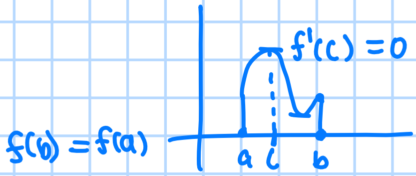
SI 507
Numerical Analysis

Midterms - 30%
Final Exam - 40%
Assignment - 30% ← Biweekly assignments
No attedence policy
Ref: 1) R. Burden and J.D Fairs - Numerical analysis
2) K.E Atkinson - An introduction to NA
Syllabus: Refer course booklet

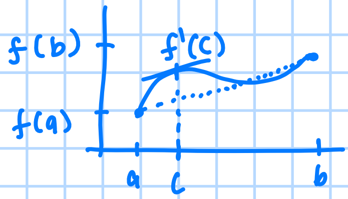
4^{th} Aug:

Theorm: (Role's theorm) suppose $f \in C([a,b])$ and f is diff on (a,b), $f(a)=f(b)$, then $\exists\, c \in (a,b)$ s.t $f'(c)=0$

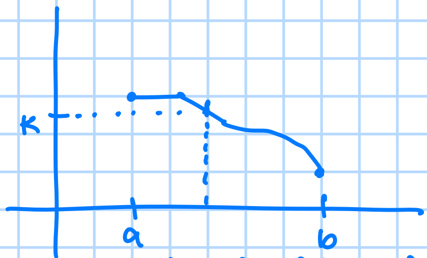


Theorm: (mean value theorm) let $f \in C([a,b])$ and diff on (a,b) then $f(b) = f(a) + (b-a) f'(c)$ for $c \in (a,b)$

or $\dfrac{f(b)-f(a)}{b-a} = f'(c)$



Theorm: (IVP/filling property) suppose $f \in C([a,b])$, k is a value between $f(a)$ and $f(b)$ then $\exists\, c \in (a,b)$ s.t $f(c)=k$



we can use the theorm if $f(a)>0$, $f(b)<0$ then $\exists\, c$ s.t $f(c)=0$

Exe: $f(x) = x^3 - 3x^2 + 2 \quad [0,2]$

$f(0) = 2$

$f(2) = 8 - 12 + 2 = -2 < 0$

so it has atleast one root in $[0,2]$ interval

Exe: $f(x) = (x-2)^2 - \ln(x)$, $x \in [e,4]$ $2<e<3$

$f(e) = (e-2)^2 - 1$

$2 < e < 3$

$\Rightarrow 0 < e-2 < 1$

$\Rightarrow 0 < (e-2)^2 < 1$

$\Rightarrow -1 < (e-2)^2 - 1 < 0$

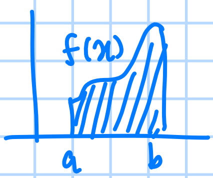
$\Rightarrow f(e) < 0$

$f(4) = 4 - \ln 4 > 0$

$\Rightarrow \exists\, c \in (e,4)$ s.t $(c-2)^2 - \ln(c) = 0$

Rieman Integration:

f is bounded on $[a,b]$, $\int_a^b f(x)dx$



let $P = \{x_0, \ldots, x_{n-1}, x_n\}$ with $x_0 = a$, $x_n = b$

s.t

$x_0 = a < x_1 < \cdots < x_{n-1} < b = x_n$

$$U(P,f) = \sum_{i=1}^{n} \sup_{x \in [x_{i-1}, x_i]} f(x)(x_i - x_{i-1})$$

$$L(P,f) = \sum_{i=1}^{n} \inf_{x \in [x_{i-1}, x_i]} f(x)(x_i - x_{i-1})$$

now $\lim_{n\to\infty} U(P_n, f) = \lim_{n\to\infty} L(P_n, f) = \int_a^b f(x)\,dx$ ← Rieman integrable

$x_i = x_0 + \frac{(x_n - x_0)}{n} i$ → interval value (one such example we can take)

Eg: $f(x) = x \quad x \in [0,1]$

$\int_0^1 x\,dx$, here $x_i = 0 + \frac{1}{n} i = \frac{i}{n}$

also $U(P_n, f) = \sum_{i=1}^{n} \left(\frac{i}{n}\right)\left[\frac{1}{n}\right]$

$= \frac{1}{n^2} \sum_{i=1}^{n} i$ ← as function is monotonic

$= \frac{1}{n^2}\left[\frac{(n+1)(n+2)}{2}\right]$ ← as function is monotonic

$L(P_n, f) = \sum_{i=1}^{n} \left(\frac{i-1}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=0}^{n-1} \frac{i}{n}\frac{1}{n} = \frac{(n)(n-1)}{2n^2}$

$\lim_{n\to\infty} U(P_n, f) = \frac{1}{2}$

$\lim_{n\to\infty} L(P_n, f) = \frac{1}{2}$ and so $\int_0^1 x\,dx = \frac{1}{2}$

Exe: $g(x) = \begin{cases} 1 \,; & x \in [0,1] \cap \mathbb{Q} \\ 0 \,; & x \in [0,1] \cap \mathbb{Q}^c \end{cases}$

$L(P_n, g) = 0$ → always as $\sup_I f(x) = 1$ as for any interval $\exists\, \mathbb{Q}$

$U(P_n, g) = 1$ → always $\inf_I f(x) = 0$ as for any interval $\exists\, \mathbb{Q}^c$

as $\lim_{n\to\infty} L(P_n, g) \neq \lim_{n\to\infty} U(P_n, g)$, g is not rieman integrable

Theorm: (taylor's theorm) suppose $f \in C([a,b])$, f is diff on (a,b),

$f(b) = f(a) + (x-a) f'(\gamma)$

$a < \gamma < x$

if f is diff n times on (a,b) then:

$$f(x) = f(a) + f'(a)\frac{(x-a)}{1!} + f^{(2)}(a)\frac{(x-a)^2}{2!} + \dots + \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a) + R_n f(x)$$

$$R_n f(x) = \frac{(x-a)^n}{n!} f^{(n)}(\gamma)$$

In $\sin x$, the taylor series converges as $n \to \infty$ so $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots$

Exe: Approximate $\sin x$ upto second order near 0?

Ans:

$\underbrace{f(0)}_{0} + f'(0)\frac{(x-0)}{1!} - \sin(\gamma(x))\frac{x^2}{2!}$

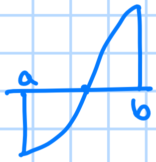
we can also do same for $\cos x$, here $g(x)=\cos x$, then

$$g(x)= g(0) + g'(0)x + g''(r(x))\frac{x^2}{2!}$$

$$= 1 + 0 - \cos(r(x))\frac{x^2}{2}$$

7th Aug:

Numerical system to find root of a non linear equation $[a,b]$
$f(a)<0, f(b)>0 \; \exists c \in (a,b)$ s.t $f(c)=0$



Methods to approximate root:

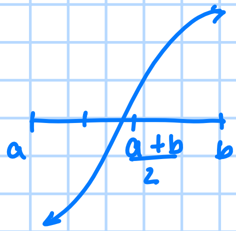
1> Bisection method
2> fixed point
} IVP

3> Newton Raphson method } taylor series

Bisection method:

given $f(a).f(b)<0$ or they have different size

so, wlog $f(a)<0, f(b)>0$



Idea is to just take midpoint and check sign again and again till interval small enough

Note: we have to pick an interval with 1 root and $[a,b]$ s.t $f(a)f(b)<0$

Step 1 is to first start with $f(a).f(b)<0$, $[a,b]$

$$c_1 = \frac{a+b}{2}$$

if $f(c_1)=0$ then done, else
$f(c_1)>0$ or $f(c_1)<0$
i.e $f(a)f(c_1)<0$ or $f(b).f(c_2)<0$
wlog $f(a)f(c_1)<0$ then

$$a_1 = a$$
$$b_1 = c_1$$

Step 2: now we do same with a_1, b_1 to get a_2, b_2
we keep on doing this to get

$\{c_1, c_2, \ldots\}$ s.t $c_n \to c$ when c is root
and if some c_i s.t $f(c_i)=0$ we get c directly

Defn: (tolerance) ε is here were to stop, $|c_n - c| < \varepsilon$ for first/smallest n we stop, so $|f(c_n)| < \varepsilon$ we stop

Note: To stop in bisection method both $|c_n - c| < \varepsilon$ and $|f(c_n)| < \varepsilon$ should be satisfied

Theorm: Suppose $f \in C[a,b]$ s.t $f(a).f(b)<0$, then bisection method generates a seq $\{c_n\}$ approximating a root c of f s.t

$$|c_n - c| < \frac{(b-a)}{2^n}, n \geq 1 \text{ so } c_n \sim c + O\left(\frac{1}{2^n}\right)$$

proof: Now $(b_n - a_n) = \frac{(b-a)}{2^{n-1}}$ $n \geq 1$ $c_n = \frac{a_n + b_n}{2}$ ← this time $a_1 = a$, $b_1 = b$

$|c_n - c| \leq \frac{b-a}{2^n}$

$a_n \leq c \leq b_n$
$a_n \leq c_n \leq b_n$

$|O(f)| \leq cf$ so, $c_n \sim c + O\left(\frac{1}{2^n}\right)$

Note: $\lim_{n\to\infty} \frac{c_{n+1} - c}{(c_n - c)^k}$ = some Scalar

then for some $k = 1, 2 \ldots$ it is how fast is the convergence for bisection method $k=1$ and so it's a slow convergence

Fixed point method:

If f satisfies $f(P) = P$ for some $p \in [a,b]$, P is a fixed point of f

Theorem: (sufficient condition for existance of fixed point) If $g \in C[a,b]$ and $g(x) \in [a,b]$ $\forall x$, then g has atleast one fixed point

additionally if g is differentiable on (a,b) and $|g'(x)| \leq k < 1$ then g has a unique fixed point.

proof:
for first part, let $f(x) = g(x) - x$ then $f(a) = g(a) - a \geq 0$ as $g(a) \in [a,b]$

$f(b) = g(b) - b \leq 0$ as $g(b) \in [a,b]$

so, $\exists$ c s.t $f(c) = 0 \Rightarrow g(c) = c$ [$\because$ IVP]

for second part, if $\exists$ 2 fixed points P & q on g $(P \neq q)$ then

$|P - q| = |g(P) - g(q)| = |g'(\gamma)(P-q)|$ ← mean value theorm

$\leq k|P-q|$

$|P-q| < |P-q|$ as $k<1$

this is a contradiction and so $\Rightarrow P = q$

Approximation:

let $g(x) = x - f(x)$, if there is one unique fixed point say P

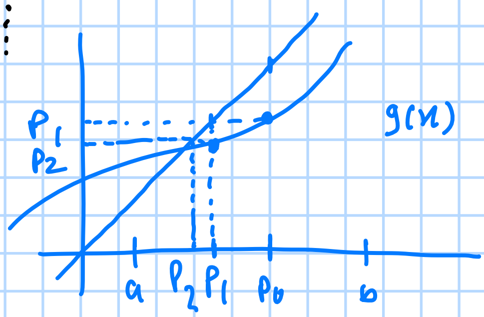
$P_0 \in [a,b]$

if $g(P_0) = P_0$ we are done, or else

$g(P_0) = P_1$

then $g(P_1) = P_2$

$\vdots$



Theorm: let $g \in C^1([a,b])$ s.t $a \leq g(x) \leq b$ and $|g'(x)| \leq k < 1$ $\forall x \in [a,b]$ then for any $p_0 \in [a,b]$ the sequence $p_n = g(p_{n-1})$ $n \geq 1$ converges to a unique fixed point of g

proof:

$$
\begin{aligned}
|p_n - p| &= |g(p_{n-1}) - g(p)| \\
&\leq k|p_{n-1} - p| \quad \text{by MVT} \\
&= k|g(p_{n-2}) - g(p)| \\
&\ \vdots \\
&\leq k^n (|p_0 - p|)
\end{aligned}
$$

$k^n \to 0$ so $p_n \to p$ as $n \to \infty$

Note: $|p_n - p_{n-1}| < \varepsilon$ then we stop, also here in this case there will always be a convergence

Eg:

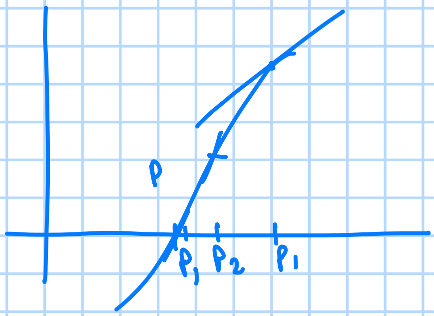
derivative > 1 so, the system fails here

seq going left but we want to reach right

11^{th} Aug :

Newton's method (taylor series):

$$0 = f(P) = f(x) + (x-P)f'(x) + \frac{(x-P)^2}{2} f''(\gamma) \qquad (x-P)^2 = \text{very small}$$



$$\Rightarrow 0 = f(x) + (x-P)f'(x)$$

$$\Rightarrow P = x - \frac{f(x)}{f'(x)}$$

now, $P_n = P_{n-1} - \dfrac{f(P_{n-1})}{f'(P_{n-1})}$

$|f'(x)| \geq m > 0$

the scheme converges if $f'(P_{n-1}) \neq 0$

Theorem: Suppose $f \in C^2[a,b]$, if $f(P)=0$ and $f'(P) \neq 0$ for some $P \in [a,b]$. Then $\exists$ a $\delta > 0$ s.t newton's method generates a seq^n $\{P_n\}$ conv to P for any interval gives $P_0 \in [P-\delta, P+\delta]$

proof: $g(x) = x - \dfrac{f(x)}{f'(x)}$, $f(P)=0$, $f'(P) \neq 0 \Rightarrow f'(x) \neq 0$ near 'p' by continuity

g is $C'[P-\delta_2, P+\delta_2]$

$$g'(x) = 1 - \left[\frac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2}\right]$$

$$= \frac{f(x)f''(x)}{(f'(x))^2}$$

$g'(P) = 0$

since g' is continious and $g'(P)=0$, $\exists \delta_1 > 0$ s.t

$|g'(x)| \leq k < 1$, $\forall x \in [P-\delta_1, P+\delta_1]$, let $\delta = \min\{\delta_1, \delta_2\}$

now from previous theorm let's show:

$$g : [P-\delta, P+\delta] \to [P-\delta, P+\delta]$$

$$|g(x) - P| = |g(x) - g(P)| = |(x-P)g'(\gamma)| \quad \forall x \in [P-\delta, P+\delta]$$

$$\leq k|x-P|$$

$$< |x-P| < \delta$$

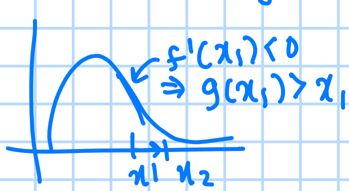
$$\Rightarrow g : [P-\delta, P+\delta] \to [P-\delta, P+\delta]$$

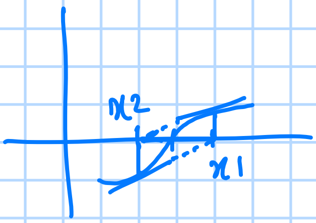
now, as $|g'(x)| \leq k < 1$ & so g has unique fixed points from previous theorm

$$\Rightarrow \underset{\substack{\downarrow \\ P}}{P_n} = \underset{\substack{\downarrow \\ g(P)}}{g(P_{n-1})} \to P$$

$\left(\left.\begin{matrix} g \in C'[a,b] \\ g : [a,b] \to [a,b] \\ |g'(x)| \leq k < 1 \end{matrix}\right\} \Rightarrow \begin{matrix}\text{fixed point} \\ \text{is unique}\end{matrix} \right)$

eg: cases where this method fails:





To fix errors, first few times we can do bisection method to make the interval smaller, and then do the same method

we can also approximate $f'(P_{n-1}) = \lim\limits_{x \to P_{n-1}} \dfrac{f(x) - f(P_{n-1})}{x - P_{n-1}}$, we can also

$$f'(p_{n-1}) \approx \frac{f(p_{n-2})-f(p_{n-1})}{p_{n-2}-p_{n-1}}$$

Note: The above method is called the secant method

Error analysis:

Defn: Suppose $\{p_n\}$ is a sequence that $\longrightarrow p$ and $p_n \neq p \ \forall n$, if $\exists$ positive constant α and λ s.t

$$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^\alpha} = \lambda$$

then we say α is the order of convergence. λ is asymptotic error

(i) $\alpha=1$, linear convergence

(ii) $\alpha=2$, quadratic

So if $\alpha=1$, $\frac{|p_{n+1}|}{|p_n|} = \lambda \Rightarrow |p_{n+1}| = \lambda^n |p_0|$

$\alpha=2$, $\frac{|a_{n+1}|}{|a_n|^2} = \lambda \Rightarrow |a_{n+1}| = \lambda |a_n|^2$

$\Rightarrow |a_{n+1}| = \lambda | \lambda |a_{n-1}|^2|^2 \cdots$

$= \lambda^{2^n-1} |a_0|^{2^n}$

order of convergence:

Fixed point:

$p_n = g(p_{n-1})$

$|p_{n+1}-p| = |g(p_n)-g(p)|$

$= |g'(\gamma_n)(p_n-p)|$ ← this is for fixed point method $g'(p)\neq 0$

$$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|} = |g'(\gamma_n)| \longrightarrow |g'(p)|$$

$$= |g'(p)|$$

Newton:

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$

$$\Rightarrow p_{n+1}-p = p_n - p - \frac{f(p_n)}{f'(p_n)} \quad \text{——} ① \quad \text{(this is from our argument)}$$

and $0 = f(p) = f(p_n) + (p-p_n)f'(p_n) + \frac{(p-p_n)^2}{2} f''(\gamma_n)$

$$f(p_n) = -(p-p_n)f'(p_n) - \frac{(p-p_n)^2}{2} f''(\gamma_n)$$

$$\frac{f(p_n)}{f'(p_n)} = -(p-p_n) - \frac{(p_n-p)^2}{2}\frac{f''(\gamma_n)}{f'(p_n)} \quad \text{——} ②$$

we put $\frac{f(p_n)}{f'(p_n)}$ in ① : (we compute $\frac{f(p_n)}{f'(p_n)}$ from taylor series expansion)

$$(p_{n+1}-p) = \cancel{(p_n-p)} + \cancel{(p-p_n)} + \frac{(p_n-p)^2}{2}\frac{f''(\gamma_n)}{f'(p_n)}$$

$$\Rightarrow \frac{p_{n+1}-p}{(p_n-p)^2} = \frac{1}{2}\frac{f''(\gamma_n)}{f'(p_n)}$$

$$\Rightarrow \frac{|p_{n+1}-p|}{|p_n-p|^2} = \frac{1}{2}\left|\frac{f''(\gamma_n)}{f'(p_n)}\right|$$

$$\Rightarrow \lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^2} = \frac{1}{2}\left|\frac{f''(p)}{f'(p)}\right| \quad \{p_n\} \text{ order of conv} = 2$$

Multiple root problem:

A solution p to $f(x)=0$ is a zero of multiplicity m if we can write $f(x)$ as

$$f(x) = (x-p)^m q(x)$$

$$\lim_{x \to p} q(x) \neq 0$$

define a new function $\mu(x) = \frac{f(x)}{f'(x)} = \frac{(x-p)^m q(x)}{m(x-p)^{m-1} q(x) + q'(x)(x-p)^m}$

$$= (x-p) \frac{q(x)}{mq(x) + (x-p)q'(x)}$$

$\mu(p) = 0$ and multiplicity $=1$

$\mu'(p) \neq 0$ as $x \to p$

$$\frac{\cancel{q(x)}}{m\cancel{q(x)} + 0} = \frac{1}{m}$$

$$\text{so } \mu'(p) = \frac{1}{m} + 0 = \frac{1}{m}$$

$g(x) = x - \frac{\mu(x)}{\mu'(x)}$, scheme will be $p_n = p_{n-1} - \frac{\mu(p_{n-1})}{\mu'(p_{n-1})}$

$$\Rightarrow g(x) = x - \frac{f(x)}{\cancel{f'(x)}} \times \frac{1}{\left[\frac{f'(x)^2 - f(x)f''(x)}{(f'(x))^2}\right]}$$

$$= x - \frac{f(x)f'(x)}{(f'(x))^2 - f(x)f''(x)}$$

this will be computationally very heavy

14th Aug:

Assignment-1 submission 28th Aug

$f(x)=0$, $p_n \longrightarrow p$ $f(p)=0$ 1) Bisection method

2) fixed point interation $p_n = g(p_{n-1})$, $g:[a,b] \longrightarrow [a,b]$ $|g'(x)|<1$

3) Newton's method

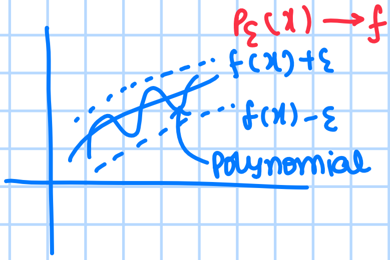
$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} \quad f \in C^2 \quad |f'(r)| \geq m > 0$$

$p_n = g(p_{n-1})$, $g(p)=p$, $|g'(x)|<1$ $p_0 \in [p-\delta, p+\delta]$ (newton's method uniqueness)

Theorm: (Weierstess approximation) suppose f is a continious funtion $[a,b]$ Then given any $\varepsilon>0$, $\exists$ a polynomial $p(x)$ s.t $|f(x)-P(x)|<\varepsilon$ $\forall x \in [a,b]$

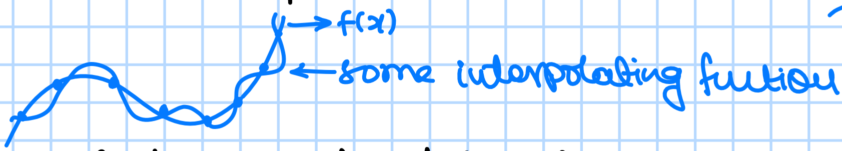
$$\sup_{x \in [a,b]} |f(x)-P(x)|<\varepsilon$$

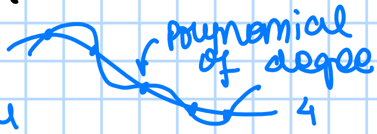
$P_\varepsilon(x) \longrightarrow f(x)$ uniformly



so, $P_n[a,b]$ is dense in $C[a,b]$, $\|\cdot\|_2$

we can use taylor series but that only approxes at a particular point





we want to find some interpolating funtion which will do this

Eg: polynomial of degree 1 s.t $f(x_0)=y_0$
$f(x_1)=y_1$

then $y-y_0 = x-x_0 \times \left(\frac{y_1-y_0}{x_1-x_0}\right)$

$$\Rightarrow y = y_0 + \frac{(x-x_0)}{(x_1-x_0)}(y_1-y_0)$$

$$\Rightarrow y = \frac{(x-x_1)}{(x_0-x_1)} f(x_0) + \frac{(x-x_0)}{(x_1-x_0)} f(x_1)$$

$$= L_0(x) f(x_0) + L_1(x) f(x_1)$$

$L_0(x_1)=0$ $\quad L_1(x_1)=1$

$L_0(x_0)=1$ $\quad L_1(x_0)=0$

let's generise the above to $L_{n,k}(x_i)$ is lagrange polynomial of degree n

$$L_{n,k}(x_i) = \begin{cases} 0 ; & i \neq k \\ 1 ; & i = k \end{cases}$$

degree n polynomial s.t

for $x_k = 1$, $x_p = 0$ $i \neq k$

$(x_0, f(x_0)), \dots (x_n, f(x_n))$

Theorm: If $x_0, \dots x_n$ are $(n+1)$ distinct points and f is a continious function whose values are given at these points then $\exists$ a unique polynomial $P(x)$ of degree 'n' s.t $f(x_k)=P(x_k)$, $0 \leq k \leq n$ and $P(x) = \sum_{k=0}^{n} L_{n,k}(x) f(x_k)$

proof: $\alpha_{n,k}(x_i) = \begin{cases} 1 & ; i = k \\ 0 & ; i \neq k \end{cases}$

$\alpha'_{n,k}(x) = (x - x_0) \cdots\cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_n)$

$\alpha'_{n,k}(x_i) = 0$ for $i \neq k$

now $\alpha'_{n,k}(x_k) = \prod_{\substack{i=0 \\ i \neq k}}^{n} (x_k - x_i)$

then $\alpha_{n,k}(x) = \dfrac{\alpha'_{n,k}(x)}{\alpha'_{n,k}(x_k)}$ (degree n as x_k not there)

then $\alpha_{n,k}(x_i) = \begin{cases} 1 ; & i = k \\ 0 ; & i \neq k \end{cases}$ (need $n+1$ points for degree n)

so now $p(x) = \sum_{k=0}^{n} \alpha_{n,k}(x) f(x_k)$

this is unique as if polynomial $p(x)$ is not unique then $\exists\, q(x)$ s.t $q \neq p$

$q(x_i) = f(x_i) \ \forall i \in \{0, \dots, n\}$

polynomial of degree n

now $h(x) = p(x) - q(x)$

then $h(x)$ is also a polynomial of degree n

it has roots at $x_i \ \forall i$

$\Rightarrow h(x_i) = 0 \quad \forall i \in \{0, \dots n\}$

it has $n+1$ roots $*$

$\Rightarrow p(x) = q(x) \ \forall x$

Theorm: suppose $x_0, \dots x_n$, $(n+1)$ distinct points on $[a,b]$, suppose $f \in C^{n+1}[a,b]$ then $\exists$ a $\xi(x) \in (a,b)$ s.t

$$f(x) = p(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

proof: $x = x_k$ $(0 \leq k \leq n)$ then we have that $f(x_k) = p(x_k)$ (P is the lagrange interpolating polynomial of degree m)

when $x \neq x_k$

Define a new fuction g s.t

$$g(t) = \underset{\in C^{n+1}}{f(t)} - \underset{\in C^{n+1}}{p(t)} - [f(x) - p(x)] \underset{\in C^{n+1}}{\prod_{i=0}^{n} \frac{(t - x_i)}{(x - x_i)}} \quad t \in [a,b]$$

$\Rightarrow g \in C^{n+1}([a,b])$

now, $g(x_k) = 0 \quad \forall \ 0 \leq k \leq n$

$g(x) = f(x) - p(x) - (f(x) - p(x))$

$= 0$ while x is fixed

t is a vaiable

so, g has $x, x_0, \dots x_n$ $(n+2)$ zer

wlog $x < x_0 < x_1 \dots < x_n$

then applyig roles theorm on every interval we get

g' has $(n+1)$ zeroes

g'' has (n) zeros $\cdots\cdots g^n$ has 2 zeroes

so, $\exists\, c, d$ s.t $g^n(c) = g^n(d) = 0$, $c, d \in (a,b)$

So, $\exists\, r \in (a,b)$ s.t $g^{n+1}(r) = 0$

↳ r depends on x so $\gamma(x) \in (a,b)$

Since $g^{(n+1)}(t) = f^{(n+1)}(t) - p^{(n+1)}(t) - [f(x) - p(x)] \dfrac{d^{n+1}}{dt^{n+1}} \left[\prod_{i=0}^{n} \dfrac{(t-x_i)}{(x-x_i)} \right]$

but p is a polynomial of degree n

$\Rightarrow p^{(n+1)}(t) = 0$

$$\Rightarrow g^{(n+1)}(t) = f^{(n+1)}(t) - [f(x) - p(x)] \frac{(n+1)!}{\prod_{i=0}^{n}(x-x_i)}$$

$g^{(n+1)}(\gamma(x)) = 0$ (By rolle's theorm)

$$\Rightarrow f^{(n+1)}(\gamma(x)) = [f(x) - p(x)] \frac{(n+1)!}{\prod_{i=0}^{n}(x-x_i)}$$

$$\Rightarrow \frac{f^{n+1}(\gamma(x)) \prod_{i=0}^{n}(x-x_i)}{(n+1)!} + p(x) = f(x)$$

and $|f(x) - p(x)| \leq \dfrac{c}{(n+1)!} |f^{(n+1)}(\gamma(x))|$

Note: only 5 questions will be in assignment, but submit all

18th Aug:

Recall, f is cont on $[a,b]$, $f(x_0),\ldots f(x_n)$

Lagrange polynomial $L_n(x)$ of degree n s.t

$$L_n(x_i) = f(x_i) \quad 0\leq i\leq n$$

$$f(x) = L_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)} \quad [f\in C^{n+1}[a,b]]$$

error

$$L_n(x) = \sum_{i=0}^{n} f(x_i)\, L_{n,i}(x)$$

$$L_{n,k}(x_i) = \delta_{i,k} = \begin{cases} 1 ; i=k \\ 0 ; i\neq k \end{cases}$$

Note: $L_{n,k}(x) = \prod_{\substack{i=0\\ i\neq k}}^{n} \frac{(x-x_i)}{(x_k-x_i)}$

$f(x_i)$ $\forall\, i\in\{0,1\ldots n-1, n\}$

$$P(x) = a_0 + a_1(x-x_0)(x-x_1) + \cdots + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

s.t $P(x_k) = f(x_k)$ $\forall\, k\in\{0,\ldots n\}$

now, $P(x_0) = f(x_0) \Rightarrow a_0 = f(x_0)$

$$f(x_1) = P(x_1) = f(x_0) + a_1(x_1-x_0)$$

$$\Rightarrow a_1 = \frac{f(x_1)-f(x_0)}{x_1-x_0} = f[x_0,x_1] \text{ (divided difference)}$$

$$f(x_2) = P(x_2) \Rightarrow a_2 = \frac{f[x_1,x_2]-f[x_1,x_0]}{x_2-x_0}$$

$\vdots$

k^{th} order divide differe

$$f[x_i,x_{i+1},\ldots x_{i+k}] = \frac{f[x_{i+1},\ldots x_{i+k}] - f[x_i,\ldots x_{i+k-1}]}{(x_n-x_0)}$$

$$a_i = f[x_0,\ldots x_i]$$

$$a_n = f[x_0,\ldots x_n]$$

Defn : $P(x) = f(x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) + \cdots + f[x_0,\ldots x_n]\prod_{i=0}^{n}(x-x_i)$

Lemma: If $f\in C^n[a,b]$ then we have

$$f[x_0,\ldots x_n] = \frac{f^n(\xi)}{n!} \text{ for some } \xi\in(a,b)$$

proof: $n=1$: $f[x_0,x_1] = \frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{\cancel{(x_1-x_0)}f'(\xi)}{\cancel{(x_1-x_0)}} = f'(\xi)$ for some $\xi\in(a,b)$

($\because$ taylor's theorm)

define $g(x) = f(x) - p(x)$ $g(x_i) = 0$ $\forall\, 0\leq i\leq n$

so g has $n+1$ zeroes

$\Rightarrow g'(x)$ has n zeroes ($\because$ rolles theorm)

$\Rightarrow g''(x)$ has $n-1$ zeroes

$\vdots$

$\Rightarrow g^n(x)$ has 1 zero say $\xi\in(a,b)$

where $g^n(x) = f^n(x) - p^n(x)$

$\Rightarrow f^n(\eta) = p^n(\eta)$

where $p^n(\eta) = f[x_0 \dots x_n]\, n!$

$\Rightarrow \dfrac{f^n(\eta)}{n!} = f[x_0 \dots x_n]$

Note: $f[x_0\, x_1] = \int_0^1 f'(t_0 x_0 + t_1 x_1)\, dt_1 \qquad t_0 + t_1 = 1$

Hermite approximation:

$f \in C^k[a,b]$, we want to find p s.t it approximates f & f' at $x_0, \dots x_m$

$P(x_i) = f(x_i) = y_i \qquad 0 \leq i \leq n$
$P'(x_i) = f'(x_i) = z_i \qquad 0 \leq i \leq n$

So, we want $\|P - f\|_{C'[a,b]} \leq \varepsilon$ i.e $\|f\|_{C'[a,b]} = \sup_{[a,b]} |f| + \sup_{[a,b]} |f'|$

Theorem: Suppose $f \in C'[a,b]$ and let $x_0, x_1 \dots x_n \in [a,b]$ are $n+1$ distinct points then $\exists$ a unique polynomial of degree atmost $(2n+1)$ s.t
$H(x_i) = f(x_i)$
$H'(x_i) = f'(x_i)$ for $0 \leq i \leq n$
H is given by

$$H(x) = \sum_{j=0}^{n} f(x_j) H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j) \hat{H}_{n,j}(x)$$

$$H_{n,j}(x_i) = \delta_{i,j} \;\&\; H'_{n,j}(x_i) = 0$$

$$\hat{H}_{n,j}(x_i) = 0 \qquad \& \; \hat{H}'_{n,j}(x_i) = \delta_{i,j}$$

$$H_{n,j}(x) = [1 - 2(x - x_j)\, l'_{n,j}(x_j)]\, l^2_{n,j}(x)$$

$$\hat{H}_{n,j}(x) = (x - x_j)\, l^2_{n,j}(x)$$

proof: $H_{n,j}(x_j) = [1 - 0] = 1$

$i \neq j$

$H_{n,j}(x_i) = [1 - 2(x_i - x_j)\, l'_{n,j}(x_j)]\, 0 = 0$

$H'_{n,j}(x) = -2\, l'_{n,j}(x_j)\, l^2_{n,j}(x) + [1 - 2(x - x_j)\, l'_{n,j}(x_j)]\, 2\, l'_{n,j}(x)\, l_{n,j}(x)$

$i \neq j$

$H'_{n,j}(x_i) = -2\, l'_{n,j}(x_j)(0) + [1 - 2(x_i - x_j)\, l'_{n,j}(x_j)]\, 0 = 0$

$i = j$

$H'_{n,j}(x_j) = -2\, l'_{n,j}(x_j)(1) + [1 - 2(0)]\, 2\, l'_{n,j}(x_j) = 0$

$H_{n,j}(x)$ satisfies the desired conditions

$\hat{H}_{n,j}(x_j) = 0$

$i \neq j$

$\hat{H}_{n,j}(x_i) = (x_i - x_j)(0) = 0$

now, $\hat{H}'_{n,j}(x) = 2\, l_{n,j}(x)\, l'_{n,j}(x)\, (x - x_j) + l^2_{n,j}(x)$

$\hat{H}'_{n,j}(x_j) = 2(1)\, l'_{n,j}(x_j)(0) + 1 = 1$

$\hat{H}_{n,i}(x_i) = 2(0) + 0 = 0$ $\quad i \neq j$

so existence part is done, for uniqueness part if there are 2 polynomial with same property G, H then

$$F(x) = H(x) - G(x) \text{ s.t } F(x_i) = 0 \quad F'(x_i) = 0, \quad i \in \{0, \ldots n\}$$

F has n+1 double roots at $x_0, x_1, \ldots, x_n$

$$\Rightarrow F(x) = g(x)\underbrace{(x-x_0)^2 \cdots (x-x_n)^2}_{\text{polynomial of degree } 2n+2}$$

$\Rightarrow$ H, G polynomial of $2n+2$

but G, H polynomial of $2n+1$ (atmost)

so, this is a contradiction and G=H

uniqueness is also done

Lemma: $f \in C^{2n+2}[a,b]$ then

$$f(x) = H(x) + \frac{f^{2n+2}(\xi)}{(2n+2)!} \prod_{i=0}^{n} (x-x_i)^2 \quad x \in (a,b)$$

$$H(x) = \sum_{j=0}^{n} f(x_j) H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j) \hat{H}_{n,j}(x)$$

proof: for fixed $x \neq x_i$

$$g(t) = f(t) - H(t) + (f(x) - H(x)) \prod_{i=0}^{n} \frac{(t-x_i)^2}{(x-x_i)^2}$$

wlog case of m+n roots: (multiplicity m, n) m+n roots

$$\frac{d}{dx}(x-a)^m (x-b)^n = m(x-a)^{m-1}(x-b)^n + n(x-a)^m (x-b)^{n-1}$$
$$= (x-a)^{m-1}(x-b)^{n-1}[m(x-b) + n(x-a)]$$

or $m-1+n-1+1 = m+n-1$ roots

s.t new root $= (mb+na)/(m+n) \in (a,b)$

wlog $g(t)$ has $2n+3$ roots $(x, x_0, x_0, \ldots x_n, x_n)$ in (a,b) $\rightarrow$ we count multiplicity as multiple roots

$g'(t)$ will have $2n+2$ roots in (a,b)

$\vdots$

$g^{2n+2}(t)$ will have 1 root in (a,b) — this follows from wlog case

so let $\gamma(x)$ = root in given interval ($\gamma(x) \in (a,b)$ and depends on x)

$$\Rightarrow g^{2n+2}(\gamma(x)) = 0$$

$$\Rightarrow f^{2n+2}(\gamma(x)) - \underbrace{H^{2n+2}(\gamma(x))}_{0} + (f(x) - H(x)) \frac{(2n+2)!}{\prod_{i=0}^{n}(x-x_i)^2} = 0$$

$\leftarrow$ Degree $2n+1$

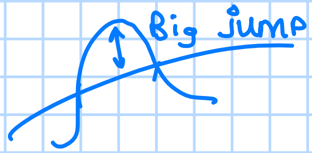
$$\Rightarrow \frac{f^{2n+2}(\gamma(x))}{(2n+2)!} \prod_{i=0}^{n} (x-x_i)^2 = f(x) - H(x)$$

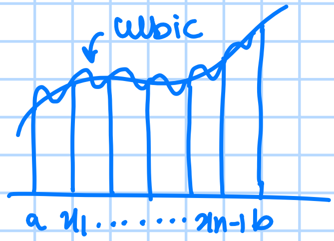
$$\Rightarrow f(x) = H(x) + \frac{f^{2n+2}(\gamma(x))}{(2n+2)!} \prod_{i=0}^{n} (x-x_i)^2$$

21^{st} Aug:

Cubic spine polynomial approximation:

so far we did lagrange approximation and Hermite polynomial, but the issue is as the polynomials are not always near





as for a subinterval $[x_0, x_1]$, if we are given derivates at same point then we can just use Hermite polynomial degree $2\times2-1=3$ and so we get a cubic polynomial

we don't want to use Hermite polynomial, different polynomial on each subinterval $[x_j, x_{j+1}]$ $0\leq j\leq n-1$

Defn: (cubic spline polynomial) A polynomial s, s_j is cubic on $[x_j, x_{j+1}]$

$$s = s_j \text{ on } [x_j, x_{j+1}] \quad 0\leq j\leq n-1$$

1) $s_j(x_j) = f(x_j)$ $\forall j\in\{0,\dots n-1\}$ and $s_{n-1}(x_n) = f(x_n)$

2) $s_j(x_{j+1}) = s_{j+1}(x_{j+1})$; $0\leq j\leq n-2$

3) $s_j'(x_{j+1}) = s_{j+1}'(x_{j+1})$; $0\leq j\leq n-2$

4) $s_j''(x_{j+1}) = s_{j+1}''(x_{j+1})$; $0\leq j\leq n-2$

} It's like the polynomial is stitched

Now let's construct $s_j(x) = a_j + b_j(x-x_j) + c_j(x-x_j)^2 + d_j(x-x_j)^3$ $[x_j, x_{j+1}]$ $0\leq j\leq n-1$

3 degree

n such polynomials

Degree of freedom:

As we have to compute $\{a_j, b_j, c_j, d_j \mid 0\leq j\leq n-1\}$ so there are $4n$ unknowns

Relations, $3(n-1)+(n+1) = 4n-2$ so atleast 2 degrees of freedom

from 2,3,4   from 1 of defn

let's impose more condition so no of condition = no of variables

5) (a) $s''(a) = s''(b) = 0$ ← 2 conditions

(b) $s'(a) = f(a)$, $s'(b) = f'(a)$

} any one of this condition gives 2 more conditions

2 conditions

Construction of cubic spine:

$s_j(x_j) = a_j$ $0\leq j\leq n-1$ a_j for $0\leq j\leq n-1$ known now

$= f(x_j)$

so a_j are known

$s_{j+1}(x_{j+1}) = a_{j+1} = s_j(x_{j+1}) = a_j + b_j h_j + c_j h_j^2 + d_j h_j^3$; $0\leq j\leq n-2$

$h_j = x_{j+1} - x_j$ $0\leq j\leq n-1$ ← defining it for all j

$s_j'(x) = b_j + 2c_j(x-x_j) + 3d_j(x-x_j)^2$

$S_j''(x) = 2c_j + 6d_j(x - x_j)$

$$S_{j+1}'(x_{j+1}) = S_j'(x_{j+1})$$

$$\Rightarrow b_{j+1} = b_j + 2c_j h_j + 3d_j h_j^2 \qquad 0 \leq j \leq n-2$$

$$\& \quad S_{j+1}''(x_{j+1}) = S_j''(x_{j+1})$$

$$\Rightarrow 2c_{j+1} = 2c_j + 6d_j h_j \qquad 0 \leq j \leq n-2$$

Define

So, total we get:
1) $a_j = f(x_j) \quad 0 \leq j \leq n-1, \quad a_n = f(x_n)$
2) $a_{j+1} = a_j + b_j h_j + c_j h_j^2 + d_j h_j^3 \qquad 0 \leq j \leq n-2$
3) $b_{j+1} = b_j + 2c_j h_j + 3d_j h_j^2 \qquad 0 \leq j \leq n-2$
4) $d_j = \dfrac{c_{j+1} - c_j}{3h_j}$ —— ① $\qquad 0 \leq j \leq n-2$

and 5) $a_n = a_{n-1} + b_{n-1}h_{n-1} + c_{n-1}h_{n-1}^2$ one more assumption

let:
6) $b_n = f'(x_n) = S_{n-1}'(x_n)$
7) $c_n = f''(x_n) = S_{n-1}''(x_n)$

now, $b_{j+1} = b_j + 2c_j h_j + \dfrac{(c_{j+1} - c_j)}{\cancel{3h_j}} \cancel{h_j \times 3}\, h_j$

$$= b_j + c_j h_j + c_{j+1} h_j$$

$$a_{j+1} = a_j + b_j h_j + h_j^2\left(\frac{c_{j+1} + 2c_j}{3}\right)$$

$$\Rightarrow a_{j+1} = a_j + b_j h_j + \frac{h_j^2}{3}(c_{j+1} + 2c_j) \qquad 0 \leq j \leq n-1$$

$$\Rightarrow b_j = \frac{(a_{j+1} - a_j)}{h_j} - \frac{h_j}{3}(c_{j+1} + 2c_j) \text{ —— ②}$$

$$\Rightarrow b_{j-1} = \frac{a_j - a_{j-1}}{h_{j-1}} - \frac{h_{j-1}}{3}(c_j + 2c_{j-1}) \qquad 1 \leq j \leq n$$

$$b_j = b_{j-1} + h_{j-1}(c_j + c_{j-1})$$

$$\frac{(a_{j+1} - a_j)}{h_j} - \frac{h_j}{3}(c_{j+1} + 2c_j) = \frac{a_j - a_{j-1}}{h_{j-1}} - \frac{h_{j-1}}{3}(c_j + 2c_{j-1}) + h_{j-1}(c_j + c_{j-1})$$

$$\Rightarrow \frac{a_{j+1} - a_j}{h_j} - \frac{(a_j - a_{j-1})}{h_{j-1}} = \frac{h_j}{3}(c_{j+1} + 2c_j) - \frac{h_{j-1}}{3}(c_j + 2c_{j-1}) + h_{j-1}(c_j + c_{j-1})$$

$$= c_{j-1}\frac{h_{j-1}}{3} + c_j\left(2\frac{h_{j-1}}{3} + 2\frac{h_j}{3}\right) + \frac{c_{j+1}h_j}{3}$$

$$\Rightarrow \frac{3}{h_j}(a_{j+1} - a_j) - \frac{3}{h_{j-1}}(a_j - a_{j-1}) = c_{j-1}h_{j-1} + 2c_j(h_j + h_{j-1}) + h_j c_{j+1} \qquad 1 \leq j \leq n-1$$

—— ③

as a_j's are known, we use it and ③ to calculate c_j's
then use ② to calculate b_j's and
① to calculate d_j's

Theorem: If f is continuous function defined on $[a,b]$ $a = x_0 < x_1 < \dots < x_n = b$ then f has a unique spline polynomial S on the nodes $\{x_0, x_1, \dots x_n\}$

s.t $S''(a) = S''(b) = 0$

proof: $S''(x_n) = c_n = 0$

$S''(x_0) = 2c_0 + 6d_0(x_0 - x_0) = c_0 = 0$

then $\quad c_{j-1}h_{j-1} + 2c_j h_j + 2c_j h_{j-1} + h_j c_{j+1} = \dfrac{3}{h_j}(a_{j+1} - a_j) - \dfrac{3}{h_{j-1}}(a_j - a_{j-1})$

so,
$$\begin{bmatrix} 1 & 0 & \cdot & \cdots & & 0 \\ h_0, & 2(h_0+h_1), & h_1 & \cdots & & 0 \\ 0 \cdots & h_1, & 2(h_1+h_2), & h_2 & \cdots & 0 \\ & \vdots & & & & \\ 0 \cdots & 0 & h_{n-2}, & \cdots & & h_{n-1} \\ 0 & \cdots & & & 0 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ \\ \\ \\ 0 \end{bmatrix}$$
known

c_n, c_0 known then we have to find $c_1 \ldots c_{n-1}$ to get all b & d

$$\det \begin{vmatrix} 1 & 0 & \cdots & & & & 0 \\ * & * & * & 0 & \cdots & & 0 \\ 0 & * & * & * & 0 & \cdots & 0 \\ \vdots & \cdots & 0 & 0 & * & * & * \\ \vdots & \cdots & & 0 & 0 & * & * & * \\ 0 & \cdots & & & & \cdots & 1 \end{vmatrix} = 1 \det \begin{vmatrix} * & * & & 0 \\ * & * & * & \\ & * & \ddots & * \\ 0 & & * & * & * \\ & & & * & * \end{vmatrix}$$

as first and last line only 1 one and so by defn mod does not change

let me show that matrix of form: $A = \begin{pmatrix} a_1 & c_1 & & & \\ b_2 & a_2 & c_2 & & \\ & b_3 & a_3 & c_3 & \\ & & \ddots & & c_{n-1} \\ & & & b_n & a_n \end{pmatrix}$

if $|a_j| \geq |b_j| + |c_j|$, $b_j, c_j \neq 0$
$j = 2 \ldots n-1$ $|a_1| > |c_1| > 0$
$|a_n| > |b_n| > 0$ are non singular

as if till $n-1 \times n-1$ non singular then by gaussian elimination getting rid of b_2 we get

$b_2 - a_1 \times x = 0$
$x = b_2 / a_2$

$$\det A = \det \begin{pmatrix} a_2 - \frac{b_2}{a_1} c_1 & c_2 & & \\ b_3 & \ddots & & \\ & & a_{n-1} & c_{n-1} \\ & & b_n & a_n \end{pmatrix}$$

s.t trivial to see above conditions satisfy, so it is non-zero as det $\neq 0$

$\Rightarrow$ our det is of same form as
$$a_j = 2(h_j + h_{j+1})$$
$$b_j = h_{j+1}$$
$$c_j = h_j$$
and so non-singular det

25th Aug:

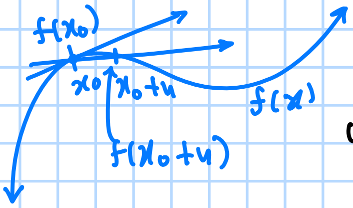
Numerical differentiation and Integration:

$f \in C'[a,b]$ $\quad f'(x_0) = \lim_{h \to 0} \dfrac{f(x_0+h) - f(x_0)}{h}$

$f(x_0+h) = f(x_0) + f'(x_0)h + \dfrac{h^2}{2} f^2(\xi(x))$ (normal taylor series expansion)

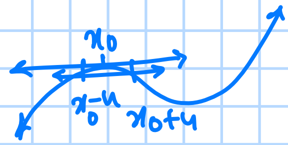
$f'(x_0) = \dfrac{f(x_0+h) - f(x_0)}{h} - \dfrac{h}{2} f^2(\xi(x))$

$f'(x_0) \sim \dfrac{f(x_0+h) - f(x_0)}{h}$ as h becomes small



as $h \to 0$ we will get slop $\to f'(x_0)$

we can also do $f'(x_0) \sim \dfrac{f(x_0+h) - f(x_0-h)}{2}$ for $h \to 0$



$f(x_0+h) = f(x_0) + hf'(x_0) + \dfrac{h^2}{2} f''(x_0) + \cdots \dfrac{f^4}{4!} f^4(\xi)$

$f(x_0-h) = f(x_0) - hf'(x_0) + \dfrac{h^2}{2!} f''(x_0) - \cdots + \dfrac{h^4}{4!} f^4(\xi)$

$f(x_0+h) - f(x_0-h) = 2hf'(x_0) + \dfrac{2h^3}{6} f^3(x_0) + O(h^4)$

$f'(x_0) \sim \dfrac{f(x_0+h) - f(x_0-h)}{2h}$

$f(x_0+h) + f(x_0-h) = 2f(x_0) + h^2 f''(x_0) + O(h^4)$

$\Rightarrow f''(x_0) \sim \dfrac{f(x_0-h) + f(x_0+h) - 2f(x_0)}{h^2}$

using lagrange polynomial:

$f \in C[a,b]$, then given $\{x_0, x_1 \ldots x_n\}$ and value of f at $\{x_0, \ldots x_n\}$

$P_n(x) \sim f(x)$

s.t

$$f(x) = \sum_{k=0}^{n} f(x_k) L_{n,k}(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

$$\Rightarrow f'(x) = \sum f(x_k) L'_{n,k}(x) + D\left(\prod_{k=0}^{n} \frac{(x-x_k)}{(n+1)!} \right) f^{(n+1)}(\xi(x))$$

$$+ \prod_{k=0}^{n} \frac{(x-x_k)}{(n+1)!} D\left(f^{n+1}(\xi(x))\right)$$

substituting x_j above:

$$f'(x_j) = \sum_{k=0}^{n} f(x_k) L'_{n,k}(x_j) + \prod_{\substack{k=0 \\ k \neq j}}^{n} \frac{(x_j - x_k)}{(n+1)!} f^{(n+1)}(\xi(x))$$ $\to$ one way to approximating $f'(x_j)$

for $0 \le j \le n$

integration:

$\int_a^b f(x)dx \sim \sum_{i=0}^{n} a_i f(x_i)$ $\leftarrow$ Quadrature formulas

1. Trapezoidal
2. Simpson
3. Newton Cotes

now $f(x) = \underbrace{P_n(x)}_{\text{lagrange polynomial}} + R(x)$

$$P_n(x) = \sum_{k=0}^{n} \mathcal{L}_{n,k}(x)\, f(x_k)$$

$$\int_a^b f(x)dx = \sum_{k=0}^{n} f(x_k) \int_a^b \underbrace{\mathcal{L}_{n,k}(x)dx}_{a_k} + \underbrace{\int_a^b R(x)dx}_{F(f)}$$

$$= \sum_{k=0}^{n} a_k f(x_k) + F(f)$$ (this means that we can have $\sum a_j f(x_j)$)

$$\Rightarrow \int_a^b f(x)dx \sim \sum_{k=0}^{n} a_k f(x_k)$$

Trapezoidal rule: (degree 1 polynomial)

$f \in C^2[a,b]$, $x_0 = a$, $b = x_1$

$f \sim P_1(x)$

$h = b - a$

$$P_1(x) = \frac{(x-x_0)}{(x_0-x_1)} f(x_0) + \frac{(x-x_0)}{(x_1-x_0)} f(x_1)$$

$$f(x) = P_1(x) + R(x)$$

$$R(x) = \frac{1}{2}(x-x_0)(x-x_1) f^2(\xi)$$

now $$\int_a^b P_1(x)\,dx = \frac{f(x_0)}{(x_0-x_1)} \left.\frac{(x-x_1)^2}{2}\right|_a^b + \frac{f(x_1)}{(x_1-x_0)} \left.\frac{(x-x_0)^2}{2}\right|_a^b$$

$$= \frac{f(a)(-1)(a-b)^{\cancel{2}}}{\cancel{(a-b)}\;2} + \frac{f(b)}{\cancel{(b-a)}} \times \frac{(b-a)^{\cancel{2}}}{2}$$

$$= \frac{(b-a)}{2}\left[f(a) + f(b)\right]$$

$$= \frac{h}{2}(f(a) + f(b))$$

Note: $\int_a^b f(x)dx = f(c)(b-a)$ $\exists\, c \in [a,b]$

$\int_a^b f \cdot g \,dx = g(c) \int_a^b f(x)dx$ if f has a constant sign i.e ≥ 0 or ≤ 0 } weighted MVT for integration

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

m ↑ minima of f $\quad$ M ↑ maxima of f

now, $$\int_a^b R(x)\,dx = \int_a^b \frac{(x-x_0)(x-x_1)}{2} f^2(\xi(x))dx$$

$(x-x_0)(x-x_1)$ has negative sign over $[x_0, x_1]$

$$= f^{(2)}(\eta) \int_a^b \frac{(x-a)(x-b)}{2}dx$$

$$= -\frac{h^3}{12} f^{(2)}(\eta)$$

total: $$\int_a^b f(x)dx = \frac{h}{2}(f(x_0) + f(x_1)) - \frac{h^3}{12} f^{(2)}(\eta)$$

$$\Rightarrow \int_a^b f(x)dx \sim \frac{h}{2}[f(x_0) + f(x_1)]$$

Eg: $f(x) = x^2$; $[0,2]$ then $\int_0^2 x^2 dx = \frac{8}{3} \approx 2.66$

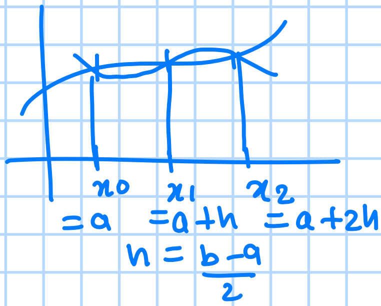
$$\frac{h}{2}[f(x_0)+f(x_1)] = \frac{2}{2} \times [0+4] = 4$$

Simpson: (degree 2 polynomial)

$f(x) = P_2(x) + R(x)$

$P_2(x) = \sum_{k=0}^{2} f(x_k) \ell_{2,k}(x)$

$$\ell_{2,0}(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$



$$\int_a^b f(x)dx \sim \int_a^b P_2(x)dx$$

$x_0 = a$, $x_1 = \frac{a+b}{2}$, $x_2 = b$

or $x_0 = a$, $x_1 = a+h$, $x_2 = a+2h$

somehow $\int_{x_0}^{x_2} P_2(x)dx$ matches taylor series integration of $f(x)$

$$f(x) = f(x_1) + (x-x_1)f'(x_1) + \frac{(x-x_1)^2}{2} f''(x_1) + \frac{(x-x_1)^3}{3!} f^{(3)}(x_1) + \frac{(x-x_1)^4}{4!} f^4(\xi(x))$$

$$\int_{x_0}^{x_2} f(x)dx = \int_{x_0} \left(f(x_1) + (x-x_1)f'(x_1) + \frac{(x-x_1)^2}{2} f''(x_1) + \frac{(x-x_1)^3}{3!} f^{(3)}(x_1) + \frac{(x-x_1)^4}{4!} f^4(\xi(x)) \right) dx$$

$$x_2 - x_1 = x_1 - x_0$$

$$= f(x_1)(x_2-x_0) + 0 + \frac{f^{(2)}(x_1)}{2!} \frac{(x-x_1)^3}{3} \Big|_{x_0}^{x_1} + \int_{x_0}^{x_2} \frac{(x-x_1)^4}{4!} f^4(\gamma(x))dx$$

$$= 2h f(x_1) + \frac{f''(x_1)}{3} h^3 + f^4(\eta) \frac{(x-x_1)^5}{5!} \Big|_{x_0}^{x_2}$$

now $f''(x_1) = \frac{f(x_1+h) + f(x_1-h) - 2f(x_1)}{h^2}$

$$\int_a^b f(x)dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] + O(h^5)$$

somehow this is same as $\int_{x_0}^{x_2} P_2(x)dx$ but there is $O(h^4)$

Eg: $\int_0^2 x^2 dx \sim \frac{h}{3}(f(0) + 4f(1) + f(2))$

$$= \frac{1}{3}(0+4+4) = \frac{8}{3}$$

this is same as $\int_0^2 x^2 dx = \frac{8}{3}$

Note: $\int_a^b f(x)dx = \frac{b-a}{2}(f(a) + f(b))$ trapezoidal

$\int_a^b f(x)dx = \frac{(b-a)}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$ simpsons rule

Newtons:

This is the general case when $f \in C^{n+2}[a,b]$, values of f at $(n+1)$ equispaced points $x_i = x_0 + ih$ $0 \le i \le n$

$x_n = x_0 + nh = b$

then $\exists \gamma \in (a,b)$ s.t

$$\int_a^b f(x)dx = \sum_{i=0}^{n} a_i f(x_i) + \frac{h^{n+3}}{(n+2)!} f^{(n+2)}(\xi) \int_0^n t^2(t-1)\cdots(t-n)\,dt \quad n \text{ is even}$$

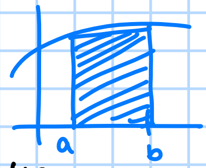
$$= \sum_{i=0}^{n} a_i f(x_i) + \underbrace{\frac{h^{(n+2)}}{(n+1)!} f^{(n+1)}(\xi) \int_0^n t(t-1)\cdots(t-n)\,dt}_{\text{this terms come from}} \quad n \text{ is even}$$

$$R(x) = \prod_{k=0}^{n} \frac{(x-x_k)}{(n+1)!} f^{(n+1)}(\xi(x))$$

28th Aug:

Numerical integration:

1) Trapezoidal rule: $\int_a^b f(x)\,dx = \frac{(b-a)}{2}(f(a)+f(b)) - \frac{(b-a)^3}{12} f''(\xi)$



2) Simpson rule: $\int_a^b f(x)dx = \frac{(b-a)}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] - \frac{(b-a)^5}{2^5(90)} f^4(\xi)$

3) Newton coten

Degree accuracy:

Defn: The degree of accuracy/precision of a quadrature formula is the largest positive integer n s.t the formula gives exact values for

$x^k \quad k=0,1,\dots,n$, P of degree $\leq n$

$P(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$

degree of precision of trapezoidal is 1, for simpsons rule is 3 as error term has 4^{th} order derivative so $(x^3)^{4} = 0$

Note: Degree of precision of newton coten is n

gaussian quadratic formula:

$\int_a^b f\,dx \sim \sum c_i f(x_i)$ ← we want to choose x_i and c_i s.t we get best possible result

$\int_a^b f(x)dx \sim \sum_{i=1}^{n} c_i f(x_i)$ n unknown $\{c_1 \dots c_n\}$ & $\{x_1 \dots x_n\}$

$n=2$, $\int_a^b f(x)dx \sim \sum_{i=1}^{2} c_i f(x_i)$ when f is a polynomial of degree 3 as $\{1, x, x^2, x^3\}$ or 4 degrees of freedom & 4 unknowns (c_1, c_2, x_1, x_2)

n=2:

lets say $[-1,1]$ $\int_{-1}^{1} f(u)dx \sim c_1f(x_1) + c_2f(x_2)$ ($2n-1$ degree polynomial) s.t the values should match for

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} \{1, x, x^2, x^3\}\,dx$$

$$c_1 + c_2 = \int_{-1}^{1} 1\,dx = 2 \quad \text{— ①}$$

$$c_1x_1 + c_2x_2 = \int_{-1}^{1} x\,dx = 0 \quad \text{— ②}$$

$$c_1x_1^2 + c_2x_2^2 = \int_{-1}^{1} x^2dx = 2/3 \quad \text{— ③}$$

$$c_1x_1^3 + c_2x_2^3 = \int_{-1}^{1} x^3dx = 0 \quad \text{— ④}$$

we get $c_1 = 1, c_2 = 1, x_1 = -x_2 = \sqrt{\frac{1}{3}}$ and formula becomes

$$\text{so } \int_{-1}^{1} f(x)dx \sim f\left(\frac{1}{\sqrt{3}}\right) + f\left(\frac{-1}{\sqrt{3}}\right)$$

this has degree of accuracy 3 from construction using $\{1, x, x^2, x^3\}$

← general case

$[a,b] \longrightarrow [-1,1]$ $\quad \varphi(t) = \alpha t + \beta$

$\varphi(a) = -1 \quad \alpha \neq 0$

$\varphi(b) = 1$

$$\int_a^b f(x)dx = \int_{-1}^1 f(\varphi(t))\varphi'(t)\,dt$$

lagendra polynomial:

A collection of polynomial $\{P_0, P_1 \dots P_n\}$ is called lagendre polynomial if they satisfies the following:

(a) each P_n is monic polynomial

(b) $\int_{-1}^1 P(x)P_n(x)\,dx = 0$ for any polynomial P of degree $(n-1)$

$P_0 = 1$, $P_1 = x$, $P_2 = x^2 - \frac{1}{3}$ as $\int_{-1}^1 P_1 P_2\,dx = \left| \frac{x^4}{4} - \frac{1}{3}\frac{x^2}{2} \right|_{-1}^1 = 0$

$$\int_{-1}^1 P_0 P_2\,dx = \left| \frac{x^3}{3} - \frac{x}{3} \right|_{-1}^1 = 0$$

$P_3(x) = x^3 - \frac{3}{5}x$ and so on, essentially we have a basis and then we want to orthogonalise it using gram smidt

The roots of P_n's are distinct, symmetric w.r.t '0' and lie on $[-1,1]$

Theorem: suppose that $\{x_1, \dots x_n\}$ are roots of n^{th} lagendre polynomial P_n then for each $i = 1, 2 \dots n$ we define

$$c_i = \int_{-1}^1 \prod_{\substack{j=1 \\ j \neq i}}^n \frac{(x - x_j)}{(x_i - x_j)}\,dx$$

when if P is any polynomial of degree $\leq 2n-1$, then

$$\int_{-1}^1 P(x)\,dx = \sum_{i=1}^n c_i P(x_i)$$

proof:

case I) P is a polynomial of degree $\leq (n-1)$

$$P(x) = \sum P(x_i) l_i(x) \quad \text{where } l_i(x) = \prod_{\substack{j=1 \\ i \neq j}}^n \frac{(x - x_j)}{(x_i - x_j)}$$

n^{th} derivate is 0, so in case of lagrange polynomial there will be no error term

$$\Rightarrow \int_{-1}^1 P(x)\,dx = \sum_{i=1}^n P(x_i) \int_{-1}^1 (l_i(x))\,dx$$

$$= \sum_{i=1}^n c_i P(x_i) \quad \text{where } c_i = \int_{-1}^1 \prod_{\substack{i=1 \\ j \neq 1}}^n \left(\frac{x - x_j}{x_i - x_j} \right) dx$$

case II) $n \leq$ degree $P \leq 2n-1$, P_n

$P(x) = Q(x)P_n(x) + R(x)$ degree $R \leq (n-1)$, $Q \leq (n-1)$

$$\Rightarrow \int_{-1}^1 P(x)\,dx = \underbrace{\int_{-1}^1 Q(x)P_n(x)\,dx}_{0 \text{ from defn-1}} + \int_{-1}^1 R(x)\,dx$$

$$= \int_{-1}^1 R(x)\,dx$$

as $R(x)$ is a polynomial of degree $\leq n-1$

$$\Rightarrow \int_{-1}^1 R(x) = \sum_{i=1}^n c_i R(x_i)$$

now as $\int_{-1}^{1} P(x)\,dx = \int_{-1}^{1} R(x)\,dx = \sum_{i=1}^{n} c_i^{\circ} R(x_i)$

for any x_i : $P(x_i) = Q(x_i)P_n(x_i) + R(x_i)$
$= R(x_i)$ ($\because x_i$ is a root of P_n)

$\Rightarrow \sum_{i=1}^{n} c_i^{\circ} R(x_i) = \sum_{i=1}^{n} c_i^{\circ} P(x_i^{\circ})$

$\Rightarrow \int_{-1}^{1} P(x)\,dx = \sum_{i=1}^{n} c_i^{\circ} P(x_i^{\circ})$

general domain:

for $[a,b]$ insted of $[-1,1]$ we can just convert $[a,b]$ to $[-1,1]$ by

$$\varphi(t) = \frac{1}{2}\left[(b-a)t + (a+b)\right]$$

$\varphi : [-1,1] \longrightarrow [a,b]$

$\varphi(1) = b$

$\varphi(-1) = a$

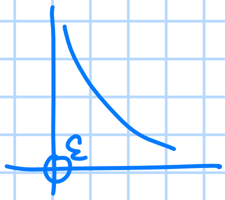
$$t = \frac{2x-a-b}{(b-a)}$$

$$\int_a^b f(x)\,dx = \int_{-1}^{1} f(\varphi(t))\varphi'(t)\,dt = \int_{-1}^{1} f\left(\frac{(b-a)t+a+b}{2}\right)\left(\frac{b-a}{2}\right)dt$$

improper Integral:

$f(x) = \frac{1}{x}$ or $\frac{1}{x^p}$ for $0<p<1$ for $[0,1)$

then $\int_0^1 \frac{1}{x}\,dx$



$= \lim_{\varepsilon \to 0} \int_0^1 \frac{1}{x}\,dx$

$= \lim_{\varepsilon \to 0} (\log(1) - \log(\varepsilon)) =$ not finite

but $\lim_{\varepsilon \to 0} \int_{\varepsilon}^{1} \frac{1}{\sqrt{x}}\,dx =$ finite

similarly $\int_{-1}^{1} \frac{1}{1-x^2}\,dx = \int_{-1}^{0} \frac{1}{1-x^2}\,dx + \int_0^1 \frac{1}{1-x^2}\,dx$

$= \lim_{\varepsilon \to 0} \left(\int_{-1+\varepsilon}^{0} \frac{1}{1-x^2}\,dx + \int_0^{1-\varepsilon} \frac{1}{1-x^2}\,dx \right)$

we should normally take ε, δ both tending to 0

suppose $f(x) = \frac{g(x)}{(x-a)^p}$ $\quad g \in C^5[a,b]$, $0<p<1$

$$F(x) = \begin{cases} \frac{g(x) - P_4(x)}{(x-a)^p} ; & a<x<b \\ 0 ; & x=a \end{cases}$$

where $P_4(x) = g(a) + (x-a)g'(a) + \cdots + \frac{(x-a)^4}{4!} g^4(a)$

$g(x) - P_4(x) = \frac{(x-a)^5}{5!} g^5(\xi)$

as $0<p<1 \Rightarrow F \in C^4[a,b]$

80, $\int_a^b f(x)dx \sim \int_a^b F(x) + \frac{P_4(x)}{(x-a)^P} dx$

1^{st} Sept:

IVP Initial value problem for ODE:

$$\left.\begin{array}{l} y'(t)=f(t,y(t)),\ t\in I=[a,b] \\ y(a)=\alpha \end{array}\right\} \text{IVP}$$

Defn: (Lipschitz) A function $f(t,y)$ is said to be Lipschitz in 'y' variable if $\exists$ constant $L>0$ s.t

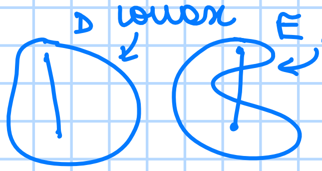
$$|f(t,y_1)-f(t,y_2)|\leq L|y_1-y_2|\ \forall\ y_1,y_2\in D \leftarrow \text{vector space}$$

Eg: $f(x)=|x|$, $x\in[-1,1]$

Note: (a) f is Lipschitz $\Rightarrow$ f is continuous
(b) f is Lipschitz $\Rightarrow$ f is integrable

Defn: (convex set) A set $D\subseteq\mathbb{R}^2$ is said to be convex if $z_1,z_2\in D \Rightarrow \lambda z_1+(1-\lambda)z_2\in D$ $\forall\ \lambda\in[0,1]$

Eg:



Lemma: Suppose $f(t,y)$ is defined on a convex set D. If $\exists$ a constant $L>0$ s.t

$$\left|\frac{\partial f}{\partial y}(t,y)\right|\leq L \quad \forall (t,y)\in D$$

then f is Lipschitz wrt y

proof: $|f(t,y_1)-f(t,y_2)|=|y_1-y_2|\,\underbrace{\frac{\partial f}{\partial y}(t,\tau)}_{\text{exist as } D \text{ is convex}}|$ ($\because$ taylor series)

$\leq L|y_1-y_2|$

Theorem: Suppose that $D=\{(t,y)\mid a\leq t\leq b,\ y\in\mathbb{R}\}$ and $f(t,y)$ is continuous on D. If f is Lipschitz on D in the y variable, then IVP

$$y'(t)=f(t,y(t)) \quad t\in[a,b]$$
$$y(a)=\alpha$$

then it has a unique solution for $a\leq x\leq b$

Note: If f is Lipschitz then global soln exist & is unique
If f is continuous (Peano's) then local solution exist $\leftarrow$ Not nec unique

Eg: $y'(t)=1+t\sin(ty)=f(t,y)$ $\quad 0\leq t\leq 2$

$\frac{df}{dy}=0+t\cos(ty)\times t=t^2\cos ty\leq 2\times 2$

$\Rightarrow \left|\frac{\partial f}{\partial y}(t,y)\right|\leq 4 \quad \forall (t,y)\in D$

$\Rightarrow$ f is Lipschitz and so $\exists$ a unique soln $y(t)$ for $y(a)=\alpha$

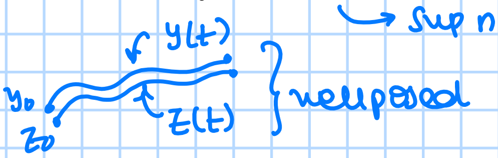
$y'(t)=-y^2 \Rightarrow y(t)=\frac{1}{x+c} \quad x\neq -c$

Gronwall's inequality: If u is differentiable on I & u satisfies

$$u'(t)\leq \beta(t)u(t)$$
$$\text{then } u(t)\leq u(a)\,e^{\int_a^t \beta(s)ds}$$

Defn: (Well posed problem) An IVP is called wellposed when
(i) IVP has a unique solution
(ii) unique soln depends cont. on the data (initial condition)

Note: If $z(t)$ satifies the IVP then
$|y(t)-z(t)|<\delta_0$ for $|y_0-z_0|<\tilde{\delta_0}$ → sup norm



Euler's method:

$y'(t)=f(t,y(t))$ $t\in[a,b]$, idea:
we divide $[a,b]$ into smaller intervals
$t_0=a<t_1<t_2\ldots<t_{n-1}<b=t_n$
where $t_i-t_{i-1}=h=\frac{(b-a)}{n}$

then $y(t_{i+1})=y(t_i)+(t_{i+1}-t_i)y'(t_i)+\frac{(t_{i+1}-t_i)}{2}y''(\xi_i)$ $\xi_i\in(t_i,t_{i+1})$
assuming $y\in C^3$
$\Rightarrow y(t_{i+1})=y(t_i)+hf(t_i,y(t_i))+\frac{h^2}{2}y''(\xi_i)$

let $w_i\sim y(t_i)$
$\Rightarrow w_{i+1}=w_i+hf(t_i,w_i)$ $i=0,\ldots n-1$
$w_0=\alpha$

Error bound:
we want to compute $y(t_i)-w_i$

Lemma: $\forall x\geq -1$ $(1+x)^m\leq e^{mx}$, $m>0$
proof:
$e^x=1+x+\frac{x^2}{2}e^{\sigma}$
$\underbrace{}_{\geq 0 \text{ as } x\geq -1}$
$\geq 1+x\geq 0$
$\Rightarrow e^{mx}\geq(1+x)^m$

Lemma: (Discrete gronwall's inequality) let s & t are two positive numbers. Let $\{a_i\}_{i=0}^{k}$ be a sequence satisfying the following

(i) $a_0\geq -\frac{t}{s}$
(ii) $a_{i+1}\leq(1+s)a_i+t$, $0\leq i\leq k-1$

then
$a_{i+1}\leq e^{(i+1)s}(a_0+\frac{t}{s})-\frac{t}{s}$

proof: fix an integer 'i'
$a_{i+1}\leq(1+s)a_i+t$
$\Rightarrow a_{i+1}\leq(1+s)^2a_{i-1}+t[1+(s+1)]$
$\vdots$ i times
$a_{i+1}\leq(1+s)^{i+1}a_0+t[1+(1+s)+(1+s)^2+\cdots+(1+s)^i]$
$\Rightarrow a_{i+1}\leq(1+s)^{i+1}a_0+t\left[\frac{1-(1+s)^{i+1}}{1-(1+s)}\right]$
$\Rightarrow a_{i+1}\leq(1+s)^{i+1}a_0+\frac{t}{s}[(1+s)^{i+1}-1]$
now $(1+x)^m\leq e^{mx}$ $x\geq -1$ so
$x=s$ $m=i+1$ to get
$\Rightarrow a_{i+1}\leq(e)^{s(i+1)}a_0+\left(\frac{t}{s}\right)e^{s(i+1)}-\frac{t}{s}$
$\Rightarrow a_{i+1}\leq e^{(i+1)s}(a_0+t/s)-t/s$

Theorm: (error estimate) let $D = \{(t,y) \mid a \le t \le b, y \in \mathbb{R}\}$ suppose $y \in C^2[a,b]$ and y is a unique solution of the IVP, then $y'(t) = f(t, y(t))$ $y(a) = \alpha$, let $w_0, w_1, \ldots, w_N$ be the approximating seq generated by euler's method, then we have and f is lipstitz for y:

L = liptitz const

$$|y(t_i) - w_i| \le \frac{hM}{2L}\left[e^{L(t_i - a)} - 1\right] \quad i = 0, \ldots, N$$

proof: for $i = 0$ $y(t_0) = y(a) = \alpha = w_0$

$$|y(t_{i+1}) - w_{i+1}| = |y(t_i) + h f(t_i, y(t_i)) + \frac{h^2}{2} y''(\xi_i) - (w_i + h f(t_i, w_i))|$$

$$= |(y(t_i) - w_i) + h\left(f(t_i, y(t_i)) - f(t_i, w_i)\right) + \frac{h^2}{2} y''(\xi_i)|$$

$$\le |y(t_i) - w_i| + h\,|f(t_i, y(t_i)) - f(t_i, w_i)| + \frac{h^2}{2} M$$

$M = \sup_I |y''|$

L = liptisz const

$$\Rightarrow \underbrace{|y(t_{i+1}) - w_{i+1}|}_{a_{i+1}} \le \underbrace{|y(t_i) - w_i|}_{a_i}(1 + hL) + \frac{h^2 M}{2}$$

let $s = hL$

$t = \frac{h^2 M}{2}$ then $\frac{t}{s} = \frac{hM}{2L}$

$$\Rightarrow a_{i+1} \le e^{(1+i)s}\left(a_0 + \frac{t}{s}\right) - \frac{t}{s} \quad (\because \text{ from prev theorm})$$

$$= \left(e^{(i+1)hL} - 1\right)\frac{Mh}{2L}$$

$$= \left(e^{(t_{i+1} - a)L} - 1\right)\frac{Mh}{2L}$$

so,

$$|y(t_{i+1}) - w_{i+1}| \le \frac{Mh}{2L}\left(e^{(t_{i+1} - a)L} - 1\right) \quad 0 \le i \le N-1$$

Note: Euler approximation $w_{i+1} = w_i + h f(t_i, w_i)$

$w_0 = \alpha$

$u_0 = \alpha + \delta_0$

$u_{i+1} = u_i + h(t_i, u_i) + \delta_{i+1} \quad 0 \le i \le N-1$

$$|y(t_i) - u_i| \le \frac{1}{L}\left(\frac{Mh}{2} + \frac{\delta}{h}\right)\left[e^{L(t_i - a)} - 1\right] + \delta_0 e^{L(t_i - a)}$$

$\delta = \min\{\delta_i, \delta_0\}$

Not necessary wellposed

as $h \to 0$ $\lim_{h \to 0}\left(\frac{Mh}{2} + \frac{\delta}{h}\right) \to \infty$

if we put $h = \sqrt{\frac{2\delta}{M}}$ then $\left(\frac{Mh}{2} + \frac{\delta}{h}\right)$ attains minimum

4^{th} Sept:

$$\begin{cases} y'(t) = f(t, y(t)) & t \in [a, b] \\ y(a) = \alpha \end{cases} \quad \text{IVP}$$

when f is Lipschitz then the above is well posed
$\hookrightarrow |f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$

then we saw Euler's approximation

$x_i = x_0 + ih \qquad x_0 = a \qquad x_N = b$

$h = \dfrac{b-a}{N}$

a b

$\{w_i\}_{i=0}^{N} \quad w_0 = \alpha \qquad w_i \sim y(t_i)$

$$w_{i+1} = w_i + hf(t_i, w_i) \quad 0 \leq i \leq N-1$$

Defn: (local truncation error) The difference method

$w_0 = \alpha$

$w_{i+1} = w_i + h\varphi(t_i, w_i) \quad 0 \leq i \leq N-1$

has local truncation error

$$\tau_{i+1}(h) = \frac{y_{i+1} - (y_i + h\varphi(t_i, y_i))}{h}$$

$$= \frac{y_{i+1} - y_i}{h} - \varphi(t_i, y_i) \qquad 0 \leq i \leq N-1$$

assuming $y_i = w_i$ as local truncation error

LTC for euler approximation:

error in taylor series

$$\frac{y_{i+1} - y_i}{h} - f(t_i, y_i) = \frac{y_{i+1} - w_i}{h} = \frac{1}{h}\frac{h^2}{2!}y''(\xi_i) \quad \xi_i \in (t_i, t_{i+1})$$

$$= O(h)$$

let $y(t_{i+1}) = y(t_i) + hy'(t_i) + \dfrac{h^2}{2!}y''(t_i) + \dots + \dfrac{h^n}{n!}y^n(t_i) + \dfrac{h^{n+1}}{(n+1)!}y^{n+1}(\xi_i)$ $\quad \xi_i \in (t_i, t_{i+1})$

$$= y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2}f'(t_i, y(t_i)) + \dots$$

where $f^{(n)}(t_i, y(t_i)) = \dfrac{d^n}{dt^n} f(t, y(t))\Big|_{t=t_i} \neq \dfrac{\partial}{\partial t} f(t, y(t))$

now let's $w_0 = \alpha$

$w_{i+1} = w_i + hT^n(t_i, w_i)\ ; \ 0 \leq i \leq N-1$

$$T^n(t_i, w_i) = f(t_i, w_i) + \frac{h}{2}f'(t_i, w_i) + \dots + \frac{h^{n-1}}{n!}f^{n-1}(t_i, w_i)$$

if we have $n = 1$, we will get back euler

Theorem: consider the IVP $y'(t) = f(t, y(t))$, $y(a) = \alpha$, $t \in I = [a, b]$
if $\{w_i\}_{i=0}^{N}$ be the taylor approximation of the IVP with step size h and $y \in C^{n+1}(I)$ then the Taylor approximation has LTE of $O(h^n)$

proof: as $y_{i+1} = y_i + hy'_i + \dfrac{h^2}{2!}y''_i + \dots + \dfrac{h^n}{(n)!}y^n_i + \dfrac{h^{n+1}}{(n+1)!}y^{n+1}(\xi)$ (i.e $\leq Ch^n$)

reminder

$$\Rightarrow y_{i+1} = y_i + hf(t_i, y_i) + \frac{h^2}{2!}f'(t_i, y_i) + \dots + \frac{h^n}{(n)!}f^{n-1}(t_i, y_i) + \frac{h^{n+1}}{(n+1)!}f^n(\xi)$$

$$\Rightarrow \frac{y_{i+1} - y_i}{h} - T^n = \frac{h^n}{(n+1)!}f^n(\xi) \leq Ch^n$$

as $f^n(\xi)$ is bounded

$$\Rightarrow \frac{y_{i+1} - y_i}{h} - T^n(t_i, y_i) = O(h^n)$$

Runze-kutta method:

Defn: Suppose $f \in C^{n+1}(D)$. $D = \{(t,y) \mid a \le t \le b,\ c \le y \le d\}$, then for any $(t_0, y_0) \in D$ we can write
$$f(t,y) = P_n(t,y) + R_n(t,y)$$
where
$$P_n(t,y) = f(t_0,y_0) + (t-t_0)\frac{\partial f}{\partial t}(t_0,y_0) + (y-y_0)\frac{\partial f}{\partial y}(t_0,y_0)$$
$$+ \left[\frac{(t-t_0)^2}{2}\frac{\partial^2 f}{\partial t^2}(t_0,y_0) + (t-t_0)(y-y_0)\frac{\partial^2 f}{\partial t \partial y}(t_0,y_0) + \frac{(y-y_0)^2}{2}\frac{\partial^2 f}{\partial y^2}\right]$$
$$+ \cdots + \left[\frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}(t-t_0)^{n-j}(y-y_0)^j \frac{\partial^n f}{\partial t^{n-j}\partial y^j}(t_0,y_0)\right]$$
$$R_n(t,y) = \frac{1}{(n+1)!}\sum_{j=0}^{n+1}\binom{n+1}{j}(t-t_0)^{n+1-j}(y-y_0)^j \frac{\partial^{n+1} f}{\partial t^{n+1-j}\partial y^j}(\xi,\eta) \quad \xi \in (t_0,t),\ \eta \in (y_0,y)$$

now, $w_{i+1} = w_i + hT^n$ in our taylor series approximation
$$T^2 = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) \quad \text{—— (1)}$$
$$a_1 f(t+\delta, y+\beta) = a_1\left[f(t,y) + \delta\frac{\partial f}{\partial t}(t,y) + \beta\frac{\partial f}{\partial y}(t,y)\right] + a_1 R$$
$$\Rightarrow a_1 f(t+\delta, y+\beta) - a_1 R = a_1\left[f(t,y) + \delta\frac{\partial f}{\partial t}(t,y) + \beta\frac{\partial f}{\partial y}(t,y)\right] \quad \text{—— (2)}$$
now, in (1): $f'(t,y) = \frac{d}{dt}(f(t,y(t)))$
$$= \frac{\partial f}{\partial t}(t,y(t)) + \frac{\partial f}{\partial y} y'(t)$$
$$= \frac{\partial f}{\partial t}(t,y(t)) + f(t,y(t))\frac{\partial f}{\partial y}$$
so (1) becomes: $T^2 = f(t,y(t)) + \frac{h}{2}\frac{\partial f}{\partial t}(t,y(t)) + \frac{h}{2} f \frac{\partial f}{\partial y}$ —— (3)

if we force (3) = (2) then we get $a_1 = 1$, $\delta = \frac{h}{2}$, $\beta = \frac{hf(t,y(t))}{2}$

so, $T^2 = f\left(t + \frac{h}{2}, y + \frac{h}{2}f(t,y)\right) - R$

now, $w_0 = \alpha$

$w_{i+1} = w_i + hT^2$

replace T^2 by $f\left(t_i + \frac{h}{2}, w_i + \frac{h}{2}f(t_i,w_i)\right)$

$\Rightarrow w_{i+1} = w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2}f(t_i, w_i)\right) \quad 0 \le i \le N-1$

RK method

Note: RK method is also known as mid point method

Note: LTE of above method is $O(h^2)$

we can do it for T^3/T^4, i.e do a taylor series expansion inside

i.e $a_1 f(t+\delta_1, y + \beta_1 f(t+\delta_2, y+\beta_2))$

where

$w_0 = \alpha$

$w_{i+1} = w_i + \frac{h}{4}\left(f(t_i,w_i) + 3f\left(t_i + \frac{2h}{3}, w_i + \frac{2h}{3}f\left(t_i + \frac{h}{3}, w_i + \frac{h}{3}f(t_i,w_i)\right)\right)\right)$

8^{th} sept:

we know that if $A_{n\times n} x = b$, $x, b \in \mathbb{R}^n$ A^{-1} exist (i.e $\det A \neq 0$) then

$x = A^{-1} b$

we want to construct $\{x_n\}_{n=1}^{\infty}$ s.t $x_n \to x$

Defn: (Lin ind & lin dep) let V be a vector space ($\mathbb{R}/\mathbb{C}$), then

(i) $\{v_1, v_2 \dots v_m\}$ is lin depent if $\exists\, \alpha_1 \dots \alpha_m$ s.t atleast one non-zero

$$\sum_{i=1}^{m} \alpha_i v_i = 0$$

(ii) $\{v_1 \dots v_m\}$ are lin ind if:

$$\sum_{i=1}^{m} \alpha_i v_i = 0 \Rightarrow \alpha_i = 0 \ \forall\, i \in \{1 \dots m\}$$

Defn: (Basis) let V be a vector space $\{v_1 \dots v_m\}$ is called basis for V if for every $v \in V$, $\exists\, \alpha_i$ s.t

$$v = \sum_{i=1}^{n} \alpha_i v_i$$

$\{v_1 \dots v_n\}$ is lin independent

Note: Any subset of linearly independent set is linearly independent and Any superset of linearly dependent set is also linearly dependent

Eg: $P_n(\mathbb{R}) = \{$ polynomials of degree $\leq n\}$

vectors of degree $n+1$

$\{1, x, \dots x^n\}$ is basis as lin independent and spans $P_n(\mathbb{R})$

$c_1 + c_2 x + \dots + c_n x^n = 0$

set $x = 0 \Rightarrow c_1 = 0$

then diff we get: $c_2 = 0$ (diff every time)

$\vdots$

$(n-1)!\, c_n = 0 \Rightarrow c_i = 0 \ \forall\, i \in \{0, \dots, n\}$

Note: To show $\{x^n\}_{n=0}^{\infty}$ is linearly independent, as if

$A = \{x^i\}_{i=0}^{\infty}$

A is lin ind if any finite subset of A is lin ind

what we have to show to show A is lin independent

sort of def for infinite sets

all polynomials, $C[a,b]$ are some examples of infinite dimension vector spaces

Defn: let $A_{n\times n}$ be a matrix then:

(1) A is called Symmetric if $A^t = A$

(2) A is called skew symmetric if $A^t = -A$

(3) A is called unitary if $A^* A = A A^* = I$ ($A^* = \bar{A}^t$)

(4) For a real matrix A, $A^t A = A A^t = I$ then A is called orthogonal

Theorm: let A be a $n\times n$ matrix, let $V=\mathbb{R}^n/\mathbb{C}^n$, then following conditions are equivalent
(1) $Ax=b$ has unique son $x\in V$, for any $b\in V$
(2) $Ax=0 \Rightarrow x=0$
(3) A^{-1} exist
(4) $\det(A)\neq 0$
(5) $\text{rank}(A)=n$

Defn: let $x,y\in\mathbb{R}^n$ $x=(x_1\ldots x_n)$, $y=(y_1\ldots y_n)$ then $\langle x,y\rangle=\sum_{i=1}^{n}x_iy_i$

$$\langle x,x\rangle=\|x\|_2^2=\sum_{i=1}^{n}x_i^2$$

Theorm: (Cauchy Schwartz) for $x,y\in\mathbb{R}^n/\mathbb{C}^n$

$$\langle x,y\rangle^2\leq\langle x,x\rangle\langle y,y\rangle$$

Eigenvalues & eigenvectors:

let A be $n\times n$ matrix $\lambda\in\mathbb{R}/\mathbb{C}$ be eignvalue of A if $\exists$ non-zero x s.t

$$Ax=\lambda x,\ x\in\mathbb{R}^n/\mathbb{C}^n$$

x is calledeigvector

$$[(A-\lambda I)x]=0$$

as $x\neq 0$

$$\Rightarrow \det(A-\lambda I)=0$$

so, $P_\lambda(A)=\underbrace{\det(A-\lambda I)}$
characteristic polynomial
roots are eigenvalues

Defn: (similar) let A,B be two $m\times m$ matrix, then A is 'similar' to B if $\exists$ a non singuar matrix P s.t

$$B=P^{-1}AP$$

Note: if λ is eigenvalue of A then
for some $x\neq 0$

$$Ax=\lambda x$$

$$\begin{aligned}P_\lambda(B)&=\det(B-\lambda I)\\&=\det(P^{-1}AP-\lambda I)\\&=\det P^{-1}(A-\lambda I)P\\&=\det(A-\lambda I)\end{aligned}$$

so λ for $A\Leftrightarrow\lambda$ for B

$$Ax=\lambda x$$

$y=P^{-1}x$ eigenvector for B

$$\begin{aligned}By&=(P^{-1}AP)y\\&=P^{-1}Ax\\&=\lambda P^{-1}x\\By&=\lambda y\end{aligned}$$

so $P^{-1}x$ is eigenvector to B

also $\text{trace}(A)=\sum\lambda_i=\text{trace}(B)$
$\det(A)=\prod\lambda_i=\det(B)$

Note: If $\det(A)=0\Rightarrow 0$ is an eigenvalue of A

Canonical forms :

If $A = A^t$ (symmetric) then $\exists P$ non-singular s.t

$$PAP^{-1} = \Lambda$$

for any A, $\exists Q$ (unitary) s.t $QQ^* = Q^*Q = I$

$$QAQ^* = U$$

upper triangular matrix

Theorem : (Schur cannonical form), let A be a $n \times n$ matrix with complex entities then $\exists$ a unitary matrix U s.t

$$T = U^* A U \text{ is upper triangular}$$

proof : for $n=1$, the result is trivial as

$$A = [a + ib]$$

$$U = U^* = [1] \quad \text{then} \quad A = T$$

lets assume result holds $\forall$ $m \times m$ matrices A when $n \leq k-1$

for $n = k$, let λ_1 be an eigenvalue of A with an eigenvector $u^{(1)}$ s.t $\|u^{(1)}\|_2 = 1$

now, let our basis be $\{u^{(1)}, u^{(2)}, \dots u^{(k)}\}$ S.t it is ortogonal

our $u^{(1)}$

By gram-smith we make them ortogonal

let $P_1 = \begin{bmatrix} u^{(1)} & u^{(2)} & \dots & u^{(k)} \end{bmatrix}$

let $B_1 = P_1^* A P_1$ where $AP_1 = \begin{bmatrix} Au^{(1)} & \dots & Au^{(k)} \end{bmatrix}$

$v^{(j)} = Au^{(j)}$

$$= \begin{bmatrix} \lambda u^{(1)} & v^{(2)} & \dots & v^{(k)} \end{bmatrix}$$

$$P_1^* = \begin{bmatrix} \bar{u}^{(1)} \\ \vdots \\ \bar{u}^{(k)} \end{bmatrix}$$

$$P_1^* A P_1 = \begin{bmatrix} \lambda & & & \\ 0 & w^{(2)} & \dots & w^{(k)} \\ \vdots & & & \\ 0 & & & \end{bmatrix} \qquad w^{(j)} = P_1^* v^{(j)}$$

so, $B_1 = \begin{bmatrix} \lambda_1 & * & * & \dots & * \\ 0 & & & & \\ \vdots & & A_2 & & \\ 0 & & & & \end{bmatrix}$ $\quad A_2$ is a $k-1 \times k-1$ matrix then by induction $\exists$ unitary matrix $\hat{P}_2$ s.t $\hat{P}_2^* A_2 \hat{P}_2 = \hat{T}_2$

$\uparrow$ upper triangular

then let $P_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \hat{P}_2 & \\ 0 & & & \end{bmatrix}$

$\downarrow$ k x k matrix, since $\hat{P}_2$ is unitary, P_2 is also unitary

then $P_2^* B_1 P_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \hat{P}_2^* & \\ 0 & & & \end{bmatrix} \begin{bmatrix} \lambda_1 & * & * & \cdots & * \\ 0 & & & & \\ \vdots & & A_2 & & \\ 0 & & & & \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \hat{P}_2 & \\ 0 & & & \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \hat{P}_2^* & \\ 0 & & & \end{bmatrix} \begin{bmatrix} \lambda_1 & * & \cdots & * \\ 0 & & & \\ \vdots & & A_2 \hat{P}_2 & \\ 0 & & & \end{bmatrix}$

$= \begin{bmatrix} \lambda_1 & * & \cdots & * \\ 0 & & & \\ \vdots & & \hat{P}_2^* A_2 \hat{P}_2 & \\ 0 & & & \end{bmatrix}$

$= \begin{bmatrix} \lambda_1 & * & \cdots & * \\ 0 & & & \\ \vdots & & \hat{T}_2 & \\ 0 & & & \end{bmatrix} = T$ $\leftarrow$ upper triangular

so, $P_2^* B_2 P_2 = P_2^* P_1^* A \, P_1 P_2 = T$

so, $\exists$ $U = P_1 P_2$

s.t $U^* = P_2^* P_1^*$ so U is unitary for k x k

$U^* A U = T$

so, by induction as true for k x k, true for general n x n

Note: If $A = A^*$ then

as $U^* A U = T$

$(U^* A U)^* = T^*$

$U^* A^* U = U^* A U = T = T^*$

$\Rightarrow T = T^*$

$\Rightarrow T = \Lambda$

so, $U^* A U = \Lambda$

11th sept :

Theorem: let A be an $n \times n$ matrix with complex entries. $\exists$ a unitary matrix U s.t $U^* A U = T$ is an upper triangular matrix

Cor: If A is Hermitian $A^* = A$, then $\exists U$ a unitary matrix s.t

$$U^* A U = \Lambda \text{ (diagonal matrix)}$$

proof: $A = A^*$

$\exists U$ s.t

$U^* A U = T$

$\Rightarrow (U^* A U)^* = T^*$

$\Rightarrow U^* A^* U = T^*$

$\Rightarrow T = T^*$

$\Rightarrow T = \Lambda$

Theorem: (singular value decomposition) let A be a $n \times m$ matrix then there are unitary matrices U and V of order m and n respectively, such that $V^* A U = \Lambda$ where Λ is a diagonal matrix of order $n \times m$

$$\Lambda = \begin{pmatrix} \mu_1 & 0 & \cdot & \cdots \\ 0 & \mu_2 & 0 & \cdots \\ \cdots & 0 & \mu_3 & 0 \cdots \\ & & \ddots & \mu_r \end{pmatrix}$$

μ_i are called singular values of A, μ_i's are real and positive

total r: $r = \text{rank}(A)$

proof: $A^*: m \times n$

$A: n \times m$

so, $A^* A: m \times m$ matrix

$(A^* A)^* = A^* A$ or $A^* A$ is Hermitian,

then $\exists U$, a unitary matrix of $m \times m$ s.t

$$U^* A^* A U = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 \cdots & 0 \\ \vdots & 0 & \ddots & \lambda_r \\ 0 & 0 & & 0_0 \end{pmatrix}$$

λ_i are eigenvalues of $A^* A$

as $A^* A$ is positive semidefinite $\langle A^* A x, x \rangle = \langle Ax, Ax \rangle \geq 0$

so λ_i's are positive

Norm

let $\underset{n \times m}{W} = A\underset{m \times m}{U}$ & $\mu_i = \sqrt{\lambda_i}$ $1 \leq i \leq r$

then

$\underset{m \times n}{W^*}\underset{n \times m}{W} = T$ and if $D = \text{diag}[\underbrace{\mu_1, \mu_2 \ldots \mu_r\ 0 \ldots 0}]$

$D^2 = T$

so, $W^* W = D^2$

$W = \begin{bmatrix} W^{(1)} & \cdots & W^{(m)} \end{bmatrix} \Big\}$ n rows

as $W^* W = \begin{bmatrix} \langle W^{(1)}, W^{(1)} \rangle, \langle W^{(2)}, W^{(1)} \rangle \ldots \\ - \quad \cdot \quad \langle W^{(2)}, W^{(2)} \rangle \ldots \\ \cdot \end{bmatrix} = D^2$ so $W^* W$ is s.t $\langle W^{(i)}, W^{(j)} \rangle = 0$ for, $i \neq j$

& $\langle W^{(i)}, W^{(i)} \rangle = \mu_i^2$ $1 \leq i \leq r$

$\langle w^{(i)}, w^{(i)} \rangle = \mu_i^2$ for $1 \le i \le r$
$\langle w^{(i)}, w^{(i)} \rangle = 0$ for $i > r$
$\Rightarrow w^{(i)} = 0 \;\; \forall \; i > r$

so, for $w^{(i)} \in \mathbb{C}^n$; $\{w^{(1)}, \dots w^{(r)}\}$ are lin ind

$\Rightarrow r \le n$

let $v^{(j)} = \frac{1}{\mu_j} w^{(j)}$; $1 \le j \le r$

and choose $v^{(r+1)}, \dots v^{(n)}$ s.t $\{v^{(1)}, \dots v^{(n)}\}$ becomes orthogonal basis by gram smith

$$V = \begin{bmatrix} v^{(1)} & \dots & v^{(n)} \end{bmatrix}_{n \times n}$$ is $n \times n$ matrix and unitary

from defn

so, now $V \underbrace{\begin{bmatrix} \mu_1 & 0 & \dots \\ 0 & \mu_2 & \\ \vdots & & \ddots \mu_r \end{bmatrix}_{n \times m}}_{\Lambda(\text{singular values})} = W_{n \times m} = AU$

$\Rightarrow V\Lambda = AU$
$\Rightarrow \Lambda = V^* A U$

Note: In previous theorm we can arrage λ_i by changing u to get $\mu_1 \ge \mu_2 \ge \dots \ge \mu_r > 0$

Defn: If $A \ge 0$ (Positive semi definate) then $\sqrt{A} = P \sqrt{\Lambda} P^t$ where $P^t A P = \Lambda$
$\Lambda = diag(\lambda_1, \dots \lambda_n)$
$\sqrt{\Lambda} = diag(\lambda_1^{1/2}, \dots \lambda_n^{1/2})$

geometric multiplicity = dim (ker $A - \lambda I$)
algebric multiplicity = power of $\det(A - \lambda I)$ for $(\lambda - \lambda_0)^r$

GM = 1
AM = 3
$\lambda = 1$ 3x3 : $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = D + N$

Jordan block form

Nilpotent

diagonal

Theorm: (Jordan canonical form) let A be a matrix of order 'n' then $\exists$ a non singular matrix P s.t

$$P^{-1} A P = \begin{bmatrix} J_{n_1}(\lambda_1) & & & \\ & J_{n_2}(\lambda_2) & & \\ & & \ddots & \\ & & & J_{n_r}(\lambda_r) \end{bmatrix}$$

multiplicity

i.e $P^{-1} A P = D + N$

diagonal Nilpotent $N^k = 0$

If $X = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ then $\det(X) = \det(A)\det(D)$

$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} A & B \\ 0 & I \end{pmatrix}$ if $\det A = 0$
we can find $(A + \varepsilon I)^{-1}$ then $\varepsilon \to 0$

as A has finite eigenvalues

never zero det

gaus elimination:

$Ax = b$, $\tilde{A} = [A ; b] \rightarrow$ augmeted matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{pmatrix} \quad b = \begin{pmatrix} 4 \\ 1 \\ -3 \\ 4 \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\tilde{A} = \left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & 4 \\ 2 & 1 & -1 & 1 & 1 \\ 3 & -1 & -1 & 2 & -3 \\ -1 & 2 & 3 & -1 & 4 \end{array}\right] \xrightarrow[\substack{R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 + R_1}]{R_2 \to R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & 4 \\ 0 & -1 & -1 & -5 & -7 \\ 0 & -4 & -1 & -7 & -15 \\ 0 & 3 & 3 & 2 & 8 \end{array}\right] \xrightarrow[R_3 \to R_3 - 4R_2]{R_4 \to R_4 + 3R_2} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & 4 \\ 0 & -1 & -1 & -5 & -7 \\ 0 & 0 & 3 & 13 & 13 \\ 0 & 0 & 0 & -13 & -13 \end{array}\right]$$

$-13x_4 = -13$
$\Rightarrow x_4 = 1$
$3x_3 + 13 = 13$
$\Rightarrow x_3 = 0$

$-x_2 + 0 - 5 = -7$
$-x_2 = -2$
$\Rightarrow x_2 = 2$

$x_1 + (2) + 0 + 3(1) = 4$
$x_1 = -1$

22nd Sept:

linear algebra:

A vector norm on $\mathbb{R}^n$ is a function $N: \mathbb{R}^n \longrightarrow \mathbb{R}$ s.t
(i) $N(x) \geq 0$ and $N(x)=0$ iff $x=0$
(ii) $N(\alpha x) = \alpha N(x)\ \forall \alpha \in \mathbb{R}^n$
(iii) $N(x+y) \leq N(x)+N(y)$

Eg: $\|x\|_2 = \left(\sum_{j=1}^{n} |x_j|^2\right)^{\frac{1}{2}}$ $\quad x \in \mathbb{R}^n$

$\|x\|_p = \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p}$ $\quad 1 \leq p < \infty$

$\|x\|_\infty = \max_{1 \leq j \leq n} |x_j|$

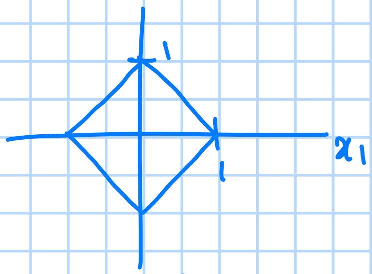
let $C[a,b]$ be continious function on $[a,b]$ then

$$\|f\|_2 = \left(\int_a^b |f|^2\right)^{1/2}$$

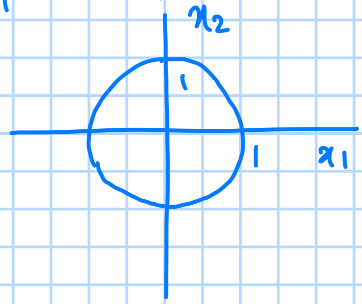
$$\|f\|_\infty = \max_{[a,b]} |f|$$

$$S_p = \{x \in \mathbb{R}^2 \mid \|x\|_p = 1\} \subseteq \mathbb{R}^2$$

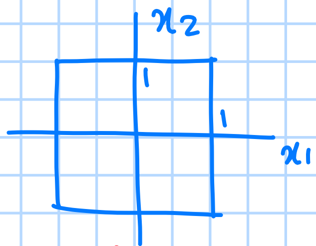
Eg: S_1 $\quad P=1$: $|x_1|+|x_2|=1$



S_2: $P=2$: $|x_1|^2+|x_2|^2=1$



S_∞: $\|x\|_\infty = 1 \Rightarrow \max\{|x_1|, |x_2|\} = 1$



Lemma: let $N(x)$ be a norm on $\mathbb{R}^n$, then $N(x)$ is a continious function

proof: $N(x)$ is continious as for any $x_n \to x$ we have

$$N(x_n) \leq N(x_n - x) + N(x)$$
$$\& \ N(x) \leq N(x - x_n) + N(x_n)$$
as $n \longrightarrow \infty$, $N(x) \leq N(x_n)$
$\& \ N(x_n) \leq N(x)$
$\Rightarrow N(x_n) = N(x)$

so as $x_n \to x$, $N(x_n) \longrightarrow N(x)$ so, N is continious by defn

Theorem: Any two norms are equivalent on $\mathbb{R}^n$, i.e. for $\|\ \|_\alpha$, $\|\ \|_\beta$ we have, $\exists c_1, c_2$ s.t

$$c_1\|\ \|_\alpha \leq \|\ \|_\beta \leq c_2\|\ \|_\alpha$$

Eg: $\exists c_1, c_2$ s.t

$$c_1\|\cdot\|_\infty \leq \|\cdot\|_2 \leq c_2\|\cdot\|_\infty$$

Defn: A sequence of vectors $\{x^{(1)}, x^{(2)}, \dots x^{(n)}\dots\}$ in $\mathbb{R}^n$ is said to be converging at x wrt some norm $\|\cdot\|$ if given some $\varepsilon>0$, $\exists$ a $N(\varepsilon)$

$$\text{s.t } \|x^{(k)} - x\| < \varepsilon \ \forall k \geq N(\varepsilon)$$

Note: Above is not same for ∞ dim as $\|f\|_\infty$ conv means uniform conv $\Rightarrow \|f\|_2$ but not otherway

Eg: $x^{(k)} = (1, 2+\frac{1}{k}, \frac{3}{k^2}, e^{-k}\sin k) \in \mathbb{R}^4$

$$x^{(k)} \xrightarrow[\|\cdot\|_\infty]{} (1, 2, 0, 0)$$

Theorem: A seq of vectors $(x^{(k)}) \longrightarrow x$ in $\|\cdot\|_\infty$ iff $\lim\limits_{k\to\infty} x_i^k = x_i \ \forall i = 1, 2 \dots n$

proof: ($\Rightarrow$) $\|x^{(k)} - x\|_\infty < \varepsilon \ \forall k \geq N(\varepsilon)$

$\Rightarrow |x_i^{(k)} - x_i| < \varepsilon \quad i = 1, 2 \dots n$ from defn of $\max\limits_{j\in\{1\dots n\}} |x_j|$

($\Leftarrow$) as, $x_i^k \longrightarrow x_i$ for all $1 \leq i \leq n$, $\exists N_i$ s.t

$$|x_i^{(k)} - x_i| < \varepsilon \quad \forall k \geq N_i(\varepsilon)$$

$$N = \max\{N_1, N_2 \dots N_N\}, \ \forall k \geq N(\varepsilon)$$

$$\Rightarrow \max_{1\leq i\leq n} |x_i^k - x_i| < \varepsilon \quad \forall k \geq N(\varepsilon)$$

$$\Rightarrow \|x^k - x\|_\infty < \varepsilon$$

Matrix norm:

let M_n denote the set of all $n\times n$ real matrices

$\|\cdot\| : M_n \longrightarrow \mathbb{R}$ which satisfies

(i) $\|A\| \geq 0$ and $\|A\| = 0 \Leftrightarrow A = 0$

(ii) $\|A+B\| \leq \|A\| + \|B\|$

(iii) $\|\alpha A\| = |\alpha|\|A\|$

(iv) $\|AB\| \leq \|A\|\|B\|$

Eg: $\|A\| = \left(\sum\limits_{1\leq i,j\leq n} |a_{ij}|^2\right)^{\frac{1}{2}}$

Theorem: $\|\ \|$ be a vector norm, then $\|A\| = \max\limits_{\|x\|=1} \|Ax\|$

Defn: (spectral radius) let $\sigma(A) = \max|\lambda|$, where λ is an eigenvalue of A

Theorem: let A be an $(n\times n)$ matrix, then

(1) $\|A\|_2 = (\sigma(A^tA))^{1/2}$

(2) $\sigma(A) \leq \|A\|$ for any norm $\|A\| = \max\limits_{\|x\|=1}\|Ax\|$

proof: for (2) let λ be an eigvalue and x be eigenvector with $\|x\|=1$ then

$$|\lambda| = \|\lambda x\| = |\lambda|\|x\| = \|Ax\| \leq \|A\|\|x\|$$
$$= \|A\|$$

$$\Rightarrow \sigma(A) \leq \|A\|$$

now for (1) as $\sigma(A) \leq \|A\|_2$

$$\Rightarrow \sigma(A^tA) \leqslant \|A^tA\|_2$$
$$\leqslant \|A^t\|_2 \|A\|_2$$
$$\leqslant \|A\|_2^2$$

$$\Rightarrow \sqrt{\sigma(A^tA)} \leqslant \|A\|_2$$

$$\|Ax\|^2 = \langle Ax, Ax\rangle = \langle A^tAx, x\rangle$$
$$Ax = \Sigma c_i \lambda_i u_i$$
$$\langle Ax, Ax\rangle = \Sigma c_i^2 \lambda_i^2$$ as eigenvector u_i are orthonormal

$$\leqslant \sigma(A^tA)\Sigma c_i^2$$
where $\|x\|_2^2 = \Sigma c_i^2$
$$\leqslant \sigma(A^tA)\|x\|_2^2$$

$$\Rightarrow \|Ax\|_2 \leqslant \|x\|_2 \sqrt{\sigma(A^tA)}$$

$$\Rightarrow \max_{\|x\|_2=1} \|Ax\|_2 \leqslant \sqrt{\sigma(A^tA}$$

$$\Rightarrow \|A\|_2 = \sqrt{\sigma(A^tA)}$$

Defn: (convergent matrix) we say an $(n\times n)$ matrix is convergent if $\lim_{k\to\infty}(A^k)_{ij} = 0 \quad \forall i,j$

eg: $A = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}$

$A^k = \begin{pmatrix} \frac{1}{2^k} & 0 \\ \frac{1}{2^{k+1}} & \frac{1}{2^k} \end{pmatrix}$ so, as $k \to \infty$, all components $\to 0$

Theorm: let A be a square matrix of order n. Then $A^m \to 0$ as $m \to \infty$ iff $\rho(A) < 1$

proof: $(\Leftarrow)$

$P^{-1}AP = J$ ← J is jordan form, P is not singular

$\Rightarrow A = PJP^{-1}$
$\Rightarrow A^m = PJ^mP^{-1}$

$J = D + N \qquad N^n = 0$ (nilpotent)

D ↓ diagonals$[\lambda_1, \ldots \lambda_n]$, $DN = ND$

$$J^m = (D+N)^m = \sum_{j=0}^{m} \binom{n}{j} D^{m-j} N^j$$

$$J^m = (D+N)^m = \sum_{j=0}^{m} \binom{m}{j} D^{m-j} N^j \qquad (N^n = 0 \Rightarrow N^j = 0 \text{ for } j \geqslant n)$$
$$= \sum_{j=0}^{n-1} \binom{m}{j} D^{m-j} N^j$$

$$\|J^m\| \leqslant \sum_{j=0}^{n-1} \frac{m!}{j!(m-j)!} \|D\|^{m-j} \|N\|^j$$

$$\leqslant c \sum_{j=0}^{n-1} \frac{m!}{j!(m-j)!} \|D\|_\infty^{m-j} \|N\|^j$$

$$\leq C\sum_{j=0}^{n-1} \frac{m!}{j!\,(m-j)!} (\sigma(A))^{m-j} \|N\|^j \longrightarrow 0 \text{ as } m\to\infty$$
as $\sigma(A)<1$

($\Rightarrow$) If $\sigma(A)\geq 1$ and $A^m \to 0$ as $m\to\infty$, then $\exists\ \lambda$ s.t $|\lambda|\geq 1$
let x be an eigenvector for λ

$$Ax=\lambda x \Rightarrow A^m x=\lambda^m x$$
as $m\to\infty$
$A^m x\to 0$
so, $\lambda^m x\to 0$, this is a contradiction, so
$\sigma(A)<1$

Theorm: let A be a square matrix of order n. If $\sigma(A)<1$ then $(I-A)^{-1}$ exist and
$(I-A)^{-1}=I+A+A^2+\cdots$

proof: let $\exists\ x\neq 0$ s.t $(I-A)x=0$
$\Rightarrow 1$ is an eigenvalue of A but $\sigma(A)<1$ so
this is a contradiction
$\Rightarrow (I-A)^{-1}$ exist

Now, if $BA=BA=I$, then $B=A^{-1}$
$$(I-A)(I+A+\cdots A^m)=I-A^{m+1}$$
as $m\to\infty$ $A^{m+1}\to 0$

so, $(I-A)^{-1}=I+A+A^2+\cdots$

29th Sept:

MATLAB: do calculations, especially matrix ones

there is command window, folder directory, workspace → stores/shows values

Help<space> function will give us what something does

calculator: we can write $4*x$, we get value of x, >> $x = 1+2*3$
$x = 7$ → enter

we have +, −, *, /, ^ ← exponents, if we add . we can get element wise operations

we can also overwrite variables, $t=5$, if $t=t+1 \Rightarrow t=6$, also adding ; will not print output

pi, i, inf, NaN
↓ ↓ ↓ ↓
3.14... $\sqrt{-1}$ ∞ Not a number

Newton's method implementation:

$f(x)=0$, given x_0
input: x_0, tolerance TOL, max itterations N

output: solution or message of failure

step 1: set $i=1$
step 2: while $i \leq N$ do 3−6
step 3: set $x = x_0 - \frac{f(x_0)}{f'(x_0)}$
step 4: if $|x-x_0| < TOL$ then
output (x); (Procedure was sucessful)
STOP
step 5: $i = i+1$
step 6: set $x_0 = x$ (update x_0) we can also plot

step 7: OUTPUT ("Method failed after N itterations")

6th oct :

The power method :

eigenvalue of largest modulus

It is an itterative method to find the dominant eigenvalue of a matrix A

$A_{n\times n}$ $\{\lambda_1, \dots \lambda_n\}$ then $|\lambda_1| > \{|\lambda_2|, \dots |\lambda_n|\}$ is dominant eigenvalue

assumptions for this method:

(I) The dominant eigenvalue has to be basic (multiplicity =1, both)

if AM=1 then GM=1

(II) The matrix A has to have n independent eigenvectors $\{v^1, v^2, \dots v^n\}$

we want to construct $\mu^m \to \lambda_1$ as $m \to \infty$

$$x = \sum_{j=1}^{n} \beta_j v^j \text{ for some } \beta_j$$

then
$$Ax = \sum \beta_j A v^j$$
$$= \sum \beta_j \lambda_j v^j$$
$$A^2 x = \sum_{j=1} \beta_j \lambda_j^2 v^j$$
$$\vdots$$
$$A^k x = \sum_{j=1}^{n} \beta_j \lambda_j^k v_j$$
$$= \lambda_1^k \Big[\beta_1 v^1 + \underbrace{\sum_{j=2}^{n} \beta_j \left(\frac{\lambda_j}{\lambda_1}\right)^k v^j}_{\to 0 \text{ as } \left|\frac{\lambda_j}{\lambda_1}\right| < 1 \text{ for } k \to \infty}\Big]$$

as $|\lambda_1| > \{|\lambda_2|, \dots |\lambda_n|\}$

$$\Rightarrow \left|\frac{\lambda_j}{\lambda_1}\right| < 1$$

so, $\lim_{k\to\infty} A^k x = \lim_{k\to\infty} \lambda_1^k \beta_1 v^1$

If, $|\lambda_1| < 1$ then $\lim_{k\to\infty} \lambda_1^k \beta_1 v^1 = 0$ and $\lim_{k\to\infty} \lambda_1^k \beta_1 v^1 \to \infty$ as $|\lambda_1| > 1$

we would like to bypass this problem by carefully choosing the initial vector x

step 1: we start with vector x^0 s.t $\|x^0\|_\infty = 1$ where $\|v\|_\infty = \max_{1\le j\le n}\{|v_1|, \dots |v_n|\}$ and so $\exists p_0$ s.t $x^0_{p_0} = 1$

step 2: we will $Ax^0 = y^1$ and define $\mu^1 = y^1_{p_0}$ for the above p_0

$$\mu^1 = y^1_{p_0} = \frac{y^1_{p_0}}{x^0_{p_0}} = \frac{(\beta_1 \lambda_1 v^1_{p_0}) + \sum^{n} \beta_j \lambda_j v^j_{p_0}}{\beta_1 v^1_{p_0} + \sum_{j=2}^{n} \beta_j v^j_{p_0}}$$

$Ax^0 = \sum \beta_j v^j$ so, $(Ax^0)_{p_0} = \sum \beta_j (v^j)_{p_0} \lambda_j$

$$= \lambda_1 \left[\frac{\beta_1 v^1_{p_0} + \sum \beta_j \left(\frac{\lambda_j}{\lambda_1}\right) v^j_{p_0}}{\beta_1 v^1_{p_0} + \sum \beta_j v^j_{p_0}}\right]$$

step 3: let p_1 be the smallest integer s.t

$$|y^1_{p_1}| = \|y^1\|_\infty \text{ and define } x^1 = \frac{y^1}{y^1_{p_1}} = \frac{Ax^0}{y^1_{p_1}} \quad \text{—①}$$

$$\|x^1\|_\infty = \frac{\|y\|_\infty}{|y^1_{p_1}|} = \frac{|y^1_{p_1}|}{|y^1_{p_1}|} = 1, \text{ so } y^2 = Ax^1 = A\left(\frac{Ax^0}{y^1_{p_1}}\right)$$

$x^1_{p_1} = 1$

$$= \frac{A^2x^0}{y^1_{p_1}} \quad \text{——— (2)}$$

then define

$$\mu^2 = y^2_{p_1} = \frac{y^2_{p_1}}{x^1_{p_1}} = \frac{\frac{(A^2x_0)_{p_1}}{\cancel{y^1_{p_1}}}}{\frac{(Ax_0)_{p_1}}{\cancel{y^1_{p_1}}}} \quad \text{from (1), (2)}$$

($x^1_{p_1} = 1$)

$$= \frac{(A^2x_0)_{p_1}}{(Ax_0)_{p_1}}$$

and as $x^0 = \sum \beta_j^0 v^j$

$A x^0 = \sum \beta_j^0 \lambda_j^0 v^j$

$A^2 x^0 = \sum \beta_j^0 \lambda_j^2 v^j$

$$= \frac{\beta_1 \lambda_1^2 v^1_{p_1} + \sum \beta_j^0 \lambda_j^2 v^j_{p_1}}{\beta_1 \lambda_1 v^1_{p_1} + \sum \beta_j^0 \lambda_j^0 v^j_{p_1}}$$

$$\mu^2 = \lambda_1 \left[\frac{\beta_1 v^1_{p_1} + \sum \beta_j^0 \left(\frac{\lambda_j}{\lambda_1}\right)^2 v^j_{p_1}}{\beta_1 v^1_{p_1} + \sum \beta_j^0 \left(\frac{\lambda_j}{\lambda_1}\right) v^j_{p_1}}\right]$$

step 4: let p_2 be the smallest integer s.t

$$|y^2_{p_2}| = \|y^2\|_\infty$$

let

$$x^2 = \frac{y^2}{y^2_{p_2}} = \frac{Ax^1}{y^2_{p_2}} = \frac{A^2x^0}{y^1_{p_1} y^2_{p_2}}$$

in this way $y^3 = Ax^2 = \frac{A^3x^0}{y^1_{p_1} y^2_{p_2}}$, and so we define

sequence of vectors $\{x^m\}, \{y^m\}, \{\mu_m\}$

$$y^m = Ax^{m-1}$$

$$\mu_m = y^m_{p_{m-1}} = \frac{(A^m x^0)_{p_{m-1}}}{(A^{m-1}x^0)_{p_{m-1}}}$$

$$= \lambda_1 \left[\frac{\beta_1 v^1_{p_{m-1}} + \sum \beta_j^0 \overbrace{\left(\frac{\lambda_j}{\lambda_1}\right)^m}^{\to 0 \text{ as } m \to \infty} v^j_{p_{m-1}}}{\beta_1 v^1_{p_{m-1}} + \sum \beta_j^0 \underbrace{\left(\frac{\lambda_j}{\lambda_1}\right)^{m-1}}_{\to 0 \text{ as } m \to \infty} v^j_{p_{m-1}}}\right]$$

where $x^m = \frac{y^m}{y^m_{p_m}} = \frac{A^m x^0}{\prod_{k=1}^{m} y^k_{p_k}}$ $\quad p_m$ is smallest integer s.t

$$|y^m_{p_m}| = \|y^m\|_\infty$$

then as $m \to \infty$

$$\mu_m \to \lambda_1 \left[\frac{\beta_1 \cancel{v^1_{p_{m-1}}}}{\beta_1 \cancel{v^1_{p_{m-1}}}}\right] = \lambda_1$$

Note: If dominant eigenvalue of a matrix A is '0' then all eigenvalues are zero

Note: Power method requires λ_1 is basic (A.M=1, i.e G.M=1)

Note: If we want to find second most dominating $|\lambda_2|$ then we want x^0 to be s.t it is orthogonal to v^1 (eigenvector of λ_1)

Inverse Power Method:

Suppose A has n-eigenvalues $\{\lambda_1, \dots \lambda_n\}, \{v_1 \dots v_n\}$

$q \neq \lambda_i \ \forall i$

$(A-qI)^{-1}$ exist and has $\{(\lambda_1-q)^{-1}, \dots (\lambda_n-q)^{-1}\}$

and $(A-qI)v = (\lambda-q)v$

$$v = (A-qI)^{-1}(\lambda-q)v$$

$$\Rightarrow \frac{v}{\lambda-q} = (A-qI)^{-1}v$$

$$\Rightarrow \frac{1}{\lambda-q} \text{ is eigenvalue}$$

if we apply power method on $(A-qI)^{-1}$, we will get value of $\frac{1}{\lambda_n-q}$ $\left(\because \frac{1}{|\lambda_n-q|} > \frac{1}{|\lambda_i-q|}\right)$

assuming λ_m is smallest

we need q to be small as $|\lambda_n-q| < |\lambda_i-q|$

changing q will get us all values of $\frac{1}{\lambda_i-q}$ assuming all eigenvalues are AM=1

Theorem: let A be a symmetric matrix with eigenvalue $\{\lambda_1, \dots \lambda_n\}$, If we have $\|Ax-\lambda x\|_2 < \varepsilon$ for some real λ and a vector x with $\|x_2\| = 1$, then

$$\min_{1\le j\le n} |\lambda_j - \lambda| < \varepsilon$$

Proof: $x = \sum_{j=1}^{n} \beta_j v_j$, v_j are orthonormal eigenvectors as A is symmetric

$$\|x\|_2^2 = 1 = \sum_{j=1}^{n} \beta_j^2$$

$$Ax - \lambda x = \sum \beta_j \lambda_j v_j - \sum \lambda \beta_j v_j$$

$$= \sum_{j=1}^{n} \beta_j (\lambda_j - \lambda) v_j$$

$$\Rightarrow \|Ax-\lambda x\|_2^2 = \sum_{j=1}^{n} \beta_j^2 (\lambda_j-\lambda)^2 \geq \min_{1\le j\le n} |\lambda_j-\lambda|^2 \left(\sum_{j=1}^{n} \beta_j^2\right)$$

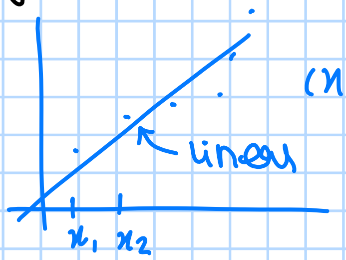
$$\Rightarrow \|Ax-\lambda x\|_2^2 \geq \min_{1\le j\le n} |\lambda_j-\lambda|^2$$

$$\Rightarrow \min_{1\le j\le n} |\lambda_j-\lambda| < \varepsilon$$

13^{th} oct:

Approximation theory:

we want to do least square linear approximation, least square polynomial approximation



(x_i, y_i) we want to find funtions, that approximate points to best possible way

we want to minimise square error,

$$\text{i.e.} \quad y_i - (a_1 x_i + a_0) = E(a_0, a_1)$$

error when line is $a_1 x + a_0$

we want to minimise $E_\infty(a_0, a_1) = \max_{1 \le i \le m} |y_i - (a_1 x_i + a_0)|$

→ min-max approximation

given $\{(x_1, y_1), \ldots (x_m, y_m)\}$

$$(a_0^*, a_1^*) = \underset{a_0, a_1}{\arg\min} \max_{1 \le i \le m} |y_i - (a_1 x_i + a_0)|$$

now in L_p norm:

$$E_p(a_0, a_1) = \sum_{i=1}^{m} |y_i - (a_1 x_i + a_0)|^p$$

we want $\frac{\partial E_p}{\partial a_0}$ and $\frac{\partial E_p}{\partial a_1} = 0$

as for $P=2$, L_2 norm is smooth:

$$E_2(a_1, a_0) = \sum_{i=1}^{m} |y_i - (a_1 x_i + a_0)|^2$$

$$= \sum_{i=1}^{m} y_i^2 - 2\sum_{i=1}^{m} y_i(a_1 x_i + a_0) + \sum_{i=1}^{m} |a_1 x_i + a_0|^2$$

$$\frac{\partial E_2}{\partial a_0} = \sum_{i=1}^{m} 2(y_i - (a_1 x_i + a_0))(-x_i) = 0$$

$$\frac{\partial E_2}{\partial a_1} = \sum_{i=1}^{m} 2(y_i - (a_1 x_i + a_0))(-1) = 0$$

$$\Rightarrow m a_0 + a_1 \sum x_i = \sum y_i \quad \text{①}$$

$$\text{and} \quad a_0 \sum x_i + a_1 \sum x_i^2 = \sum x_i y_i \quad \text{②}$$

solving above, we get

$$a_0 = \frac{\sum x_i^2 \sum y_i - (\sum x_i y_i) \sum x_i}{m \sum x_i^2 - (\sum x_i)^2}$$

$$a_1 = \frac{m \sum x_i y_i - \sum x_i \sum y_i}{m \sum x_i^2 - (\sum x_i)^2}$$

Polynomial least square approximation:

The idea is to approximate the dataset $\{(x_i, y_i)\}_{i=1}^{m}$ with an algebric polynomial of degree 'n' $P_n(x)$

$$P_n(x) = a_0 + a_1 x + \dots + a_n x^n$$

$$E(a_0, \dots a_n) = \sum_{i=1}^{m} (y_i - P_n(x_i))^2$$

$$\frac{\partial E}{\partial a_j} = 0 \quad \forall \quad 0 \leq j \leq n$$

$$\frac{\partial E}{\partial a_j} = \sum_{i=1}^{m} 2(y_i - P_n(x_i))(-x_i^j) = 0$$, if we expand this

$$\Rightarrow \sum y_i x_i^j = \sum P_n(x_i)(x_i)^j$$

$$P_n(x_i) = \sum_{k=0}^{n} a_k x_i^k$$

$$\Rightarrow \sum_{i=1}^{m} y_i x_i^j = \sum_{i=1}^{m} \left(\sum_{k=0}^{n} a_k x_i^k \right)(x_i)^j$$

$$\Rightarrow \sum_{k=0}^{n} a_k \sum_{i=1}^{m} x_i^{k+j} = \sum_{i=1}^{m} y_i x_i^j \qquad 0 \leq j \leq n$$

for $j=0$: $a_0 \sum_{i=1}^{m} x_i^0 + a_1 \sum_{i=1}^{m} x_i^1 + a_2 \sum_{i=1}^{m} x_i^2 + \dots + a_n \sum_{i=1}^{m} x_i^n = \sum_{i=1}^{m} y_i x_i^0$

$j=1$: $a_0 \sum_{i=1}^{m} x_i^1 + a_1 \sum_{i=1}^{m} x_i^2 + \dots + a_n \sum_{i=1}^{m} x_i^{n+1} = \sum_{i=1}^{m} y_i x_i^1$

$Ax = b$ form

$$\begin{bmatrix} \sum x_i^0 & \sum x_i^1 & \cdots & \sum x_i^n \\ \sum x_i^1 & \sum x_i^2 & \cdots & \sum x_i^{n+1} \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \sum y_i x_i^0 \\ \sum y_i x_i^1 \\ \vdots \end{bmatrix}$$

this matrix has det $\neq 0$

Exe: show that matrix A is invertible → done down

now, we want to try to use some idea for function $f \in C[a,b]$

$$E(a_0, \dots a_n) = \int_a^b |f(x) - P_n(x)|^2 dx$$

$$P_n(x) = \sum a_k x^k$$

we will try $\frac{\partial E}{\partial a_j} = 0$ for each $0 \leq j \leq n$

$$\frac{\partial E}{\partial a_j} = \int_a^b 2(f(x) - P_n(x))(x^j)\, dx = 0$$

$$\Rightarrow \int_a^b f(x) x^j dx = \int_a^b P_n(x) x^j dx \quad \forall \ 0 \leq j \leq n$$

$$\Rightarrow \int_a^b \left(\sum a_k x^k\right) x^j = \int_a^b f(x) x^j dx$$

$$\Rightarrow \sum_{k=0}^{n} a_k \int_a^b x^{k+j}\,dx = \int_a^b f(x) x^j\,dx$$ n+1 equations and n+1 unknowns

$$\Rightarrow \sum_{k=0}^{n} a_k \underbrace{\left(\frac{b^{k+j+1}-a^{k+j+1}}{k+j+1}\right)}_{\text{quality we get on integration}} = \int_a^b f(x)\, x^j\,dx$$

$$H_{kj} = \frac{b^{k+j+1}-a^{k+j+1}}{k+j+1} \; ; \; 0\le k\le n,\; 0\le j\le n$$

$\uparrow$
Hilbert matrix

Defn: (Hilbert matrix) H s.t $H_{kj} = \dfrac{b^{k+j+1}-a^{k+j+1}}{k+j+1}$, $0\le k,j\le n$

Eg: $f(x) = \sin(\pi x)$ $[0,1]$
$P(x) = a_0 + a_1 x + a_2 x^2$

$$a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 = \frac{2}{\pi}$$
$$\frac{1}{2}a_0 + \frac{1}{3}a_1 + \frac{1}{4}a_2 = \frac{1}{\pi}$$
$$\frac{1}{3}a_0 + \frac{1}{4}a_1 + \frac{1}{5}a_2 = \frac{\pi^2-4}{\pi^3}$$

Defn: (weight function) An integrable function w is called a weight function if $w(x)\ge 0\ \forall x\in I$ and $w(x)\not\equiv 0$ on any subinterval of I

Eg: $w(x) = \dfrac{1}{\sqrt{1-x^2}}$ $|x|<1$

then $\int_a^b w(x)|f|^2\,dx$ ← weited L_2 norm

now, $E(a_0,\dots a_n) = \displaystyle\int_a^b w(x)\,(f(x)-P(x))^2\,dx$ where $P(x) = \sum_{k=0}^{n} a_k \varphi_k$ where $\{\varphi_1,\varphi_2,\dots\varphi_n\}$ are lin independent functions

now, $\dfrac{\partial E}{\partial a_j} = 0$ then $\displaystyle\int_a^b w(x)\,2[f(x)-P(x)](-\varphi_j(x)\,dx) = 0$

$$\Rightarrow \int_a^b w(x) f(x)\,\varphi_j(x)\,dx = \int_a^b w(x)\,P(x)\,\varphi_j(x)\,dx$$

$$\Rightarrow \int_a^b w(x)\left(\sum_{k=0}^{n} a_k \varphi_k(x)\right)\varphi_j(x)\,dx = \int_a^b w(x) f(x)\varphi_j(x)\,dx$$

if φ_j satisfy:

$$\int w(x)\varphi_j(x)\,\varphi_k(x)\,dx = \begin{cases} 0 & j\ne k \\ \alpha & j=k \end{cases}$$

so we get: $a_j\alpha_j = \displaystyle\int_a^b w(x) f(x)\,\varphi_j(x)\,dx$

$$\Rightarrow a_j = \frac{1}{\alpha_j}\int_a^b w(x) f(x)\varphi_j(x)\,dx$$

Theorm: If $\{Q_0, \dots Q_n\}$ are orthogonal set of functions on $[a,b]$ wrt 'w' then the least square approximation of f on $[a,b]$ wrt w is

$$P(x) = \sum_{k=0}^{n} a_k Q_k(x)$$

$$a_k = \frac{\int_a^b w(x) f(x) Q_k(x)}{\int_a^b w(x) Q_k^2(x) dx}$$

proof: This follows trivially from calculations done above

Theorm: Define the polynomiales Q_i's in the following way

$$Q_0(x) = 1, \quad Q_1(x) = x - B_1(x)$$

$$B_1(x) = \frac{\int_a^b x w(x) Q_0^2}{\int_a^b w(x) Q_0^2}$$

for $k \geq 2$:

$$Q_k(x) = (x - B_k) Q_{k-1} - C_k Q_{k-2}$$

$$B_k = \frac{\int_a^b x w(x) (Q_{k-1})^2}{\int_a^b w(x) (Q_{k-1})^2} \qquad C_k = \frac{\int_a^b x w(x) Q_{k-1} Q_{k-2}}{\int_a^b w(x) (Q_{k-2})^2}$$

$\{Q_1, Q_2, \dots Q_n\}$ are orthogonal wrt w

Exe: show that matrix A is invertible

Ans:

$$A = \begin{bmatrix} \sum x_i^0 & \sum x_i^1 & \sum x_i^2 & \cdots & \sum x_i^n \\ \sum x_i^1 & \cdots & & \cdots & \sum x_i^{n+1} \\ \vdots & & & & \end{bmatrix}$$

lets assume A does not have full rank, then $\exists$

$$\begin{pmatrix} \sum x_i^j \\ \vdots \\ \sum x_i^{j+n} \end{pmatrix} = \sum_{\substack{r=0 \\ r \neq j}}^{n+1} \alpha_r \begin{pmatrix} \sum x_i^r \\ \vdots \\ \sum x_i^{r+n} \end{pmatrix}$$

$\exists$ r s.t $\alpha_r \neq 0$

$$\sum x_i^j = \alpha_0 \sum x_i^0 + \alpha_1 \sum x_i^1 + \cdots \alpha_{j-1} \sum x_i^{j-1} + \alpha_{j+1} \sum x_i^{j+1} + \cdots \alpha_{n+1} \sum x_i^{n+1}$$

$$\sum x_i^{j+1} = \alpha_0 \sum x_i^1 + \alpha_1 \sum x_i^2 + \cdots \alpha_{j-1} \underset{\substack{\uparrow \\ \text{this row}}}{\sum x_i^j} + \alpha_{j+1} \sum x_i^{j+2} + \cdots + \alpha_{n+1} \sum x_i^{n+2}$$

$\vdots$

$\sum x_i^{j+n}$ in terms of $\alpha_0, \alpha_1 \dots \alpha_{n+1}$

as $\{1, x, ...\}$ is lin ind $\rightarrow$ A is invertible

16th oct :

orthogonal polynomials :

Also called chebyshev polynomial, o.g polynomial on interval

$[-1,1]$ wrt $w(x) = \frac{1}{\sqrt{1-x^2}}$ ← weight

$T_n(x) = \cos(n\cos^{-1}x)$ for $n \geq 0$

$T_0(x) = \cos(0) = 1$

$T_1(x) = \cos(\cos^{-1}x) = x$

$$\int_{-1}^{1} w(x) T_n(x) T_m(x) = 0 \text{ for } n \neq m$$

if $\cos^{-1}x = \theta$

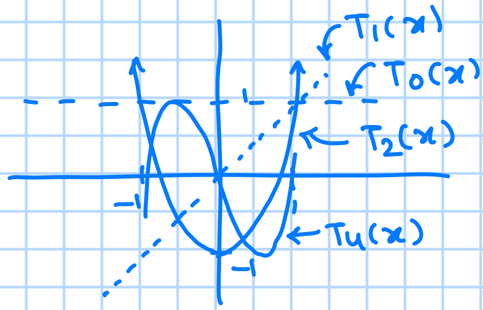
$T_n(x) = \cos(n\theta)$

$T_{n+1}(x) = \cos(n+1)\theta$

$= \cos n\theta \cos\theta - \sin n\theta \sin\theta$

$T_{n-1}(x) = \cos(n-1)\theta$

$= \cos n\theta \cos\theta + \sin n\theta \sin\theta$



$T_{n+1}(x) + T_{n-1}(x) = 2\cos\theta\cos n\theta$

$= 2\cos\theta \, T_n(x)$

$T_{n+1}(x) + T_{n-1}(x) = 2x\, T_n(x)$

← recurrence relation of T_n

so, $T_{n+1}(x) = 2x\, T_n(x) - T_{n-1}(x)$

$T_0(x) = 1$

$T_1(x) = x$

$T_2(x) = 2x^2 - 1 \qquad T_3(x) = 4x^3 - 3x \quad \ldots$

now, let's show T_n's are o.g wrt $w(x) = \frac{1}{\sqrt{1-x^2}}$

let $\cos^{-1}x = \theta$

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_n(x) T_m(x)\, dx = \int_0^{\pi} \frac{1}{\cancel{\sqrt{1-\cos^2\theta}}} \cos(n\theta)\cos m\theta \cancel{\sin\theta}\, d\theta$$

$$= \int_0^{\pi} \cos n\theta \cos m\theta \, d\theta$$

$$= \begin{cases} 0 & ; n \neq m \\ \frac{\pi}{2} & ; n = m \end{cases}$$

Theorm: The chebyshev polynomial $T_n(x)$ of degree 'n' has n real zeroes of multiplicity = 1 in $[-1,1]$ at

$$x_k = \cos\left(\frac{(2k-1)\pi}{2n}\right) ; \; k = 1, 2 \ldots n$$

moreover $T_n(x)$ attains its extrema at $x_k' = \cos\left(\frac{k\pi}{n}\right)$

for $k = 0, 1 \ldots n$

$T_n(x_k') = (-1)^k \quad k = 0, 1 \ldots n$

proof : $x_k = \cos\left(\frac{(2k-1)\pi}{2n}\right)$

$T_n(x_k) = \cos(n\cos^{-1}(x_k)) = \cos\left(n\frac{(2k-1)\pi}{2n}\right) = \cos\left((2k-1)\frac{\pi}{2}\right) = 0$

so, T_n has zero at x_k for $k=1,2\ldots n$

for second part, we compute $T_n'(x)$

$$T_n'(x) = n\frac{\sin(n\cos^{-1}x)}{\sqrt{1-x^2}} = 0$$

$$\Rightarrow \sin(n\cos^{-1}x) = 0$$
$$\Rightarrow n\cos^{-1}x = k\pi$$
$$\Rightarrow x = \cos\frac{k\pi}{n} \quad \text{for } k=1\ldots n-1$$

$$T_n(x_k') = \cos\left(n\cancel{\cos^{-1}}\cancel{\cos}\frac{k\pi}{n}\right) = \cos k\pi = (-1)^k \quad \text{for } k=1,\ldots n-1$$

for end points its trivial (i.e $k=0,n$ case)

$\tilde{T}_n(x) = \frac{1}{2^{n-1}}T_n(x)$, $\tilde{T}_0(x) = 1$ where $\tilde{T}_n(x)$ are monic polynomial

Exe: show that $\tilde{T}_n(x)$ is monic $\forall n \geq 0 \rightarrow$ <u>done down</u>

now, $\tilde{\Pi}_n$ set of all monic polynomial of degree n

Theorm: The polynomials of form $\tilde{T}_n(x)$ has the following properties

$$\frac{1}{2^{n-1}} = \max_{[-1,1]}|\tilde{T}_n(x)| \leq \max_{x\in[-1,1]}|P_n(x)| \quad \forall P_n \in \tilde{\Pi}_n$$

equality holds only if $P_n = \tilde{T}_n$

Lagrange interpolation:

$$f(x) = P(x) \quad \{x_0, x_1 \ldots x_n\}$$
$$+ \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)\ldots(x-x_n) \text{ for } f\in C^{n+1}[-1,1]$$

$$|f(x)-P(x)| = \left|\frac{f^{(n+1)}(\xi(x))}{(n+1)!}\right| \underbrace{|(x-x_0)\ldots\ldots(x-x_m)|}_{\text{we want to minimise this error}}$$

degree n

$|(x-x_0)\cdots(x-x_m)| = \tilde{T}_{n+1}(x)$ then minimum from previous theorm

$$\sup_{[-1,1]}|f(x)-P(x)| \leq \frac{1}{2^n(n+1)!}\sup_{x\in[-1,1]}|f^{n+1}\gamma(x)|$$

Reducing degree of $P(n)$:

if $P_n(x) = a_nx^n + \cdots + a_0$ $(a_n \neq 0)$
we want to represent this as $b_{n-1}x^{n-1}+\cdots b_0$
S.t we minimise the square error

new polynomial

$\left(\frac{P_n - P_{n-1}}{a_n}\right)$ now $\frac{P_n}{a_n}$ becomes monic polynomial of degree n

$$\tilde{T}_n(x) = \frac{P_n - P_{n-1}}{a_n} \text{ then } P_{n-1} = P_n - a_n\tilde{T}_n(x)$$

we get $\max_{[-1,1]} |P_n - P_{n-1}| = |a_n| \max_{[-1,1]} \left| \frac{(P_n - P_{n-1})}{a_n} \right| = |a_n| \max |\tilde{T}_n(x)|$

$$= \frac{|a_n|}{2^{n-1}}$$

Theorm: The polynomials of form $\tilde{T}_n(x)$ has the following properties

$$\frac{1}{2^{n-1}} = \max_{[-1,1]} |\tilde{T}_n(x)| \leq \max_{x \in [-1,1]} |P_n(x)| \quad \forall P_n \in \tilde{\Pi}_n$$

equality holds only if $P_n = \tilde{T}_n$

proof: we will argue with contradiction

if not, then $\exists$ a $P_n \in \tilde{\Pi}_n$ s.t $P_n \neq \tilde{T}_n$

and

$$\frac{1}{2^{n-1}} = \max \tilde{T}_n(x) \geq \max |P_n| \geq |P_n|$$

let $Q(x) = \tilde{T}_n(x) - P_n(x)$ - polynomial degree $(n-1)$

then $x_k = \cos\left(\frac{k\pi}{n}\right)$ $\tilde{T}_n(x_k) = \frac{(-1)^k}{2^{n-1}}$ $\quad k = 0, 1 \ldots n$

abs max condition

$$Q(x_k) = \frac{(-1)^k}{2^{n-1}} - P_n(x_k) \quad k = 0, 1 \ldots n$$

$Q(x_k) \geq 0$ when k is even as $\frac{1}{2^{n-1}} \geq |P_n|$

$Q(x_k) \leq 0$ when k is odd

$\Rightarrow Q$ has atleast n roots

this is a contradiction as $P_n \neq \tilde{T}_n$

Exe: show that $\tilde{T}_n(x)$ is monic $\forall n \geq 0$

Ans:

$$T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x)$$
$$T_0(x) = 1$$
$$T_1(x) = x$$

$$\tilde{T}_n(x) = \frac{1}{2^{n-1}} T_n(x)$$
$$\tilde{T}_0(x) = 1$$

now, $\tilde{T}_1(x) = x \rightarrow$ monic so true for $n=1$

if true for n

then

$$\tilde{T}_n(x) = \underbrace{x^n}_{\text{monic}} + \cdots$$

$$T_{n+1}(x) = 2x\, T_n(x) - T_{n-1}(x)$$
$$= 2x\left(2^{n-1}\tilde{T}_n(x)\right) - T_{n-1}(x)$$

$$T_{n+1}(x) = 2^n x^{n+1} + \cdots$$
$$\Rightarrow \frac{1}{2^n} T_{n+1}(x) = x^{n+1} + \cdots$$

so, $\tilde{T}_{n+1}(x)$ is monic, ie true for all n

27th oct :

so far we have seen $f \sim$ polynomial (Lagrange / chebyshev polynomial)
now we will approximate f (continious)
with rational polynomials

$$f \sim r(x) = \frac{p(x)}{q(x)}, \quad q(x) \neq 0$$

rational polynomial

Note: To reduce error, we use rational polynomials

Defn: (Rational function) A rational r of degree $N = m+n$ will be of form $r(x) = \frac{p(x)}{q(x)}$, $\deg(p) = m$, $\deg(q) = n$

p, q are polynomials

now, we want $f^K(0) = r^K(0)$ for $K = 0,1,2 \dots N$, if we assume

derivatives of f matches that of r at $N+1$ many $\frac{d}{dx}$

$f \sim r$ on I

interval

if $0 \in I$, $q(x) = q_0 + q_1 x + \cdots q_n x^n$

$q(0) = q_0 \neq 0$ then it will be non-zero in some interval

wlog we will assume that $q_0 \neq 0$

Near 0

we want $(f - r) = O(x^{N+1})$

say $f(x) - r(x) = \alpha(x) x^{N+1}$

then $f^K(0) - r^K(0) = 0 \quad K = 0,1,2 \dots N$

so, $f(x) - r(x) = \frac{f(x) q(x) - p(x)}{q(x)}$

let $f(x) = \sum_{K=0}^{\infty} a_K x^K$ ← local taylor series at 0

$$A = \left(\frac{\left(\sum_{K=0}^{\infty} a_K x^K\right)\left(\sum_{l=0}^{n} q_l x^l\right) - \left(\sum_{l=0}^{m} p_l x^l\right)}{q(x)} \right) \leftarrow O(x^{N+1})$$

then $x^0, x^1 \dots x^N$ coeff $= 0$ for $A = O(x^{N+1})$

$$A = (a_0 q_0 - p_0) x^0 + (a_1 q_0 + a_0 q_1 - p_1) x^1 + (a_2 q_0 + a_1 q_1 + a_0 q_2 - p_2) x^2 + \cdots$$

$a_0 q_0 = p_0$

$a_1 q_0 + a_0 q_1 = p_1$

$a_2 q_0 + a_1 q_1 + a_0 q_2 = p_2$

$\vdots$

$\sum_{i=0}^{K} a_i q_{K-i} = p_K \qquad K = 0,1 \dots N$

we already know a_i's, so total $N+1$ equations and $N+1$

$p_k = 0$ for $k \geq n+1$, $q_k = 0$ for $k \geq m+1$

so total $N+1$ unknowns

wlog $q_0 = 1$ then $N+1$, and we find p_i, q_i

Chebyshev rational approximation:

we want to approximate $f \sim r(x) = \dfrac{\sum_{k=0}^{m} p_k T_k(x)}{\sum_{k=0}^{n} q_k T_k(x)}$ — Chebyshev polynomial of degree k

we are just taking different basis of $\{T_0, T_1, \ldots T_k\}$ and not $\{1, x, x^2, \ldots x^k\}$

now, $f(x) - r(x) = \dfrac{f(x) \sum_{k=0}^{n} q_k T_k(x) - \sum_{k=0}^{m} p_k T_k(x)}{\sum_{k=0}^{n} q_k T_k(x)}$

and so now if $f(x) = \sum_{k=0}^{\infty} a_k T_k(x)$

now, $fq - p = (\sum a_k T_k)(\sum q_k T_k) - (\sum p_k T_k)$

$$= (a_0 T_0 + a_1 T_1 + a_2 T_2 + \cdots)(q_0 T_0 + q_1 T_1 + q_2 T_2 + \cdots q_n T_n)$$

$$= (p_0 T_0 + p_1 T_1 + \cdots p_m T_m)$$

then we can compute but we won't get straight forward basis

Trigonometric polynomial approximation (Fourier transformation):

$f \in C[-\pi, \pi]$ $\{\sin nx, \cos nx\}$

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n-1} (a_k \cos kx + b_k \sin kx) + a_n \cos nx$$

$$S_n(x) \xrightarrow{n \to \infty} f(x)$$

so, $f(x) = \dfrac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$

and $\int_{-\pi}^{\pi} \sin nx \cos mx = 0$, $n \neq m$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \quad k = 0, 1, \ldots n$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \quad , k = 1, 2 \ldots n-1$$

eg: $f(x) = |x|,\ x \in [-\pi, \pi]$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \sin kx \, dx = 0 \text{ as odd function}$$

$$f(x) = \frac{a_0}{2} + \sum_{i=1}^{K} a_i \cos ix$$

$$a_i = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos(ix) \, dx$$

3rd Nov :

Multivariable Calculus :

Continuity and differentiability: $f: \mathbb{R}^n \to \mathbb{R}^m,\ n, m > 1$

$$f(x) = (f_1(x), f_2(x), \dots f_m(x))$$

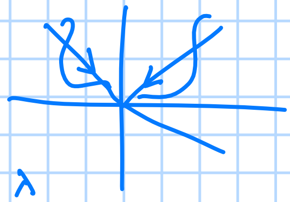
we say f is continious at $x_0 \in \mathbb{R}^n$ if given $\varepsilon > 0$, $\exists\, \delta > 0$ s.t

$$\| f(x) - f(x_0) \|_m < \varepsilon \text{ when } \| x - x_0 \|_n < \delta$$

$$\|x\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

Eg: $(x, y) \in \mathbb{R}^2$

$$f(x,y) = \begin{cases} 0 & ; (x, y) = 0 \\ \dfrac{xy}{x^2+y^2} & ; (x, y) \neq 0 \end{cases}$$



for $y = \lambda x$ $\quad f(x,y) = \dfrac{\lambda \cancel{x^2}}{\cancel{x^2} + \lambda^2 \cancel{x^2}} = \dfrac{\lambda}{1+\lambda^2}$

as $x, y \to 0$ $\quad f(x, y) = 0\ \forall$ path $(x, y) \to 0$
then we can say f is continious

were not free, so f is not continious at $(0,0)$

Eg: $f(x,y) = \begin{cases} \dfrac{x^2}{x^2+y^4} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$

not continious
as for $x = y^2$: $\quad f(y^2, y) = \dfrac{y^4}{y^4+y^4} = \dfrac{1}{2} \neq 0$ as $y \to 0$

$f: \mathbb{R}^n \to \mathbb{R}^m$, f is differtiable at some point at x_0 if $\exists$ a linear operator $A(x_0)$ s.t

$$\lim_{h \to 0} \frac{\| f(x_0+h) - f(x_0) - A(x_0)h \|}{\|h\|} = 0$$

$$A(x_0) \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$$

Eg: $f(x) = Bx$, B-matrix $\quad A(x_0) = B$

$f(A) = \det(A)$

we will get this

$$\lim_{h \to 0} \frac{\| \det(A+h) - \det(A) - T(A)h \|}{\|h\|} = 0$$

$\det(A) = \sum \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$,
$\det(A+h) = \sum \operatorname{sgn}(\sigma)(a_{1\sigma(1)} + h_{1\sigma(1)})(\ )\cdots(\ )$

$\det(e^A) = e^{\operatorname{trace}(A)}$
as $e^A = P^{-1} e^D P$ $\quad$ as
$$e^D = \begin{pmatrix} e^{\lambda_1} & 0 & \cdots \\ 0 & e^{\lambda_2} & 0 \\ \vdots & 0 & \ddots \end{pmatrix}$$

$\det(A+h) - \det(A) \approx \sum_\sigma \sum_i \operatorname{sgn}(\sigma) [a_{1\sigma(1)} \cdots a_{n\sigma(n)} h_{i\sigma(i)}]$

$a_{i\sigma(i)}$ not preset

so putting $T(A): M_{n \times n} \to \mathbb{R}$ as
$$T(A)h = \sum_\sigma \sum_i \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} \cdots a_{n\sigma(n)} h_{i\sigma(i)}$$

now, $g(x) = x - \frac{f(x)}{f'(x)}$ by newton's method

$f : \mathbb{R}^n \to \mathbb{R}^n$

then $\det f'(x) \neq 0$ $f'(x) =$ jacobian of f

let F be a function from $\mathbb{R}^n \to \mathbb{R}^n$ s.t $f(x) = (f_1(x), \dots f_n(x))$

Defn: The function $f : \mathbb{R}^n \to \mathbb{R}^n$ has a fixed point if $f(P) = P$ for some $P \in \mathbb{R}^n$

Theorem: let $D = \{(x_1, \dots x_n) \in \mathbb{R}^n \mid a_i \leq x_i \leq b_i, 1 \leq i \leq n\}$ suppose F is continuous on D with the property that $F(x) \in D$, when $x \in D$

Then F has a fixed point in D.

moreover, if $\exists$ a $k < 1$ and F is C^1

$$\left| \frac{\partial F_i}{\partial x_j} \right| \leq \frac{k}{n} \quad x \in D$$, then F has a unique fixed point

Newton's method:

$$X_{n+1} = X_n - (J(F)(x_n))^{-1} F(x_n)$$

Jacobian

so when $\det(J(F)(x_n)) \neq 0$ we have above scheme

$X_{n+1} \to P$ quadratically

Fixed point

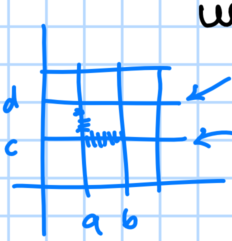
Derivative approximation:

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h}$$

$$f''(x_0) \sim \frac{f(x_0 + h) + f(x_0 - h) - 2f(x_0)}{h^2}$$

in $\mathbb{R}^2$: $\Delta u = \frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u = 0$

we approximate both terms seperately



$y_0 = c$ $y_j = c + jk \to$ size of interval

$x_0 = a$

$x_i = a + ih \to$ size of interval

$0 \leq i \leq l$

$0 \leq j \leq m$

$\Delta \sim$ invariant under orthonormal change of coordinates

we evaulte at every (i,j) Δ

$$\frac{\partial^2 u}{\partial x^2}(x_i, y_j) + \frac{\partial^2 u}{\partial^2 y}(x_i, y_j) = \Delta^2 u_{i,j}$$

$$\frac{\partial^2 u}{\partial x^2}(x_i, y_j) = \frac{u(x_{i+1}, y_j) - 2u(x_i, y_i) + u(x_{i-1}, y_i)}{h^2} + \text{error}$$

same for $\frac{\partial^2 u}{\partial y^2}(x_i, y_i)$

we are trying to find u

$$f(x_i, y_i) = \frac{u(x_{i+1}, y_i) - 2u(x_i, y_i) + u(x_{i-1}, y_i)}{h^2} + \frac{u(x_i, y_{i+1}) - 2u(x_i, y_i) + u(x_i, y_{i+1})}{k^2}$$

$$\Delta u = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right) u = 0 \quad \text{(Laplacian, elliptic)}$$

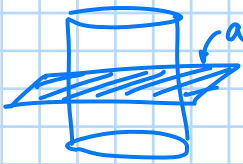
$u(t, x)$

$\partial_t u - \Delta u = 0$ (parabolic, heat equation)

$\partial_t^2 u - \Delta u = 0$ (wave equation, hyperbolic)

Eg: $\begin{cases} \Delta u = f \; ; \; \Omega \\ u = g \; ; \; \partial\Omega \end{cases}$

$\partial_t u - \Delta u = 0$ on $(0,T) \times \Omega$
$u = f \quad (0,T) \times \partial\Omega$
$u(0, x) = u_0$



at each time step

Note: Above we get a matrix trying to find U, we get u as solution to PDE