CS 405

Game theory and Algorithmic Mechanism Design

2 x 15% – two quizes
10% – scribing in group
30% midsem
30% endsem

Saturday sept 6 : Quiz1 – 2:30 to 4 pm
Oct 22 : Quiz2 during class

problem sets: week before every exam
(2 quizes and 2 exams)
supposed to solve all problems

Tutorials – some Sat/Sun before exam
Specific problem on Piazza atleast 2 days before

Join piazza and course webpage

Res: game theory – Michael Maschler, Eilon Solan
Lec notes on mech design – Debasis Mishra, ISI

30th July:

Eg: game of king, queen and kindoms

		King: Agric	War
queen	Ag	5,5	0,6
	war	6,0	1,1

king's satisfaction

queen satisfaction

either select war or Agriculture

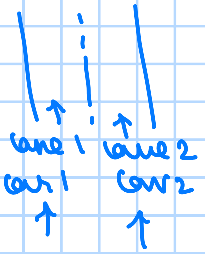
we want to take a decision for the player, the decision has to be simultanious (let's assume at start of the year)

The solution is to choose war, as we will never have satisfaction 0 and if Agri chosen, we can loose everything

Note: The above choice is called a 'dominant' choice, and the above strategy is called 'dominant strategy'. (war, war) is a 'dominant strategy equilibrium.

There is a differece between socially optimal and equilibrium. we want to design games s.t socially optimal behavior becomes equilibrium.

Eg: 2 cars



		car 2: left	right
car 1	left	5,5	-10,-10
	right	-10,-10	5,5

→ collision

collision

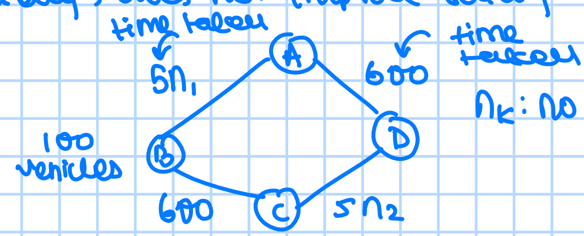
The above game does not have a dominant strategy as there is no 'best' choice.

Nash equilibrium

Equilibrium here is a strategy profile from when no player wants to unilaterally deviate

Note: dominant strategy ⇒ Nash equilibrium
Nash equilibrium ⇏ dominant strategy

Eg: Adding resources (blindly) does not improve society

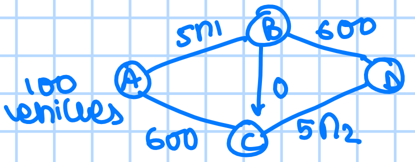


n_k: no of vehicles on road k

game strategy:
wlog
1st player: B → A → D
2nd player: B → C → D
3rd player: B → A → D
⋮
2k player: B → C → D
2k+1 player: B → A → D

↓ one by one

Now equilibrium: 50 vehicles on each path
time for each vehicle = 50 × 5 + 600 = 850



$5 \times 99 < 600$ so always Ⓐ → Ⓑ → Ⓒ → Ⓓ is cheaper path

so, if there are 100 vehicles, this strategy becomes dominant strategy.
$5 \times 100 + 5 \times 100 = 1000$ is new time taken where Ⓐ⟷Ⓒ and Ⓑ⟷Ⓓ are wasted.

Eg: Auction

	player 1	player 2	player 3
value:	35	20	75

→ this values are private information

we want to ensure that the painting goes to the customer with highest value.

we ask everyone to bid in an envelop and submit, find out which is highest but payment will be the second price

The above game is called the second price aution, the above game will be played fairly as:

	player 1	player 2	player 3
value	35	20	75
priced	80	20	75

↑ if player 1 wins they pay 75 which is 40 more than valuation so loss of player 1.

priced	30	20	35

↓ less value then they does not win but if placed actual value they would have won at 35, so always better to place actual valuation

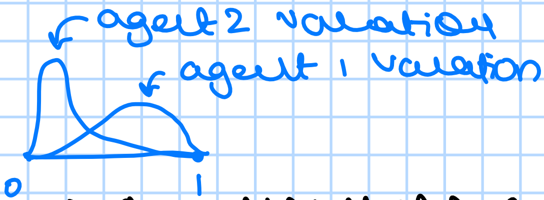
fair division:
we want to design algoritms for a better society

B Day cake

Hetrogeneous: equal amount may have different values for an agent
Divisible: any fraction allocated is feasable
Differing prefernce: different agents may have different prefernce for the same price

Normalisation: $\forall i, \; V_i([0,1]) = 1$ ← whole cake; V_i → valuation for agent i

$\forall i, \; V_i(A_i) \geq \frac{1}{n}$ as each agent gets atleast the average (PROP)



Envy free world: $\forall i \neq j \;\; V_i(A_i) \geq V_i(A_j)$ for agent i and j
(EF)

here $V_i(A_i) \geq V_i(A_j) \; \forall j$
$\Rightarrow V_i(A_i) \geq \frac{1}{n}[V_i(A_1) + V_i(A_2) + \dots V_i(A_n)]$

$\Rightarrow v_i(A_i) \geq \frac{1}{n}[v_i([0,1])]$

$\Rightarrow v_i(A_i) \geq \frac{1}{n}$

ED $\Rightarrow$ PROP

Note: propotionality is implied if EF is achieved

I cut, you choose algorithm also ensures envyfree algoritm

PROP: yes, Agent 1 cuts 1/2 and agent 2 picks larger

EF: yes, as $v_j(A_j) \geq v_j(A_i)$ $\forall i \neq j$ ensured

1st Aug.

In an eng. course we are given a circuit and we want to get certain outcome but in game theory we are given game and we have to predict the outcomes.

Game :

For our war v/s Agri game we want to find reasonable outcome.

Defn: A game is a formal rep of strategic interaction between players

Defn: A choice available to players/Agents is called action (As where $s \in S$)

Defn: The mapping from state of the game to action is called strategy ($d : S \to A$)

Note: In single-state games, strategy and action are equivalent, but not in multi-state games

games can be of many kinds and representation:
Normal form, Extensive form, static, dynamic, repeated, stochastic....

we will mostly do this

Defn: (game theory) Field of study of strategic interaction b/w players, who are rational and intelligent

A player is rational if she picks an action to achieve her most desired outcome to maximise their own happiness.

A player is intelligent if she knows the rules of the game perfectly and can pick the best action considering other rational and intl. players.

our goal is to predict the outcomes of the game.

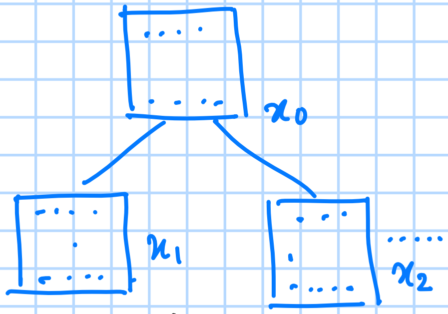
Defn: (common knowledge) A fact is CK if all players know the fact all players know that all players know the fact, and this in recursion goes on.

Eg: an isolated island, no reflecting device, see only other people, 3 men (eye color Blue or Black) but not talk about eye color, 4th person: True Statement: Blue eye are bad and should leave and atleast one blue eyed man present.

Case 1: 1 blue eyed man: first day they leave
Case 2: 2 blue eyed man: first day nothing, second day both leaves
Case 3: 3 blue eyed man: third day all 3 leaves

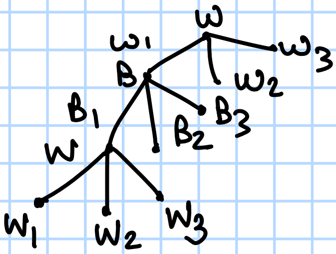
8X8 game :

W, B Agent, every piece has some action, starts with W
Ends: win for W, if W captures B's King
B captures W's Kings
Draw: No King is captured
we want to find if W has a winning strategy or B or draw
board position is x_k



Just by looking at board it is not possible to see where it came from, so we record history:

$(x_0, x_1 \dots x_k)$ is called game situation



Defn: (Strategy) A strategy for W is a fuction S_W which associates every game situation $(x_0, \dots x_k) \in H$, k even, with a board position x_{k+1} s.t $x_k \to x_{k+1}$ is a single valid move for W.

If we have S_W and S_B, we can determine our outcome

$$S_W(x_0) = x_1$$
$$S_B((x_0, x_1)) = x_2 \quad \text{(this is using history)}$$
$$S_W((x_0, x_1, x_2)) = x_3$$
$$\vdots$$

Now we want to find some S_W s.t it always wins.

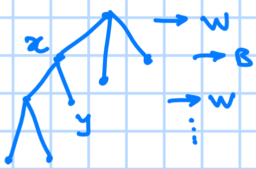
Defn: (winning strategy) is a strategy S_W^* s.t $\forall S_B$, (S_W^*, S_B) ends in win for W.

Similarly let S_W' be strategy which W wins or draws

Theorm: In our 8x8 board game one of the following is true:

either W has a winning strategy
either B has a winning strategy
Each player has a draw guarenteeing strategy

proof:



Each vertex in tree rep a game situation
$\Gamma(x)$: subtree rooted at x (including x)
n_x: no. of vertices in subtree $\Gamma(x)$

y: one of the nodes s.t $y \in \Gamma(x)$ but $y \neq x$

as $y \in \Gamma(x)$ but $y \neq x \Rightarrow n_y < n_x$ as $\Gamma(y) \subseteq \Gamma(x)$
for case $n_x = 1$ its trivial that theorm is true (only 3 possible option)
for $n_x > 1$, for all $y \neq x$, s.t $\Gamma(y) \subseteq \Gamma(x)$ the teorm holds (assumig) then $C(x)$ = children of x s.t

WLOG assume W moves at x, now:
case 1: $\exists y \in C(x)$ s.t condition 1 is true then
$x \to y$ by W and by induction we have a winning st

(case 1 is first as agents are rational)

case 2: $\forall y \in C(x)$, condition 2 is true, then every move by B wins and thorm holds

(otherwise all moves are bad)

case 3: Not case 1, Not case 2

$\forall y \in C(x)$, W does not have a winning strategy, so B has winning or both draw

(just basic not statements)

and $\exists y' \in C(x)$ s.t case 2 is not true

$\Rightarrow$ from $\sim$ case 1 $\wedge$ $\sim$ cas 2, y' is draw

$\Rightarrow$ W picks y' where B can only gaurtee a draw

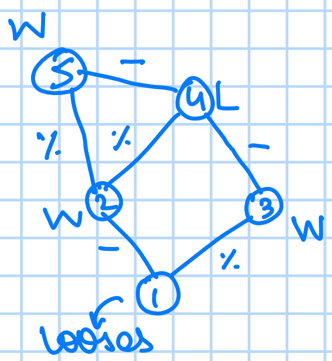
$(C_1 \cup C_2 \cup C_3 = 1 \Rightarrow C_3 = 1 - (C_1 \cup C_2)$
$= (C_1 \cup C_2)^c$
$C_3 = C_1^c \cap C_2^c)$

Halving game:

Two players, start with $n \rightarrow$ currNum
player +1 moves first
player -1 moves then

each player: decrement - i.e currNum $\rightarrow$ currNum-1
divide / ie currNum $\rightarrow \left\lfloor \frac{\text{currNum}}{2} \right\rfloor$

player who leaves board with 0 looses i.e currNum = 0



Note: In the above game we canot apply / prove the same theorm as our result depends on the initial number n.
(so theorm cannot be applied to games which share states)

6th Aug :

Normal form games:

it's a rep technique for games, particularly suitable for static games
players interact only once with each other

$N = \{1, 2 \dots n\} \rightarrow n$ set of players

$S_i \rightarrow$ player i set of strategies $s_i \in S_i$

$S = S_1 \times S_2 \times \dots S_n = \underset{i \in N}{X} S_i$ = set of strategy profiles $= S$

$s = (s_1, s_2 \dots s_n) \in S \rightarrow$ strategy profile

$s_{-i} = (s_1 \dots s_{i-1}, s_{i+1}, \dots s_n) \rightarrow$ strategy profile without player i

outcomes = strat profile $s = (s_i, s_{-i}) \rightarrow$ abuse of notation but good way of rep

$u_i : S \rightarrow \mathbb{R}$ is called the utility function of player i

s is the outcome of the game, utility function takes s as input and outputs in $\mathbb{R}$

strategy space

Normal form game representation will be $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$
all utility functions

Note: If S_i is finite, $\forall i \in N$ then the game is called finite game.

Eg: penalty shootout game

goalkeeper

shooter		L	C	R
	L	-1,1	1,-1	1,-1
	C	1,-1	-1, 1	1,-1
	R	1,-1	1,-1	-1, 1

diagonal is where shooter looses

$N = \{1, 2\}$ 1 = shooter 2 = goal keeper

$S_1 = S_2 = \{L, C, R\}$

$U_1(L, L) = -1$

$u_1(L, C) = u_1(L, R) = 1$

similarly all else

Domination in NFGs:

player 2

player 1	L	C	R
U	1,0	1,3	3,2
D	-1,6	0,5	3,3

C is a choice which gives player 2 best utility/Happiness

Defn: (strictly dominated strategy) $s_i' \in S_i$ is strictly dominated if $\exists s_i \in S_i$ s.t $\forall s_{-i} \in S_{-i}$

$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$

s_i strictly dominates s_i'

Defn: (weakly dominated strategy) A strategy $s_i' \in S_i$ of player i is weakly dominated if $\exists\, s_i \in S_i$ s.t $\forall s_{-i} \in S_{-i}$ $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$, $\exists\, \tilde{s}_{-i} \in S_{-i}$ s.t $u_i(s_i, \tilde{s}_{-i}) > u_i(s_i', \tilde{s}_{-i})$

Note: $\tilde{s}_{-i}$ is specific to pair of strategies (s_i, s_i') so $\tilde{s}_{-i}$ is a function of (s_i, s_i')

Defn: (Dominant strategy) s_i is (weakly) dominant strategy for i if s_i is strictly (weakly) dominates all other $s_i' \in S_i \setminus \{s_i\}$

Eg: Kindom's delema

	Agri	War
Agri	5,5	0,6
War	6,0	1,1

strictly better

strictly better

Indivisible items, two players have value V_1, V_2

$[0, M] \geq V_1, V_2$

Quote any $\mathbb{R}$ from it, player quoting larger wins and pay's loosing player's quote

utility of winning = wining θ - value by looser
utility of loosing = 0

$N = \{1,2\}$ $S_1 = S_2 = [0, M]$

$$u_1(s_1, s_2) = \begin{cases} v_1 - s_2 \,; & \text{if } s_1 \geq s_2 \\ 0 \,; & \text{otherwise} \end{cases}$$

$$u_2(s_1, s_2) = \begin{cases} v_2 - s_1 \,; & s_1 < s_2 \\ 0 \,; & \text{otherwise} \end{cases}$$

In the above example we want to show $s_1 = v_1$ is weakly dominant

① $u_1(v_1, s_2) \geq u_1(s_1, s_2)$ $\forall s_2, \forall s_1 \neq v_1$

② $u_1(v_1, \tilde{s}_2) > u_1(s_1, \tilde{s}_2)$, $\exists\, \underbrace{\tilde{s}_2(v_1, s_1)}_{\text{function of } (v_1, s_1)}$ $\forall s_1 \neq v_1$

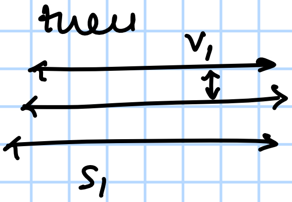
for ① case I: $v_1 \geq s_2$ then
$u_1(v_1, s_2) \geq u_1(s_1, s_2)$ is trivial

case II : $v_1 < s_2$ then
$u_1(v_1, s_2) \geq u_1(s_1, s_2)$ as now utility becomes negative

for ② let $\tilde{s}_2$ be between v_1 and s_1 or let $s_2 = \frac{v_1 + s_1}{2}$

then $u_1(v_1, \tilde{s}_2) > u_1(s_1, \tilde{s}_2)$ in both the cases

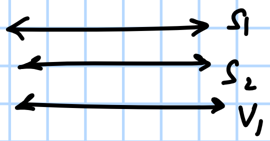
case I: $v_1 \geq s_1$ then



have $u_1(v_1, s_2) = v_1 - \left(\frac{v_1 + s_1}{2}\right) > 0$

$u_1(s_1, s_2) = 0$

case II : $v_1 < s_1$ then



$u_1(v_1, s_2) = 0$

$u_2(s_1, s_2) = v_1 - s_2 < 0 = u_1(v_1, s_2)$

so in both cases strictly inequality

Defn: (Dominant strategy equilibrium) A strategy profile $(s_1^*, \dots, s_n^*)$ is strictly (weakly) dominant strategy equilibrium (SDSE/WDSE) if s_i^* is strictly (weakly) dominant $\forall i \in N$

Eg:

	D	E
A	5,5	0,5
B	5,0	1,1
C	4,0	1,1

WD WD

Strategy player 1 B
player 2 E
$(B, E) = $ WDSE

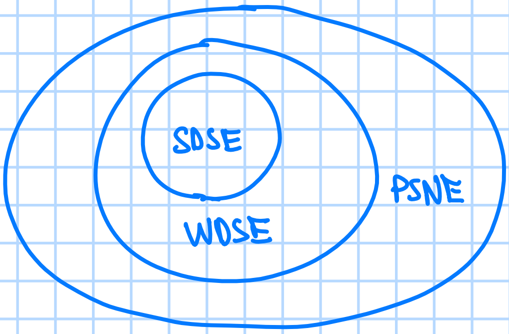
Nash Equilibrium:

No player gain by a unilateral deviation

Defn: (Nash Eq) A strategy profile (s_i^*, s_{-i}^*) is a pure strategy Nash Eq (PSNE) if $\forall i \in N$ and $\forall s_i \in S_i$

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$$

Note: SDSE $\Rightarrow$ WDSE $\Rightarrow$ PSNE



Best response of player i against the strategy profile s_{-i} of other players is a strategy that gives maximum utility i.e

$$B_i(s_{-i}) = \{ s_i \in S_i \mid u_i(s_i, s_{-i}) \geq u(s_i', s_{-i}), \forall s_i' \in S_i \}$$

PSNE is s.t (s_i^*, s_{-i}^*)

$$s_i^* \in B_i(s_{-i}^*) \quad \forall i \in N \text{ (By defination)}$$

so rational players do not play dominated strategies, and so to obtain rational outcomes eliminate dominated strategies, for strictly dominated strategies order of elimination does not matter but for weakly it does as $S_{-i}(s_i, s_i')$

8th Aug:

DSE: $u_i(s_i^*, s_{-i}) \geq u_i(s_i', s_{-i}) \ \forall s_{-i}, \forall s_i', \forall i$

($\forall s_{-i}, \forall s_i'$ is the diff b/w DSE and NE)

$(s_i^*) = $ DSE

PSNE: (s_i^*, s_{-i}^*) $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \ \forall s_i \in S_i$

Note: order of elimination matters in case of weakly dominated strategies

Eg:

WD by R

		L	C	R
WD by M	T	1,2	2,3	0,3
	M	2,2	2,1	3,2
WD by M	B	2,1	0,0	1,0

$\rightarrow$

	L	C	R
~~T~~	~~1,2~~	~~2,3~~	~~0,3~~
M	2,2	2,1	3,2
B	2,1	0,0	1,0

$\rightarrow$

	L	C	R
~~T~~	~~1,2~~	~~2,3~~	~~0,3~~
M	2,2	2,1	~~3,2~~
B	2,1	0,0	~~1,0~~

$\rightarrow$

	L	C	R
~~T~~	~~1,2~~	~~2,3~~	~~0,3~~
M	2,2	~~2,1~~	~~3,2~~
~~B~~	~~2,1~~	~~0,0~~	~~1,0~~

T, R, B, C order $\rightarrow$ (M, L) : (2,2)

one more way: B, L, C, T $\rightarrow$ (M, R) : (3,2)

} Both are nash equilibrium

Max-Min:

Risk aversion of players:

	L	R
T	2,1	1,-20 ← so we select this
M	3,0	-10,1
B	-100,2	3,3 → Nash Equilibrium

↑ But too risky (-100,2)

Defn: (max-min strategy) worst case optimal choice is maxmin strategy

$$s_i^{max\,min} \in \arg\max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$$

max; min of v_i from every row

Note: $s_{-i}^{min}(s_i) \in \arg\min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$ so is a function of s_i, as s_i changes minimizer keeps changing

max min value (utility of of max-min strategy) of player i is given by:

$$v_{-i} = \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$$

$$u_i(s_i^{maxmin}, t_{-i}) \geq v_{-i} \quad \forall t_{-i} \in S_{-i}$$

for given example

$v_{-1} = 1 \quad s_1^{maxmin} = T$

$v_{-2} = 0 \quad s_2^{maxmin} = L$

Note: $u_i(s_i, t_{-i}) \geq \min_{\tilde{s}_{-i} \in S_{-i}} u_i(s_i, \tilde{s}_{-i}) \ \forall s_i$

$$\Rightarrow u_i(s_i^{maxmin}, t_{-i}) \geq \min_{\tilde{s}_{-i} \in S_{-i}} u_i(s_i^{maxmin}, \tilde{s}_{-i})$$

$$= \max_{s_i} \min_{\tilde{s}_{-i}} u_i(s_i, \tilde{s}_{-i}) = v_{-i}$$

$$\Rightarrow u_i(s_i^{maxmin}, t_{-i}) \geq v_{-i} \quad \forall t_{-i} \in S_{-i}$$

Theorm: If s_i^* is dominant strategy for player i, then it will be a max-min strategy for player i as well. (DS ⇒ MM)

proof:
s_i^* = dominant strategy, so

$$u_i(s_i^*, s_{-i}) \geq u_i(s_i', s_{-i}) \; \forall s_i', s_{-i}$$

then let $s_{-i}^{min}(s_i') \in \underset{s_{-i} \in S_{-i}}{\arg\min}\, u_i(s_i', s_{-i})$

as equation holds $\forall s_{-i}$, so for $s_{-i}^{min}(s_i')$ also holds

$$\Rightarrow u_i(s_i^*, s_{-i}^{min}(s_i')) \geq u_i(s_i', s_{-i}^{min}(s_i')) \; \forall s_i' \in S_i$$

so as s_i^* is better than whatever oppoent plays as worst case for us, still s_i^* is more

$$\Rightarrow s_i^* \in \underset{s_i' \in S_i}{\arg\max}\, u(s_i', s_{-i}^{min}(s_i')) = \underset{s_i \in S_i}{\arg\max} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$$

$\Rightarrow s_i^*$ = max-min strategy

Theorm: Every PSNE $s^* = (s_i^*)$ of a normal form game satisfies $u_i(s^*) \geq \underline{v}_i \; \forall i \in N$

proof:

$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \; \forall s_i, \forall i \in N$ (this is by defination)

now, $u_i(s_i, s_{-i}^*) \geq \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) \; \forall s_i \in S_i$

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \; \forall s_i \in S_i$$

$$\Rightarrow u_i(s_i^*, s_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, s_{-i}^*) \geq \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$$

$$\Rightarrow u_i(s_i^*, s_{-i}^*) \geq \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) = \underline{v}_i$$

$$\Rightarrow u_i(s^*) \geq \underline{v}_i \; \forall i \in N$$

Elimination of strategies:

Iterated elimination of dominated strategies:

PSNE gives stability while maxmin gives security

In the given removal of weakly dominated stradegies the maxmin values:

	maxmin	player 1	player 2
	Before	2	0
Remove B:	After	2	2

player who is removing's maxmin value does not get affected

Theorm: Consider NFG $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ and let $s_j' \in S_j$ be a dominated strategy let G' be the residual game after removing s_j'. Then, the maximum value of j in G' is equal to her maxmin value of G.

proof: column wise j v/s i column, if j is dominating i
then
$\min_j \geq \min_i$ → this is because element wise

$j_k \geq i_k \geq \min i \Rightarrow j_k \geq \min i \Rightarrow \min j \geq \min i$
⇒ maxmin does not change

formally: maxmin value of j in G, $\underline{v}_j = \max \min u_j(s_j, s_{-j})$
maxmin value of j in G' $\underline{v}_j' = \max_{s_j \in S_j \setminus \{s_j'\}} \min u_j(s_j, s_{-j})$

suppose t_j dominates s_j' in G, $t_j \in S_j \setminus \{s_j'\}$ then

$$u_j(t_j, s_{-j}) \geq u_j(s_j', s_{-j}) \ \forall s_{-j} \in S_{-j} \text{ (from defn)}$$

$$\Rightarrow \min_{s_{-j} \in S_{-j}} u_j(t_j, s_{-j}) = u_j(t_j, \tilde{s}_{-j}) \geq u_j(s_j', \tilde{s}_{-j}) \geq \min_{s_{-j} \in S_{-j}} u_j(s_j', s_{-j})$$

this is v_{-j}

$$\Rightarrow \max_{s_j \in S_j \setminus \{s_j'\}} \min u_j(s_j, s_{-j}) \geq \min_{s_{-j} \in S_{-j}} u_j(s_j', s_{-j})$$

minima of removed

so $v_{-j} = \max\{ v_{-j}', \min_{s_{-j} \in S_{-j}} u_j(s_j', s_{-j})\}$

$= v_{-j}'$ from previous inequality $(v_{-j}' \geq \min_{s_{-j} \in S_{-j}} U_j(s_j', s_{-j}))$

Equilibrium after iterated elimination:

Theorm: consider $G, \hat{G}$ are games before and after eliminating a strategy (not nec dominated). If s^* is a PSNE in G and survives in $\hat{G}$, then s^* is PSNE in $\hat{G}$ too

proof: let us remove $s_i' \neq s_i^*$ from G, then from defination of PSNE

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \ \forall s_i \in S_i \setminus \{s_i'\}$$

and for $j \neq i$

$$u_j(s_j^*, s_{-j}^*) \geq u_j(s_j, s_{-j}^*) \ \forall s_j \in S_j$$ as this is same as previous inequality for PSNE of G

so, if s^* is PSNE of G, player i is playing, if player does not remove s_i^*, then

PSNE of $\hat{G} = s^*$

$\Rightarrow$ PSNE of $\hat{G}$ = PSNE of G

Theorm: Consider NFG G, let $\hat{s}_j$ be a weekly dominated strategy of j, if $\hat{G}$ is obtained from G eliminating $\hat{s}_j$, then every PSNE of $\hat{G}$ is a PSNE of G.

proof: in $\hat{G}$

$$\hat{S}_j = S_j \setminus \{\hat{s}_j\} \qquad \hat{S}_i = S_i \ \ \forall i \neq j$$

we have to show

$$s^* = (s_j^*, s_{-j}^*) \text{ is a PSNE } \hat{G} \Rightarrow \text{PSNE of } G$$

as, $u_i(s^*) \geq u_i(s_i, s_{-i}^*) \ \forall i \neq j, \forall s_i \in \hat{S}_i = S_i$

$$u_j(s^*) \geq u_j(s_j, s_{-j}^*) \ \forall s_j \in \hat{S}_j$$

for j as $\hat{s}_j$ is dominated, $\exists t_j \in \hat{S}_j$ s.t

$$u_j(t_j, s_{-j}) \geq u_j(\hat{s}_j, s_{-j}) \ \forall s_{-j} \in S_{-j}$$

$$\Rightarrow u_j(t_j, s_{-j}^*) \geq u_j(\hat{s}_j, s_{-j}^*)$$

as s^* is PSNE

$$\Rightarrow u_j(s_j^*, s_{-j}^*) \geq u_j(t_j, s_{-j}^*) \geq u_j(\hat{s}_j, s_{-j}^*)$$

$\Rightarrow$ so PSNE of G^* is PSNE of G as $u_j(\hat{s}_j, s_{-j}^*)$ is less and only thing removed

for $i \neq j$ trivial that PSNE $\Rightarrow$ PSNE of G

Matrix games:

A special class with certain nice requirities and stability properties

Defn: (two player zero-sum game) A NFG $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ $N = \{1, 2\}$ and $u_1 + u_2 = 0$

we call them matrix game as only one matrix can rep utilities as they are additive inverses.

13th Aug:

Matrix games:

two player zero-sum game

Defn: NFG $\langle N, (S_i)_{i\in N}, (u_i)_{i\in N}\rangle$ $N=\{1,2\}$ $u_1+u_2=0$

	L	R
L	-1,1	1,-1
R	1,-1	-1,1

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = u$$

row wise taken

	L	R	maxmin
L	-1	1	-1
R	1	-1	-1
minmax	1	1	

$\}$ -1

column wise taken

$\underbrace{\;}_{1}$

	L	C	R	maxmin
T	3	-5	-2	-5
M	1	4	1	1
B	6	-3	-5	-5
minmax	6	4	1	

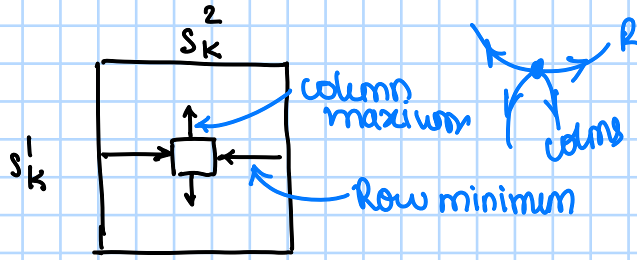
$\}$ max of all this: 1

min of all: 1

Note: PSNE of the penalty shoot game: Not exist

of the second game: (M, R)

Defn: (saddle point) The value is simultaneously the maximum in its column and minium in row, i.e maximum for player 1, minimum for player 2



$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ NO saddle point

$\begin{bmatrix} 3 & -5 & -2 \\ 1 & 4 & 1 \\ 6 & -3 & -5 \end{bmatrix}$ ← column max, row min (M, R)

Theorm: In a matrix game with utility matrix u (s_1^*, s_2^*) is a saddle point iff it is a PSNE

proof: consider (s_1^*, s_2^*) to be saddle point

$$\Leftrightarrow u(s_1^*, s_2^*) \geq u(s_1, s_2^*) \;\forall s_1 \in S_1$$

$$u(s_1^*, s_2^*) \leq u(s_1^*, s_2) \;\forall s_2 \in S_2$$

as $u \equiv u_1 \equiv -u_2$ (from defn of zero sum game)

$$\Leftrightarrow u_1(s_1^*, s_2^*) \geq u_1(s_1, s_2^*) \quad \forall s_1 \in S_1$$

$$u_2(s_1^*, s_2^*) \geq u_2(s_1^*, s_2) \quad \forall s_2 \in S_2$$

Saddle point $\Leftrightarrow$ PSNE

Note: $\underline{v} = \max_{s_1\in S_1} \min_{s_2 \in S_2} u(s_1, s_2)$ maxmin $\qquad s_1^* = \arg\max_{s_1 \in S_1} \min_{s_2\in S_2} u(s_1, s_2)$

$\overline{v} = \min_{s_2 \in S_2} \max_{s_1 \in S_1} u(s_1, s_2)$ minmax

Lemma: For a matrix game $\overline{v} \geq \underline{v}$

proof:

$$u(s_1, s_2) \geq \min_{t_2 \in S_2} u(s_1, t_2) \;\forall s_1, s_2$$

for fixed s_2: $f(s_1) \geq g(s_1) \;\forall s_1$

$$\max_{s_1 \in S_1} f(s_1) \geq \max_{\tilde{s}_1 \in S_1} g(\tilde{s}_1)$$

$\Rightarrow \max_{t_1 \in S_1} u(t_1, s_2) \geq \max_{t_1 \in S_1} \min_{t_2 \in S_2} u(t_1, t_2)$, $\forall s_2 \in S_2$

$\Rightarrow \min_{t_2 \in S_2} \max_{t_1 \in S_1} u(t_1, s_2) \geq \max_{t_1 \in S_1} \min_{t_2 \in S_2} u(t_1, t_2)$

$\Rightarrow \overline{v} \geq \underline{v}$

Relation b/w v and PSNE:

	L	R	maxmin
L	-1	1	-1
R	1	-1	-1
min max	1	1	

$\overline{v} = 1$

$\underline{v} = -1$

$\overline{v} > \underline{v}$ PSNE does not exist

	L	C	R
T	3	-5	-2
M	1	4	1
B	6	-3	-5

maxmin = 1

minmax = 1

$\overline{v} = 1 = \underline{v}$ PSNE exist

Note: $s_1^* \in \arg\max_{s_1 \in S_1} \min_{s_2 \in S_2} u(s_1, s_2)$

$s_2^* \in \arg\min_{s_2 \in S_2} \max_{s_1 \in S_1} u(s_1, s_2)$

Theorm: A game has PSNE (or saddle point) iff $\overline{v} = \underline{v} = u(s_1^*, s_2^*)$, also (s_1^*, s_2^*) = PSNE

proof:

$(\Rightarrow)$ let (s_1^*, s_2^*) be a PSNE, then

$u(s_1^*, s_2^*) \geq u(s_1, s_2^*)\ \forall s_2 \in S_2$

$\Rightarrow u(s_1^*, s_2^*) \geq \max_{t_1 \in S_1} u(t_1, s_2^*) = f(s_2^*)$

$\Rightarrow u(s_1^*, s_2^*) \geq \min_{t_2 \in S_2} \max_{t_1 \in S_1} u(t_1, t_2)$ as s_2^* is a specific choice $(\because f(s_2^*) \geq \min_{s_2} f(s_2))$

$\Rightarrow u(s_1^*, s_2^*) \geq \overline{v}$

for player 2: $u(s_1^*, s_2^*) \leq u(s_1^*, s_2)$, $\forall s_2 \in S_2$ $(\because u = -u_2)$

$\Rightarrow u(s_1^*, s_2^*) \leq \min_{t_2 \in S_2} u(s_1^*, t_2)$

$\Rightarrow u(s_1^*, s_2^*) \leq \max_{t_1 \in S_1} \min_{t_2 \in S_2} u(t_1, t_2) = \underline{v}$

now, $\overline{v} \leq u(s_1^*, s_2^*) \leq \underline{v}$

and $\underline{v} \leq \overline{v}$ $\Rightarrow$ $\underline{v} = \overline{v} = u(s_1^*, s_2^*)$

from previous theorm

$(\Leftarrow)$ $\overline{v} = \underline{v} = u(s_1^*, s_2^*)$ and s_1^*, s_2^* are maxmin and minmax

$u(s_1^*, s_2) \geq \min_{t_2 \in S_2} u(s_1^*, t_2)$ by defn

$= \max_{t_1 \in S_1} \min_{t_2 \in S_2} u(t_1, t_2)$ as s_1^* is maxmin strategy for 1

(1) — $\Rightarrow u(s_1^*, s_2) \geq \underline{v}\ \forall s_2 \in S_2$ $(\underline{v} = u(s_1^*, s_2^*))$

similarly $u(s_1, s_2^*) \leq \max_{t_1 \in S_1} u(t_1, s_2^*)$

$\Rightarrow u(s_1, s_2^*) \leq \min_{t_2 \in S_2} \max_{t_1 \in S_1} u(t_1, t_2)$ from defn of s_2^*

② — $\Rightarrow u(s_1, s_2^*) \leq \bar{v} \quad \forall s_1 \in S_1$ $(\bar{v} = u(s_1^*, s_2^*))$

now as $\left.\begin{aligned} &u(s_1, s_2^*) \leq u(s_1^*, s_2^*) \ \forall s_1 \in S_1 \\ &\& \ u(s_1^*, s_2) \geq u(s_1^*, s_2^*) \ \forall s_2 \in S_2 \end{aligned}\right\}$ from ①, ②

$\Rightarrow (s_1^*, s_2^*)$ is PSNE

Mixed strategy:

Probability distribution over set of that player

		Player 2	
Player 1		$\frac{4}{5}$ L	$\frac{1}{5}$ R
$\frac{2}{3}$	L	(-1,1)	(1,-1)
$\frac{1}{3}$	R	(1,-1)	(-1,1)

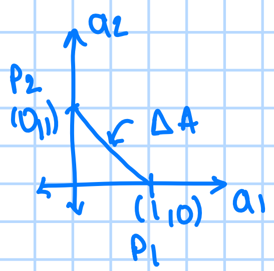
consider a fixed set A, define

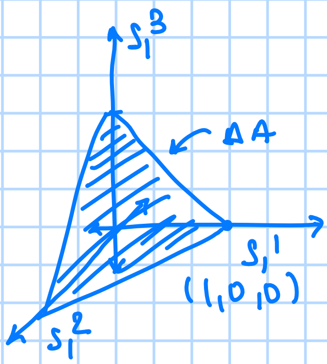
$$\Delta A = \{ p \in [0,1]^{|A|} \mid \sum_{a \in A} p(a) = 1 \}$$

strategy options / actions

$A = \{a_1, a_2, \dots, a_m\}$
$\quad\ \downarrow \quad \downarrow \quad \dots \quad \downarrow$
$\quad p_1 \quad p_2 \quad\ \ \ p_m$

$$\Delta A = \left\{ \begin{bmatrix} p_1 \\ \vdots \\ p_m \end{bmatrix} \in [0,1]^{[A]} \ \middle| \ \sum p_i = 1 \right\}$$





if player had 3 choices $\{s_1^1, s_1^2, s_1^3\}$ then ΔA is all probability

Note: σ_i is a mixed strategy for player i
$\sigma_i \in \Delta S_i$, $\sigma_i : S_i \to [0,1]$ s.t $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$

$$\sigma_i = \begin{bmatrix} \sigma_i(s_i^1) \\ \vdots \\ \sigma_i(s_i^{|S|}) \end{bmatrix} \in [0,1]^{|S|}$$

Note: we are discussing non-coorporative games, players choose strategy independently

joint probability of player 1 picking s_1 and player 2 picking $s_2 = \sigma_1(s_1)\sigma_2(s_2)$

utility of player i at a mixed strategy profile (σ_i, σ_{-i})

$$u_i(\sigma_i, \sigma_{-i}) = \sum_{s_1 \in S_1} \sum \cdots \sum_{s_n \in S_n} \sigma_1(s_1) \cdots \sigma_n(s_n) u_i(s_1 \dots s_n)$$

expection of utities at pure strategy

so, we are overloading u_i to deote the utility at pure and mixed stratgies

Mixed Strategies Nash equilibrium:

Defn: (MSNE) MSNE is a mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ s.t

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*), \ \forall \sigma_i \in \Delta S_i, \forall i \in N$$

Degenerate (prob masses → 1)

Note: PSNE ⇒ MSNE as $\bar{\sigma}_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = s_i \Rightarrow u_i(\bar{\sigma}_i, \bar{\sigma}_{-i}) \geq u_i(\sigma_i, \bar{\sigma}_{-i})$
fits in MSNE defination

$$u_i(s_i, \sigma_{-i}) = \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i})\, u_i(s_i, s_{-i}) \quad \text{where } \sigma_{-i}(s_{-i}) = \prod_{j \neq i} \sigma_j(s_j)$$

Theorm: $(\sigma_i^*, \sigma_{-i}^*)$ is MSNE iff $\forall s_i \in S_i$ and $\forall i \in N$ $u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*)$

proof:

(⇒) s_i is special case of mixed strategy with s_i prob = 1, then from defn of MSNE this is trivial

(⇐) for any σ_i of i:

$$u_i(\sigma_i, \sigma_{-i}^*) = \sum_{s_i \in S_i} \sigma_i(s_i)\, u_i(s_i, \sigma_{-i}^*)$$
$$\leq \sum_{s_i \in S_i} \sigma_i(s_i)\, u_i(\sigma_i^*, \sigma_{-i}^*)$$
$$= u_i(\sigma_i^*, \sigma_{-i}^*) \left[\sum_{s_i \in S_i} \sigma_i(s_i)\right]$$
$$= u_i(\sigma_i^*, \sigma_{-i}^*)$$

$$\Rightarrow u_i(\sigma_i, \sigma_{-i}^*) \leq u_i(\sigma_i^*, \sigma_{-i}^*) \ \forall \sigma_i \in \Delta S_i$$

eg:

	$\frac{4}{5}$ L	$\frac{1}{5}$ R
$\frac{2}{3}$ L	-1, 1	1, -1
$\frac{1}{3}$ R	1, -1	-1, 1

$$u_1(\sigma_1, \sigma_2) = \frac{2}{3} \cdot \frac{4}{5}(-1) + \frac{2}{3} \cdot \frac{1}{5}(1) + \frac{1}{3} \cdot \frac{4}{5}(1) + \frac{1}{3} \cdot \frac{1}{5}(-1) = -0.2$$

$$u_1\left(L, \left(\frac{4}{5}, \frac{1}{5}\right)\right) = (-1)\frac{4}{5} + (1) \times \frac{1}{5} < 0$$

$$u_1\left(R, \left(\frac{4}{5}, \frac{1}{5}\right)\right) = 1 \times \frac{4}{5} + (-1) \times \frac{1}{5} > 0$$

this mixed strategy profile is not MSNE
as $u_1(s_1, \sigma_2) > u_1(\sigma_1, \sigma_2)$ from prev theorm
also its trivial that R will be picked as most utility

Defn: (Support of mixed strategy) For mixed strategy σ_i, subset of strategy set of i on which σ_i has positive mass is called support of σ_i denoted as:

$$\delta(\sigma_i) = \{s_i \in S_i \mid \sigma_i(s_i) > 0\}$$

20th Aug:

Defn: (Support) For mixed strategy σ_i, $\delta(\sigma_i) = \{ s_i \in S_i \mid \sigma_i(s_i) > 0 \}$

Theorm: (characterization of MSNE) A mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ is an MSNE iff $\forall i \in N$,

1) $u_i(s_i, \sigma_{-i}^*)$ is same $\forall s_i \in \delta(\sigma_i^*)$

2) $u_i(s_i, \sigma_{-i}^*) \geq u_i(s_i', \sigma_{-i}^*)$, $\forall s_i \in \delta(\sigma_i^*)$, $s_i' \notin \delta(\sigma_i^*)$

Note: The above theorm also can be written as

$u_i(s_i, \sigma_{-i}^*) \geq u_i(s_i', \sigma_{-i}^*)$, $\forall s_i \in \delta(\sigma_i^*)$, $\forall s_i' \in S_i$

Eg:

	L	R
L	(-1,1)	(1,-1)
R	(1,-1)	(-1,1)

Case I: support profile $(\{L\},\{L\})$
then $u_1(L,L) = -1$
but $u_1(R,L) = 1$ so not support

Case II: support: $(\{L,R\},\{L\})$ - symmetrical for other cases
$u_1(L,L) \neq u_1(R,L)$, not support

Case III: Support: $(\{L,R\},\{L,R\})$

for this case

	L (q)	R ($1-q$)
L (p)		
R ($1-p$)		

$u_1(L,(q,1-q)) = u_1(R,(q,1-q)) \Rightarrow (-1)q + 1(1-q) = (1)q + (-1)(1-q)$
$\Rightarrow q = 1/2$

Similarly $p = 1/2$

$$MSNE = \left(\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right) \right)$$

Exe: solve the following and find MSNE

	F	C
F	(2,1)	(0,0)
C	(0,0)	(1,2)

	F	C	D
F	(2,1)	(0,0)	(1,1)
C	(0,0)	(1,2)	(2,0)

Ans: First game:

Case I: $\{F\}, \{F\}$ symmetrically other 1-1 case
then not possible as $U_2(F,F) < U_2(F,C)$

Case II: $\{F,C\}, \{F\}$ not possible as 2 diffent U_1

Case III: $\{F,C\}, \{F,C\}$ then let $((p,1-p),(q,1-q))$

now, $U_1(F,(q,1-q)) = U_1(C,(q,1-q))$
$q \times 2 + (1-q)0 = q(0) + (1-q)(1)$
$2q = 1-q$
$\Rightarrow q = 1/3$

$U_2((p,1-p),F) = U_2((p,1-p),C)$
$\Rightarrow p \times 1 + (1-p) \times 0 = p \times 0 + (1-p) \times 2$
$\Rightarrow p = 2 - 2p$
$\Rightarrow p = 2/3$

so $\left(\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right) \right)$ is MSNE second game: trivial to see only thing that will work is

$\{F, C\}$ for player 1 and $\{F, C\}$ for player 2

$U_2((p, 1-p), F) = p$

$U_2((p, 1-p), C) = 2(1-p) = 2-2p$

$\Rightarrow p = 2-2p$

$\Rightarrow p = 2/3$

$U_1(F, (q, 1-q)) = U_1(C, (q, 1-q))$

$\Rightarrow 2 \times q + 0 = 0 + 1 - q$

$\Rightarrow q = 1/3$

so $\left(\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, 0\right)\right) = \text{MSNE}$

<u>Theorem</u>: (characterization of MSNE) A mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ is an MSNE iff

$\forall i \in N$, 1) $u_i(s_i, \sigma_{-i}^*)$ is same $\forall s_i \in \delta(\sigma_i^*)$

2) $u_i(s_i, \sigma_{-i}^*) \geq u_i(s_i', \sigma_{-i}^*)$, $\forall s_i \in \delta(\sigma_i^*)$, $s_i' \notin \delta(\sigma_i^*)$

<u>proof</u>: firstly we see $\max_{\sigma_i \in \Delta S_i} u_i(\sigma_i, \sigma_{-i}) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i})$

as maximizing w.r.t a distribution $\Leftrightarrow$ whole prob mass at max

if $(\sigma_i^*, \sigma_{-i}^*)$ is MSNE then

$$\max_{\sigma_i \in \Delta S_i} u_i(\sigma_i, \sigma_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*) = \max_{s_i \in \delta(\sigma_i^*)} u_i(s_i, \sigma_{-i}^*)$$

as maximizer must lie in $\delta(\sigma_i^*)$ if not then we put all prob mass on $s_i' \notin \delta(\sigma_i^*)$ that has maximum value of utility $(\sigma_i^*, \sigma_{-i}^*)$ not MSNE

$(\Rightarrow)$ given $(\sigma_i^*, \sigma_{-i}^*)$ is MSNE

$$u_i(\sigma_i^*, \sigma_{-i}^*) = \max_{\sigma_i \in \Delta S_i} u_i(\sigma_i, \sigma_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*) = \max_{s_i \in \delta(\sigma_i^*)} u_i(s_i, \sigma_{-i}^*) \quad \text{— ①}$$

by defn of Utility

$$u_i(\sigma_i^*, \sigma_{-i}^*) = \sum_{s_i \in S_i} \sigma_i^*(s_i) u_i(s_i, \sigma_{-i}^*) = \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i) u_i(s_i, \sigma_{-i}^*) \quad \text{— ②}$$

as equation 1 and 2 are equal, this can only happen when all same as max = positive weighted average

and condition 1 is done

for second condition, if $\exists s_i \in \delta(\sigma_i^*)$ and $s_i' \notin \delta(\sigma_i^*)$ s.t

$u_i(s_i, \sigma_{-i}^*) < u_i(s_i', \sigma_{-i}^*)$

then we can shift prob. mass $\sigma^*(s_i)$ to s_i', this new mixed strategy gives strict higher utility and contradicts MSNE

$(\Leftarrow)$ given 2 conditions we want to show $(\sigma_i^*, \sigma_{-i}^*)$ is MSNE

let $u_i(s_i, \sigma_{-i}^*) = m_i(\sigma_{-i}^*)$ $\forall s_i \in \delta(\sigma_i^*)$ from 1)

$m_i(\sigma_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*)$ from 2)

$$u_i(\sigma_i^*, \sigma_{-i}^*) = \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i)\, u_i(s_i, \sigma_{-i}^*)$$

$$= m_i(\sigma_{-i}^*)$$

$$= \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*)$$

$$= \max_{\sigma_i \in \Delta S_i} u_i(\sigma_i, \sigma_{-i}^*)$$

$$\geq u_i(\sigma_i, \sigma_{-i}^*) \quad \forall \sigma_i \in \Delta S_i$$

$\Rightarrow (\sigma_i^*, \sigma_{-i}^*)$ is MSNE

Algorithm to find MSNE :

NFG $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$

total no of supports for $S_1 \times \cdots \times S_n$ is

$$K = (2^{|S_1|}-1) \times (2^{|S_2|}-1) \times \cdots \times (2^{|S_n|}-1)$$

for $X_1 \times X_2 \times \cdots \times X_n \subseteq S_1 \times \cdots \times S_n$ is a support profile

we solve:

$$w_i = \sum_{s_{-i} \in S_{-i}} \left(\prod_{j \neq i} \sigma_j(s_j)\, u_i(s_i, s_{-i}) \right) \quad \forall s_i \in X_i, \forall i \in N$$

$$w_i \geq \sum_{s_{-i} \in S_{-i}} \left(\prod_{j \neq i} \sigma_j(s_j)\, u_i(s_i, s_{-i}) \right), \forall s_i \in S_i \setminus X_i, \forall i \in N$$

$$\sigma_j(s_j) \geq 0, \forall s_j \in S_j, \forall j \in N, \quad \sum_{s_j \in X_j} \sigma_j(s_j) = 1 \quad \forall j \in N$$

Note : This is not linear unless $n=2$

for general game no poly-time algorithm

problem of finding MSNE is PPAD-complete (polynomial parity Arg of Directed graphs)

Eg:

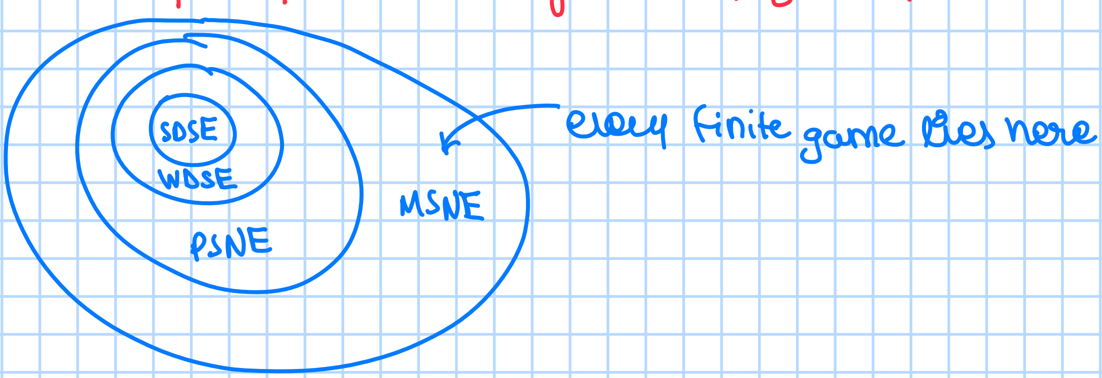
	L	R
T	(4,1)	(2,5)
M	(1,3)	(6,2)
B	(2,2)	(3,3)

$\frac{1}{2}T \quad \frac{1}{2}M$ v/s B

$$\begin{pmatrix} 2.5 \\ 4 \end{pmatrix} \text{ v/s } \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$ so, B is dominated by a weak strategy,

we can remove it to compute algoritm faster

Theorm: If a pure strategy s_i, is strictly dominated by a mixed stratgy $\sigma_i \in \Delta S_i$ then every MSNE of game s_i is chosen with prob 0.



Defn: (Finite games) A game is said to be finite when no of players are finite, and each player has finite set of strategies.

Theorm: (Nash) Every finite game has a MSNE

proof uses Brouwer's fixed point theorm

22nd Aug:

Correlated strategy and Eq:

enter - mediating agent / device
- → alternative explanation of players rationality
- → utilities of all players may get better
- → computational tractable

		player 2 wait	go
player 1	wait	(0,0)	(1,2)
	go	(2,1)	(-10,-10)

PNE: (W,G) or (G,W)

MSNE: large prob of waiting

MSNE $\Leftrightarrow$ ① $u_i(s_i, \sigma_{-i}^*)$ same $\forall s_i \in \delta(\sigma_i^*)$
② $u_i(s_i, \sigma_{-i}^*) \geq u_i(s_i', \sigma_{-i}^*)$ $\forall s_i \in \delta(\sigma_i^*)$ $\forall s_i' \notin \delta(\sigma_i^*)$

only case: {W,G}, {W,G}

$$u_1(W,(q,1-q)) = u_1(G,(q,1-q))$$
$$q \times 0 + (1)(1-q) = 2\times q + (-10)(1-q)$$
$$1-q = 2q - 10 + 10q$$
$$\Rightarrow 11 = 13q$$
$$\Rightarrow q = \frac{11}{13}$$

by symmetry $p = \frac{11}{13}$ so, $\left(\left(\frac{11}{13}, \frac{2}{13}\right), \left(\frac{11}{13}, \frac{2}{13}\right)\right)$

In practice something else happens, traffic light exist, trusted third party is called mediator

Role: randomise over the strategy profiles (not individual strategies)
Suggest that corresponding strategy

P_1	P_2
P_3	P_4

→ Equilibrium from (P_1, P_2, P_3, P_4)
P_i = prob of strategy profile

If strategies are enforceable then it's an equilibrium (correlated)

Defn: (correlated strategy) $\pi : S_1 \times \dots \times S_n \to [0,1]$ s.t $\sum_{s \in S} \pi(s) = 1$ (π is a mapping)

Eg: $\pi(W,W) = 0$
$\pi(W,G) = \frac{1}{2}$
$\pi(G,W) = \frac{1}{2}$
$\pi(G,G) = 0$

Correlated equilibrium: π is a correlated equilibrium when no player gains from dev while others follow suggested strategy

π is common knowledge

Defn: (corr. eq) corr π s.t $\sum_{s_{-i} \in S_{-i}} \pi(s_i, s_{-i}) \cdot u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} \pi(s_i, s_{-i}) u_i(s_i', s_{-i})$ $\forall s_i, s_i' \in S_i$ $\forall i \in N$

Note: The above defn has expected value, if it holds for all players then it's equilibrium

	F	C
F	(2,1)	(0,0)
C	(0,0)	(1,2)

MSNE : $\left(\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right)\right)$

$\pi(C,C) = \frac{1}{2}$ $\quad \pi(F,F) = \frac{1}{2}$ $\quad$ CE $\mathbb{E}[\text{Utility}] = \frac{3}{2} > \mathbb{E}[\text{Utility MSNE}]$

	W	G
W	(0,0)	(1,2)
G	(2,1)	(-10,-10)

MSNE: $\left(\left(\frac{11}{13}, \frac{2}{13}\right), \left(\frac{11}{13}, \frac{2}{13}\right)\right)$

Exe: is $\pi(W,G) = \pi(W,W) = \pi(G,W) = \frac{1}{3}$ a CE? other CE? → done down

Computing CE:

$$\sum_{s_{-i} \in S_{-i}} \underbrace{\pi(s_i, s_{-i})}_{\text{variable}} \underbrace{[u_i(s_i, s_{-i}) - u_i(s_i', s_{-i})]}_{\text{constants}} \geq 0 \quad \forall s_i, s_i', \forall i \in N$$

CE finding is to solve a set of linear equations

total if $|S_i| = m$ then ${}^mC_2$ options for s_i, s_i'
and as $\forall i \in N$
total inequalities $= O(nm^2)$

also $\pi(s) \geq 0, \forall s \in S$
$\sum_{s \in S} \pi(s) = 1$
→ $O(m^n)$ inequalities
so $O(m^n + nm^2) \approx O(m^n)$ inequalities

Now, Feasibility linear program is faster than MSNE
as MSNE total support profiles
$K = (2^{|S_1|} - 1) \times (2^{|S_2|} - 1) \times \cdots \times (2^{|S_n|} - 1)$
$= O(2^{mn})$
CE: inequality order $= O(m^n)$
$O(m^n)$ is exponentially smaller than $O(2^{mn})$

we can also:

$$\max_{\pi} \sum_{i \in N} \sum_{s \in S} \pi(s_i, s_{-i}) u_i(s_i, s_{-i})$$

Theorm: For every MSNE σ^*, $\exists$ CE π^*

Proof: $(\sigma_1^*, \sigma_2^*, \ldots \sigma_n^*)$ is a MSNE

let $\pi^*(s_1, s_2 \ldots s_n) = \prod_{i=1}^{n} \sigma_i^*(s_i)$ and let's use MSNE characterization theorm

① $u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i', \sigma_{-i}^*) \; \forall s_i' \in S_i, i \in N$

② $\forall s_i \in \delta(\sigma_i^*)$, $u_i(\sigma_i^*, \sigma_{-i}^*) = \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i) u_i(s_i, \sigma_{-i}^*) = u_i(s_i, \sigma_{-i}^*)$

③ $\forall s_i \in \delta(\sigma_i^*)$, $s_i' \in S_i$

$u_i(s_i, \sigma_{-i}^*) \geq u_i(s_i', \sigma_{-i}^*)$ (from ①, ②)

$\Rightarrow \sigma_i^*(s_i) u_i(s_i, \sigma_{-i}^*) \geq \sigma_i^*(s_i) u_i(s_i', \sigma_{-i}^*)$

$\Rightarrow \sum_{s_{-i} \in S_{-i}} \sigma_i^*(s_i) \sigma_{-i}^*(s_{-i}) u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} \sigma_i^*(s_i) \sigma_{-i}^*(s_{-i}) u_i(s_i', s_{-i})$

$= \sum_{s_{-i} \in S_{-i}} \pi^*(s_i, s_{-i}) u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} \pi^*(s_i, s_{-i}) u_i(s_i', s_{-i})$

so, π^* is CE from defn

SDSE ⇒ WDSE ⇒ PSNE ⇒ MSNE ⇒ CE

	Agri	War
Agri	(5,5)	(0,6)
War	(6,0)	(1,1)

Ene: CE of this game?

Ans: as above $\exists$ a dominant strategy $\Rightarrow$ MSNE $\Rightarrow$ CE $P(W,W)=1$

Till now we saw normal form games, $\langle N, (S_i)_{i\in N}, (U_i)_{i\in N} \rangle$, we saw rational intelligent and common knowledge

Ene: Is $\pi(W,G)=\pi(W,W)=\pi(G,W)=\frac{1}{3}$ a CE? other CE?

Ans:

	W	G
W	(0,0)	(1,2)
G	(2,1)	(-10,-10)

$$\sum_{s_{-i}} \pi(s_i, s_{-i}) u_i(s_i, s_{-i}) \geq \sum_{s_{-i}} \pi(s_i, s_{-i}) u_i(s_i', s_{-i})$$

$i=1$, $s_i=W$ then

$$\frac{1}{3}(1) + \frac{1}{3}(0) > \frac{1}{3}(-10) + \frac{1}{3}(2)$$

$s_i = G$:

$$\frac{1}{3}(2) > \frac{1}{3}(0)$$

Same for 2, so the game is CE

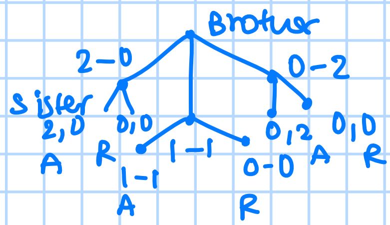
29th Aug:

Richer representation of games:

multi stage games (Board games)
players interact in seq (seq is history) (like the 8x8 game)

PIEFG:

Perfect information extensive form games
eg: 8x8 is perfect information game



A = accept
R = reject

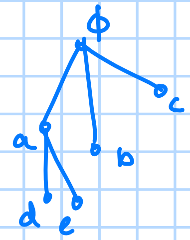
when R then both chocolates taken away

Formal capture:

PIEFG $\langle N, A, H, X, P, (u_i)_{i\in N}\rangle$

N: set of players, A: set of possible actions (all players)
H: set of all sequences of actions satisfying $\phi \in H$
if $h \in H$

$h = (a^0, a^1 \ldots a^{T-1})$ terminates if $\nexists\, a^{(T)} \in A$ s.t $(a^0, \ldots a^T) \in H$



X: action mapping fuction $Z \subseteq H$ be set of terminal history

$X(a,d) = \phi$
$X(a) = \{d, e\}$
$X: H \setminus Z \longrightarrow 2^A$

P: player function which tells us which player is playing

$P(\phi) = 1$ $P: H\setminus Z \longrightarrow N$
$P(a) = 2$

u_i: utility of i $u_i: Z \longrightarrow \mathbb{R}$

eg: $N = \{1, 2\}$
$A = \{2\text{-}0, 1\text{-}1, 0\text{-}2, A, R\}$
$H = \{\phi, (2\text{-}0), (1\text{-}1), (0\text{-}2), \underbrace{(2\text{-}0, A), (2\text{-}0, R), (1\text{-}1, A), (1\text{-}1, R), (0\text{-}2, A), (0\text{-}2, R)}_{Z}\}$

$X(\phi) = \{(2\text{-}0), (1\text{-}1), (0\text{-}2)\}$

$X(2\text{-}0) = X(1\text{-}1) = X(0\text{-}2) = \{A, R\}$

$P(\phi) = 1$ $P(2\text{-}0) = 2$

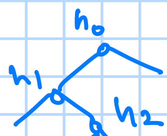
$u_1(2\text{-}0, A) = 2$

strategy:

action at history h

$s_i = \times_{\{h \in H \mid P(h) = i\}} X(h)$

player i will play



$s_i = X(h_1) \times X(h_2) \times X(h_3) \times \cdots$
$= \times_{\{h \in H \mid P(h) = i\}} X(h)$

eg: $S_1 = \{2\text{-}0, 1\text{-}1, 0\text{-}2\}$

$S_2 = X(2\text{-}0) \times X(1\text{-}1) \times X(0\text{-}2)$
$= \{A,R\} \times \{A,R\} \times \{A,R\}$

$= \{AAA, AAR, \ldots RRR\}$

Sister

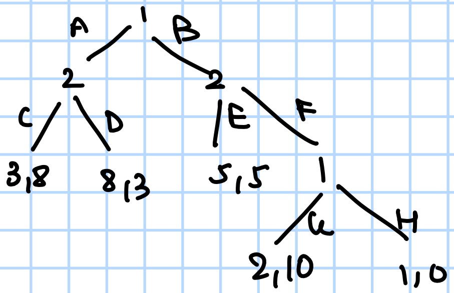
Brother	AAA	AAR	ARA	ARR	RAA	RAR	RRA	RRR
2-0	2,0	2,0	2,0	2,0	0,0	0,0	0,0	0,0
1-1	1,1	1,1	0,0	0,0	1,1	1,1	0,0	0,0
0-2	0,2	0,0	0,2	0,0	0,2	0,0	0,2	0,0

NE but R at 1-1 is not possible

NE

From above concept of PSNE is not good enough to predict outcomes of PIEFGs

PIEFG to NFG:



$S_1 = \{AG, AH, BG, BH\}$
$S_2 = \{CE, CF, DE, DF\}$

(AG, CF), (AH, CF), (BH, CE) are NE

So better notion will consider history and ensure utility maximization

	CE	CF	DE	DF
AG	3,8	3,8	8,3	8,3
AH	3,8	3,8	8,3	8,3
BG	5,5	2,10	5,5	2,10
BH	5,5	1,0	5,5	1,0

NE

(AG, CF) is best

(AH, CF)
(BH, CE) } both have issues as H is not played and similar (subgame is not perfect)

Subgame perfection:

subgame is game rooted at an immediate vertex

Defn: The subgame of a PIEFG, G is rooted at history h is the restriction of G to descendents of h

Subgame perfection is best response for every subgame

Defn: (Subgame perfect NE) SPNE of PIEFG G is a strategy profile $s \in S$ s.t $\forall G'$ of G the restriction of s to G' is a PSNE of G'

← Returns best possible action

Backind (hist h):
    If $h \in Z$ then:
        return $u(h), \phi$
    best_util$_{P(h)} = -\infty$
    $\forall a \in X(h)$:
        util_at_child$_{P(h)}$ = Backind ((h, a))

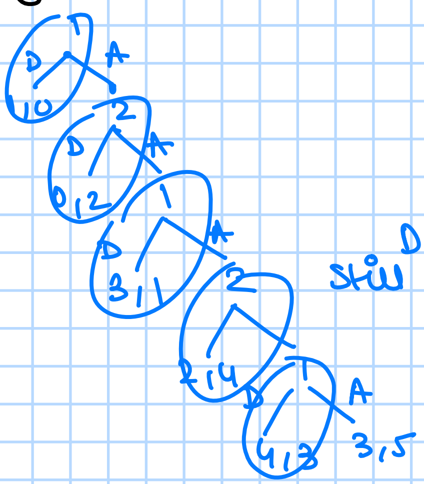
if util_at_child > best_util then
best_util = util_at_child
best_action = a
return best_util, best action

Limitations for SPNE:

Idea of subgame perfection is based on backward induction

Adv: SPNE is garanteed to exist in finite PIEFGs
SPNE is PSNE (when PSNE is garanteed to exist)
algo to find SPNE is doable

dis: 8x8 board ~ 10^{150} vertices (more than no of molecules)
cognitive limits of real player may prohibit playing SPNE



D after every turn is SPNE
still most subjects who play are they to do A so that
they continue the few rounds

Issue with SPNE is if game reaches this history by equilibrium what to do

we have to use idea of belief

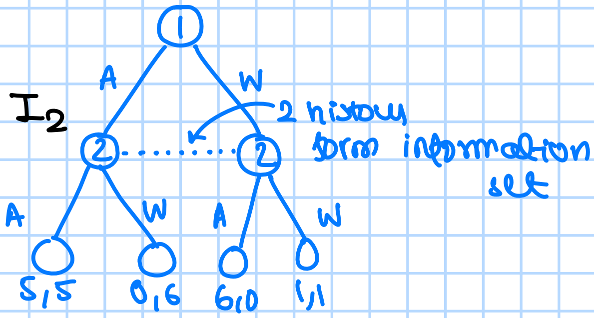
30th Aug:

games with imperfect information:

games discussed so far (EFGs) are perfect info games

every player has perfect knowledge about all developments in the game until that round - limited practical use - several game have states that are unknown to certain agents - eg card games

	Agri	War
Agri	(5,5)	(0,6)
War	(6,0)	(1,1)

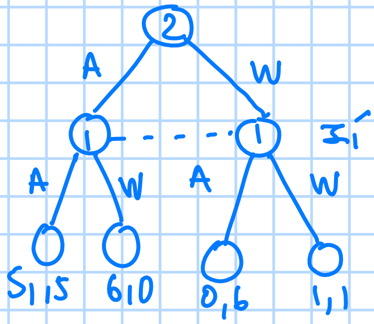


imperfect information EFGs, indistinguishable nodes connected via dotted line

player 2 does not know which node/history the game is in

These indistinguishable histories form 'information set' for player 2, more general than PIEFGs, as information sets can be singletons

IIEFGs are not unique for a given simultaneous move game



Defn: (Imperfect information extensive form games) An IIEFG is a tuple

$$\langle N, A, H, X, P, (u_i)_{i\in N}, (I_i)_{i \in N} \rangle \text{ where}$$

$\langle N, A, H, X, P, (u_i)_{i\in N}\rangle$ is PIEFG, $\forall i \in N$, $I_i = (I_i^1, I_i^2, \ldots I_i^{k(i)})$ is a partition of $\{h \in H \setminus Z, P(h) = i\}$ with property that

$$X(h) = X(h')$$
$$P(h) = P(h') = i \text{ when } \exists j \text{ s.t } h, h' \in I_i^j$$

Note: I_i^j's are called information sets of player i, I_i is information sets of i

At an information set, player and their available actions are same.
Player is uncertain about which history in the information set is reached

Differences with PIEFG:

Since actions at an information sets are identical, X can be defined over I_i^j s.t
$X(h) = X(h') = X(I_i^j) \ \forall h, h' \in I_i^j$

Strategies can also be defined over information sets
strategy set of player $i \in N$ is defined as Cartesian product of the

$$S_i = \times_{\tilde{I} \in I_i} X(\tilde{I}) = \times_{j=1}^{k(i)} X(I_i^j)$$

So, IIEFGs help NFGs to be represented using EFGs

Eg: two player zero sum game

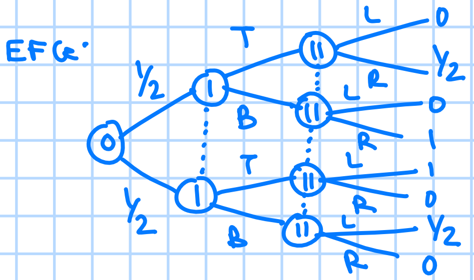
		player 2 L	player 2 R
player 1	T	0	$1/2$
	B	0	1

		player 2 L	player 2 R
player 1	T	1	0
	B	$1/2$	0

matrix G_1 ⟷ matrix G_2

selected with probability $1/2$

EFG:



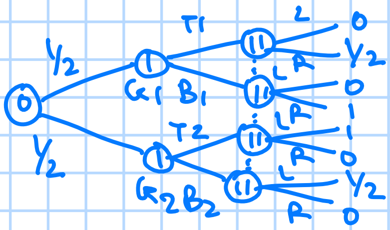
NFG:

	L	R
T	$1/2$	$1/4$
B	$1/4$	$1/2$

$MSNE = \left(\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{2}\right)\right)$

$value = \frac{3}{8}$

now if player I is informed but II is not:



NFG:

	L	R
T_1T_2	$1/2$	$1/4$
T_1B_2	$1/4$	$1/4$
B_2T_1	$1/2$	$1/2$
B_1B_2	$1/4$	$1/2$

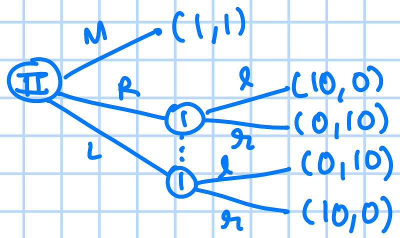
MSNE: $(1(B_1T_2), (1/2, 1/2))$

value: $1/2$

Theorem: Let Γ be a two-player zero sum game in extensive form and let Γ' be the game derived from Γ by splitting several information sets of player I. Then the value of the game Γ' in mixed strategies is greater than or equal to the value of Γ in mixed strategies.

Note: Not true for general-sum games

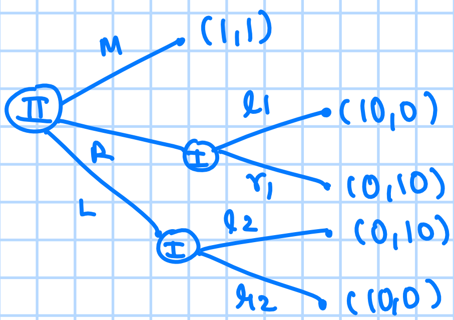
Eg: general game:



MSNE: $\left(\left(\frac{1}{2}(l), \frac{1}{2}(r)\right), \left(\frac{1}{2}(L), \frac{1}{2}(R), 0(M)\right)\right)$

expected payoff $= (5,5)$

if player I gets more information:



MSNE: $(1(l_1 r_2), (0(L), 0(R), 1(M)))$

expected payoff $= (1,1)$

Strategies in IIEFGs:

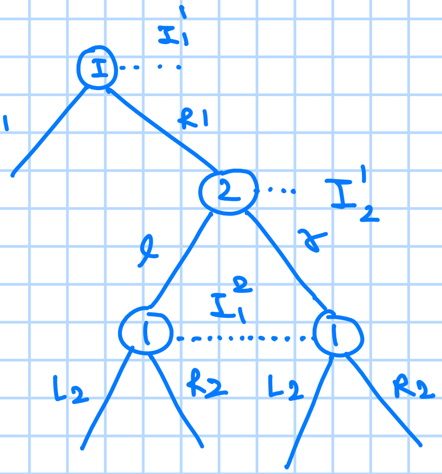
Randomised strategies in IIEFGs:

$$S_i = \underset{j=1}{\overset{K(i)}{\times}} X(I_i^j) \quad \text{for } i$$

in NFGs: mixed strategies randomise over pure strategies

in EFGs: ① Randomise over strategies define at begining of game

② randomise over the action at an information set ← Behavioral strategies



Pure strategies: $(\underbrace{L_1L_2, L_1R_2, R_1L_2, R_1R_2}_{S_1})$

mixed strategies: σ_1

s.t $\sigma_1 \in \Delta(S_1)$

i.e $\sigma_1(L_1L_2), \dots \sigma_1(R_1R_2)$

Behavioral strategy:

b_1 s.t

$b_1(I_1^1) \in \Delta(L_1, R_1)$

$b_1(I_2^1) \in \Delta(L_2, R_2)$

b_2 s.t

$b_2(I_2^1) \in \Delta(l, r)$

Defn: (Behavioural Strategy) A behavioural strategy of a player is an IIEFG is a function that maps each information sets to a probability distribution over the set of possible action at that information set.

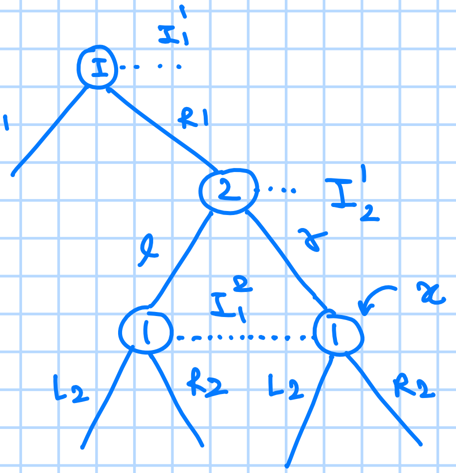
Note: In above example, mixed strategies (MS) live in $\mathbb{R}^4$, but BSs live in two $\mathbb{R}^2$

Note: Equivalence in terms of the probability of reaching a vertex/history x

say $p(x;\sigma)$ = prob of reaching node x under MS σ

$p(x;b)$ = prob of reaching node x under BS b

Eg:



$$P(x;\sigma) = \sigma_1(R_1)\sigma_2(r)$$
$$= (\sigma_1(R_1L_2) + \sigma_1(R_1R_2))\,\sigma_2(r)$$

$$P(x;b) = b_1(I_1^1)(R_1) \cdot b_2(I_2^1)(r)$$

Note: players can choose different kinds of strategies

$$p(x|\sigma_1, b_2) = (\sigma_1(R_1L_2) + \sigma_1(R_1R_2))\, b_2(I_2^1)(r)$$

Defn: (Equivalence) A mixed strategy σ_i and behavioural strategy b_i of a player i in an IIEFG are equivalent if for every mixed/behavior strategy ξ_{-i} of other players and every vertex x in the game tree:

$$P(x|\sigma_i, \xi_{-i}) = P(x|b_i, \xi_{-i})$$

Eg: if in above game,

$$b_1(I_1^1)(L_1) = \sigma_1(L_1L_2) + \sigma_1(L_1R_2)$$
$$b_1(I_1^1)(R_1) = \sigma_1(R_1L_2) + \sigma_1(R_1R_2)$$
$$b_1(I_1^2)(L_2) = \sigma_1(L_2|R_1)$$
$$b_1(I_1^2)(R_2) = \sigma_1(R_2|R_1)$$

then b_1, σ_1 are equivalent

Note: The equivalence holds at all leaf nodes

Claim: It is enough to check the equivalence only at the leaf nodes.
proof: pick an arbitrary non leaf node, then prob of reaching that is equal to sum of all leaf nodes in subtree

Theorem: (Utility Equivalence) If σ_i, b_i are equivalent, then for every mixed/behavioral strategy vector of other player ξ_{-i}, the following holds

$$u_j(\sigma_i, \xi_{-i}) = u_j(b_i, \xi_{-i}) \quad \forall j \in N$$

Cor: let σ, b be equivalent i.e σ_i, b_i equivalent $\forall i \in N$, then $u_i(\sigma) = u_i(b) \;\forall i \in N$

3rd Sept:

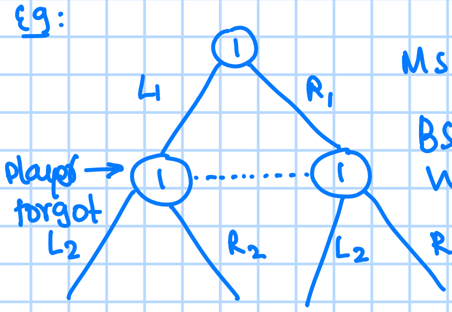
Equivalent Strategies in IIEFGs:

we need behavioral strategies as they are more natural for large IIEFGs as players plan at every stage (information set) of the game rather than a masterplan

A small number of variables to deal with
consider a player having 4 information sets with 2 actions each
$2\times2\times2\times2$ variables for mixed strategies
but $2+2+2+2$ for behavioral strategies

we want to see if equivalence always holds or not i.e for given σ_i, does $\exists\, b_i$ s.t $P(x|\sigma_i, \xi_{-i}) = P(x|b_i, \xi_{-i})\ \forall x, \forall \xi_{-i}$
and same for if given b_i $\exists\, \sigma_i$

Eg:



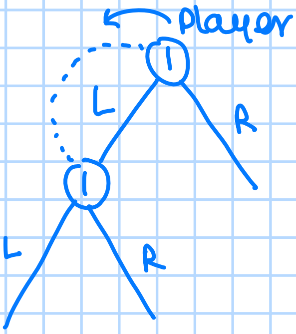
MS: $\sigma_1(L_1L_2) = \frac{1}{2}$ $\sigma_1(L_1R_2) = 0$ $\sigma_1(R_1L_2) = 0$ $\sigma_1(R_1R_2) = \frac{1}{2}$

BS: $b_1(L_1)$, $b_1(R_1)$, $b_1(L_2)$, $b_1(R_2)$, if player remember it has made a move but forgets which move, then we can see BS not possible but as

$\sigma_1(L_1R_2) = \sigma_1(R_1L_2) = 0$ MS has more control

Note: Above is an example of MS with no equivalent BS

Eg:



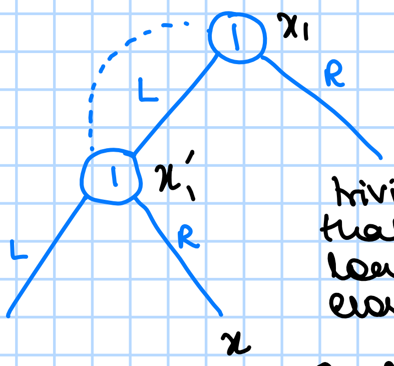
player forgot whether it made a move or not

BS: positive mass on the second leaf node possible, not for mixed strategy

Note: The above is an example that $\exists$ BS with no equivalent MS

so, from above two examples, we see that equivalence does not hold if players are forgetful

BS no equivalent MS:



x: non root node
action at x_1 leading to x

$\exists$ node that has path from root that crosses same information set (x) (x_1)

trivial to see if path from root to x passes through x_1 and x_1' that are in same information set of player i, and the action leading to x at x_1 and x_1' is different, then no pure strategy can ever lead to x.

as mixed strategies is a randomisation over pure strategies, every mixed strategy will put zero probability mass on x but behavioral strategy randomise on every vertex independently, hence x may be reached in behavioral strategies with positive probability

we convert our observations into lemma

Lemma: If $\exists$ a path from the root to some vertex x that passes through same information set atleast twice, and action leading to x is not same at each of those vertices, then player at the information set has a behavioral strategy with no eq MS

Theorm: Consider an IIEFG s.t every vertex has atleast two actions. Every BS has eq MS iff each I_i^j's of player intersects every path emanating from root atmost once

To formalise (i.e set condition of equivalence) we formalise forgetfulness of players

Defn: (Choice of same action at Inf. set) let $x = (x^0, x^1, \dots x^K)$
$\hat{x} = (x^0, \hat{x}^1, \dots \hat{x}^L)$ be two paths

let I_i^j be s.t $x \cap I_i^j = x_k$
$\hat{x} \cap I_i^j = \hat{x}_l$

these two paths choose same action at I_i^j if:
① $k < K$
$l < L$
② action $x^k \to x^{k+1}$
$\hat{x}^l \to \hat{x}^{l+1}$ are identical
i.e $a_i(x^k \to x^{k+1}) = a_i(\hat{x}^l \to \hat{x}^{l+1})$

Defn: (perfect recall) Player i has perfect recall if
① every Inf. set of player i intersects every path from root to leaf atmost once
② every two path that end in the same inf set of player i pass thru same inf set in same order and in every such inf set the two path choose same action.

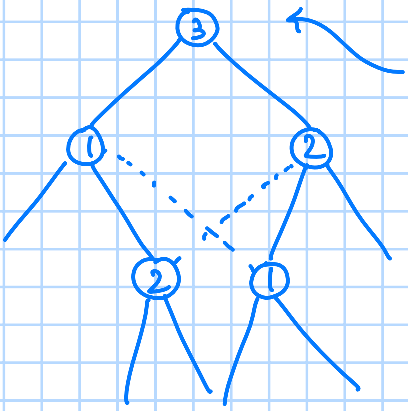
so, $\forall I_i^j$, every pair $x, y \in I_i^j$ of vertices, if decision vertices of i are $x_i^1, x_i^2, \dots x_i^L = x$ and $y_i^1, y_i^2, \dots y_i^{L'} = y$ for two paths from root to y then

decision vertices not paths in I_i^j

① $L = L'$
② $x_i^l = y_i^l \in I_i^k$ for some k
③ $a_i(x_i^l \to x_i^{l+1}) = a_i(y_i^l \to y_i^{l+1})$ $\forall l = 1, 2, \dots L-1$

Defn: A game has perfect recall if every player has a perfect recall

Eg:



game with perfect recall

as ① $\forall$ Player i, every path from root leaf intersects all I_i^j's atmost once

② every two path end in same I_i^j pass thru same Inf set in same order and two path choose same action at every path

2 diff action
so not perfect recall

let $S_i^*(x)$ be set of pure strategies of i which lead to x

Theorm: If player i has perfect recall, then for x, x' vertices in same inf set $S_i^*(x) = S_i^*(x')$

Theorm: (Kuhn) If every IIEFG, if i is a player with perfect recall then $\forall$ mixed strategy of i $\exists$ eq BS.

10th sept:

Equilibrium in IIEFGs:

we can extend subgame perfection (from PIEFG) but as nodes/histories are uncertain extend mix strategy.

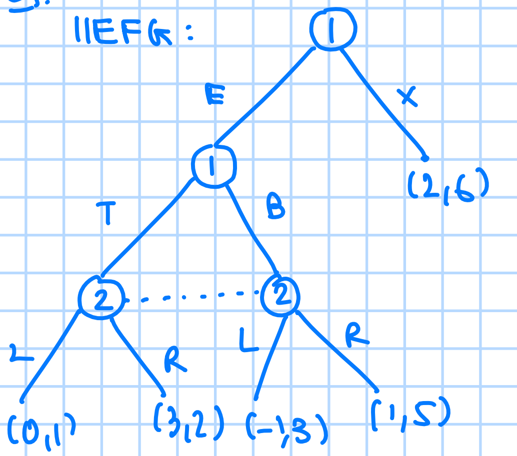
we need belief of each player (by information set)

conditional prob distribution over history in inf set conditional on reaching the inf set

Belief + MS forms equilibrium

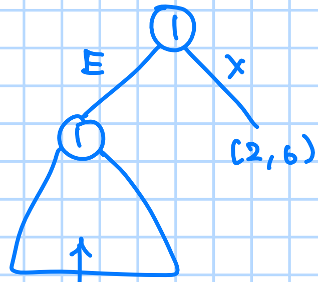
eg:

IIEFG:



we can define this as subgame - simultaneous

⇒



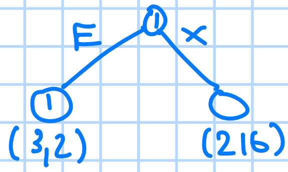
we assume simultaneous

1 \ 2	L	R
T	(0,1)	(3,2)
B	(-1,3)	(1,5)

← Dominating Strategy (R)

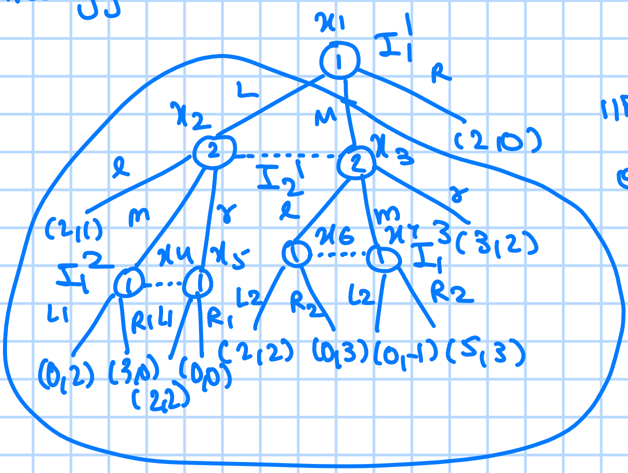
dominating strategy (T)

so



so subgame perfection:
player 1 → E → T
player 2 → R
(ET, R) = equilibrium

Eg:



IIEFG with perfect recall

only one subtree (tree itself)

In above example if we know $P(L) = 5/12$, $P(M) = 4/12$, $P(R) = 3/12$ } $b_1(I_1^1)$

$b_2(I_2^1)$: $l = 1$, $m = 0$, $r = 0$

$b_1(I_1^2)$: $R_1 = 1$

$b_1(I_1^3)$: $L_2 = 1$

} pure strategies, we want to see if equilibrium

$$P(x_2 | I_2^1) = \frac{\frac{5}{12}}{\frac{5}{12} + \frac{4}{12}} = \frac{5}{9}$$

$P(x_6 | I_1^3) = 1$ as player 2 will pick l, if belief $> 5/7$ still pick L2 as

$P(x_6 | I_1^3) \times 2 + (1 - P(x_6 | I_1^3)) 0 > (1 - P(x_6 | I_1^3)) 5$
$\Rightarrow P(x_6 | I_1^3) > \frac{5}{7}$ then we pick L_2

for player 2: Expected utility by picking $l = \frac{5}{9} \times 1 + \frac{4}{9} \times 2 = \frac{13}{9}$ larger than any choice

Note: Above when player plays L/M/R, the expected utility are same for all three choices, so they are a mixed strategy and so, we need prob dist on picking L/M/R to make beliefs s.t we attain equilibrium

Note: Above MS/BS is sequentially rational

Defn: (Belief) let Inf set of i be $\mathcal{I}_i = \{I_i^1, \dots I_i^{K(i)}\}$, belief of i is a mapping

$$\mu_i^j : I_i^j \longrightarrow [0,1] \text{ s.t } \sum_{x \in I_i^j} \mu_i^j(x) = 1$$

Defn: (Bayesian Belief) given μ_i, belief is bayesian wrt σ if it is derived from σ using bayes rule:

$$\mu_i^j(x) = \frac{P_\sigma(x)}{\sum_{y \in I_i^j} P_\sigma(y)} \quad \forall x \in I_i^j, \forall j = 1,2 \dots K(i)$$

Defn: Sequential Rationality) A strategy σ_i of player i at an inf set I_i^j is seq rational given σ_{-i} partial belief μ_i^j if:

$$\sum_{x \in I_i^j} \mu_i^j(x) u_i(\sigma_i, \sigma_{-i} | x) \geq \sum_{x \in I_i^j} \mu_i^j(x) u_i(\sigma_i', \sigma_{-i} | x), \quad \forall \sigma_i' \in \Delta S_i$$

The triple $\underbrace{(\sigma, \mu)}_{\text{assessment}}$ is sequentially rational if it is seq rational $\forall$ player at $\forall$ inf set

seq rational is redefination of NE

Theorm: In PIEFG, a BS σ is SPNE iff $(\sigma, \hat{\mu})$ is seq rational

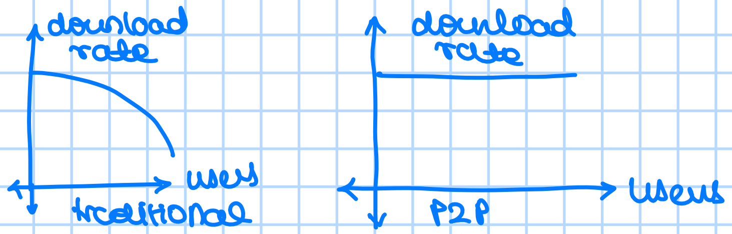
$\hat{\mu}$ is degenerate dist, and as in PIEFG all inf set singletons

Defn: (Perfect Bayesian Eq) (σ, μ) is PBE if $\forall i \in N$, μ_i is bayesian wrt σ, σ_i is seq rational given σ_{-i} and μ_i

Note: (μ, σ) is (σ) as μ comes from σ if Bayesian equilibrium

P2P file Sharing:

Peer to Peer



Idea of server and client is decentralised, moment we have file, we become server and so download rate does not change

Scalable, Failure Resilience

some terminology: protocol: messages that can be sent/taken over
client: Particular process for sending message

Reference client: particular implementation
peer: Clients/servers are peers

Napster (1999-2001) centralized database, users download music, ran into copyright issues

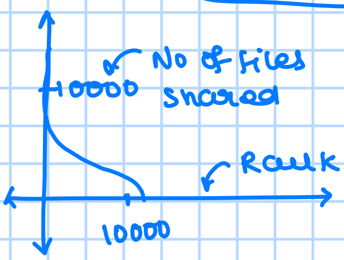
Gnutella (2000-) get IP address from peers from set to known peers, to get file send query to broadcast by peer to known peers, then if some peer want to give file peer gets it.

Gnutella File sharing game

1 \ 2	Share	Free-Ride
share	(2,2)	(-1,3)
Free-ride	(3,-1)	(0,0)

→ DS
→ DS

so both free-Ride



85% free riding by 2005
<1% world wide P2P traffic by Gnutella

BitTorrent (2001-) Approx 85% P2P traffic in US, also for Linux distribution

Break file into pieces - Repeated game, we want "if you let me download, I reciprocate"

.torrent file: Name, SHA1, size, Tracker URL → list of peers
↓
Tracker

Tracker is a centralized entity that controls the traffic. tracks location & speed of upload.

Reference client protocol: set threshold r for uploading speed (~1/3 of max speed)
if peer j uploaded to i at rate ≥ r unchoke j
if peer j uploaded to i at rate < r } choke j for 3 time periods

tit-for-tat / Repeated prisoners dilemma is equilibrium

Strategy behaviour:

client's goal is to max download/upload speed: Attack bittorrent

BitThief: goal is to download files without upload, keep asking peers from trackers, exploit optimistic unchoking by going neighbourhood quickly
Fix is to block same IP address within 30 mins

Second: "rare-first" to request, reveal most common piece that reciprocating peer does not have, try to protect a monopoly, keep others interested

we want rarest piece of file, we show tracker we are uploading (cheap common file) so no one downloads, but ask for rare file

~12% faster this method / this strategy

Note: The above strategy is fully logical

we have now seen how game theory is useful in computer systems, early systems did not have this, so exploitable

12^{th} Sept:

Non-coop games: complete info, players know which games they are playing

Normal form | Extensive form

- even IIEFG as players know

Non-coop incomplete info: Players do not deterministically know which game they are playing

Coorperative games: Player form coalitions and utilities are defined over coalition

Repeated prisoners dilemma

other types: repeated, stochastic, etc

games with incomple info:

Players don't deterministically know
They recieve private signals/types

Harsanyi: common prior distribution from which all types are drawn

Note: Also called bayesian

Eg: Football – 2 teams, each team : aim to win (Aggressive) } types
aim to draw (Passive)

types: private signal by external factors (weather conditions, player injuries, ground condition)

4 possible profiles: AA, AP, PA, PP

AA profile: (A = Attack, D = Defend)

	A	D
A	1,1	2,0
D	0,2	0,0

AP:

	A	D
A	2,0	2,1
D	0,1	0,0

PA is symmetric

PP:

	A	D
A	0,0	1,0
D	0,1	-1,-1

The probability of choosing different games (or type profiles) comes from a common prior distribution.

Common prior & common knowledge.

Defn: Bayesian game: $\langle N, (\Theta_i)_{i\in N}, P, (\Gamma_\theta)_{\theta\in(\times_{i\in N}\Theta_i)}\rangle$

N = set of players
Θ_i = set of types of player i
P = common prior dist over $\Theta = \times_{i\in N}\Theta_i$
s.t.
$\sum_{\theta_{-i}\in\Theta_{-i}} P(\theta_i,\theta_{-i}) > 0 \ \forall \theta_i\in\Theta_i \ \forall i \in$ as otherwise we can drop it

Γ_θ: NFG for the type of profile $\theta\in\Theta$ i.e. $\Gamma_\theta = \langle N, (A_i(\theta))_{i\in N}, (u_i(\theta))_{i\in N}\rangle$

$u_i : A \times \Theta \longrightarrow \mathbb{R}$

$A = \times_{i \in N} A_i$ (we assume $A_i(\Theta) = A_i \ \forall \Theta$)

Stages of Bayesian games:

$\Theta = (\Theta_i, \Theta_{-i})$ is chosen randomly according to P

Nature $\longrightarrow \theta_1 \rightarrow$ only P1 knows $\theta_1 \longrightarrow a_1$

$\vdots$

$\theta_n \rightarrow$ only Pn knows $\theta_n \rightarrow a_n$

$u_i(\Theta)[a_1, a_2 \dots a_n]$ = utility of player i → types

depends on $(\theta_1, \theta_2 \dots \theta_n)$ & $(a_1, a_2, \dots a_n)$

$u_i(a_i, a_{-i}, \theta_i, \theta_{-i})$

Defn: Strategy is a plan to map type to action

$s_i : \Theta_i \longrightarrow A_i$ pure

$\sigma_i : \Theta_i \longrightarrow \Delta A_i$ mixed

as $s_i : \Theta_i \longrightarrow A_i$

$\theta_i \in \Theta_i$

$s_i(\theta_i)$ = some action given θ_i type

player i before θ_i — ex-ante

own θ_i — ex-interim (after θ_i realised)

once θ_{-i} is known — θ_{-i} — ex-post (θ_i, θ_{-i})

ex-ante $\xrightarrow{\theta_i}$ ex-interim $\xrightarrow{\theta_{-i}}$ ex-post

Player does not know anything

maximise this in Bayesian probability

Defn: (Ex-ante utility) Expected utility before observing θ_i

$$u_i(\sigma) = \sum_{\theta \in \Theta} P(\theta)\, u_i(\sigma(\theta); \theta)$$

Expected value

$$= \sum_{\theta \in \Theta} P(\theta) \sum_{(a_1, a_2 \dots a_n) \in A} \prod_{j \in N} \sigma_j(\theta_j)[a_j]\, u_j(a_1, a_2 \dots a_n / \theta_1, \theta_2 \dots \theta_n)$$

belief of player i over other people type

$$P(\theta_{-i} | \theta_i) = \frac{P(\theta_i, \theta_{-i})}{\sum_{\tilde{\theta}_{-i} \in \Theta_{-i}} P(\theta_i, \tilde{\theta}_{-i})}$$ Bayes rule

this is why we need all marginals > 0

Defn: (Ex-interim) Expected utility after observing one's type

$$u_i(\sigma | \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} P(\theta_{-i} | \theta_i)\, u_i(\sigma(\theta); \theta)$$

For independent types, observing θ_i does not give info on θ_{-i}

$$u_i(\sigma) = \sum_{\theta \in \Theta} P(\theta_i | \theta_{-i})\, u_i(\sigma(\theta), \theta)$$

$$= \sum_{\theta_i \in \Theta_i} \sum_{\theta_{-i} \in \Theta_{-i}} P(\theta_i) P(\theta_{-i} | \theta_i)\, u_i(\sigma(\theta), \theta)$$

$$= \sum_{\theta_i \in \Theta_i} P(\theta_i) \sum_{\theta_{-i} \in \Theta_{-i}} \underbrace{P(\theta_{-i} | \theta_i) u_i(\sigma(\theta), \theta)}_{u_i(\sigma | \theta_i)}$$

$$u_i(\sigma) = \sum_{\theta_i \in \Theta_i} P(\theta_i) u_i(\sigma | \theta_i)$$

eg: seller type: price he is willing to sell
buyer type: price at which he is willing to buy

$$\Theta_1 = \Theta_2 = \{1, 2, \ldots 100\} \qquad A_1 = A_2 = \{1, 2, \ldots 100\}$$

if ask of seller < bid of buyer, if type is independent

$$P(\theta_2 | \theta_1) = P(\theta_2) = \frac{1}{100} \quad \forall \theta_1, \theta_2 \in \Theta$$

$$P(\theta_1 | \theta_2) = P(\theta_1) = \frac{1}{100} \quad \forall \theta_1, \theta_2 \in \Theta$$

$$P(\theta_1, \theta_2) = \frac{1}{10000} \quad \forall \theta_1, \theta_2 \in \Theta$$

$$\Theta = \Theta_1 \times \Theta_2 \qquad u_1(a_1, a_2 | \theta_1, \theta_2) = \begin{cases} \frac{a_1 + a_2}{2} - \theta_1 & ; a_2 \geq a_1 \\ 0 & ; \text{otherwise} \end{cases}$$

a_1: seller ask, a_2: Buyer bid

$$u_2(a_1, a_2 | \theta_1, \theta_2) = \begin{cases} \theta_2 - \left(\frac{a_1 + a_2}{2}\right) & ; a_2 \geq a_1 \\ 0 & ; \text{otherwise} \end{cases}$$

eg: Sealed Bid Auction:
Two players, both willing to buy an object. Their values θ_i and bid b_i lie in $[0, 1]$

type	θ_1	θ_2	...	θ_n
action	b_1	b_2	...	b_n

Allocation function

$$O_1(b_1, b_2) = \begin{cases} 0 & ; b_1 \geq b_2 \\ 1 & ; o\omega \end{cases} \qquad O_2(b_1, b_2) = \begin{cases} 1 & ; b_2 > b_1 \\ 0 & ; o\omega \end{cases}$$

Beliefs:

$$\left.\begin{aligned} f(\theta_2 | \theta_1) &= 1 \quad \forall \theta_1, \theta_2 \\ f(\theta_1 | \theta_2) &= 1 \quad \forall \theta_1, \theta_2 \\ f(\theta_1, \theta_2) &= 1 \quad \forall \theta_1, \theta_2 \end{aligned}\right\} \text{Distribution, uniform, independent}$$

winner pays his/her bid

$$u_i(b_1, b_2 | \theta_1, \theta_2) = O_i(b_1, b_2)(\theta_i - b_i)$$

Equilibrium:

Nash equilibrium: Ex-ante as $(\sigma^*, P): u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i', \sigma_{-i}^*)$

$$\forall \sigma_i', \forall i \in N$$

can be replaced with pure strategy

Bayesian eq: ex-interim

$$(\sigma^*, P): u_i(\sigma_i^*(\theta_i), \sigma_{-i}^* | \theta_i) \geq u_i(\sigma_i'(\theta_i), \sigma_{-i}^* | \theta_i)$$

σ^*: strategy

$$\forall \sigma_i', \forall \theta_i \in \Theta_i \; \forall i \in N$$

$$u_i(\sigma) = \sum_{\theta \in \Theta} P(\theta) u_i(\sigma(\theta) | \theta) \qquad u_i(\sigma | \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} P(\theta_{-i} | \theta_i) u_i(\sigma(\theta) | \theta)$$

Theorem: In finite Bayesian games (σ^*, ρ) is Bayesian Eq $\Updownarrow$ Nash Eq

proof: $(\Rightarrow)$ trivial

$(\Leftarrow)$ By contradiction, (σ^*, ρ) is not bayesian, $\exists i \in N, \theta_i \in \Theta_i$, some $a_i \in A_i$ s.t

$$u_i(a_i, \sigma_{-i}^* \mid \theta_i) > u_i(\sigma_i^*(\theta_i), \sigma_{-i}^* \mid \theta_i)$$

$\forall \tilde{\theta}_i \neq \theta_i \quad \hat{\sigma}_i$

$u_i(\hat{\sigma}_i, \sigma_{-i}^*) > u_i(\sigma_i^*, \sigma_{-i}^*)$ is what we want to show

let $\hat{\sigma}_i$ s.t $\hat{\sigma}_i(\tilde{\theta}_i) = \sigma_i^*(\tilde{\theta}_i) \ \forall \tilde{\theta}_i \in \Theta_i \setminus \{\theta_i\}$

$$\hat{\sigma}_i(\tilde{\theta}_i)(a_i) = 1 \quad \text{when } \tilde{\theta}_i = \theta_i$$
$$\hat{\sigma}_i(\tilde{\theta}_i)(b_i) = 0 \quad \forall b_i \in A_i \setminus \{a_i\}$$

now, $u_i(\hat{\sigma}_i, \hat{\sigma}_{-i}) = \sum_{\hat{\theta}_i \in \Theta_i} p(\hat{\theta}_i) u_i(\hat{\sigma}_i(\hat{\theta}_i), \hat{\sigma}_{-i} \mid \hat{\theta}_i)$

$$= \sum_{\hat{\theta}_i \in \Theta_i \setminus \{\theta_i\}} p(\hat{\theta}_i) u_i(\sigma_i^*(\hat{\theta}_i), \sigma_{-i}^* \mid \hat{\theta}_i) + p(\theta_i) u_i(a_i, \sigma_{-i}^* \mid \theta_i)$$

$$> u_i(\sigma_i^*, \sigma_{-i}^*)$$

Theorem: Every finite Bayesian game has a bayesian equilibrium (set of players, action sets, type sets finite) → Harsani reduction

proof: proof follows from coverting it into complete inf game

Eq: Sealed bid auction

$$u_1(b_1, b_2, \theta_1, \theta_2) = (\theta_1 - b_1)\, \mathbb{I}_{\{b_1 \geq b_2\}}$$
$$u_2(b_1, b_2, \theta_1, \theta_2) = (\theta_2 - b_2)\, \mathbb{I}_{\{b_2 > b_1\}}$$

$b_1 = s_1(\theta_1) \quad b_2 = s_2(\theta_2)$ assume $s_i(\theta_i) = \alpha_i \theta_i, \ \alpha_i > 0$

BE: s_i^* / α_i^* that maximises $\max_{\sigma_i} u_i(\sigma_i, \sigma_{-i}^* \mid \theta_i)$

for 1: $\max_{\sigma_1} u_1(\sigma_1, \sigma_{-1}^* \mid \theta_1) = \max_{b_1 \in [0,1]} \int_0^1 f(\theta_2 \mid \theta_1)(\theta_1 - \theta_2)\, \mathbb{I}_{b_1 \geq \alpha_2 \theta_2} d\theta_2$ (By defn)

$$= \max_{b_1 \in [0,1]} (\theta_1 - b_1) \frac{b_1}{\alpha_2}$$

maximise we $b_1 = \frac{\theta_1}{2}$

we get: $b_1 = \theta_1/2, \ b_2 = \theta_2/2$

$$s_1^*(\theta_1) = \frac{\theta_1}{2}$$
$$s_2^*(\theta_2) = \frac{\theta_2}{2}$$

Eq: Second price auction:

$$u_1(b_1, b_2, \theta_1, \theta_2) = (\theta_1 - b_2)\, \mathbb{I}\{b_1 \geq b_2\}$$

$$u_2(b_1, b_2, \theta_1, \theta_2) = (\theta_2 - b_1)\, \mathbb{I}\{b_1 < b_2\}$$

then $\int_0^1 f(\theta_2|\theta_1)(\theta_1 - s_2(\theta_2))\, \mathbb{I}\{b_1 \geq s_2(\theta_2)\}\, d\theta_2$

$$= \int_0^1 (\theta_1 - \alpha_2\theta_2)\, \mathbb{I}\{\theta_2 \leq \frac{b_1}{\alpha_2}\}\, d\theta_2$$

$$= \frac{1}{\alpha_2}\left(b_1\theta_1 - \frac{\theta_1^2}{2}\right)$$ maximise this wrt $b_1 \to b_1 = \theta_1$

similarly $b_2 = \theta_2$

if θ_1, θ_2 were arbitrary and independent

$$\int_0^{b_1/\alpha_2} f(\theta_2)(\theta_1 - \alpha_2\theta_2)\, d\theta_2 = \theta_1 F\left(\frac{b_1}{\alpha_2}\right) - \alpha_2 \int_0^{b_1/\alpha_2} \theta_2 f(\theta_2)\, d\theta_2$$

diff wrt b_1:

$$\theta_1 \frac{1}{\alpha_2} f\left(\frac{b_1}{\alpha_2}\right) - \alpha_2 \cdot \frac{b_1}{\alpha_2} f\left(\frac{b_1}{\alpha_2}\right)\frac{1}{\alpha_2} = 0 \Rightarrow f\left(\frac{b_1}{\alpha_2}\right)(b_1 - \theta_1) = 0$$

$$\Rightarrow b_1 = \theta_1 \text{ if } f\left(\frac{b_1}{\alpha_2}\right) > 0$$

similarly for player 2

24^{th} sept:

mechanism design:

Task is to set the rules

eg: elections, license scare resources, matching students to universities

model will have: N = set of players

X = set of outcomes ← like winners in an election

Θ_i = set of private info of i (type) $\theta_i \in \Theta_i$

ordinal : θ_i defines an ordering over outcome 1, 2, 3...

cardinal : $u_i : X \times \Theta_i \longrightarrow \mathbb{R}$ utility function

$u_i : X \times \Theta \longrightarrow \mathbb{R}$

Note : $u_i : X \times \Theta_i \longrightarrow \mathbb{R}$ private value model

$u_i : X \times \Theta \longrightarrow \mathbb{R}$ interdependent value model

eg: voting : X = set of candidates

θ_i = ranking over candidates

single object allocation : $x = (a, p) \in X$

$a = (a_1, a_2, \dots a_n)$ $a_i \in \{0,1\}$ → allocation

$\sum_{i \in N} a_i \leq 1$

$p = (p_1, \dots p_n)$ p_i is payment charged to i

θ_i = value of i for object

$$u_i(x, \theta_i) = \underbrace{a_i \theta_i}_{\text{value for player } i} - p_i$$

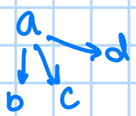
Social choice function:

designer has an objective and is captured by SCF

$$f : \Theta_1 \times \Theta_2 \times \dots \times \Theta_n \longrightarrow X$$

eg: in voting, if a candidate beats everyone else in pairwise contest then he/she shall be chosen a winner

a	b	c
b	a	d
c	d	a
d	c	b



eg: public project choice

$\theta_i : X \rightarrow \mathbb{R}$ value of each project

$f(\theta) \in \arg\max_{a \in X} \sum_{i \in X} \theta_i(a)$

↓

action maximises θ_i

if $f(\theta) \in \max_{a \in A} \min_{i \in N} \theta_i(a)$ egalitarian approach

we want to create a game $f(\theta)$ emerges as an outcome of an equilibrium

Mechanism:

Defn: indirect mechanism is a collection of message spaces and decision rule $\langle M_1, M_2, \dots M_n, g\rangle$

M_i is message space of i

$g : M_1 \times M_2 \times \dots \times M_n \longrightarrow X$

Eg: giving every agent card deck M_i, ask to pick some m_i

Note: a direct mechanism is same as above with $M_i = \Theta_i \;\; \forall i \in N, \; g \equiv f$

indirect mechanisms are not very commonplace as any indirect mech can be solved with direct mechanism.

Defn: (weak dominance) $\langle M_1, \dots M_n, g\rangle$, a message m_i is weakly dominant for i at θ_i if

$$u_i(g(m_i, \tilde{m}_{-i}), \theta_i) \geq u_i(g(m_i', \tilde{m}_{-i}), \theta_i) \quad \forall \tilde{m}_{-i}, m_i'$$

Note: above is for cardinal, it can be seen for ordinal also:

$$g : M_1 \times \dots \times M_n \longrightarrow X$$

$$\underbrace{g(m_i, \tilde{m}_{-i})}_{= x_1} \in X$$

$$\underbrace{g(m_i', \tilde{m}_{-i})}_{= x_2} \in X$$

then x_1 is more preferred than x_2 by θ_i

i.e x_1 over x_2 or x_1, x_2

we write this as:

$$x_1 \, \theta_i \, x_2$$

so, $g(m_i, \tilde{m}_{-i}) \, \theta_i \, g(m_i', \tilde{m}_{-i}) \quad \forall \tilde{m}_{-i}, \forall m_i'$

Defn: (Dominant strategy implementable) An SCF $f : \Theta \to X$ is implemented in dominant strategy by $\langle M_1, \dots M_n, g\rangle$ if

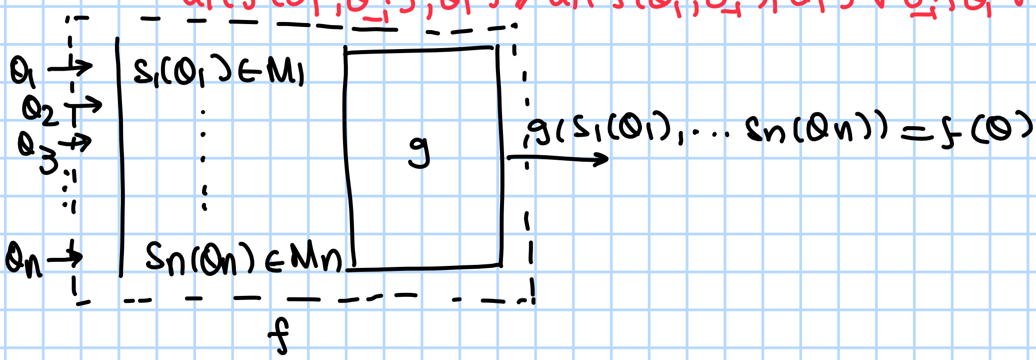
① $\exists \; s_i : \Theta_i \to M_i$ s.t $s_i(\theta_i)$ is dominant strategy for i at θ_i, $\forall \theta_i \in \Theta_i$ $\forall N$

② $g(s_1(\theta_1), \dots s_n(\theta_n)) = f(\theta) \;\; \forall \theta \in \Theta$

we call this an indirect implementation, SCF f is DSI by $\langle M_1, M_2, \dots M_n, g\rangle$

Defn: (Dominant strategy incentive compatible) $\langle \Theta_1, \dots \Theta_n, f\rangle$ is DSIC if

$$u_i(f(\theta_i, \tilde{\theta}_{-i}), \theta_i) \geq u_i(f(\theta_i', \tilde{\theta}_{-i}), \theta_i) \;\; \forall \tilde{\theta}_{-i}, \theta_i' \;\; \forall i \in N$$



Revelation Principle for DSI SCFs

Theorem: If $\exists$ an indirect mechanism that implements f in dominated strategies then f is DSIC proof: let f be implemented by $\langle M_1, M_2, \dots M_n, g\rangle$ so, $\exists s_i : \Theta_i \to M_i \ \forall i \in N$ s.t

$$u_i(g(s_i(\theta_i), \tilde{m}_{-i}), \theta_i) \geq u_i(g(m_i', \tilde{m}_{-i}), \theta_i) \ \forall m_i', \tilde{m}_{-i}, \theta_i, \forall i \in N \quad \text{—— ①}$$

$$\& \ g(s_i(\theta_i), s_{-i}(\theta_{-i})) = f(\theta_i, \theta_{-i}) \ \forall \theta_i, \theta_{-i} \quad \text{—— ②}$$

in ①: let $m_i' = s_i(\theta_i')$
$\tilde{m}_{-i} = s_{-i}(\tilde{\theta}_{-i})$ for $\forall \theta_i', \tilde{\theta}_{-i}$ (arbitrary)

so, $u_i(g(s_i(\theta_i), s_{-i}(\tilde{\theta}_{-i})), \theta_i) \geq u_i(g(s_i(\theta_i'), s_{-i}(\tilde{\theta}_{-i})), \theta_i)$

$\Rightarrow u_i(f(\theta_i, \tilde{\theta}_{-i}), \theta_i) \geq u_i(f(\theta_i', \tilde{\theta}_{-i}), \theta_i)$ ($\because$ ②)

$\Rightarrow f$ is DSIC

Bayesian Extension:

Agents may have probabilistic information about other's types

Types generated from a common prior, where Bayesian game:

$$\langle N, (M_i)_{i\in N}, (\Theta_i)_{i\in N}, P, (\Gamma_\theta)_{\theta\in\Theta} \rangle$$

Defn: An indirect mechanism $\langle M_1, M_2, \dots M_n, g\rangle$ implements an SCF f in Bayesian strategies if

$\exists (s_1, \dots s_n)$ s.t $f(\theta_i)$ maximizes ex-interim utility $\forall i \in N$

$$\mathbb{E}_{\theta_{-i}|\theta_i}[u_i(g(s_i(\theta_i), s_{-i}(\theta_{-i})), \theta_i)] \geq \mathbb{E}_{\theta_{-i}|\theta_i}[u_i(g(m_i', s_{-i}(\theta_{-i})), \theta_i)] \quad \forall m_i', \theta_i, \forall i \in N$$

expectation with common prior P
and $g(s_i(\theta_i), s_{-i}(\theta_{-i})) = f(\theta_i, \theta_{-i}) \ \forall \theta$

we call f a Bayesian implementable via $\langle M_1 \dots M_n, g\rangle$ under P

Note: If an SCF f dominant strategy implementable, then its bayesian implementable

Defn: $\langle \Theta_1, \dots \Theta_n, f\rangle$ is Bayesian incentive compatible (BIC) if

$$\mathbb{E}_{\theta_{-i}|\theta_i}[u_i(f(\theta_i, \theta_{-i}), \theta_i)] \geq \mathbb{E}_{\theta_{-i}|\theta_i}[u_i(f(\theta_i', \theta_{-i}), \theta_i)] \quad \forall \theta_i, \theta_i', \forall i \in N$$

Theorem: (Revelation Principle for Bayesian implementable SCFs) If an SCF f is implemented in Bayesian equilibrium, then f is BIC

proof: let f be implemented by $\langle M_1, M_2, \dots M_n, g\rangle$ so, $\exists s_i : \Theta_i \to M_i \ \forall i \in N$ s.t

$$\mathbb{E}_{\theta_{-i}|\theta_i} u_i(g(s_i(\theta_i), s_{-i}(\theta_{-i})), \theta_i) \geq \mathbb{E}_{\theta_{-i}|\theta_i}[u_i(g(s(\theta_i'), s_{-i}(\theta_{-i})), \theta_i)] \ \forall \theta_i, \theta_{-i} \quad \text{—— ①}$$

as $g(s_i(\theta_i), s_{-i}(\theta_{-i})) = f(\theta_i, \theta_{-i}) \ \forall \theta_i, \theta_{-i}$

in ① let
$g(s_i(\theta_i), s_{-i}(\theta_{-i})) = f(\theta_i, \theta_{-i})$
$g(s_i(\theta_i'), s_{-i}(\theta_{-i})) = f(\theta_i', \theta_{-i})$

$$\Rightarrow \mathbb{E}_{\theta_{-i}|\theta_i} u_i(f(\theta_i, \theta_{-i}), \theta_i) \geq \mathbb{E}_{\theta_{-i}|\theta_i}[u_i(f(\theta_i', \theta_{-i}), \theta_i)] \ \forall \theta_i, \theta_i' \ \forall i \in N$$

$\Rightarrow f$ is BIC by defn

Arrow's social welfare funtion:

can we ignore truthful revelation, we want to resonably aggregate opinions for a general setup

we want to make social pref from individual preference

Eg: $\begin{bmatrix} a & & d \\ c & & a \\ d & \cdots & b \\ b & & c \end{bmatrix} \rightarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ where a, b, c, d are candidates preferences for every player i

$\theta_1 \ldots\ldots \theta_n$

a, b or a/b is $a R_i b$

for $A = \{a_1 \ldots a_m\}$ alternatives
$N = \{1, 2 \ldots n\}$
every player i has pref relation R_i over A (Binary relation $a R_i b$)

a_i is atleast as good as b for i

properties of R_i:

Completness: for $a, b \in A$ either $a R_i b$ or $b R_i a$
Reflexivity: $\forall a \in A, a R_i a$
Transitivity: if $a R_i b$ & $b R_i c \Rightarrow a R_i c$

let R be set of all preference ordering
ordering R_i is linear if for every $a, b \in A$, s.t $a R_i b$ and $b R_i a$ implies $a = b$ (asymmetric)

set of all linear preference orders is denoted by P

Note: Any arbitrary ordering R_i can be decomposed into
① asymmetric part P_i
② symmetric part I_i

Eg: $R_i = \begin{bmatrix} a \\ b, c \\ d \end{bmatrix} = \{(a,b), (a,c), (a,d), (b,c), (c,b), (b,d), (c,d)\}$

$P_i = \begin{bmatrix} a & a \\ b & c \\ d & d \end{bmatrix} = \{(a,b), (a,c), (a,d), (b,d), (c,d)\}$ $I_i = \{(b,c), (c,b)\}$

Arrovian social welfare function (ASWF):

$F: R^n \longrightarrow R$ domain and codomain both rankings
$F(R) = F(R_1, R_2 \ldots R_n)$, $\hat{F}(R)$ is asymmetric part of $F(R)$
$\bar{F}(R)$ is symmetric part of $F(R)$

Note: pareto is also the unanimity

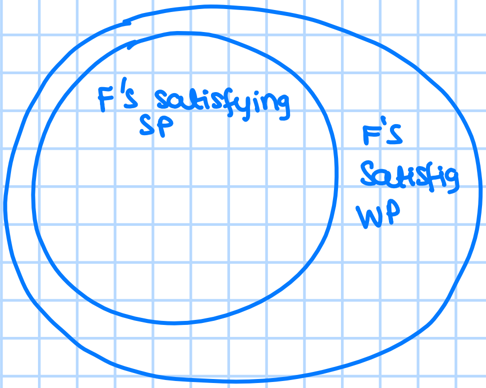
Defn: (weak pareto) An ASWF F satisfies weak pareto if the following holds for all $a, b \in A$, and for every strict preference profile R:

$$[a P_i b, \forall i \in N] \Rightarrow [a \hat{F}(R) b]$$

Eg: $\begin{bmatrix} a, b & a \\ & b \end{bmatrix}$ is weak pareto, vacuously true, this is as $a P_i b$ i=2, for i=1 we does not count as does not exist in P_1

Defn: (strong pareto) An ASWF F satisfies strong pareto if following holds for all $a, b \in A$ and for every preference profile R:

$$[a R_i b, \forall i \in N \text{ and } a P_j b \text{ for some } j \in N] \Rightarrow [a \hat{F}(R) b]$$



as SP ⇒ WP