* multivariable- calculus *

- closed bounded rectangle:
  $R = [a,b] \times [c,d]$
  ↳ cartesian product of two integrals $[a,b] \times [c,d]$

- graph of f:
  $\{(x, y, f(x,y)) \in \mathbb{R}^3 \mid (x,y) \in R\}$ in $\mathbb{R}^3$ is called graph of f.

- contour line:
  fix $c \in \mathbb{R}$ then
  $\{(x, y, c) \in \mathbb{R}^3 \mid f(x,y) = c, \ (x,y) \in R\}$ in $\mathbb{R}^3$
  i.e $z = c$ in $\mathbb{R}^3$

* double integral of non-negative function:

  $f(x,y) = x^2 + y^2$
  in $[-3,3] \times [-3,3]$
  volume below graph of f
  over $R = [-3,3] \times [-3,3]$

  $$V := \{(x,y,z) \mid (x,y) \in [-3,3] \times [-3,3], \ 0 \leq z \leq f(x,y)\}.$$

  $$\iint_R f(x,y)\,dx\,dy = \text{vol of } V.$$

+ volume of cuboid:
  $f(x,y) = \alpha$

  $$\iint_{[0,b] \times [0,d]} \alpha \, dx\, dy = \boxed{bd\alpha}$$
  → area

* Partition in rectangles: Pation P of R is cart. prodt of P_1 and P_2.

$R = [a,b] \times [c,d]$

$P_1 = \{x_0, x_1, \dots x_m\}$

with $a = x_0 < x_1 < x_2 < \dots < x_{m-1} < x_m = b$

$P_2 = \{y_0, y_1, \dots y_n\}$

with $c = y_0 < y_1 < \dots < y_{n-1} < y_n = d$

and $P = P_1 \times P_2$

$P = \{(x_i, y_j) \mid i \in \{0,1, \dots m\}, j \in \{0,1, \dots n\}\}$

- note: P divides R into nm non-overlaping sub-rectagles denoted by:

$$R_{ij} := [x_i, x_{i+1}] \times [y_j, y_{j+1}]$$

$\forall\ i = 0, \dots m-1$

$\ \ j = 0, \dots n-1$

$$\boxed{R = \bigcup_{ij} R_{ij}}$$

- Area (Δ_{ij}) of each rectagle R_{ij}:

$\Delta_{ij} = (x_{i+1} - x_i) \times (y_{j+1} - y_j)$

$\forall\ i = 0, 1, \dots m-1$

$\&\ j = 0, 1, \dots n-1$

- norm:

$\|P\| := \max \{(x_{i+1} - x_i), (y_{i+1} - y_i) \mid i = 0, \dots m-1, j = 0, \dots n-1\}$

Note: norm $\rightarrow$ Area not present as $(x_{i+1} - x_i) \uparrow\uparrow$ ~~that~~ and $(y_{i+1} - y_i) \downarrow\downarrow$

* darboux integral :
let $f: R \to \mathbb{R}$ be bounded where R is rectagle.
$m(f) = \inf\{f(x,y) \mid (x,y) \in R\}$
$M(f) = \sup\{f(x,y) \mid (x,y) \in R\}$
$\forall\ i = 0,1,\dots m-1,\ j = 0,1,\dots n-1$

$m_{ij}(f) := \inf\{f(x,y) \mid (x,y) \in R_{ij}\}$
$M_{ij}(f) := \sup\{f(x,y) \mid (x,y) \in R_{ij}\}$

- lower ~~darboux~~ double sum :
$L(f,P) := \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} m_{ij}(f)\,\Delta_{ij}$

- upper ~~darboux~~ double sum :
$U(f,P) := \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} M_{ij}(f)\,\Delta_{ij}$

- Note :
$m(f)(b-a)(d-c) \le L(f,P) \le U(f,P) \le M(f)(b-a)(d-c)$

- lower darboux integral :
$L(f) := \sup\{L(f,P) \mid P$ is any partition of $R\}$

upper darboux integral :
$U(f) := \inf\{U(f,P) \mid P$ is any partion of $R\}$

- Note : $L(f) \le U(f)$

* defination of darboux integral:
A bounded function $f: R \to \mathbb{R}$ is said to be darboux integrable if
$L(f) = U(f)$.

The darboux integral of f is the common value $U(f) = L(f)$ and is denoted by:

$$\iint_R f, \quad \iint_R f(x,y)\,dA \quad \text{or} \quad \iint_R f(x,y)\,dx\,dy.$$

* Theorm: (Riemann condition)
let $f: R \to \mathbb{R}$ be a bounded function, then f is integrable if and only if $\forall \varepsilon > 0$, $\exists P_\varepsilon$ of R s.t

$$|U(f, P_\varepsilon) - L(f, P_\varepsilon)| < \varepsilon$$

* Riemann Integral:
tagged partion (P, t)
$t = \{t_{ij} \mid t_{ij} \in R_{ij},\ i = 0, \dots m-1,\ j = 0, \dots n-1\}$

Riemann sum of f
$$S(f, P, t) = \sum_{j=0}^{n-1} \sum_{i=0}^{m-1} f(t_{ij})\Delta_{ij}$$

Defination: (Riemann integral)
A bounded $f: R \to \mathbb{R}$ is said to be Riemann int. if $\exists\ S \in \mathbb{R}$, s.t $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t
$$|S(f, P, t) - S| < \varepsilon$$
$\forall\ \|P\| < \delta$ and S is value of Riemann integral of f.

- Note:
Darboux integrability and Riemann integrability are equivalent.

- $f \to$ integrable if Darboux / Riemann condition hold.

- $\iint_R f(x,y)\,dx\,dy := S = L(f) = U(f)$

- Regular partition:

$$x_{i+1} = x_i + \frac{(b-a)}{n}$$

$$y_{j+1} = y_j + \frac{(d-c)}{n}$$

take
$$t = \{ t_{ij} \in R_{ij} \mid i, j \in \{0, 1, \ldots n-1\} \}$$

- Theorm:
A bounded function $f : R \to \mathbb{R}$ is Riemann integrable if and only if
Riemann sum
$$S(f, P_n, t) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} f(t_{ij})\, \Delta_{ij}$$
tends to same limit $S \in \mathbb{R}$ as $n \to \infty$ for any choice of tag t.

- Note: given $f \to$ integrable, then use

$R = [0,1] \times [0,1]$

$t = \left\{ \left(\frac{i}{n}, \frac{j}{n} \right) \middle| \; i = 0, \ldots n-1, \; j = 0, \ldots n-1 \right\}$

$$S(f, P_n, t) = \left(\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \left(\frac{i}{n}\right)^2 + \left(\frac{j}{n}\right)^2 \right) \frac{1}{n^2}$$

$$\lim_{n \to \infty} S(f, P_n, t) = S$$

~~* properties~~ ~~* propert~~

- Domain additive property –
  - $f : R \to \mathbb{R}$
    ↳ Partition R into finite non-overlaping sub rectangles.
  - f is int on $R \iff f$ is int. on each sub-rect.

- Algebric properties:

$$\iint_R f = \alpha A(R)$$

$$f+g \quad \iint_R f+g = \iint_R f + \iint_R g$$

$$\iint_R \alpha f = \alpha \iint_R f$$

if $f(x,y) \leq g(x,y)$

$$\Rightarrow \iint_R f \leq \iint_R g$$

$|f|$ is int. and $\left|\iint_R f\right| \leq \iint_R |f|$

- Iterated Integrals:

$$h(y) = \int_a^b f(x,y)\,dx \qquad g(x) = \int_c^d f(x,y)\,dy$$

$$\int_c^d h(y)\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$$

$$\int_a^b g(x)\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$$

* Note:
- If f is integrable on rectangle $R = [a,b] \times [c,d]$ and if either one of the iterated integrals exist, then it equals to double int

$$\iint_R f(x,y)\,dx\,dy$$

- if both exist → they must be equal.

* Fubini theorm:

$\int_c^d f(x,y)\,dy$ for each $x \in [a,b]$, exist

then $\int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$ exist and equal to I.

* Algebric property proof:

$|f(x,y)| \leq M_1$, $|g(x,y)| \leq M_2$

then

$M = \max\{M_1, M_2\}$

$|f(x,y)| \leq M$ and

$|g(x,y)| \leq M$

$\forall\ x,y \in R$

now for any $\theta_1, \theta_2 \in R_{ij}$

$$|f(\theta_1)g(\theta_1) - f(\theta_2)g(\theta_2)|$$
$$= |f(\theta_1)g(\theta_1) + f(\theta_1)g(\theta_2) - f(\theta_1)g(\theta_2) - f(\theta_2)g(\theta_2)|$$

$$\cancel{\leq |g(\theta_1)||f(\theta_1) - f(\theta_2)| + |g(\theta_2)||g(\theta_2}$$

$$\leq |f(\theta_1)||g(\theta_1) - g(\theta_2)| + |g(\theta_2)||f(\theta_1) - f(\theta_2)|$$
$$\leq M|M_{ij}g - m_{ij}g| + M|M_{ij}f - M_{ij}f|$$
$$< \frac{M\varepsilon}{2M} + \frac{M\varepsilon}{2M} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \boxed{\varepsilon}$$

$< \varepsilon$

* Proof of fubini theorem:

given $\forall \varepsilon > 0$, $\exists P_\varepsilon = \{(x_i, y_j) \mid i = 0, 1, \dots k-1; \; j = 0, \dots n-1\}$ of R

s.t

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$$

also $\int_c^d f(x,y)\,dy$ exist

$$A(x) = \int_c^d f(x,y)\,dy, \; \forall x \in [a,b]$$

$m(f)(d-c) \leq A(x) \leq M(f)(d-c)$
$\therefore A(x)$ is bounded.

also

$$A(x) = \sum_{j=0}^{n-1} \int_{y_j}^{y_{j+1}} f(x,y)\,dy \quad \forall x \in [a,b]$$

domain addition.

$$\sum_{j=0}^{n-1} m_{ij}(f)(y_{j+1}-y_j) \leq A(x) \leq \sum_{j=0}^{n-1} M_{ij}(f)(y_{j+1}-y_j)$$

$$\sum_{j=0}^{n-1} m_{ij}(f)(y_{j+1}-y_j) \leq m_i(A) \leq M_i(A) \leq \sum_{j=0}^{n-1} M_{ij}(f)(y_{j+1}-y_j)$$

multiply by $(x_{i+1}-x_i)$
and sum over $i=0, \dots k-1$

$$L(f, P_\varepsilon) \leq \sum_{i=0}^{k-1} m_i(A)(x_{i+1}-x_i) \leq \sum_{i=0}^{k-1} M_i(A)(x_{i+1}-x_i)$$
$$\leq U(f, P_\varepsilon)$$

cancels

$$\boxed{U(A, P_1) - L(A, P_1) < \varepsilon}$$

* Note:
The both iterated integrals may exist but function f may not be double integrable.

Eg: $R = [0,1] \times [0,1]$

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{(x^2+y^2)^3}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$\cancel{\int_0^1 \frac{xy(x^2-y^2)}{(x^2+y^2)^3}\, dy}$$

* **Note:**

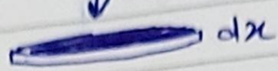
$f(x,y) = \phi(x)\psi(y)$

$$\iint_R f(x,y)\,dx\,dy = \left(\int \phi(x)\,dx\right)\left(\int \psi(y)\,dy\right)$$

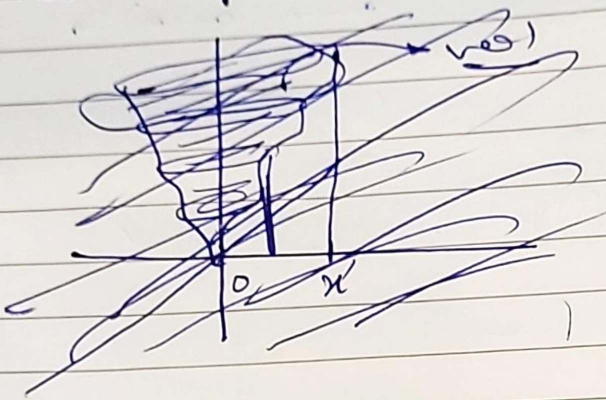
* Cavalieri's principle:

volume of two solid are equal if the areas of corresponding cross sections are equal.

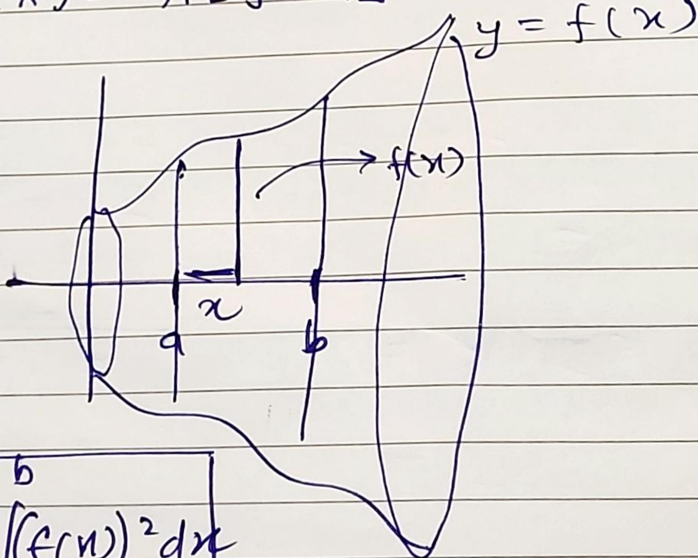
- **Slice method** :
- slice in dish
- add all
- $\int_a^b \underbrace{A(x)\,dx}_{\text{disk (small vol)}}$



- for solid of revolution



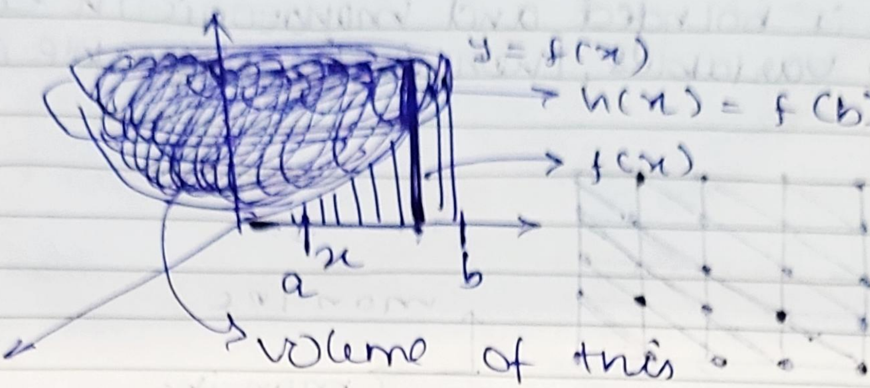
$$A(x) = \pi[f(x)]^2$$



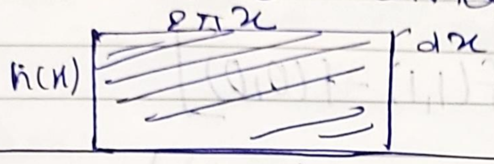
$$\boxed{V = \pi \int_a^b (f(x))^2\,dx}$$

for y axis → inverse.

* Shell method :



$2\pi h(x) x \rightarrow$ small area



$$\boxed{\int_a^b 2\pi h(x) x \, dx = \text{vol}}$$

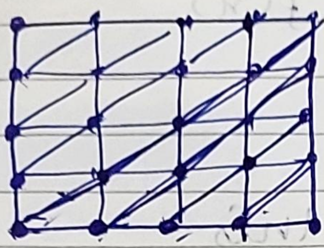
* washer method :

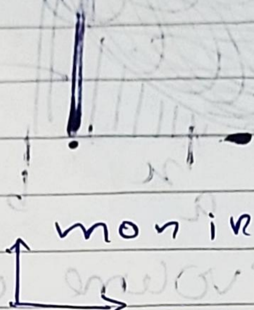
$$\pi \int_a^b [f_2(x)^2 - f_1(x)^2] dx$$

$$2\pi \int_a^b x [h_2(x) - h_1(x)] dx$$

* Theorm:

If f is bounded and monotonic in each two variables, then f is integrable in R.





$$U(f, P_n) - L(f, P_n)$$

$$\leq \frac{1}{n^2} \sum_{i=1}^{2n-1} [f(1,1) - f(0,0)]$$

$$= \frac{2n-1}{n^2} (f(1,1) - f(0,0))$$

$$\to 0 \quad \text{as } n \to \infty$$

• Theorm:

If a function $f : R \to \mathbb{R}$ is bounded and cont on R except possibly finitely many points in R, then f is integrable on R.

"Zero areas"

Bounded subset E of $\mathbb{R}^2$ has 0 area if $\forall \varepsilon > 0$, finitely many rects whose union contains E and sum $< \varepsilon$.

- Graph of cont function — 0 area.

* Theorm:

If a fuction f is bounded and cont on a rectagle $R = [a,b] \times [c,d]$ except possibly along finite number of graphs of cont. fuction, then f is integrable on R.

* Theorm:

given a rectagle R and a bounded fuction $f: R \to \mathbb{R}$, the function is integrable over R if the points of discontinuity of f is a set of 'content zero'.
$\hookrightarrow$ or has zero area.

converse:

There are integrable fuctions whose point of discontinuity is not a set of 'content zero'.

Counter example:

Bivaridte Thomae function

$f: [0,1] \times [0,1] \to \mathbb{R}$ is defined by

$$f(x,y) = \begin{cases} 1 ; & \text{if } x = 0, y \in \mathbb{Q} \cap [0,1] \\ \frac{1}{q} ; & x, y \in \mathbb{Q}[0,1] \text{ and } x = \frac{p}{q} \\ & p, q \text{ rel prime} \\ 0 ; & \text{otherwise} \end{cases}$$

* integrals over any bound subset

$$f^*(x,y) = \begin{cases} f(x,y), & (x,y) \in D \\ 0, & (x,y) \notin D \end{cases}$$

as $f(x,y)$ bounded

$x^2+y^2 \leq a^2$

$D = [-a,a] \times [-a,a]$

$$\iint_D f(x,y)\,dx\,dy = \iint_R f^*(x,y)\,dx\,dy$$

* Domain additive property :

$$\iint_D f = \iint_{D_1} f + \iint_{D_2} f$$

$D = D_1 \cup D_2$

$D_1 \cap D_2$ content zero.

* Boundry :

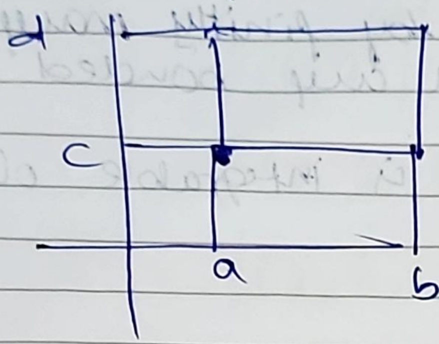
Let $D \subseteq \mathbb{R}^2$ be a bounded set.
A point in the boundry of D is one which has a seq in D and a seq in $\mathbb{R}^2 - D$ conv to it.

$D = \{(x,y) \mid x^2+y^2 \leq r^2\}$

$\partial D = \{(x,y) \mid x^2+y^2 = r^2\}$

* Boundry of rectangle:

$R = [a, b] \times [c, d]$



$\partial R = \{(a, y) \in \mathbb{R}^2 \mid c \leq y \leq d\} \cup \{(b, y) \in \mathbb{R}^2 \mid c \leq y \leq d\}$
$\cup \{(x, c) \in \mathbb{R}^2 \mid a \leq x \leq b\} \cup \{(x, d) \in \mathbb{R}^2 \mid a \leq x \leq b\}$

$S = \{(x, y) \mid x, y \in \mathbb{Q}\}$ $\boxed{\partial S = \mathbb{R}^2}$

* Note: For $f: D \to \mathbb{R}$ to be integrable we need ∂D to be content zero.

* Path and a curve:

path
φ in $\mathbb{R}^2$ (or $\mathbb{R}^3$) is a cont function

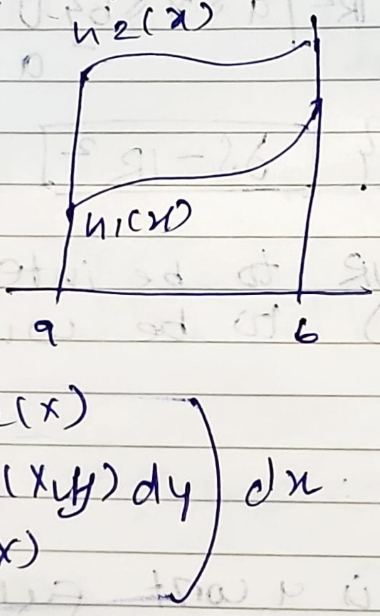
$\varphi: [a, b] \to \mathbb{R}^2$ (or $\varphi: [a, b] \to \mathbb{R}^3$)

closed if
$\varphi(a) = \varphi(b)$

* **Theorm:**
let $D \subset \mathbb{R}^2$ be a bouded set whose boundry ∂D is given by finitly many continous curves then any bounded and cont. function
$f : D \to \mathbb{R}$ is integrable over D.

* **Elementary regions**
- Type 1:

$$D_1 = \{(x,y) \mid a \leq x \leq b \text{ and } h_1(x) \leq y \leq h_2(x)\}$$



$$\int_a^b \left(\int_{h_1(x)}^{h_2(x)} f(x,y)\, dy \right) dx$$

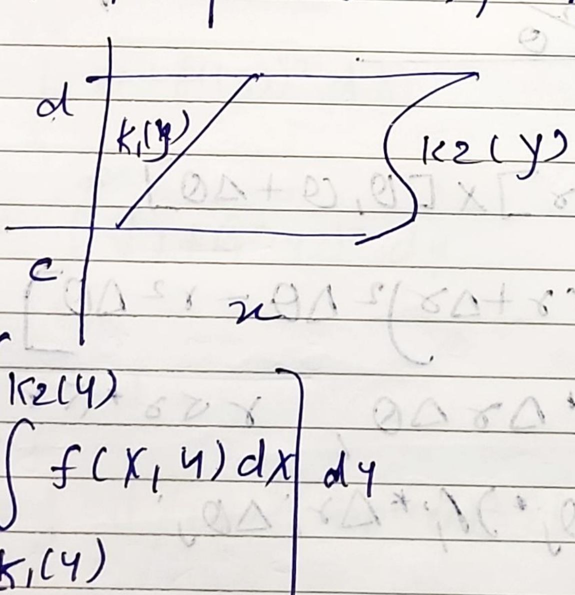
note:

$$\partial D_1 = \{(a,y) \mid h_1(a) \le y \le h_2(a)\} \cup$$
$$\{(b,y) \mid h_1(b) \le y \le h_2(b)\} \cup$$
$$\{(x, h_1(x)) \mid a \le x \le b\} \cup$$
$$\{(x, h_2(x)) \mid a \le x \le b\}$$

$\partial D_1 \to$ content zero.

- Type 2 :

$$D_2 = \{(x,y) \mid c \le y \le d,\ k_1(y) \le x \le k_2(y)\}$$



$$\int_c^d \left[\int_{k_1(y)}^{k_2(y)} f(x,y)\,dx \right] dy$$

- Type 3 :

step shaped subset of $\mathbb{R}^2$

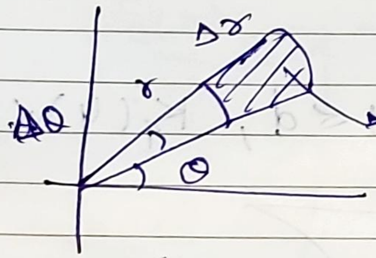
Annulus ◎ → mix/subtract/
divide in 4 quadrants.

* Polar coordinates:

$x = r\cos(\theta)$
$y = r\sin(\theta)$
$r > 0, \quad \theta \in [0, 2\pi]$.

$D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le a^2\}$
$\downarrow$
$D^* = \{(r,\theta) \mid 0 \le r \le a, \theta \in [0,2\pi]\}$

$g(r,\theta) = f(r\cos\theta, r\sin\theta)$



$[r, r+\Delta r] \times [\theta, \theta+\Delta\theta]$

area $= \frac{1}{2}\left[(r+\Delta r)^2 \Delta\theta - r^2\Delta\theta\right]$
$\approx r^* \Delta r \Delta\theta, \quad r \le r^* \le r+\Delta r$

$\sum_i \sum_j g(r_i^*, \theta_j^*) r_i^* \Delta r_i \Delta\theta_j$

$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$

- Parametrized curve:
  $(x(t), y(t))$ path C
  $x, y : [\alpha, \beta] \to \mathbb{R}$ are cont. functions
  $\underbrace{[\alpha, \beta]}_{\text{paramater interval}}$

length C
- smooth

$$l(C) = \int_{\alpha}^{\beta} \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

as $\sum \sqrt{(x(t_i) - x(t_{i-1}))^2 + (y(t_i) - y(t_{i-1}))^2}$

$= \sum_{i=1}^{n} \sqrt{(x'(s_i))^2 + (y'(u_i))^2}\, (t_i - t_{i-1})$

$$l(C) = \int_{a}^{b} \sqrt{1 + (f'(x))^2}\, dx$$

$$l(C) = \int_{c}^{d} \sqrt{1 + (g'(y))^2}\, dy$$

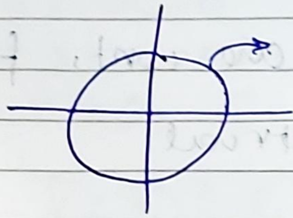
Polar word:

$$l(C) = \int_{\alpha}^{\beta} \sqrt{\rho(\theta)^2 + \rho'(\theta)^2}\, d\theta$$

r

$x(\theta) := \rho(\theta) \cos\theta$
$y(\theta) := \rho(\theta) \sin\theta$

* length of circle of radius :



$x(\theta) = a\cos\theta$

$y(\theta) = a\sin\theta$

$$\int_{-\pi}^{\pi} \sqrt{(P(\theta))^2 + (P'(\theta))^2}\, d\theta$$

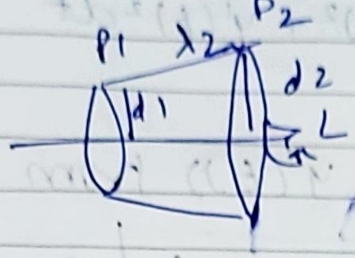
$$= \int_{-\pi}^{\pi} \sqrt{a^2}\, d\theta$$

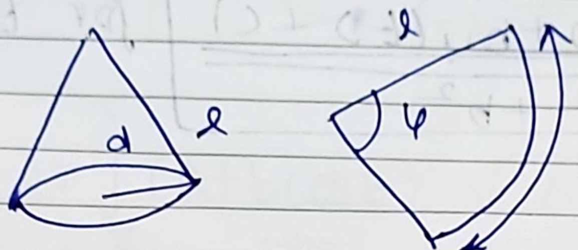
$$= |a|(2\pi) = \boxed{2\pi a}$$

* for $\mathbb{R}^3$:

$$l(C) = \int_{\alpha}^{\beta} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$$

* Surface of revolution :-
curve C in $\mathbb{R}^2$ is revolved about line L in $\mathbb{R}^2$.





$2\pi d = l\varphi$

$\frac{1}{2} l^2 \varphi = \boxed{\pi l d}$

S.A of frustum:

$\pi(d_1+d_2)(l_2-l_1) = \boxed{\pi(d_1+d_2)\lambda_2}$

$P_i = (x(t_i), y(t_i))$ for $i = 0,1,\dots n$

$\alpha = t_0 < t_1 < \dots < t_n = \beta$

$d_0, d_1, d_2 \dots d_n$
dist for $P_0, P_1, \dots P_n$
from L.
Let $\lambda_1, \lambda_2, \dots \lambda_n$ be line seg

$\pi \sum_{i=1}^{n} (\rho(t_{i-1}) + \rho(t_i))\lambda_i$

$\boxed{d_i = \rho(t_i)}$

$\lambda_i = \sqrt{x'(s_i)^2 + y'(u_i)^2}\ (t_i - t_{i-1})$

$$A(S) = 2\pi \int_{\alpha}^{\beta} \rho(t)\sqrt{x'(t)^2 + y'(t)^2}\,dt$$

$\rho(t)$ dist of $(x(t), y(t))$ from L.

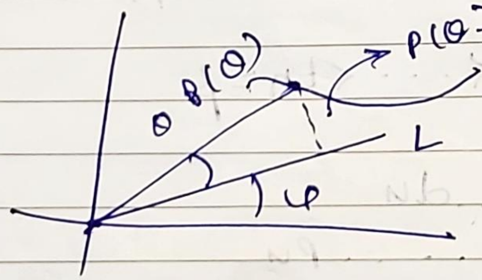
$$\boxed{\rho(t) = \frac{|ax(t) + by(t) + c|}{\sqrt{a^2+b^2}}}$$ for $t \in [a, b]$

note

$$A(S) = 2\pi \int_{\alpha}^{\beta} \rho(t)\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

$$A(S) = 2\pi \int_{\alpha}^{\beta} |f(x)|\sqrt{1 + (f'(x))^2}\,dx$$

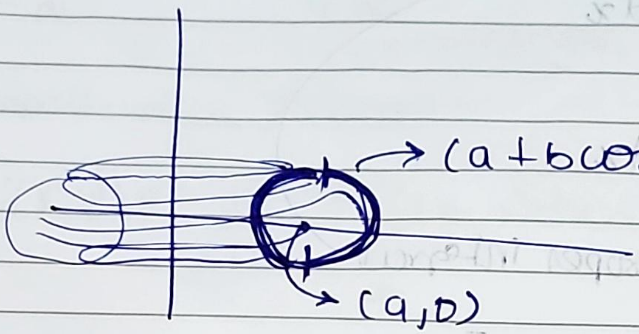
* Note:



$$\boxed{\rho(\theta) = P(\theta)\,|\sin(\theta - \varphi)|}$$

$$\boxed{A(S) = 2\pi \int_{\alpha}^{\beta} P(\theta)\,|\sin(\theta - \varphi)|\,\sqrt{P'(\theta) + P(\theta)^2}\,d\theta}$$

* Surface area of torus :

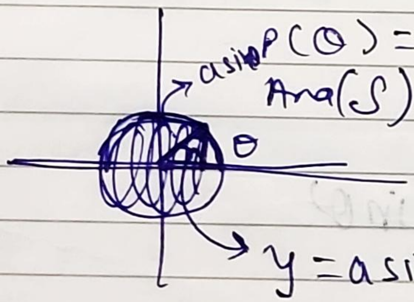


$$2\pi\int_{-\pi}^{\pi}(a+b\cos t)\sqrt{b^2}\,dt$$

$$= 2\pi b\int_{-\pi}^{\pi}(a+b\cos t)\,dt$$

$$= \boxed{4\pi^2 ab}$$

* Surface area of sphere :



$\rho(\theta) = a$

$$\text{Area}(S) = 2\pi\int_0^{\pi} a\sin\theta\sqrt{a^2+0}\,d\theta$$

$$= \boxed{4\pi a^2}$$

* Integral of gaussian :

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx$$

$$\lim_{T \to \infty} \int_{-T}^{T} e^{-x^2} dx$$

improper integral

$$I^2 = \int_{-T}^{T} e^{-x^2} dx \int_{-T}^{T} e^{-x^2} dx$$

$$= \int_{-T}^{T} e^{-x^2-y^2} dx\, dy$$

$$= \int_{-T}^{T}\int_{-T}^{T} e^{-(x^2+y^2)} dx\, dy$$

$x = r\cos\theta$

$y = r\sin\theta$

$e^{-x^2-y^2} = f(x,y)$

$g(r,\theta) = f(r\cos\theta, r\sin\theta)$

$g(r,\theta) = e^{-r^2}$

$$\int_0^{2\pi}\int_0^{\infty} e^{-r^2}.\, r.\, dr.\, d\theta$$

$$= \int_0^{2\pi} \left[-\frac{1}{2} e^{-r^2}\right]_0^{\infty} d\theta = \pi \Rightarrow I = \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$$

* Mean-value theorem -

$$f(x', y') = \frac{1}{A(D)} \iint_D f(x, y)\, dx\, dy$$

proof: since f is bounded, cont
min-max at some point $(x_0, y_0) \in D$
and $(x_1, y_1) \in D$. since D is elementary
$\exists\ \varphi : [0,1] \to \mathbb{R}^2$
s.t $\varphi(0) = (x_0, y_0)$
$\varphi(1) = (x_1, y_1)$

now $f \circ \varphi : [0,1] \to \mathbb{R}$
IVT on this

$$f(x_0, y_0) \times A(D) = \iint_D f(x, y)\, dx\, dy$$

Application:
centre of mass (wt. avg)

$$\bar{x} = \frac{\iint_D x\, \rho(x, y)\, dx\, dy}{\iint_D \rho(x, y)\, dx\, dy}$$

$$\bar{y} = \frac{\iint_D y\, \rho(x, y)\, dx\, dy}{\iint_D \rho(x, y)\, dx\, dy}$$

* analogy to 3 variable :

$f : B = [a,b] \times [c,d] \times [e,f] \longrightarrow \mathbb{R}$

integrale over rectagle cuboid.

$S(f, P_n, t) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} f(t_{ijk}) \underbrace{\Delta B_{ijk}}_{\text{vol}}$

if integrable if
as $n \to \infty$
$\lim_{n \to \infty} S(f, P_n, t)$ convgy to some S
for any t.

$\iiint_B f \, dv$, $\iiint_B f(x,y,z) \, dV$ or $\iiint_B f(x,y,z) \, dx\,dy\,dz$

- Note:
$f : B \subset \mathbb{R}^3 \to \mathbb{R}$ is bounded + cont in B
except possibally (finit / union of finit)
graphs of form:
$y = b(x,z)$
$z = a(x,y)$
$x = c(y,z)$ then integrable.

contet zero

$\iiint_{B^*} f^* = \iiint_B f$

* <u>Fubini's theorm</u> :

$$\iiint_B f(x,y,z)\,dx\,dy\,dz = \int_a^b \int_c^d \int_e^f f(x,y,z)\,dz\,dy\,dx$$

<u>RHS</u>: iterated integral exist

<u>Note</u>: If f is integrable, whenever any of these iterated integral exist, it is equal to the value of the integral of f over B. If f is continuous on B, then f is integrable on B and all iterated integrals exist and their values are equal to integral of f on B.

* <u>Elementry regions in $\mathbb{R}^3$</u>:

$$W = \{(x,y,z) \in \mathbb{R}^3 \mid \gamma_1(x,y) \leq z \leq \gamma_2(x,y), (x,y) \in D\}$$

where D is <u>elementry</u> region in $\mathbb{R}^2$.

$$\iiint_W f(x,y,z)\,dx\,dy\,dz = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

* Change of variables in $\mathbb{R}^2$:

Ω open subset of $\mathbb{R}^2$ and

$u : \Omega \to \mathbb{R}^2$

one-one transformation

$u(u,v) = (h_1(u,v), h_2(u,v))$

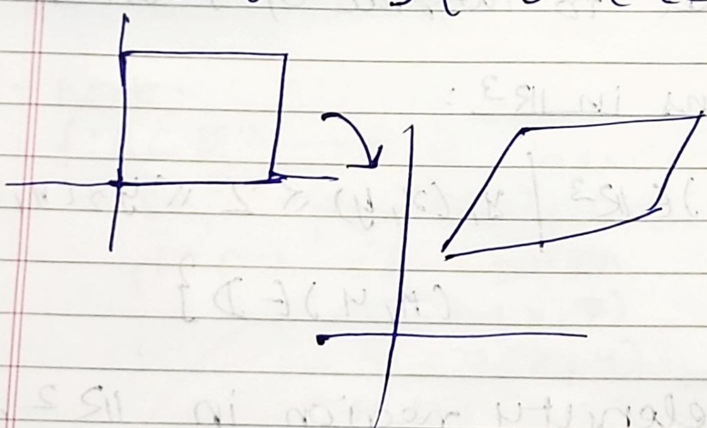
$\forall (u,v) \in \Omega$

$x = h_1(u,v)$

$y = h_2(u,v)$

$$\left.\begin{aligned} x &= au + bv + t_1 \\ y &= cu + dv + t_2 \end{aligned}\right\} \to \text{affine linear functions}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$



$$(a, c, 0) \times (b, d, 0) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \cdot \hat{k}$$

* general:

$x = h_1(u,v)$

$y = h_2(u,v)$

$$\Delta x = h_1(u+\Delta u, v+\Delta v) - h_1(u,v) \approx \frac{\partial h_1}{\partial u}\Delta u + \frac{\partial h_1}{\partial v}\Delta v$$

$$\Delta y = h_2(u+\Delta u, v+\Delta v) - h_2(u,v) \approx \frac{\partial h_2}{\partial u}\Delta u + \frac{\partial h_2}{\partial v}\Delta v$$ (taylor's theorm)

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v} \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix}$$

$$J(h) = \begin{pmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v} \end{pmatrix}$$

* Theorm:

D be closed and bonded $\subseteq \mathbb{R}^2$ s.t ∂D has cont. zero.

let $f: D \to \mathbb{R}$ be cont.

$h: \Omega \to \mathbb{R}^2$ one-one diff funtion

s.t $\det(J(h)) \neq 0$

$D^* \subset \Omega$ s.t $\boxed{h(D^*) = D}$

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} (f \circ h)(u,v) \left|\det J h\right| du\,dv$$

Note:

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

$$\iint_D f(x,y)\,dx\,dy$$

$$= \iint_{D^*} f(x(u,v), y(u,v)) \cancel{du\,dv} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

* Bivariate thome function:
(counter example)

$$f(x,y) = \begin{cases} 1; & \text{if } x=0,\ y \in \mathbb{Q} \cap [0,1] \\ \frac{1}{q}; & \text{if } x,y \in \mathbb{Q} \cap [0,1],\ \text{and } x = \frac{p}{q} \\ 0; & \text{otherwise} \end{cases}$$

~~Proof:~~

~~first $\forall \varepsilon > 0$,~~

~~$f(x,y) \geq \frac{\varepsilon}{2}$~~ $\varepsilon \geq 2$ for some $y \in [0,1]$ and $x \in [0,1]$

~~as if $\cancel{I \geq \frac{\varepsilon}{2}}$~~

~~then~~

and $\frac{1}{q} \geq \frac{\varepsilon}{2}$ $\quad \frac{2}{\varepsilon} \geq q$

$q \leq \frac{2}{\varepsilon}$

integers

so finit values. $\quad q \geq 1$

~~suppose~~
~~$B = \{ x \in (0,1] : f(x,y) \geq \frac{\varepsilon}{2}$ for some $y \in [0,1] \}$~~
~~$|B| = m$ suppose~~

Proof:
first for some q

$$f(x) = \begin{cases} \frac{1}{q} ; & x \in \mathbb{Q} \cap [0,1] \\ 0 ; & \text{otherwise} \end{cases}$$

now,
$\forall \varepsilon > 0$,
if $\boxed{\exists N \in \mathbb{N}, \text{s.t. } \frac{1}{N} < \varepsilon}$ $\quad$ Bounded $\quad A_N = \{ x \mid q \leq N \}$

now,
if there is a partition P in $(0,1]$
and $\max\{x_i - x_{i-1}\} < \frac{\varepsilon}{|A_N|}$

note: max $2|A_N|$ elements b/w (x_i, x_i)
then

$$U(f,P) = \sum_{i=1}^{n} (x_i - x_{i-1}) \sup (f(x))$$
in $[x_{i-1}, x_i]$

$$= \sum_{[x_{i-1}, x_i] \cap A_N = \phi} (x_i - x_{i-1}) \sup (f(x))$$
in $(\quad)$

$$+ \sum_{[x_{i-1}, x_i] \cap A_N \neq \phi} (x_i - x_{i-1}) \sup (f(x))$$
in $(\quad)$

$$< \frac{\varepsilon}{|A_N|} (2|A_N|) + (1)\left(\frac{1}{N}\right) < 2\varepsilon + \varepsilon = 3\varepsilon$$

so $\int_0^1 f(x,y)\,dx \rightarrow$ exist ~~and~~

then $\int_0^1 \int_0^1 f(x,y)\,dx\,dy \rightarrow$ exist