Tutorial-7:

$L := \frac{d^2}{dx^2}$

then $L\phi = \lambda\phi$ is the eigenvalue

Exe: $y'' - y = x$ in $[0,1]$ $y(0) = 0 = y(1)$ find the green's function

Ans:

$$y'' - y = x \quad \text{in } [0,1]$$

$$L = \frac{d^2}{dx^2} - 1 = x \quad Ly = y'' - y = x$$

$y_g = c_1 e^{x} + c_2 e^{-x}$ ← general solution $y'' = y$

$y_p = -x$

so, $y = -x + \dfrac{e^x - e^{-x}}{e - \frac{1}{e}}$ ∵ $y(0) = 0$

$$y(1) = -1 + \frac{e - \frac{1}{e}}{e - \frac{1}{e}} = 0$$

Boundary $y(0) = 0, y(1) = 0$ (Dirichlet Boundary)

$Lu = f$

$u = \int G f = L^{-1} f$

for each fixed $s \in (0,1)$ (property of green's function)

$v(x) = G(x,s)$ to satisfy $\partial_x^2 v - v = 0 \quad \forall x \neq s$

$v(0) = v(1) = 0$

$$v(x) = \begin{cases} A(s) e^{x} + B(s) e^{-x} ; & 0 \le x < s \\ C(s) e^{x} + D(s) e^{-x} ; & 0 \le s < x \le 1 \end{cases}$$

$v(0) = 0 \Rightarrow A(s) + B(s) = 0$

$v(1) = 0 \Rightarrow C(s) e + \frac{D(s)}{e} = 0$

$$v(x) = \begin{cases} A(s)(e^{x} - e^{-x}) \\ C(s)(e^{x} - e^{2-x}) \end{cases}$$

now, $G(x,s)$ is continuous, so $G(s^+,s) = G(s^-,s)$

$$A(s)(e^{s} - e^{-s}) = C(s)(e^{s} - e^{2-s}) \quad \text{①}$$

and jump condition: ← we are diff

$$G_x(s^+,s) - G_x(s^-,s) = 1$$

← diff

$$A(s)(e^{s} + e^{-s}) - C(s)(e^{s} + e^{2-s}) = 1 \quad \text{②}$$

①, ② gets us: $A(s)(e^{s} + e^{s}) = 1 + C(s)(e^{s} + e^{s})$

$A(s)(e^{-s} + e^{-s}) = 1 + C(s)(e^{2-s} + e^{2-s})$

$\Rightarrow \quad A(s) = \dfrac{e^{s} - e^{2-s}}{2(e^2 - 1)} \qquad C(s) = \dfrac{e^{s} - e^{-s}}{2(e^2 - 1)}$

So we get $G(x,s) = \begin{cases} \frac{e^{s} - e^{2-s}}{2(e^2-1)}(e^x - e^{-x}) ; & 0 \le x \le s \\ \underline{\quad\quad} ; & s \le x \end{cases}$

$$y(x) = \int_0^1 G(x,s)\, s\, ds$$

Now, $L = \frac{d^2}{dx^2} - 1$; look at the eigenvalue problem

$$L\phi = \lambda\phi \quad \text{with} \quad \phi(0) = 0 = \phi(1)$$

$$\phi'' = (1+\lambda)\phi$$

$\phi(x) = \sin n\pi x$ which satisfies boundary condition for $\lambda = -(1 + (n\pi)^2)$ $\forall n \geq 0$

$(\lambda_n, \phi_n) \to$ eigenvalue eigen pairs

$$\int_0^1 \phi_n \phi_m = \int_0^1 \sin(n\pi x) \sin(m\pi x)\, dx = \begin{cases} 0 ; & m \neq n \\ \frac{1}{2} ; & m = n \end{cases}$$

$\langle f, g \rangle = \int_0^1 fg$ then wrt inner product, ϕ_m, ϕ_n are ortogonal

for :

$f = \sum_{n \geq 0} c_n \phi_n \qquad f \in C^2$

we want to solve $Lu = f$

then $u = \sum_{n \geq 0} c_n \phi_n$

$$Lu = \sum_{n \geq 0} c_n L\phi_n = \sum_{n \geq 0} c_n \lambda_n \phi_n = f$$

$$\langle f, \phi_i \rangle = \langle \sum c_n \lambda_n \phi_n, \phi_i \rangle$$

$$= \langle c_i \lambda_i \phi_i, \phi_i \rangle$$

$$= \frac{c_i \lambda_i}{2}$$

$$c_m = \frac{\langle f, \phi_m \rangle}{\lambda_m \|\phi_m\|^2}$$

then $u(x) = \sum_{n \geq 0} \frac{\phi_n(x) \langle f, \phi_n \rangle}{\lambda_n \|\phi_n\|^2}$

$$= \sum_{n \geq 0} \frac{\phi_n(x) \int_0 f \phi_n dx}{\lambda_n \|\phi_n\|^2}$$

$$= \int_0^1 \left(\sum_{n \geq 0} \frac{\phi_n(x) \phi_n(s)}{\lambda_n \|\phi_n\|^2} \right) f(s)\, ds$$

$$= \int_0^1 H(x,s)\, f(s)\, ds$$

$$H(x,s) = \sum_{n \geq 0} \frac{Q_n(x)\, Q_n(s)}{\lambda_n \| Q_n \|^2}$$

$$= \sum_{n \geq 0} \frac{2}{-(1+(n\pi)^2)} \sin(n\pi x) \sin(n\pi s)$$

this should be the green's function

so, $K(x,s) = \sum_{n \geq 0} \frac{Q_n(x) Q_n(s)}{\lambda_n \| Q_n \|^2}$

Tutorial-8 :

$$e^A = I + A + \frac{1}{2!}A^2 + \dots$$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \text{ then } A = PDP^{-1} \quad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$\text{so, } e^A = e^{PDP^{-1}} = I + PDP^{-1} + \frac{1}{2!}(PDP^{-1})^2 + \dots$$

$$= P\left(I + D + \frac{1}{2!}D^2 + \dots\right)P^{-1}$$

$$= P e^{\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}} P^{-1}$$

$$e^{\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}} = I + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix} + \dots = \begin{pmatrix} 1+\lambda_1+\frac{1}{2!}\lambda_1^2+\dots & 0 \\ 0 & 1+\lambda_2+\frac{1}{2!}\lambda_2^2+\dots \end{pmatrix}$$

$$Ax = \lambda x, \ (Ax - \lambda x) = 0 \Rightarrow (A - I\lambda)x = 0$$

$$\Rightarrow \det(A - \lambda I) = 0$$

$$\Rightarrow \det\left(\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$$

$$\Rightarrow \det\begin{pmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = 0 \Rightarrow (2-\lambda)(1-\lambda) - 1 = 0$$

$$\Rightarrow 2 - 2\lambda - \lambda + \lambda^2 - 1 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda + 1 = 0$$

$$\Rightarrow \lambda = \frac{3 \pm \sqrt{9-4}}{2} \Rightarrow \lambda = \frac{3 \pm \sqrt{5}}{2}$$

$$D = \begin{pmatrix} \frac{3+\sqrt{5}}{2} & 0 \\ 0 & \frac{3-\sqrt{5}}{2} \end{pmatrix}$$

$$Ax = \left(\frac{3+\sqrt{5}}{2}\right)x \Rightarrow \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{3+\sqrt{5}}{2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow \left(2 - \frac{3+\sqrt{5}}{2}\right)x_1 + x_2 = 0 \text{ and } x_1 + \left(1 - \frac{3+\sqrt{5}}{2}\right)x_2 = 0$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1-\sqrt{5} \\ 2 \end{pmatrix} \qquad \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+\sqrt{5} \\ 2 \end{pmatrix}$$

$$P = \begin{pmatrix} x_1 & x_1' \\ x_2 & x_2' \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1-\sqrt{5} & 1+\sqrt{5} \\ 2 & 2 \end{pmatrix}$$

Note: A can also be $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \rightarrow$ Not diagonisable

$$\text{so, } A = I + N, \ e^A = e^{I+N} = e^I(I+N)$$

$$= \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}(I + N)$$

$$= \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e & 2e \\ 0 & e \end{pmatrix}$$

Tutorial-9 :

$$L(y) = -\frac{d}{dt}\left(p(t)y'(t)\right) + q(t)y(t) = \lambda r(t)y(t) + f(t) \quad \text{--- (1)}$$

$$U_1(y) = \alpha_1 y(a) + \alpha_2 y'(a) = 0$$

$$U_2(y) = \beta_1 y(b) + \beta_2 y'(b) = 0$$

Suppose λ is not a eigenvalue of $L(y)$

ϕ_1 s.t $L(\phi_1)=0$, $\phi_1(a)=\alpha_2$, $\phi_1'(a)=-\alpha_1$
$U_1(\phi_1)=0$,

similarly choose

ϕ_2 s.t $L(\phi_2)=0$, $\phi_2(b)=\beta_2$, $\phi_2'(b)=-\beta_1$

so $U_2(\phi_2)=0$

check that ϕ_1 and ϕ_2 are linearly independent

let $\phi = c_1(t)\phi_1(t) + c_2(t)\phi_2(t)$ be particular solution of (1), use variation of parameters

$$c_1' = \frac{\phi_2(t) f(t)}{p(t) W(\phi_1,\phi_2)}$$

$$c_2' = \frac{-\phi_1(t) f(t)}{p(t) W(\phi_1,\phi_2)}$$

To make $U_1(\phi)=0$, we choose $c_2(a)=0$

$U_2(\phi)=0$, we choose $c_1(b)=0$

$$c_1(t) = -\int_t^b \frac{\phi_2(s) f(s)}{p(s) W(\phi_1,\phi_2)(s)}\, ds$$

$$c_2(t) = -\int_a^t \frac{\phi_1(s) f(s)}{p(s) W(\phi_1,\phi_2)(s)}\, ds$$

$$\phi(t) = -\int_t^b \frac{\phi_1(t)\phi_2(s) f(s)}{p(s) W(\phi_1,\phi_2)(s)}\, ds - \int_a^t \left(\quad \right)$$

$$= \int_a^b G(t,s) f(s)\, ds \qquad G(t,s) = \begin{cases} \dfrac{-\phi_1(s)\,\phi_2(t)}{p(s) W(\phi_1,\phi_2)(s)} & \\ \dfrac{-\phi_2(s)\,\phi_1(t)}{p(s)\, W(\phi_1,\phi_2)} & t \leq s \leq b \end{cases}$$

Assignment-4

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Assignment - 4

1.1 (a) $y''(t) + \lambda y(t) = 0$ on $(0,1)$, $y(0)=0$, $y(1)+y'(1)=0$

Case I : $\lambda > 0$

then characteristic equation becomes

$$m^2 + \lambda = 0$$
$$r_1, r_2 = \pm \sqrt{\lambda}\, i$$

so, $y(t) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$ is solution

now, $y(0) = c_1 \cos(\sqrt{\lambda}\,0) + 0 = c_1 = 0$

$$y(1) + y'(1) = c_2 \sin\sqrt{\lambda} + \frac{d}{dx} c_2 \sin\sqrt{\lambda}x \Big|_{x=1}$$
$$= c_2 \sin\sqrt{\lambda} + \sqrt{\lambda}\, c_2 \cos\sqrt{\lambda}$$
$$= 0$$

$$c_2(\sin\sqrt{\lambda} + \sqrt{\lambda}\cos\sqrt{\lambda}) = 0$$

now if $c_2 = 0$ then we will get trivial solution

so, $\sin\sqrt{\lambda} + \sqrt{\lambda}\cos\sqrt{\lambda} = 0$

if $\cos\sqrt{\lambda} = 0$

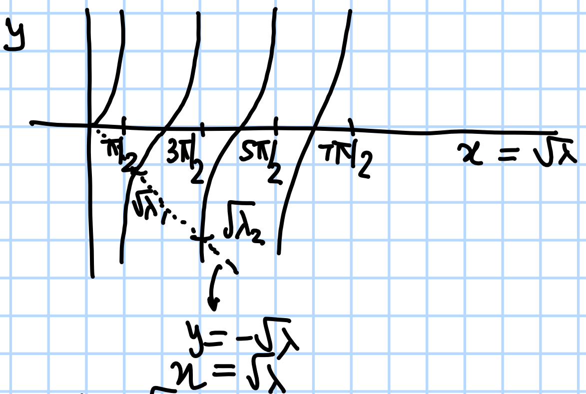
this makes $\sqrt{\lambda} = \frac{(2n+1)\pi}{2}$ for $n = 0,1,2\ldots$

then $\sin\sqrt{\lambda} = 1$ for $n = 0,2,4,\ldots$
$\sin\sqrt{\lambda} = -1$ for $n = 1,3,5,\ldots$

but if $\cos\sqrt{\lambda} = 0$ then
$\sin\sqrt{\lambda} = 0$ this is a contradiction
so, $\cos\sqrt{\lambda} \neq 0$

then dividing by $\cos\sqrt{\lambda}$: $\tan\sqrt{\lambda} + \sqrt{\lambda} = 0$
$\Rightarrow \tan\sqrt{\lambda} = -\sqrt{\lambda}$

letting x axis be $\sqrt{\lambda}$ value, graph looks like :



Asymptotes of $\tan\sqrt{\lambda}$

we see that $\sqrt{\lambda_n}$ has value very close to $(2n+1)\pi$ for $n = 0,1,\ldots$
but not equal to $\frac{(2n+1)\pi}{2}$ (so $\cos\sqrt{\lambda} \neq 0$)

let $\sqrt{\lambda_n} \approx \frac{(2n+1)\pi}{2}$ $n = 0,1,\ldots$

$$\Rightarrow \lambda_n \approx \frac{(2n+1)^2\pi^2}{4} \quad n = 0,1,\ldots$$

and corresponding $y(t) = \sin\sqrt{\lambda_n}\, t$ for $n = 0,1,2\ldots$

$y(t) = \sin\sqrt{\lambda_n}t$ are the eigenfuctions

$\lambda_n \approx \frac{(2n+1)^2\pi^2}{4}$ $n = 0,1,2\ldots$ are eigenvalues

Case II : $\lambda = 0$:

$y''(t) = 0 \Rightarrow y(t) = c_1 t + c_0$

now, as $y(0) = 0$

$\Rightarrow c_1(0) + c_0 = 0$

$\Rightarrow c_0 = 0$

$y(t) = c_1 t$

now $y(1) + y'(1) = c_1(1) + c_1 = 0$

$\Rightarrow c_1 = 0$

so, $y(t) = 0$

or for $\lambda = 0$, we get trivial solution

so, $\lambda = 0$ not an eigenvalue

Case III : $\lambda < 0$:

$y''(t) + \lambda y(t) = 0$, charactestic equation becomes:

$m^2 + \lambda = 0$

$\Rightarrow r_1, r_2 = \pm\sqrt{-\lambda}$

so, $y(t) = c_1\cosh(\sqrt{\lambda}t) + c_2\sinh(\sqrt{\lambda}t)$

where $\cosh x = \frac{e^x + e^{-x}}{2}$

$\sinh x = \frac{e^x - e^{-x}}{2}$

now, $y(0) = c_1(1) + c_2(0) = c_1 = 0$

$y(1) + y'(1) = c_2\sinh(\sqrt{-\lambda}) + \sqrt{-\lambda}c_2\cosh(\sqrt{-\lambda}) = 0$

$= c_2(\sinh(\sqrt{-\lambda}) + \sqrt{-\lambda}\cosh\sqrt{-\lambda}) = 0$

as $\lambda < 0 \Rightarrow -\lambda > 0$

$\Rightarrow \sqrt{-\lambda} > 0$

true for $x = \sqrt{-\lambda}$, $e^{2x} > 1 \Rightarrow \frac{e^{2x} - 1}{2e^x} > 0$

$\Rightarrow \frac{e^x - e^{-x}}{2} > 0$

$\Rightarrow \sinh\sqrt{-\lambda} > 0$

trivial to see $\cosh\sqrt{-\lambda} > 0$

and so, $\sqrt{-\lambda}\cosh\sqrt{-\lambda} + \sinh\sqrt{-\lambda} > 0$

so, $c_2 = 0$

as $c_1 = 0, c_2 = 0 \Rightarrow y(t) = 0$

this is the trivial solution,

so, $\forall \lambda \in (-\infty, 0)$, λ is not an eigenvalue

(b) $y''(t) + \lambda y(t) = 0$ on $(0, 2\pi)$, $y(0) = y(2\pi)$ and $y'(0) = y'(2\pi)$

Case I : $\lambda > 0$:

In this case, the charetistic polynomial we get

$m^2 + \lambda = 0$

i.e $m = \pm\sqrt{\lambda}i$ roots

so, $y(t) = c_1\sin\sqrt{\lambda}t + c_2\cos\sqrt{\lambda}t$ form of solution now from BVP: $y(0)=y(2\pi)$

$$C_2 = C_1 \sin 2\pi\sqrt{\lambda} + C_2 \cos 2\pi\sqrt{\lambda} \quad \text{--- (1)}$$

and $y'(t) = \sqrt{\lambda} c_1 \cos\sqrt{\lambda} t - \sqrt{\lambda} c_2 \sin\sqrt{\lambda} t$

$$y'(0) = \sqrt{\lambda} c_1$$

$$y'(2\pi) = \sqrt{\lambda} c_1 \cos\sqrt{\lambda} 2\pi - \sqrt{\lambda} c_2 \sin\sqrt{\lambda} 2\pi$$

so, $\cancel{\sqrt{\lambda}} c_1 = \cancel{\sqrt{\lambda}} c_1 \cos\sqrt{\lambda} 2\pi - \cancel{\sqrt{\lambda}} c_2 \sin\sqrt{\lambda} 2\pi$

$$c_1 = c_1 \cos\sqrt{\lambda}\, 2\pi - c_2 \sin\sqrt{\lambda}\, 2\pi \quad \text{--- (2)}$$

from (1) and (2) we get

$$\underbrace{\begin{bmatrix} \sin 2\pi\sqrt{\lambda} & \cos 2\pi\sqrt{\lambda} - 1 \\ \cos 2\pi\sqrt{\lambda} - 1 & -\sin 2\pi\sqrt{\lambda} \end{bmatrix}}_{W(\lambda)} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

if $\det(W(\lambda)) \neq 0$ then $W(\lambda)$ is invertible and we see

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = W^{-1}(\lambda) \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow c_1 = 0, c_2 = 0$, y becomes trivial

so for λ as eigenvalue we need $\det(W(\lambda)) = 0$

$$\det \begin{bmatrix} \sin 2\pi\sqrt{\lambda} & \cos 2\pi\sqrt{\lambda} - 1 \\ \cos 2\pi\sqrt{\lambda} - 1 & -\sin 2\pi\sqrt{\lambda} \end{bmatrix} = -\sin^2 2\pi\sqrt{\lambda} - (\cos^2 2\pi\sqrt{\lambda} + 1 - 2\cos 2\pi\sqrt{\lambda})$$

$$= -1 - 1 + 2\cos 2\pi\sqrt{\lambda}$$

$$= -2 + 2\cos 2\pi\sqrt{\lambda} = 0$$

so, $\cos 2\pi\sqrt{\lambda} = 1$

i.e $\cancel{2}\cancel{\pi}\sqrt{\lambda} = \cancel{2}n\cancel{\pi}$ for $n = 1, 2, \ldots$

$\Rightarrow \lambda = n^2$ for $n = 1, 2, \ldots$

let $\lambda_n = n^2$ then $\det(W(\lambda_n)) = 0$, so

$$y(t) = c_1 \sin\sqrt{\lambda_n} t + c_2 \cos\sqrt{\lambda_n} t$$

so, eigenfunction corresponding to $\lambda_n = n$ will be $\sin 2\pi n$ and $\cos 2\pi n$, we

<u>case II</u>: $\lambda = 0$:

in this case the characteristic polynomial

$$m^2 = 0 \Rightarrow m = 0, 0 \quad \text{i.e}$$

$$y(t) = c_1 t + c_0$$

from BVP: $y(0) = y(2\pi) \Rightarrow c_1(\cancel{0}) + \cancel{c_0} = c_1(2\pi) \cancel{+ c_0}$

$\Rightarrow c_1(2\pi) = 0$

$\Rightarrow c_1 = 0$

and $y'(t) = c_1$ as $y'(0) = y'(2\pi) \Rightarrow c_1 = c_1$

so, $y(t) = c_0$ or constant
and so, $\lambda = 0$ is an eigenvalue and corresponding eigenfunction is $y(t) = 1$

Case III : $\lambda < 0$:

in this case charactestic polynomial
$$m^2 + \lambda = 0$$
$$\Rightarrow m = \pm\sqrt{-\lambda}$$

so, $y(t) = c_1 e^{\sqrt{-\lambda}t} + c_2 e^{-\sqrt{-\lambda}t}$

$y'(t) = \sqrt{-\lambda}c_1 e^{\sqrt{-\lambda}t} - \sqrt{-\lambda}c_2 e^{-\sqrt{-\lambda}t}$

By BVP:
$$y(0) = y(2\pi)$$
$$\Rightarrow c_1 + c_2 = c_1 e^{2\pi\sqrt{-\lambda}} + c_2 e^{-2\pi\sqrt{-\lambda}} \quad \text{— (1)}$$

$$y'(0) = y'(2\pi)$$
$$\Rightarrow c_1 - c_2 = c_1 e^{2\pi\sqrt{-\lambda}} - c_2 e^{-2\pi\sqrt{-\lambda}} \quad \text{— (2)}$$

adding (1), (2): $2c_1 = 2c_1 e^{2\pi\sqrt{-\lambda}}$

so $c_1 = 0$ or $1 = e^{2\pi\sqrt{-\lambda}}$
but as $\lambda < 0 \Rightarrow 2\pi\sqrt{-\lambda} > 0$
$\Rightarrow e^{2\pi\sqrt{-\lambda}} > 1$

$\Rightarrow c_1 = 0$

similarly subtracting (1) and (2) gives us $c_2 = 0$
then
$y(t) = 0$ or trivial, $\forall \lambda \in (-\infty, 0)$ no eigenvalue

1.2 $y''(t) + \lambda y(t) = 0$
$$y(0) = y(\pi)$$
$$y'(0) = -y'(\pi)$$

now, $L(y) = \left(\frac{\partial^2}{\partial t^2} + \lambda\right) y$

then for any $u, v \in C^2[0,\pi]$ s.t they satisfy BV
$$\int_0^\pi vLu - uLv\,dt = \int_0^\pi v(u'' + \cancel{\lambda u}) - u(v'' + \cancel{\lambda v})\,dt$$
$$= \int_0^\pi v u'' - u v''\,dt$$
$$= vu'\Big|_0^\pi - \int_0^\pi \cancel{v'u'}\,dt - u'v\Big|_0^\pi + \int_0^\pi \cancel{v'u'}\,dt$$
$$= vu' - uv'\Big|_0^\pi$$
$$= v(\pi)u'(\pi) - u(\pi)v'(\pi) - v(0)u'(0) - u(0)v'(0)$$
$$= -v(0)u'(0) + u(0)v'(0) - v(0)u'(0) - u(0)v'(0)$$
$$= 2v(\pi)u'(\pi)$$
(∵ By using Boundary condition)

$2v(\pi)u'(\pi)$ is not necessarily 0
so, L is not self-adjoint given Boundary condition

Case I: $\lambda \in \mathbb{C}, \lambda \neq 0$

characteristic polynomial is $m^2 + \lambda = 0$
$\Rightarrow m = \pm\sqrt{-\lambda}$

and so solution is $y(t) = c_1 e^{\sqrt{-\lambda}\,t} + c_2 e^{-\sqrt{-\lambda}\,t}$

$(\because \lambda \neq 0, \lambda \in \mathbb{C})$

then from BV: $y(0) = y(\pi)$, $y'(0) = -y'(\pi)$

$y(0) = c_1 + c_2$

$y(\pi) = c_1 e^{\pi\sqrt{-\lambda}} + c_2 e^{-\pi\sqrt{-\lambda}}$

$y'(t) = \sqrt{-\lambda}\, c_1 e^{\sqrt{-\lambda}t} - \sqrt{-\lambda}\, c_2 e^{-\sqrt{-\lambda}\,t}$

$y'(0) = \sqrt{-\lambda}\, c_1 - \sqrt{-\lambda}\, c_2$

$y'(\pi) = \sqrt{-\lambda}\, c_1 e^{\sqrt{-\lambda}\pi} - \sqrt{-\lambda}\, c_2 e^{-\sqrt{-\lambda}t}$

from BV: $y(0) = y(\pi)$

and $y'(0) = -y'(\pi)$

we get:
$$\underbrace{\begin{bmatrix} e^{\pi\sqrt{-\lambda}} - 1 & e^{-\pi\sqrt{-\lambda}} - 1 \\ e^{\pi\sqrt{-\lambda}} - 1 & 1 + e^{-\pi\sqrt{-\lambda}} \end{bmatrix}}_{W(\lambda)} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\det(W(\lambda)) = e^{\pi\sqrt{-\lambda}} - 1 + 1 - e^{-\pi\sqrt{-\lambda}} - \left(-1 + e^{\pi\sqrt{-\lambda}} - e^{-\pi\sqrt{-\lambda}} + 1\right)$$
$$= 0$$

$\forall \lambda \in \mathbb{C}\setminus\{0\}$ $\det(W(\lambda)) = 0$, so, system is not invertible and for all values of c_1, c_2, it works

so $\lambda \in \mathbb{C}\setminus\{0\}$ is an eigenvalue
corresponding $e^{\sqrt{\lambda}t}$, $e^{-\sqrt{-\lambda}t}$ are eigenfunctions

Case II: $\lambda = 0$:

then $y'' = 0 \Rightarrow y(t) = c_1 t + c_2$
$y(0) = c_2 = y(\pi) = c_1\pi + c_2$
so, $c_1\pi = 0 \Rightarrow c_1 = 0$

and $y'(t) = c_1$ and $y'(0) + y'(\pi) = c_1 + c_1 = 0$
$\Rightarrow c_1 = 0$

so, for any value of c_2 $y(t) = c_2$ is a solution

so, $\lambda = 0$ is eigenvalue, corresponding eigenfunction $y(t) = 1$

from case I, II $\forall \lambda \in \mathbb{C}$ is an eigenvalue

1.3 $[py']' - qy + \lambda r y = 0 \quad 0<x<1$

p, q, r are real-valued and cont on $[0,1]$
$p, r > 0 \ \forall x \in [0,1]$

$\alpha_1 y(0) + \alpha_2 y'(0) = 0$
$\beta_1 y(1) + \beta_2 y'(1) = 0$ } Boudry condition

$\alpha_1^2 + \alpha_2^2 \neq 0, \quad \beta_1^2 + \beta_2^2 \neq 0$

(a) now $Ly = [py']' - qy + \lambda r y$

then for any u, v satisfying boudry condition

$$\int_0^1 Lu\,v - Lv\,u\,dt = [p(vu' - uv')]_0^1$$

(∵ done in class)

so, $\int_0^1 Lu\,v - Lv\,u\,dt = p(1)[v(1)u'(1) - u(1)v'(1)] - p(0)[v(0)u'(0) - u(0)v'(0)]$

case I: $\beta_1 \neq 0$ then $v(1)u'(1) - u(1)v'(1)$
$= -\frac{\beta_2 v'(1)}{\beta_1}u'(1) + \frac{\beta_2 u'(1) v'(1)}{\beta_1}$ (∵ Boudry condition)
$= 0$

case II: $\beta_2 \neq 0$ then $v(1)u'(1) - u(1)v'(1)$
$= -\frac{\beta_1 v(1)\cancel{u(1)}}{\cancel{\beta_2}} + \frac{\beta_1 \cancel{v(1)}u(1)}{\cancel{\beta_2}}$ (∵ Boudry condition)
$= 0$

case III: $\beta_1 = \beta_2 = 0$
this is not possible as $\beta_1^2 + \beta_2^2 \neq 0$ given

so, from three cases, we get $v(1)u'(1) - v'(1)u(1) = 0$
similarly if we have $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$ we will get
$v(0)u'(0) - v'(0)u(0) = 0$

so, $\int_0^1 (Lu)v - (Lv)u\,dt = 0$, and L is then self-adjoint

(b) if λ is an eigenvalue, then

$$(py')' - qy + \lambda r y = 0 \quad \text{——— ①}$$

taking conjugate:

$$(p\bar{y}')' - q\bar{y} + \bar{\lambda} r \bar{y} = 0 \quad \text{——— ②}$$

multiplying ① with $\bar{y}$ and ② with y

$$\bar{y}(py')' - qy\bar{y} + \lambda r y\bar{y} = 0 \quad \text{——— ③}$$
$$y(p\bar{y}')' - qy\bar{y} + \bar{\lambda} r y\bar{y} = 0 \quad \text{——— ④}$$

subtracting ③ and ④ we get:

$$\bar{y}(py')' - y(p\bar{y}')' = \bar{\lambda} r y\bar{y} - \lambda r y\bar{y}$$

let $L(y) = (py')'$

then

$$\int_0^1 \bar{y} L(y) - y L(\bar{y}) = \int_0^1 (\bar{\lambda}-\lambda) r y \bar{y} \, dt \quad \text{—— ⑤}$$

as y is non-trivial (as eigenfunction)
$\exists \alpha \in [0,1]$ s.t
$y(\alpha) \neq 0$
$\Rightarrow y(\alpha)\bar{y}(\alpha) \neq 0$
$\Rightarrow |y(\alpha)|^2 > 0$ for some α

and as $r > 0 \ \forall x \in [0,1]$

$$\int_0^1 r|y|^2 dt > 0$$

and so, ⑤ becomes:

$$\int_0^1 \bar{y} L(y) - y L(\bar{y}) \, dt = (\bar{\lambda}-\lambda) \underbrace{\int_0^1 r|y|^2 dt}_{>0 \text{ term}}$$

and as L is sturm-liouville type

$$\int_0^1 \bar{y} L(y) - y L(\bar{y}) \, dt = p[(\bar{y})'y - (y)'\bar{y}]_0^1$$

and as y is eigenfunction of λ it follow boundry condition

as $\bar{y}$ is conjugate of y, it will also follow boundy condition

then from (a) $\int_0^1 \bar{y} L(y) - y L(\bar{y}) \, dt = 0$

so, $0 = (\bar{\lambda}-\lambda) \underbrace{\int_0^1 r|y|^2 dt}_{>0}$

$\Rightarrow \bar{\lambda} - \lambda = 0$
$\Rightarrow \bar{\lambda} = \lambda$

so, λ is real, as this is true for any λ eigenvalue every eigenvalue is real

now if λ_1, λ_2 are two eigenvalues, corresponding y_1, y_2 are two eigenfunctions

$$(py_1')' - q y_1 + \lambda_1 r y_1 = 0$$
$$(py_2')' - q y_2 + \lambda_2 r y_2 = 0$$

multiplying both by y_2 and y_1 we get

$$y_2 (py_1')' - q y_1 y_2 + \lambda_1 r y_1 y_2 = 0$$
$$y_1 (py_2')' - q y_1 y_2 + \lambda_2 r y_1 y_2 = 0$$

Subtracting both, $y_2(py_1')' - y_1(py_2')' = \lambda_2 r y_1 y_2 - \lambda_1 r y_1 y_2$

$$\Rightarrow \int_0^1 y_2(py_1')' - y_1(py_2')' dt = (\lambda_2-\lambda_1)\int_0^1 r y_1 y_2 dt$$

let $Ly = (py')'$ with boundary condition same as given

then $\int_0^1 y_2(py_1')' - y_1(py_2')' dt$

$$= p[y_2 y_1' - y_2' y_1]_0^1 \quad (\because \text{done in class})$$

$$= 0 \quad (\because \text{(a) and } y_1, y_2 \text{ are eigenvalues so they follow boundary condition})$$

so, $0 = (\lambda_1 - \lambda_2)\int_0^1 r y_1 y_2 dt$ —— ⑥

$$\Rightarrow \lambda_1 = \lambda_2 \text{ or } \int_0^1 r y_1 y_2 dt = 0$$

also, if y_1, y_2 are eigenfunctions of same λ then

wronskian $W = p(y_1 y_2' - y_2 y_1')$

as $W' = (q_2 - q_1) y_1 y_2$ ($\because$ theorm done in class)

as same λ, $q_2 = q_1$

$\Rightarrow W' = 0$

so, $W = C$ some constant

$$W(0) = P(0)[y_1(0)y_2'(0) - y_2(0)y_1'(0)] = 0$$

($\because$ from (a) calculation)

so, $W \equiv 0$ and so y_1, y_2 are linearly dependent —— ⑦

so from ⑥, ⑦ for $\lambda \neq \lambda^*$, $\int_0^1 r y y^* = 0$, λ, λ^* are eigenvalues y, y^* are corresponding eigenfunctions

and, if $\lambda = \lambda^*$ then y, y^* are linearly dependent so, λ is distinct (eigenspace of dim 1)

1.4 (a) Hermite equation: $y''(t) - 2ty'(t) + \lambda y(t) = 0$ with $y(0)=0, y(1)=0$

$$P(t) = e^{\int_0^t (-2s) ds}$$

$$= e^{[-s^2]_0^t} = e^{-t^2} > 0 \ \forall t \in [0,1]$$

then $(py')'' = py'' + p'y'$

so, multiplying given equation with $P(t)$ gives us

$$\underbrace{py'' - 2tpy'}_{} + \lambda p y = 0$$

$$-(py')' = \lambda p y$$

$$\Rightarrow -(e^{-t^2} y')' = \lambda e^{-t^2} y \quad (q \equiv 0, w = e^{-t^2})$$

so, $P(t)=e^{-t^2}$, $q(t)\equiv 0$, $w(t)=e^{-t^2}$

$P, w>0$, P, q, w all satisfy sturm-liouille condition

and $\int_0^1 vLu - uLv\,dt = P[vu'-uv']_0^1$

$= e^{-1}[v(1)\cancel{u'(1)} - \cancel{u(1)}v'(1)]$

$- e^0[v(0)\cancel{u'(0)} - \cancel{u(0)}v'(0)]$

$=0$

so, BVP becomes self-adjoint BVP form

(b) Euler equation: $t^2y''(t)+ty'(t)+\lambda y(t)=0$ with $y(1)=0$, $y(e)=0$

now, as $t\in[1,e]$ $t\neq 0$, so dividing by t^2 we get

$$y''(t)+\frac{y'(t)}{t}+\lambda\frac{y(t)}{t^2}=0$$

now $P(t)=e^{\int_1^t \frac{1}{s}ds}=e^{\ln(t)}=t$

and so multiplying by t:

$$ty''+y'+\lambda\frac{y}{t}=0$$

$(ty')'=ty''+y'$, so

$$(ty')'+\lambda\frac{y}{t}=0$$

$\Rightarrow -(ty')'=\frac{\lambda y}{t}$ where $w(t)=\frac{1}{t}$ $q(t)\equiv 0$

$P(t)=t$ $\forall\, t\in[1,e]$

$w, P>0$ $\forall\, t\in[1,e]$

above satisfy sturm-liouille condition

and $\int_1^e vLu-uLv\,dt = P[vu'-uv']_1^e$

$= P(e)[v(e)\cancel{u'(e)} - \cancel{u(e)}v'(e)]$

$-P(1)[v(1)\cancel{u'(1)} - \cancel{u(1)}v'(1)]$

$=0$

so, BVP becomes self-adjoint BVP form

1.5 $y''+\lambda y=0$ in $[0,\pi]$ with

$$m_{11}y(0)+m_{12}y'(0)+n_{11}y(\pi)+n_{12}y'(\pi)=0$$

$$m_{21}y(0)+m_{22}y'(0)+n_{21}y(\pi)+n_{22}y'(\pi)=0$$

$$m_{11}m_{22}-m_{12}m_{21}\neq 0 \text{ or } n_{11}n_{22}-n_{12}n_{21}\neq 0$$

(a) now $\int_0^\pi vLu-uLv\,dt=[vu'-uv']_0^\pi$ where u, v satisfy given boundary

$(\because$ sturm-Liouville$)$

$$v(\pi)u'(\pi) - u(\pi)v'(\pi) - v(0)u'(0) + u(0)v'(0) = 0$$

then self adjoint

now $(\Rightarrow)$ if $v(\pi)u'(\pi) - u(\pi)v'(\pi) - v(0)u'(0) + u(0)v'(0) = 0$ — ①

then

as Boundry condition is

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} + \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix}\begin{bmatrix} y(\pi) \\ y'(\pi) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

wlog $m_{11}m_{22} - m_{12}m_{21} \neq 0$ then

$\det\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \neq 0 \Rightarrow \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ is invertible

multiplying by $\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}^{-1}$:

$$\begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} + \underbrace{\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}^{-1}}_{M^{-1}}\underbrace{\begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix}}_{N}\begin{bmatrix} y(\pi) \\ y'(\pi) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} = -M^{-1}N\begin{bmatrix} y(\pi) \\ y'(\pi) \end{bmatrix} \quad \text{— ②}$$

now $\begin{bmatrix} v(0) & u(0) \\ v'(0) & u'(0) \end{bmatrix} = \begin{bmatrix} (-M^{-1}N)\begin{matrix} v(\pi) \\ v'(\pi) \end{matrix} & (-M^{-1}N)\begin{matrix} u(\pi) \\ u'(\pi) \end{matrix} \end{bmatrix}$ $(\because ②)$

$$= -M^{-1}N\begin{bmatrix} v(\pi) & u(\pi) \\ v'(\pi) & u'(\pi) \end{bmatrix}$$

taking det on both sides we get

$$v(0)u'(0) - v'(0)u(0) = \det(-M^{-1}N)\,(v(\pi)u'(\pi) - u(\pi)v'(\pi))$$

as from ① $v(0)u'(0) - v'(0)u(0) = v(\pi)u'(\pi) - u(\pi)v'(\pi)$

$\Rightarrow 1 = \det(-M^{-1}N)$

as 2×2 matrix

$\det(-I) = (-1)^2 = 1$

so, $1 = \det(M^{-1})\det(N)$

$\Rightarrow \det(M) = \det(N)$

$\Rightarrow m_{11}m_{22} - m_{12}m_{21} = n_{11}n_{22} - n_{21}n_{12}$

$(\Leftarrow)$ Now if $m_{11}m_{22} - m_{12}m_{21} = n_{11}n_{22} - n_{21}n_{12} \neq 0$ — ③

given as $() \neq 0$ or $() \neq 0$

then as Boundry condition:

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} + \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix}\begin{bmatrix} y(\pi) \\ y'(\pi) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and as $\det\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \neq 0 \Rightarrow M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ is invertible

then

$$\begin{bmatrix} v(0) & u(0) \\ v'(0) & u'(0) \end{bmatrix} = \begin{bmatrix} -M^{-1}N \begin{matrix} v(\pi) \\ v'(\pi) \end{matrix} & -M^{-1}N \begin{matrix} u(\pi) \\ u'(\pi) \end{matrix} \end{bmatrix}$$

($\because$ previous result)

$$= -M^{-1}N \begin{bmatrix} v(\pi) & u(\pi) \\ v'(\pi) & u'(\pi) \end{bmatrix}$$

taking det on both sides:

$$v(0)u'(0) - u(0)v'(0) = \det(-M^{-1}N)\ (v(\pi)u'(\pi) - u(\pi)v'(\pi))$$

as $\det(-M^{-1}N) = \underbrace{\det(-I)}_{(-1)^2} \frac{1}{\det(M)} \det(N)$

$$= \frac{1}{\det(M)} \det(N)$$

$$= 1 \quad (\because ③)$$

so, ① is satisfied and

$$\int_0^\pi vLu - uLv\, dt = 0$$

(b) under $m_{11}m_{22} - m_{12}m_{21} = n_{11}n_{22} - n_{12}n_{21}$

if $y'' + \lambda y = 0$ where λ is eigenvalue, y is corresponding eigenfunction satisfying Boundary condition ——①

then taking conjugate:

$$\bar{y}'' + \bar{\lambda}\bar{y} = 0 \text{ ——②}$$

multiplying ① by $\bar{y}$ and ② by y:

$$\bar{y}(y'') + \lambda y\bar{y} = 0$$

$$y(\bar{y}'') + \bar{\lambda} y\bar{y} = 0$$

subtracting we get:

$$\bar{y}(y'') - y(\bar{y}'') = (\bar{\lambda} - \lambda) y\bar{y}$$

let $Ly = y''$ then By sterm-liouville from (a) we know BVP is self-adjoint

integrating we get:

$$\int_0^\pi \bar{y}\,Ly - y\,L\bar{y}\,dt = 0 = \int_0^\pi (\bar{\lambda}-\lambda)\,y\bar{y}\,dt$$

as $\bar{y}$ will also satisfy same boudry condition ($\because$ taking conjugate on boudry condition)

now as y is eigenfunction, it is non-trivial
so, $\exists\, \alpha \in [0,\pi]$ s.t

$$y(\alpha) \neq 0$$

$$\Rightarrow y(\alpha)\bar{y}(\alpha) > 0$$

$$\Rightarrow \int_0^\pi |y|^2\,dt > 0$$

so, $$0 = (\bar{\lambda}-\lambda)\underbrace{\int_0^\pi |y|^2\,dt}_{>0}$$

$$\Rightarrow \bar{\lambda}-\lambda = 0$$

$$\Rightarrow \bar{\lambda} = \lambda \text{ or } \lambda \text{ is real}$$

or this problem does not have any complex eigenvalues

1.6 $-(y''+y)=1 \quad y(0)=0,\ y(1)=1$

now let $y''+y=-1$ and $Ly = \left(\frac{\partial^2}{\partial t^2}+1\right)y$

then $Ly=0,\ y(0)=0,\ y(1)=0$ —— ①
has solution:

$m^2+1=0$ (characteristic roots)
$\Rightarrow m = \pm i$

or

$y_1(t) = \cos t \qquad y_2(t) = \sin t$

this are the two lin ind solutions of $Ly=0$

$y(t) = c_1\cos t + c_2\sin t$

$y(0) = c_1 = 0 \qquad y(1) = c_2\sin(1) = 0$
$\Rightarrow c_2 = 0$

so, $y \equiv 0$ is the solution of ①
or ① has trivial solution,
then by theorm done in class

① will have unique green's function

now, we also know that any green's function should satisfy the three properties given in class let $u(x,t)=\begin{cases}\alpha_1(t)\,y_1(x)+\alpha_2(t)\,y_2(x)\ ; & 0\le x\le t\le 1\\ \beta_1(t)\,y_1(x)+\beta_2(t)\,y_2(x)\ ; & 0\le t\le x\le 1\end{cases}$

α_i, β_i are continious,

if we show, α_i, β_i are unique, above will be our green's function

from (1) property (continuity of u)

we will get $\alpha_1(t)y_1(t)+\alpha_2(t)y_2(t)=\beta_1(t)y_1(t)+\beta_2(t)\,y_2(t)$

let $\gamma_i(t)=\alpha_i(t)-\beta_i(t)$ then

$$\gamma_1(t)\,y_1(t)+\gamma_2(t)\,y_2(t)=0 \quad \text{—— (2)}$$

from (2) trivial to see $\frac{\partial u}{\partial x}$ is continious for $t\neq x$

then $\lim\limits_{x\to t^+}\frac{\partial u}{\partial x}-\lim\limits_{x\to t^-}\frac{\partial u}{\partial x}=\frac{-1}{a_2(t)}=-1$

$a_2(t)=1$ as $Ly=y''+y$

so, $\beta_1(t)u_1'(t)+\beta_2(t)u_2'(t)-\alpha_1(t)u_1'(t)-\alpha_2(t)\,u_2'(t)=1$

$$\Rightarrow \gamma_1(t)u_1'(t)+\gamma_2(t)\,u_2'(t)=-1 \quad \text{—— (3)}$$

$u_1=y_1,\ y_2=u_2$

so from (2),(3):

$$W(u_1,u_2)(t)\begin{bmatrix}\gamma_1(t)\\ \gamma_2(t)\end{bmatrix}=\begin{bmatrix}0\\ -1\end{bmatrix}$$

as $W(u_1,u_2)(t)=\begin{bmatrix}u_1 & u_2\\ u_1' & u_2'\end{bmatrix}=\begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}$

$\det W=\cos^2 t+\sin^2 t=1\neq 0$

so W^{-1} exist

$$W^{-1}(u_1,u_2)(t)=\begin{bmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{bmatrix}$$

so, $\begin{bmatrix}\gamma_1(t)\\ \gamma_2(t)\end{bmatrix}=\begin{bmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{bmatrix}\begin{bmatrix}0\\ -1\end{bmatrix}$

$=\begin{bmatrix}\sin t\\ -\cos t\end{bmatrix}$

$\gamma_1(t)=\sin t,\quad \gamma_2(t)=-\cos t$ —— (4)

now from (3) fixing $t \in [0,1]$

$$G(x,t) = \xi(x)$$

$\xi(0)=0, \quad \xi(1)=0 \rightarrow$ Boundry condition

so, $\xi(0) = \alpha_1(t) y_1(0) + \alpha_2(t) y_2(0)$

$$0 = \alpha_1(t)\cos(0)$$

$$\Rightarrow \alpha_1(t) = 0 \Rightarrow \alpha_1 \equiv 0 \quad \text{—— (5)}$$

and $\xi(1) = \beta_1(t) y_1(1) + \beta_2(t) y_2(1)$

$$0 = \beta_1(t)\cos(1) + \beta_2(t)\sin(1) \quad \text{—— (6)}$$

now as $\nu_1(t) = \sin(t) = \alpha_1(t) - \beta_1(t) \quad (\because (4))$

but $\alpha_1(t)=0$ from (5)

so, $\sin(t) = -\beta_1(t) \Rightarrow \beta_1(t) = -\sin(t)$

then (6) becomes: $+\sin(t)\cos(1) = \beta_2(t)\sin(1)$

$$\Rightarrow \beta_2(t) = \sin(t)\frac{\cos(1)}{\sin(1)}$$

and $\nu_2(t) = \alpha_2(t) - \beta_2(t) = -\cos t \quad (\because (4))$

$$\alpha_2(t) = \sin(t)\frac{\cos(1)}{\sin(1)} - \cos(t)$$

so, we got $\alpha_1, \alpha_2, \beta_1, \beta_2$ all unique values as we got α_1 which gave β_1 and then got β_2 which gave α_2

$$\text{so, } G(x,t) = \begin{cases} \left(\sin(t)\dfrac{\cos(1)}{\sin(1)} - \cos(t)\right)\sin(x) ; & 0 \leq x \leq t \leq 1 \\ -\sin(t)\cos(x) + \sin(t)\dfrac{\cos(1)}{\sin(1)}\sin(x) ; & 0 \leq t \leq x \leq 1 \end{cases}$$

moreover $\sin(t)\cos(1) - \cos(t)\sin(1)$

$$= \sin(t-1)$$

and $\sin(x)\cos(1) - \cos(x)\sin(1) = \sin(x-1)$

$$\text{so, } u(x,t) = \begin{cases} \dfrac{\sin(t-1)\sin x}{\sin(1)} ; & 0 \leq x \leq t \leq 1 \\ \dfrac{\sin(t)\sin(x-1)}{\sin(1)} ; & 0 \leq t \leq x \leq 1 \end{cases}$$

$$\text{also, } \xi(x) = \begin{cases} \dfrac{\sin(t-1)\sin x}{\sin(1)} ; & 0 \leq x < t \leq 1 \\ \dfrac{\sin(t)\sin(x-1)}{\sin(1)} ; & 0 \leq t < x \leq 1 \end{cases}$$

$$\xi'(x) = \begin{cases} \dfrac{\sin(t-1)\cos x}{\sin(1)} ; & 0 \leq x < t \leq 1 \\ \dfrac{\sin(t)\cos(x-1)}{\sin(1)} ; & 0 \leq t < x \leq 1 \end{cases}$$

$$\xi''(x) = \begin{cases} -\dfrac{\sin(t-1)\sin x}{\sin(1)} \ ; & 0 \le x < t \le 1 \\ -\dfrac{\sin(t)\sin(x-1)}{\sin(1)} \ ; & 0 \le t < x \le 1 \end{cases}$$

and $\xi''(x) + \xi(x) = 0 \ \forall \ x \in [0,t) \cup (t,1]$

so constructed $G(x,t)$ satisfies all 3 properties and so $G(x,t)$ is the unique green function of ①

now $y'' + y = 0$ and $y(0) = 0, \ y(1) = 1$ — ⑦

then solution to ⑦:

$m^2 + 1 = 0 \Rightarrow m = \pm i$ roots of characteristic polynomial

so

$$y(t) = c_1 \cos(t) + c_2 \sin(t)$$

$$y(0) = c_1 = 0$$

$$y(1) = c_2 \sin(1) = 1$$

$$\Rightarrow c_2 = \frac{1}{\sin(1)}$$

so, $y(t) = \dfrac{\sin(t)}{\sin(1)}$ is solution to ⑦

solution to ① is:

$$y(x) = \int_0^1 G(x,t) f(t)\,dt \quad \text{for } Ly = f$$ (∵ theorm done in class)

here $f(t) = -1$ so

$$y(x) = \int_0^1 -G(x,t)\,dt$$

combining ①, ⑦ we get solution of

$$Ly = -1, \ y(0) = 0, \ y(1) = 1$$

as $y(x) = \dfrac{\sin(x)}{\sin(1)} + \displaystyle\int_0^1 -G(x,t)\,dt$

$G(x,t)$ is well defined above

$$y(x) = \frac{\sin(x)}{\sin(1)} + \int_0^x -G(x,t)\,dt + \int_x^1 -G(x,t)\,dt$$

$$= \frac{\sin(x)}{\sin(1)} - \int_0^x \frac{\sin(t)\sin(x-1)}{\sin(1)}\,dt - \int_x^1 \frac{\sin(t-1)\sin(x)}{\sin(1)}\,dt$$

$$= \frac{\sin(x)}{\sin(1)} + \int_0^x \frac{\sin(t)\sin(1-x)}{\sin(1)}\,dt + \int_x^1 \frac{\sin(1-t)\sin(x)}{\sin(1)}\,dt$$

$$= \frac{\sin(x)}{\sin(1)} + \frac{\sin(1-x)}{\sin(1)}\left[-\cos t\right]_0^x + \frac{\sin(x)}{\sin(1)}\left[+\cos(1-t)\right]_x^1$$

$$y(x) = \frac{\sin(x)}{\sin(1)} + \frac{\sin(1-x)}{\sin(1)}\left[1-\cos x\right] + \frac{\sin(x)}{\sin(1)}\left[1-\cos(1-x)\right]$$

1.7 $Ly = -\frac{d}{dt}\left(p\frac{dy}{dt}\right) + qy = f(t) \quad t \in [0,1]$

$p \in C^1[0,1] \quad p(t)>0 \ \forall t \in [0,1]$

$q, f \in C[0,1]$

$U_1(y) = \alpha_1 y(0) + \alpha_2 y'(0) = 0$

$U_2(y) = \beta_1 y(1) + \beta_2 y'(1) = 0$

$\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ s.t. $\alpha_1^2 + \alpha_2^2 \neq 0, \ \beta_1^2 + \beta_2^2 \neq 0$

and $\lambda = 0$ not an eigenvalue of Homogeneous problem

so, for $Ly = \lambda y = 0$, $U_1(y)=0$, $U_2(y)=0$ only trivial solution (as $\lambda=0$ is not an eigenvalue)

(a) as $Ly=0$, $U_1(y)=0$, $U_2(y)=0$ has only trivial solution, we know that operator will have unique green function, moreover

for $Ly=f$, $U_1(y)=0$, $U_2(y)=0$

$y(x) = \int_0^1 G(x,t) f(t)\,dt$ is the unique solution

now if $G(t,s) = \begin{cases} \dfrac{-y_1(s)y_2(t)}{p(t)W(y_1,y_2)(t)} & ; 0 \le s \le t \le 1 \\ \dfrac{-y_1(t)y_2(s)}{p(s)W(y_1,y_2)(s)} & ; 0 \le t \le s \le 1 \end{cases}$ satisfies all those condition of green function, then

y_1, y_2 nontrivial soln of $Ly=0$, at $t=0, t=1$ Boundary condition (respectively), W is wronskian

from uniqueness, $G(t,s)$ will be the green function

(1) $G: [0,1]\times[0,1] \longrightarrow \mathbb{R}$ is continuous:

this is trivial for $s<t$ or $s>t$ as y_1, y_2 are C^2 function as they satisfy differential equation

$p \in C^1[0,1]$ is given and $W(y_1,y_2)$ is also C^1 as $y_1y_2' - y_2y_1'$ and y_1, y_2 are lin ind (otherwise $W(y_1,y_2) \neq 0$) and $p \neq 0$ (given)

so, G is continuous for $t<s$ or $t>s$

for $t=s$ case: G will be continuous as

$$G(t,t) = \begin{cases} \dfrac{-y_1(t)y_2(t)}{p(t)W(y_1,y_2)(t)} \\ \dfrac{-y_1(t)y_2(t)}{p(t)W(y_1,y_2)(t)} \end{cases} \text{Both same}$$

$\Rightarrow G(t,t) = \dfrac{-y_1(t)\, y_2(t)}{P(t) W(y_1,y_2)(t)}$ similar to $t<s, t>s$ case this is continious

so, $G : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ is continious

(2) $\frac{\partial G}{\partial t}$ condition:

for $t > s$:

$$\frac{\partial G}{\partial t} = \frac{\partial}{\partial t}\left(\frac{-y_1(s)\, y_2(t)}{P(t) W(y_1,y_2)(t)} \right)$$

as $\underset{\uparrow}{y_2}$ is C^2, $P \in C^1[0,1]$ and $\underset{\uparrow}{W(y_1,y_2)(t)} \in C^1$

as solution of given $Ly=0$ as $\underbrace{y_1 y_2'}_{\in C^1} - \underbrace{y_2 y_1'}_{\in C^1}$

$$\text{so, } \frac{\partial G}{\partial t} = \frac{-y_1(s)\, y_2'(t)}{P(t) W(y_1,y_2)(t)}$$

$$+ y_1(s)\, y_2(t) \frac{(+1)\left(P(t) W'(y_1,y_2)(t) + P'(t) W(y_1,y_2)(t)\right)}{\left(P(t) W(y_1,y_2)(t)\right)^2}$$

$$= \frac{-y_1(s)}{\left(P(t) W(y_1,y_2)(t)\right)^2} \Big[y_2'(t) P(t) W(y_1,y_2)(t) - y_2(t) P(t) W'(y_1,y_2)(t) - y_2(t) P'(t) W(y_1,y_2)(t) \Big]$$

now, $W(y_1,y_2)(t)$
$$= y_1 y_2' - y_2 y_1'$$

$$W'(y_1,y_2)(t) = \cancel{y_1' y_2'} + y_1 y_2'' - \cancel{y_2' y_1'} - y_2 y_1''$$
$$= y_1 y_2'' - y_2 y_1''$$

now, $PW' + P'W = P y_1 y_2'' - P y_2 y_1'' + P' y_1 y_2' - P' y_2 y_1'$
$$= y_1 [P y_2'' + P' y_2'] - y_2 [P y_1'' + P' y_1']$$

as $Ly = 0$ i.e $-[Py']' + qy = 0$
$$\Rightarrow Py'' + P'y' = qy$$

so, $PW' + P'W = y_1[q y_2] - y_2[q y_1] = 0$

$$\text{so, } \frac{\partial G}{\partial t} = \frac{-y_1(s)\, y_2'(t) \cancel{P(t) W(y_1,y_2)}(t)}{\left(P(t) W(y_1,y_2)(t)\right)^{\cancel{2}}}$$

$$= -\frac{y_1(s)\, y_2'(t)}{P(t) W(y_1,y_2)(t)} \quad \text{——— (1)}$$

, $y_2 \in C^2$ as solution of $Ly=0$

for $t<s$:

$$\frac{\partial u}{\partial t} = \frac{d}{dt}\left(\frac{-y_1(t)\,y_2(s)}{p(s)\,W(y_1,y_2)(s)}\right)$$

$$= \frac{-y_1'(t)\,y_2(s)}{p(s)W(y_1,y_2)(s)} \quad \text{——— ②}$$

so from ①, ② trivial to see both continuous as diff of all c^1, c^2 function

so, $\frac{\partial u}{\partial t}$ is cont on each $\{(t,s) \mid 0 \leq t < s \leq 1\}$
and $\{(t,s) \mid 0 \leq s < t \leq 1\}$

now, $\lim_{t\to s^+} \frac{\partial u}{\partial t}(t,s) - \lim_{t\to s^-} \frac{\partial u}{\partial t}(t,s)$

$$= -\frac{y_1(t)y_2'(t)}{P(t)W(y_1,y_2)(t)} + \frac{y_1'(t)\,y_2(t)}{P(t)W(y_1,y_2)(t)} \quad (\because ①, ②)$$

$$= -\frac{1}{P(t)} \quad (\because W(y_1,y_2) = y_1y_2' - y_2y_1')$$

so, condition 2 is satisfied

(3) condition fixing s:

for $s\in[0,1]$, $\xi(t) = u(t,s)$

$$\xi(t) = \begin{cases} -\dfrac{y_1(s)\,y_2(t)}{P(t)W(y_1,y_2)(t)} & ; 0 \leq s < t \leq 1 \\ -\dfrac{y_1(t)\,y_2(s)}{P(s)W(y_1,y_2)(s)} & ; 0 \leq t < s \leq 1 \end{cases}$$

$$\xi'(t) = \begin{cases} -\dfrac{y_1(s)y_2'(t)}{P(t)W(y_1,y_2)(t)} & ; 0 \leq s < t \leq 1 \\ -\dfrac{y_1'(t)y_2(s)}{P(s)W(y_1,y_2)(s)} & ; 0 \leq t < s \leq 1 \end{cases}$$

$(\because ①, ②)$

now, $\frac{d}{dt}\left(\frac{-y_1(s)y_2'(t)}{P(t)W(y_1,y_2)(t)}\right) = -\frac{y_1(s)\,y_2''(t)}{P(t)W(y_1,y_2)(t)}$

$\left(\because \frac{d}{dt}(PW) = 0, \text{ shown above}\right)$

and $\frac{d}{dt}\left(-\frac{y_1'(t)\,y_2(s)}{P(s)W(y_1,y_2)(s)}\right) = \frac{-y_1''(t)\,y_2(s)}{P(s)W(y_1,y_2)(s)}$

so, $$\xi''(t) = \begin{cases} \dfrac{-y_1(s)\, y_2''(t)}{p(t)\, W(y_1, y_2)(t)} & ; 0 \le s < t \le 1 \\ \dfrac{-y_1''(t)\, y_2(s)}{p(s)\, W(y_1, y_2)(s)} & ; 0 \le t < s \le 1 \end{cases}$$

now, $Ly = -[py']' + qy$

$= -py'' - p'y' + qy$

now, $L\xi = -p\xi'' - p'\xi' + q\xi$

$$= \begin{cases} \dfrac{-p(t)\, y_1(s) y_2''(t)}{p(t)\, W(y_1,y_2)(t)} + \dfrac{p'(t)\, y_1(s) y_2'(t)}{p(t)\, W(y_1,y_2)(t)} - \dfrac{q(t) y_1(s) y_2(t)}{p(t)\, W(y_1,y_2)(t)} & 0 \le s < t \le 1 \\ -\dfrac{p(t)\, y_2(s)\, y_1''(t)}{p(s)\, W(y_1, y_2)(s)} + \dfrac{p'(t)\, y_2(s)\, y_1'(t)}{p(s)\, W(y_1,y_2)(s)} - \dfrac{q(t)\, y_2(s) y_1(t)}{p(s)\, W(y_1,y_2)(s)} & 0 \le t < s \le 1 \end{cases}$$

$$= \begin{cases} \dfrac{y_1(s)\; L y_2(t)}{p(t) W(y_1, y_2)(t)} & ; 0 \le s < t \le 1 \\ \dfrac{y_2(s)\, L y_1(t)}{p(s)\, W(y_1, y_2)(s)} & ; 0 \le t < s \le 1 \end{cases}$$

$$= \begin{cases} 0 & ; 0 \le s < t \le 1 \\ 0 & ; 0 \le t < s \le 1 \end{cases} \quad \text{as } Ly_1 = 0,\ Ly_2 = 0$$

$= 0$ for $t \in [0, s) \cup (s, 1]$

and $U_1(\xi(t)) = U_1(G(t,s))$

$= \alpha_1 G(0,s) + \alpha_2 \dfrac{d}{dt} G(0,s)$

$= \alpha_1 \left(- \dfrac{y_1(0)\, y_2(s)}{p(s) W(y_1, y_2)(s)} \right) + \alpha_2 \left(\dfrac{-y_1'(0) y_2(s)}{p(s) W(y_1, y_2)(s)} \right)$

$(\because ②)$

$= -\dfrac{y_2(s)}{p(s) W(y_1, y_2)(s)} \left[\alpha_1 y_1(0) + \alpha_2 y_1'(0) \right] = 0$

as $U_1(y) = 0$ given

$U_2(\xi(t)) = U_2(G(t,s))$

$= \beta_1 G(1,s) + \beta_2 \dfrac{\partial}{\partial t} G(1,s)$

$= -\dfrac{\beta_1\, y_1(s) y_2(1)}{p(1)\, W(y_1, y_2)(1)} - \dfrac{\beta_2\, y_1(s)\, y_2'(1)}{p(1)\, W(y_1, y_2)(1)} \quad (\because ①)$

$= -\dfrac{y_1(s)}{p(1) W(y_1, y_2)(1)} \left(\beta_1 y_2(1) + \beta_2 y_2'(1) \right) = 0 \quad (\because U_2(y_2) = 0)$

so, $G(t,s)$ satisfies all three condition, and as greens function is unique

$G(t,s)$ is required green's function

(b) This part follows from condition (2) of greens function in (a)

(c) now let $p(t)W(y_1,y_2)(t) = c$ (some constant) $\forall t \in [0,1]$ ($\because \frac{d}{dt}(pW)=0$ shown above in (a))

as $p(t) > 0$, $\forall t \in [0,1]$ and $W(y_1,y_2)(t)$ non-zero otherwise denominator zero so G not defined

$\Rightarrow c \neq 0$

then $G(t,s) = \begin{cases} -\dfrac{y_1(s)y_2(t)}{c} & ; 0 \leq s \leq t \leq 1 \\ -\dfrac{y_1(t)y_2(s)}{c} & ; 0 \leq t \leq s \leq 1 \end{cases}$

and so, $G(s,t) = \begin{cases} -\dfrac{y_1(t)y_2(s)}{c} & ; 0 \leq t \leq s \leq 1 \\ -\dfrac{y_1(s)y_2(t)}{c} & ; 0 \leq s \leq t \leq 1 \end{cases}$ ($\because$ switching t and s)

$= \begin{cases} -\dfrac{y_1(s)y_2(t)}{c} & ; 0 \leq s \leq t \leq 1 \\ -\dfrac{y_1(t)y_2(s)}{c} & ; 0 \leq t \leq s \leq 1 \end{cases} = G(t,s)$

so, $G(s,t) = G(t,s)$ or green's function is symmetric

Assignment - 5

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Assignment - 5

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1.1 $Y'(t) = AY(t)$

e^{At} = exponent operator

(a) $A = \begin{bmatrix} 2 & -2 \\ 1 & 0 \end{bmatrix}$ then by 2×2 trace-determinant form (done in notes)

$\mu = \frac{1}{2}\mathrm{tr}(A) = \frac{1}{2}(2+0) = 1$

$\delta = \det(A) = 2$

now, $\omega = \sqrt{\mu^2 - \delta} = \sqrt{1-2} = i$, so by 2×2 trace-determinant formula as $\mu^2 < \delta$:

$\omega = i\beta \quad \beta = \sqrt{\delta - \mu^2} = \sqrt{2-1} = 1$

$\Rightarrow \omega = i$ and $e^{At} = e^{\mu t}\left[\cos(\beta t) I + \frac{\sin(\beta t)}{\beta}(A - \mu I)\right]$

$= e^{t}\left[\cos(t) I + \sin(t)(A - I)\right]$

$(\because \beta = 1, \mu = 1)$

so, $e^{At} = e^{t}\begin{bmatrix} \cos t & 0 \\ 0 & \cos t \end{bmatrix} + e^{t}\sin t \begin{bmatrix} 2-1 & -2 \\ 1 & 0-1 \end{bmatrix}$

$= \begin{bmatrix} e^{t}\cos t & 0 \\ 0 & e^{t}\cos t \end{bmatrix} + \begin{bmatrix} e^{t}\sin t & -2e^{t}\sin t \\ e^{t}\sin t & -e^{t}\sin t \end{bmatrix}$

$e^{At} = \begin{bmatrix} e^{t}(\cos t + \sin t) & -2e^{t}\sin t \\ e^{t}(\sin t) & e^{t}(\cos t - \sin t) \end{bmatrix}$

now let $Y(t) = e^{At+c}$, c is some constant

then $Y'(t) = Ae^{At}e^{c}$ $\left(\because \frac{d}{dt}e^{At} = Ae^{At}\right.$ done in class$\left.\right)$

or $Y'(t) = AY(t)$

so, $Y(t) = e^{At+c}$ satisfy given equation, $c \in \mathbb{R}$ or $\mathbb{C}$ (depends on problem)

moreover for any $Y(t_0) = Y_0$ (IVP) the solution is unique ($\because$ done in class)

$\Rightarrow Y(t) = e^{At+c}$ is the genral solution

$Y(t) = e^{At+c}$

$= \begin{bmatrix} e^{t+c}(\cos t + \sin t) & -2e^{t+c}(\sin t) \\ e^{t+c}(\sin t) & e^{t+c}(\cos t - \sin t) \end{bmatrix}$

(b) $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, similar to (a), we will use the 2×2 trace-determinant formula for computing e^{At}:

$\mu = \frac{1}{2}\mathrm{tr}(A) = \frac{1}{2}(-1-1) = -1$

$\delta = \det(A) = (1-1) = 0$, so $\omega = \sqrt{\mu^2 - \delta} = \sqrt{1-0} = 1$

as $\mu^2 > \delta$:

$$e^{At} = e^{\mu t}\left[\cos(\omega t) I + \frac{\sin(\omega t)}{\omega}(A - \mu I)\right]$$

$$= e^{-t}\left[\cos(t) I + \sin(t)(A + I)\right]$$

$$e^{At} = e^{-t}\cos(t)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + e^{-t}\sin(t)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (\because \mu = -1, \omega = 1)$$

$$= \begin{bmatrix} e^{-t}\cos t & e^{-t}\sin t \\ e^{-t}\sin t & e^{-t}\cos t \end{bmatrix}$$

now, similar to (a), the general solution becomes

$$Y(t) = e^{At + c}, \quad c \text{ is some constant}$$

$c \in \mathbb{R}$ or $\mathbb{C}$ (depends on problem)

$$\Rightarrow Y(t) = \begin{bmatrix} e^{c-t}\cos t & e^{c-t}\sin t \\ e^{c-t}\sin t & e^{c-t}\cos t \end{bmatrix}$$

1.2 $ay'' + by' + cy = 0$ $a \neq 0$, b, c are constants

then let $Y(t) = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$ $\quad Y'(t) = \begin{bmatrix} y'(t) \\ y''(t) \end{bmatrix}$

let $A = \begin{bmatrix} \alpha & \beta \\ \varphi & \delta \end{bmatrix}$

then forcing $AY = Y'$

$$\Rightarrow \begin{bmatrix} \alpha & \beta \\ \varphi & \delta \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \end{bmatrix}$$

$\alpha y + \beta y' = y'$

$\Rightarrow (1-\beta)y' = \alpha y$

as not given relation of y' and y

for any general case this is true if

$\alpha = 0$, $1 - \beta = 0$

i.e $\alpha = 0$, $\beta = 1$

also, $\varphi y + \delta y' = y''$

as $ay'' + by' + cy = 0$ and $a \neq 0$

$$\Rightarrow y'' = -\frac{c}{a}y - \frac{b}{a}y'$$

so, $\varphi = -\frac{c}{a}$, $\delta = -\frac{b}{a}$

so, $A = \begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix}$

now, eigenvalues of A are roots of

$$\det(A - \lambda I) = 0$$

i.e $\det\begin{bmatrix} -\lambda & 1 \\ -c/a & -b/a - \lambda \end{bmatrix} = 0$

$$\Rightarrow \lambda\left(\frac{b}{a}+\lambda\right)+\frac{c}{a}=0$$
$$\Rightarrow \lambda(b+\lambda a)+c=0$$
$$\Rightarrow \lambda^2 a+\lambda b+c=0$$

now charecteristic equation roots are: ar^2+br+c

so, trivial to see that eigenvalues of A and roots of charecteristic equation are same

1.3 $A_{n\times n}\in M_{n\times n}(\mathbb{R})$

To prove : $\det(e^{tA})=e^{t(\text{trace } A)}$

proof :

as A is a square matrix

A can be decomposed as

$A=VJV^{-1}$ where V is an invertible matrix and J is the jordan matrix

$J_k(\lambda)=\lambda I+N$, N is nilpotent

the above is from jordan decomposition of a matrix (works for any $A_{n\times n}\in M_{n\times n}(\mathbb{R})$)

now J is a matrix s.t its diagonal is just the eigenvalues with its multiplicity (or $\lambda=0$ also works)

now, $e^{J_k(\lambda)t}=e^{\lambda t}\left(I+Nt+\frac{N^2t^2}{2!}+\dots+\frac{N^{k-1}t^{k-1}}{(k-1)!}\right)$

($\because$ done in notes)

and $e^{At}=ve^{Jt}v^{-1}$ ($\because$ done in notes)

now, $\det(e^{At})=\det(ve^{Jt}v^{-1})$
$=\cancel{\det(v)}\det(e^{Jt})\cancel{\det(v^{-1})}$
$=\det(e^{Jt})$

and for $J=\begin{bmatrix} [\;]_{J_1} & 0 & 0 \\ 0 & [\;]_{J_2} & \vdots \\ & & \ddots \end{bmatrix}$ then $\det(J)=\det(J_1)\det(J_2)\dots\det(J_n)$

now $e^J=\begin{bmatrix} [\;]_{e^{J_1}} & 0 & \\ 0 & [\;]_{e^{J_2}} & \\ & & \ddots \end{bmatrix}$ then similar to above

$\det(e^J)=\det(e^{J_1})\det(e^{J_2})\dots\det(e^{J_n})$ ——— ①

now, as $e^{J_k(\lambda)t}=e^{\lambda t}\left(I+Nt+\frac{N^2t^2}{2!}+\frac{N^3t^3}{3!}+\dots\frac{N^{k-1}t^{k-1}}{(k-1)!}\right)$

$$= \begin{bmatrix} e^{\lambda t} & * & 0 & \cdots & 0 \\ 0 & e^{\lambda t} & * & \cdots & 0 \\ 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & * \\ & & & & e^{\lambda t} \end{bmatrix}_{k\times k}$$

the superdiagonal will have some values and diagonals will be $e^{\lambda t}$

claim: $\det \begin{bmatrix} e^{\lambda t} & * & 0 & \cdots & 0 \\ 0 & e^{\lambda t} & * & \cdots & 0 \\ 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & * \\ & & & & e^{\lambda t} \end{bmatrix}_{k\times k} = e^{k\lambda t}$ for any values on superdiagonals ——— ②

trivial for $k=1$ as $\det [e^{\lambda t}]_{1\times 1} = e^{\lambda t}$

let's say true for some $k=n-1$, then

$$\det \begin{bmatrix} e^{\lambda t} & \overset{\text{some } \alpha}{\alpha} & 0 & 0 & \cdots \\ & e^{\lambda t} & * & & \\ & & \ddots & & \end{bmatrix}_{n\times n} = e^{\lambda t} \det \begin{bmatrix} e^{\lambda t} & * & \\ & e^{\lambda t} & * \\ & & \ddots \end{bmatrix}_{n-1\times n-1}$$

$$- \alpha \det \begin{bmatrix} 0 & * & \\ 0 & & * \\ \vdots & & \\ 0 & & \end{bmatrix}_{n-1\times n-1} + 0 \det(\ ) + \cdots + \det(\ )$$

full column 0

$$= e^{\lambda t} \det \begin{bmatrix} e^{\lambda t} & * \\ & \ddots \end{bmatrix}_{n-1\times n-1}$$

$$= e^{\lambda t} e^{(n-1)\lambda t}$$

$$= e^{n\lambda t} \quad (\because \text{induction Hypothesis})$$

so, $\det(e^{J_k(\lambda) t}) = e^{k\lambda t}$ $(\because ②)$

now, from ① let $Jt = \begin{bmatrix} J_{k_1}(\lambda_1)t & & \\ & \ddots & \\ & & J_{k_n}(\lambda_n)t \end{bmatrix}$

then $\det e^{Jt} = \det(e^{J_{k_1}(\lambda_1)t}) \det(e^{J_{k_2}(\lambda_2)t}) \cdots \det(e^{J_{k_n}(\lambda_n)t})$

$$= e^{k_1\lambda_1 t} e^{k_2\lambda_2 t} \cdots e^{k_n\lambda_n t} \quad (\because ②)$$

$$= e^{(\sum_{i=1}^{n} k_i\lambda_i)t}$$

then $\det(e^{At}) = \det(e^{Jt}) = e^{(\sum_{i=1}^{n} k_i\lambda_i)t}$

now as λ_p is eigenvalue (assuming it can be 0) and k_p are multiplicities of it

as $\sum_{i=1}^{n} \lambda_p k_p = \text{trace}(A)$ ($\because$ from linear algebra

so, $\det(e^{At}) = e^{t\,\text{trace}(A)}$

now, $\det(e^{At}) = e^{t\,\text{trace}(A)} > 0$ as $e^x > 0 \ \forall x \in \mathbb{R}$

$\Rightarrow \det(e^{At}) \neq 0$

$\Rightarrow e^{At}$ is non-singular

To prove: $e^{tA}: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continious

proof: for t fixed, let $\|e^{tA}\| = c$, then for $y_0 \in \mathbb{R}^n$

$\forall \varepsilon > 0,\ \exists \delta = \frac{\varepsilon}{c}$ s.t $\|y - y_0\| < \delta = \frac{\varepsilon}{c}$ for $y \in \mathbb{R}^n$

we have $\|e^{tA}y - e^{tA}y_0\|$

$= \|e^{tA}(y - y_0)\|$

$\leq \|e^{tA}\| \|y - y_0\|$

$< \cancel{c}\, \frac{\varepsilon}{\cancel{c}} = \varepsilon$

so, $\|e^{tA}y - e^{tA}y_0\| < \varepsilon$, so

$$\lim_{y \to y_0} e^{tA}y = e^{tA}y_0$$

or $e^{tA}: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continious

To prove: $\forall t \in \mathbb{R}$, mapping $e^{tA}: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continiously differtiable

proof: Now fixing t makes e^{tA} a matrix, also we have shown $\det(e^{tA}) \neq 0$

now, let $T = e^{tA}$ matrix, then
from multivariable calculus,

$T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is differtiable at some $x_0 \in \mathbb{R}^n$ if $\exists$ a linear operator $A(x_0)$ s.t

$$\lim_{h \to 0} \frac{\|T(x_0 + h) - T(x_0) - A(x_0)h\|}{\|h\|} = 0$$

now for any value of x_0, $A(x_0) = T$ then

$$\lim_{h \to 0} \frac{\|T(\cancel{x_0} + h) - T(\cancel{x_0}) - T(h)\|}{\|h\|} = 0$$

so, $\forall x_0$, T is differtiable

now, the jacobian of $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$

let $T = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ \vdots & & & \\ T_{n1} & \cdots & \cdots & T_{nn} \end{bmatrix}$

and $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ then $Tx = (\Sigma T_{1i} x_i, \Sigma T_{2i} x_i, \ldots \Sigma T_{ni} x_i)$

now let $f_i(x) = \sum_{k=1}^{n} T_{ik} x_k$

$f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$

then $Tx = (f_1(x), f_2(x), \dots f_n(x))$

now, $\left.\frac{\partial f_i(x)}{\partial x_j}\right|_{x_0} = \frac{d}{dx_j}\left(\sum T_{ik} x_k\right)$

$= T_{ij}$

or Jacobian of $T = [T_{ij}] = T$
at x_0

as all $T_{ij} \in \mathbb{R}$ are constant, all T_{ij} are continious ($\because$ constat)

or all componets of Jacobian of T contnious
$\forall x_0$ (as x_0 choice does not matter)
and derivatie exist $\forall x_0$

$\Rightarrow T = e^{tA}$ is continiously differtiable ($\because$ multivariable calculus result)

1.4 $\phi(t) = \begin{bmatrix} \sin t & \cos t \\ -\cos t & \sin t \end{bmatrix}$ is fundamental matrix for system

$Y'(t) = AY(t)$

as $\phi(t)$ is fundamental matrix, its colums are linearly independent solution to a system of linear differtial equations forming basis of all possible solution (in notes)

so $Y(t) = c_1 \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix} + c_2 \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$

then $Y'(t) = c_1 \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + c_2 \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$

and as system is $Y'(t) = A\,Y(t)$

and $Y(t)$ is 2×1 and $Y'(t)$ is 2×1
A will be 2×2

let $A = \begin{bmatrix} \alpha & \beta \\ \varphi & \delta \end{bmatrix}$

then forcing $Y'(t) = AY(t)$

makes $c_1 \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + c_2 \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \varphi & \delta \end{bmatrix} Y(t)$

or $c_1 \cos t - c_2 \sin t = \alpha(c_1 \sin t + c_2 \cos t) + \beta(-c_1 \cos t + c_2 \sin t)$ —— ①

and $c_1 \sin t + c_2 \cos t = \varphi(c_1 \sin t + c_2 \cos t) + \delta(-c_1 \cos t + c_2 \sin t)$ —— ②

$\Rightarrow \cos t\,[c_1 - \alpha c_2 + \beta c_1] = \sin t\,[c_2 + \alpha c_1 + \beta c_2]$ ($\because$ ①)

as $\sin t, \cos t$ linearly independent ($\because W(\sin t, \cos t) = 1$)

we need $c_1 - \alpha c_2 + \beta c_1 = 0$ and $c_2 + \alpha c_1 + \beta c_2 = 0$

$c_1[\beta+1] = c_2[\alpha]$ and $c_2[1+\beta] = -c_1[\alpha]$

as we can vary c_1, c_2 to any value both equation will be satisfied if

$\beta + 1 = 0,\ \alpha = 0 \Rightarrow \beta = -1,\ \alpha = 0$ —— ③

and $\sin t\,[c_1 - \varphi c_1 - \delta c_2] = \cos t\,[-c_2 + \varphi c_2 - \delta c_1]$ ($\because$ ②)

so, similar to above

$0 = c_1 - \varphi c_1 - \delta c_2 = -c_2 + \varphi c_2 - \delta c_1$

$\Rightarrow [1-\varphi]c_1 - \delta c_2 = 0$ and $[\varphi - 1]c_2 - \delta c_1 = 0$

$\Rightarrow 1 = \varphi,\ \delta = 0$ ($\because$ similar to before) —— ④

so, from ③, ④:

$$A = \begin{bmatrix} \alpha & \beta \\ \varphi & \delta \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and now we will use the 2x2 trace-determinant formula to compute e^{At}

$\mu = \frac{1}{2}\operatorname{tr}(A) = \frac{1}{2}(0+0) = 0$

$\delta = \det(A) = 1$

so, $\mu^2 < \delta$ then, $\omega = i\beta$ where

$\beta = \sqrt{\delta - \mu^2} = \sqrt{1-0} = 1$

so, $e^{At} = e^{\mu t}\left[\cos(\beta t) I + \frac{\sin(\beta t)}{\beta}(A - \mu I)\right]$

$= e^0\left[\cos(t) I + \sin(t) A\right]$ ($\because \mu = 0, \beta = 1$)

$= \cos(t)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin(t)\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$

1.5 $Y'(t) = AY(t)$ A is $n \times n$ matrix $\Phi(\cdot)$ is fundamental matrix of system

<u>To prove</u>: Any matrix $\Psi(\cdot) \in M_{n\times n}$ is a fundamental matrix iff $\exists C \in M_{n\times n}$ s.t $\Psi(t) = \Phi(t)C$ $\det(C) \neq 0$

proof : ($\Leftarrow$) as $\exists C \in M_{n\times n}$ s.t $\det(C) \neq 0$, $\psi(t) = \phi(t)C$

now fundamental matrix is s.t

$\phi'(t) = A\phi(t)$ and columns linearly independent (in notes)

so, $\phi'(t)C = A\phi(t)C$ ($\because$ multiplying both sides by C)

and $\psi'(t) = \phi'(t)C$ ($\because C$ is constant)

so, $\psi'(t) = A\psi(t)$ and ——— ①

as columns are linearly independent for $\phi(t)$

$\det(\phi(t)) \neq 0$
and as $\det(C) \neq 0$
$\Rightarrow \det(\phi(t)C) \neq 0$
$\Rightarrow \det(\psi(t)) \neq 0$

($\det(\phi(t)) \equiv 0$ is not true but, there can be some t_0 s.t $\det(\phi(t_0)) = 0$) same for $\psi(\cdot)$

so, columns of $\psi(t)$ are also linearly independent ——— ②

from ①, ② $\psi(\cdot) \in M_{n\times n}$ is a fundamental matrix

($\Rightarrow$) $\psi(\cdot)$ is a fundamental matrix, $\phi(\cdot)$ is a fundamental matrix then

$\psi'(t) = A\psi(t)$ and $\phi'(t) = A\phi(t)$

also, both ϕ, ψ's columns will be linearly independent (in notes)

and if $\psi(t) = \begin{bmatrix} \psi_1(t) & \cdots & \psi_n(t) \end{bmatrix}$ then every column is solution of

$$\dot{y}(t) = Ay(t)$$

moreover $\{\psi_1(t), \ldots \psi_n(t)\}$ is solution space

similarly $\phi(t) = \begin{bmatrix} \phi_1(t), & \ldots & \phi_n(t) \end{bmatrix}$ means

$\{\phi_1(t), \ldots \phi_n(t)\}$ as solution space

now, $\forall \psi_i(t)$ as its a solution of $\dot{y}(t) = Ay(t)$
$\exists c_{1i}, c_{2i}, \ldots c_{ni}$ constants s.t

$$\psi_i(t) = \sum_{k=1}^{n} c_{ki}\phi_k(t) \quad (\because \{\phi_1(t), \ldots \phi_n(t)\} \text{ is basis of solution space of } \dot{y}(t) = Ay(t))$$

$\forall i \in \{1, 2, \ldots n\}$

so,
$$\psi_1(t) = c_{11}\phi_1(t) + c_{21}\phi_2(t) + \cdots c_{n1}\phi_n(t)$$
$$\psi_2(t) = c_{12}\phi_1(t) + c_{22}\phi_2(t) + \cdots c_{n2}\phi_n(t)$$
$$\vdots$$
$$\psi_n(t) = c_{1n}\phi_1(t) + c_{2n}\phi_2(t) + \cdots c_{nn}\phi_n(t)$$

or $\Psi(t) = \begin{bmatrix} \psi_1(t) \cdots \psi_n(t) \end{bmatrix} = \begin{bmatrix} \phi_1(t) \cdots \phi_n(t) \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}$

$\Rightarrow \Psi(t) = \Phi(t) C$ where $C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ \vdots & & & \\ c_{n1} & \cdots & \cdots & c_{nn} \end{bmatrix}$

also $\det(C) \neq 0$ as if $\det(C) = 0$ then

$\det(\Psi(t)) = \det(\Phi(t)) \det(C)$
$\equiv 0 \ \forall t$

but $\det(\Psi(t)) \neq 0$ for some values of t
(this is a contradiction) t as $\Psi(t)$ is fundamental matrix

so, $\det(C) \neq 0$

So, $\exists C \in M_{n\times n}$ s.t $\Psi(t) = \Phi(t) C, \forall t \in \mathbb{R}$
s.t $\det(C) \neq 0$

1.6 (a) $f(t) = t^a e^{-bt}$ for some $a, b > 0$ constants, $c > 0$ s.t $c < b$

<u>To prove</u>: $\exists K > 0$ constat s.t $|f(t)| \leq K e^{-ct}$ for $0 \leq t < \infty$

<u>proof</u>: for $K = 0$: $f(0) = 0$

so, $\forall\, 0 < c < b \quad e^{-c(0)} = e^{-1}$
any $K > 0$ works
then $0 \leq \frac{K}{e}$ (true)

for $K \in (0, \infty)$: $f(t) = t^a e^{-bt}$

$= e^{\ln(t^a) - bt}$

$(\because t > 0, a > 0$ so $t^a > 0$ and $\ln(t^a))$ can be written

$= e^{a\ln(t) - bt}$

now, as given c is s.t $0 < c < b$
$\Rightarrow -b < -c < 0$
as $t \in (0, \infty)$
$\Rightarrow -bt < -ct < 0$

let $b - c = \varepsilon > 0$ then

$e^{-bt} = e^{-(\varepsilon + c)t}$

so, $f(t) = e^{a\ln(t)} e^{-bt}$

$= e^{a\ln(t)} e^{-\varepsilon t - ct}$

$= e^{(a\ln(t) - \varepsilon t)} e^{-ct}$

now let $g(t) = a\ln(t) - \varepsilon t$ for $t \in (0, \infty)$

then $g'(t) = \frac{a}{t} - \varepsilon$ ($\because$ chain rule)

and so

for $t = \frac{a}{\varepsilon}$

$g'\left(\frac{a}{\varepsilon}\right) = 0$ —— ①

also, $g''(t) = -\frac{a}{t^2}$ ($\because$ chain rule)

as $t \in (0,\infty)$, $a > 0$ (given)

$\Rightarrow g''(t) < 0 \ \forall t \in (0,\infty)$

$\Rightarrow g''\left(\frac{a}{\varepsilon}\right) < 0$ —— ②

from ①, ② we get $g\left(\frac{a}{\varepsilon}\right)$ is maxima of g i.e

$$g(t) \leq g\left(\frac{a}{\varepsilon}\right) \ \forall t \in (0,\infty)$$

$$\Rightarrow g(t) \leq a\ln\left(\frac{a}{\varepsilon}\right) - \cancel{\varepsilon}\left(\frac{a}{\cancel{\varepsilon}}\right)$$

$$\leq a\ln\left(\frac{a}{\varepsilon}\right) - a$$

then $f(t) = e^{g(t)}e^{-ct} \leq e^{g(a/\varepsilon)}e^{-ct} \ \forall t \in (0,\infty)$

as $\forall t \in (0,\infty)$ f is positive ($\because$ trivial to see)

$\Rightarrow |f(t)| \leq e^{a\ln(a/\varepsilon) - a}e^{-ct}$

let $K = e^{a\ln(a/\varepsilon) - a}$ then $K > 0$ ($\because e^x > 0 \ \forall x$)

moreover from both the cases, $\exists K > 0$ s.t

$|f(t)| \leq Ke^{-ct}$

(b) $A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, $\lambda < 0$

now using triangular 2x2 shortcut of calculating e^{At} (in notes)

A is upper triangular, $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ with $\underset{(\lambda=\lambda)}{a = d}$:

$$e^{At} = \begin{bmatrix} e^{at} & bte^{at} \\ 0 & e^{at} \end{bmatrix}$$

$a = \lambda$, $b = 1$, $d = \lambda$

$$\Rightarrow e^{At} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} = e^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

now, let $\|e^{At}\| = \max_{x_0 \in \mathbb{R}^2} \frac{\|e^{At}x_0\|}{\|x_0\|} = \max_{\|x_0\|=1} \|e^{At}x_0\|$

then, $\|e^{At}x_0\| \leq \|x_0\| \|e^{At}\|$ —— ①

and from (a) we know that

$|te^{\lambda t}| \leq Ke^{-ct}$ as $b = -\lambda > 0$
$0 < c < b$
$0 < c < -\lambda$

$\exists K > 0$ s.t above is true for $t \in [0, \infty)$

and for any $x_0 \in \mathbb{R}^2$ s.t $\|x_0\| = 1 \Rightarrow x_0 = (\alpha, \beta)$
then $\alpha^2 + \beta^2 = 1$

$$e^{At}x_0 = e^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = e^{\lambda t}\begin{bmatrix} \alpha + t\beta \\ \beta \end{bmatrix}$$

$$\|e^{At}x_0\| = e^{\lambda t}\left[\sqrt{(\alpha + t\beta)^2 + \beta^2}\right]$$

$$= e^{\lambda t}\left[\sqrt{1 + 2\alpha t\beta + t^2\beta^2}\right]$$

as $\alpha \leq 1$, $\beta \leq 1$ (otherwise $\|x_0\| > 1$ if $\alpha > 1$ or $\beta > 1$)

so, $\|e^{At}x_0\| \leq e^{\lambda t}\left[\sqrt{1 + 2t + t^2}\right]$

$\leq e^{\lambda t}\left[\sqrt{(t+1)^2}\right]$

$\leq e^{\lambda t}|t+1|$

$\max\limits_{\|x_0\|=1} \|e^{At}x_0\| \leq e^{\lambda t}|t+1|$

$\Rightarrow \|e^{At}\| \leq e^{\lambda t}|t+1|$

and as $t \in [0, \infty)$

$\|e^{At}\| \leq e^{\lambda t}(t+1)$

so, ① becomes: $\|e^{At}x_0\| \leq \|x_0\| e^{\lambda t}(t+1)$ $\forall x_0 \in \mathbb{R}^2$

let $x = t+1$ then $x \in [1, \infty)$
and
$f(x) = e^{\lambda x}(x^1)$ from (a) becomes
s.t $a = 1$
$b = -\lambda > 0$
then $\forall c$ s.t $0 < c < b = -\lambda$
$\exists M > 0$ s.t
$f(x) \leq Me^{-cx}$

so, $\|e^{At}x_0\| \leq \|x_0\| e^{-\lambda} e^{\lambda x} x$

$\leq \|x_0\| e^{-\lambda} Me^{-c(t+1)}$

$\leq \|x_0\| e^{-\lambda - c} Me^{-ct}$

fixing $c = \frac{-\lambda}{2}$ as $0 < \frac{-\lambda}{2} < -\lambda$, $\exists M > 0$ s.t

$\|e^{At}x_0\| \leq \|x_0\| e^{-\lambda+\lambda/2} M e^{+\lambda/2 t}$

so, $\exists K = e^{-\lambda+\lambda/2} M > 0$
as, $e^{-\lambda/2} > 0$ and $M > 0$
and $\alpha = -\frac{\lambda}{2} > 0$ s.t

$\|e^{At}x_0\| \leq K e^{-\alpha t} \|x_0\| \; \forall \, x_0 \in \mathbb{R}^2$

1.7 $A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & a \end{bmatrix}$, $a < 0$

then $\det(A-\lambda I) = \det \begin{bmatrix} -\lambda & 1 & 0 \\ -1 & -\lambda & 0 \\ 0 & 0 & a-\lambda \end{bmatrix}$

$= (\lambda)^2(a-\lambda) + (+1)(a-\lambda)$
$= (\lambda^2+1)(a-\lambda)$

so, eigenvalues of A are $\pm i$, a as roots of cubic equation

as all roots are unique and

$\det(A) = a \neq 0$

we have, $\exists P$ (3×3) matrix s.t P^{-1} exist and

$A = P \Lambda P^{-1}$ $\quad \Lambda = \begin{bmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & a \end{bmatrix}$

or jordan form of A is

$\Lambda = \begin{bmatrix} J_1(i) & 0 & 0 \\ 0 & J_1(-i) & 0 \\ 0 & 0 & J_1(a) \end{bmatrix}$

$\alpha(A) = \max\limits_{\forall \lambda_i}(\mathrm{Re}(\lambda_i)) = \max\{0, 0, a\} = 0$
as $a < 0$ given

then from theorm done in class (Jordan-stock estimate and growth trajectory)

$\exists M \geq 1$ (constat)

$m = \max\{k_j - 1 \,|\, \mathrm{Re}(\lambda_j) = \alpha(A)\}$

$= \max\{1-1, 1-1\} = 0$ ($\because$ one-multiplicity)

s.t $\forall t \geq 0$, $\|e^{At}\| \leq M e^{\alpha(A)t}(1+t)^m$

$\leq M e^{0t}(1+t)^0 = M$

$\Rightarrow \|e^{At}\| \leq M \quad \forall \, t \geq 0$ —— ①

now, $\|e^{At}\| = \max_{\|x_0\|=1} \|e^{At}x_0\|$

$= \max_{x_0 \in \mathbb{R}^2} \frac{\|e^{At}x_0\|}{\|x_0\|}$

$\Rightarrow \|e^{At}\| \geq \frac{\|e^{At}x_0\|}{\|x_0\|} \quad \forall x_0 \in \mathbb{R}^2$

$\Rightarrow \|e^{At}x_0\| \leq \|e^{At}\|\|x_0\|$

$\leq M\|x_0\| \ \forall t \geq 0 \ (\because ①)$

1.8 $\quad 0 \leq a < \infty$

$u''(t) - 2tu'(t) + 2au(t) = 0 \quad t \in \mathbb{R}$

$v(t) = e^{-t^2/2}u(t)$

$v'(t) = u'(t)e^{-t^2/2} - te^{-t^2/2}u(t)$

$= u'(t)e^{-t^2/2} - tv(t)$

so, $v''(t) = u''(t)e^{-t^2/2} - tu'(t)e^{-t^2/2} - v(t) - tv'(t)$

$= u''(t)e^{-t^2/2} - tu'(t)e^{-t^2/2} - u(t)e^{-t^2/2} - t[u'(t)e^{-t^2/2} - te^{-t^2/2}u(t)]$

$= (2\cancel{tu'}(t) - 2au(t))e^{-t^2/2} - \cancel{tu'(t)e^{-t^2/2}} - u(t)e^{-t^2/2} - \cancel{tu'(t)e^{-t^2/2}} + t^2e^{-t^2/2}u(t)$

$= v(t)[-2a - 1 + t^2]$

$v''(t) + \underbrace{(1 + 2a - t^2)}_{q(t)} v(t) = 0$

then $q(t) = 1 + 2a - t^2$

then let $R = \sqrt{2a+1+c}$ for some fixed $c > 0$

then for $|t| \geq R$

i.e $\quad t^2 \geq R^2 \Rightarrow -t^2 \leq -R^2$

$q(t) = 1 + 2a - t^2 \leq 1 + 2a - R^2$

$= 1 + 2a - 2a - 1 - c$

$\Rightarrow q(t) \leq -c$

$\forall |t| \geq R$

and $v''(t) - cv(t) = 0$ has a non-trivial solution e^{ct} on interval $(-\infty, -R) \cup (R, \infty)$

so by sturm comparison theorm, as e^{cx} has non ossilatory zero on $(-\infty, -R) \cup (R, \infty)$

solution of $v''(t) + q(t)v(t) = 0$ will have atmost one zero on $(-\infty, -R)$ or (R, ∞)

on $[-R, R]$ as $q(t)$ is analytic, we will have $v(t)$ as analytic solution, so it will have finitely many roots on $[-R, R]$

$\Rightarrow$ any non-trivial solution of $v''(t) + q(t)v(t) = 0$ has finite zeros

$\Rightarrow e^{t^2/2}u(t)$ has finite zeros on $\mathbb{R}$

$\Rightarrow u(t)$ has finite zeros on $\mathbb{R}$ ($\because e^{t^2/2} > 0 \ \forall t \in \mathbb{R}$)

1.9 $q : (0,\infty) \longrightarrow \mathbb{R}$ is continious funtion

$$q(t) \geq \frac{c}{t^2} \quad \text{for some } c > \frac{1}{4}$$

To prove: any non trivial solution u of the differential equation

$$u''(t) + q(t)u(t) = 0$$

has infinitely many zeros in $(0,\infty)$

proof: as $q(t) \geq \frac{c}{t^2}$ for some $c > \frac{1}{4}$

assuming $u''(t) + \left(\frac{c}{t^2}\right)u(t) = 0$

let $v(t)e^{t/2} = u(e^t)$

where $x = e^t$ as $t \in \mathbb{R}$, $x \in (0,\infty)$ one-one corresponderce

$$v(t)e^{t/2} = u(e^t)$$

where $dx = e^t\,dt$

$$e^{t/2}\frac{d}{dt}v(t) + v(t)\frac{1}{2}e^{t/2} = u'(e^t)e^t$$

also $\frac{d}{dt}(u(e^t)) = u'(e^t)e^t$

$$= e^t\frac{d}{dx}(u(x)) = xu'(x)$$

$$\Rightarrow u'(e^t) = u'(x)$$

also $\frac{d}{dt}(u'(e^t)) = u''(e^t)e^t = e^t\frac{d}{dx}u'(x) = xu''(x)$

$$\Rightarrow u''(e^t) = u''(x)$$

now, $u''(e^t) + \frac{c}{(e^t)^2}u(e^t) = 0$

then $e^{2t}u''(e^t) + cu(e^t) = 0$

$$\Rightarrow e^{3t/2}u''(e^t) + e^{-t/2}cu(e^t) = 0 \quad \text{——— (1)}$$

as $v(t) = e^{-t/2}u(e^t)$

$$v''(t) = e^{-t/2}u'(e^t)e^t - \frac{1}{2}e^{-t/2}u(e^t)$$

$$v''(t) = \frac{d}{dt}\left(e^{t/2}u'(e^t) - \frac{1}{2}e^{-t/2}u(e^t)\right)$$

$$= \frac{1}{2}e^{t/2}\cancel{u'(e^t)} + e^{t/2}u''(e^t)e^t + \frac{1}{4}e^{-t/2}u(e^t) - \frac{1}{2}e^{-t/2}\cancel{e^t u'(e^t)}$$

$= -e^{-t/2} c\, u(e^t) + \frac{1}{4} e^{-t/2} u(e^t) \quad (\because ①)$

$\Rightarrow V''(t) + cV(t) - \frac{1}{4}V(t) = 0$

$\Rightarrow V''(t) + \left(c - \frac{1}{4}\right)v(t) = 0$ ——— ②

as $c > \frac{1}{4}$ $\quad V''(t) + \left(c - \frac{1}{4}\right)v(t) = 0$ for $t \in \mathbb{R}$

will have solution as ossilatory

$v(t) = A\cos(\omega t) + B\sin(\omega t)$

$\omega = \sqrt{c - \frac{1}{4}} > 0 \quad (\because ②)$

so, v will have infinitely many solutions

then $u(e^t) = e^{t/2}v(t)$ will have zeros corresponding to zeros of $e(x)$ $x = e^t$ ($\because$ one-one corresponde)

$\Rightarrow u'' + \frac{c}{x^2}u = 0$ will have infinely many zeros in $(0,\infty)$ ——— ③

now as $u''(t) + q(t)u(t) = 0$

$q(t) \geq \frac{c}{t^2}$ given

as $u''(t) + \frac{c}{t^2}u(t) = 0$ has infinite zeros

from sturm comparison, between any two zeros of ③ there is minimum one zero of u satisfying

$u'' + qu = 0$ (u not trivial)

$\Rightarrow u$ has infinite many zeros on $(0,\infty$

1.10 $q : [a,b] \longrightarrow \mathbb{R}$ continious funtion

$u, V \in C^1[a,b]$ twice differentiable on (a,b)

$u(0) = V(0) = 0$

$u'(0) = V'(0) > 0$

$u(t) > 0$ for $t \in (a,b)$

$u''(t) + q(t)u(t) = 0 \quad t \in [a,b]$

$V''(t) + q(t)V(t) \geq 0 \quad t \in [a,b]$

<u>To prove</u>: $V(t) \geq u(t) \quad \forall t \in [a,b]$

<u>proof</u>: $(V(t) - u(t))'' + q(t)(V(t) - u(t)) \geq 0$
$t \in [a,b]$

let $\omega(t) = v(t) - u(t$

$\omega(0) = 0$

$\omega'(0) = 0$

$$\omega'' + q\omega \geq 0$$

let $\phi(t) = \omega'(t)u(t) - \omega(t)u'(t)$

$$\phi'(t) = \omega''u + \cancel{\omega'u'} - \cancel{\omega'u'} - \omega u''$$

$$= \omega''u + \omega(+qu) \quad (\because u'' + qu = 0)$$

$$\phi'(t) = \omega''u + q\omega u$$

$$= \underbrace{u}_{>0}(\underbrace{\omega'' + q\omega}_{\geq 0})$$

$$\Rightarrow \phi'(t) \geq 0 \quad \forall t \in (a,b)$$

$\phi(0) = \omega'(0)u(0) - \omega(0)u'(0)$
$= 0$

as $\phi'(t) \geq 0 \quad \forall t \in (a,b)$

as $\underset{t \in (a,b)}{u(t) > 0}$ but $u(0) = 0$

$0 \notin (a,b)$

<u>case I</u>: $0 \leq a$:

as $u(0) = 0$

$\phi'(t) \geq 0 \quad \forall t \in (a,b)$

$\phi(0) = 0$

$\Rightarrow \phi(t) \geq 0 \quad \forall t \in (a,b)$

as $u(t) > 0 \quad \forall t \in (a,b)$

$$\left(\frac{\omega}{u}\right)' = \frac{\omega'u - u\omega'}{u^2} = \frac{\phi}{u^2} \quad (\because u > 0 \text{ denominator})$$

$$\Rightarrow \left(\frac{\omega}{u}\right)' \geq 0$$

$\Rightarrow \left(\frac{\omega}{u}\right)$ is non decreasing

now $\lim_{t \to 0^+} \frac{\omega(t)}{u(t)} = \lim_{t \to 0^+} \frac{\omega'(t)}{u'(t)}$ ($\because$ L-Hospital)

$= 0 \quad (\because \omega'(0) = 0, u'(0) > 0)$

as ω, u continious

$$\Rightarrow \left(\frac{\omega}{u}\right) \geq 0 \quad \forall t \in (a,b)$$

$$\Rightarrow \omega \geq 0 \quad \forall t \in (a,b)$$

$$\Rightarrow v - u \geq 0 \quad \forall t \in (a,b)$$

$\Rightarrow v \geq u \quad \forall t \in (a,b)$

<u>case II</u>: $0 \geq b$:

$$\phi'(t) \geq 0 \quad \forall t \in (a,b)$$
$$\phi(0) = 0$$
$$\Rightarrow \quad \phi(t) \leq 0 \quad \forall t \in (a,b)$$

$$\Rightarrow \left(\frac{w}{u}\right)' = \frac{\phi}{u^2} \leq 0 \quad \forall t \in (a,b)$$

$$\Rightarrow \left(\frac{w}{u}\right)' \leq 0 \quad \forall t \in (a,b)$$

$\lim_{t \to 0^-} \frac{w(t)}{u(t)} = 0$ ($\because$ similar argument)

$\Rightarrow \frac{w(t)}{u(t)} \geq 0 \quad \forall t \in (a,b)$ ($\because$ $\frac{w}{u}$ decreasing, near zero $\frac{w}{u} = 0$)

$\Rightarrow w(t) \geq 0 \quad \forall t \in (a,b)$

$\Rightarrow v(t) \geq u(t) \quad \forall t \in (a,b)$

as u, v are continious

$$\lim_{t \to a^-} v(t) \geq \lim_{t \to a^-} u(t)$$
$$\Rightarrow v(a) \geq u(a)$$
similarly
$$v(b) \geq u(b)$$

so, $\forall t \in [a,b]$, $v(t) \geq u(t)$