sum_{i \in N} p_i < 1$ similarly $f_i(P) \geq p_i \; \forall i \in N$

Defn: (ANON) An SCF is ANON if $\forall \sigma : N \to N$ the task get permuted accordingly $\forall \sigma$, $f_{\sigma(i)}(P^\sigma) = f_i(P)$, $\forall i \in N$

Eg: $N = \{1,2,3\}$ $\sigma\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}2\\3\\1\end{pmatrix}$

$P = (0.7, 0.4, 0.3) \Rightarrow P^\sigma = (0.3, 0.7, 0.4)$

$f_1(P) = f_2(P^\sigma)$
$f_2(P) = f_3(P^\sigma)$
$f_3(P) = f_1(P^\sigma)$

Defn: (serial dictatorship) A predetermined seq of agent is fixed, each agent gets her pick or leftover share

if $\sum p_i < 1 \to$ last agent gets leftover share

Note: serial dictatorship is not ANON, but is PE (trivial) and SP as agent has no need to misreport

Defn: (proportional) Every player is assigned a share that is c times their peak i.e. $c \sum_{i \in N} p_i = 1$

10th oct:

$\nearrow$ [ ] among n play
unit task

Player i – p_i ← Peak

$\sum p_i = 1$, <1, or >1 and if $\neq 1$ then someone over/underworked

we saw $f: S^n \longrightarrow A$ if $c \sum_{i \in N} p_i = 1$ and every player cp_i, then not SP as

$$\left.\begin{array}{l} p_1 = 0.2 \\ p_2 = 0.3 \\ p_3 = 0.1 \end{array}\right\} \Rightarrow c = \frac{1}{0.6} \qquad \text{player 1: } \frac{1}{3} \text{ (more than 0.2)}$$

$\downarrow$

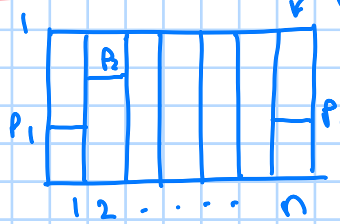
0.1 (misreport) $\Rightarrow c' = 1/0.5$

then player 1 gets 0.2

} case when a player misreports

Note: we want to ensure PE, ANON and SP SCF in the task allocation domain

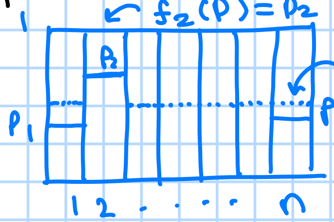
The uniform rule:



← volume of every container is 1

$\sum p_i = 1$ ← then we are done

we slowly remove water for every cylinder i.e $f_i(P) \downarrow$ till $\sum_{i \in N} f_i(P) = 1$
when $f_i(P) \downarrow$ if some p_j is realised, we don't reduce further
i.e $f_j(P) = p_j$ $f_i(P) \downarrow$ $i \neq j$ and we keep doing this



$f_i(P) = \alpha \quad i \neq 2$

α s.t $(n-1)\alpha + p_2 = 1$

$\Rightarrow \alpha = \dfrac{1 - p_2}{n-1}$

Defn: for $f^u: S^n \to A$

$\sum p_i = 1 \Rightarrow f_i^u(P) = p_i$

$\sum p_i < 1 \Rightarrow f_i^u(P) = \max\{p_i, \mu(P)\}$ where $\mu(P)$ solves $\sum_{i \in N} \max\{p_i, \mu\} = 1$

$\sum p_i > 1 \Rightarrow f_i^u(P) = \min\{p_i, \mu(P)\}$ where $\mu(P)$ solves $\sum \min\{p_i, \mu\} = 1$

$x_i \rightarrow$ allocation of player i, $\mu \rightarrow$ final water level ($\sum p_i < 1$)

if $p_i \geq \mu \Rightarrow f_i^u(P) = p_i$
if $\mu \geq p_i \Rightarrow f_i^u(P) = \mu$

$x_i = \max\{p_i, \mu\}$ $\left.\sum x_i = 1,\ x_i \geq 0,\ x_i \geq p_i,\ x_i \geq \mu\right\}$ LP

similarly, we will get LP for $\sum p_i > 1$ case, all inequalities flip

now, f^u is ANON as only the peaks matter and not their owner

f^u is also PE as ① $f_i^u(P) = p_i$ $\forall i \in N$, $\sum_{i \in N} p_i = 1$
② $f_i^u(P) \geq p_i$ $\forall i \in N$ if $\sum_{i \in N} p_i < 1$
③ $f_i^u(P) \leq p_i$ $\forall i \in N$ if $\sum_{i \in N} p_i > 1$

} so this is also PE

eg: we can use this for gpu sharing, in gpu cluster we need a nice way to divide computation power we can also impose 'envy free' i.e if $\underbrace{u_i(f_i(P))}_{\text{How much share}} < \underset{\uparrow}{u_i}(f_j(P))$ we don't want

utility of i^{th} player

we see in f^u case, we get $u_i(f_i(P)) \geq u_i(f_j(P))$ $\forall i,j$
so f^u is envyfree

if $\sum P_i = 1$, SP
$\sum P_i < 1 \rightarrow f_i^u(P) \geq P_i$ $\forall i \in N$
manipulation only for $i \in N$ s.t $f_i^u(P) > P_i \Rightarrow \mu(P) > P_i$
if $P_i' > \mu(P) > P_i$

leads to worse outcome for i than $\mu(P)$

same for $\sum P_i > 1$, so we get SP

Theorm: An SCF in the task allocation domain is SP + PE + ANON iff it is uniform rule

Note: we saw f^u is EF (envy free) and all SP, PE, ANON, EF are polynomial-time computable

Mechanism design with transfers:

X: space of all outcomes $\quad F: \Theta \longrightarrow X$
$x \in X$ has two componets, allocation a
payment $\pi = (\pi_1, \ldots \pi_n)$
$\pi_i \in \mathbb{R}$

$x = (a, \pi) = (a, \underbrace{(\pi_1, \pi_2, \ldots \pi_n)}_{\text{meaning of money is same to everyone}})$

Eg: $a = \{$park, bridge, ... $\} \rightarrow$ public decision

divisible : $a = (a_1, a_2, \ldots a_n)$
$a_i \in [0,1]$ $\sum_{i \in N} a_i = 1$ $\quad a_i$ = fraction of resources i gets

single indivisible: painting
$a = (a_1, \ldots a_n)$
$a_i \in \{0,1\}$, $\sum a_i \leq 1$

indivisible objects: S = set of objects

$A = \{(A_1, \ldots A_n) \mid A_i \subseteq S, A_i \cap A_j = \phi, \forall i \neq j\}$

Type of agent i is $\theta_i \in \Theta_i$ private information of i

benifit = valuation fution
$v_i : \underset{\uparrow \text{ allocation}}{A} \times \underset{\nwarrow \text{ type of player}}{\Theta_i} \longrightarrow \mathbb{R}$

If i is s.t θ_i^{env} then $a \in \{$Bridge, Park$\}$ $\quad v_i(B, \theta_i^{env}) < v_i(P, \theta_i^{env})$

we also have money / payment / transfer $\pi_i \in \mathbb{R}$ $\forall i \in N$
payment vetor $\pi = (\pi_1, \pi_2, \ldots \pi_n)$
utilities $\quad \underset{\nwarrow \text{ utility funtion, depends on } \pi_i \text{ payment}}{u_i} : X \times \Theta_i \longrightarrow \mathbb{R}$ s.t $u_i((a, \pi), \theta_i) = v_i(a, \theta_i) - \pi_i$ (quasi-linear)

Note: The domain is restricted as
$(a, (\pi_i, \pi_{-i})), (a, (\pi_i', \pi_{-i}))$
if $\pi_i < \pi_i'$ then (a, π) always preferred
but in complete domain — both pref order would have been feasible

$f: \Theta \to A$ together $(f, P): \Theta \to X$
$P_i: \Theta \to \mathbb{R}$

$F = (f, (P_1, \dots P_n)) = (f, P)$ $\quad f: \Theta_1 \times \Theta_2 \times \cdots \times \Theta_n \to A$

$P_i: \Theta_1 \times \Theta_2 \times \cdots \Theta_n \to \mathbb{R} \quad \forall i \in N$

Eg: constant rule: $f^c(\theta) = a \quad \forall \theta \in \Theta$
dictator rule: $f^D(\theta) \in \underset{a \in A}{\operatorname{argmax}}\, v_d(a, \theta_d), \forall \theta \in \Theta$
some $d \in N$

Allocative / utilitarian:

$f^{AE}(\theta) \in \underset{a \in A}{\operatorname{argmax}} \sum_{i \in N} v_i(a, \theta_i)$
together happiness, but some person very unhappy, some extremely happy

Affine maximizer:

$f^{AM}(\theta) \in \underset{a \in A}{\operatorname{argmax}} \left(\sum_{i \in N} \lambda_i v_i(a, \theta_i) + K(a) \right)$ where $\lambda_i \geq 0$ not all zero

$\lambda_i = 1, \forall i \in N, K \equiv 0 \Rightarrow f^{AM}(\theta) = f^{AE}(\theta)$

$\lambda_d = 1, \lambda_j = 0 \ \forall j \neq d, K \equiv 0 \Rightarrow f^{AM}(\theta) = f^D(\theta)$

max-min / egalitarian:

$f^{MM}(\theta) \in \underset{a \in A}{\operatorname{argmax}} \min_{i \in N} v_i(a, \theta_i)$

we want to design P_i's s.t people play fair (also f)

No deficit: $\sum_{i \in N} P_i(\theta) \geq 0 \ \forall \theta \in \Theta$ (not negative)

No subsidy: $P_i(\theta) \geq 0 \ \forall \theta \in \Theta, \forall i \in N$ (every π_i is non-neg)

Budget balance: $\sum_{i \in N} P_i(\theta) = 0, \forall \theta \in \Theta$, i.e overall F makes has net 0 monetary
the mechanism design does not have money in hand

Defn: (DSIC) A mechanism (f, P) is dominant strategy incentive comp

$v_i(f(\theta_i, \tilde{\theta}_{-i}), \theta_i) - P_i(\theta_i, \tilde{\theta}_{-i}) \geq v_i(f(\theta_i', \tilde{\theta}_{-i}), \theta_i) - P_i(\theta_i', \tilde{\theta}_{-i})$

$\forall \tilde{\theta}_{-i} \in \Theta_{-i}, \ \theta_i', \theta_i \in \Theta_i, \ \forall i \in N$

15^{th} oct :

we have seen $F: \Theta_1 \times \Theta_2 \times \cdots \Theta_n \longrightarrow X$

$(f, P_1, \ldots P_n)$ where $X = \{(a, \pi) \mid a \in A, \pi \in \mathbb{R}^n\}$

$F = (f, P)$ $\qquad \Theta = \Theta_1 \times \Theta_2 \times \cdots \times \Theta_n$

$f: \Theta \longrightarrow A$

$P_i: \Theta \longrightarrow \mathbb{R} \quad \forall i \in [n]$

we also saw we need u_i s.t valuation of $a \in A$ $\quad v_i: X \times \Theta_i \to \mathbb{R}$

$$u_i((f,P), \theta_i) = v_i(f(\theta), \theta_i) - \underbrace{P_i(\theta)}_{\text{Price paid for } a \in A} \qquad P_i: \Theta \to \mathbb{R}$$

DSIC:

Reported $\qquad$ Reported $\qquad$ Reported

$$v_i(f(\theta_i, \tilde{\theta}_{-i}), \theta_i) - P_i(\theta_i, \tilde{\theta}_{-i}) \geq v_i(f(\theta_i', \tilde{\theta}_{-i}), \theta_i) - P_i(\theta_i', \tilde{\theta}_{-i})$$

true type $\qquad$ true θ_i $\qquad$ Reported

$\forall \theta_i, \theta_i' \in \Theta_i, \forall \tilde{\theta}_{-i} \in \Theta_{-i}, \forall i \in N$

Eg: $N = \{1,2\}$ $\quad \Theta_1 = \Theta_2 = \{\theta^H, \theta^L\}$

$f: \Theta_1 \times \Theta_2 \longrightarrow A$

for $i=1$:

$v_1(f(\theta^H, \theta_2), \theta^H) - P_1(\theta^H, \theta_2) \geq v_1(f(\theta^L, \theta_2), \theta^H) - P_1(\theta^L, \theta_2) \quad \forall \theta_2 \in \Theta_2$

same for θ^L

and same for player 2

now if we find (f, P) s.t DSIC is satisfied, then

new payment, so payment is changable

let $q_i(\theta_i, \theta_{-i}) = P_i(\theta_i, \theta_{-i}) + h_i(\theta_{-i}) \quad \forall \theta \in \Theta, \forall i \in N$

then, (f, q) is also DSIC $\quad (\because h_i(\tilde{\theta}_{-i})$ same on both sides $)$

Note: we dont know if converse is true or not, as well paymets that implement f, only differ by factor of $h_i(\theta_{-i})$

Note: we will say f is our SCF, not $F = (f, P)$

Exe: let $\theta, \tilde{\theta} \in \Theta$ s.t $f(\theta) = f(\tilde{\theta}) = a$ and p implements f, then $P_i(\theta) \Rightarrow P_i(\tilde{\theta})$

where $\theta = (\theta_i, \theta_{-i})$

$\tilde{\theta} = (\tilde{\theta}_i, \theta_{-i})$ assuming DSIC $\rightarrow$ done down

Defn: (Pareto optimal) A mechanism $(f, (P_1, \ldots P_n))$ is PO if at any $\theta \in \Theta$ there does not exist $b \in A$, $b \neq f(\theta)$, $(\pi_1 \ldots \pi_n)$ with $\sum_{i \in N} \pi_i \geq \sum_{i \in N} P_i(\theta)$

$$v_i(b, \theta_i) - \pi_i \geq v_i(f(\theta), \theta_i) - P_i(\theta), \ \forall i \in N$$

with inequalities being strict for some $i \in N$

Note: If we reduce all $P_i(\theta)$ then π_i same or PO does not make sense, we can always reduce π_i

so, we need some condition to spend atleast the same budget

Theorm: $(f, (P_1 \ldots P_n))$ is PO iff it is allocatively efficient

proof:

$$f^{AE}(\theta_i, \theta_{-i}) \in \underset{a \in A}{\operatorname{argmax}} \sum_{i \in N} v_i(a, \theta_i)$$

($\Leftarrow$) we will show $\sim PO \Rightarrow \sim AE$

as $\sim PO$ means, $\exists b, \pi, \theta$ s.t $\sum_{i\in N} \pi^i \geq \sum_{i \in N} P_i(\theta)$

$v_i(b,\theta_i) - \pi^i \geq v_i(f(\theta),\theta_i) - P_i(\theta)$, $\forall i \in N$, strict for some $i \in N$

if we sum all ineq:

$$\sum_{i\in N} v_i(b,\theta_i) - \sum_{i\in N} v_i(f(\theta),\theta_i) > \sum_{i\in N}\pi^i - \sum_{i\in N} P_i(\theta) \geq 0$$

$$\Rightarrow \sum_{i\in N} v_i(b,\theta_i) > \sum_{i\in N} v_i(f(\theta),\theta_i)$$

so, f is $\sim AE$

($\Rightarrow$) we will show $\sim AE \Rightarrow \sim PO$

as, $\sim AE \Rightarrow \exists \theta, b \neq f(\theta)$ s.t

$$\sum_{i\in N} v_i(b,\theta_i) > \sum_{i\in N} v_i(f(\theta),\theta_i)$$

now let $\delta = \sum_{i\in N} v_i(b,\theta_i) - \sum_{i\in N} v_i(f(\theta),\theta_i) > 0$

let $\pi^i = v_i(b,\theta_i) - v_i(f(\theta),\theta_i) + P_i(\theta) - \delta/n \quad \forall i \in N$

so, $v_i(b,\theta_i) - \pi^i = v_i(f(\theta),\theta_i) - P_i(\theta) + \delta/n > v_i(f(\theta),\theta_i) - P_i(\theta) \quad \forall i \in N$

and $\sum_{i\in N}\pi^i = \sum_{i\in N} v_i(b,\theta_i) - \sum_{i\in N} v_i(f(\theta),\theta_i) + \sum_{i\in N} P_i(\theta) - \delta$

$$= \delta + \sum_{i\in N} P_i(\theta) - \delta$$

$$= \sum_{i\in N} P_i(\theta)$$

so f is $\sim PO$

Defn: (Groves payment) consider the following payment

$$P_i^G(\theta_i, \theta_{-i}) = h_i(\theta_{-i}) - \sum_{i\neq j} v_j(f^{AE}(\theta_i,\theta_{-i}),\theta_j)$$

where $h_i: \Theta_{-i} \to \mathbb{R}$ is an arbitrary function

If i wins, not much subtraction but if looses, much subtraction

where $P_i^G(\theta_i,\theta_{-i}) = h_i(\theta_{-i}) - \underbrace{\sum_{i\neq j} v_j(f^{AE}(\theta_i,\theta_{-i}),\theta_j)}_{\text{welfare of the rest of society in i's presence}}$

eg: $N=\{1,2,3,4\}$ single item allocation

$\theta_1=10, \theta_2=8, \theta_3=6, \theta_4=4$

$h_i(\theta_{-i}) = \min \theta_{-i}$

$h_1=4, h_2=4, h_3=4, h_4=6$

$P_1 = 4-0, P_2 = 4-10, P_3 = 4-10, P_4 = 6-10$ so only player 1 pays, others get paid

groves mechanism is f^{AE} and their payment is P_i^G's $\forall i \in N$

Theorm: Groves mechanism are DSIC, f^{AE}

proof: for some $i \in N$

$$f^{AE}(\theta_i, \tilde{\theta}_{-i}) = a$$

$$f^{AE}(\theta_i', \tilde{\theta}_{-i}) = b$$

as $a = \arg\max_{a \in A} \sum_{i \in N} v_i(a, \theta_i)$

so, $\sum_{i \in N} v_i(a, \theta_i) \geq \sum_{i \in N} v_i(c, \theta_i) \quad \forall c \in A$

$\Rightarrow \sum_{i \in N} v_i(a, \theta_i) \geq \sum_{i \in N} v_i(b, \theta_i)$ for $a = f^{AE}(\theta_i, \tilde{\theta}_{-i})$
$b = f^{AE}(\theta_i', \tilde{\theta}_{-i})$

then, $v_i(f^{AE}(\theta_i, \tilde{\theta}_{-i}), \theta_i) - p_i(\theta_i, \tilde{\theta}_{-i})$

$$= v_i(f^{AE}(\theta_i, \tilde{\theta}_{-i}), \theta_i) - h_i(\tilde{\theta}_{-i}) + \sum_{i \neq j} v_j(f^{AE}(\theta_i, \tilde{\theta}_{-i}), \tilde{\theta}_j)$$

$$= \sum_{i \in N} v_i(a, \theta_i) - h_i(\tilde{\theta}_{-i})$$

$$\geq \sum_{i \in N} v_i(b, \theta_i) - h_i(\tilde{\theta}_{-i})$$

$$\geq v_i(f^{AE}(\theta_i', \tilde{\theta}_{-i}), \theta_i) - p_i(\theta_i', \tilde{\theta}_{-i})$$

as i was arbitary, this holds for every $i \in N$

so, Groves mechanism is DSIC

Vickrey-Clarke-Groves Mechanism (VCG)

we have f^{AE}, p_i^{G} but

$$h_i(\theta_{-i}) = \max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j)$$

$$p_i^{VCG}(\theta_i, \theta_{-i}) = \underbrace{\max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j)}_{\text{sum of values of others in absence of } i} - \underbrace{\sum_{j \neq i} v_j(f^{AE}(\theta_i, \theta_{-i}), \theta_j)}_{\text{sum of values of of others in presence of } i}$$

Note: in case of auction, max term is second highest payment all else is 0 for winner, for others its just 0

utility is the marginal contribution of i in social welfare

Eg:

	Football	Library	Museum
A	0	70	50
B	95	10	50
C	10	50	50
	105	130	150

150 → Argmax ⇒ Museum is $a \in A$

$p_A^{VCG} = 105 - 100 = 5$
$p_B^{VCG} = 120 - 100 = 20$
Pivotal as their presence change

$p_C^{VCG} = 100 - 100 = 0$ ← Non pivotal as c change → still museum

Exe: let $\theta, \tilde{\theta} \in \Theta$ s.t. $f(\theta) = f(\tilde{\theta}) = a$ and p implements f, then $p_i(\theta) \Rightarrow p_i(\tilde{\theta})$

where $\theta = (\theta_i, \theta_{-i})$

$\tilde{\theta} = (\tilde{\theta}_i, \theta_{-i})$ assuming DSIC

Ans: $v_i(f(\theta), \theta_i) - p_i(\theta) \geq v_i(f(\tilde{\theta}), \theta_i) - p_i(\tilde{\theta})$

$\Rightarrow p_i(\tilde{\theta}) \geq p_i(\theta)$

similarly $p_i(\theta) \geq p_i(\tilde{\theta})$

$\Rightarrow p_i(\theta) = p_i(\tilde{\theta})$

Exe: pVCG for a multiple item auction

Ans:

$A = (a_1, a_2, \ldots a_n) \rightarrow$ items

it will be individual pVCG for all items

17th oct :

Combinatorial allocation :

sales of multiple objects

	ϕ	$\{1\}$	$\{2\}$	$\{1 \& 2\}$
θ_1	0	8	6	12
θ_2	0	9	4	14

$v_i(a, \theta_i) = \theta_i(a)$

let $a^*: \{1\} \rightarrow 2, \{2\} \rightarrow 1$

$$p_1^{VCG}(\theta_1, \theta_2) = \max_{a \in A} \sum_{j \neq 1} \theta_j(a) - \sum_{j \neq 1} \theta_j(a^*)$$

$= 14 - 9 = 5$ payoff $= 6 - 5 = 1$

$p_2^{VCG}(\theta_1, \theta_2) = 12 - 6 = 6$ payoff $= 9 - 6 = 3$

$M = \{1, 2 \ldots m\}$ set of objects

$\Omega = \{ S \mid S \subseteq M \}$ is set of bundles

$\theta_i : \Omega \rightarrow \mathbb{R}$ type/value of agent i

$\theta_i(S \cup \{j\}) - \theta_i(S) \geq 0 \quad \forall S \in \Omega$ s.t $j \notin S \quad \forall i, j$ objects are goods

let $a = \{a_0, \ldots a_n\}$, $a_i \in \Omega$, $a_i \cap a_j = \phi$, $\forall i \neq j$

a_0: set of unallocated objects $\bigcup_{i=0}^{n} a_i = M$

let A be set of all such allocations

Note: we assume selfish valuations, i.e $\theta_i(a) = \theta_i(a_i)$ or agent i's valuation does not depend on the allocation of others

Lemma: In the allocation of goods, the VCG payment for an agent, that gets no object in the efficient allocation, is zero

proof :

$$a^* \in \operatorname{argmax}_{a \in A} \sum_{j \in N} \theta_j(a)$$

$a_i^* = \phi$

$$a_{-i}^* \in \operatorname{argmax}_{b \in A_{-i}} \sum_{j \in N \setminus \{i\}} \theta_j(b)$$

$p_i^{VCG}(\theta) \geq 0$ also $p_i^{VCG}(\theta) = \sum_{j \neq i} \theta_j(a_{-i}^*) - \sum_{j \neq i} \theta_j(a^*)$

$\theta_i(a_{-i}^*) = 0$ and $\theta_i(a^*) = \theta_i(a_i^*) = 0$

as a_i is considered ϕ

then $p_i^{VCG}(\theta) = \sum_{j \in N} \theta_j(a_{-i}^*) - \sum_{j \in N} \theta_j(a^*) \leq 0 \Rightarrow p_i^{VCG}(\theta) = 0$

as $p_i^{VCG}(\theta) \geq 0$

Defn: (Ex-post individual Rationality) A mechanism (f, p) is ex-post individual rational (ex-post IR) if

$$v_i(f(\theta), \theta_i) - p_i(\theta) \geq 0, \quad \forall \theta \in \Theta, \forall i \in N$$

If ex-post IR, then agents will have incentive to participate

Defn: (choice set monotonicity) $\forall i \in N$, $A_{-i} \subseteq A$ i.e addition of agents weakly increases choices

Defn : (No Negative Externality) $\forall i \in N, \theta \in \Theta, v_i(a^*_{-i}(\theta_{-i}), \theta_i) \geq 0$, i.e efficient allocation without an 'agent i' yields non-negative value to that agent

Lemma : If the allocations satisfy choice set monotonicity and the valuations have no negative externality, then VCG mechanism is individually rational

proof :

$$v_i(f^{AE}(\theta_i, \theta_{-i}), \theta_i) - p_i^{VCG}(\theta_i, \theta_{-i}) = \text{utility of } i$$

by defn of VCG payment:

$$\text{Utility of } i = \sum_{j \in N} v_j(a^*(\theta_i, \theta_{-i}), \theta_j) - \sum_{i \neq j} v_j(a^*_{-i}(\theta_{-i}), \theta_j)$$

as $A_{-i} \subseteq A \Rightarrow$ Utility of $i \geq \sum_{j \in N} v_j(a^*_{-i}(\theta_{-i}), \theta_j) - \sum_{i \neq j} v_j(a^*_{-i}(\theta_{-i}), \theta_j)$

$$\geq v_i(a^*_{-i}(\theta_{-i}), \theta_i) \geq 0 \;(\because \text{no neg ext.})$$

Cor : VCG is Ex-post IR for combinatoral allocation

proof : As in VCG we have choice set monotonicity and also no Neg Externality

Internet advertising :

It is very sucessful as : ① User data
② measurable actions
③ low latency

Types : ① sponsored search ads
② contextual ads
③ display ads

position auctions : auctions to sell multiple ad position on a page

$N = \{1, 2 \dots n\}$ set of advertisers
$M = \{1, 2 \dots m\}$ set of slots, assume $m \geq n$ i.e every ad shown
$\uparrow$ Best $\quad \uparrow$ worst

$v_{ij} = CTR_{ij} \cdot v_i$

v_i: value of a click

CTR_{ij}: $[0,1]$ click through rate (prob of click on i's ad, j^{th} pos)

$CTR_{ij} = \rho_i \cdot P_j$

P_j: prob of position comp

ρ_i is quality of i

then $v_{ij} = \underbrace{P_j}_{\text{Position effect}} \underbrace{(\rho_i v_i)}_{\text{agent effect}}$

position effect : $P_1 = 1$, $P_j > P_{j+1}$, $\forall i = 1, 2 \dots m-1$

slot allocation : $a = (a_1 \dots a_n)$ $\quad a_i$ is slot allocated to i

then

$$v_i(a, \theta_i) = P_{a_i} \cdot (\hat{\rho}_i \cdot \theta_i)$$

$\hat{\rho}_i$: estimated by auction holder

$$a^* \in \underset{a \in A}{\operatorname{argmax}} \sum_{i \in N} v_i(a, \theta_i)$$

a^*: efficient allocation

Note: allocation a will be efficient iff it is "rank-by-expected revenue" ($\hat{p}_i\,\theta_i$) mechanism

as moment maximising problem: sum maximised when maximum weight on maximum value → $a_i = 1$ if $\hat{p}_i\,\theta_i$ maximum

so, slot allocation is a sorting problem and hence computationally tractable

Natural candidate for payment: VCG

so, $(b_1, b_2, \dots b_n)$ are bids $\hat{\theta}_i = b_i$

wlog: $\hat{p}_1 b_1 \geq \hat{p}_2 b_2 \geq \dots \geq \hat{p}_n b_n$

a^* is s.t $a_i^* = i$

$a_{-i}^* \in \underset{a \in A}{\operatorname{argmax}} \sum_{j \neq i} v_j(a, \theta_j)$ here agents after i get one better slot than a^*

so, $p_i^{VCG} = \sum_{j \neq i} v_j(a_{-i}^*, \theta_j) - \sum_{j=1}^{n} v_i(a^*, \theta_j)$

$= \sum_{j=i}^{n-1} (p_j - p_{j+1})(\hat{p}_{j+1}\, b_{j+1}) \quad \forall i = 1, 2 \dots n-1$

$p_n^{VCG}(b) = 0$

then pay-per-click $= \frac{1}{p_i \hat{p}_i}\, p_i^{VCG}(b) \quad \forall i = 1, 2 \dots n$

Pros and cons of VCG:

Pros: ① DSIC
② No subsidy
③ Never charge agent with no item
④ no one looses money

Cons: ① Privacy and transparency – it reveals a lot
② susceptibility to collusion – public good decision of A or B

	A	B	Payment
1	200	0	150
2	100	0	50
3	0	250	0

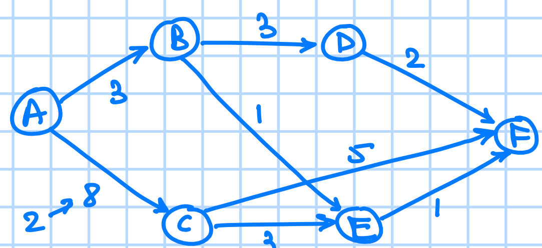
if 1,2 bid higher →

	A	B	Pay
1	250	0	100
2	150	0	0
3	0	250	0

they reduce their payments

③ Not frugal: payments could be very large

eg:



item delivery
source = A
destination = F

$P_{AB} = C(F \to A) - C(F \to B)$

$A \to B \to E \to F$

if 2: $P_{AB} = (-2-3-1) - (-1-1) = -4$

if 8: $P_{AB} = -4 + 2 - 8 = -10$

④ Revenue monotonicity violation: revenue should weakly inc with no of players

⑤ Not budget balanced: this is no-deficit mechanism but almost always surplus

surplus must be taken away / destroyed

Note: Nath and Sandholm gave efficiency and budget balance to know how much we burn and what efficiency we compromise

25th oct :

generalization of VCG mechanism:

we want to incorporate larger class of DSIC mechanism in quasi-linear domain

Defn: (Affine maximizer (AM) allocation rule) $w_i \geq 0, \forall i \in N$ (not all zero)

w_i - different weight for players

$K: A \to \mathbb{R}$ is any arbitrary function - translation

$$f^{AM}(\theta) \in \underset{a \in A}{\operatorname{argmax}} \left(\sum_{i \in N} w_i \theta_i(a) + K(a) \right)$$

$\theta_i(a)$ = valuation of player i on allocation

Note: $K \equiv 0$, $w_i = 1$, $\forall i \in N$ is efficient

$K \equiv 0$, $w_d = 1$, $w_i = 0$ $\forall i \neq 0$ is dictatorial

w_i's are different - not ANON

K is non-constant - different importance to different allocation

so, AM is superclass of VCG (efficient) allocation, and hence satisfy more properties

we can ask a characterisation question (like GS theorem) in quasi-linear setting for public goods

Defn: An AM rule f^{AM} with weights w_i $\forall i \in N$ and the function K satisfies independence of non-influencial agents (INA) if $\forall i \in N$ where $w_i = 0$

$$f^{AM}(\theta_i, \theta_{-i}) = f^{AM}(\theta_i', \theta_{-i}) \quad \forall \theta_i, \theta_i', \theta_{-i}$$

as zero wt agent should not break any tie either

Eg: if INA not satisfied then

$w_i = 0$

if there is a tie but allocation less preferred one for agent i

so, AM can be manipulated

Theorem: An AM rule satisfying INA is implementable in dominant strategies

proof:

we have to construct p^{AM} s.t (f^{AM}, p^{AM}) becomes DSIC

let
$$p_i^{AM}(\theta_i, \theta_{-i}) = \begin{cases} \frac{1}{w_i}\left[h_i(\theta_{-i}) - \left\{\sum_{j \neq i} w_j \theta_j(f^{AM}(\theta)) + K(f^{AM}(\theta))\right\}\right] & ; \forall i : w_i > 0 \\ 0 & ; \forall i : w_i = 0 \end{cases}$$

payoff of i if $w_i > 0$:

$$\text{payoff} = \theta_i(f^{AM}(\theta_i, \theta_{-i})) - p_i^{AM}(\theta_i, \theta_{-i})$$

$$= \frac{w_i}{w_i}\theta_i(f^{AM}(\theta_i, \theta_{-i})) + \frac{1}{w_i}\sum_{i \neq j} w_j \theta_j(f^{AM}(\theta)) + \frac{1}{w_i}K(f^{AM}(\theta)) - \frac{1}{w_i}h_i(\underline{\theta})$$

$$= \frac{1}{w_i}\left(\sum w_j \theta_j(f^{AM}(\underline{\theta})) + K(f^{AM}(\underline{\theta})) - h_i(\underline{\theta}) \right)$$

$$\geq \frac{1}{w_i}\Big[\Big\{\sum w_j \theta_j (f^{AM}(\theta_i', \theta_{-i})) + K(f^{AM}(\theta_i', \theta_{-i}))\Big\} - h_i(\theta_{-i})\Big]$$

$$= \theta_i (f^{AM}(\theta_i', \theta_{-i})) - p_i^{AM}(\theta_i', \theta_{-i})$$

for $w_i = 0$: $f^{AM}(\theta_i, \theta_{-i}) = f^{AM}(\theta_i', \theta_{-i}) \ \forall \theta_i, \theta_i', \theta_{-i}$

now, θ_i valuation contain all possible $\theta_i : A \to \mathbb{R}$ (no restriction)
with this unrestricted space of valuation, we can characterize the class DSIC mechanism in quasi-linear domain

Theorem: (Roberts) let A be finite $|A| \geq 3$. If the type of space is unrestricted, then every ONTO and dominant strategy implementable allocation rule must be an affine maximizer

Note: ONTO + DSIC $\Rightarrow$ AM given unrestricted domain, $|A| \geq 3$

Single object allocation:

$T_i \subseteq \mathbb{R}$ T_i – type set of agent i

$t_i \in T_i$ – value of agent i if she wins the object

an allocation a is a vector of length n that represents probability of winning the object by respective agent (a_0 = pro of not selling)

$$\Delta A = \{a \in [0,1]^{n+1}, \sum_{i=0}^{n} a_i = 1\}$$

$f: T_1 \times T_2 \times \cdots T_n \to \Delta A$ allocation rule

$v_i(a, t_i) = a_i t_i$ (allocation valuation)

$f(t_i, t_{-i}) = i$'s probability of winning object when type profile (t_i, t_{-i})

$f_0(t)$ = prob of not selling

Vickrey (second price auction):

let $t_{-i}^{(2)} = \max\limits_{j \neq i} \{v_j\}$

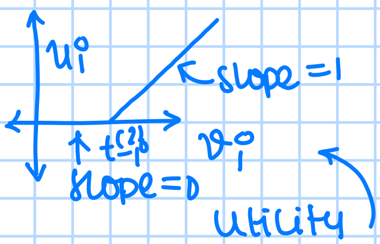
now agent i wins if $v_i > t_{-i}^{(2)}$

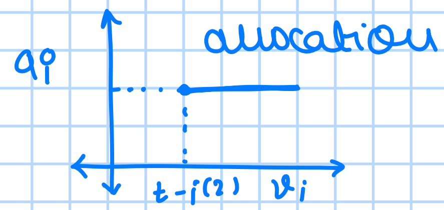
looses: $v_i < t_{-i}^{(2)}$

if $v_i = t_{-i}^{(2)}$ then tie breaking rule decides

$t_{-i}^{(2)}$ if i is winner is the payment

$$u_i(v_i, v_{-i}) = \begin{cases} 0 & : v_i < t_{-i}^{(2)} \\ v_i - t_{-i}^{(2)}; & v_i > t_{-i}^{(2)} \end{cases}$$





$g: I \to \mathbb{R}$ is convex $\forall x, y \in I$, $\lambda \in [0,1]$

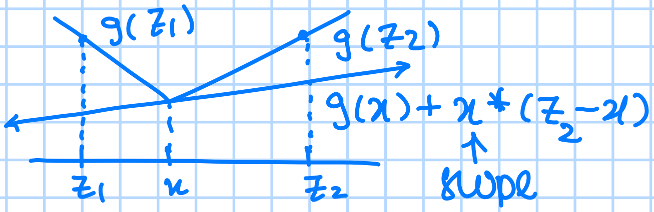
$$\lambda g(x) + (1-\lambda) g(y) \geq g(\lambda x + (1-\lambda) y)$$

Note: convex functions are continuous in their interior of its domain

Note: convex functions are differentiable almost everywhere

so, the point where convex function is not diff form a countable set

Defn: (subgradient) $\forall x \in I$, x^* is subgradient at x if $g(z) \geq g(x) + x^*(z-x)$ $\forall z \in I$



lemma: let $g: I \to \mathbb{R}$ be convex, if x is interior of I and g is diff at x then $g'(x)$ is unique subgradient of g

lemma: $g: I \to \mathbb{R}$ be convex function. $\forall x \in I$, a subgradient of g at x exist

Note: $I' \subseteq I$ be set of points where g is diff. Then $I \setminus I'$ has measure 0. Set of subgradients at a point form convex set.

$$g'_+(x) = \lim_{z \to x, z > x} g'(z)$$

$$g'_-(x) = \lim_{z \to x, z < x} g'(z)$$

Note: subgradients at $x \in I \setminus I'$ is $[g'_-(x), g'_+(x)]$
now let subgradients of g at $x \in I$ be $\partial g(x)$

then ① $\partial g(x) = \{g'(x)\}$ $\forall x \in I'$
② $\partial g(x) \neq \phi$ $\forall x \in I$

lemma: let $g: I \to \mathbb{R}$ be convex. let $\phi(z) \in \partial g(z)$ $\forall z \in I$, $\forall x, y \in I$ s.t $x > y$ we have $\phi(x) \geq \phi(y)$

so, subgradient functions are monotone

lemma: let $g: I \to \mathbb{R}$ be convex function, $\forall x, y \in I$

$$g(x) = g(y) + \int_y^x \phi(z)\,dz, \quad \phi: I \to \mathbb{R} \text{ s.t } \phi(z) \in \partial g(z) \ \forall z \in I$$

Myerson's lemma:

Defn: An allocation rule is non-decreasing if $\forall$ agent $i \in N$, $t_{-i} \in T_{-i}$ we have $f_i(t_i, t_{-i}) \geq f_i(s_i, t_{-i})$ $\forall s_i, t_i \in T_i$, $t_i > s_i$

If other agents fixed, probability of allocation never decreases with valuation

Theorm: (myerson) If $T_i = [0, b_i]$ $\forall i \in N$, and valuations are in product form then $f: T \to \Delta A$, $(P_1, P_2, \ldots P_n)$ are DSIC iff
f is non-decreasing
$P_i(t_i, t_{-i}) = P_i(0, t_{-i}) + t_i f_i(t_i, t_{-i}) - \int_0^{t_i} f_i(x, t_{-i})\,dx$ $\forall t_i \in T_i, \forall t_{-i} \in T_{-i}$ $\forall i \in N$

proof: (⇒) (f,p) is DSIC $T_i = [0, b_i]$ $v_i(a, t_i) = a_i t_i$

$$u_i(t_i, t_{-i}) = t_i f_i(t_i, t_{-i}) - p_i(t_i, t_{-i})$$
$$u_i(s_i, t_{-i}) = s_i f_i(s_i, t_{-i}) - p_i(s_i, t_{-i})$$

as (f,p) is DSIC

$$u_i(t_i,t_{-i}) = t_i f_i(t_i,t_{-i}) - p_i(t_i,t_{-i}) \geq t_i f_i(s_i,t_{-i}) - p_i(s_i,t_{-i})$$
$$= s_i f_i(s_i,t_{-i}) + (t_i - s_i) f_i(s_i,t_{-i}) - p_i(s_i,t_{-i})$$
$$= u_i(s_i,t_{-i}) + (t_i - s_i) f_i(s_i,t_{-i})$$
$$\Rightarrow u_i(t_i,t_{-i}) \geq u_i(s_i,t_{-i}) + (t_i - s_i) f_i(s_i,t_{-i}) \quad \text{— ①}$$

fixing t_{-i}:

$$g(t_i) = u_i(t_i,t_{-i}) \quad , \quad \phi(t_i) = f_i(t_i,t_{-i})$$

$\Rightarrow g(t_i) \geq g(s_i) + \phi(s_i)(t_i - s_i)$

$\Rightarrow \phi(s_i)$ is subgradient of g at s_i — ②

$\forall x_i, z_i \in T_i$, $y_i = \lambda x_i + (1-\lambda) z_i$ $\lambda \in [0,1]$

then from ①:

$$g(x_i) \geq g(y_i) + \phi(y_i)(x_i - y_i)$$
$$g(z_i) \geq g(y_i) + \phi(y_i)(z_i - y_i)$$

$\Rightarrow \lambda g(x_i) + (1-\lambda) g(z_i) \geq g(y_i) + \phi(y_i)(\lambda x_i + (1-\lambda) z_i - y_i)$

$\Rightarrow \lambda g(x_i) + (1-\lambda) g(z_i) \geq g(\lambda x_i + (1-\lambda) z_i)$

$\Rightarrow g$ is convex

then in ② $\phi(s_i)$ is subgradient at s_i

then

$\phi(x) \geq \phi(y)$ $\forall x > y$ ← lemma done above

i.e. $\phi(\cdot) = f_i(\cdot, t_{-i})$ is non-decreasing

and, $g(x) = g(y) + \int_y^x \phi(z)\,dz$ ← lemma done above

here $g(t_i) = g(0) + \int_0^{t_i} \phi(x)\,dx$

$\Rightarrow u_i(t_i,t_{-i}) = u_i(0,t_{-i}) + \int_0^{t_i} f_i(x,t_{-i})\,dx$

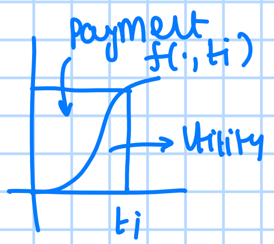
$\Rightarrow t_i f_i(t_i,t_{-i}) - p_i(t_i,t_{-i}) = -p_i(0,t_{-i}) + \int_0^{t_i} f_i(x,t_{-i})\,dx$

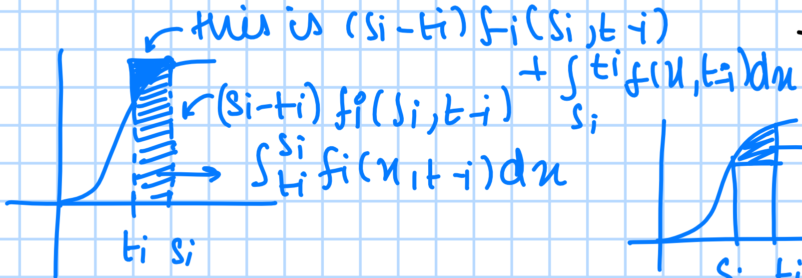
$\Rightarrow p_i(t_i,t_{-i}) = p_i(0,t_{-i}) + t_i f_i(t_i,t_{-i}) - \int_0^{t_i} f_i(x,t_{-i})\,dx$

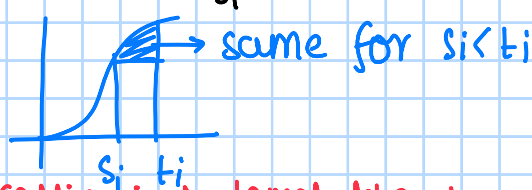
(⇐) f is non-decreasing, payment formula given

wlog $p_i(0,t_{-i}) = 0$

$$\big(t_i f_i(t_i,t_{-i}) - p_i(t_i,t_{-i})\big) - \big(t_i f_i(s_i,t_{-i}) - p_i(s_i,t_{-i})\big) = (s_i - t_i) f_i(s_i,t_{-i}) + \int_{s_i}^{t_i} f_i(x,t_{-i})\,dx \geq 0$$







conc: An allocation rule in a single object allocation setting is implementable in dominant strategies iff it is non-decreasing

28^{th} Oct:

Illustration of Myerson's lemma:

second price auction is a single object allocation

$$P_i^{\circ}(0,t_{-i}^{\circ}) + t_i f_i(t_i,t_{-i}) - \int_0^{t_i} f_i(x,t_i)\,dx$$

(non-dec f)

Efficient allocation with reserve is also non-dec

highest value < r (reserve price) no one gets object

otherwise highest bidder

$v_i > \max(t_{-i}^{(2)}, r)$ payment $= \max\{t_{-i}^{(2)}, r\}$

Not so common allocation rule:

$N = \{1,2\}$

$A = \{a_0, a_1, a_2\}$

$t = (t_1, t_2)$

$u(t) = \max\{2, t_1^2, t_2^3\}$ $A = \{a_0, a_1, a_2\}$

$0 > 1 > 2$ tie break

select a_0, a_1, a_2 on which the expression maxima

P_1 gets: $t_1 > \sqrt{\max\{2, t_2^3\}}$

P_2 gets: $t_2 > \sqrt[3]{\max\{2, t_1^2\}}$

Defn: (Ex-post individually rational) A mechanism (f,P) s.t

$t_i^{\circ} f_i(t_i, t_{-i}) - P_i(t_i, t_{-i}) \geq 0 \ \forall t_i \in T_i, t_{-i} \in T_{-i}, \forall i \in N$

so even after everyone revealed their type, participation weakly prefered

Lemma: In the single object allocation setting, DSIC mechanism (f,P)

① It is IR iff $\forall i \in N$, $\forall t_{-i}^{\circ} \in T_{-i}$, $P_i^{\circ}(0, t_{-i}^{\circ}) \leq 0$

② It is IR and satisfy no subsidy i.e $P_i(t_i, t_{-i}) \geq 0 \ \forall t_i \in T_i$, $t_{-i} \in T_{-i}$, $\forall i \in N$

iff $\forall i \in N, t_{-i} \in T_{-i}$, $P_i(0, t_{-i}^{\circ}) = 0$

proof: ① (f,P) is IR

then $0 - P_i(0, t_{-i}) \geq 0$

$\Rightarrow P_i(0, t_{-i}) \leq 0$

if $P_i(0, t_{-i}) \leq 0$ then

$t_i^{\circ} f_i^{\circ}(t_i^{\circ}, t_{-i}^{\circ}) - P_i^{\circ}(t_i, t_{-i}^{\circ}) = t_i f_i(t_i, t_{-i}) - P_i(0, t_{-i}^{\circ}) - t_i f_i(t_i, t_{-i})$

$+ \int_0^{t_i} f_i(x, t_{-i})\,dx \geq 0$

② IR $\Rightarrow P_i(0, t_{-i}) \leq 0$, if $P_i(0, t_{-i}) \geq 0$

$\Rightarrow P_i(0, t_{-i}) = 0$

clearly if $P_i(0, t_{-i}) = 0$ then (f,P) is IR and no-subsidy

Eg: object goes to highest bidder, but payment is st everyone is compensated some amount

wlog $t_1 > t_2 > \dots > t_n$

if highest, second highest get $\frac{1}{n}$ of third highest

$p_1(0,t_{-1}) = p_2(0,t_{-2}) = -\frac{1}{n} t_3$

everyone else gets $\frac{1}{n} t_2$

$p_i(0,t_{-i}) = -\frac{1}{n} t_2 \quad (i \neq 1,2)$

so, if $t_1 > t_2 > \cdots > t_n$

$1 \text{ pays} = -\frac{1}{n} t_3 + t_1 - \int_0^{t_1} f_1(x, t_{-1})\, dx$

$= -\frac{1}{n} t_3 + \cancel{t_1} - (\cancel{t_1} - t_2) = t_2 - \frac{1}{n} t_3$

$2 \text{ pays} = -\frac{1}{n} t_3$

$i \text{ pays} = -\frac{1}{n} t_2 \quad \text{for } i = 3, 4, \ldots n$

so, $\text{Total} = -\frac{1}{n} t_3 + t_2 - \frac{1}{n} t_3 - \frac{(n-2)}{n} t_2 = \frac{2}{n}(t_2 - t_3)$

as $n \to \infty$ total $\to 0$

Note: above was an example of deterministic mechanism which redistributes money

Eg: allocate object $\left(1 - \frac{1}{n}\right)$ to highest bidder

$\frac{1}{n}$ to second highest bidder

$p_i(0, t_{-i}) = -\frac{1}{n} \times$ second highest bid in $\{t_j, j \neq i\}$

wlog $t_1 > t_2 > t_3 > \cdots t_n$

$1 \text{ pays} = -\frac{1}{n} t_3 + \left(1 - \frac{1}{n}\right) t_1 - \left[\frac{1}{n}(t_2 - t_3) + \left(1 - \frac{1}{n}\right)(t_1 - t_2)\right]$

$= \left(1 - \frac{2}{n}\right) t_2$

$2 \text{ pays} = -\frac{1}{n} t_3 + \frac{1}{n} t_2 - \frac{1}{n}(t_2 - t_3) = 0$

$\text{every other} = -\frac{1}{n} t_2$

so, Together $= 0$

Note: Above is an example of randomised mechanism which redistributes money

Optimal mechanism design:

we want to maximise revenue earned by the auctioneer. The maximisation has to be done on the common prior distribution over types

$T_i = [0, b_i] \quad G = \text{common prior over } T = \bigtimes_{i=1}^{n} T_i$

distribution

g is density of G

$G_{-i}(s_{-i} \mid s_i)$ = condition dist over s_{-i} given i's type is s_i

$g_{-i}(s_{-i} \mid s_i)$ = density derived from g and bayes rules

every mechanism $(f, p_1, p_2, \ldots p_n)$ induces expected allocation and payment (α, π)

$$\alpha_i(\underbrace{s_i}_{\text{Reported}} ; \underbrace{t_i}_{\text{true}}) = \int_{s_{-i} \in T_{-i}} \underbrace{f_i(s_i, s_{-i})}_{\text{probabilistic allocation}} \times \underbrace{g_{-i}(s_{-i} \mid t_i)}_{\text{prior on other types}} \, ds_{-i}$$

Expected allocation

$$\pi_i(s_i ; t_i) = \int_{s_{-i} \in T_{-i}} p_i(s_i, s_{-i}) \, g_{-i}(s_{-i} \mid t_i) \, ds_{-i}$$

Expected payment

so, expected utility of agent i $u_i = t_i \alpha_i(s_i ; t_i) - \pi_i(s_i ; t_i)$

Defn: (Bayesian incentive compatibility (BIC)) A mechanism (f, p) is BIC if $\forall i \in N, \forall s_i, t_i \in T_i$

$$t_i \alpha_i(t_i ; t_i) - \pi_i(t_i ; t_i) \geq t_i \alpha_i(s_i ; t_i) - \pi_i(s_i ; t_i)$$

Similarly f is baysian implementable if $\exists p$ s.t (f, p) is BIC

assuming prior are independent

$$G(s_1, s_2, \ldots s_n) = \prod_{i \in N} G_i(s_i)$$

$$G(s_{-i} \mid t_i) = \prod_{j \neq i} G_j(s_j)$$

also $\alpha_i(t_i) = \alpha_i(t_i ; t_i)$

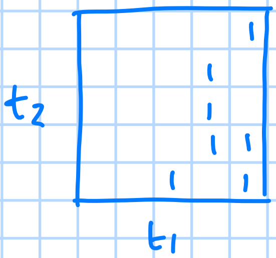
Assumptions

Defn: (Non-decreasing in expectation (NDE)) An allocation rule is NDE if $\forall i \in N$, $\forall s_i, t_i \in T_i$, $s_i < t_i$ we have

$$\alpha_i(s_i) \leq \alpha_i(t_i)$$

Note: ND $\Rightarrow$ NDE as $f_i(t_i ; t_{-i}) \geq f_i(s_i ; t_{-i})$ for $t_i > s_i$

now, NDE $\not\Rightarrow$ ND as we have a lot of counter examples



Theorem: (Myerson) A mechanism (f, P) in the independent prior setting is BIC iff ① f is NDE
② p_i satisfies $\pi_i(t_i) = \pi_i(0) + t_i \alpha_i(t_i) - \int_0^{t_i} \alpha_i(x) \, dx$

$\forall t_i \in T_i, \forall i \in N$

proof: same as Myerson theorem

Defn: A mechanism (f, P) is interim individually rational (IIR) if $\forall$ bidder $i \in N$ we have $t_i \alpha_i(t_i) - \pi_i(t_i) \geq 0 \; \forall t_i \in T_i$

lemma: A mechanism (f,p) is BIC and IIR iff

① f is NDE

② p_i satisfy $\pi_i(t_i)=\pi_i(0)+t_i\alpha_i(t_i)-\int_0^{t_i}\alpha_i(x)\,dx \quad \forall i\in N$

③ $\forall i\in N$, $\pi_i(0)\leq 0$

proof: ①, ② uniquely identifies BIC mechanism

then applying IIR at $t_i=0$ and ② gives us $\pi_i(0)\leq 0$

also ② ≥ 0 and $\pi_i(0)\leq 0$ gives us IIR

31st Oct :

Single agent optimal mechanism design :

$T = [0, \beta]$ $\quad M = (f, p)$

$$f : [0,\beta] \to [0,1]$$
$$p : [0, \beta] \to \mathbb{R}$$

Incentive compatibility [BIC and DSIC equivalent]

$$t f(t) - p(t) \geq t f(s) - p(s) \quad \forall\, t, s \in T$$

Individual rationality [IR and IIR equivalent]

$$t f(t) - p(t) \geq 0 \quad \forall\, t, s \in T$$

Note: Expected revenue earned by a mechanism M is given by

$$\pi^M = \int_0^\beta p(t) g(t)\, dt$$

Defn: (Optimal mechanism) An optimal mechanism M^* for a single agent is a mechanism in class of all IC and IR mechanism s.t

$$\pi^{M^*} \geq \pi^M \quad \forall M$$

IC, IR $M = (f, p)$ then from characterisation results we know f is monotone and

① $p(t) = p(0) + t f(t) - \int_0^t f(x)\, dx \quad (\because IC)$

② $p(0) \leq 0 \quad (\because IR)$

as we want to maximise revenue, we need $p(0) = 0$

$$\Rightarrow p(t) = t f(t) - \int_0^t f(x)\, dx$$

Note: In optimal mechanism, payment is completely given once the allocation is fixed, so we need to optimise only on f

$$\pi^f = \int_0^\beta p(t) g(t)\, dt \qquad \text{maximise this wrt } f$$
$$= \int_0^\beta \left(t f(t) - \int_0^t f(x)\, dx \right) g(t)\, dt$$
$$= \int_0^\beta \left(t f(t) - \int_0^t f(x)\, dx \right) g(t)\, dt$$
$$= \int_0^\beta t f(t) g(t)\, dt - \int_0^\beta \int_0^t f(x)\, dx\, g(t)\, dt \qquad \text{changing order}$$
$$= \int_0^\beta t f(t) g(t)\, dt - \int_0^\beta \int_x^\beta g(t)\, dt\, f(x)\, dx \quad (\because \text{fubini's theorem})$$
$$= \int_0^\beta t f(t) g(t)\, dt - \int_0^\beta \int_t^\beta g(x)\, dx\, f(t)\, dt$$
$$= \int_0^\beta \left(t f(t) g(t) - (1 - G(t)) f(t) \right) dt$$
$$= \int_0^\beta \left(t - \frac{(1 - G(t))}{g(t)} \right) g(t) f(t)\, dt$$

Lemma: For any implementable allocation rule f, we have

$$\pi^f = \int_0^\beta \left(t - \frac{(1-G(t))}{g(t)}\right) g(t) f(t)\,dt$$

Note: $w(t) = \left(t - \frac{(1-G(t))}{g(t)}\right)$ is also called virtual valuation

so now the optimisation problem becomes

OTP 1 : $$f = \underset{f \in \text{Non dec}}{\arg\max} \int_0^\beta w(t) g(t) f(t)\,dt$$

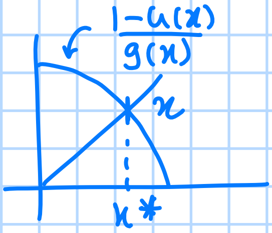
Assumptions :

we will assume G satisfies monotone hazard rate condition (MHR)
i.e $\frac{g(x)}{1-G(x)}$ is non decreasing in x

Note: Standard distributions like uniform and exponential satisfy MHR condition

Note: we can see that if G satisfies MHR then $\exists\, x^*$ s.t
$$x^* = \frac{1-G(x^*)}{g(x^*)}$$

then $w(x) = x - \frac{(1-G(x))}{g(x)}$



is s.t
$w(x) \geq 0 \ \forall\, x > x^*$
$w(x) \leq 0 \ \forall\, x < x^*$

so, unrestricted solution to OTP 1 is

$$f(t) = \begin{cases} 0 \ ; & t < x^* \\ 1 \ ; & t > x^* \\ \alpha \ ; & t = x^* \end{cases} \quad \alpha \in [0,1]$$

but this f is non-decreasing, so
f is solution of OTP 1

Theorem: A mechanism (f,P) under MHR condition is optimal iff

① f is given by $f(t) = \begin{cases} 0 \ ; & t < x^* \\ 1 \ ; & t > x^* \\ \alpha \ ; & t = x^* \end{cases}$, $\alpha \in [0,1]$

where x^* is solution of $x = \frac{1-G(x)}{g(x)}$

② $\forall\, t \in T$, $P(t) = \begin{cases} x^* \ ; & t \geq x^* \\ 0 \ ; & \text{otherwise} \end{cases}$

Optimal mechanism design with multiple agents:

Here we will call a mechanism optimal if its BIC, IIR and maximizes revenue, by previous results this reduces to

① f_i's are NDE $\forall\, i \in N$

② $\pi_i(t_i)$ has a specific integral formula, $\pi_i(0) = 0$

so, Expected payment made by agent i $= \int_{T_i} \pi_i(t_i) g_i(t_i) dt_i$ $\quad T_i = [0, b_i]$

$$= \int_0^{b_i} w_i(t_i) g_i(t_i) \alpha_i(t_i) dt_i$$
($\because$ similar to previous)

$w_i(t_i) = t_i - \frac{(1 - G(t_i))}{g(t_i)}$ (virtual valuation of player i)

$$\alpha_i(t_i) = \int_{T_{-i}} f_i(t_i, t_{-i}) g_{-i}(t_{-i}) dt_{-i}$$

so expected payment becomes

$$= \int_T w_i(t_i) f_i(t) g(t) dt$$

total revenue by all players

$$= \sum_{i \in N} \int_T w_i(t_i) f_i(t) g(t) dt$$

$$= \int_T \underbrace{\sum_{i \in N} (w_i(t_i) f_i(t))}_{\text{Expected total virtual valuation}} g(t) dt$$

so, OTP 2 : $f \in \underset{f \text{ is NDE}}{\operatorname{argmax}} \int_T \left(\sum_{i \in N} w_i(t_i) f_i(t)\right) g(t) dt$

unconstrained optimization problem OTP 2 is:

sold

$$f_i(t) = \begin{cases} 1 ; & w_i(t_i) \geq w_j(t_j) \ \forall j, \text{ break ties arbitrary} \\ 0 ; & \text{otherwise} \end{cases}$$

$f_i(t) = 0 \ \forall i \in N$, if $w_i(t_i) < 0 \quad \forall i \in N$

unsold

but above f can be not NDE

Defn : A virtual valuation w_i is regular if $\forall s_i, t_i \in T_i$ with $s_i < t_i$ it holds $w_i(s_i) \leq w_i(t_i)$

Note : MHR $\Rightarrow$ virtual valuation

Lemma : Suppose every agent's valuation are regular. The allocation rule of the optimal mechanism is same as the solution of the unconstrained problem

proof :

OTP 2 unconstrained f solution

$$f_i(t) = \begin{cases} 1 ; & w_i(t_i) \geq w_j(t_j) \ \forall j, \text{ break ties arbitrary} \\ 0 ; & \text{otherwise} \end{cases} \quad \text{(SOLD)}$$

$f_i(t) = 0 \ \forall i \in N$, if $w_i(t_i) < 0 \quad \forall i \in N$ (UNSOLD)

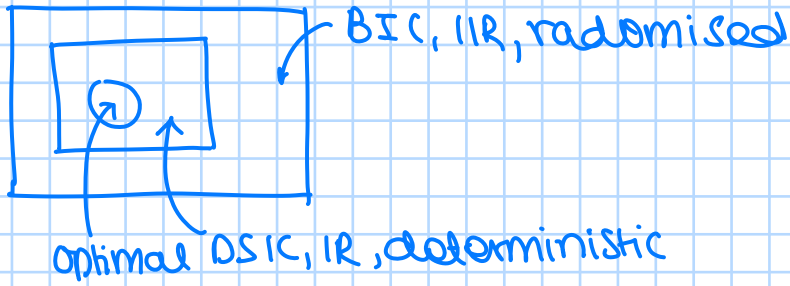
$w_i(t_i) \geq w_i(s_i) \quad \forall t_i > s_i \Rightarrow f_i(t_i, t_{-i}) \geq f_i(s_i, t_{-i}) \ \forall t_{-i} \in T_{-i}, \forall s_i < t_i$
$\Rightarrow f$ is ND $\Rightarrow f$ is NDE

so, OTP 2 solution when w_i are regular:

$$f_i(t)=\begin{cases}1; & w_i(t_i)\geq w_j(t_j)\ \forall j, \text{ break ties arbitrarily}\\ 0; & \text{otherwise}\end{cases} \quad \text{(sold)}$$

$$f_i(t)=0\ \forall i\in N, \text{ if } w_i(t_i)<0\ \forall i\in N \quad \text{(unsold)}$$

Note: above f is not only NDE, but also ND and deterministic



above is space of mechanisms with regular virtual valuations

Theorem: suppose every agent's valuation is regular. Then, for every type profile t, if $w_i(t_i)<0,\ \forall i\in N$, $f_i(t)=0,\ \forall i\in N$

otherwise $f_i(t)=\begin{cases}1; & \text{if } w_i(t_i)\geq w_j(t_j)\ \forall j\in N\\ 0; & \text{otherwise}\end{cases}$

with tie broken arbitrarily

payment: $p_i(t)=\begin{cases}0 & ; f_i(t)=0\\ \max(w_i^{-1}(0), \kappa_i^*(t_{-i})) & ; f_i(t)=1\end{cases}$

where $w_i^{-1}(0)$: value of t_i when $w_i(t_i)=0$

$\kappa_i^*(t_{-i})=\inf\{t_i \mid f_i(t_i,t_{-i})=1\}$

then (f,p) is an optimal mechanism

7th Nov:

Eg: $T_1 = [0,12]$
$T_2 = [0,18]$

$$\omega_1(t_1) = t_1 - \left(\frac{1-G(t)}{g(t)}\right) = t_1 - \frac{\left(1 - \frac{t_1}{12}\right)}{\frac{1}{12}} = 2t_1 - 12$$

$$\omega_2(t_2) = 2t_2 - 18$$

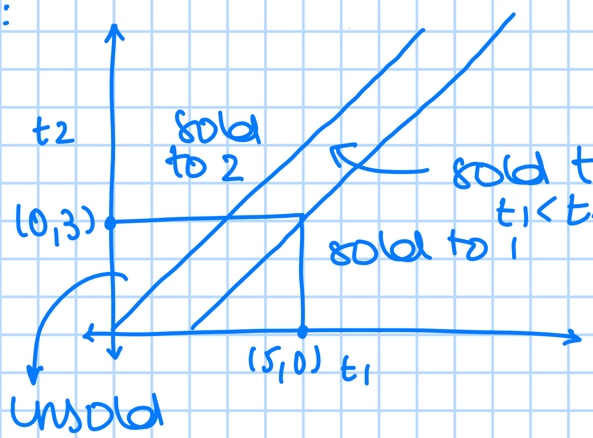
t_1	t_2	ω_1	ω_2	auction	p_1	p_2
4	8	-4	-2	unsold	0	0
2	12	-8	6	2	0	9
6	6	0	-6	1	6	0
9	9	6	0	1	6	0
8	15	4	12	2	0	11

Eg: symmetric bidders, valuation from same dist
$g_i = g,\ T_i = T\ \forall i \in N$
$\omega_i = \omega$
$\omega(t_i) > \omega(t_j)$ iff $t_i > t_j$
object to highest bidder, Not sold if $\omega^{-1}(0) > t_i\ \forall i \in N$

$$p_i = \max\{\omega^{-1}(0), \max_{j \neq i} t_j\}$$

Note: above is the second price auction with reserve price, it is efficient when object is sold

Eg:



$T_1 = [0,10]$
$T_2 = [0,6]$

$\omega_1(t_1) = 2t_1 - 10$
$\omega_2(t_2) = 2t_2 - 6$

$\omega_1(t_1) > \omega_2(t_2)$
$2t_1 - 10 > 2t_2 - 6$
$\Rightarrow t_1 - 5 > t_2 - 3$
$\Rightarrow t_1 > t_2 + 2$

$\omega_1(t_1) > 0,\ \omega_2(t_2) > 0$

unsold is inefficient above, also region where $t_1 > t_2$ but sold to 2

Efficiency and groves mechanism:

uniqueness of groves for efficiency $f^{eff}(t) \in \arg\max_{a \in A} \sum_{i \in N} t_i(a)$

Theorm: (Green and Laffont, Holmstrom) If the type space is 'sufficiently rich', every efficient and DSIC mechanism is a groves mechanism

proof: $A = \{a, b\}$ $\sum_{i \in N} t_i(a)$ and $\sum_{i \in N} t_i(b)$ welfare

suppose $\sum_{i \in N} t_i(a) \geq \sum_{i \in N} t_i(b)$ i.e a is chosen $\left(\because f^{eff} \in \arg\max_{a \in A} \sum t_i(a)\right)$

fix valuation of other agents to t_{-i}
fix value of i at alternative b as $t_i(b)$

$\exists\ t_i^*(a)$ s.t $\forall t_i(a) > t_i^*(a)$, a is outcome
and $\forall t_i(a) < t_i^*(a)$, b is outcome

using DSIC, $t_i^*(a)+\varepsilon = t_i(a)$ $\varepsilon > 0$

$$t_i^*(a)+\varepsilon - P_{ia} \geq t_i(b) - P_{ib}$$

payment has to be same for all allocati

similarly $t_i'(a) = t_i^*(a) - \delta$, $\delta > 0$

$$t_i(b) - P_{ib} \geq t_i^*(a) - \delta - P_{ia}$$

ε, δ are arbitrary

$$t_i^*(a) - P_{ia} = t_i(b) - P_{ib} \quad (\because \delta \to 0, \varepsilon \to 0) \quad \text{—— (1)}$$

but $t_i^*(a)$ is threshold of efficient outcome

$$\Rightarrow t_i^*(a) + \sum_{j\neq i} t_j(a) = t_i(b) + \sum_{j \neq i} t_j(b) \quad \text{—— (2)}$$

then from (1), (2): $P_{ia} - P_{ib} = \sum_{j\neq i} t_j(b) - \sum_{j\neq i} t_j(a)$

$$\Rightarrow P_{ix} = h_i(t_{-i}) - \sum_{j\neq i} t_j(x)$$

$\Rightarrow$ Payment are groves

Theorem: (Green and Laffont) No groves mechanism is budget balanced ie $\nexists\, P_i^G$ s.t

$$\sum_{i\in N} P_i^G(t) = 0 \quad \forall t \in T$$

Cor: If the valuation space is sufficiently large, no efficient mechanism can be both DSIC and BB

Theorem: (Green and Laffont) No groves mechanism is budget balanced ie $\nexists\, P_i^G$ s.t

$$\sum_{i\in N} P_i^G(t) = 0 \quad \forall t \in T$$

proof: consider two alternative $\{0,1\}$

0 : project not taken

1 : project is undertaken

outcome 0, every agent has zero value

if $\exists h_i, \forall i \in N$ s.t $\sum_{i\in N} P_i(t) = 0$ ← groves

now let ω_1^+, ω_1^- for player 1

ω_2 for player 2

s.t $\omega_1^+ + \omega_2 > 0$; project is build

$\omega_1^- + \omega_2 < 0$; project is not build

by budget balance: $h_1(\omega_2) - \omega_2 + h_2(\omega_1^+) - \omega_1^+ = 0$ for (ω_1^+, ω_2)

and $h_1(\omega_2) + h_2(\omega_1^-) = 0$ for (ω_1^-, ω_2)

from both we get $\omega_2 = h_2(\omega_1^+) - h_2(\omega_1^-) - \omega_1^+$

so, RHS on w_1 so if we sligtly alter w_2 we should get same inequalities

but then $\tilde{w}_2 \neq h_2(w_1^+) - h_2(w_1^-) - w_1^+$

weakening DSIC for budget balance:

$$a^*(t) \in \arg\max_{a \in A} \sum_{i \in N} t_i(a)$$

allocation still efficient

$$\delta_i(t_i) = \mathbb{E}_{t_{-i}|t_i}\Big[\sum_{j \neq i} t_j(a^*(t))\Big]$$

payment

d'Aspremont, Gerard-Varet, Arrow

$$p_i^{dAGVA}(t) = \frac{1}{n-1} \sum_{j \neq i} \delta_j(t_j) - \delta_i(t_i)$$

This payment implements efficient allocation rule in bayes Nash equilibrium

$$\mathbb{E}_{t_{-i}|t_i}[t_i(a^*(t)) - p_i^{dAGVA}(t)] = \mathbb{E}_{t_{-i}|t_i} \sum t_j(a^*(t)) - \mathbb{E}_{t_{-i}|t_i}\Big[\frac{1}{n-1}\sum_{j \neq i} \delta_j(t_j)\Big]$$

$$\geq \mathbb{E}_{t_{-i}|t_i} \sum t_j(a^*(t_i', t_{-i})) - \mathbb{E}_{t_{-i}|t_i}\Big[\frac{1}{n-1}\sum_{j \neq i} \delta_j(t_j)\Big]$$

$$= \mathbb{E}_{t_{-i}|t_i}[t_i(a^*(t_i', t_{-i})) - p_i^{dAGVA}(t_i', t_{-i})]$$

and budget is balanced as:

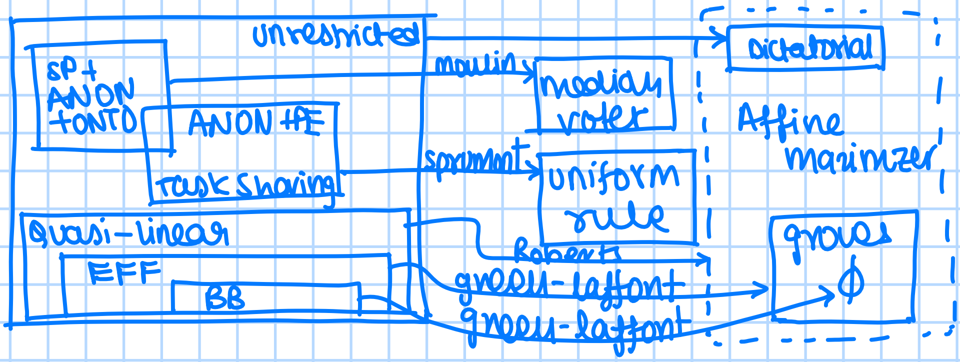
$$\sum p_i^{dAGVA}(t) = \frac{1}{n-1} \sum_{i \in N} \sum_{j \neq i} \delta_j(t_j) - \sum_{i \in N} \delta_i(t_i)$$

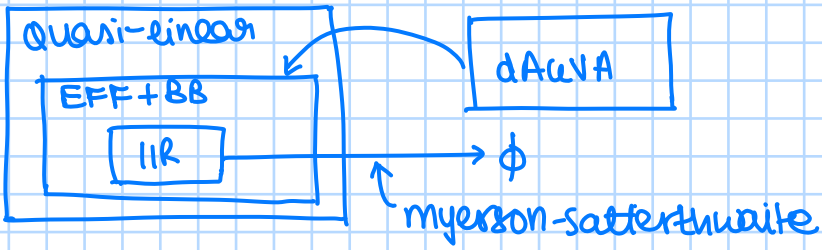
$$= \frac{n-1}{n-1} \sum_{j \in N} \delta_j(t_j) - \sum_{i \in N} \delta_i(t_i) = 0$$

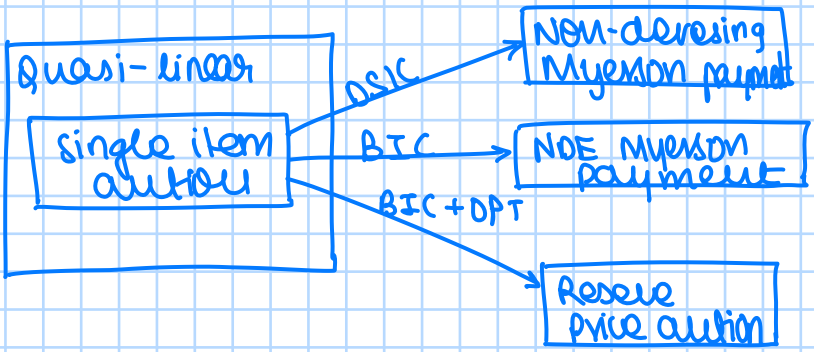
Theorem: dAGVA mechanism is efficient, BIC, and BB

Note: dAGVA is not IIR

Theorem: (Myerson, Satterthwaite) In a bilateral trade (that involves two type of agents: seller and buyer) no mechanism can be simultaniously BIC, efficient, IIR and budget balanced




Quasi-linear
EFF + BB
IIR
dAGVA
ϕ
myerson-satterthwaite


Quasi-linear
single item auction
DSIC
BIC
BIC + OPT
Non-decreasing Myerson payment
NDE Myerson payment
Reserve price auction