MA 417
Ordinary differential equations

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grading: 50% Endsem, 30% Midsem
20% assignments

Attendance: No attendance policy

Ref: L. Perko, diff equat & dinamic systems
ordinary diff equations and dynamic systems G. Teschl

Notes: ME Taylor, available on his website
ODE

Notes: Simon Brendon ODE Notes

7.35 is guaranteed passing

1^{st} Aug :

Defn: An m^{th} order ordinary differential equation is a relation of form

$$F(t, u(t), u'(t), \dots u^{(m)}(t)) = 0 \quad t \in I$$

$I \subseteq \mathbb{R}$ is an open set and

$F : \underbrace{I \times \mathbb{R} \times \mathbb{R} \cdots \times \mathbb{R}}_{m+1 \text{ times}} \longrightarrow \mathbb{R}$ is a given function and

u is the unknown function.

Note: sometimes this is refered to as an ODE in implicit form.

implicit is $y^2 + x^2 = 5$
not $y =$ some function

Defn: (Explicit form of ODE) A relation of the form

$$u^{(m)}(t) = F(t, u(t), \dots u^{(m-1)}(t)), \; t \in I \subseteq \mathbb{R}$$

Defn: A function $u : I \to \mathbb{R}$ is said to be a solution of m^{th} order ODE if it is m times differential and verifies.

preliminaries:

FTC: let $f : [a,b] \longrightarrow \mathbb{R}$ cont differentiable, then (there is a extension at a, b where it is diff)

$$f(x) - f(a) = \int_a^x f'(t)\,dt \quad x \in [a,b]$$

let $\delta : [a,b] \longrightarrow \mathbb{R}$ be cont. Define

$$f(x) = \int_a^x \delta(t)\,dt \text{ then } f \text{ is cont differentiable}$$

and $f'(x) = \delta(x) \;\; \forall x \in (a,b)$

Gronwall's inequality: let $g, h : [a,b] \longrightarrow [0, \infty)$ be continious function satisf. (both have range $\mathbb{R}^+$)

$$g(t) \leq c + \int_a^t g(s)\,h(s)\,ds \;\; \forall t \in [a,b]$$

where c is a non-negative constant, then

$$g(t) \leq c e^{\int_a^t h(s)ds} \; \forall t \in [a,b]$$

and if $c = 0$ then $g(t) = 0$

Theorm: (Gronwall's inequality) $g, h : [a,b] \longrightarrow [0, \infty)$ cont and $\exists\, c \geq 0 \in \mathbb{R}$ s.t

$$g(t) \leq c + \int_a^t g(s)\,h(s)\,ds \quad \forall t \in [a,b]$$

then $g(t) \leq c\, e^{\int_a^t h(s)ds} \; \forall t \in [a,b]$

and if $c = 0 \Rightarrow g(t) = 0$

proof: let $K(t) = c + \int_a^t g(s)\,h(s)\,ds$ for $t \in [a,b]$

then K is cont. differentiable (FTC) and as $g(s), h(s), c \geq 0$
$\Rightarrow K(t) \geq 0$

Case I : $(c > 0)$ here $K'(t) = g(t)\,h(t)$

but as $g(t) \leq K(t)$ (from construction of K)

$\Rightarrow g(t)h(t) \leq K(t)h(t)$

$\Rightarrow K'(t) \leq K(t)h(t)$

and $K(t) \geq C > 0 \Rightarrow K(t) > 0$

$\Rightarrow \dfrac{K'(t)}{K(t)} \leq h(t) \quad \forall t \in [a,b]$

$\Rightarrow (\ln K(t))' \leq h(t)$

$\Rightarrow \int_a^t (\ln K(s))' ds \leq \int_a^t h(s) ds$

$\Rightarrow \ln K(t) - \ln \underbrace{K(a)}_{= c} \leq \int_a^t h(s) ds$

$\Rightarrow K(t) \leq c e^{\int_a^t h(s) ds}$

$\Rightarrow \underbrace{g(t) \leq K(t)}_{\text{from construction}} \leq c e^{\int_a^t h(s) ds}$

$\Rightarrow g(t) \leq c e^{\int_a^t h(s) ds}$

case 2: $C = 0$, $\forall \varepsilon > 0$, we have

this trick can be used at many other places

$$g(t) \leq \varepsilon + \int_a^t g(s) h(s) ds$$

then by case I

$\Rightarrow g(t) \leq \varepsilon e^{\int_a^t h(s) ds}$

$\Rightarrow g(t) = 0$ as g is cont and $\forall \varepsilon > 0$, $g(t) \leq \varepsilon e^{\int_a^t h(s) ds}$

Linear ODE :

Defn: An m^{th} order linear ODE is a relation of the form:

$$a_m(t) u^m(t) + a_{m-1}(t) u^{m-1}(t) + \dots + a_1(t) u'(t) + a_0(t) u(t) = b(t) \quad \forall t \in I$$

where $\forall t \in I$, $a_m(t) \neq 0$

Defn: we say linear ODE is homogenious if $b(t) = 0 \ \forall t \in I$, otherwise we say its inhomogenious or non-homogenious.

Theorm: Consider the linear homogenious ODE
$a_m(t) u^m(t) + \dots + a_0(t) u(t) = 0 \ \forall t \in I$, $a_m(t) \neq 0$
let $X = \{ u : I \to \mathbb{R} \mid u$ is a solution$\}$ then X is a real vector space with usual addition of function & scalar multiplication by real numbers.

proof: It is trivial to prove all the identities of vector space, however let's show for $u, v \in X, \alpha \in \mathbb{R}$ $u + \alpha v \in X$

as $u \in X, v \in X$ we have $a_m(t) u^m(t) + \dots a_0(t) u(t) = 0$ —— ①

$a_m(t) v^m(t) + \dots a_0(t) v(t) = 0$ —— ②

① $+ \alpha \times$ ②: $a_m(t)[u^m(t) + \alpha v^m(t)] + \dots a_0(t)[u(t) + \alpha v(t)] = 0$

if $u(t) + \alpha v(t) = z(t)$ then $a_m(t) z^m(t) + \dots a_0(t) z(t) = 0$

$\Rightarrow z \in X \Rightarrow u + \alpha v \in X$

linear operator

$Lu(t) = a_m(t)u^m(t) + \dots + a_0(t)u(t)$

$L.\, C^m(I) \longrightarrow C^0(I)$

$Lu = u'' + u^2$

this operator is not linear

one way to say a fuction is small that $u \in C^0(I)$ closed interval
then let $\sup|u|$ define now small a fuction is on a given $C^0(I)$ vector space.

$C^2(I) \rightarrow$ normed vector space, but as $C^2(I) \subseteq C^0(I)$ we want to say two fuction in $C^2(I)$ are close to each other, for this we use:

$$\sup_I |u| + \sup_I |u'| + \sup_I |u''| = \text{norm}$$

$$d(g,h) = \sup_I |g-h| + \sup_I |(g-h)'| + \sup_I |(g-h)''|$$

some form of distance for $g, h \in C^2(I)$

Exe: what is the dimension of X in the above theorm for $m=1$?

Ans: for $m=1$

$$a_1(t)u'(t) + a_0(t)u(t) = 0$$

$$\Rightarrow u'(t) + a(t)u(t) = 0 \quad (\text{dividing by } a_1(t))$$

A differtiable fuction $u: I \to \mathbb{R}$ satisfyes the ab (here it is iff)

$$\Updownarrow$$

$$[u'(t) + a(t)u(t)]\underbrace{e^{\int_{t_0}^t a(s)ds}}_{\text{this is never } 0} = 0 \quad \text{for some } t_0 \in I$$

(multiplying by both sides)

$u \in X \Leftrightarrow [u(t)e^{\int_{t_0}^t a(s)ds}]' = 0$

$\Leftrightarrow u(t)e^{\int_{t_0}^t a(s)ds} = c$ constant

$c = u(t_0)$

$u \in X \Leftrightarrow u(t) = u(t_0)e^{-\int_{t_0}^t a(s)ds}$

so X has dim 1. (trivial)

Theorm: let $I \subseteq \mathbb{R}$, $t_0 \in I$, $\alpha_0, \dots \alpha_m \in \mathbb{R}$, $a_0, a_1, \dots a_{m-1}, b$ are cont. fuctions: $I \to \mathbb{R}$ then the linear inhomogeious ODE

$$u^m(t) + a_{m-1}u^{m-1}(t) + \dots a_0(t)u(t) = b(t)$$

$u(t_0) = \alpha_0$, $u'(t_0) = \alpha_1, \dots u^{m-1}(t_0) = \alpha_{m-1}$, $\exists$ a solution which is unique.

we will only do uniqueness, existance will be proved later but see how intuition wise the solution space has m degrees of freedom and fixing all dim we get unique solution

proof: suppose $u_1, u_2: I \to \mathbb{R}$ are the solution of the given equation then define

$$u(t) = u_1(t) - u_2(t)$$

then u satisfies

$$u^m(t) + a_{m-1}u^{m-1}(t) + \dots + u(t)a_0(t) = 0$$

and $\forall i = 0, 1 \dots m-1$

$$u^{(i)}(t_0) = 0$$

now as $u^{(i)}(t_0)=0 \ \forall i=0,1\dots m-1$
$\Rightarrow u^m(t_0)=0$

now let $g(t)=\sum_{i=0}^{m-1}|u^i(t)|$, we want to first show $u(t)=0$ $\forall t>t_0$

fix $T>t_0$, then as a_i, b are continuous, $\exists M>0$ s.t

$\max_{t\in[t_0,T]} |a_i(t)|\leq M_i,\ i=0,\dots m-1$ (we don't know about $u^{(m)}(t)$)

$\max_{t\in[t_0,T]} |b(t)|\leq M_b \quad M=\max_{i\in\{0,\dots m-1\}}\{M_i, M_B\}$

we have, $u^i(t)=u^i(t_0)+\int_{t_0}^{t} u^{i+1}(s)ds=\int_{t_0}^{t} u^{i+1}(s)ds$
($u^i(t_0)=0$)

$|u^i(t)|\leq \left|\int_{t_0}^{t} u^{i+1}(s)ds\right|\leq \int_{t_0}^{t}|u^{i+1}(s)|ds$ if $i\leq m-2$ then

$|u^i(t)|\leq \int_{t_0}^{t} g(s)ds$ (By formula of $g(s)$ but not for $m-1$)

Now, $|u^{m-1}(t)|\leq \int_{t_0}^{t}|u^m(s)|ds$ (FTC)

$\Rightarrow |u^{m-1}(t)|\leq \int_{t_0}^{t}|-a_{m-1}u^{m-1}(s)\cdots - a_0 u(s)|ds$
(we use the mod principle)

$\leq \int_{t_0}^{t}[|a_{m-1}(s)||u^{m-1}(s)|+\cdots |a_0(s)||u(s)|]ds$

$\leq M\int_{t_0}^{t}[|u^{m-1}|+|u^{m-2}|+\dots+|u|]ds$

$\leq M\int_{t_0}^{t} g(s)ds$

now, $g(t)\leq \sum_{i=0}^{m-2}|u^i(t)|+M\int_{t_0}^{t} g(s)ds$

$g(t)\leq (m-1)\int_{t_0}^{t} g(s)ds+M\int_{t_0}^{t} g(s)ds$

$\Rightarrow g(t)\leq (m-1+M)\int_{t_0}^{t} g(s)ds$

$g(t)\leq c+\int_{t_0}^{t} g(s)h(s)ds$

$c=0 \quad h(s)=m-1+M$

and so $\Rightarrow g(t)=0$ (from gronwall)

$\Rightarrow \forall t>t_0$
$g(t)=0$

$\sum_{i=0}^{m-1}|u^i(t)|=0\Rightarrow u^i(t)=0$
$\forall i\in\{0,\dots m-1\}$

5^{th} Aug:

Theorm: let $I \subseteq \mathbb{R}$, $t_0 \in I$, $\alpha_0, \ldots \alpha_m \in \mathbb{R}$, $a_0, a_1, \ldots a_{m-1}, b$ are cont. functions: $I \to \mathbb{R}$ then the linear inhomogeious ODE

$$u^m(t) + a_{m-1} u^{m-1}(t) + \cdots a_0(t) u(t) = b(t)$$
$u(t_0) = \alpha_0$, $u'(t_0) = \alpha_1, \ldots u^{m-1}(t_0) = \alpha_{m-1}$, $\exists$ a solution which is unique.

proof: we will continue the proof, as alway shown that $\forall\ t > t_0$

$$g(t) = 0 \Rightarrow \sum_{i=0}^{m-1} |u^i(t)| = 0$$

we got $g(t) = 0$ by choosing $u^i(t)$ and then $g(t) \leq c + \int_t^{t_0} g(s) h(s) ds$

now let $v(t) = u(t_0 - t)$ for $t \in t_0 - I = \{t_0 - s \mid s \in I\} = \tilde{I}$

now, $v(0) = u(t_0)$, $v^i(t) = (-1)^i u^i(t_0 - t)$ and then

$$(-1)^m v^m(t) + (-1)^{m-1} a_{m-1}(t_0 - t) v^{m-1}(t) + \cdots + a_0(t - t_0) v(t) = 0$$

and then as $v(0) = v'(0) = \cdots = v^{m-1}(0) = 0$

$\Rightarrow \forall t > t_0$ $v(t) = 0$ (similar proof to previous one)

$\Rightarrow \forall t < t_0$ $u(t) = 0$

so $u(t) = 0$ $\forall t \in I$

$\Rightarrow u_1(t) = u_2(t)$

and so it is unique

linear independence and wronskian:

defn: let $I \subseteq \mathbb{R}$ be an interval and $u_1, u_2, \ldots u_m$ be real valued functions defined on I. we say that $u_1, \ldots u_m$ are

(I) linearly dependent if $\exists\ c_1, c_2 \ldots c_m$ (not all zero) s.t
$$c_1 u_1(t) + \cdots c_m u_m(t) = 0 \quad \forall t \in I$$

(II) linearly independent if not linear dependent
$$\sum_{j=0}^{m} c_i u_i(t) = 0 \ \forall\ t \in I \Rightarrow c_i = 0 \ \forall i \in \{1, \ldots m\}$$

Defn: (wronskian) $I \to$ interval u_j, $i = 1, 2 \ldots m$ are $(m-1)$ differentiable, we define the wronskian of $u_1, \ldots u_m$ as:

$$W[u_1(t), \ldots u_m(t)] = \det \begin{bmatrix} u_1^0(t) & \cdots & u_m^0(t) \\ \vdots & & \vdots \\ u_1^{m-1}(t) & \cdots & u_m^{m-1}(t) \end{bmatrix} = \det[u_{j+1}^i(t)]$$

$0 \leq i, j \leq m-1$
$i \to$ row
$j \to$ column

propn: If $W[u_1(t_0), \ldots u_m(t_0)] \neq 0$ for some $t_0 \in I$ then $u_1, \ldots u_m$ are linearly ind

proof:

If $u_1 \ldots u_m$ are linearly dependent then $c_1 u_1 + \cdots + c_m u_m = 0 \ \forall t \in I$
not all c_i are zero
$$\vdots$$
$$c_1 u_1^{m-1} + \cdots + c_m u_m^{m-1} = 0$$

so, $\exists$ a non zero solution of the above system of equations

$$\begin{bmatrix} u_1(t_0) & \cdots & u_m(t_0) \\ \vdots & & \\ u_1^{m-1}(t_0) & \cdots & u_m^{m-1}(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

now if the Wronskian $\neq 0$ the matrix is invertable and so, $\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ as $\text{matrix}^{-1} \times 0 = 0$

$\Rightarrow c_i = 0 \ \forall i \in \{1 \ldots m\}$ this contradicts that u_i's are lin dep

$\Rightarrow \{u_1, u_2 \ldots u_m\}$ are linearly independent

Note: If $u_1 \ldots u_m$ are linearly dependent then $W[u_1(t), \ldots u_m(t)] = 0 \ \forall t \in I$ (this is just contrapositive of the proposition)

Note: The converse is not true, as $\exists u_1, \ldots u_m$ linearly independent, but $W[u_1(t), \ldots u_m(t)] = 0 \ \forall t \in I$

Exe: $u_1(t) = t^2$
$u_2(t) = t|t|$

$u_1'(t) = 2t$
$u_2'(t) = \begin{cases} 2t & : t > 0 \\ 0 & : t = 0 \\ -2t & : t < 0 \end{cases}$

$$W[u_1(t), u_2(t)]_{t>0} = \det \begin{bmatrix} t^2 & t^2 \\ 2t & 2t \end{bmatrix} = 0$$

$$W[u_1(t), u_2(t)]_{t<0} = \det \begin{bmatrix} t^2 & -t^2 \\ 2t & -2t \end{bmatrix} = 0$$

$$W[u_1(0), u_2(0)] = \det \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Theorm: $I \subseteq \mathbb{R}$ be an interval, $a_i : I \to \mathbb{R}$ all continious. let $u_1, \ldots u_m$ be m solns of the ODE

$$u^m + a_{m-1}u^{m-1} + \cdots a_0 u = 0 \ \forall t \in I$$

suppose $u_1, \ldots u_m$ are lin ind, then $W[u_1, \ldots u_m] \neq 0 \ \forall t \in I$

proof: suppose its not true then given $\{u_1 \ldots u_m\}$ lin independent, we will have $t_0 \in I$ s.t

$$W[u_1(t_0) \ldots u_m(t_0)] = 0$$

$$\Rightarrow \begin{bmatrix} u_1(t_0) & \cdots & u_m(t_0) \\ \vdots & & \\ u_1^{m-1}(t_0) & \cdots & u_m^{m-1}(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ s.t } \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} \neq \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

let $v(t) = c_1 u_1(t) + c_2 u_2(t) + \cdots c_m u_m(t) \ \forall t \in I$
then v also solves the given differential equation
moreover $v(t_0) = 0$
similarly $v'(t_0) = 0, \ldots v^{m-1}(t_0) = 0$
then by the uniqueness theorm $\Rightarrow v(t) = 0 \ \forall t \in I$

$\Rightarrow c_1 u_1 + \dots c_m u_m = 0 \;\forall t \in I$

and as $\begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} \neq \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

$\Rightarrow \{u_1, \dots u_m\}$ are linearly dependent, this is a contradiction

so, $W[u_1, \dots u_m] \neq 0 \;\forall t \in I$

Cor: If $u_1, u_2 \dots u_m$ are m solutions of the linear homogenous ODE then $\{u_1, \dots u_m\}$ are linearly independent iff $W[u_1(t_0), \dots, u_m(t_0)] \neq 0$ for some $t_0 \in I$

proof: same as the two proof above, here we add a stronger condition.

Theorm: $a_i : I \to \mathbb{R}$ are continious. let X be the vector space of the solution of the linear mth ODE

$$u^m + a_{m-1} u^{m-1} + \dots a_0 u = 0 \;\forall t \in I$$

then $\dim X = m$

proof: let u_1 be a solution s.t

$$u_1(t_0) = 1 \quad u_1'(t_0) = 0, \dots \; u_1^{m-1}(t_0) = 0$$

similarly let u_2 be a solution s.t

$$u_2(t_0) = 0 \quad u_2'(t_0) = 1, \dots u_2^{m-1}(t_0) = 0$$

so $u_i^j(t_0) = \begin{cases} 0 \; ; \; i - j \neq 1 \\ 1 \; ; \; i - j = 1 \end{cases}$

now, $\det \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = \det \begin{bmatrix} u_1(t_0) & \dots & u_m(t_0) \\ \vdots & & \vdots \\ u_1^{m-1}(t_0) & \dots & u_m^{m-1}(t_0) \end{bmatrix} = 1 \neq$

$\Rightarrow \{u_1 \dots u_m\}$ are linearly independent

now let $u \in X$, we want to find c_j s.t $u = \sum_{j=1} c_j u_j$

let $c_j = u^{j-1}(t_0)$ (this is by our construction)

let $v(t) = u(t) - c_1 u_1(t) - c_2 u_2(t) \dots - c_m u_m(t)$

now $v(t_0) = u(t_0) - u(t_0) = 0$

$v'(t_0) = u'(t_0) - u'(t_0) = 0$

$\vdots$

$v^{m-1}(t_0) = 0$ and also $v(t)$ is a solution to the given diff equation

$\Leftrightarrow v(t) = 0 \;\forall t \in I$ (By uniqueness)

$\Leftrightarrow u(t) = \sum_{j=1}^{m} c_i u_i(t)$

so we showed if $u \in X$ then $u = \sum c_i u_i$ (By showing $u - \sum c_i u_i = 0$) and if $u = \sum c_i u_i$ then $u \in X$

Cor: If u_p is any solution of the inhomogenious ODE $u^m + a_{m-1} u^{m-1} + \dots + a_0 u = b$ then any solution u can be written as $u = v + u_p$ where v is a solution of $u^m + a_{m-1} u^{m-1} + \dots a_0 u = 0$

proof: as u_p is a soln to given inhomogeneous ODE $u^m + a_{m-1} u^{m-1} + \dots a_0 u = b$

and u is another solution, then $(u-u_p)^m + a_{m-1}(u-u_p)^{m-1} + \dots + a_0(u-u_p) = 0$

$\Rightarrow u - u_p$ is a solution to $u^m + a_{m-1}u^{m-1} + \dots a_0 u = 0$

$\Rightarrow u - u_p \in X$ where X = vector space of solution of the liner ODE

so $\exists v \in X$ s.t $v = u - u_p$ (this is from previous theorm)

$\Rightarrow u = v + u_p$

so $\forall u \in X_p$ = set of solutions of given ODE

$\exists v \in X$ s.t $u = v + u_p$

$\therefore$ Every solution of the given ODE can be written as $v + u_p$ where $v \in X$

8^{th} Aug:

Linear homogenious equation with constant coefficients:

we are interested in the ODE $a_m u^{m}(t) + a_{m-1} u^{m-1}(t) + \cdots + a_1 u'(t) + a_0 u(t) = 0$
$a_i \in \mathbb{R},\ t \in \mathbb{R}$

define a linear differential operator L with const coefficients as

$$L = \sum_{i=0}^{m} a_i \left(\frac{d}{dt}\right)^i \quad a_i \in \mathbb{R} \quad \text{is s.t } L: C^m(\mathbb{R}) \longrightarrow C^1(\mathbb{R})$$

In this notation we want to find kernal.

Defn: ($P(\lambda)$) we define a polynomial $P(\lambda) = a_m \lambda^m + \cdots a_1 \lambda^1 + a_0$, this is called the characteristics polynomial of L

conversely given a polynomial of degree m, we can allocate a constant coefficient differential operator L_P s.t P is the characteristic polynomial of L_P

Lemma: let P be a polynomial of degree m, suppose $P = P_1 P_2$, where P_i are polynomial with real coeff then for any m times diff function u, then

$$L_P(u) = L_{P_1} L_{P_2}(u) = L_{P_2} L_{P_1}(u)$$

proof: now, let $P_1 = a_{k_1} \lambda^{k_1} + \cdots a_1 \lambda^1 + a_0$

$$P_2 = b_{k_2} \lambda^{k_2} + \cdots b_1 \lambda^1 + b_0$$

where $a_i, b_i \in \mathbb{R}\ \forall i$ and $k_1 + k_2 = m$

now
$$L_{P_1} L_{P_2}(u) = L_{P_1}\left[b_{k_2} u^{k_2} + \cdots\cdots b_1 u' + b_0\right]$$
$$= b_{k_2} L_{P_1} u^{k_2} + \cdots b_1 L_{P_1} u' + L_{P_1} b_0$$
$$= b_{k_2}\left[a_{k_1} u^m + a_{k_1 - 1} u^{m-1} + \cdots\cdots a_0 u^{m-k_2}\right] + \cdots + a_0 b_0$$
$$= \sum_{i=0}^{k_2} b_i \sum_{j=0}^{k_1} [a_j u^{i+j}] = \sum_{i=0}^{k_2} \sum_{j=0}^{k_1} a_j b_i u^i u^j$$
$$= \sum_{j=0}^{k_1} \sum_{i=0}^{k_2} a_j b_i u^i u^j = L_{P_2}(L_{P_1}(u)) = L_{P_2} L_{P_1}(u)$$

$\Rightarrow L_{P_1} L_{P_2} = L_{P_2} L_{P_1}$

and from $P = P_1 P_2 = \sum\sum a_j b_i \lambda^{i+j}$ it is trivial to see that it is equal to $L_{P_1} L_{P_2}$ & $L_{P_2} L_{P_1}$

Note: If v is a solution of $L_{P_1}(u) = 0$ or $L_{P_2}(v) = 0$, then v is a solution of $L_P(v) = 0$
v solution of L_{P_1} or $L_{P_2} \Rightarrow v$ solution of L_P

let us consider the characteristic polynomial

$$P(\lambda) = a_m \lambda^m + \cdots\cdots + a_1 \lambda^1 + a_0$$

By fundamental theorm of algebra, we can write

$$P(\lambda) = a_m (\lambda - \lambda_1) \cdots (\lambda - \lambda_m) \quad \lambda_i \in \mathbb{C}$$ ← Not real

Note: λ_i need not be distinct

suppose λ_i is not real then $\bar{\lambda}_i$ is a root of P & hence $\bar{\lambda}_i = \lambda_j$ for some j

we can write

$$P(\lambda) = a_m (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_{k_1})^{m_{k_1}} \times (\lambda - z_1)^{n_1} (\lambda - \bar{z}_1)^{n_1} \cdots (\lambda - \bar{z}_{k_2})^{n_{k_2}} (\lambda - z_{k_2})^{n_{k_2}}$$

so, $\sum_{i=1}^{k_1} m_i + \sum_{i=1}^{k_2} n_i \times 2 = m$

$p(\lambda) = a_m p_1 \dots p_k q_1 \dots q_{k_2}$

$$p_i = (\lambda - \lambda_i)^{m_i}$$
$$q_i = [(\lambda - z_i)(\lambda - \bar{z}_i)]^{n_i}$$

If we find solution to $Lp_i(u)=0$, $Lq_i(u)=0$ then u solves $Lp(u)=0$

$Lp_i(u)$ solution:

$$Lp_i(u) = \left(\frac{d}{dt} - \lambda_i\right)^{m_i} u = 0$$
$$= \underbrace{\left(\frac{d}{dt} - \lambda_i\right) \times \cdots \times \left(\frac{d}{dt} - \lambda_i\right)}_{m_i \text{ times}} = 0$$

now, $\left(\frac{d}{dt} - \lambda_i\right) u = 0$

$$\Rightarrow u' - \lambda_i u = 0$$
$$\Rightarrow u' = \lambda_i u$$
$$\Rightarrow u = e^{\lambda_i t}$$

now for second order

$$\left(\frac{d}{dt} - \lambda_i\right)^2 u = 0$$

let $\left(\frac{d}{dt} - \lambda_i\right) u = v$

$$\Rightarrow \left(\frac{d}{dt} - \lambda_i\right) v = 0$$
$$\Rightarrow v = e^{\lambda_i t} \Rightarrow u' - \lambda_i u = e^{\lambda_i t}$$
$$\Rightarrow (u' - \lambda_i u) e^{-\lambda_i t} = 1$$
$$\Rightarrow (u e^{-\lambda_i t})' = 1$$
$$\Rightarrow u e^{-\lambda_i t} = t + c_1$$
$$\Rightarrow u(t) = t e^{\lambda_i t} + c_1 e^{\lambda_i t}$$

Theorm: The function $e^{\lambda_i t}, t e^{\lambda_i t}, \dots t^{m_i - 1} e^{\lambda_i t}$ are m_i linearly independent solutions of $Lp_i v = 0$

proof: let's first check that

$$Lp_i(t^j e^{\lambda_i t}) = 0 \text{ if } j \leq m_i - 1$$

for $j=0$ $\quad Lp_i(e^{\lambda_i t}) = 0$

as $Lp_i v = \left(\frac{d}{dt} - \lambda_i\right)^{m_i - 1} \left(\frac{d}{dt} - \lambda_i\right)(e^{\lambda_i t}) = \left(\frac{d}{dt} - \lambda_i\right)^{m_i - 1} \times 0 = 0$

if $\forall j \leq n$ true then for $n+1$:

$$Lp_i(t^{n+1} e^{\lambda_i t}) = \left(\frac{d}{dt} - \lambda_i\right)^{m_i - 1} \left[(n+1) t^n e^{\lambda_i t} + \cancel{\lambda_i t^{n+1} e^{\lambda_i t}} - \cancel{\lambda_i t^{n+1} e^{\lambda_i t}}\right]$$
$$= \left(\frac{d}{dt} - \lambda_i\right)^{m_i - 1} (n+1) t^n e^{\lambda_i t}$$ as true for $j=n$, we get this to be zero

$= 0$, so $\forall j \leq m_i - 1$ true

Now that we know $t^j e^{\lambda_i t}$ is a solution $\forall j \leq m_i - 1$, lets show that $\{e^{\lambda_i t}, \dots t^{m_i - 1} e^{\lambda_i t}\}$ are linearly independent

$$c_1 e^{\lambda_i t} + \dots c_{m_i} t^{m_i - 1} e^{\lambda_i t} = 0 \Rightarrow c_1 + c_2 t + \dots c_{m_i} t^{m_i - 1} = 0$$

for $t=0$ $c_1=0$
then again divide by t
then $c_2=0$ and so on
so $c_i=0$ $\forall i \in \{1 \dots m_i\}$
$\Rightarrow \{e^{\lambda_i t}, \dots t^{m_i-1}e^{\lambda_i t}\}$ is linearly independent
are lin ind and soln to m_i order

Note: as $\{e^{\lambda_i t}, te^{\lambda_i t}, \dots t^{m_i-1}e^{\lambda_i t}\}$ are lin ind and soln to m_i order ODE, it is also basis to X (soln space of ODE)

$Lq_j(u)$ solution:

$$q_j = [(\lambda - z_j)(\lambda - \bar{z}_j)]^{n_j}$$

for $n_j = 1 \Rightarrow \left(\frac{d}{dt} - z_j\right)\left(\frac{d}{dt} - \bar{z}_j\right) u = 0$

now we can plug in $u = e^{z_j t}$ and its a solution

$$u = e^{(x_j + i y_j)t}$$
$$= e^{x_j t}[\cos y_j t + i \sin y_j t]$$

now, $L_{q_j} e^{z_j t} = 0 \Leftrightarrow L_{q_j}(e^{x_j t}\cos y_j t) = 0$
$L_{q_j}(e^{x_j t}\sin y_j t) = 0$

now, $\{e^{xt}\cos yt, e^{xt}\sin yt\}$ is linearly indepedent as plugging differt values of t will give $c_1, c_2 = 0$

so $L_{q_j}(u) = 0$ has degree 2 solution if $n_j = 1$
$$\{e^{x_j t}\cos y_j t, e^{x_j t}\sin y_j t\}$$

Theorm: The funtions $e^{x_j t}\sin y_j t, te^{x_j t}\sin y_j t, \dots t^{n_j-1}e^{x_j t}\sin y_j t,$
$e^{x_j t}\cos y_j t, te^{x_j t}\cos y_j t, \dots t^{n_j-1}e^{x_j t}\cos y_j t$
are $2n_j$ linearly independent solutions of $L_{q_j} u = 0$

proof: This proof is similar to the previous one, we just replace $\lambda = x_j + i y_j$, seperation to real not needed as operator does not have i, it will be same.

Theorm: A basis for the space of solution of $Lu = 0$ is given by

$$\underbrace{t^j e^{\lambda_i t}}_{\substack{i=1,2\dots k_1 \\ j=0,\dots m_i-1}}, \quad \underbrace{t^j e^{x_i t}\sin y_i t, \; t^j e^{x_i t}\cos y_i t}_{\substack{i=1,2\dots k_2 \\ j=0,\dots n_i-1}}$$

proof: It follows from proof of two theorms

Solution of inhomogenious equations:

Consider ODE $u^m(t) + a_{m-1}(t)u^{m-1}(t) + \dots + a_0(t)u(t) = f(t)$

$I \to$ open intervel, a_i are cont, $t \in I$

Variation of parameters:

let $u_1, \dots u_m$ be m linearly independent solution of the homogenious equation
then any soution u of the homogenious solution
$u = \sum c_i u_i$ $c_i \in \mathbb{R}$

we are looking for a particular solution of the given inhomogenious equation

we are saying particular solution looks like this

$$u_p(t) = c_1(t)u_1(t) + \cdots c_m(t)u_m(t)$$

$$u'_p(t) = \sum_{i=1}^{m} (\underbrace{c_i'(t)u_i(t)}_{} + c_i(t)u_i'(t))$$

$$u'_p(t) = \sum_{i=1}^{m} c_i(t)u_i'(t)$$ we want c_i's s.t $\sum c_i'(t)u_i(t) = 0$

$$0 = \sum c_i'(t)u_i(t)$$

we want c_i s.t $\sum c_i'(t)u_i(t) = 0$,

$$u_p''(t) = \sum(\underbrace{c_i'(t)u_i^2(t)}_{} + c_i(t)u_i^2(t)) = 0$$

find c_i s.t $\sum c_i' u_i^2 = 0$

$$u_p''(t) = \sum c_i(t)u_i^2(t)$$

$$0 = \sum c_i'(t)u_i^2(t)$$

Similarly:

$$0 = \sum c_i'(t)u_i^r(t) \quad \text{for } r = 0,1,\ldots m-2$$

and $$u_p^m(t) = \sum_{i=1}^{m} [\underbrace{c_i'(t)u_i^{m-1}(t)}_{} + c_i(t)u_i^m(t)]$$

Here we cannot say this is zero

u_p solves inhomogenous equation iff

$$\sum_{i=1}^{m} [c_i'(t)u_i^{m-1}(t) + c_i(t)u_i^m(t)] + a_{m-1}(t)\sum c_i(t)u_i^{m-1}(t)$$
$$\vdots$$
$$+ a_0(t)\sum c_i(t)u_i(t) = f(t)$$

$$\Leftrightarrow \sum_{i=1}^{m} c_i'(t)u_i^{m-1}(t) = f(t)$$ ($\because$ everything else is of form $u^m + a_{m-1}u^{m-1} + \cdots a_0 u = 0$)

Theorm: The function $u_p(t) = \sum_{i=1}^{m} c_i(t)u_i(t)$ is a particular solution of above ODE if $c_1, \ldots c_m$ satisfy the equations

$$W[u_1 \ldots u_m]\begin{bmatrix} c_1' \\ \vdots \\ c_m' \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ f(t) \end{bmatrix}$$ and find c_i solution

proof:

$$c_1'(t)u_1(t) + \cdots\cdots + c_m'(t)u_m(t) = 0$$
$$\vdots$$
$$c_1'(t)u_1^{m-2}(t) + \cdots\cdots + c_m'(t)u_m^{m-2}(t) = 0$$
$$c_1'(t)u_1^{m-1}(t) + \cdots\cdots + c_m'(t)u_m^{m-1}(t) = f(t)$$

are the equations we get from our calculation above

also as $\{u_1, \ldots u_m\}$ are linearly independent

$$W[u_1 \ldots u_m] \neq 0 \quad \forall t \in I$$

now,

$$\begin{bmatrix} u_1 & \cdots & u_m \\ \vdots & & \\ u_1^{m-1} & \cdots & u_m^{m-1} \end{bmatrix}\begin{bmatrix} c_1' \\ \vdots \\ c_m' \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ f(t) \end{bmatrix}$$

now, $c_i(t) = \int_{t_0}^{t} c_i'(s)\,ds$ for some $t_0 \in I$

Note: So what is happening is that given ODE, we find $\{u_1, \dots u_m\}$ L.I solutions by putting $f(t)=0$ $\forall t$, then we find u_p by the above known method to find

$v = u + u_p$ $\forall u \in X$ (solution space of $\{u_1 \dots u_m\}$)

12^{th} Aug:

Tutorial will cover more facts about real analytic functions

Theorem: let I be an open interval $\subseteq \mathbb{R}$ and $f: I \to \mathbb{R}$ be a C^∞ function then
f is Real analytic in I $\Leftrightarrow$ $\forall$ Cpt set $K \subseteq I$ $\exists r>0, M>0$ s.t $|f^j(x)| \leq M j! r^{-j}$ $\forall x \in K, \forall j = 0,1,2.....$

linear ODEs with real analytic coefficients:

$$L(u) = u^m(t) + a_{m-1}(t)u^{m-1}(t) + \cdots + a_0(t)u(t)$$

a_j for $j = \{0,1 \ldots m-1\}$ are Real analytic funtions

Theorm: Any solution of $Lu=0$ is real analytic

Eg: $u'' - tu = 0 \quad t \in \mathbb{R}$

let's look at solution of form $u(t) = \sum_{k=0}^{\infty} c_k t^k$

if u is like this then

$$u'(t) = \sum_{k=1} k c_k t^{k-1} = \sum_{k=0}^{\infty} (k+1)c_{k+1} t^k \quad R \text{ same}$$

$$u''(t) = \sum_{k=1} (k+1)(k) c_{k+1} t^{k+1} = \sum_{k=0} (k+2)(k+1) c_{k+2} t^k$$

$$\Rightarrow \sum_{k=0} (k+2)(k+1) c_{k+2} t^k - t \sum_{k=0} c_k t^k = 0$$

$$2c_2 + \sum_{k=1} [(k+2)(k+1) c_{k+2} - c_{k-1}] t^k = 0$$

$$\Rightarrow c_2 = 0 \qquad (k+2)(k+1)c_{k+2} = c_{k-1}$$

$$\Rightarrow c_5 = c_8 \cdots = 0 \quad \text{or} \quad c_{2+3k} = 0 \ \forall k \in \mathbb{N} \cup \{0\}$$

$$c_3 = \frac{c_0}{6}, \quad c_{3k} = \frac{c_0}{2.3.5 \ldots (3m-1)(3m)}$$

$$c_{3k+1} = \frac{c_1}{3.4.6 \ldots 3k(3k+1)}$$

$$u(t) = \sum_{m=0} c_0 d_{3m} t^{3m} + c_1 d_{3m+1} t^{3m+1} \qquad d_0 = 1, \ d_1 = 1$$

where $\lim \sqrt[n]{|a_n|} = 0$ in all cases as $n \to \infty$

$\Rightarrow R \to \infty$ $\therefore$ series converge

Theorm: let $I \subseteq \mathbb{R}$ open interval $a_i : I \to \mathbb{R}$ be real analytic for $i = 0, 1, \ldots. m-1$ and let $t_0 \in I$ and $\alpha_0, \ldots \alpha_{m-1} \in \mathbb{R}$ then ODE

$$Lu = u^m + a_{m-1} u^{m-1} + \cdots a_0 u = 0$$
$$u^i(t_0) = \alpha_i \quad i = 0,1 \ldots. m-1$$

has unique analytic solution

proof: For $m=2$ we will prove, same for general case we have to show below:
assume a_i has a power series expansion along t_0 in (t_0-R, t_0+R) for some $R>0$, then ODE $Lu=0$ in (t_0-R, t_0+R) has solution of form

$$u(t) = \sum_{k=0}^{\infty} c_k (t-t_0)^k \text{ in } (t_0-R, t_0+R)$$

if $m=2$ along $t_0=0$ $\quad u^2 + a_1 u' + a_0 u = 0$ —— ①
$u(0) = \alpha_0$
$u'(0) = \alpha_1$

$$a_i(t) = \sum_{k=0}^{\infty} b_k^i t^k \qquad t \in (-R, R), \ i = 0,1$$

$\forall r \in (0,R)$, $a_i(t)$ conv at $t = r$, $\exists M>0$ s.t

$|b_k^i| \le M\gamma^{-k}$ (from prev theorm) (k! not there as b_k^i and not f^i)
$\forall k = 0, 1, \dots$

lets see what the assumption
$u(t) = \sum_{k=0}^{\infty} c_k t^k$ leads to

$$u'(t) = \sum_{k=0} (k+1) c_{k+1} t^k$$

$$u^2(t) = \sum_{k=0} (k+2)(k+1) c_{k+2} t^k$$

u solves ① $\Leftrightarrow$ $\displaystyle\sum_{k=0} (k+2)(k+1) c_{k+2} t^k + \sum_{k=0}^{\infty} [\sum_{j=0}^{k} b_j^1 c_{k-j+1} (k-j+1)] t^k + \sum_{k=0}^{\infty} [\sum_{j=0}^{k} b_j^0 c_{k-j}] t^k = 0$

$$\Leftrightarrow (k+2)(k+1) c_{k+2} = -\sum_{j=0}^{k} [b_j^1 (k-j+1) c_{k-j+1} + b_j^0 c_{k-j}] \quad \text{②} \quad \forall k = 0, 1, \dots$$

$$\left.\begin{aligned} u(0) &= \alpha_0 \Rightarrow c_0 = \alpha_0 \\ u'(0) &= \alpha_1 \Rightarrow c_1 = \alpha_1 \end{aligned}\right\} \text{③}$$

now, to show $\sum_{k=0}^{\infty} c_k t^k$ conv for any $t \in (-R, R)$

we will find some constants d_k
$k = 0, 1, \dots$ s.t
$|c_k| \le d_k$ and radius of conv $\sum d_k t^k$ is $\ge R$

choose $d_0 = |\alpha_0|$ $d_1 = |\alpha_1|$ from ③

$$(k+2)(k+1)|c_{k+2}| \le \sum_{j=0}^{k} \left[|b_j^1||c_{k-j+1}|(k-j+1) + |b_j^0||c_{k-j}| \right]$$

$$\le \sum_{j=0}^{k} [|b_{k-j}^1||c_{j+1}|(j+1) + |b_{k-j}^0||c_j|]$$

as $|b_k^j| \le M\gamma^{-k}$

$$\le \sum_{j=0}^{k} M\gamma^{j-k} [(j+1)|c_{j+1}| + |c_j|]$$

so, $\displaystyle (k+2)(k+1)|c_{k+2}| \le M \sum_{j=0}^{k} \gamma^{j-k} [(j+1)|c_{j+1}| + |c_j|]$

now for $k = 2, 3, \dots$ using relation

$$(k+2)(k+1) d_{k+2} = \frac{M}{\gamma^k} \sum_{j=0}^{k} \gamma^j [(j+1) d_{j+1} + d_j] + M\gamma d_{k+1}$$

now, as $d_0 = |\alpha_0| \ge |c_0|$
$d_1 = |\alpha_1| \ge |c_1|$
true for 0, 1 now if true till $k+1$ then for $k+2$:

$$(k+3)(k+2) d_{k+3} = \frac{M}{\gamma^{k+1}} \sum_{j=0}^{k+1} \gamma^j [(j+1) d_{j+1} + d_j] + M\gamma d_{k+2}$$

and $|c_i| \le d_i$ $\forall i = 0, 1, \dots k+1$

$$(k+2)(k+1)|c_{k+2}| \le M \sum_{j=0}^{k} \gamma^{j-k} [(j+1)|c_{j+1}| + |c_j|]$$

$$\le M \sum_{j=0}^{k} \gamma^{j-k} [(j+1) d_{j+1} + d_j]$$

$$\le (k+2)(k+1) d_{k+2} - M\gamma d_{k+1} \le (k+2)(k+1) d_{k+2}$$

$\Rightarrow |c_{k+2}| \le d_{k+2}$, so by induction
$|c_i| \le d_i$ $\forall i \in \mathbb{N} \cup \{0\}$

now we have to show $d_k \le$ something nice to get conv with $R' \ge R$

19th Aug:

Theorem: let $I \subseteq \mathbb{R}$ open interval $a_i : I \to \mathbb{R}$ be real analytic for $i = 0, 1, \dots, m-1$ and let $t_0 \in I$ and $\alpha_0, \dots, \alpha_{m-1} \in \mathbb{R}$ then ODE

$$Lu = u^m + a_{m-1}u^{m-1} + \dots a_0 u = 0$$
$$u^i(t_0) = \alpha_i \quad i = 0, 1, \dots, m-1$$

has unique analytic solution

proof: we have already showed that $m=2$ (not wlog)

let $u(t) = \sum_{i=0}^{\infty} c_i t^i$ ← wlog case of $t_0 = 0$, we have to show that it converges $t \in (-R, R)$

also $|c_i| \leq d_i \ \forall i \in \mathbb{N} \cup \{0\}$ shown using induction

where $a_i(t) = \sum_{k=0}^{\infty} b_k^i t^k$, and we defined d_j:

$$(k+3)(k+2)d_{k+3} = \frac{M}{\gamma^{k+1}} \sum_{j=0}^{k+1} \gamma^j \left[(j+1)d_{j+1} + d_j\right] + M\gamma d_{k+2}$$

when we got

$$u(0) = \alpha_0 = c_0$$
$$u'(0) = \alpha_1 = c_1$$

$$(k+2)(k+1)|c_{k+2}| \leq \sum_{j=0}^{k} \left[|b'_{k-j}|\,|c_{j+1}|(j+1) + |b^0_{k-j}|\,|c_j|\right]$$

$$|b_k^i| \leq M\gamma^{-k}$$

$$(k+2)(k+1)|c_{k+2}| \leq \frac{M}{\gamma^k} \sum \gamma^j \left((j+1)|c_{j+1}| + |c_j|\right)$$

we showed $|c_k| \leq d_k$ by induction

now we want to show that $\sum_{k=0}^{\infty} d_k t^k$ will converge $\forall \gamma < R$

$$(k+1)k\,d_{k+1} = \frac{M}{\gamma^{k-1}}\left[\sum_{j=0}^{k-1} \gamma^j \left((j+1)d_{j+1} + d_j\right) + \cancel{\frac{M\gamma d_k \times \gamma^{k-1}}{M}}\right]$$

$$(k)(k-1)d_k = \frac{M}{\gamma^{k-2}}\left[\sum_{j=0}^{k-2} \gamma^j \left((j+1)d_{j+1} + d_j\right) + \gamma^{k-1} d_{k-1}\right]$$

↙ multiplying by γ

$$\text{now } (k)(k+1)\gamma d_{k+1} = \frac{M}{\gamma^{k-2}} \sum_{j=0}^{k-1} \left[\gamma^j (j+1) d_{j+1} + \gamma^j d_j\right] + \gamma^2 M d_k$$

$$= \left(\frac{M}{\gamma^{k-2}} \sum_{j=0}^{k-2} \left[\gamma^j (j+1) d_{j+1} + \gamma^j d_j\right] + M\gamma d_{k-1}\right)$$
$$- M\gamma d_{k-1} + \frac{M}{\gamma^{k-2}}\left(\gamma^{k-1}(k) d_k + \gamma^{k-1} d_{k-1}\right) + \gamma^2 M d_k$$

$$= (k)(k-1)d_k - M\gamma \cancel{d_{k-1}} + M\gamma k d_k \cancel{+ M\gamma d_{k-1}} + \gamma^2 M d_k$$

$$= (k)(k-1)d_k + M\gamma k d_k + \gamma^2 M d_k$$

$$\text{now, } \frac{d_{k+1}}{d_k} = \frac{(k)(k-1)}{(k)(k+1)\gamma} + \frac{M\gamma k}{(k)(k+1)\gamma} + \frac{\gamma^2 M}{(k)(k+1)\gamma}$$

$$\lim_{k \to \infty} \left|\frac{d_{k+1}}{d_k}\right| = \frac{1}{|\gamma|} \Rightarrow \forall\, t < |\gamma| \text{ the seq converges}$$

for $|\gamma| \in (0, R)$ series converges for $t_0 = 0$, this can be done for any t from $\mathbb{R}$, so $v(t)$ is also real analytic

Legendre Equation:

The legendre equation is $(1-t^2)u^{(2)} - 2tu^{(1)} + \alpha(\alpha+1)u = 0$, where $\alpha \in \mathbb{R}$, $|t| < 1$

22nd Aug:

Legendre Equation:

$$(1-t^2)u'' - 2tu' + \alpha(1+\alpha)u = 0 \quad \alpha \in \mathbb{R}$$

assuming $|t|<1$:

$$u'' - \frac{2t}{1-t^2}u' + \alpha\frac{(1+\alpha)}{1-t^2}u = 0$$

now $-\frac{2t}{1-t^2} = -2t\sum_{k=0}^{\infty} t^{2k} \quad |t|<1$

$$\frac{\alpha(1+\alpha)}{1-t^2} = \alpha(1+\alpha)\sum_{k=0}^{\infty} t^{2k} \quad |k|<1$$

for our theorem, any solution will be of form

$$u(t) = \sum_{k=0}^{\infty} c_k t^k \quad \text{as coeff are real analytic (theorem done)}$$

$$u''(t) = \sum_{k=1}^{\infty}(k+1)kc_{k+1}t^{k-1} = \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}t^k$$

$$u'(t) = \sum_{k=0}^{\infty}(k+1)c_{k+1}t^k$$

now, $(1-t^2)u'' - 2tu' + \alpha(\alpha+1)u = 0$

$$\Rightarrow (1-t^2)\left(\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}t^k\right) - 2t\left(\sum_{k=0}(k+1)c_{k+1}t^k\right) + \alpha(\alpha+1)u = 0$$

$$\Rightarrow \sum(k+2)(k+1)c_{k+2}t^k - \sum(k+2)(k+1)c_{k+2}t^{k+2}$$
$$- 2\sum(k+1)c_{k+1}t^{k+1} + \alpha(\alpha+1)u = 0$$

on calculation (in uploaded notes online)

$$\Rightarrow \sum_{k=0}^{\infty}(k+1)(k+2)c_{k+2}t^k - \sum_{k=0}^{\infty}k(k-1)c_kt^k - \sum_{k=0}^{\infty}2kc_kt^k + \alpha(\alpha+1)\sum_{k=0}^{\infty}c_kt^k = 0$$

on computation we get: (calculation in prof's notes)

$$(k+1)(k+2)c_{k+2} - k(k-1)c_k - 2kc_k + \alpha(\alpha+1)c_k = 0$$
$$k = 0, 1, 2 \dots$$

$$\Rightarrow c_{2n+2} = \frac{(-1)^n c_0(\alpha+2n-1)\cdots(\alpha+1)\,\alpha\cdots(\alpha-2n+2)}{(2n)!}$$

$$c_{2n+1} = \frac{(-1)^n c_1(\alpha+2n)(\alpha+2n-2)\cdots(\alpha+2)(\alpha-1)(\alpha-3)\cdots(\alpha-2n+1)}{(2n+1)!}$$

we can get two lin ind solutions as

$$u_1(t) = \sum_{k=0}^{\infty}c_{2k}t^{2k} \quad c_1 = 0 \leftarrow \text{when } c_1 = 0$$

$$u_2(t) = \sum_{k=0}^{\infty}c_{2k+1}t^{2k+1} \quad c_0 = 0 \leftarrow \text{when } c_0 = 0$$

the case when $\alpha \in \mathbb{N}$,

Case I: $\alpha = 2m$: then in this case

$$c_{2(m+j)} = 0 \quad \forall j = 1, 2 \dots$$

$\Rightarrow u_1(t)$ is polynomial of degree $2m$

Case II: $\alpha = 2m+1$

$u_1(t)$ is power series

$u_2(t)$ is degree $(2m+1)$

Defn: The polynomial solution of the degree n, Legendre ODE

$$(1-t^2)u'' - 2tu' + (n)(n+1) = 0 \qquad (\alpha = n)$$

satisfying $P_n(1) = 1$, is called the n^{th} Legendre polynomial

$V(t) = \frac{d^n}{dt}(t^2-1)^n$, $P_n(t) = \frac{1}{2^n n!} V(t) \rightarrow$ this is the nth Legendre polynomial

$W(t) = (t^2-1)^n$, $W(t)$ satisfies ODE $(t^2-1)W' - 2ntW = 0$

now if we diff ODE $(n+1$ times$)$ (calculation in prof's notes)

$$\frac{d^k}{dt^k}[(t^2-1)W'(t)] = (t^2-1)W^{k+1}(t) + 2ktW^k(t) + k(k-1)W^{k-1}(t)$$

$\frac{d^k}{dt^k}[tW(t)] = tW^k(t) + kW^{(t-1)}(t)$, then we put $k = n$ and one term will cancel out, after differentiating 1 more time to get $(t^2-1)W^{(n+2)}(t) + 2tW^{(n+1)}(t) - n(n+1)W^{(n)}(t) = 0$

Linear ODE with singular point:

$$a_m(t)u^m + a_{m-1}(t)u^{m-1} + \dots + a_0(t)u = 0$$

Defn: (singular point of ODE) A point t_0 s.t $a_m(t_0) = 0$ is called singular point of ODE

eg: $t^2u'' - u' - \frac{3}{4}u = 0$ $t = 0$ singular point, if we assume there is a power series expansion then

$$u = \sum_{k=0}^{\infty} c_k t^k$$

on calculating $\left((k)(k-1) - \frac{3}{4}\right)c_k = (k+1)c_{k+1}$

if $c_0 = 0 \Rightarrow c_k = 0 \ \forall k \Rightarrow u \equiv 0$

if $c_0 \neq 0 \Rightarrow c_k \neq 0 \ \forall k$ as $(k)(k+1) \neq 3/4$ for any $k \in \mathbb{N} \cup \{0\}$

the power series

$$\left|\frac{c_{k+1}}{c_k}\right| = \left|\frac{(k)(k-1) - 3/4}{(k+1)}\right| \longrightarrow \infty \text{ as } k \to \infty$$ this is problematic

Defn: t_0 is said to be a regular singular point for ODE if it can be written in the form

$$(t-t_0)^m u^m(t) + a_{m-1}(t)(t-t_0)^{m-1}u^{m-1}(t) + \dots + (t-t_0)a_1(t)u'(t) + a_0(t)u(t) = 0$$

where $a_0, a_1 \dots a_{m-1}$ are real analytic

Euler's equation:

$a, b \in \mathbb{R}$

$Lu = t^2u'' + atu' + bu = 0$ this has regular singular point at $t = 0$

let us look at the solution of form $u(t) = t^\gamma$

$$L(t^\gamma) = [\gamma(\gamma-1) + a\gamma + b]t^\gamma = q(\gamma)t^\gamma$$

where $q(\gamma) = \gamma(\gamma-1) + a\gamma + b$

if $q(\gamma) = 0$ then $L(t^\gamma) = 0$, this q is called indicial polynomial

let γ_1, γ_2 be roots of $q(\gamma)$

Case I: $\gamma_1, \gamma_2 \in \mathbb{R}$ & $\gamma_1 \neq \gamma_2$

then $u_1(t) = t^{\gamma_1}$, $u_2(t) = t^{\gamma_2}$ and they are lin independent are solutions on $(0, \infty)$

Case II: $r_1, r_2 \in \mathbb{C}$, $r_1 = a+ib$
$r_2 = a-ib$

$$u_1(t) = t^{r_1} = t^{a+ib}$$
$$= e^{(a+ib)\log t}$$
$$= e^{a\log t}\cos(b\log t)$$
$$u_2(t) = e^{a\log t}\sin(b\log t)$$

for $t \in (0,\infty)$

<u>Case III</u>: $r_1 = r_2 \in \mathbb{R}$ then $q(r) = (r-r_1)^2$ $\quad q(r_1)=0$
$q'(r_1)=0$ as roots repeated

$$L(t^r) = q(r)t^r$$
$$\frac{d}{dr}L(t^r) = L\left(\frac{d}{dr}t^r\right) \quad \left(\because f(x,y) \text{ then } \partial_x \partial_y f = \partial_y \partial_x f \text{ and so similar to this as } t^r \text{ is smooth}\right)$$
$$= L(e^{r\log t}\log t)$$
$$= L(t^r \log t)$$
$$q'(r)t^r + q(r)t^r \log t = 0 \text{ for } r = r_1$$

$\Rightarrow L(t^{r_1}\log t) = 0$
$\Rightarrow t^{r_1}\log t$ is also a soluti

t^{r_1} is a solution (trivial)

so, $u_1(t) = t^{r_1}$
$u_2(t) = t^{r_1}\log t$

26^{th} Aug:

Second order ODEs with singular points:

$$(t-t_0)^2u^2(t)+(t-t_0)\,a(t)u'(t)+b(t)u(t)=0$$

$a(t)$, $b(t)$ → Real analytic

wlog $t_0=0$ then:

$L(u)=t^2u^2(t)+t\,a(t)u'(t)+b(t)u(t)=0$ where a & b have power series expansions

$$a(t)=\sum_{k=0}^{\infty} a_k t^k \qquad |t|<R$$
$$b(t)=\sum_{k=0}^{\infty} b_k t^k$$

$$u(t)=t^r\sum_{k=0}^{\infty} c_k t^k=\sum_{k=0}^{\infty} c_k t^{r+k}$$

we calculate $u'(t)$, $u^2(t)$ (calculations in notes) we get

$$c_k(k+r)(k+r-1)+\sum_{j=0}^{k}(a_{k-j}c_j(j+r))+\sum_{j=0}^{k}c_jb_{k-j}=0$$ ← Recurrence relation

at $k=0$:

$$c_0(r)(r-1)+c_0(a_0r)+c_0(b_0)=0$$

$$\Rightarrow c_0[r(r-1)+ra_0+b_0]=0$$

now let

$$q(r)=r(r-1)+a_0r+b_0$$ ← Indicial polynomial

so now the recursive relation become:

$$q(r)c_0=0$$
$$q(r+k)c_k=-\sum_{j=0}^{k-1}[(r+j)a_{k-j}+b_{k-j}]c_j \quad \text{—— (1)}$$

let r_1, r_2 be roots of $q(r)=0$

wlog $\mathrm{Re}\, r_1 \geq \mathrm{Re}\, r_2$, then if we take $r=r_1$ then for any given c_0 we can define c_k

Note: we are considering $t\neq 0$, $r(r-1)+a_0r+b_0=0$, $1-a_0\geq 0$ for $\mathrm{Re}(r_1),\mathrm{Re}(r_2)\geq 0$ (notes)

$u(t)=t^{r_1}\sum_{k=0} c_kt^k$ converges in $|t|<R$, this is a solution is trivial as by construction it solves

converges is provable using c_k, d_k method (proof in prof's notes)

Case I: $r_1-r_2\notin \mathbb{N}\cup\{0\}$ then

start with any c_0, we can define c_k and can get

$$u(t)=t^{r_2}\sum_{k=0}^{\infty}c_kt^k \qquad |t|<R$$ (proof of conv in prof's notes)

Case II: $r_1=r_2 \Rightarrow q(r_1)=q'(r_1)=0$

If we take $c_0=1$, then we can find a solution of form $u_1(t)=t^{r_1}\sum_{k=0}c_kt^k \quad |t|<R$

and now let $r\in(r_1-\varepsilon, r_1+\varepsilon)$ let $c_0=1$ (wlog)

$$q(r+k)c_k(r)=-\sum_{j=0}^{k-1}[a_{k-j}(j+r)+b_{k-j}]c_j(r)$$

c_k depends on r

then $L(t^r\sum_{k\geq 0} c_k(r)t^k)=q(r)t^r$ ($\because$ because of recursion everything else vanishes)

$$L\left(\frac{d}{dr}\bigg|_{r=r_1}\left[t^r\sum c_k(r)t^k\right]\right)=q'(r)t^r+q(r)\,r\,t^{r-1} \quad \text{at } r=r_1$$

$q'(r)t^r \to 0$, $q(r)\,r\,t^{r-1} \to 0$

$=0$

so, second solution is given by

$$u_2(t) = \frac{d}{dr}\Big|_{r=r_1} \left(t^r \sum_{k\geq 0} c_k(r) t^k \right)$$

$$= t^{r_1} \sum_{k\geq 0} c_k'(r_1) t^k + \log(t)\, u_1(t)$$

this has to converge (see notes)

Case III: $r_1 = r_2 + m$ where $m \in \mathbb{N}$

$$u_1(t) = t^{r_1} \sum_{k\geq 0} c_k t^k$$

$$\sum (r+k) c_k = -\sum_{j=0}^{k-1} \left((r+j) a_{k-j} + b_{k-j} \right) c_j$$

if $c_0 = 0$ then $c_0 = 0 \cdots = c_{m-1}$

but $\underbrace{q(r_2+m)}_{0} c_m = 0$

$0 \times c_m = 0$

c_m will be undefined

so we can choose any value of c_m

if we do not choose $c_0 = 0$ i.e let $c_0 = 1$ (wlog)

$c_1, c_2, \ldots, c_{m-1}$ are determined but

$0 \times c_m = f(\underbrace{c_0, \ldots c_{m-1}}_{\text{known values}})$ if $\neq 0$ then

$0 \times c_m \neq 0$

this is an issue and no solution this way

29^{th} Aug:

Fix to prev issues: we used uniform convergence theorem, and by diff $\sum \partial_\gamma c_k(\gamma) t^k$ to get a new recurrence formula and use induction to show convergence (see notes)

Existence / Uniqueness for the initial value:

consider the initial value problem

$$u^m(t) = f(t, u(t), u'(t), \dots u^{m-1}(t)) \text{ (not a vector)}$$
$$u(t_0) = \alpha_0, \dots u^{m-1}(t_0) = \alpha_{m-1}$$

let's define $v_1 = u$, $v_2 = u'$, ... $v_m = u^{m-1}$

$$v_1' = v_2$$
$$v_2' = v_3$$
$$v_3' = v_4$$
$$\vdots$$
$$v_{m-1}' = v_m$$
$$v_m' = f(t, v_1, v_2, v_3 \dots v_{m-1})$$
$$v_1(t_0) = \alpha_0$$
$$v_2(t_0) = \alpha_1$$
$$\vdots$$
$$v_m(t_0) = \alpha_{m-1}$$

this converts given into system of first order ODEs from defn and

vector form conversion

conversely, suppose $v_1, \dots v_n$ are differentiable & solve given, then let $u(t) = v_1(t)$ then u is m times diff & solves given.

Defn: A first order system of ODE of form $u'(t) = f(t, u(t))$ where $u(t) = (u_1(t), u_2(t), u_3(t), \dots u_m(t))$

$u_i : I \to \mathbb{R}$ and $f(t, u(t)) = (f_1(t, u(t)), f_2(t, u(t)), \dots f_m(t, u(t)))$

$$f_i : I \times \Omega \to \mathbb{R}$$
where $\Omega \subseteq \mathbb{R}^m$

$f(t, u) = u' \Rightarrow f(t, u_1, u_2 \dots, u_m) = (u_1', u_2', \dots u_m')$ i.e $f: I \times \Omega \to \mathbb{R}^m$, $\Omega \subseteq \mathbb{R}^m$

Notation: $X = (x_1, \dots x_m)$

$$|X| = \left(\sum_{j=1}^{m} x_j^2\right)^{1/2}$$

$$u' = (u_1', u_2', \dots u_m')$$

$$\int_a^b u(t)\,dt = \left(\int_a^b u_1(t)\,dt, \dots, \int_a^b u_m(t)\,dt\right)$$

Defn: let X be a non-empty set and $\{f_\alpha\}_{\alpha \in J}$ be a family of maps $f_\alpha : X \to \mathbb{R}^m$ then the family $\{f_\alpha\}_{\alpha \in J}$ is said to be

(a) pointwise bounded if $\forall x \in X$, $\exists M_x > 0$ s.t

$$|f_\alpha(x)| \leq M_x \quad \forall \alpha \in J$$

(b) uniformly bounded if $\exists M > 0$ s.t
$$|f_\alpha(x)| \leq M \quad \forall x \in X, \alpha \in J$$

Defn: let (X,d) be a metric space & $\{f_\alpha\}_{\alpha\in J}$ be a family of maps $f_\alpha : X \longrightarrow \mathbb{R}$ then the family f_α is said to be equicontinious if $\forall \varepsilon>0, \exists \delta>0$ s.t

$$\forall x,y\in X \text{ s.t } d(x,y)<\delta \Rightarrow |f_\alpha(x)-f_\alpha(y)|<\varepsilon \quad \forall \alpha\in J$$

Eg: $\{x^n\}_{n\in\mathbb{N}}$ is a family of function which is <u>not</u> equicontinious

Theorm : (Arzela Ascoli) let (X,d) be a compact metric space & $f_n : X\longrightarrow \mathbb{R}^m$ $n\in\mathbb{N}$, be a sequence of cont. function, suppose f_n is pointwise bdd & equicontinious, then

(I) f_n is uniformly bdd
(II) f is a continious function $f: X\longrightarrow\mathbb{R}^m$ & $\exists$ a subsequence f_{n_i} s.t

$$f_{n_i}\longrightarrow f \text{ uniformly}$$

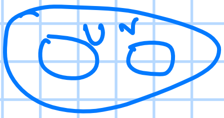
let $u' = f(t,u(t)), u(t_0)=x$, f is cont —— ①

By cont we mean $x_t\to x$ then $f(x_t)\longrightarrow f(x)$

lemma: $\Omega\subseteq\mathbb{R}^m$ is an open set, $u: I\longrightarrow \Omega$ is a solution of $u' = f(t,u(t))$ iff u is continious and satisfies

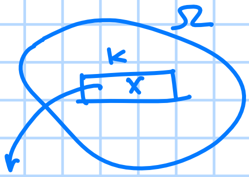
$$u(t) = x + \int_{t_0}^{t} f(s,u(s))ds \quad \forall t\in I$$

proof : Follows from fundamental theorm of calculus



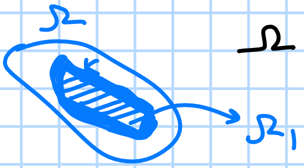
$\text{dist}(U,V) = \inf\{d(x,y) \mid x\in U, y\in V\}$

Theorm : (Peano's existance) $\Omega\subseteq\mathbb{R}^m$ be open, I open on $\mathbb{R}$, $f: I\times\Omega\longrightarrow\mathbb{R}^m$ continious, let K be a compact subset of Ω. $[a,b]\subseteq I$ where $-\infty<a<b<\infty$, then $\exists\varepsilon>0$ s.t the IVP $u' = f(t,u(t))$ has a solution for $t\in(t_0-\varepsilon, t_0+\varepsilon)$ & $x\in K$, ε depends on $[a,b]$ and K but not on x.



Region of initial values

proof : given that K is a compact subset of Ω, we have $d(K,\mathbb{R}^m\setminus\Omega)>0$ choose $R>0$ s.t $R<d(K,\mathbb{R}^m\setminus\Omega)$, define



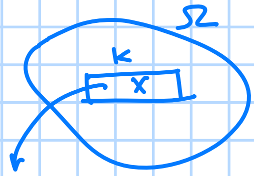
$\Omega_1 := \{x\in\Omega \mid d(x,K)<R\}$ then Ω_1 is open and $\bar{\Omega}_1$ is compact subset of Ω

choose $\varepsilon_0>0$ s.t $[a-\varepsilon_0, b+\varepsilon_0]\subseteq I$
let $M = \max\{|f(t,x)| \mid t\in[a-\varepsilon_0, b+\varepsilon_0], x\in\bar{\Omega}_1\}$

now let $\varepsilon = \min\left(\frac{R}{M},\varepsilon_0\right)$, now we will show picard's itteration

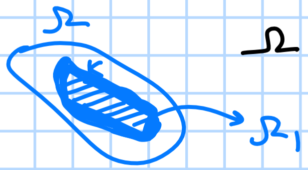
2nd sept :

Theorem: (Peano's existence) $\Omega \subseteq \mathbb{R}^m$ be open, I open on $\mathbb{R}$, $f: I \times \Omega \longrightarrow \mathbb{R}^m$ continuous, let K be a compact subset of Ω. $[a,b] \subseteq I$ where $-\infty < a < b < \infty$, then $\exists \varepsilon > 0$ s.t the IVP $u' = f(t, u(t))$ has a solution for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ & $x \in K$, ε depends on $[a,b]$ and K but not on x.



Region of initial values

Proof: given that K is a compact subset of Ω, we have $d(K, \mathbb{R}^m \setminus \Omega) > 0$
choose $R > 0$ s.t $R < d(K, \mathbb{R}^m \setminus \Omega)$, define



$\Omega_1 := \{ x \in \Omega \mid d(x, K) < R \}$ then Ω_1 is open and $\bar{\Omega}_1$ is compact subset of Ω

choose $\varepsilon_0 > 0$ s.t $[a - \varepsilon_0, b + \varepsilon_0] \subseteq I$
let $M = \max\{ |f(t,x)| \mid t \in [a - \varepsilon_0, b + \varepsilon_0], x \in \bar{\Omega}_1 \}$

now let $\varepsilon = \min(\frac{R}{M}, \varepsilon_0)$, now we will show picard's iteration

now we will construct a sequence u_n of "approximate solution"
let $t_i^n = t_0 + \frac{i\varepsilon}{n}$ $0 \quad \varepsilon/n$

now we define $u_n : [t_0, t_0 + \varepsilon] \longrightarrow \mathbb{R}^m$ as

$$u_n(t) = \begin{cases} \underbrace{u(t_0^n)}_{x} + f(t_0^n, u(t_0^n))(t - t_0^n) ; & t \in [t_0^n, t_1^n] \\ u(t_1^n) + f(t_1^n, u(t_1^n))(t - t_1^n) ; & t \in [t_1^n, t_2^n] \\ \vdots \end{cases}$$

more compactly $u_n(t) = u_n(t_i^n) + f(t_i^n, u(t_i^n))(t - t_i^n)$ for $t \in [t_i^n, t_{i+1}^n]$
now, we claim for any $n \in \mathbb{N}$,
claim 1: $u_n(t) \in \bar{\Omega}_1 \ \forall t \in [t_0, t_0 + \varepsilon]$ ← we get Bounded ness
claim 2: $|u_n(t) - u_n(s)| \leq M|t - s| \ \forall t, s \in [t_0, t_0 + \varepsilon]$
← we get equicontinuity

so once shown we can apply Arzela-Ascoli

for case I: let $t \in [t_0^n, t_1^n]$ then $|u_n(t) - x| = |f(t_0^n, x)||t - t_0|$

$$\leq M\varepsilon/n$$
$$\leq \frac{\cancel{M}}{n} \times \frac{R}{\cancel{M}} \quad \text{as } \varepsilon \leq R/M$$

so $|u_n(t) - x| \leq R/n < R$
$\Rightarrow u_n(t) \in \bar{\Omega}_1$

now, for $t \in [t_1^n, t_2^n]$ then

$$|u_n(t) - x| \leq |u_n(t) - u_n(t_1^n)| + \underbrace{|u_n(t_1^n) - u_n(t_0^n)|}_{\leq R/n}$$
$$\leq |f(t_1^n, u(t_1^n))||t - t_1^n| + R/n$$
$$\leq R/n + R/n$$
$$\leq R$$

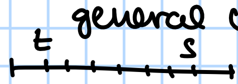
$\Rightarrow u_n(t) \in \bar{\Omega}_1$

Similarly $\forall t \in [t_0, t_0 + \varepsilon]$ we get
$u_n(t) \in \bar{\Omega}_1$

now, case II: for $|u_n(t)-u_n(s)| \leq M|t-s|$, $\forall t,s \in [t_0, t_0+\varepsilon]$

suppose $t,s \in [t_i^n, t_{i+1}^n]$ for some $i \leq n-1$

general case



$$|u_n(t)-u_n(s)| = |u_n(\cancel{t_i^n}) + f(t_i^n, u(t_i^n))(t-t_i^n) - u_n(\cancel{t_i^n}) - f(t_i^n, u(t_i^n))(s-t_i^n)|$$

$$= |t-s|\,|f(t_i^n, u(t_i^n))| \leq M|t-s|$$

$$\Rightarrow |u_n(t)-u_n(s)| \leq M|t-s|$$

now let $t \in [t_i^n, t_{i+1}^n]$ & $s \in [t_j^n, t_{j+1}^n]$, $i<j$ (wlog) then

$$|u_n(t)-u_n(s)| \leq |u_n(t)-u_n(t_{i+1}^n)| + |u_n(t_{i+1}^n)-u_n(t_{i+2}^n)| + \cdots + |u_n(t_j^n)-u_n(s)|$$

$$\leq M[|t-t_{i+1}^n| + \cdots + |t_j^n - s|]$$

$$= M[s-\cancel{t_j^n}+\cancel{t_j^n}-\cancel{t_{j-1}^n}+\cdots+\cancel{t_{i+1}^n}-t]$$

$$= M[s-t]$$

$$\Rightarrow |u_n(t)-u_n(s)| \leq M|t-s|$$

from two claims done above

By Arzela-Ascoli, $\exists$ a subsequence u_{n_i} s.t

$u_{n_i} \longrightarrow u$ uniformly, u is continious

where $u:[t_0, t_0+\varepsilon] \longrightarrow \Omega_1$ is continious

as $u_{n_i} : [t_0, t_0+\varepsilon] \longrightarrow \Omega_1$

now if we show $u(t) = x + \int_{t_0}^{t} f(s, u(s))\,ds$ $\forall t \in [t_0, t_0+\varepsilon]$ ($\because$ theorm done previously)

for $t \in [t_0, t+\varepsilon]$

$\forall n_i$, $t \in [t_k^{n_i}, t_{k+1}^{n_i}]$

as per defn

$$u_{n_i}(t) = u_{n_i}(t_k^{n_i}) + f(t_k^{n_i}, u(t_k^{n_i}))(t-t_k^{n_i})$$

by telescopic sum

$$u_{n_i}(t) - x = u_{n_i}(t) - u_{n_i}(t_k^{n_i}) + u_{n_i}(t_k^{n_i}) - u_{n_i}(t_{k-1}^{n_i}) + \cdots + u_{n_i}(t_1^{n_i}) - \underbrace{u_{n_i}(t_0^{n_i})}_{x}$$

$$= \int_{t_k^{n_i}}^{t} u_{n_i}'(s)\,ds + \sum_{j=1}^{k} \int_{t_{j-1}^{n_i}}^{t_j^{n_i}} u_{n_i}'(s)\,ds$$

$$\Rightarrow u_{n_i}(t) = x + \sum_{j=1}^{k} \int_{t_{j-1}^{n_i}}^{t_j^{n_i}} u_{n_i}'(s)\,ds + \int_{t_k^{n_i}}^{t} u_{n_i}'(s)\,ds$$

$$\Rightarrow u_{n_i}(t) = x + \sum_{j=1}^{k} \int_{t_{j-1}^{n_i}}^{t_j} f(t_{j-1}^{n_i}, u_{n_i}(t_{j-1}^{n_i}))\,ds + \int_{t_k^{n_i}}^{t} f(t_k^{n_i}, u_{n_i}(t_k^{n_i}))\,ds$$

$$\Rightarrow u_{n_i}(t) = x + \sum_{j=1}^{k} \int_{t_k^{n_i}}^{t} f(t_k^{n_i}, u_{n_i}(t_k^{n_i})) - f(t_k^{n_i}, u(t_k^{n_i}))\,ds \quad A_i$$

$$+ \int_{t_k^{n_i}}^{t} f(t_k^{n_i}, u_{n_i}(t_k^{n_i})) - f(t_k^{n_i}, u(t_k^{n_i}))\,ds \quad B_i$$

$$+\sum_{i=1}^{k}\int_{t_k^{n_i}}^{t} f(t_k^{n_i},u(t_k^{n_i}))\,ds + \int_{t_k^{n_i}}^{t} f(t_k^{n_i},u(t_k^{n_i}))\,ds \to C_i^\circ$$

$f:[a,b]\times\bar{\Omega}_1$ is uniformly cont so (proper proof in notes)
as $n_i\to\infty$ we get $A_i^\circ\to 0$, $B_i^\circ\to 0$ and by riemman

$$C_i^\circ \to \int_{t_0}^{t} f(s,u(s))\,ds$$

so, $u(t) = x+\int_{t_0}^{t} f(s,u(s))\,ds$ and so we get
$u'(t)=f(t,u(t))$, observe this proofs that is a solution in $[t_0,t_0+\varepsilon]$
we do same for $[t_0-\varepsilon,t_0]$ to get same result

Non-uniqueness of solution:

$u'(t)=\sqrt{u(t)}$ $u(0)=0$ then $u(t)\equiv 0$ is a solution
& $u_\alpha(t)=\begin{cases}\frac{(t-\alpha)^2}{4}; & t\geq\alpha\\ 0; & t<\alpha\end{cases}$ are solutions

as $u_\alpha'(t)=\frac{(t-\alpha)}{2}$ $\sqrt{u_\alpha(t)}=\left|\frac{t-\alpha}{2}\right|=\frac{t-\alpha}{2}$ as $t\geq\alpha$
so $u_\alpha'(t)=(u_\alpha(t))^{1/2}$

even if f is continious
we didnt get unique solution
even two solution exist

12th sept:

IVP: $u'(t) = f(t, u(t))$, $u(t_0) = x_0$

$u: I \to \mathbb{R}^m$, $t_0 \in I$, $f: D \to \mathbb{R}^m$ is a given function on open set $D = \mathbb{R} \times \mathbb{R}^m$ —— ①
$\uparrow$ interval

Note: we have seen ① $\Leftrightarrow u(t) = x_0 + \int_{t_0}^{t} f(s, u(s))\,ds$

let (X, d) when $d(f,g) = \|f-g\|_\infty$, then we can show $\|\ \|_\infty$ is a norm i.e $\|x\|_\infty = 0 \Leftrightarrow x = 0$, triangular inequality
$X = C([a,b], \mathbb{R}^m)$ is a complete metric space / Banach space

$(X, \|\ \|_\infty)$ is a complete non-linear space
→ distance preset, not angles, Hilbert space (angles)

where $\|f\|_\infty = \max_{t \in [a,b]} \|f(t)\|$ (not sup as compact $[a,b]$)
sup is weaker than max

Banach's contraction principle:

Defn: A map $T: (X,d) \to (X,d)$ is a contraction if $\exists q \in [0,1)$ s.t

$$d(Tu, Tv) \leq q\, d(u,v) \quad \forall u, v \in X$$

Theorm: (Banach fixed point) let (X,d) be complete and $T: X \to X$ be a contraction, then

(i) $\exists$ a unique fixed point x^* and $Tx^* = x^*$

(ii) $\forall x_0 \in X$, $x_{k+1} = Tx_k$ converge to x^*

(iii) error estimate:
$$d(x_k, x^*) \leq \frac{q^k}{1-q} d(x_1, x_0)$$

proof: Define $x_{k+1} = Tx_k$ then

$$d(x_{k+1}, x_k) \leq q\, d(x_k, x_{k-1}) \cdots \leq q^k d(x_1, x_0)$$
for $m > k$
$$d(x_m, x_k) \leq \sum_{j=k}^{m-1} d(x_{j+1}, x_j) \leq d(x_1, x_0) \sum_{j=k}^{m-1} q^j$$

$$= d(x_1, x_0) \underbrace{\frac{q^k (1-q^{m-k})}{1-q}}_{\text{fixed}} \to 0 \text{ as } k \to \infty$$

so, $\{x_k\}$ is a cauchy sequence
so completness yields $x_k \to x^*$

continuity of T gives: $Tx^* = \lim_{k\to\infty} Tx_k = \lim_{k\to\infty} x_{k+1}$
$= x^*$
$\Rightarrow Tx^* = x^*$

Exe: show error estimate of banach fixed point theorm
$d(x_k, x^*) \leq \frac{q^k}{1-q} d(x_1, x_0)$ → done down

Gronwall:

we have seen $u(t) \leq A + \int_{t_0}^{t} u(s) g(s) ds \Rightarrow u(t) \leq A e^{\int_{t_0}^{t} g(t) dt}$

Differential gronwall:

Lemma: let $z : [t_0, T] \to [0, \infty)$ be C^1,
$b, f : [t_0, T] \to [0, \infty)$ be integrable and assume

$$z'(t) \leq b(t) z(t) + f(t) \text{ then}$$

$$z(t) \leq z(t_0) e^{\int_{t_0}^{t} b} + \int_{t_0}^{t} e^{\int_s^t b(\tau) d\tau} f(s) ds$$

proof: set $\beta(t) = \int_{t_0}^{t} b$ $\quad \frac{d}{dt}\left(e^{-\beta(t)} z(t)\right) = e^{-\beta(t)}\left(z'(t) - b(t) z(t)\right)$

$$\leq e^{-\beta(t)} \left[\cancel{b(t) z(t)} + f(t) - \cancel{b(t) z(t)} \right]$$

$$\frac{d}{dt}\left(e^{-\beta(t)} z(t)\right) \leq e^{-\beta(t)} f(t)$$

$$\Rightarrow \int_{t_0}^{t} \left(e^{-\beta(s)} z(s)\right)' \leq \int_{t_0}^{t} e^{-\beta(s)} f(s) ds$$

$$\Rightarrow e^{-\beta(t)} z(t) - z(t_0) \leq \int_{t_0}^{t} e^{-\beta(s)} f(s) ds$$

$$\Rightarrow z(t) \leq z(t_0) e^{\beta(t)} + \underbrace{e^{\beta(t)} \int_{t_0}^{t} e^{-\beta(s)} f(s) ds}_{e^{\int_{t_0}^{t} b(\tilde{t}) d\tilde{t}} \int_{t_0}^{t} e^{-\int_{t_0}^{s} b(\tilde{t}) d\tilde{t}} f(s) ds}$$

$$= \int_{t_0}^{t} e^{\int_s^t b(\tilde{t}) d\tilde{t}} f(s) ds$$

$$\Rightarrow z(t) \leq z(t_0) e^{\int_{t_0}^{t} b(t) dt} + \int_{t_0}^{t} e^{\int_s^t b(\tilde{t}) d\tilde{t}} f(s) ds$$

Theorem: (Picard-Lindelöf) suppose $(t_0, x_0) \in D \leftarrow$ open s.t $D \subseteq \mathbb{R}^m$
and there is a Rectangle

$$R = [t_0 - a, t_0 + a] \times B_R(x_0) \subseteq D \text{ s.t}$$

(a) f is continious on R and $\|f(t,x)\| \leq M$ on R
(b) f is Lipschitz on R with constants
$L \geq 0$
in particular this means $\|f(t,x) - f(t,y)\| \leq L \|x - y\|$
$\forall (t,x), (t,y) \in R$

choose $T = \min\{a, \frac{R}{M}, \frac{1}{2L}\}$ when $L = 0$ drop $\frac{1}{2L}$ term, then

$\exists$ unique solution $u : [t_0 - T, t_0 + T] \to \mathbb{R}^m$ to $u' = f(t,u)$
moreover, the Picard iterates $u^{(0)}(t) = x_0$, $u^{(k+1)}(t) = x_0 + \int_{t_0}^{t} f(s, u^{(k)}(s)) ds$

converges uniformly to u with

$$\|u^{(k)}-u\|_{L^\infty} \leq \frac{q^k}{1-q}\|u^{(1)}-x_0\| \quad q=LT\leq\frac{1}{2}$$

proof: let $x_{k+1} = u^{(k+1)}(t)$

$$x_{k+1} = x_0 + \int_{t_0}^{t} f(s, x_k)\,ds$$

let $Tx_k = x_0 + \int_{t_0}^{t} f(s,x_k)\,ds$

i.e $Tg\cdot = x_0 + \int_{t_0}^{t} f(s,g(s))\,ds$

if we show T is a contraction, then as $C(\underbrace{[t_0-T, t_0+T]}_{I}, \mathbb{R}^m)$ a complete metric space

$$\text{let } X := \{u \in C(I,\mathbb{R}^m) \mid \|u(t)-u(t_0)\|_{L^\infty} \leq R\}$$

we have to show

① X is closed, so X is complete

② $T: X \longrightarrow X$ is a contraction

③ use banach fixed point theorem to show unique solution

X is complete is from choice of $(X, \|\ \|_\infty)$, so from ①, ②, ③ if we show T is a contraction, we are done

$$T: X \rightarrow X \quad T(x)-T(y) = \int_{t_0}^{t}(f(s,x(s))-f(s,y(s)))\,ds$$

$$\Rightarrow |T(x)-T(y)| \leq \int_{t_0}^{t} L\|x(s)-y(s)\|\,ds$$

$$\leq LM\|x-y\| \leq RL/T\|x-y\|$$

Exe: show error estimate of banach fixed point theorem

$$d(x_k, x^*) \leq \frac{q^k}{1-q} d(x_1, x_0)$$

Ans: $d(x_m, x_k) \leq \sum_{j=k}^{m-1} d(x_{j+1}, x_j) \leq d(x_1, x_0) \sum_{j=1}^{m-1} q^j = \frac{q^k(1-q^{m-k})}{1-q} d(x_0, x_1)$
$m>k$

as $m \to \infty$: $d(x_k, x^*) \leq \frac{q^k}{1-q} d(x_1, x_0)$

23rd Sept:

Picard - Lindelof:

Theorem: Suppose $(t_0, x_0) \in D \to$ open set in $\mathbb{R}^m$ and there is a rectangle R s.t $R = [t_0 - a, t_0 + a] \times B_R(x_0) \subseteq D$ s.t f is continuous on rectangle R and $\|f(t,x)\| \leq M$ on R, and f is lipschitz in x on R with constant $L \geq 0$

$$\|f(t,x) - f(t,y)\| \leq L\|x-y\| \quad \forall (t,x),(t,y) \in R$$

choose $T = \min\{a, \frac{R}{M}, \frac{1}{2L}\}$ where we drop $\frac{1}{2L}$ if $L = 0$, then

$\exists$ unique solution in $u: [t_0 - T, t_0 + T] \longrightarrow \mathbb{R}^m$ to $u' = f(t, u(t))$ $u(t_0) = x_0$
moreover the picard iterates $u_0(t) \equiv x_0$ $u_{k+1}(t) = x_0 + \int_{t_0}^{t} f(s, u_k(s))ds$
converge uniformly with error estimate

$$\|u_k - u\|_{L^\infty} \leq \frac{q^k}{1-q}\|u_1 - x_0\|_{L^\infty} \quad L^\infty = \text{sup norm}, \; q = LT \leq \frac{1}{2}$$

proof: let $I = [t_0 - T, t_0 + T]$, consider the complete metric space $C(I, \mathbb{R}^m)$ with $\|\cdot\|_\infty$-norm i.e $f \in C(I, \mathbb{R}^m)$ (Banach space)

let $X = \{g \in C(I, \mathbb{R}^m) \mid \sup_{t \in I}\|g(t) - x_0\| \leq R$ i.e $\|g - x_0\|_{L^\infty(I)} \leq R\}$, X is closed and so complete

let $\Gamma: X \longrightarrow C(I, \mathbb{R}^m)$ be an operator s.t
$$(\Gamma h)(t) = x_0 + \int_{t_0}^{t} f(s, h(s))ds$$

Γ is continuous in t (trivial)

step 1: for any $h \in X$, $t \in I$, $\|\Gamma h - x_0\| = \|\int_{t_0}^{t} f(x, h(s))ds\| \leq M|t - t_0|$
$\leq MT \leq R$
as $T = \min\{a, \frac{R}{M}, \frac{1}{2L}\}$

$\Rightarrow \Gamma h \in X$, so $\Gamma: X \longrightarrow X$ where X is a complete space

step 2: (contraction) For $h_1, h_2 \in X$, $t \in I$,
$$\|(\Gamma h_1)(t) - (\Gamma h_2)(t)\| \leq \int_{t_0}^{t} \|f(s, h_1(s)) - f(s, h_2(s))\| ds$$
$$\leq L\|h - h_2\|_{L^\infty} |t - t_0|$$
$$\leq LT\|h_1 - h_2\|_{L^\infty}$$
$$\leq \frac{1}{2}\|h_1 - h_2\|_{L^\infty}$$

let $q = LT$ as $LT \leq \frac{1}{2}$

$\Rightarrow q \leq \frac{1}{2}$

now, we get $\Gamma: X \longrightarrow X$, then everything else follows from banach fixed point theorem

Exe: show that $C(I, \mathbb{R}^m)$ is complete with $\|\cdot\|_\infty$ norm. $\to$ done down
Exe: show that X is a closed set and hence complete set. $\to$ done down

proposition: (continuous dependency on initial data) let u solve $u' = f(t,u)$ $u(t_0) = u_0$, v solve $v' = f(t, v)$, $v(t_0) = v_0$, then for $t \in I$ (under hypothesis of theorem)

$$\|u(t) - v(t)\| \leq e^{L|t - t_0|}\|u_0 - v_0\|$$

proof: From the integral version of the ODEs we have:

$$\|u(t) - v(t)\| \leq \|u_0 - v_0\| + L\int_{t_0}^{t} \|u(s) - v(s)\| ds$$

$$\Rightarrow \|u(t)-v(t)\| \leq \|u_0-v_0\| e^{L\int_{t_0}^{t}(1)ds} \quad (\because \text{gronwalls})$$

$$\leq \|u_0-v_0\| e^{L|t-t_0|}$$

Note: There are some conditions like Osgood condition on f that will weaken lipschitz condition (proof in prof notes) (not part of syllabus)

Osgood condition: $\|f(t,x)-f(t,y)\| \leq w(\|x-y\|)$ ← some fuction

$$\int_0^{\infty} \frac{dr}{w(r)} = \infty$$

Continuation, maximum intervals and Blowup:

Lemma: let f be continious & locally lipschitz in x on D (constant depends on an open set $\subset X$), let $u:(\alpha,\beta) \longrightarrow \mathbb{R}^m$ be a solution to $u' = f(t,u(t))$ with $(t,u(t)) \in D \ \forall t \in (\alpha,\beta)$ and $t_0 \in (\alpha,\beta)$ if $\beta < \infty$ and $\exists$ compact set $K \subseteq \mathbb{R}^m$ s.t

$$\{(t,u(t)) \mid t \in [t_0,\beta)\} \subseteq [t_0,\beta)\} \subseteq [t_0,\beta] \times K \subseteq D$$

then $u(t)$ extends continiously to $t=\beta$ and so solution exist past β

proof: By continuity of f on $[\alpha,\beta] \times K$, $\exists M$ s.t $\|f\| \leq M$, now for $t,s \in [t_0,\beta)$

$$\|u(t)-u(s)\| = \|\int_s^t f(\tau,u(\tau))d\tau\| \leq M|t-s|$$

$\Rightarrow u(t_n)$ is a cauchy sequence as $t_n \uparrow \beta$

as K is compact every cauchy seq is convergent

let $u(t_n) \longrightarrow u_\beta \in K$, let $u(\beta) = u_\beta$ then we get $u(t)$ entends to $t=\beta$

Exe: find a fuction wich is locally lipschitz but not lipschitz

Ans: x^2 is one such fuction

Exe: show that $C(I,\mathbb{R}^m)$ is complete with $\|\cdot\|_\infty$ norm.

Ans:

$$C(I,\mathbb{R}^m) = \{ f \mid f: I \longrightarrow \mathbb{R}^m, \ f \text{ is cont} \}$$

for $\forall \varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}$ s.t $\forall n,m > n_\varepsilon$

$$\|f_n - f_m\|_\infty < \varepsilon$$

if $\varepsilon = 1 \Rightarrow n_1 \in \mathbb{N}$ s.t

$$\|f_n - f_m\|_\infty < 1$$

as $\|f_n\|_\infty \leq \|f_n - f_{n_1+1}\| + \|f_{n_1+1}\|$

$$\Rightarrow \|f_n\|_\infty \leq 1 + \|f_{n+1}\|$$

$$M = \max\{\|f_1\|, \ldots \|f_{n_1}\|, 1 + \|f_{n+1}\|\}$$

then $\|f_i\| \leq M \ \forall i$ $\Rightarrow$ or seq is bounded, as bounded seq has conv subsequence $\{f_{p_i}\}_{i=1}^{\infty}$, and let $\alpha = \lim_{i\to\infty} f_{p_i}$

we get $\forall \frac{\varepsilon}{2} > 0$, $\exists n'_{\varepsilon/2} \in \mathbb{N}$ s.t $\|f_{p_i} - \alpha\| < \frac{\varepsilon}{2} \ \forall i > n'_{\varepsilon/2}$

as f_i is cauchy $\exists n_{\varepsilon/2} \in \mathbb{N}$ s.t

$$\|f_n - f_m\| < \varepsilon/2 \quad \forall n,m > n_{\varepsilon/2}$$

$n_\varepsilon = \max\{n_{\varepsilon/2}, n'_{\varepsilon/2}\}$ then

$$\forall \varepsilon > 0, \ \|f_n - \alpha\| \leq \|f_n - f_{n_i}\| + \|f_{n_i} - \alpha\| < \varepsilon/2 + \varepsilon/2$$

$$\forall n > n_\varepsilon \quad \Rightarrow \alpha = \lim_{i\to\infty} f_i$$

Exe: show that X is a closed set and hence complete set.

Ans: as

$$X = \{ g \in C(I,\mathbb{R}^m) \mid \sup_{t\in I} \|g(t) - x_0\| \leq R \}$$

$\Rightarrow X = \{ g \in C(I, \mathbb{R}^m) \mid \|g - x_0\|_{L^\infty} \leq R \}$

for any $\{g_i\}_{i=1}^{\infty}$ s.t $g_i \longrightarrow g$ where $g_i \in X \ \forall i$

we get $\|g - x_0\| \leq \|g - g_n + g_n - x_0\|$

$\leq \varepsilon + \|g_n - x_0\|$

as $\varepsilon \longrightarrow 0 \Rightarrow \|g - x_0\| \leq R$

$\Rightarrow g \in X$

so, X is closed (and so complete) subspace of $C(I, \mathbb{R}^m)$

30th Sept:

Unique continuation / Maximal interval of existence / Blowups / Escapes:

Defn: $\Omega \subset \mathbb{R}^m$ be open, $f: \Omega \to \mathbb{R}^m$ then

(a) f is called Lipschitz cont on Ω if $\exists L > 0$ s.t

$$|f(x) - f(y)| \leq L|x-y| \quad \forall x, y \in \Omega$$

(b) f is called locally Lipschitz if for every compact set $K \subset \Omega$, $\exists L_K$, $L_K > 0$ s.t

$$|f(x) - f(y)| \leq L_K |x-y| \quad \forall x, y \in K$$

Note: $f = (f_1, .. f_m)$ f is Lipschitz (or locally Lipschitz) iff f_i is Lipschitz (or locally Lipschitz) $\forall i \in [K]$

Exe: $\Omega \subset \mathbb{R}^n$, Ω is convex, $f: \Omega \to \mathbb{R}$ is C^1 and $|\nabla f(x)| \leq L$ $\forall x \in \Omega$, then f is Lipschitz

Ans: $g(t) = f(y + t(x-y))$

$$g: [0,1] \longrightarrow \mathbb{R}^n$$

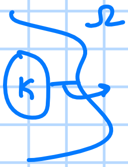
$$|f(x) - f(y)| = |g(1) - g(0)| = \left|\int_0^1 g'(t)\,dt\right| \leq L|x-y|$$

as $|g'(t)| = |\nabla f(y + t(x-y))(x-y)|$

Exe: $f: \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is C^1, f is locally Lipschitz

Ans: Fix $\underset{\text{compact}}{K \subset \Omega}$, $\forall x, y \in K$ we have to show $\exists L_K > 0$ s.t $|f(x) - f(y)| \leq L_K|x-y|$

K is compact, Ω is open



$d(K, \Omega^c) > 0$ as K is compact

choose $0 < R < d(K, \Omega^c)$, let $K \subset \bigcup_{x \in K} B(x, R/2) \subset \Omega$

as $\{B(x, R/2) \mid \forall x \in K\}$ covers K

$\exists x_1, x_2 \ldots x_K$ s.t

$\{B(x_i, R/2) \mid \forall i \in [K]\}$ covers K ($\because$ K is compact)

s.t $K \subset \bigcup_{i=1}^{K} B(x_i, R/2)$

fix $x, y \in K$ then:

<u>Case I</u>: $|x-y| \geq \frac{R}{4}$

$$|f(x) - f(y)| \leq 2 \sup_K |f|$$

$$\leq 2 \|f\|_{L^\infty(K)} \times \frac{\frac{R}{4}}{\frac{R}{4}}$$

$$\leq \frac{8}{R} \|f\|_{L^\infty(K)} |x-y|$$

<u>Case II</u>: $|x-y| < \frac{R}{4}$

as $|x-y| < \frac{R}{4}$, $\exists i$ s.t $x, y \in B(x_i, R)$

we shoot up the radius so balls overlap

$\Rightarrow |f(x) - f(y)| \leq L_i |x-y|$

from previous result as open ball $B(x_i, R)$ is convex

Uniqueness:

Theorem: let $f: I\times\Omega \to \mathbb{R}^m$ where Ω: open $\subseteq \mathbb{R}^m$, I is open interval of $\mathbb{R}$
$(t,x)\to f(t,x)$
f is continuous, f is locally lipschitz in second variable

suppose for some $t_0\in I, x_0\in\Omega$ the IVP

$u(t)=x_0+\int_{t_0}^{t} f(s,u(s))ds \Leftrightarrow \begin{cases} u'(t)=f(t,u(t)) \\ u(t_0)=x_0 \end{cases}$ has two solutions

u_1 defined on I_1, u_2 defined on I_2 then
$u_1=u_2$ on $I_1\cap I_2 \neq \phi$

proof: Assume
$I_1\cap I_2=(a,b)$ $a<t_0<b$, then for $\forall t\in[t_0,b)$

$$u_1(t)-u_2(t)=\int_{t_0}^{t}(f(s,u_1(s))-f(s,u_2(s)))ds$$

$$\Rightarrow |u_1(t)-u_2(t)|\leq L\int_{t_0}^{t}|u_1(s)-u_2(s)|ds$$

$\Rightarrow u_1(t)=u_2(t)$ from gronwall's inequality
similarly for $(a,t_0]$

Note: under the assumption of previous theorm, $u(t)=\begin{cases} u_1(t) & \text{on } I_1 \\ u_2(t) & \text{on } I_2 \end{cases}$
is a C^1 solution to IVP

Defn: $f: I\times\Omega \to \mathbb{R}^m$ be continious and locally lipschitz in the second variable, we say (a,b) is the maximal interval of existance for IVP if there is a u defined on (a,b) that satisfies IVP and no other solution is defined on any larger interval

Theorm: $f: I\times\Omega \to \mathbb{R}^m$ be continious and locally lipschitz in second variable, let $(t_0,x_0)\in I\times\Omega$ and $u:(a,b)\subset I\to\Omega$ be a solution to IVP

$u(t)=x_0+\int_{t_0}^{t} f(s,u(s)) \Leftrightarrow \begin{cases} u'(t)=f(t,u(t)) \\ u(t_0)=x_0 \end{cases}$

suppose, $\exists$ a compact set $K\subset\Omega$ s.t $\{u(t)\,|\,t\in[t_0,b)\}\subset K$ and if $\exists\,\delta>0$ s.t $[t_0,b+\delta)\subset I$, then u can be extended to a larger interval $[t_0,b_1)$ for some $b_1>b$
$[t_0,b]\subset I$

proof: let's take ε as in peano's theorm
$[t_0,b]\times K$
$\downarrow$
any (t_1,x_1) then IVP: $u'(t)=f(t,u(t))$
$u(t_1)=x_1$
has a solution $\tilde{u}$ on $(t_1-\varepsilon,t_1+\varepsilon)$

$I_1=(a,b)$
$I_2=(t_1-\varepsilon,t_1+\varepsilon)$ then $u=\tilde{u}$ from previous theorm
the extension will be unique from picards itteration

Theorm: $\Omega=\mathbb{R}^m$, $f: I\times\mathbb{R}^m\to\mathbb{R}^m$ is cont, lipschitz in second variable, f is bounded then for any $t_0\in I$, $x_0\in\mathbb{R}^m$, the maximal interval of existance is I

proof: fix $[a,b]\subset I$, s.t $t_0\in(a,b)$, we claim that IVP has solution defined on (a,b)
suppose $(a,b)\supset J\ni t_0$, $u:J\to\mathbb{R}^m$ be solution

$$|u(t)-x_0| \leq \left| \int_{t_0}^{t} f(s,u(s))\,ds \right|$$
$$\leq |t-t_0| \, \|f\|_{L^\infty}$$
$$\leq |b-a| \, \|f\|_{L^\infty}$$
$$\Rightarrow \{u(t) \mid t\in J\} \subseteq \overline{B(x_0, \|f\|_\infty (b-a))} \subset \mathbb{R}^m$$

we can extend solution to I

Theorem: Assume $f: I\times \Omega \to \mathbb{R}^m$ is continuous, locally Lips in 2^{nd} variable, for any initial data $(t_0, x_0) \in I\times\Omega$, if $\exists$ unique maximal solution $u:(a,b) \to \mathbb{R}^m$ to IVP, for positivity we consider only $[t_0, b)$ then one of following happen:

(I) u is unbounded on $[t_0, b)$, $b \leq \sup I$
$$\limsup_{t\to b} |u(t)| = \infty$$

(II) u is bounded on $[t_0, b)$ and $b = \sup I$

(III) u is bounded on $[t_0, b)$ and $b < \sup I$ and $\lim_{t\to b} \operatorname{dist}(u(t), \partial\Omega) = 0$

Theorem: $f: I\times\mathbb{R}^m \to \mathbb{R}^m$ defined as $f = (f_i)$; $f_i(t,x) = \sum_{j=1}^{m} a_{ij}(t) x_j + b_i(t)$
a_{ij}, b_i are continuous on I, then for any $(t_0, x_0) \in I\times\mathbb{R}^m$ IVP has a unique solution I

Cor: Consider IVP $u^{(m)}(t) + a_{m-1}(t) u^{(m-1)}(t) + \cdots + a_0(t) u(t) = b(t)$
$u(t_0) = \alpha_0, \ldots\ u^{(m-1)}(t_0) = \alpha_{m-1}$, has a unique solution

7^{th} oct:

Theorm: $f: I \times \mathbb{R}^m \to \mathbb{R}^m$ be continious, and locally Lipschitz in 2^{nd} variable, or global
if f satisfies

$$|f(t,x)| \leq a(t)|x_0| + b(t)$$
$$a, b: I \to \mathbb{R} \text{ are cont}$$

then IVP $\begin{cases} u'(t) = f(t, u(t)) \\ u(t_0) = x_0 \end{cases}$ has unique solution on I for every $(t_0, x_0) \in I \times \mathbb{R}^m$

linear boundry value problem:

we start with ODE $a_2(t)u''(t) + a_1(t)u'(t) + a_0(t)u = f$, $a_2(t) \neq 0$, $\forall t \in I$
a_0, a_1, a_2 are continious

$$u(t_0) = \alpha \quad \alpha, \beta \in \mathbb{R}$$
$$u'(t_0) = \beta$$

this can be trasformed into vectors, so we know there will exist a solution

now if:

$$Lu = a_2u'' + a_1u' + a_0u = 0 \quad I = [a,b]$$

$\begin{cases} u(a) = \alpha \\ u(b) = \beta \end{cases}$ this becomes a boundry value problem

$$\begin{cases} a_1u(a) + a_2u'(a) + b_1u(b) + b_2u'(b) = c_1 \\ \tilde{a}_1u(a) + \tilde{a}_2u'(a) + \tilde{b}_1u(b) + \tilde{b}_2u'(b) = c_2 \end{cases}$$

Defn: let $[a,b] \subset \mathbb{R}$, $a_i: [a,b] \to \mathbb{R}$ $i = 0, 1, \dots m$ be continious fuctions and assume $a_m(t) \neq 0$ $\forall t \in [a,b]$

let $L := \sum_{i=0}^{m} a_i(t) \frac{d^i}{dt^i}$ on space $C^m[a,b]$

For real numbers, a_{ij}, b_{ij}, $i = 1, 2 \dots m$, $j = 0, \dots m-1$

let $B_i u = \sum_{j=0}^{m-1} a_{ij} u^{(j)}(a) + \sum_{j=0}^{m-1} b_{ij} u^{(j)}(b)$
$\forall i = 1, 2, \dots m$

Then the non-Homogenious boundry value problem (NH BVP) is to find u satisfying

$$\begin{cases} Lu = f \\ B_i u = c_i \quad \forall i = 1, 2 \dots m \end{cases}$$

Defn: (Homogenious BVP)

m dimmension vector space (seen before)

$$Lu = 0$$
$$B_i u = 0 \quad \forall i = 1, 2 \dots m$$

Theorm: let $u_1, u_2, \dots, u_m$ be linearly independent solution to $Lu = 0$
then, HBVP has only the trivial solution $u \equiv 0$
iff $\det[B_i u_j] \neq 0$

proof:

$u = \sum_{i=1}^{m} \alpha_i u_i$, $B_i u = 0$ $\forall i = 1, \dots m$, now

$$0 = B_j u = B_j\left(\sum \alpha_i u_i\right) = \sum_{i=1}^{m} \alpha_i B_j u_i$$

$$= [\beta_j u_1 \cdots \beta_j u_m]\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_m\end{bmatrix}$$

$$= \left[(\beta_j u_i)\right]\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_m\end{bmatrix}$$

$$= 0$$

so if $\det[(\beta_j u_i)] \neq 0$ then $\alpha_1 = \cdots = \alpha_m = 0$
if $\alpha_1, \alpha_2 \ldots \alpha_m = 0$ then $\det[(\beta_j u_i)] \neq 0$

NHBVP:

if $f : [a,b] \longrightarrow \mathbb{R}$, $c_i \in \mathbb{R}$, $i = 1 \ldots m$

$$Lu = f \text{ in } [a,b]$$
$$\beta_i u = c_i \quad \forall i$$

Note: If u_1 solves $Lu = f$, $\beta_i u_1 = 0$
u_2 solves $Lu = 0$, $\beta_i u_2 = c_i \ \forall i$

then $u = u_1 + u_2$ is the u that solves

Now for case $m = 2$,

$$Lu = a_2(t)u''(t) + a_1(t)u'(t) + a_0(t)u(t) \qquad a_2 \neq 0 \text{ on } [a,b]$$
$$B_1 u = a_{10}u(a) + a_{11}u'(a) + b_{10}u(b) + b_{11}u'(b)$$
$$B_2 u = a_{20}u(a) + a_{21}u'(a) + b_{20}u(b) + b_{21}u'(b)$$

where $\begin{cases} Lu = f \\ \beta_i u = 0 \end{cases}$

let's first find particular solution of $Lu = f$
let u_1, u_2 are LI solutions to $Lu = 0$
$$u_p(t) = c_1(t)u_1(t) + c_2(t)u_2(t)$$

using variation of parameters

$$\text{VOP}\begin{cases} c_1'(t)u_1(t) + c_2'(t)u_2(t) = 0 \\ c_1'(t)u_1'(t) + c_2'(t)u_2'(t) = \dfrac{f(t)}{a_2(t)} \end{cases}$$

$$c_1'(t) = -\frac{f(t)}{a_2(t)}\frac{u_2(t)}{W(u_1,u_2)(t)}$$

$$c_2'(t) = \frac{f(t)}{a_2(t)}\frac{u_1(t)}{W(u_1,u_2)(t)}$$

now, $u_{p_1} = u_1(t)\int_a^t c_1'(s)\,ds + u_2(t)\int_a^t c_2'(s)\,ds = \int_a^t \dfrac{u_1(s)u_2(t) - u_1(t)u_2(s)}{a_2(s)\,W(s)} f(s)\,ds$

$$-u_{p_2} = u_1(t)\int_t^b c_1'(s)\,ds + u_2(t)\int_t^b c_2'(s)\,ds$$

$$\Rightarrow -u_{p_2} = \int_t^b \frac{u_1(s)u_2(t) - u_2(s)u_1(t)}{a_2 W} f(s)\,ds$$

let $F(t,s) = \dfrac{u_1(s)u_2(t) - u_2(s)u_1(t)}{a_2(s)W(s)}$, then:

$$u_{p_1} = \int_a^t F(t,s)f(s)\,ds$$

$$u_{p_2} = \int_t^b -F(t,s)f(s)\,ds$$

so, $u_p(t) = \int_a^b G(t,s)f(s)\,ds$ where

$$G(t,s) = \begin{cases} \frac{1}{2}F(t,s) \;;\; a \le s \le t \le b \\ -\frac{1}{2}F(t,s) \;;\; a \le t \le s \le b \end{cases}$$

as $Lu_{p_1} = f$
$Lu_{p_2} = f$ $\quad L\left(\dfrac{u_{p_1} + u_{p_2}}{2}\right) = f$

Defn: (green's function) for $m=2$, A function $G: [a,b]\times[a,b] \to \mathbb{R}$ is said to be green's funtion of diff operator $L\left(\sum_{i=0} a_i \frac{d^i}{dt^i}\right)$ with boundry condition $B_i u = 0$; $i=1,2$ if it satisfies

(i) $G: [a,b]\times[a,b] \to \mathbb{R}$ is continious

(ii) $\frac{\partial f}{\partial G}$ exist on $[a,b]\times[a,b]$ and $\frac{\partial G}{\partial t}$ is cont in each $\{(t,s) \mid a \le t < s \le b\}$ and $\{(t,s) \mid a \le s < t \le b\}$

and $\lim\limits_{t\to s^+} \frac{\partial G}{\partial t}(t,s) - \lim\limits_{t\to s^-} \frac{\partial G}{\partial t}(t,s) = \frac{1}{a_2(s)}$

(iii) For each, $s \in [a,b]$ fixed, $\zeta(t) = G(t,s)$ solves $L\zeta = 0$ on $[a,s)\cup(s,b]$ and $B_i\zeta = 0 \;\forall i = 1,2$

Note: Above in case of $m=2$, G is a green's funtion and $F(t,s)$ is cont

Theorm: suppose HBVP $\begin{cases} Lu = 0 \\ B_i u = 0 \end{cases}$ has only trivial solution, then

(1) The operator L with BC $B_i u = 0$ $i=1,2$ has unique green's fuction

(2) For any $f \in C[a,b]$ the NHBVP with HBC $Lu = f$, $B_i u = 0$ $i=1,2$ has unique solution

$$u(x) = \int_a^b G(x,t)f(t)\,dt$$

proof: let u_1, u_2 be LI solution to $Lu = 0$,

Ansatz: $G(x,t) = \begin{cases} \alpha_1(t)u_1(x) + \alpha_2(t)u_2(x) \;;\; a \le x < t \le b \\ \beta_1(t)u_1(x) + \beta_2(t)u_2(x) ;\; a \le t < x \le b \end{cases}$

α_i, β_i are continious fuctions if we can show $\alpha_1, \alpha_2, \beta_1, \beta_2$ are unique satisfying above properties

(i) define $\gamma_i(t) = \alpha_i(t) - \beta_i(t)$

$$\gamma_1(t)u_1(t) + \gamma_2(t)u_2(t) = 0 \quad \text{(from continuity of green's function)} \quad ①$$

(ii) $\beta_1(t)u_1'(t) + \beta_2(t)u_2'(t) - \alpha_1(t)u_1'(t) - \alpha_2(t)u_2'(t) = \frac{1}{a_2(t)}$

$$\Rightarrow \gamma_1(t)u_1'(t) + \gamma_2(t)u_2'(t) = -\frac{1}{a_2(t)} \quad ②$$

so, $W(u_1, u_2)(t) \begin{bmatrix} \gamma_1(t) \\ \gamma_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{a_2(t)} \end{bmatrix}$ from ①, ②

as u_1, u_2 are lin ind $\Rightarrow \exists$ unique $\gamma_1(t)$ and $\gamma_2(t)$

(iii) $\xi(x) = G(x,t)$

$B_i\xi = 0 \Leftrightarrow a_{i0}(\alpha_1(t)u_1(a) + \alpha_2(t)u_2(a))$
$+ a_{i1}(\alpha_1(t)u_1'(a) + \alpha_2(t)u_2'(a))$
$+ b_{i0}(\beta_1(t)u_1(b) + \beta_2(t)u_2(b))$
$+ b_{i1}(\beta_1(t)u_1'(t) + \beta_2(t)u_2'(b)) = 0$

$\Leftrightarrow \alpha_1(t)B_iu_1 + \alpha_2(t)B_iu_2$
$= \gamma_1(t)[b_{i0}u_1(b) + b_{i1}u_1'(b)]$
$+ \gamma_2(t)[b_{i0}u_2(b) + b_{i1}u_2'(b)]$

$$\Leftrightarrow \underbrace{\begin{bmatrix} B_1u_1 & B_2u_2 \\ B_2u_1 & B_2u_2 \end{bmatrix}}_{\text{as } \gamma \text{ unique, non zero}} \begin{bmatrix} \alpha_1(t) \\ \alpha_2(t) \end{bmatrix} = \begin{bmatrix} * \\ * \end{bmatrix}$$

$\Leftrightarrow \alpha_1, \alpha_2$ are unique

so, $\beta_i = \alpha_i - \gamma_i$ are unique

so, G is unique

for second part: $u(x) = \int_a^b G(x,t)f(t)\,dt$ to solve $\begin{cases} Lu = f \\ B_iu = 0 \end{cases}$

as $u(x) = \int_a^x G(x,t)f(t)\,dt + \int_x^b G(x,t)f(t)\,dt$

pickup the boundary condition

$$u'(x) = G(x,x)f(x) + \int_a^x \frac{\partial G}{\partial x}(x,t)f(t)\,dt - G(x,x)f(x) + \int_x^b \frac{\partial G}{\partial x}(x,t)f(t)\,dt$$

and $u''(x) = \int_a^b \frac{\partial^2 u}{\partial x^2} f + \lim_{t\to x^-} \frac{\partial u}{\partial x}(x,t)f(t) - \lim_{t\to x^+} \frac{\partial u}{\partial x}(x,t)f(t)$

$\Rightarrow \; u''(x) = \int_a^b \frac{\partial^2 u}{\partial x^2} f + \frac{f(x)}{a_2(x)}$

now, $Lu(x) = a_2 u'' + a_1 u' + a_0 u$

$= \left[\int_a^b a_2 \frac{\partial^2 u}{\partial x^2} f \right] + f(x) + \int_a^b a_1 \frac{\partial u}{\partial x} f + a_0 \int_a^b u f$

$= f(x) + \int_a^b L(\cancel{u(x,t)}) f(t) dt$

$= f(x) + 0$

$= f(x)$

$B_i u = 0$ for $i = 1,2$ (this can be seen by putting in equation)

Defn: The linear operator $L = \sum_{i=0}^{m} a_i(t) \frac{d^i}{dt^i}$ together with BC $B_i u = 0$, $i = 1, \dots m$ is said to be self-adjoint if:

$$\int_a^b Lu(t)v(t)dt = \int_a^t u(t)Lv(t)dt \quad \forall u, v \in C^m[a,b]$$

$$B_i u = 0 = B_i v \quad \forall i = 1, \dots m$$

Theorm: let L with BC $B_i u = 0$ $\forall i = 1, \dots m$ be self-adjoint and also suppose the HBVP $\begin{cases} Lu = 0 \\ B_i u = 0 \; \forall i \end{cases}$ has only the trivial solution

then, green function $\sim L$ has this property

$$G(x,t) = G(t,x) \quad \text{where } G \sim L$$
$$\forall x, t \in [a,b]$$

proof: let $f, g \in C[a,b]$

let u, v s.t $Lu = f$

$Lv = g$

and $B_i u = 0 = B_i v \;\; \forall i$

L is self adjoint i.e

$$\int_a^b uLv = \int_a^b (Lu)v$$

$u(x) = \int_a^b G(x,t) f(t) dt$ as Green function is unique

$\Rightarrow \int_a^b Lu v = \int_a^b f(x) \int_a^b G(x,t) g(t) \, dx dt$

$\int_a^b uLv = \int_a^b \int_a^b G(x,t) f(t) g(x) dt dx$

$= \int_a^b \int_a^b G(t,x) f(x) g(t) dx dt$

as $\int_a^b uLv = \int_a^b Luv \Rightarrow \int_a^b \left[\underbrace{\int_a^b (G(x,t) - G(t,x)) f(x) dx}_{h(t)} \right] g(t) dt = 0$

$\Rightarrow \int_a^b h(t) g(t) = 0 \quad \forall g \in C[a,b]$

$\Rightarrow h(t) = 0$ as polynomials dense in $C[a,b]$

$\Rightarrow u(x,t) = u(t,x)$

17th oct:

Matrix exponentiation and linear time-invariant systems:

$$\dot{x}(t) = Ax(t) + f(t) \quad \text{——} (1)$$

$x(t) \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times m}$ ← time invariant, $f(t) \in \mathbb{R}^m$

eg: $\Delta \to$ laplacian

$e^{t\Delta} \to$ solution of heat equation, $e^{it\sqrt{\Delta}} \to$ solution operator of wave equations

Note: This method is like a gateway to e^L operators

now homogeneous if $f(t) \equiv 0$ in (1)

Defn: $e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \dots$

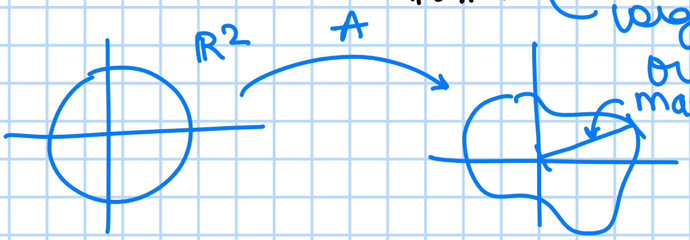
operator norm is one such

If we take any "submultiplicative norm" of A, then $\|e^{At}\| \leq I + \|A\|t + \frac{t^2}{2}\|A\|^2 + \dots$

if $I + t\|A\| + \frac{t^2}{2!}\|A\|^2 + \dots$ converges, then e^{At} converges

operator norm is $A: \mathbb{R}^m \to \mathbb{R}^m$

$\|A\| = \sup_{\|v\|=1} \|Av\|$ ← length 1, we take sup of all one sphere



now if $A = P \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} P^{-1}$ i.e if A is invertible

then $f(A) = P \begin{pmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{pmatrix} P^{-1}$

characterisation:

Theorem: $Y(t) = e^{At}$ is the unique matrix satisfying $Y'(t) = AY(t)$, $Y(0) = I$

proof: Pick any submultiplicative matrix norm $\|\cdot\|$ Then $\sum_{k=0}^{\infty} \left\| \frac{(At)^k}{k!} \right\| \leq \sum_{k=0}^{\infty} \frac{|t|^k \|A\|^k}{k!}$ $\left(\because \|At\| \leq \|A\||t| \right)$

$= e^{|t|\|A\|} < \infty$

→ so, series converges absolutely for each t and uniformly on compact intervals

similarly, $\sum_{k=1}^{\infty} \left\| \frac{A(At)^{k-1}}{(k-1)!} \right\| \leq \|A\| e^{|t|\|A\|} < \infty$, so series of termwise derivate also converges and so, $\frac{d}{dt}\left((At)^k\right) = Ak(At)^{k-1}$

$$\Rightarrow \frac{d}{dt}\left(\sum_{k=0}^{\infty}\frac{(At)^k}{k!}\right) = \sum_{k=1}^{\infty} \frac{A(At)^{k-1}}{(k-1)!}$$

$$\Rightarrow \frac{d}{dt}(e^{At}) = Ae^{At}$$

$\Rightarrow Y(t) = e^{At}$ is unique solution for $Y'(t) = AY(t)$

Key properties:

$\frac{d}{dt} e^{At} = Ae^{At} = e^{At}A$ (∵ it's a series so multiplication from both sides is same)

but in wave form we see group also

$e^{(t+s)A} = e^{tA} e^{sA}$ is semigroup property by multiplication of series

Invertibility: $(e^{At})^{-1} = e^{-At}$

$\det(e^{At}) = e^{(\mathrm{tr}A)t}$

Homogenious IVP:

$\dot{x} = Ax \quad x(t_0) = x_0 \Rightarrow x(t) = e^{A(t-t_0)}x_0$ trivial to see

computing e^{tA} in practice:

(a) Diagonalise: $A = V\Lambda V^{-1}$ with $\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{pmatrix}$ then

$e^{tA} = Ve^{t\Lambda}V^{-1}$ where $e^{t\Lambda} = \begin{pmatrix} e^{t\lambda_1} & & \\ & \ddots & \\ & & e^{t\lambda_m} \end{pmatrix}$

Jordan forms: $A = VJV^{-1}$ ← Jordan block

$k \times k$: $J_k(\lambda) = \lambda I + N$ where N is Nilpotent, $N^k = 0$

then, $e^{J_k(\lambda)t} = e^{\lambda t}\left(I + Nt + \frac{N^2t^2}{2!} + \dots + \frac{N^{k-1}t^{k-1}}{(k-1)!}\right)$

Inhomogenious system:

$\dot{x}(t) = Ax(t) + f(t)$, $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^m$, $f \in$ some general class —— (2) $x_0 = x(t_0)$

(A) constant

(f) L^1_{loc} (local integrable funtions) is one such funtional class

Theorm: unique solution of inhomogenious equation is

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-s)} f(s)\,ds$$

Note: $\int_{t_0}^{t} A(t-s) f(s)ds$ part is called Duhamel

Theorm: unique solution of inhomogenious equation is
$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-s)} f(s)ds$

proof:
$$(e^{-At}x(t))' = (e^{-At})'x + e^{-At}x' \quad (\because \text{leibiz})$$
$$= -Ae^{-At}x + e^{-At}x'$$
$$= e^{-At}(x' - Ax)$$
$$= e^{-At}f(t)$$
$$e^{-At}x(t) - e^{-At_0}x_0 = \int_{t_0}^{t} e^{-As}f(s)ds$$
$$\Rightarrow x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-s)}f(s)ds$$

Note: if $f \in C^k$ we get $x \in C^{k+1}$

Holder spaces:

let $\Omega \subseteq \mathbb{R}^m$ be bdd domain, $f: \Omega \to \mathbb{R}$ is bdd

Holder norm

let $$\|f\|_{C^{0,\alpha}} \triangleq \sup_{x,y \in X} \frac{|f(x)-f(y)|}{d(x,y)^\alpha}$$

fuction can osilate upto some α factor

$f \in C^k(\Omega)$

$\|\cdot\|_{C^{0,\alpha}}$ induce banach spaces

$$\|f\|_{C^{k,\alpha}} = \|f\|_{C^k(\Omega)} + \sup_{x,y \in \Omega} \sum_{|\beta|=k} \frac{|D^\beta f(x) - D^\beta f(y)|}{|x-y|^\alpha}$$

where $\|f\|_{C^k(\Omega)} = \sum_{|\beta| \le k} \sup |D^\beta f|$

$\|\cdot\|_{C^k(\Omega)}$ induce banach spaces

laplace

now if $\Delta u = f$ and $f \in C^2$ then we want to show u is C^4
but there are counterexamples

so, just continuity of f is not enough

but if $f \in C^{k,\alpha}$ then $u \in C^{k+2,\alpha}$

α can be small

also, $|D^\beta f(x)| = |\partial_{x_1}^{\beta_1} \partial_{x_2}^{\beta_2} \cdots \partial_{x_m}^{\beta_m} f|$ $\quad \beta = (\beta_1 \ldots \beta_m)$, $|\beta| = \sum \beta_j = k$ for $\|\cdot\|_{C^{k,\alpha}}$ norm

proposition: (Holder gain) let $I = [a,b]$, $f \in C^{0,\alpha}(I, \mathbb{R}^m)$ with $\alpha \in (0,1)$ then $x \in C^{1,\alpha}(I, \mathbb{R}^m)$ and $\exists$ constant $C(A, I) > 0$ s.t
$$\|x'\|_{C^{0,\alpha}(I)} \le C(\|x\|_{C^0(I)} + \|f\|_{C^{0,\alpha}(I)})$$

proof: we will take it for granted that $x \in C^1$ as if $f \in C^k$ then $x \in C^{k+1}$ (in prof's notes)

now, $x'(t) = Ax(t) + f(t)$

$$= Ae^{A(t-t_0)}x_0 + \int_{t_0}^{t} Ae^{A(t-s)}f(s)\,ds + f(t)$$

fix $t, u \in I$, wlog $t > u$:

$$\text{so then, } x'(t) - x'(u) = \underbrace{(f(t) - f(u))}_{T_1} + \underbrace{\int_u^t Ae^{A(t-s)}f(s)\,ds}_{T_2}$$

$$+ \underbrace{A(e^{A(t-t_0)} - e^{A(u-t_0)})x_0}_{T_3}$$

T_1: since $f \in C^{0,\alpha}$

$$\|T_1\| = \|f(t) - f(u)\| \leq \|f\|_{C^{0,\alpha}} |t-u|^\alpha$$

T_2: $\exists M = \sup_{(r,s) \in I^2} \|Ae^{A(r-s)}\| < \infty$

so, $\|T_2\| \leq \int_u^t M \|f(s)\|\,ds \leq M \|f\|_{C^0(I)} |t-u|$

$|t-u| = |t-u|^\alpha |t-u|^{1-\alpha} \leq (b-a)^{1-\alpha} (t-u)^\alpha$

now, $\|T_3\| = \|A(e^{A(t-t_0)} - e^{A(u-t_0)})\|$

The map $r \longmapsto Ae^{A(r-s)}$ in C^1 with derivative $A^2e^{A(r-s)}$

by MVT: $\|T_3\| \leq L|t-u| \Rightarrow L'|t-u|^\alpha$

so, all terms controlled by $|t-u|^\alpha$ form

21^{st} Oct:

Long-time behavior:

let $\lambda_1, \lambda_2, \dots \lambda_m$ are the eigenvalues of A
$\lambda_i \in \mathbb{C}$
$x' = Ax + f(t)$
solution is of form e^{At}, if λ_j very large then solution $\to \infty$
$\lambda_i \to -\infty$ then solution $\to 0$

let $\alpha(A) = \max_j \operatorname{Re} \lambda_j$

if $\alpha(A) < 0$ then $e^{At} \to 0$ as $t \to \infty$ (stable)

$\alpha(A) > 0$ then $e^{At} \to \infty$ as $t \to \infty$ (unstable)

$\alpha(A) = 0$, then e^{At} growth can be said to be polynomial $\times$ trignometric osilatory

Proposition: (Jordan-scork estimate and growth trajectory) let $A \in \mathbb{C}^{m \times m}$ and write it jordan-decomposition

$$A = VJV^{-1} \text{ where } J = \operatorname{diag}(J_{k_1}(\lambda), \dots J_{k_s}(\lambda_s))$$

where $J_k(\lambda) = \lambda I + N$ where $N^k = 0$

$$\alpha(A) = \max_{1 \le j \le s} \operatorname{Re}(\lambda_j) \text{ and let } m = \max\{k_j - 1 \mid \operatorname{Re}\lambda_j = \alpha(A)\}$$

then $\exists$ a constant $K \geq 1$ (depending on A and the chosen operator norm)

(i) s.t $\forall t \geq 0$, $\|e^{tA}\| \leq Ke^{\alpha(A)t}(1+t)^m$

moreover if $\exists$ a jordan block of $k \geq 2$ with $\operatorname{Re}\lambda = \alpha(A)$
then $\exists$ non-zero vector v and constant $c > 0$ $t_0 \geq 0$

(ii) $\|e^{tA}v\| \geq ce^{\alpha(A)t}t^{k-1}$ $\forall t \geq t_0$

proof: fix $k \geq 1$, $\lambda \in \mathbb{C}$ as $N^k = 0$

$$e^{J_k(\lambda)t} = e^{(\lambda I + N)t} = e^{\lambda t} \sum_{r=0}^{k-1} \frac{(Nt)^r}{r!}$$

$\|\cdot\| \to$ submultiplicative matrix norm

$\exists C_k \geq 1$, with $\|N^r\| \leq C_k$ $\forall$ $r \leq k-1$ as N^k are finitely many
then
$$\|e^{J_k(\lambda)t}\| \leq C_k e^{(\operatorname{Re}\lambda)t} \sum_{r=0}^{k-1} \frac{t^r}{r!}$$

$$\leq C_k' e^{(\operatorname{Re}\lambda)t}(1+t)^{k-1}$$

now, e^{Jt} is block diagonal

$e^{J_{k_i}(\lambda_i)t} \leftarrow$ Blocks like this

$\exists C \geq 1$ s.t
$\|e^{Jt}\| \leq C \max_j \|e^{J_{k_j}(\lambda_j)t}\|$ (as the maximal shell is controlled by $J_{k_j}(\lambda_j)$)

then, $\|e^{tA}\| = \|Ve^{Jt}V^{-1}\|$
$\leq \|V\|\|V^{-1}\|\|e^{Jt}\|$

so we have upper bound for all of this

$$\|e^{tA}\| \leq K e^{\alpha(A)}(1+t)^m$$

now for lower bound calculation

assume J has a block $J_k(\lambda)$ with $Re\lambda = \alpha(A)$ and $k \geq 2$ in Jordan basis, let $e_1, \dots e_k$ be the standard chain in that block, so that

$$Ne_j = e_{j-1}, \; e_0 = 0 \quad (\text{for } k=1, \text{ this will make } e_0 = 0)$$

then

$$e^{J_k(\lambda)t} \underset{\substack{\uparrow \\ \text{vector}}}{e_k} = e^{\lambda t} \sum_{\gamma=0}^{k-1} \frac{t^\gamma}{\gamma!} N^\gamma e_k$$

$$= e^{\lambda t}\left(e_k + t e_{k-1} + \dots + \frac{t^{k-1}}{(k-1)!} e_1\right)$$

then one can say that

$$\| e^{J_k(\lambda)t} e_k \| \geq C_0 e^{\alpha(A)t} t^{k-1}$$

Exe: why do we need $k \geq 2$ in previous theorem part (ii)

Ans: For $k=1$, the bound (lower) goes as $k=1$, $\Rightarrow \| \; \| \geq (\;) t^0$

Stern-Liouville theory:

let $a<b$, $p, q, w \in C^1([a,b])$, $p(x)>0$, $w(x)>0$ ← density

$$\int_c^d f(x) \underset{\substack{\uparrow \\ \text{density is } 1}}{dx}$$

$$\int_c^d f(x) \underset{\substack{\uparrow \\ \text{density is } e^{-x^2}}}{e^{-x^2}} dx$$

now, the stern-Liouville operator:

$$Ly = -(p(x)y'(x))' + q(x)y(x)$$

this operator is not exactly second order

now, $\int Ly\, y = \int -(py')'y = \int p(y')^2$

i.e $\langle Ly, y\rangle \geq 0 \to$ Bi linear form is semidefinite

A S.L problem askes for non-trivial solutions of

$$\begin{cases} Ly = \lambda w y & a<x<b \\ y(a)\cos\alpha + p(a)y'(a)\sin\alpha = 0 & 0 \leq \alpha < \pi \text{ (some } \alpha) \\ y(b)\cos\beta + p(b)y'(b)\sin\beta = 0 & 0 \leq \beta < \pi \text{ (some } \beta) \end{cases}$$

Exe: $\int Luv - uLv\, dx = 0$ i.e operator is symmetric

Ans:

$$\int_a^b (Lu)v = \int (-(pu')' + qu)(v)$$

$$= -\int_a^b (pu')'v + \int_a^b quv$$

$$= \left. -v(pu')\right|_a^b + \int_a^b v'(pu') + \int_a^b quv$$

so, $\int_a^b (Lu)v - \int_a^b (Lv)u = \left| -v(Pu') \right|_a^b + \left| u'(Pv) \right|_a^b$

$= u'(b)P(b)v(b)$
$\quad - u'(a)P(a)v(a)$
$\quad - v(b)P(b)u'(b) + v(a)P(a)u'(a)$

if $\beta \neq 0$ then

$u'(b)P(b) = -u(b)\dfrac{\cos\beta}{\sin\beta}$

and $v'(b)P(b) = -v(b)\dfrac{\cos\beta}{\sin\beta}$

similarly $\alpha \neq 0$ then we get Right side 0

so, if $\alpha \neq 0$ & $\beta \neq 0$ $\langle Lu, v\rangle - \langle Lv, u\rangle = 0$

if $\alpha = 0$ or $\beta = 0$ i.e $u(a) = 0$, $v(a) = 0$
or $u(b) = 0$, $v(b) = 0$

then also trivial to see right side 0

24th Oct :

Sturm-Liouville:

$Ly = -(py')' + qy$, Dirichlet boundary condition

$$y\big|_{\partial\Omega} = 0$$

Neumann boundary condition

$$\frac{dy}{dx}\Big|_{\partial\Omega} = 0$$

} 2 most famous boundary condition

$y\cos + y'\sin$ is linear combination of both

self adjoint BC
$$\begin{cases} Ly = \lambda \omega y \\ y(a)\cos\alpha + y'(a)\sin\alpha = 0 \\ y(b)\cos\beta + y'(b)\sin\beta = 0 \end{cases} \qquad 0 \le \alpha, \beta < \pi$$

Recall, our inner product

$$\langle f, g\rangle_\omega = \int_a^b f\,\bar{g}\,\omega\,dx$$

(ω ← background density)

Integration by parts and symmetry:

Lemma: for $u, v \in C^2[a,b]$

$$\int_a^b (vLu - uLv)\,dx = \left[p(x)(vu' - uv')\right]\Big|_a^b$$

proof: $\int_a^b vLu - \int_a^b uLv = \int_a^b v(-(pu')' + \cancel{quv})$
$$- \int_a^b u(-(pv')' + \cancel{quv})$$
$$= -v(pu')\Big|_a^b - \int_a^b u'pv' + \int_a^b u'pv' + u(pv')\Big|_a^b$$
$$= p(x)(vu' - uv')\Big|_a^b$$

proposition: (symmetry) If u, v satisfy the self-adjoint BC, then the boundary term in previous lemma vanishes

and $\int_a^b vLu = \int_a^b uLv$

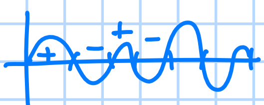
proof: proof is trivial, follows from previous results

Note: Zeroes are basic

lemma: If y solves self-adjoint BC and $y(x_0) = y'(x_0) = 0$ for some $x_0 \in [a,b]$, then $y \equiv 0$

proof: as value and diff both zero, uniqueness makes everything 0 as locally goes to global

we want to study "Nodal-structure" i.e



and we want to see comparison, if y_1, y_2 solve, we want to see how change

Nodal-structure:

Seperation of zeros: we first state and prove the theorm in the classical Schrodinger form $y'' + Q(x)y = 0$ (the case $p \equiv w \equiv 1$)

Theorm: (Stern seperation) let y_1, y_2 be lineanly independent solution of $y'' + Q(x)y = 0$ on (a,b) with $Q \in C[a,b]$ between any two connecitive zeroes of y_1 there lie exactly one zero of y_2, visa-versa for y_2 and y_1

proof:

let $\alpha < \beta$ be the connective zeros of y_1, as zeros are basic $y_1'(\alpha)$, $y_1'(\beta)$ are non-zero and have opposite sign

The wronskian $W = y_1 y_2' - y_1' y_2$

$$\Rightarrow W' = \cancel{y_1' y_2'} + y_1 y_2'' - \cancel{y_1' y_2'} - y_1'' y_2$$

$$= y_1 y_2'' - y_2 y_1''$$

$$= y_1(-Q y_2) - (-Q y_1) y_2$$

$$= 0$$

$\Rightarrow W' = 0$ and so wronskian is constat

$$W(\alpha) = W(\beta)$$

where

$$W(\alpha) = -y_1'(\alpha) y_2(\alpha)$$

$$W(\beta) = -y_1'(\beta) y_2(\beta)$$

$$\Rightarrow \underbrace{y_1'(\alpha) \, y_2(\alpha) = y_1'(\beta)}_{\text{opposite signs}} y_2(\beta)$$

$y_2(\alpha), y_2(\beta)$ must have same sign if $y_2(\alpha), y_2(\beta)$ does not vanish

so $y_2(\alpha), y_2(\beta) = 0$ but if 0 then $W = 0 \rightarrow$ this is a contradiction as linear independent

so, $\exists\, c \in (\alpha, \beta)$ s.t $y_2(c) = 0$
(our asumption of y_2 not vanishing was wrong)

Liouville transform to schrodinger

$$p, q, w \in C^2[a,b], \quad p > 0, \quad w > 0$$

let the chage of coordinate be

$$t \triangleq t(x) = \int_{x_0}^{x} \sqrt{\frac{w(s)}{p(s)}}\, ds \rightarrow \text{bijetive funtion}$$

$x = t^{-1}$ funtion

$$K(x) = (p(x) w(x))^{1/4}$$

and $u(t) = K(x) y(x)$ $\quad t = x^{-1}$

t we are writing x in terms of t

$$-\frac{d^2u}{dt^2} + \underbrace{v(t)}u(t) = \lambda u(t)$$

$$\frac{q(x)}{w(x)} + \frac{K_{tt}(t)}{K(t)} \quad \leftarrow \text{double derivative}$$

Note: The seperation of zeroes theorm, it suffices to prove it for schrodinger case

$$-(py')' + Qy = \lambda w x$$

$$-(u')' + v(t)u(t) = \lambda u(t)$$

Picone identity and comparison theorm:

Lemma: let $p>0$, be cont on $[a,b]$ and let y_1, y_2 be non-trivial solutions of $-(py_1')' + q_1 y_1 = 0$

$$-(py_2')' + q_2 y_2 = 0$$

on an interval where y_2 has no zeros
then $\left[p\left(\frac{y_1}{y_2}\right)(y_1'y_2 - y_1 y_2')\right]' = (q_2 - q_1)y_1^2 + p\left(y_1' - \frac{y_1}{y_2}y_2'\right)^2 \geq (q_2-q_1)y_1^2$

proof: It follows from combing both and differentiating as denominator $\neq 0$

Theorm: let $q_1, q_2 \in C[a,b]$ with $q_2 \geq q_1$ and non identical on any subinterval. If y_1 solves $y'' + q_1 y_1 = 0$ and has consecutive zeroes at $\alpha < \beta$, then every non-trivial solution of $y_2'' + q_2 y_2 = 0$ has atleast one zero in (α, β)
if $q_2 > q_1$ on (α, β) then exactly one zero occurs

we will see more general case of above theorm

Theorm: (Sturm-Picone composition) y_1 solves $-(py_1')' + q_1 y_1 = 0$ and consecutive zeros at $\alpha < \beta$

y_2 solves $-(py_2')' + q_2 y_2 = 0$
$q_2 \geq q_1$ and $q_2 \not\equiv q_1$ on any sub-interval of (α, β), then y_2 has atleast one zero in (α, β)

proof: If y_2 has a zero in (α, β) then there is nothing to prove, otherwise y_2 keep a fixed sign on (α, β)

$$\text{wlog } y_2|_{(\alpha,\beta)} > 0$$

$$\left[p\left(\frac{y_1}{y_2}\right)(y_1'y_2 - y_2'y_1)\right]' \geq (q_1 - q_2)y_1^2 \quad (\because \text{inequality above})$$

integrating from α to β

$$p\left(\frac{y_1}{y_2}\right)(y_1'y_2 - y_1y_2')\Big|_\alpha^\beta \geq \int_\alpha^\beta (q_2-q_1)y_1^2$$

$\Rightarrow 0 \geq \int_\alpha^\beta (q_2-q_1)\,y_1^2$ as $q_2 \geq q_1$ and not identically equal

this is a contradiction

so, $y_1^2\Big|_{(\alpha,\beta)} > 0$ as consecutive roots

so, $\exists$ atleast one root of y_2 on (α,β)

Defn: (p-weighted wronskian) $W_p = p(y_1y_2' - y_1'y_2)$

Lemma: (wronskian derivative) $-(py_1')' + q_1y_1 = 0$ and $-(py_2')' + q_2y_2 = 0$

then $W_p'(x) = (q_2-q_1)y_1y_2 \quad \forall x \in (a,b)$

proof:

$W_p(x) = p(y_1y_2' - y_1'y_2)$ $\qquad -(py_1')' + q_1y_1 = 0$
$\qquad -(py_2')' + q_2y_2 = 0$

$$\begin{aligned}
W_p'(x) &= p'(y_1y_2' - y_1'y_2) \\
&\quad + p(\cancel{y_1'y_2'} + y_1y_2'' - \cancel{y_1'y_2'} - y_1''y_2) \\
&= y_1(p'y_2' + py_2'') - y_2(p'y_1' + y_1''p) \\
&= y_1(py_2')' - y_2(py_1')' \\
&= y_1(q_2y_2) - y_2(q_1y_1) \\
&= (q_2-q_1)(y_1y_2)
\end{aligned}$$

28th oct:

Δ is laplace operator, if we take dirichlet boundary condition

$$u|_{\partial\Omega} = 0$$

$$\Delta u = \lambda u$$

in the experiment, $\sqrt{\lambda}$ = frequency

$\frac{1}{\sqrt{\lambda}}$ = wavelength

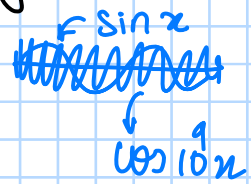
sand will collect on eigenfunctions $\rightarrow$ not vibrating

Nodal domain - eigenfunction zero at some points, we see that as freq $\uparrow$, the nodal set becomes dense

Note: Nodal set is wavelength dense $B(x, \frac{c}{\sqrt{\lambda}})$ so, more freq nodes $\uparrow$

Note: $c_1\sqrt{\lambda} \geq H^{n-1}(N_\lambda) \geq c_2\sqrt{\lambda}$

There are a lot of open problems in this, sturm-liouville is 1-D toy model of above problem

Eg: 

atleast one zero is fine, not exactly one zero

Note: in previous theorem, exactly one zero theorm is false, from above example

but between any two zeroes of ϕ_n (n^{th} eigenfunction) there is exactly one zero of ϕ_{n+1}

Theorm: (Quotient monotonicity) let $p \in C[a,b]$ with $p > 0$ and $q_1, q_2 \in C[a,b]$

suppose y_1, y_2 are non-trivial solutions of

$$-(p y_i')' + q_i y_i = 0 \text{ on } (a,b)$$

let $J = (\alpha, \beta) \subseteq (a,b)$ be an open interval on which both y_1, y_2 has no zero

let $\gamma = \frac{y_1}{y_2}$ on J, then the identity

$$(p y_2^2 \gamma')' = (q_1 - q_2) y_1 y_2 = (q_1 - q_2)\gamma y_2^2$$

holds pointwise on J. in particular if

(i) $q_2 \geq q_1$ on J & $y_1, y_2 \geq 0$ on J then γ is decreasing on J. If $q_2 > q_1$ on some point, then γ is strictly decreasing

(ii) If $q_2 \leq q_1$ on J & $y_1 y_2 \geq 0$ on J then γ is increasing on J. If $q_2 < q_1$ on some point, then γ is strictly increasing

proof: $\gamma = \frac{y_1}{y_2}$

$$\Rightarrow \gamma' = \frac{y_1' y_2 - y_1 y_2'}{y_2^2}$$

$$\Rightarrow p y_2^2 \gamma' = p(y_1' y_2 - y_1 y_2')$$

$$\Rightarrow (p y_2^2 \gamma')' = (p y_1')' y_2 + \cancel{p y_1' y_2'} - \cancel{p y_1' y_2'} - y_1 (p y_2')'$$

$$= (q_1 - q_2) y_1 y_2$$

$$= (q_1 - q_2) \gamma y_2^2 \quad \text{from defn of } \gamma$$

now, rest follows from trivial calculation on our identity as if $q_2 \geq q_1$ and $y_1 y_2 \geq 0$

$$\Rightarrow (p y_2^2 \gamma')' \leq 0$$

i.e $p y_2^2 \gamma'$ is decreasing

but as $y_2^2 > 0$, $p > 0$

rest of proof in notes, but we would need one more condition of

$$\left.\begin{array}{l} y_2(\alpha) = 0 \\ y_2'(\alpha) = 1 \end{array}\right\} \to \text{some kind of boundry condition}$$

$$y_1(\alpha) \geq 0$$

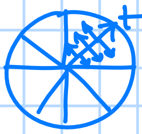
so, $(p y_2^2 \gamma')' = (q_1 - q_2) y_1 y_2$

$$\Rightarrow \int_\alpha^t (p y_2^2 \gamma')' = \int_\alpha^t \underbrace{(q_1 - q_2) y_1 y_2}_{\leq 0}$$

$$\Rightarrow \gamma'(t) - \gamma'(\alpha) \leq 0$$

$$\Rightarrow \gamma'(t) \leq \gamma'(\alpha) \quad \text{we need } \gamma'(\alpha) \leq 0$$

Note: Below lemma is for geodesic divergence



of a vector, if we project them down

the geodesics follow ode $y'' = -Ky$

lemma: let $K \in C[0, \tau]$ and y solve $y''(t) = -K(t) y(t)$

$$\left\{\begin{array}{l} y(0) = 0 \\ y'(0) = 1 \end{array}\right.$$

Then, $\forall t \in [0, \tau]$, $y(t) = t - \int^t (t-s) K(s) y(s)\, ds$

more generally, if $y(0) = a$ and $y'(0) = b$, then

$$y(t) = a + bt - \int_0^t (t-s) K(s) y(s)\, ds$$

proof: Integrate $y'' = -Ky$

$$y'(t) - y'(0) = -\int_0^t K(s) y(s)\, ds$$

$$\Rightarrow y'(t) = 1 - \int_0^t K(s)y(s)ds$$

now $\int_0^t y'(t)dt = y(t) - y(0) = y(t)$

and, $\int_0^t (1 - \int_0^\tau K(s)y(s)ds)d\tau = t - \int_0^t \left(\int_0^\tau K(s)y(s)ds\right)d\tau$

fourbini's theorm $\longrightarrow$ $= t - \int_0^t \int_s^t K(s)y(s)d\tau\, ds$

$= t - \int_0^t (t-s)K(s)y(s)ds$

so, $y(t) = t - \int_0^t (t-s)K(s)y(s)ds$

lemma: (Near zero behaviour) let $K \in [0,\tau]$ and let y solve $y'' = -Ky$
$y(0)=0$, $y'(0)=1$, set $M_0 = \sup_{[0,\tau]} |K| < \infty$

then $\exists\, t_0 \in \left(0, \min\left\{\tau, \sqrt{\frac{3}{2M_0}}\right\}\right]$ s.t $\forall t \in [0, t_0]$

$$|y(t) - t| \le \frac{M_0}{3} t^3$$

in particular $y(t) = t + o(t^3)$ as $t \to 0$

proof: $y(t) = t - \int_0^t (t-s)K(s)y(s)ds$

for $t>0$ let $\varphi(t) = \sup_{0<s\le t} \frac{|y(s)|}{s}$

finite as $y(0)=0$, $y'(0)=1$ so cannot blow up very fast near 0

$$|y(t) - t| \le \int_0^t (t-s)|K(s)||y(s)|ds$$
$$\le M_0\varphi(t)\int_0^t (t-s)s\,ds$$
$$\le M_0\varphi(t)\frac{t^3}{6}$$

Fix $t>0$, and repeat same estimate with t replaced by any $s \in (0,t]$

$$\frac{|y(s)|}{s} \le 1 + \frac{M_0}{6}\varphi(s)s^2 \quad (\because \text{above putting } t=s)$$

take sup on both sides to get

$$\varphi(t) \le 1 + \frac{M_0}{6}t^2\varphi(t)$$

then rest follows from assumption as it gives upper bound $\varphi(t)$ and so it follows

as $t_0 \in \left(0, \min\left(\tau, \sqrt{\frac{3}{2M_0}}\right)\right) \Rightarrow |y(t) - t| \le \frac{M_0 t^3}{3}$

or $y(t) = t + o(t^3)$

4^{th} Nov:

Sturm Osillation Theorm:

setting is the following: let $p \in C^1[a,b]$
$q, r \in C[a,b]$

$p > 0$, $r > 0$ on $[a,b]$

consider the strum-liouille problem:

$$(SL) \quad -(py')' + qy = \lambda r y \quad x \in [a,b]$$

$$(BC) \quad \begin{aligned} & y(a)\cos\alpha + p(a)y'(a)\sin\alpha = 0 \\ & y(b)\cos\beta + p(b)y'(b)\sin\beta = 0, \quad \alpha, \beta \in [0, \pi) \end{aligned}$$

for each fixed λ, define left-normalized solution $y(x,\lambda)$ satisfying

$$\left.\begin{aligned} y(a,\lambda) &= -\sin\alpha \\ y'(a,\lambda) &= \frac{\cos\alpha}{p(a)} \end{aligned}\right\} \text{satisfies BC at } x = a$$

Prüfer Transformation (coordinates):

let $u = y$, $v = py'$ then $u' = y' = \frac{v}{p}$

$$v' = (py')' = (q - \lambda v)y = (q - \lambda v)u$$

For a non-trivial solution define prüfer variable:

(ρ, θ) as: $u = \rho\sin\theta$
$v = \rho\cos\theta$

$\rho = \sqrt{v^2 + u^2} > 0$ as if $\exists x_0$ $\rho(x_0) = 0$
$\Rightarrow v = 0, u = 0$
$\Rightarrow y(x_0) = 0$
$y'(x_0) = 0$
$\Rightarrow y \equiv 0$ on $[a,b]$
as non-trivial solution this not possible

θ = phase
ρ = amplitude

Phase-Amplitude Equations:

$u = \rho\sin\theta \qquad v = \rho\cos\theta$

$u' = \frac{v}{p} = \frac{\rho\cos\theta}{p} \qquad v' = (q - \lambda r)u = (q - \lambda r)\rho\sin\theta$

then $u' = \rho'\sin\theta + \rho\theta'\cos\theta$
$v' = \rho'\cos\theta - \rho\theta'\sin\theta$

$$\Rightarrow \rho'\sin\theta + \rho\theta'\cos\theta = \frac{\rho\cos\theta}{p} \quad \text{——— ①}$$

$$\rho'\cos\theta - \rho\theta'\sin\theta = (q - \lambda r)\rho\sin\theta \quad \text{——— ②}$$

Solve for ρ' and θ':

multiply ① by $\sin\theta$, ② by $\cos\theta$, add them up to get

$$\rho' = \rho \sin\theta \cos\theta \left(\frac{1}{p} + q - \lambda r \right)$$

Amplitude equation

and ① $\times \cos\theta$ − ② $\times \sin\theta$ gives us:

$$\theta' = \frac{1}{p}\cos^2\theta + (\lambda r - q)\sin^2\theta$$ phase equation

now, from initial condition

$$\left.\begin{array}{l} y(a,\lambda) = -\sin\alpha \\ y'(a,\lambda) = \frac{\cos\alpha}{p} \end{array}\right\} \xRightarrow{\rho,\theta} \begin{array}{l} \rho(a,\lambda) = 1 \\ \theta(a,\lambda) = -\alpha \end{array}$$

(1) Since $p \in C^1$; $q, r \in C$, ρ, θ are C^1, $\rho > 0$ on $[a,b]$

(2) Behaviour at zeros of y

if $x_0 \in (a,b)$ s.t $y(x_0,\lambda) = 0 \Leftrightarrow u(x_0,\lambda) = 0$
$\Leftrightarrow \sin\theta(x_0,\lambda) = 0$
$\Leftrightarrow \theta(x_0,\lambda) = n\pi$
$n \in \mathbb{Z}$

at $x_0 \in (a,b)$

$\theta'(x_0,\lambda) = \frac{1}{p(x_0)} > 0$

phase equation

so, zeros of $y(\cdot,\lambda)$ are basic

(3) Boundary condition at $x = b$:

$$y(b,\lambda)\cos\beta + p(b)y'(b,\lambda)\sin\beta = 0$$

$$\xrightarrow{\rho,\theta} \rho(b,\lambda)\sin(\theta(b,\lambda) + \beta) = 0$$

Note: $\lambda \in \mathbb{C}$ is eigenvalue of (SL) iff $\sin(\theta(b,\lambda) + \beta) = 0$
i.e $\theta(b,\lambda) = -\beta + n\pi$, $n \in \mathbb{Z}$

(4) The pair $(u,v) = (y, py')$ traces a curve in u-v plane
ρ is the distance to origin and θ is the angle

The phase ODE says, how fast the angle rotates

zeros of y, corr to this curve, crossing v-axis $(u=0)$ at angles $\theta = n\pi$, $n \in \mathbb{Z}$

Laurance Perko reference

Theorem: (cont. dependence of parameters) $u' = \frac{v}{p}$, $v' = (q - \lambda r)u$

If a vector field (u,v) is solution to our system and initial data depends smoothly on a parameter, the solution depends smoothly on that parameters

$$w = \begin{pmatrix} u \\ v \end{pmatrix} \qquad w' = \underset{\substack{\uparrow \\ \text{smooth wrt } x,\lambda}}{A(x,\lambda)}\, w$$

by smooth we have C^1

Differential (SL) wrt λ:

we will follow $\partial_\lambda y = y_\lambda$ notation

$$\left.\begin{array}{ll} -(py_\lambda')' + qy_\lambda = \lambda r y_\lambda + ry & \text{(diff wrt } \lambda) \\ y_\lambda(a,\lambda) = 0 = y_\lambda'(a,\lambda) & \text{(left normalised)} \end{array}\right\} ③$$

Lagrange Identity and monotonicity in λ:

let $J(x,\lambda)=p(x)(y y_\lambda' - y' y_\lambda)$

where $y_\lambda' = \partial_\lambda \partial_x y$

$y' = \partial_x y$

$y_\lambda = \partial_\lambda y$

(SL): $(py')' = qy - \lambda r y$

③ : $(py_\lambda')' = q y_\lambda - \lambda r y_\lambda - r y$

$$\partial_x J(x,\lambda) = p'(y y_\lambda' - y' y_\lambda) + p(y' y_\lambda' + y y_\lambda'' - y'' y_\lambda - y' y_\lambda')$$
$$= y(p y_\lambda')' - y_\lambda (p y')'$$
$$= y(q y_\lambda - \lambda r y_\lambda - r y) - y_\lambda (q y - \lambda r y)$$
$$= -r y^2 \leq 0$$

so, $J(x,\lambda) = -\int_a^x r(t) y^2(t,\lambda)\,dt$

and $J(a,\lambda)=0$

Relation of J to θ_λ:

$u = \rho \sin\theta$
$v = \rho\cos\theta$

$\sec^2\theta = \frac{\rho^2}{v^2}$ $\quad \tan\theta = \frac{u}{v}$

$$\sec^2\theta\, \theta_\lambda = \frac{u_\lambda}{v} - \frac{u v_\lambda}{v^2} = \frac{-(u v_\lambda - u_\lambda v)}{v^2} = \frac{\rho^2}{v^2}\theta_\lambda$$

$\Rightarrow \theta_\lambda = -\frac{J}{\rho^2} > 0$ for $x \in (a,b]$

$\Rightarrow \theta$ is strictly increasing in λ

now, λ is an eigenvalue for (SL) iff $\theta(b,\lambda) = -\beta + k\pi$, $k \in \mathbb{Z}$

so, for λ_n, λ_{n+1}: $\theta(b,\lambda_{n+1}) - \theta(b,\lambda_n) = \pi$

Note: let $L: \mathcal{D}(L) \subset H \to H$, L is linear self-adjoint

Domain of L $\quad$ seperable Hilbert space

Theorem: For $L:\mathcal{D}(L)\subset H \to H$, then

(1) Eigenvalues are real

(2) $E = \{$set of all eigenvalues$\}$ is atmost countable

(3) E has no limit point

This theorm tells us that $< \lambda_k \le \lambda_{k+1} \le \cdots$ $k \in \mathbb{Z}$, it has no limit point (cannot stop) and atmost countable

$Ly = (-py')' + qy$ will have eigenvalues $\lambda_1, \lambda_2, \ldots\ldots$

$\lambda_1 < \lambda_2 < \ldots\ldots$

atmost countable so λ_1 start and no repeats as θ is strictly increasing

and $\theta(b, \lambda_{n+1}) - \theta(b, \lambda_n) = \pi$

Theorm : (SOT) $\phi_n = y(x, \lambda_n)$ be the eigenfunction corresponding to λ_n, then ϕ_n has exactly $n-1$ zeros in (a, b)

proof :

step 0: $\theta_n(x) = \theta(x, \lambda_n)$
$\phi_n(x_0) = 0 \Leftrightarrow \theta_n(x_0) \in \pi\mathbb{Z}$

also each zero x_0 is basic, $\theta_n'(x_0) > 0$

step 1: ϕ_1 corr to λ_1 does not vanish in (a, b)
wlog $\phi_1 > 0$ in (a, b)
ϕ_1 has 0 zeroes

step 2: induction $n > 1$
assume ϕ_n has $n-1$ zeros

$a < \xi_1 < \xi_2 < \xi_3 \ldots < \xi_{n-1} < b$

step 3 : $d_n(x) = \theta_{n+1}(x) - \theta_n(x)$
$> 0 \ \forall x > 0$ as $\partial_\lambda \theta > 0$

$d_n(a) = -\alpha - (-\alpha) = 0$
$d_n(b) = \pi$
$d_n > 0$

step 4 : Atleast n zeroes for ϕ_{n+1}:

Fix $\xi_j < \xi_{j+1}$ zeroes of ϕ_n

$\phi_{n+1}(\xi_j) = \phi_n(\xi_j) + d_n(\xi_j)$
$= k_j\pi + d_n(\xi_j)$ some $k_j \in \mathbb{Z}$

$\sin(\theta_{n+1}(\xi_j)) = (-1)^{k_j} \sin(d_n(\xi_j))$

similarly $\sin(\theta_{n+1}(\xi_{j+1})) = (-1)^{k_j+1} \sin(d_n(\xi_{j+1}))$

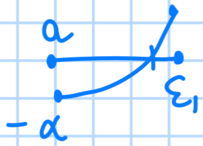
as d_n is between 0 and π it does not chge sign

so, $\sin(\theta_{n+1}(\xi_j)) \sin(\theta_{n+1}(\xi_{j+1})) < 0$

$\Rightarrow$ Between any two zeroes of ϕ_n, ϕ_{n+1} has atleast one zeroes

or $n-2$ zeroes b/w ξ_1 to ξ_{n-1}, and at

Endpoints, b/w a and ε_1 and ε_{n-1} and b there are two other roots



so, atleast n roots

Step 5: φ_{n+1} has atmost n roots

$$\theta_{n+1}(b)-\theta_{n+1}(a) = \alpha_n(b)+\theta_n(b)-\alpha_n(a)-\theta_n(a)$$
$$= \pi + \underbrace{\theta_n(b)-\theta_n(a)}_{\text{atmost } n-1 \text{ crossings}}$$

for each interior zero, corresponding to a crossing of level $k\pi$, as all crossing are strict (θ_n is inc) this increases the total phase by exactly π

so from above φ_{n+1} has n roots

7^{th} Nov :

Rayleigh quotient :

The Rayleigh Quotient on the "admissible class" of functions

$$R(y) = \frac{\int_a^b p(y')^2 + qy^2\,dx}{\int_a^b wy^2\,dx}$$

we are trying to capture: $\dfrac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2}$

lemma: If y is an eigenfunction with eigenvalue λ, then $R(y)=\lambda$

the lemma can be proved by

$$-\Delta y = \lambda y$$

$\Rightarrow \int -\Delta y\, y = \int \lambda y^2$ (multiply both sides by y and integrate)

$$\Rightarrow \frac{\int |\nabla y|^2}{\int y^2} = \lambda$$

this is what we get

Note: The opposite thing is also true

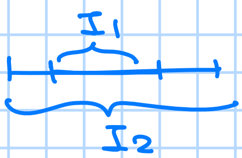
lemma: let $p, q, w \in C[a,b]$ with $p, w > 0$ and consider the regular (SL) problem with S.L. b.c. Let λ_1 be first eigenvalue

$$\lambda_1 = \inf_{0 \neq y \in D} \frac{\int_a^b p(y')^2 + qy^2}{\int_a^b wy^2}$$

D = admissible class

if u^* attains $R(u^*) = \lambda_1$, then u^* is the first eigenfunction

Domain monotonicity:

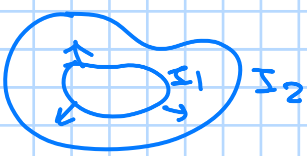


$I_1 \subseteq I_2$, dirichlet boundary condition

$$y\big|_{\Omega} = 0$$

then $\lambda_1(I_1) \geq \lambda_1(I_2)$

This holds for Higher dimensions also, but not for $y\big|_{d\Omega} = 0$ case



Theorm : (Weyl asymptotes) $-(py')' + qy = \lambda w y$ on $[a, b]$. The eigenvalues λ_n of the (SL) problem satisfy

$$\lambda_n \sim \frac{(n\pi)^2}{T^2}, \quad n \to \infty$$

Note : Above theorm is also true in higher domains which becomes

$\lambda_n \sim |\Omega|^{2/n}$ $|\Omega|$ = volume of domain, n = dimension

So, volume of domain can be captured by the frequencies of vibration

Note : One of the open problems are from laplace eigenvalues, we want to determine geometry (not only volume)

Fix a number γ

$N(\gamma) = \{\lambda_j \mid \lambda_j \leq \gamma\}$ this is more exact than $\lambda_n \sim |\Omega|^{2/n}$

$N(\gamma) \sim |\Omega| \gamma^{n/2} + O(\gamma^{n/2-1})$