SI 419
Combinatorics

2 parts : ① enumerative combinatorics
② graph theory part

Ref: A walk through comb, Miklos Bona — 1 to 10 chap
combinatorics techniques, Sharad S. Sane

aim - first 10 chapters of first book

2 quizzes, 1 midsem, 1 endsem, maybe surprise quiz
10% + 10%, 30%, 50%, then endsem goes down

No attendance policy

28th July:

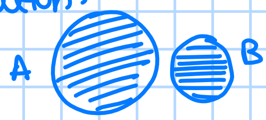
Let S be a finite set.
$|S|$ = No of elements in S

In the first part it's going to be find $|S|$

Defn: (Additive principle) let A, B be finite disjoint set, then $|A \cup B| = |A| + |B|$

Defn: (Generalised additive priciple) let $A_1, A_2, \dots A_n$ be finite sets and pairwise disjoint ($A_i \cap A_j = \phi \ \forall i \neq j$)
then $|\bigcup_{i=1}^{n} A_i| = \sum_{i=1}^{n} |A_i|$ (proof using induction)

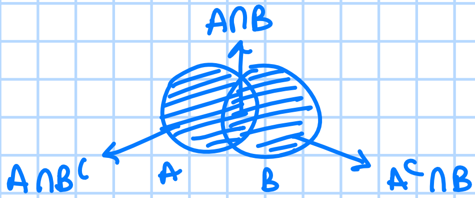
Eg: $A = \{1,2,3\}$ $|A| = 3$
$B = \{4\}$ $|B| = 1$
$A \cup B = \{1,2,3,4\} \Rightarrow |A \cup B| = 4$
and $|A \cup B| = |A| + |B|$



Note: $P(A \cup B) = P(A) + P(B)$ when $A \cap B = \phi$

Suppose A, B are two finite sets (i) $|A \cup B| = |A| + |B| - |A \cap B|$ (this can be provable)
(ii) $|A \cup B| \leq |A| + |B|$ (this can also be proved)

Eg: $A = \{1,2,3,4\}$
$B = \{2,4,6\}$
$A \cup B = \{1,2,3,4,6\}$
$|A| = 4$
$|B| = 3$
$|A \cup B| = 5$ so, $|A \cup B| \leq |A| + |B|$ is true



To prove: $|A \cup B| \leq |A| + |B|$

proof: $|A \cup B| = |(A \cap B^c) \cup (A \cap B) \cup (A^c \cap B)|$
$= \underbrace{|A \cap B^c| + |A \cap B|}_{|A|} + \underbrace{|A^c \cap B|}_{\leq |B|}$ ($\because$ additive priniple)

$\Rightarrow |A \cup B| \leq |A| + |B|$

Exe: Suppose $A_1, A_2 \dots A_n$ n-finite sets
$|\bigcup_{i=1}^{n} A_i| \leq \sum_{i=1}^{n} |A_i|$ → done see down

Defn: let A, B be two sets, we define $A - B = \{x \mid x \in A, \text{ but } x \notin B\}$, collection of elements in A but not in B

Eg: $A = \{1,2 \dots 10\}$
$B = \{6,7, \dots 15\}$

then $A - B = \{1,2, \dots 5\}$

Note: $A \setminus B = A - B = A \cap B^c$
$[n] = \{1,2 \dots n\}$

Defn: (subtraction principle) let A, B be two finite sets and $B \subseteq A$, then $|A - B| = |A| - |B|$

To prove: if $B \subseteq A$, then $|A - B| = |A| - |B|$

proof:
$A = B \cup (A - B)$
↑
disjoint

$\Rightarrow |A| = |B| + |A - B|$ ($\because$ Additive priniple)
$\Rightarrow |A| - |B| = |A - B|$

Exe: Find no. of positive integers ≤ 1000 s.t they have atleast two different digits

Ans: $S = \{x \mid 0 < x \leq 1000, x \text{ has atleast 2 different digits}\}$

$S = S_2 \cup S_3 \cup \{1000\}$

$\leftarrow$ 2 digit numbers $S_2 = \{ab \mid a \neq b\}$

$\leftarrow$ 3 digit numbers $S_3 = \{abc \mid a \neq b \text{ or } a \neq c \text{ or } b \neq c\}$

$S = [1000] - B$

$B = \{x \mid x > 0,\ x \leq 1000,\ x \text{ has one digit}\}$
$= \{1, 2, 3, \ldots, 9, 11, 22, 33, \ldots, 99, 111, 222, \ldots, 999\}$

$\Rightarrow |S| = |[1000]| - 27$
$= 1000 - 27$
$= 973$

Theorm: (Product priniple) Let A, B be finite sets, $|\{(a,b) \mid a \in A,\ b \in B\}| = |A| \times |B|$ ← tuples

proof: $|A| = n$
$|B| = m$

$A = \{a_1, a_2, \ldots, a_n\}$
$B = \{b_1, b_2, \ldots, b_m\}$

$$A \times B = \{(a_1, b_1), \ldots (a_1, b_m), \rightarrow m \text{ colmns}$$
$$(a_2, b_1), \ldots (a_2, b_m),$$
$$\vdots$$
$$(a_n, b_1), \ldots \ldots (a_n, b_m)\}$$
} n rows

then $|A \times B| = n \times m$ (no of colmns $\times$ no of rows)
$= |A| \times |B|$
$\Rightarrow |A \times B| = |A| \times |B|$

Theorm: (generalised product principle) $A_1, A_2 \ldots A_k$ are finite set

$$|\{(a_1, a_2 \ldots a_k) \mid a_i \in A_i\ \forall\ 1 \leq i \leq k\}| = |A_1| \times |A_2| \times \ldots |A_k|$$

proof: for $k = 2$, $|\{(a_1, a_2) \mid a_i \in A_i\ \forall\ 1 \leq i \leq 2\}| = |A_1| \times |A_2|$ (already proved)

now let's assume that for $k = n$ true then for $k = n+1$:

$$|\{(a_1, a_2 \ldots a_{n+1}) \mid a_i \in A_i\ \forall 1 \leq i \leq n+1\}| = |\{(a_1, a_2 \ldots a_n) \mid a_i \in A_i, \forall 1 \leq i \leq n\} \times A_{n+1}|$$
$$= |\{(a_1, \ldots a_n) \mid a_i \in A_i, \forall 1 \leq i \leq n\}| \times |A_{n+1}| \text{ (from case } k=2)$$
$$= |A_1| \times |A_2| \times \ldots |A_n| \times |A_{n+1}|$$

$\therefore$ By induction, for $k = n+1$ this is true, $\therefore$ true overall

Exe: The number of 4-digits positive integers which start and end with even digits

Ans:

$S = \{abcd \mid a, d \text{ even}, abcd > 0\}$

$S_a = \{2, 4, 6, 8\}$

$S_b = \{0, 1, 2, \ldots, 9\}$

$S_c = \{0, 1, 2, \ldots 9\}$

$S_d = \{0, 2, 4, 6, 8\}$

$S = S_a \times S_b \times S_c \times S_d \Rightarrow |S| = |S_a \times S_b \times S_c \times S_d|$
$= 4 \times 10 \times 10 \times 5$
$= 2000$

Ex: How many ways to choose password for ATM pin, given size is b/w 4 and 6, can use 0,1... 9 without any restriction

Ans: $S = S_4 \cup S_5 \cup S_6$

$\hookrightarrow$ size 4,5,6

$|S_4| = 10^4$

$|S_5| = 10^5$

$|S_6| = 10^6$

$\Rightarrow |S| = 10^4 + 10^5 + 10^6$

$= 1.11$ million

Exe: If password does not start with 0, and length = 5, it contains digit 8, how many max attempts needed

Ans:

$\underline{0}$ _ _ _ _ $\leftarrow$ 4 digits unknown, case of only one 8 digit

$4C_1 \times 10^3 + 4C_2 \times 10^2 + 4C_3 \times 10^1 + 4C_4 \times 10^0$

1 way of choosing $\leftarrow$ 2 ways of choosing 8

So total $4000 + 1200 + 40 + 1$

$= 5241$ ways for 4 differt cases of 1;8, 2;8, 3;8 or 4;8

Exe: Suppose $A_1, A_2 \ldots\ldots A_n$ n-finite sets

$$\left|\bigcup_{i=1}^{n} A_i\right| \leq \sum_{i=1}^{n} |A_i|$$

Ans: Now as $|A_1 \cup A_2| \leq |A_1| + |A_2|$ (already proved)

then for $n=2$ true

let's assume true for $n=k$

then for $n=k+1$:

$$\left|\bigcup_{i=1}^{k+1} A_i\right| = \left|\bigcup_{i=1}^{k} A_i \cup A_{k+1}\right| \leq \left|\bigcup_{i=1}^{k} A_i\right| + |A_{k+1}| \quad (\text{case of } n=2)$$

$$\Rightarrow \left|\bigcup_{i=1}^{k+1} A_i\right| \leq \sum_{i=1}^{k+1} |A_i|$$

$\therefore$ true for $n=k+1$

$\therefore$ By induction true

31^{st} July:

Exe: Password can't 0 as first digit, has 8, len is 5
(i) 0 not first, (ii), (iii)

Ans:
S = set of all passwords satisfying these
= A − B

A = set of passwords (i), (iii)
B = set of passwords (i), (iii), ~(ii)

$|A| = 9\times10^4$ ~0 _ _ _ _

$|B| = 8\times 9^4$

~0 ~8 ~8 ~8 ~8 ~8

$|A-B| = |A| - |B| = 9\times10^4 - 8\times9^4$

Method of bijection:

let S,T be two finite set, and $f: S\to T$ is a bijection, then $|S|=|T|$

($f: S\to T$ map, f is one-one and onto)

Defn: (bijection) given a map $f: S\to T$, then if f is one-one or $\forall x\neq y\in S$ $f(x)\neq f(y)$ and f is onto i.e $\forall y\in T, \exists x\in S$ s.t $f(x)=y$
then we say $f: S\to T$ is a bijection

To prove: $f: S\to T$ is bijective map and S,T are finite, then $|S|=|T|$

proof: As f is one-to-one $\Rightarrow |S|\leq|T|$ as $\forall x\in S, \exists$ a distinct $y\in T$ (def of one to one)
and as $\forall y\in T, \exists x$ s.t $f(x)=y$ or f is onto $\Rightarrow |T|\leq|S|$

therefore $|S|\leq|T|$ and $|T|\leq|S|$
$\Rightarrow |S|=|T|$

Note: To find $|S|$, we may find a bijection with set T and $|T|$ is known or calculatable compared to $|S|$

Eg: $S = \{(x_1, x_2 \ldots x_n) \mid x_i\in\{0,1\}\}$

$|S| = 2^n$ ($\because$ multiplication principle)

Exe: find number of subsets of [n]. Find $|\mathcal{P}([n])|$

Ans:

$[n] = \{1,2,3,\ldots\ldots n\}$

$f: \mathcal{P}([n]) \longrightarrow S$

$f(X) = (x_1, x_2 \ldots x_n)$

where

$x_i = \begin{cases} 1 & \text{if } i\in X \\ 0 & \text{if } i\notin X\end{cases}$

$f(\emptyset) = (0,0\ldots0)$

now $f: \mathcal{P}([n]) \to S$ is one-one as

if $f(X)=f(Y)$

$\Rightarrow (x_1, x_2 \ldots x_n) = (y_1, y_2 \ldots y_n)$

$\Rightarrow x_i = y_i\ \forall i\in\{1,2\ldots n\}$

$\Rightarrow (i\in X \text{ and } i\in Y)$ or $(i\notin X \text{ and } i\notin Y)$ $\forall i\in\{1\ldots n\}$

$\Rightarrow \quad X = Y$

$\Rightarrow f$ is one-one

f is onto as:

$\forall\, Y \in S$ say $Y = (x_1, x_2 \ldots x_n)$

then

let $X_y = \{i \mid x_i = 1\}$

then $f(X_y) = Y$

$\therefore f$ is onto

$\Rightarrow f$ is a bijection

$\Rightarrow |S| = |\mathcal{P}([n])|$

$\Rightarrow |\mathcal{P}([n])| = 2^n$

Exe: In how many ways can you choose subsets S and T of $[n]$ s.t $S \subseteq T$?

Ans:

$$A = \{(S,T) \mid S \subseteq T \subseteq [n]\}$$

$|A|$ is what we want to find

$$A' = \{(x_1, \ldots x_n) \mid x_i \in \{0,1,2\}\}$$

belongs to T not S

belongs to S and T

does not belong to S or T

let $f: A \to A'$

$$f(X) = \left\{(x_1, x_2 \ldots x_n) \;\middle|\; \begin{array}{l} x_i = 0 \text{ if } i \notin S \text{ or } T \\ x_i = 1 \text{ if } i \notin S,\ i \in T \\ x_i = 2 \text{ if } i \in S,\ i \in T \end{array}\right\}$$

$|A'| = 3^n$

f is one-one:

$f(X) = f(Y) \qquad X, Y \in A$

$(x_1, x_2 \ldots x_n) = (y_1, y_2 \ldots y_n)$

$\Rightarrow x_i = y_i \quad \forall i \in \{1, 2 \ldots n\}$

$\Rightarrow (S_x, T_x) = (S_y, T_y)$

$\Rightarrow X = Y$ so one-one

f is onto:

$\forall\, Y \in A'$, let $Y = (y_1, \ldots y_n)$

then from defination of A' we can consut S, T s.t

$S = \{i \mid y_i = 2\}$

$T = \{i \mid y_i = 1 \text{ or } 2\}$

$\Rightarrow \exists\, X_y = (S,T) \in A$ s.t

$f(X_y) = Y$

$\Rightarrow f$ is onto

$\therefore |A| = |A'| = 3^n$

Note: In some situations if A, B have same number of elements then it can be shown that $\exists$ a bijection $f: S \to T$. So we dont have to calulate no. of elements in these sets.

Exe: show that number of odd sizes is same as number of subsets of $[n]$ of size even.

Ans: $S = \{A \subseteq [n] \mid |A| \text{ is odd}\}$ supposing n is odd

$T = \{A \subseteq [n] \mid |A| \text{ is even}\}$

$f : S \longrightarrow T$

$f(A) = [n] \setminus A$ as n is odd, $|A|$ is odd

$\Rightarrow |[n] \setminus A| = n - |A| \rightarrow$ even

so $[n] \setminus A \in T$

$f(A) = [n] \setminus A$ is a bijection (trivial to see one-one and onto mapping)

$\Rightarrow |S| = |T|$

and as $|S| + |T| = |\mathcal{P}([n])| = 2^n$

$\Rightarrow |S| = |T| = 2^{n-1}$

if n is odd: odd $n-1$ so no of odd = no of even

or use $\sqcup$ to denote disjoint

$$\mathcal{P}([n]) = \mathcal{P}([n-1]) \sqcup C$$

$$C = \{A \subseteq [n] \mid n \in A\}$$

for C: $\exists$ a bijection b/w C and $\mathcal{P}([n-1])$

$f : C \longrightarrow \mathcal{P}([n-1])$

$f(A) = A - \{n\}$ (trivially bijective)

$\Rightarrow |C| = |\mathcal{P}([n-1])|$

as $\mathcal{P}[n-1]$ has same number of even and odd so C will have same number of odd and even

# of odd sets in C = # of even sets in $\mathcal{P}[n-1]$

# of even sets in C = # of odd sets in $\mathcal{P}[n-1]$

$g : S \longrightarrow T$

$$g(A) = \begin{cases} A - \{n\} & \text{if } n \in A \\ A \cup \{n\} & \text{if } n \notin A \end{cases}$$

then g is one-one and onto (trivial) and so g is bijection and $|A| = |T|$ for n even case.

one more method for n = even:

$$V = \{(x_1 \ldots x_n) \mid x_i \in \{0,1\}, \sum_{i=1}^{n} x_i \text{ is odd}\}$$

x_i is odd or even for $i \in \{1, 2 \ldots n-1\}$ but x_n is fixed so 2^{n-1} options

$f : S \longrightarrow V$ s.t f is a bijection (trivial)

$\Rightarrow |S| = 2^{n-1}$

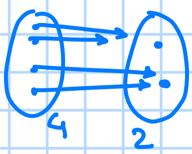
$\Rightarrow |T| = 2^{n-1}$

$\Rightarrow |S| = |T|$

Division principle:

Defn: (d to one map) let S, T be two sets with fixed positive integers, we say a map $f : T \rightarrow S$ is d-to-one map if $\forall s \in S$, $\exists d$ elements in T s.t $f(t) = s$

Eg:



2-to-one map

Defn:(Divison principle) If $f: T \to S$ is a d-to-one map then $|T| = d|S|$

our aim was to count no. of finite elements in a finite set S.
we can:

1) count directly
2) Additive principle
$S = S_1 \cup S_2 \cup \dots \cup S_m$
$\Rightarrow |S| = \sum_{i=1}^{m} |S_i|$
3) Subtraction priciple
$S = A - B \quad B \subseteq A$
$\Rightarrow |S| = |A| - |B|$
4) multiplication priciple
5) combination of addition/subtraction/multiplication
6) Bijective method
7) division priciple

pigeonhole principle:

we have to distribute n balls into k boxes when $n > k$, then there is a box which gets atleast 2 balls.
(proof of this is trivial)

generalised pigeonhole principle:

let $A_1, \dots, A_k$ are finite sets which are pair-wise disjoint

if $|\bigcup_{i=1}^{k} A_i| > kr$, then $\exists$ a set A_j s.t

$|A_j| > r$

To prove: $A_1 \dots A_k$ s.t $\forall i \neq j \; A_i \cap A_j = \phi$
$|\bigcup_{i=1}^{k} A_i| > rk$
then $\exists j \in \{1 \dots k\}$ s.t $|A_j| > r$

proof: if $\forall j \in \{1 \dots k\} \; |A_j| \leq r \Rightarrow |\bigcup_{j=1}^{k} A_j| = \sum_{i=1}^{k} |A_i| \leq r \times k$
$\Rightarrow |\bigcup A_j| \leq rk$
contradiction
so $\exists j \in \{1 \dots k\}$ s.t
$|A_j| > r$

Ex: let $x_1, x_2, \dots, x_n \in \mathbb{N}$, then $\exists k, l$ $(0 < k \leq l \leq n)$ s.t $(x_k + x_{k+1} + \dots + x_{k+l})$ is divisible by n.

Ans: let $y_j = \sum_{i=1}^{j} x_i$, $j = 1, \dots, n$
$y_j \bmod n \neq 0$ then $\{1, \dots, n-1\}$ reminder
so for $j = 1, \dots, n$ if none have mod $n = 0$, then
$\exists j_1, j_2$ s.t $y_{j_1} \bmod n = y_{j_2} \bmod n$ (pigeonhole principle)
$j_1 \neq j_2$ then $y_{j_2} - y_{j_1} = \sum_{i=j_1+1}^{j_2} x_i$ (wlog $j_2 > j_1$)

$y_{j_2} - y_{j_1} \bmod n = 0 \Rightarrow$ so we get $j_2 = \ell$ $j_1 = k$ and
$j_2 < n$ so done.

4^{th} Aug :

P-H-P :

Distributing n balls into k boxes when $k<n$, then there is atleast one box which gets more than one ball.

A_i, $i \in \{1, \dots n\}$ s.t $A_i \cap A_j = \phi$ for $i \neq j$, if $\sum_{i=1}^{n} |A_i| > nk$ then $\exists$ atleast one i s.t $|A_i| > k$

Exe : n player tournament (any two player play one game against each other) prove that at any given point of time there are two players who have played same number of games.

Ans : At time point t, let m_i be the number of games played by i^{th} player.

$m_i \in \{0, 1, \dots, n-1\}$ for $i \in [n]$ (n players so $i \in \{1, 2 \dots n\}$)

but $m_i \in \{0, \dots, n-2\}$ as if some player has not played any game the max can be $n-2$

or $m_i \in \{1, \dots n-1\}$ (other case) (only two cases $\{0, \dots n-2\}$ or $\{1, \dots n-1\}$)

now using PHP, for case I : $m_i \in \{0, \dots n-2\}$ ← $n-1$ choices

$i \in [n] \rightarrow n$ players, so, $\exists i \neq j$ s.t

$m_i = m_j$

case II : $m_i \in \{1, \dots n-1\}$ $i \in [n]$, we have n numbers and no of possible values is $n-1$, then by PHP

$\exists i \neq j \in [n]$ s.t

$m_i = m_j$

so in both cases at a time, $\exists$ 2 players who have played same number of games

Exe : select $(n+1)$ distinct integers from $[2n]$. Then there is a pair of integers from the selected one whose sum is $(2n+1)$

Ans : $[2n] = \{1, 2 \dots 2n\}$, now lets make n subsets s.t

$A_i = \{i, 2n-i+1\} \ \forall i \in [n]$

A_i's are s.t sum of their elements is $i + (2n-i+1) = 2n+1$

so,

$[2n] = \bigcup_{i=1}^{n} A_i$

by selecting $n+1$ numbers from n subgroups, by PHP

$\exists i \in [n]$ s.t 2 numbers from same subgroup

so by selecting $n+1$, $\exists$ 2 numbers s.t they are from same subset, so sum $2n+1$

Exe : A student has 6 weeks to prepare. He decides to study in following way

① Atleast one hour everyday

② Study in multiple of hours

③ total of 70 hours in 6 weeks

Show that no matter how he plans, he will study exactly 13 hours during some consecutive days.

Ans :

$6 \times 7 = 42$ days, let x_i = hours studied on i^{th} day

so

$1 \leq x_i \leq 24$ $i \in [42]$

$\sum_{i=1}^{42} x_i = 70$

we have to show $\exists i < j$ s.t $\sum_{k=i}^{j} x_k = 13$

so let $y_i = \sum_{k=1}^{i} x_k$ for $i \in [42]$, if we can show $\exists\, i<j$ s.t $y_j - y_i = 13$ we are done

$S = \{y_1, \ldots, y_{42}\}$

$1 \leq y_1 < y_2 < \cdots < y_{42} = 70$

$T = \{y_1 + 13, \ldots, y_{42} + 13\}$

$14 = 1+13 \leq y_1 + 13 \leq \cdots \leq y_{42} + 13 = 83$

$|S| = 42$

$|T| = 42$

collective range of elements in S and T is s.t $\forall\, y \in S \cup T$

$1 \leq y \leq 83$, so collective range is $[83]$

also no two elements of S are same and no two elements of T are same so total $|S| + |T| = 84$ elements, $\therefore \exists\, 2$ elements with same value say x, y. But as no two elements same inside, wlog $x \in S$, $y \in T$ and so $\exists\, i, j$ s.t $i \neq j$

$y_i = 13 + y_j \Rightarrow y_i - y_j = 13$

and $i > j$ as y_i is strictly inc

permutation:

Arrangement of n-distinct objects in a line is one such example of permutation. we want to find 'total permutations' of n distinct objects.

$\underset{n \text{ options}}{\_\_}\ \underset{n-1 \text{ options}}{\_\_}\ \cdots\ \underset{1 \text{ option}}{\_\_}$

so by multiplication principle: $(n) \times (n-1) \times \cdots (1) = n!$

^nP_r:

Number of permutations of n-distinct objects taking r at a time

$\underset{n \text{ choices}}{\_\_}\ \underset{n-1 \text{ choices}}{\_\_}\ \cdots\ \underset{n-r+1 \text{ choices}}{\_\_}$

r positions

total: $(n) \times (n-1) \times \cdots (n-r+1) = \dfrac{n!}{(n-r)!} = {^nP_r}$

permuting r objects from set of n distinct objects

so, $^nP_r = \dfrac{n!}{(n-r)!}$

^nC_r:

Number of subsets of size k of $[n]$, let this be N

$N \times k! =$ choosing k size subsets $\times$ $k!$ = permutations of chosen subset

choosing a subset of size k

permuting it to k!

no. of ways of choosing x inside permutations = no of permutations of n distinct objects taking k at time $= {^nP_k}$

$\Rightarrow N\, k! = {^nP_k}$

$\Rightarrow N = \dfrac{^nP_k}{k!} = {^nC_k} = \dfrac{n!}{(n-k)!\, k!}$

$S = \{$set of all permutations of k distinct elements taken from $[n]\}$

$|S| = N \times K!$ ← one method of counting

$|S| = {}^nP_K$ ← one more way of counting it

so $|S| = N \times K! = {}^nP_K$
$\Rightarrow N = {}^nC_K$

Exe: We have K-different types of objects suppose x_i = no of i^{th} object $i \in [K]$ $\forall i \in [K]$, $x_i > 0$ and $\sum_{i=1}^{K} x_i = n$, find total permutations of n-objects.

Ans: denote i^{th} type object as a_i, make them distinct by:

$$\left.\begin{matrix} a_1^1, a_1^2, \ldots a_1^{x_1} \\ a_2^1, a_2^2, \ldots a_2^{x_2} \\ \vdots \\ a_K^1, \ldots \ldots a_K^{x_K} \end{matrix}\right\}$$ total n, if all distinct then $n!$ = total permutation

$S = \{$set of permutations of a_i^j's$\}$

$|S| = n!$

if the answer is N, then lets fix all the permutations of this group by making them unique:

$T = \{$set of perms when x_1 many a_1, x_2 many a_2, ... x_K many $a_K\}$

if we fix an element in T, we can permut them as unique by total of $x_1! \times x_2! \times \ldots x_K!$

then $|T| \times x_1! \times \ldots x_K! = |S|$

$\Rightarrow N \times x_1! \times \ldots x_K! = n!$

$$\Rightarrow N = \frac{n!}{x_1!\, x_2! \ldots x_K!}$$

Note: The above question can also be done by fixing a map d-to-one where $f: S \longrightarrow T$ $|T|d = |S|$ and $d = x_1! \times x_2! \ldots x_K!$. map is remove the superfix from a_i^j so exactly d many corresponds to one in T and so $|S| = d|T|$

one more proof is that we take n positions:

_ _ _ . . . . . _ → n positions

we can find x_1 places from n by finding number of ways of making x_1 size subsets from n positions $= {}^nC_{x_1}$

$$({}^nC_{x_1}) \times ({}^{\underbrace{n-x_1}_{\text{positions left}}}C_{x_2}) \times ({}^{n-x_1-x_2}C_{x_3}) \times \ldots \times ({}^{n-x_1-\ldots-x_{K-1}}C_{x_K})$$

$$= \frac{n!}{\cancel{(n-x_1)!}\, x_1!} \times \frac{\cancel{(n-x_1)!}}{\cancel{(n-x_1-x_2)!}\,(x_2)!} \cancel{\times \ldots} \ldots \times \frac{\cancel{1}}{\cancel{1}\,(x_K)!}$$

$$= \frac{n!}{x_1! \ldots x_K!}$$

Theorm: $\binom{n}{k}+\binom{n}{k+1}=\binom{n+1}{k+1}$

proof: coubinotarl proof of this will iucude us making two sets and cordinality is equal for both.

$$S=\{A \mid A\subseteq[n+1] \text{ and } |A|=k+1\}$$

$|S|=\binom{n+1}{k+1}$ from our previous proofs

now, $S=S_1\cup S_2$ s.t

$S_1=\{A \mid A\subseteq[n], |A|=k+1\}$

$S_2=\{A \mid A\subseteq[n+1], |A|=k+1, n+1\in A\}$

$|S_2|=\binom{n}{k}$ $\quad |S_1|=\binom{n}{k+1}$

Exe: $1+2+\cdots n=\frac{(n)(n+1)}{2}$ → done

Exe: $\sum_{k=0}^{n}\binom{n}{k}^2=\binom{2n}{n}$

Ans: $S=\{A \mid A\subseteq[2n], |A|=n\}$

$|S|=\binom{2n}{n}$

$[2n]=\{1,3,\ldots 2n-1\}\cup\{2,4,\ldots 2n\}$

choose k odd $n-k$ even

so

$$|S|=\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}=\sum_{n=0}^{k}\binom{n}{k}^2$$

Exe: For all positive integers $k\leq n$, $\binom{k}{2}+\binom{n-k}{2}+k(n-k)=\binom{n}{2}$

Ans: if $k\geq 2$ then we make 2 subsets of size k and $n-k$ then choose 2 from k, or choose 2 from $n-k$ or choose 1 from each, we get same result, if $k<2$ then we cosider $\binom{k}{2}=0$

so $\binom{k}{2}+\binom{n-k}{2}+(k)(n-k)$

$=\binom{n}{2}$

Exe: $\sum_{k=1}^{n}k\binom{n}{k}=n2^{n-1}$ or making groups of size k and one caption

Ans: Right side is $n\times 2^{n-1}$ or no of ways of finding a caption b/w n differnt people $=n$ and for $n-1$ players choice of being in team or not, so

Right $=n\times 2^{n-1}$

on left we have k people in team, and choosing 1 caption from them, so

$$\sum_{k=1}^{n}k\binom{n}{k}=n\times 2^{n-1}$$

Exe: $1+2+\cdots n=\frac{(n)(n+1)}{2}$

Ans: on right $S=\{A\subseteq[n+1] \mid |A|=2\}$ or $|S|=\binom{n+1}{2}=\frac{(n+1)!}{(n-1)!\,2!}=\frac{(n)(n+1)}{2}$

now, $S = S_1 \cup S_2 \cup \cdots \cup S_n \cup S_{n+1}$

where

$S_i = \{$choosen i and 1 from all numbers before $i\}$

so $|S_1| = 0$ as no way to make a pair

$|S_2| = 1$

$|S_3| = 2$

$\vdots$

$|S_{n+1}| = n$

and $|S| = \sum_{i=1}^{n+1} |S_i| = \sum_{i=1}^{n} i = {}^nC_2 \Rightarrow 1+2+\cdots+n = {}^nC_2$

7^{th} Aug:

Multiset:

set was a collection of distinct objects, so $S=[n]$, $S=\{a,b,c,d\}$

Note: In a set no chosen elements appear more than once. No repetition of elements.

Note: ordering of elements does not matter

$$\{1,2,3\}=\{1,3,2\}$$

Defn: (multiset) finite collection of elements where each element is coming from a set S say

(i) repetation of elements are allowed

(ii) odering of elements does not matter

Such elements is called a multiset over the set S.

Exe: $S=[5]$, $\{1,1,1\}$, $\underbrace{\{1,2,2\}=\{2,1,2\}}_{\text{Same multisets}}$ → all this are called multisets of S

Note: A multiset of a set S not necassarily contains all element of S.

Exe: How many multiset are there of size n over a set of size k

Ans: If $S=[k]$ $M=\{\text{multiset } m \mid |m|=n \text{ and } m \text{ is a multiset of } S\}$

$$T=\{(x_1,\dots x_k) \mid x_i\geq 0, \sum_{i=1}^{k} x_i=n\}$$

$f:T\to M$ is a bijection

then, $|T|=|M|$, now $x_i\geq 0$

$$x_1+\cdots x_k=n$$
$$\underbrace{x_1+1}+\cdots x_k+1=n+k$$
$$y_1+y_2+\cdots y_k=n+k$$
s.t $y_i\geq 1$

$n+k-1$ options

select $k-1 \to \binom{n+k-1}{k-1}$

Defn: (weak composition) A tuple $(x_1\dots x_k)$ of non-negative integers is called a weak composition of n into k-parts

$$x_i\geq 0, \sum_{i=1}^{k} x_i=n$$

Eg:
$$\left.\begin{array}{l}(1,1,1,1)\\(2,0,2,0)\\(1,2,1,0)\\(4,0,0,0)\end{array}\right\}\text{composition of 4 in 4 parts}$$
$\vdots$

Note: Two compositions of n are diffent:

(i) If sizes are diffent

(ii) $(x_1,\dots x_k)$, $(y_1\dots y_k)$ are diffent if $\exists i$ s.t $x_i\neq y_i$

Defn: (composition) $(x_1, \dots x_k)$ is called composition of n into k parts if $x_i\in\mathbb{N}$ and $\sum_{i=1}^{k} x_i=n$

Eg: $(1,1,1,1)$ → composition of 4 into 4 parts

Note: There is no composition of n into k parts if $k>n$

Theorm: Number of weak composition of n into k parts is ${}^{n+k-1}C_{k-1}$

proof: we want to distribute n identical objects (in our case the number 1) into k identical boxes (in our case the x_i's)

for n balls
k boxes, we need to permute $(n+k-1)$ things where $n, k-1$ things are same
so our answer will be equal to no. of permutation of n identical balls and $(k-1)$ bars

$$= \frac{(n+k-1)!}{n!\,(k-1)!} = \binom{n+k-1}{n} = \binom{n+k-1}{k-1}$$

Theorm: No. of compositions of n into $k \leq n$ parts is ${}^{n-1}C_{k-1}$

proof: The number above will be similar to choose $k-1$ spots from $n-1$ spots
$= \binom{n-1}{k-1}$

It is also same as no. of distribution of n identical balls into k-distinct boxes so that no box is empty
we can also give one ball to each box so now $n-k$ balls left
and so $\binom{n-k+k-1}{k-1} = \binom{n-1}{k-1}$ is from weak composition

Defn: (set partition) It is a partition of a set S in non-empty disjoint blocks s.t $\bigcup_{i=1}^{k} S_i = S$

Eg: $S = [5] = \{1,2,3,4,5\}$

$\{1,3\}$ $\{2,4,5\}$ set partion of $[5]$ of size 2

5 size set partion

Note: In a set partition, size denotes no of blocks

Defn: $(S(n,k))$ denotes the number of set partion of $[n]$ into k-blocks.

Note: Odering of blocks does not matter

$\{1,2\}, \{3,4\}, \{5\}$ and $\{5\}, \{1,2\}, \{3,4\}$ are same partitions

Now, as $S(n,k)$ is number of set partions of $[n]$ of size k, we have

1. If $k > n$, $S(n,k) = 0$
2. $S(n,1) = 1$
3. $S(n,n) = 1$

Exe: find $S(n,2)$ → done

Theorm: For $k, n \in \mathbb{N}$, $k \leq n$ show that $S(n,k) = S(n-1,k-1) + k\,S(n-1,k)$

proof: $T \rightarrow$ set of all set partitions of $[n]$ into k blocks

$$T = T_1 \cup T_2$$

$T_1 \rightarrow$ set of all set partition of $[n]$ into k blocks where $\{n\}$ is a block

$T_2 = T - T_1$, $|T_1| = S(n-1, k-1)$ [$\because$ $[n-1]$ will be partitioned into $(k-1)$ blocks]

$X \in T_2$, then $[n-1]$ is a partition into k-blocks and 'n' can sit in any of them so

$\leftarrow k$ choices for $\{n\}$

$k \times S(n-1, k) = |T_2|$

so $|T| = k \times S(n-1, k) + S(n-1, k-1)$

Exe: find $S(n, 2)$

Ans: 2 blocks: $\dfrac{{}^nC_1 + {}^nC_2 + \cdots {}^nC_{n-1}}{2}$ = total as ${}^nC_k$ = selecting k and $k-1$ otherwise and divide by 2 to account for double count

$= \dfrac{2^n - 2}{2}$

$S(n, 2) = 2^{n-1} - 1$

11^{th} Aug:

weak composition $(a_1, \dots, a_k)$, $a_i \geq 0$, $\sum a_i = n$ → distribution of n-identical balls into k-distinct boxes $({}^{n+k-1}C_{k-1})$

composition: $(a_1 \dots a_k)$, $a_i > 0$, $\sum a_i = n$ → ... where no box is empty $({}^{n-1}C_{k-1})$

set partition: $\{S_1, \dots, S_k\}$ is a set partition of $[n]$ s.t $\bigcup_{i=1}^{k} S_i = [n]$, $S_i \neq \phi$, $S_i \cap S_j = \phi \ \forall i \neq j$

↪ distribution of distguble balls into k identical boxes

Note: $(2,1,2), (1,2,2), (2,2,1)$ are diffent composition of 5 as: $x_1 + x_2 + x_3 = 5$

Now, $S(n,k)$ = No of set partion of $[n]$ into k-blocks

$\tilde{S}(n,k) = \{P_1, \dots\}$ P_i is a set partion of $[n]$ into k blocks

$P_1 = \underbrace{\{1\}\{2\}\{3\}\dots\{k-1\}\{k \dots n\}}_{k \text{ partions of } [n]}$

Note: $S(n,k) = |\tilde{S}(n,k)|$

Theorm: For $k \leq n$, $k, n \in \mathbb{N}$, $S(n,k) = S(n-1,k-1) + kS(n-1,k)$

proof: $\tilde{S}(n,k) = T_1 \sqcup T_2$ $\quad T_1 = \{P \in \tilde{S}(n,k) \mid \{n\} \text{ is a block in } P\}$

$T_2 = \{P \in \tilde{S}(n,k) \mid \{n\} \text{ is not a block in } P\}$

$|T_1| = |\tilde{S}(n-1,k-1)|$ as there is a bijection b/w T_1 and $\tilde{S}(n-1,k-1)$

$|T_2| = k \times |\tilde{S}(n-1,k)|$ as there is a d-to-one mapping from T_2 to $\tilde{S}(n-1,k)$ with $d = k$

given a set partition of $[n-1]$ into k blocks, we have k choices for n to place. So we have k diffent set partition of $[n]$ into k blocks, given a set partion of $[n-1]$ into k blocks

$|\tilde{S}(n,k)| = S(n,k) = |T_1| + |T_2|$

$S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$

Exe: No of funtions from $[n]$ to $[m]$

Ans:

$f(1) \to m$ choices

$f(2) \to m$ choices

$\vdots$

$f(n) \to m$ choices

So $m \times m \times \cdots m = m^n$ = total f from $[n] \to [m]$

Exe: no of one-to-one map from $[n]$ to $[m]$

Ans:

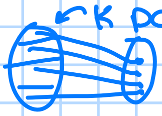
no of one-to-one $= \begin{cases} 0; & \text{if } n > m \\ {}^{m}P_{n}; & \text{if } n \leq m \end{cases}$

choose n and permute it to get the number

Exe: No of onto maps from $[n]$ to $[m]$

Ans:

value $= \begin{cases} 0; & n < m \\ v; & n \geq m \end{cases}$,



← k partitions $\tilde{S}(n,k)$ and we have $k!$ to arrange so

$v = S(n,k) \times k!$ here $k = m$

Theorem: For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, $x^n = \sum^{n} S(n,k)(x)_k$ where $(x)_k = x(x-1)\dots(x-(k-1))$

Proof: $S(n,0)=0$ so $k=0$ does $k=1$ not matter

$$P(x)=x^n$$
$$Q(x)=\sum_{k=1}^{n} S(n,k)(x)_k$$

we will show $P(x)=Q(x)$ $\forall x=n,n+1,\dots\dots$

$x=m\in\mathbb{N}$, $m\geq n$

$P(m)=m^n$ = no of functions from $[n]$ to $[m]$ = LHS

$$RHS=\sum_{k=0}^{n} S(n,k)(m)_k$$
$$=\sum_{k=0}^{n} S(n,k)\times\frac{m!}{(m-k)!}$$

now $f:[n]\longrightarrow[m]$ s.t size of range is k

${}^mC_k \rightarrow$ choose k from m

$T_k=\{f:[n]\longrightarrow[m] \mid |f[n]|=k\}$

Range of f

choosing k from $m={}^mC_k$
no. of onto maps $=S(n,k)\times k!$
total $=S(n,k)\times k!\times{}^mC_k$
$=S(n,k)\,{}^mP_k$

now

$T_1\sqcup T_2\sqcup\cdots T_m$ = no of maps

$\Rightarrow \sum|T_i|=m^n$
$\Rightarrow \sum S(n,i)\,{}^mP_i=m^n$

Defn: $S(n,k)$ is called Stirling number of second kind

Defn: $B(n)=\sum_{k=0}^{n} S(n,k)$

Bell number

Theorem: show that $B(n+1)=\sum_{i=0}^{n}\binom{n}{i}B(i)$, conventionally $S(0,0)=1$

Proof:
$B(n+1)$ = set of all set partions of $[n+1]$ of different sizes
now, size of block which contains $(n+1)$ is ≥ 1 and $\leq n+1$

S = set of all set partions of $(n+1)$

$|S|=B(n+1)$

$S=\bigcup_{k=1}^{n+1} T_k$ $\quad T_k$ = set of all partitions of $[n+1]$ where size of the block containg $n+1$ is k

$$|S|=\sum_{k=1}^{n+1}|T_k|$$

now $|T_k|=\binom{n}{k-1}\times\underbrace{\text{set of all other partions}}_{B(n+1-k)}$

choosing $k-1$ from $\{1,2\dots n\}$

$$=\binom{n}{k-1}\times B(n+1-k)$$

$$B(n+1)=\sum_{k=1}^{n+1}\binom{n}{k-1}B(n+1-k)=\sum_{l=0}^{n}\binom{n}{l}B(n-l)=\sum_{i=0}^{n}\binom{n}{i}B(i)$$

Integer partition:

$a_1 \geq a_2 \geq \dots \geq a_k > 0$

Suppose $a_i \in \mathbb{N}$ and $\sum_{i=1}^{k} a_i = n$, then $(a_1, \dots a_k)$ is called an integer partition of n

Eg: $(2,2,1), (2,1,2)$ but $(2,2,1)$ is integer partition, so we can also write it as multiset $\{1,2,2\}$

Analogy will be distribution of n identical balls in k identical boxes, so that no box is empty

14^{th} Aug:

Integer partition:

$a_1 \geq a_2 \geq \dots \geq a_k \geq 1$

$a_1 + a_2 + \dots a_k = n$

$(a_1, a_2 .. a_k)$ is called integer partion of n
$a_i \in \mathbb{N}, \forall i \in [k]$

Composition of n into k parts is a tuple $(x_1 \dots x_k)$ s.t $x_i \in \mathbb{N}, \forall i \in [k], \sum_{i=1}^{k} x_i = n$

Note: $(2,1,2), (2,2,1)$ are diffent composition of 5 in 3 parts but $(2,2,1)$ is an integer partition, so order of elements matter

In integer partition we write the elements in decreasing order, so we can represent integer partion in a multiset as well. Conventionally to write integer partition as a tuple whose elemet appear in decreasing order.

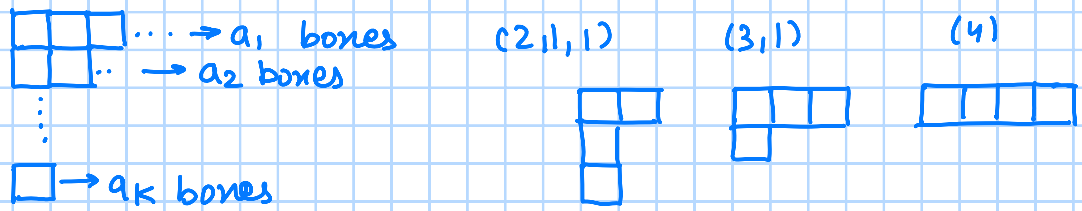
$$(a_1, a_2 \dots a_k), a_1 \geq a_2 \geq \dots \geq a_k$$

Eg: $(1,1), (2) \to$ IP of 2
$(1,1,1), (2,1), (3) \to$ IP of 3
$(1,1,1,1), (2,1,1), (2,2), (3,1), (4) \to$ IP of 4

Young diagram / Ferrer's shape:

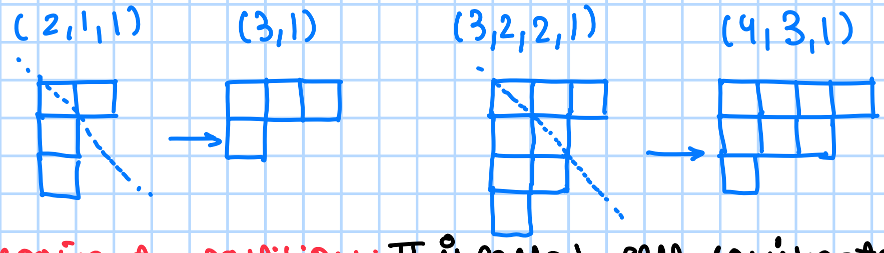
Defn: Young diagram correspondig to an integer partition $(a_1 \dots a_k)$ of n is a collection of n boxes drawn row wise, when i^{th} row as a_i many boxes

Eg:



Conjugate partition: let $(a_1 \dots a_k)$ be an int partion, its young diagram and reflect it wrt to its diagonal, new diagram correspondsto an integer partition and is called conjugate partition of $(a_1, \dots a_k)$

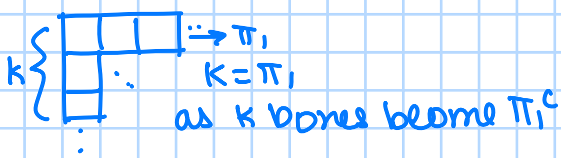
Eg:



Self conjugate partition: π is called self conjugate if π is equal to its conjugate partition.

If $\pi = (\pi_1 \dots \pi_k)$ is a self conjugate partition

$\pi^c = (\pi_1^c \dots \pi_l^c)$ whe $l = k$ as $\pi^c = \pi$



$\pi_1 = k$ as $\pi_1 = \pi_1^c = k$ from trivial observation (no. of rows $= k$)

Notation: $P_k(n) =$ No of integer partition of n into k parts

$$P(n) = \sum_{k=1}^{n} P_k(n) \to \text{total no. of integer partition of } n$$

Theorm: The number of partition of n into at most k parts is equal to no of partitions of n into parts not larger than k (each part $\leq k$)

proof:
$$\sum_{i=1}^{k} p_i(n) = \text{no of partition of } n \text{ into atmost } k \text{ parts}$$

set of all int partitions of n into atmost k parts = set of all Young diagrams with n boxes no of rows $\leq k$

set of all int. parts of n where each part $\leq k$
= set of all young diagrams with n boxes s.t no of columns $\leq k$

now there is a bijection b/w no of columns $\leq k$ and no of rows $\leq k$

Theorm: let $q(n)$ be the number of int partion of n in which each part is atleast 2

$$|\{(a_1 \dots a_k) \mid \sum a_i = n,\ a_1 \geq a_2 \geq \dots \geq a_k \geq 2\}| = q(n)$$

then
$$q(n) = p(n) - p(n-1) \quad \text{for } n \geq 2$$
i.e $p(n) = p(n-1) + q(n)$

proof:
$$S = \{(\pi_1 \dots \pi_k) \mid \sum_{i=1}^{k} \pi_i = n,\ \pi_1 \geq \dots \geq \pi_k \geq 1,\ 1 \leq k \leq n\}$$
$$T_1 = \{(\pi_1 \dots \pi_k) \in S \mid \pi_k = 1\}$$
$$T_2 = \{(\pi_1 \dots \pi_k) \in S \mid \pi_k \geq 2\}$$

then $S = T_1 \sqcup T_2$ as $T_1 \cap T_2 = \phi$
$\Rightarrow |S| = |T_1| + |T_2|$
$\Rightarrow p(n) = p(n-1) + q(n)$

$|T_1| = p(n-1)$ as there is a bijection

X = set of all integer partitions of $n-1$
$$= \{(\pi_1 \dots \pi_k) \mid \sum_{i=1}^{n} \pi_i = n-1,\ \pi_1 \geq \dots \pi_k \geq 1;\ 1 \leq k \leq n-1\}$$

$f: X \longrightarrow T_1$
$(\pi_1 \dots \pi_k) \longrightarrow (\pi_1 \dots \pi_k, 1)$
$\Rightarrow |X| = |T_1|$
$\Rightarrow |T_1| = p(n-1)$

Exe: Prove that for $n \geq 2$, the number of integer partition in which the first two parts are equal is $p(n) - p(n-1)$

Ans:
$T_3 = \{\pi \in S \mid \pi_1 = \pi_2\}$ now $T_1 \leftrightarrow T_3$ by conjugation

$$T_3^c = \{\pi^c \mid \pi \in T_3\}$$

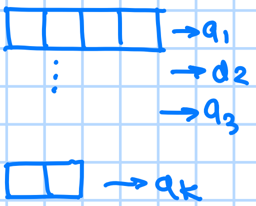
T_3^c is all the integer partions s.t they have atleast 2 columns in the young diagram = no of partions with atleast 2 in each part

$\Rightarrow |T_3|^c = |T_1| \Rightarrow |T_3| = p(n) - p(n-1)$ from previous theorm

18^{th} Aug:

$$\begin{cases} (a_1 \dots a_k) \\ \sum_{i=1}^{k} a_i = n, \ a_1 \geq a_2 \geq \dots \geq a_k \geq 1 \end{cases} \rightarrow \text{integer partition of } n \text{ into } k \text{ parts}$$

Younge diagram:



$\pi = (a_1, a_2 \dots a_k)$, then π^c is conjugate partition

$\pi = \pi^c$ is called self conjugate partition

$P_k(n)$ = no. of integer partion of n in k parts

$P(n) = \sum_{i=1}^{n} P_i(k)$ = total no of int partion of n

Exe: Find the no of int. partition of $2n$ with no odd parts (in terms of $P(n)$)

Ans:

$(a_1 \dots a_k)$ is integer partion with no odd part

$\Rightarrow \sum_{i=1}^{k} a_i = 2n$ ← all even $(a_i = 2b_i)$

$\Rightarrow \sum_{i=1}^{k} \frac{a_i}{2} = n \Rightarrow \sum_{i=1}^{k} b_i = n$ ← all integers

T_{2n} = set of int part with no odd part

S_n = set of all int part

$f: T_{2n} \rightarrow S_n$

$f((a_1 \dots a_k)) = \left(\frac{a_1}{2}, \dots \frac{a_k}{2}\right)$

$f^{-1}((b_1 \dots b_k)) = (2b_1 \dots 2b_k)$

f is a bijection as f^{-1} exist

$\Rightarrow |T_{2n}| = |S_n| = P(n)$

Exe: show that $\sum_{k=1}^{n} P(k) \leq P(2n)$

Ans: If $\exists\ g: S \longrightarrow S_{2n}$ $\ S = \bigcup_{k=1}^{n} S_k$ and g is one-one

then $|S| \leq |S_{2n}|$

$(a_1 \dots a_i) \in S_k$, $k \leq n$

let $g((a_1 \dots a_i)) = (2n-k, a_1 \dots a_i)$

as

$\sum a_i = k$

$2n - k + \sum a_i = 2n$

& as $a_1 \leq k \leq n$

$\Rightarrow 2n - k \geq 2n - a_1 \geq n$

$\Rightarrow 2n - k \geq n \geq a_1$

so $(2n-k, a_1 \dots a_i)$ is a integer partion of size $k+1$ of $2n$

$\in S_{2n}$

now for $g(x)=g(y)$
$x=(a_1 \ldots a_i)$
$y=(b_1 \ldots b_k)$

$\Rightarrow (2n-i, a_1, \ldots a_i) = (2n-k, b_1 \ldots b_k)$

$\Rightarrow 2n-i = 2n-k$ & $a_m = b_m \; \forall m \in \{1,2 \ldots i\}$

$\Rightarrow i=k$ & $a_m = b_m \; \forall m \in [i]$

$\Rightarrow x=y$

so, g is one-one

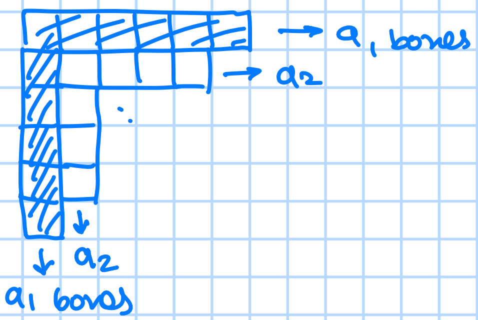
$\therefore |S| \leq |S_{2n}|$

Theorm: The no of self conjugate position of n is equal to the number of position of n in which all parts are odd and distinct

proof:

$S=$ set of all self-conjugate partions of n

$T=$ set of all int partions of n in which all parts odd and distinct



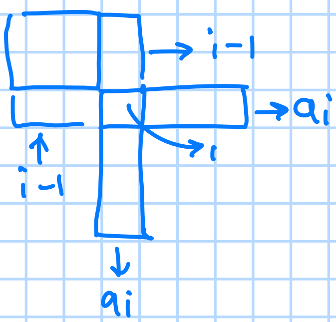
$a_1 + a_1 - 1 =$ no of boxes in first row + colmn
$2a_1 - 1 =$ no of boxes in first row + col

so $f: S \longrightarrow T$ s.t

$f(a_1) = 2a_1 - 1$

$a_2 + a_2 - 1 - 2 =$ no of boxes of 2nd row + col − common from first

$f(a_2) = 2a_2 - 3$



similarly $f(a_i)$
$= (2a_i - i + 1 - i + \cancel{1} - \cancel{1})$
$f(a_i) = (2a_i - 2i + 1)$

$f(a) = (2a_1 - 1, 2a_2 - 3, 2a_3 - 5, \ldots\ldots)$
as $a_i \geq a_{i+1}$

$2a_i - (2i) > 2a_{i+1} - (2i)$
$\quad -1 \qquad\qquad -1-2$

$\Rightarrow 2a_i - 2i + 1 > 2a_{i+1} - 2i - 1$

so $f(a)$ parts are distinct and odd, and

$\Sigma f(a)_i = n$ by consution

let $(b_1 \ldots b_k) \in T$, if $\exists$ unique $a \in S$ s.t $f(a) = (b_1 \ldots b_k)$ then we show f^{-1} exist and we are done

$f(a) = (b_1 \ldots b_k)$ i.e $\Sigma b_i = n$
$b_1 > b_2 > \cdots b_k \geq 1$
& all b_i are odd

we have to show $(b_1 \cdots b_k) = (2a_1 - 1, 2a_2 - 3, \ldots)$ where $a_1 \geq a_2 \geq \cdots$

wlog if $a_1 < a_2$

$\Rightarrow a_2 - a_1 \geq 1$

$\Rightarrow 2a_2 - 2a_1 \geq 2$

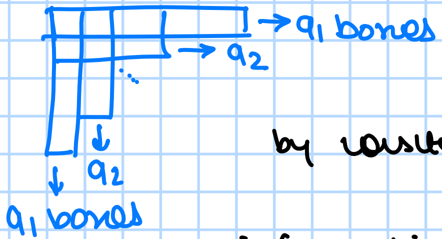
$\Rightarrow 2a_2 - 3 \geq 2a_1 - 1$

$\Rightarrow b_2 = 2a_2 - 3 \geq 2a_1 + 2 - 3 = b_1$

$\Rightarrow b_2 \geq b_1$ this is a contradiction

so, $a_1 \geq a_2$

now, $(b_1 \ldots b_k) = (2a_1 - 1, \ldots)$ is true



by condition $(a_1 \ldots\ldots)$ is conjugate

$f(a_1 \ldots) = (b_1, b_2 \ldots)$

by condition this is unique so, $\exists$ unique a s.t for $b \in T$ $a \in S$ and $f(a) = b$

$\Rightarrow f$ is a bijection

$\Rightarrow |S| = |T|$

Inclusion-Exclusion principle:

let $A_1, \ldots A_n$ be finite sets, then (version 1)

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \leq i_1 < i_2 \leq n} |A_{i_1} \cap A_{i_2}| + \cdots + (-1)^{n-1} \left|\bigcap_{i=1}^{n} A_i\right|$$

$$= \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} |A_{i_1} \cap \cdots \cap A_{i_k}|$$

In terms of probability: $(\Omega, \mathcal{F}, P)$ (version 2 proved using induction)

$A_1 \ldots A_n \in \mathcal{F}$

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} P(A_{i_1} \cap \cdots \cap A_{i_k})$$

version 2 $\Rightarrow$ version 1

$\Omega = \bigcup_{i=1}^{n} A_i$ $\quad |\Omega| = N$

$\mathcal{F} = P(\Omega)$

$P : \mathcal{F} \longrightarrow [0,1]$

let $P(A) = \frac{|A|}{|\Omega|} = \frac{|A|}{N}$

$\Rightarrow |A| = N P(A)$

as version 2 is true:

$$P\left(\bigcup_{i=1}^{k} A_i\right) = \sum (-1)^{k-1} \sum P(A_{i_1} \ldots \cap A_{i_k})$$

$$\Rightarrow N P\left(\bigcup_{i=1}^{k} A_i\right) = \sum (-1)^{k-1} N \sum P(A_{i_1} \ldots \cap A_{i_k})$$

$$\Rightarrow \left|\bigcup_{i=1}^{k} A_i\right| = \sum (-1)^{k-1} \sum |A_{i_1} \cap \cdots \cap A_{i_k}|$$

combinatorial way:

let $x \in \bigcup_{i=1}^{n} A_i$, then on LHS x has contributed 1

on RHS we need x contributes 1

if $x \in A_i$ for j many i's then

$x \notin A_{i_1} \cap \cdots \cap A_{i_\ell}$ for $\ell > j$

$S = \{i \mid x \in A_i\}$ $\quad |S| = j$

now $^nC_1 - {^nC_2} + \cdots (-1)^{n-1}\, {^nC_n} = {^nC_0} = 1$

so, $\sum |A_i| - \sum |A_i \cap A_j| + \cdots = |\cup A_i|$

Exe: prove that $S(n,k)$ = no of set partions of $[n]$ into k parts

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n = \sum_{i=0}^{k} (-1)^i \frac{(k-i)^n}{i!\,(k-i)!}$$

Ans: $k!\,S(n,k)$ = no of surjective maps from $[n]$ to $[k]$ $\quad n \geq k$

$i \in [k]$ $\quad A_i = \{f : [n] \to [k] \mid i \notin \text{Range of } f\}$

Set of all surjective maps from $[n]$ to $[k] = \bigcap_{i=1}^{k} A_i^c$

$$\Rightarrow \left|\bigcap_{i=1}^{k} A_i^c\right| = \left|\left(\bigcup_{i=1}^{k} A_i\right)^c\right| = k^n - \left|\bigcup_{i=1}^{k} A_i\right|$$

$$\text{now } \left|\bigcup_{i=1}^{k} A_i\right| = \sum_{i=1}^{k} |A_i| - \sum_{i<j} |A_i \cap A_j| + \cdots + (-1)^{k-1} \sum_{i_1 < i_2 < \cdots < i_k} |A_{i_1} \cap A_{i_2} \cdots \cap A_{i_k}|$$

$$|A_i| = |\{f : [n] \to [k] \mid i \notin \text{range of } f\}|$$
$$= (k-1) \times (k-1) \times \cdots (k-1)$$
$$= (k-1)^n$$

$$\underset{i<j}{|A_i \cap A_j|} = |\{f \mid [n] \to [k] \mid i, j \notin \text{range of } f\}| = (k-2)^n$$

$i_1 < i_2 < \cdots < i_r$ $\quad |A_{i_1} \cap A_{i_2} \cdots \cap A_{i_r}| = (k-r)^n$

$$\text{now, } \sum_{i_1 < \cdots < i_r} |A_{i_1} \cap \cdots \cap A_{i_r}| = \binom{k}{r}(k-r)^n$$

choosing r from k

$$\text{then } \left|\bigcup_{i=1}^{k} A_i\right| = \binom{k}{1}(k-1)^n - \binom{k}{2}(k-2)^n \cdots + (-1)^{k-1}\binom{k}{k}(k-k)^n$$

$$k^n - \left|\bigcup_{i=1}^{k} A_i\right| = k^n - \binom{k}{1}(k-1)^n + \binom{k}{2}(k-2)^n + \cdots + (-1)^k \binom{k}{k}(k-k)^n$$

$$= \sum_{i=0}^{k} (-1)^i \binom{k}{i}(k-i)^n$$

$$\Rightarrow k!\,S(n,k) = \sum_{i=0}^{k} (-1)^i \binom{k}{i}(k-i)^n$$

$$\Rightarrow S(n,k) = \frac{1}{k!}\sum_{i=0}^{k} (-1)^i \binom{k}{i}(k-i)^n$$

21st Aug:

$$S(n,k) = \sum_{i=0}^{k} (-1)^i \frac{(k-i)^n}{i!\,(k-i)!}$$

No of set partitions of [n] of [k] parts

$k!\,S(n,k)$ = no of surjective maps $f : [n] \longrightarrow [k]$

$A_i = \{ f : [n] \longrightarrow [k] \mid i \notin \text{Range of } f \}$ $\quad i = 1, 2 \ldots k$

onto maps from $[n] \longrightarrow [k]$ = All maps $\setminus \cup A_i$

$\Rightarrow$ get $\bigcap_{i=1}^{k} A_i^c$

no of onto maps from $[n] \longrightarrow [k]$ $= |\bigcap_{i=1}^{k} A_i^c|$

= total no. of functions from $[n] \longrightarrow [k]$

$= k^n - |\bigcup_{i=1}^{k} A_i|$

total functions from multiplication rule

$$|\bigcup_{i=1}^{k} A_i| = \sum_{j=1}^{k} (-1)^{j-1} \sum_{i_1 < i_2 < \cdots < i_j} |\bigcap_{l=1}^{j} A_{i_l}|$$

choose 1 from [k]

$$\sum |A_i| = \binom{k}{1} (k-1)^n$$

k-1 values $\times$ $\underbrace{k-1 \times \cdots k-1}_{n \text{ times}}$

$$\sum_{i \neq j} |A_i \cap A_j| = \binom{k}{2} (k-2)^n \qquad |A_{i_1} \cap \cdots \cap A_{i_j}| = (k-j)^n$$

value of $|A_i \cap A_j|$

ways of choosing

$|\bigcup_{i=1}^{k} A_i| = \sum (-1)^{i-1} \binom{k}{i} (k-i)^n$ from above calculations

Derangement / matching Problem:

Suppose that there are n chairs from 1, 2 .... n and n-persons $P_1, P_2 \ldots P_n$ will be seated in these chairs. How many ways to assign seats to them so that P_i does not get i-th seat $\forall i = 1, 2 \ldots n$

$A_i = \{ P_i \text{ get } i^{th} \text{ seat} \}$

then

$|A_i| = (n-1)!$

$|A_i \cap A_j| = (n-2)!$, $i \neq j$

$\vdots$

$|A_{i_1} \cap \cdots \cap A_{i_k}| = (n-k)!$

now $|\bigcup_{i=1}^{k} A_i|$ = no of sets s.t atleast one P_i gets i^{th} seat

$\Rightarrow |(\bigcup_{i=1}^{k} A_i)^c|$ = no P_i get i set

$\Rightarrow$ total $- |\bigcup_{i=1}^{k} A_i| = (n)! - |\bigcup_{i=1}^{k} A_i|$

$$\left|\bigcup_{i=1}^{k} A_i\right| = \sum_{i=1}^{n}(-1)^{i-1}\sum_{l=1}^{i}\left|\bigcap A_{i_l}\right|$$

$$= \sum_{i=1}^{n}(-1)^{i-1}\binom{n}{i}(n-i)!$$

$$\Rightarrow \text{Total} = n! + \sum_{i=1}^{n}(-1)^{i}\binom{n}{i}(n-i)!$$

$$\text{Total} = \sum_{i=0}^{n}(-1)^{i}\binom{n}{i}(n-i)!$$

Exe: How many permutations of numbers $\{1,1,2,2,3,3,4,5,6\}$ are there so that identical digits are not in consecutive position?

Ans:

Total $= \dfrac{9!}{(2!)^3}$ $\quad A_1 = \{$permutations s.t 1s are consecutive$\}$

$|A_1| = \dfrac{8!}{(2!)^2} = |A_2| = |A_3|$

$i \neq j \quad i,j \in \{1,2,3\}$

$|A_i \cap A_j| = \dfrac{7!}{(2)!}$

$|A_i \cap A_j \cap A_k| = 6!$

$\text{total} - \left|\bigcup_{i=1}^{3} A_i\right| = \text{given case}$

$$\Rightarrow \frac{9!}{(2!)^3} - \binom{3}{1}\frac{8!}{(2!)^2} + \binom{3}{2}\frac{7!}{(2!)^1} - \binom{3}{3}6! = \text{final permutations}$$

Exe: Find the number of integers b/w 0 and 9999 inclusive so that both 2 and 5 appear in integer

Ans:

Total cases $= |\{0,1,2\dots 9999\}| = 9999+1 = 10000$

$xyzw$ where $x,y,z,w \in \{0,\dots 9\}$

$S = \{xyzw;\ x,y,z,w \in \{0,\dots 9\}\}$

$\Rightarrow |S| = 10000$

$A = \{xyzw \in S \mid x,y,z,w \neq 2\}$

$|A| = 9^4$

$B = \{xyzw \in S \mid x,y,z,w \neq 5\}$

$|B| = 9^4$

$A \cup B = \{xyzw \mid x,y,z,w \neq 5 \text{ or } x,y,z,w \neq 2\}$

$(A\cup B)^c = \{xyzw \mid \exists\ 2 \text{ digits s.t } d_1 = 2,\ d_2 = 5\}$

$$\Rightarrow |(A\cup B)^c| = |S| - \binom{2}{1}|A| + \binom{2}{2}|A\cap B|$$

$$= 10^4 - 2\times 9^4 + 8^4$$

Restricted compositions of integers:

find the number of all possible compositions of n into k parts $(x_1 \dots x_k)$ s.t

$a_i \leq x_i \leq b_i$ for $i = 1, \ldots k$

Eg: No of compositions of 18 into 4 parts

$1 \leq x_1 \leq 5$
$-2 \leq x_2 \leq 4$
$0 \leq x_3 \leq 5$
$3 \leq x_4 \leq 9$

$x_1 + x_2 + x_3 + x_4 = 18$

$y_1 = x_1 - 1$
$y_2 = x_2 + 2$
$y_3 = x_3$
$y_4 = x_4 - 3$ then $\sum y_i - 1 + 2 - 3 = 18$
$\Rightarrow \sum y_i = 16$

where, $0 \leq y_1 \leq 4$
$0 \leq y_2 \leq 6$
$0 \leq y_3 \leq 5$
$0 \leq y_4 \leq 6$

S = set of all weak partitions of 16 into 4 parts

$= \binom{16 + 4 - 1}{4 - 1}$

$= \binom{19}{3}$

$A_1 = \{(y_1, \ldots y_4) \mid \sum y_i = 16, \ y_1 \geq 5\}$

$A_2 = \{(y_1 \ldots y_4) \mid \sum y_i = 16, \ y_2 \geq 7\}$

$A_3 = \{(y_1 \ldots y_4) \mid \sum y_i = 16, \ y_3 \geq 6\}$

$A_4 = \{(y_1 \ldots y_4) \mid \sum y_i = 16, \ y_4 \geq 7\}$

$|A_1^c \cap A_2^c \cdots \cap A_4^c| = |S \setminus \bigcup_{i=1}^{4} A_i|$
$= |S| - |\bigcup_{i=1}^{4} A_i|$
$= |S| - \sum |A_i| + \sum |A_i \cap A_j| - \sum |A_i \cap A_j \cap A_k| + \sum |A_1 \cap A_2 \cap A_3 \cap A_4|$

Multisets of size n from [k]:

No of multisets of size n from [k]

$x_1 + x_2 + \cdots x_k = n$
$\downarrow$
x_i = no of i in multiset

$x_i \geq 0$, so answer is same as no. of weak composition

$= \binom{n + k - 1}{k - 1}$

let $S = \{x_1, x_2 \ldots, x_n \mid 1 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq k\}$, $x_i \in [k]$

$1 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq k$
$\Rightarrow 1 \leq x_1 < x_2 + 1 < x_3 + 2 \cdots < x_{n-1} + n - 2 < x_n + n - 1 \leq k + n - 1$

$\Rightarrow 1 \leq y_1 < y_2 \cdots < y_n \leq k+n-1$

A = multiset of size n from set [k]

B = set of all subsets of n of [n+k-1] of size n

and $\exists$ Bijection $f: A \longrightarrow B$

$$f(x_1, x_2 \ldots x_n) = (x_1, x_2+1, \ldots x_n+k-1)$$

$$|A| = |B|$$
$$|A| = \binom{n+k-1}{n} = \binom{n+k-1}{k-1}$$

Theorm: (Binomial) for $n \in \mathbb{Z}_{\geq 0}$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

proof: $(x+y)^n = \underset{1}{(x+y)}\underset{2}{(x+y)} \cdots \underset{n}{(x+y)}$

term looks like: $x^k y^{n-k}$, now how many ways to choose

$\binom{n}{1}$ choose for $k=1$

$\binom{n}{0}$ choose for $k=0$

and so $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} (x)^k (y)^{n-k}$

Theorm: (Multinomial law) $(x_1+x_2+\cdots x_k)^n = \sum_{\substack{i_1+\cdots+i_k=n \\ 0 \leq i_j \leq n}} \frac{n!}{i_1! i_2! \ldots i_k!} x_1^{i_1} \ldots x_k^{i_k}$

proof: $\binom{n}{i_1 \cdots i_k} = \frac{n!}{i_1! \ldots i_k!}$

now, $(x_1 + \cdots x_k)^n = (x_1+\cdots+x_k)(x_1+\cdots x_k)\cdots(x_1+\cdots+x_k)$

now, $x_1^{i_1} x_2^{i_2} \ldots x_k^{i_k}$ s.t $i_1+\cdots+i_k = n$

choose i_1 from n in $\binom{n}{i_1}$

i_2 from $n-i_1$ $\binom{n}{i_1}\binom{n-i_1}{i_2}$

$\vdots$

i_k: $\binom{n}{i_1}\binom{n-i_1}{i_2}\cdots\binom{i_k}{i_k}$

$$\text{total} = \frac{n!}{\cancel{(n-i_1)!}(i_1)!} \times \frac{\cancel{(n-i_1)!}}{(i_2)!\cancel{(n-i_1-i_2)!}} \cdots \frac{\cancel{(n-i_1-i_2 \cdots -i_{k-1})!}}{(i_k!)(0)!} = \frac{n!}{i_1! i_2! \ldots i_k!}$$

Exe: Find no. of different terms that occur in a multinomial expansion of $(x_1 \ldots x_k)^n$

Ans:

$i_1+i_2+\cdots+i_k = n$ s.t $i_j \geq 0$ $\forall j$, so

$\binom{n+k-1}{k-1}$ = no of such weak partition of n of k parts

28th Aug:

Catalan numbers:

It is a sequence of numbers $\{C_0, C_1, \dots\}$ s.t $C_n = \frac{1}{n+1}\binom{2n}{n}$ $\left(C_0 = \frac{1}{0+1}\binom{0}{0} = 1\right)$

Defn: (n^{th} Catalan number) $C_n = \frac{1}{n+1}\binom{2n}{n}$ where $n \in \mathbb{Z}_{\geq 0}$

The number has been seen in many problems, they appear naturally

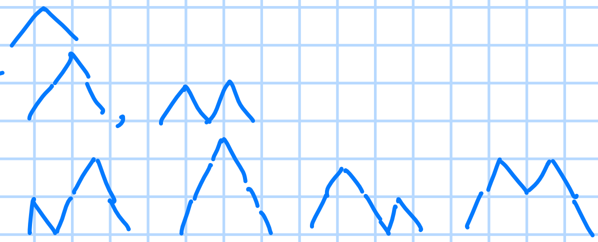
Problem 1: We have n-pairs of parentheses and we want a 'valid' grouping of it, where 'valid' group means that each open para matches with closer.
We want to find out total many grouping are there for each value of n.

Eg: valid: () $n=1$
not valid:)(
$n=2$: ()(), (()))(() not valid

By convention for $n=0$ we consider 1 $(C_0 = 1)$

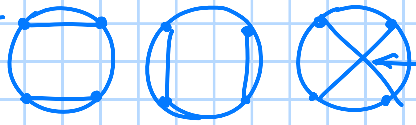
Problem 2: How many "mountain ranges" can we find with n-upstrokes and n-downstrokes s.t all stay above the ground. → cannot go underground

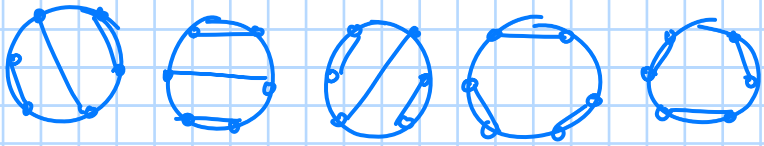
Eg: $n=1$
$n=2$
$n=3$



Problem 3: If $2n$ person seated around a circular table, then we want to find in how many ways all of them be simultaneously, shaking hands with other person at the table s.t none of the arms cross each other
(Non-crossing partitions)

Eg: $n=2$ ← not allowed
$n=3$

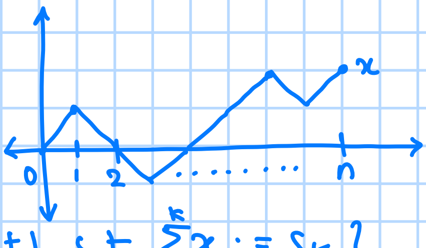




Reflection principle:

Defn: (Path) Let $n \in \mathbb{N}$, $x \in \mathbb{Z}$, a path from $(0,0)$ to (n,x) is a polygonal line whose vertices have abscissas $0,1,2..n$ and ordinates $S_0, S_1 \dots S_n$ s.t $S_0 = 0$, $S_n = x$ and $S_i - S_{i-1} = \pm 1$

Eg: $\{(0,0), (1,S_1), (2,S_2), \dots (n, S_n)\}$, $S_n = x$



$\{(0,0),(1,S_1),\dots(n,S_n)\} \leftrightarrow \{(x_1, \dots x_n) \mid x_i = \pm 1, \text{ s.t } \sum_{i=1}^{k} x_i = S_k, \ S_n = x\}$

$p = \#$ of $+1$
$q = \#$ of -1 → $p+q = n$, $p - q = x$ so, $\binom{n}{p}$ = choose p from n

when $p+q=n$
$p-q=x$
$\Rightarrow p=\frac{n+x}{2}$ so, $\binom{n}{\frac{n+x}{2}}$ is total paths

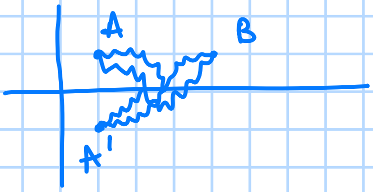
Note: $\frac{n+x}{2}$ has to be an integer as $x \leq n$, $x \geq -n$ and if $(0,0) \longrightarrow (5,0)$ then $x=0$, $n=5$

were either odd even $\left.\begin{matrix} \\ +1 & -1 \end{matrix}\right\}$ Not possible to reach 0
(n,x) both odd / both even needed: even odd $\left.\begin{matrix} \\ +1 & -1 \end{matrix}\right\}$

So, number of paths from $(0,0)$ to (n,x) is $= \begin{cases} \binom{n}{\frac{n+x}{2}} ; & \frac{n+x}{2} \in \mathbb{Z} \\ 0 ; & \text{otherwise} \end{cases}$

Feller vol.1 is a nice book for ref

Reflection principle:
Let $A=(a,\alpha)$, $B=(b,\beta)$ be integral points s.t $b>a\geq 0$ and $\alpha>0$, $\beta>0$, then the number of paths from A to B that touches or crosses x-axis is same as the number of paths from A' to B where where $A'=(a,-\alpha)$ reflection of A wrt x-axis.



Let $S = \{$paths from A to B, which touches or crosses x-axis$\}$
$T = \{$Paths from A' to $B\}$

path in S is of form $\{(a,\underset{s_\alpha}{\alpha}), (a+1, s_{\alpha+1}), \dots (b, \underset{\beta}{s_b})\} = P_1$, now let

$i = \min\{k \mid s_k = 0\}$ for a path in S
then lets make a new path

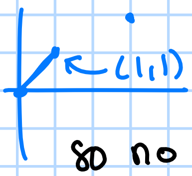
$P_1' = \{(a,-s_a), (a+1,-s_{a+1}), \dots (i,-s_i), (i+1, s_{i+1}), \dots (b, s_k)\}$
then $f: S \longrightarrow T$ s.t $f(P_1)=P_1'$ is a bijection

Note: $b>a$ above

Exe: Show that $f: S \longrightarrow T$ s.t $f(P_1)=P_1'$ is a bijection

Ans: It is trivial as f changes sign of first i & rest is same $f(f(P_1))=P_1 \Rightarrow f=f^{-1}$ or invertible

Ballot problem: let $n,x \in \mathbb{N}$, we want to find the number of paths from $(0,0)$ to (n,x) s.t $s_i>0$ for $i=1,2\dots n$, path: $\{(0,0),(1,s_1),\dots (n,s_n=x)\}$



we can observe that $s_1=1$ or else $s_1=-1$ and it is not >0

so no of path from $(1,1)$ to (n,x) that does not cross x-axis
$=$ total paths from $(1,1)$ to (n,x) $-$ $(1,1)$ to (n,x) paths which cross x axis

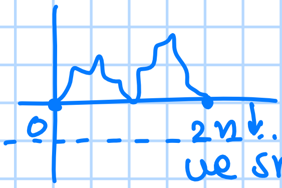
$=$ total from $(1,1)$ to (n,x) $-$ $(1,-1)$ to (n,x) ($\because$ Reflection principle)

$$= \binom{n-1}{\frac{(n-1)+(x-1)}{2}} - \binom{n-1}{\frac{(n-1)+(x+1)}{2}}$$

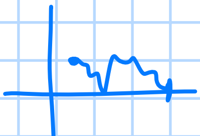
$$= \binom{n-1}{\frac{n+x}{2}-1} - \binom{n-1}{\frac{n+x}{2}}$$

Exe: Find no. of paths from $(0,0)$ to $(2n,0)$ s.t $s_i \geq 0$, $\forall i$

Ans: $T=\{(0,0),(1,s_1)\dots(2n,0)\}$



$S_1 = 1$



we shift x-axis then $(0,0) \to (2n,0)$
becomes
$(0,1) \to (2n,1)$
s.t. $S_i > 0 \ \forall i$

then total $(0,1) \to (2n,1)$ − cases where it touches/crosses x axis

$$= \binom{2n}{\frac{2n+(1-1)}{2}} - \binom{2n}{\frac{2n+(1+1)}{2}}$$

$$= \binom{2n}{n} - \binom{2n}{n+1}$$

$$= \binom{2n}{n} - \frac{2n!}{(n-1)!} \times \frac{1}{(n+1)!}$$

$$= \binom{2n}{n} - \frac{2n!}{(n)!\,(n)!} \times \frac{n}{n+1}$$

$$= \frac{1}{n+1}\binom{2n}{n}$$

so solution is $C_n = \frac{1}{n+1}\binom{2n}{n}$

for our brackets problem we have $(A)B$ then

$$b_n = \sum_{i=0}^{n-1} b_i \, b_{n-i-1}$$

where $b_0 = 1$, $b_1 = 1$ $()$, $b_2 = |\{()(),\ (())\}| = 2$

$$b_2 = b_0 b_1 + b_1 b_0 = 1\times 1 + 1 \times 1 = 2$$

$$b_3 = b_0 b_2 + b_1 b_1 + b_2 b_0 = 2 + 1 + 2 = 5 \qquad C_3 = \frac{1}{\cancel{4}} \times \frac{6 \times 5 \times \cancel{4}}{3.2.1} = 5$$

$$f(z) = \sum_{n=0}^{\infty} b_n z^n \qquad \text{now } f(z)^2 = b_0 b_0 + (b_0 b_1 + b_1 b_0) z + \cdots$$

as $f(z) = \sum_{n=0}^{\infty} b_n z^n$

$$\Rightarrow (f(z))^2 = \left(\sum_{n=0}^{\infty} b_n z^n\right)\left(\sum_{n=0}^{\infty} b_n z^n\right)$$

$$= (b_0 + b_1 z + b_2 z^2 + \cdots)(b_0 + b_1 z + b_2 z^2 + \cdots)$$

$$= b_0 b_0 + z^1 [b_1 b_0 + b_0 b_1] + z^2 [b_0 b_2 + b_1 b_1 + b_2 b_0] + \cdots$$

$$= b_1 + z^1 [b_2] + z^2 [b_3] + \cdots$$

$$= \frac{1}{z} [z b_1 + z^2 b_2 + \cdots]$$

$$(f(z))^2 = \frac{1}{z} [f(z) - b_0]$$

$$\Rightarrow (f(z))^2 = \frac{1}{z} (f(z) - 1)$$

solving for $f(z)$ we get explicit formula for $b_i \ \forall i \in \mathbb{Z}_{\geq 0}$

1st Sept :

parantheses problem :

we have n-pairs of parentheses , we call it valid grouping if following holds:

(i) Every open para has a matching closed para
(ii) the para in between a matched pair of para form a valid group

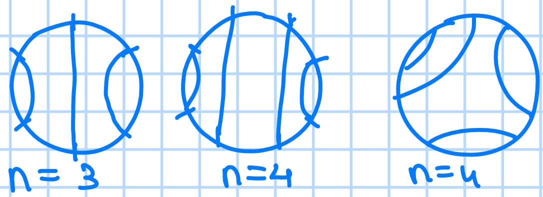
Eg: $n=1$ ()
$n=2$: ()(), (())
$n=3$: ()()(), (())(), ()(()), ((())), (()())

The grouping like)(is not valid grouping

Hand shaking problem :

2n people are sitting on a round table and shaking hand simultaniously in such a way that no hand cross each other

Eg:



} valid ones for $n=3,4,4$ respectively

Now for our para question if for (we put +1 &) we put -1 then the answer will be same for no of paths from (0,0) to (2n,0) s.t every y coord of path ≥ 0 i.e does not cross x axis (but can touch)

$S = \{$valid grouping n-pairs of paras$\}$

$T = \{$path from (0,0) to (2n,0) which does not cross x-axis$\}$
Bijection
$\approx \{(x_1, x_2, \ldots, x_{2n}) \mid x_i = \pm 1, \sum_{i=1}^{j} x_i \geq 0, \sum_{i=1}^{2n} x_i = 0\}$

Recursive relation :
satisfyde by Catalyn number

$C_0 = 1$
$C_1 = 1$ ()
⋮ $\quad$ inside para form a valid grouping , suppose i pairs of para
: $n-1$ (A) B ← n-1-i pairs of paras

$C_n = \sum_{i=0} C_i C_{n-i-1} \Rightarrow C_n = C_0 C_{n-1} + C_1 C_{n-2} + \cdots C_{n-1} C_0$

now, $C_1 = C_0 C_0 = 1$
$C_2 = C_0 C_1 + C_1 C_0 = 1+1 = 2$
$C_3 = C_0 C_2 + C_1 C_1 + C_2 C_0 = 2\times1 + 1\times1 + 1\times2 = 5$
()(()) (())() (()())
()()() ((()))

For our hadshaking problem if we fix two people then on either side we need even no of people or else not valid and so
if i pairs on left & n-1-i pairs on right
handshakes for n pairs
then $h_n = \sum_{i=0} h_i h_{n-1-i}$
$\Rightarrow h_n = C_n$ (Catlyn number)

now if we define $f(x) = \sum_{i=0}^{\infty} c_i x^i = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots\ldots$

$$(f(x))^2 = c_0 c_0 + (c_0 c_1 + c_1 c_0) x + (c_0 c_2 + c_1 c_1 + c_2 c_0) x^2 + \ldots$$
$$= c_1 + c_2 x + c_3 x^2 + \ldots$$
$$x(f(x))^2 = c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$
$$= f(x) - c_0$$
$$\Rightarrow x(f(x))^2 = f(x) - 1$$
$$\Rightarrow y^2 x - y + 1 = 0$$
$$y = \frac{1 \pm \sqrt{1-4x}}{2x}$$
$$\Rightarrow f(x) = \frac{1 \pm \sqrt{1-4x}}{2x} \quad \text{as } x \to 0$$

$f(x) \to \infty$ if we take $+$

valid solution of $f(x)$:
$$f(x) = \frac{1 - \sqrt{1-4x}}{2x}$$

Binomial theorem:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \quad n \in \mathbb{Z}_{\geq 0}$$

if n is any real number and $|x|<1$

$$(1+x)^n = 1 + \frac{n}{1} \cdot x + \frac{(n)(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \ldots\ldots$$ proof is from taylor series expansion

Exe: show that $(1+x)^n = 1 + \frac{n}{1} x + \ldots$ given $n \in \mathbb{R}$, $|x|<1$ → done down

now, $$(1-4x)^{1/2} = 1 + \frac{\frac{1}{2}}{1}(-4x) + \frac{\frac{1}{2}(\frac{1}{2}-1)(-4x)^2}{2!} + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(-4x)^3}{3!} + \ldots\ldots$$

$$\frac{1-(1-4x)^{1/2}}{2x} = \frac{1}{2x}\left[2x + \left(\frac{1}{4}\right) \times \frac{1}{2} \times 4 \times 4 \times x^2 + 4 \cdot 4 \cdot 4 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{1}{3 \cdot 2 \cdot 1} x^3 + \ldots\right]$$

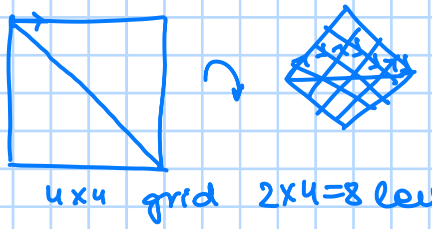
$$f(x) = 1 + x + 2x^2 + \ldots\ldots$$

x^n coeff for $f(x)$
$$= \frac{\left|\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-(n)\right)(4)^{n+1}\right|}{2}$$
$$= \frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\cdots\left(\frac{2n-1}{2}\right)(4)^{n+1}$$
$$= \prod_{i=1}^{n-1}(2i+1) \times \frac{1}{(2)^{n-1}} \times \frac{1}{2^3} \times (2)^{2n+2}$$
$$= \prod_{i=1}^{n-1}(2i+1) \times (2)^n$$
$$= \frac{(2n)(2n-2)\cdots(2n-n)}{n+1}$$
$$= \frac{1}{n+1}\binom{2n}{n}$$

Diagonal avoiding path:

we have a grid of $n \times n$ squares, we want to find no of path of $2n$ from upper left corner to lower right corner which does not cross the diagonal line

Eg:



so $2\times C_4$ ← no of mountain range

2×4=8 length path s.t we cannot cross diagonal line

from above example it is trivial to see that no of path = 2× catalyn number

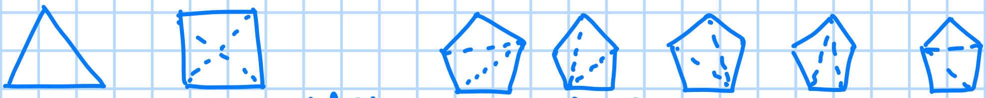
$= 2C_n$

for $n\times n = 2C_n$

polygon triangulation:

we want the number of ways to triangulate a regular polygon with $(n+2)$ sides

Eg:



$n=1 \Rightarrow 1$

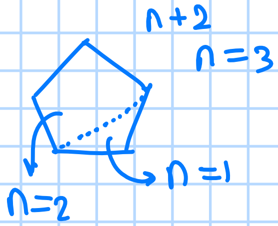
$n=2 \Rightarrow 4$ sides

2

$n=3 \Rightarrow 5$ sides

5

this comes from the way that if we cut one then left & right can be cut in $f_i \times f_{n-1-i}$ ways, so we get C_n as our answer

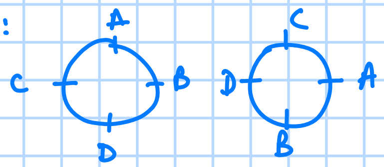


$\rightarrow f_3 = f_2 f_1 + f_1 f_2$

linear permutation:

Arranging n-objects on a line, circular permutation of n-objects is arranging them in a circle. we say two circular permutation are same if one can be arrived by rotation of the other permutation.

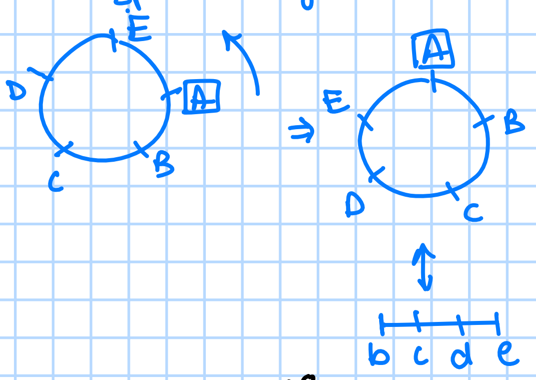
Eg:



these are two circular permutation (same)

Now we want to find no. of circular permutations of distinct objects, i.e we want to mark objects as special case, then rotate the circle to bring the special object at top position.

Eg: A, B, C, D, E
← special object



this permutation has a bijection with linear permutation of B, C, D, E

Note: There is a bijection b/w the set of circular permutation of n distinct objects and the set of linear permutations of $(n-1)$ distinct objects.

Exe: 10 people, $P_1 \ldots P_{10}$ are seated on a round table where P_1 & P_2 are not sitting to each other, how many arrangements possible?

Ans: $9! =$ total cases

If we make (P_1P_2) then $8!$ (letting P_{10} = special)

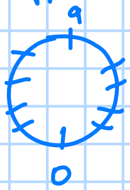
but (P_2P_1) also valid so

$9! - 8! - 8! =$ total

Exe: Find the number of circular permutation of $\{0,1 \ldots 9\}$ in which 0 & 9 are not opposite

Ans: total $= 9!$ (special = 0)

if 0 & 9 are opposite



$\} \rightarrow 8!$

so total $= 9! - 8!$

Defn: (Simple Graph) A graph $G = (V, E)$ is s.t V is a set of elements $\{v_1, v_2 \ldots\}$ called vertex set and v_i's are called vertices

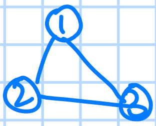
E is called the edge set (not a multiset) and pair of distinct vertices

$E = \{\{v_1, v_2\}, \{v_4, v_6\} \ldots\}$ so $\{v_i, v_i\} \notin E$

Eg:

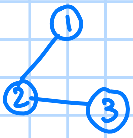
$V = \{1\}$

$E = \phi$ so $G = (\{1\}, \phi)$



$V = \{1,2,3\}$

$E = \{\{1,2\}, \{1,3\}, \{2,3\}\}$

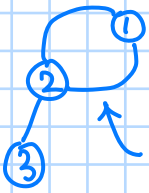


$V = \{1,2,3\}$

$E = \{\{1,2\}, \{2,3\}\}$

Note: Above graph also called simple graph

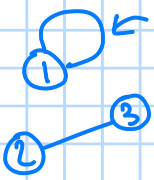
Eg:



$V = \{1,2,3\}$

$E = \{\{1,2\}, \{1,2\}, \{2,3\}\}$ ← multi set

double edge present so not a basic graph



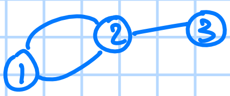
$V = \{1,2,3\}$

$E = \{\{1,1\}, \{2,3\}\}$

not a basic graph

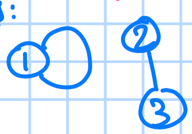
Defn: (multigraph) graph where multiple edges are present

Eg:

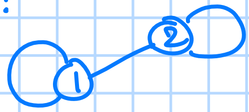


Defn: (general graph) A graph where loop/multiple edges allowed

Eg:



Eg:



Note: Different authors use different naming of graphs as sometimes basic graph is called graph, general graph as graph

for a basic graph with $|V|=n$, we have $\binom{n}{2}$ maximum edges and so $2^{\binom{n}{2}}$ is total diff basic graphs

Exe: show that $(1+x)^n = 1 + \frac{n}{1}x + \cdots$ given $n \in \mathbb{R}$, $|x|<1$

Ans:

now, $(1+x)^n = f(x)$

then

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + f''\frac{(x_0)(x-x_0)^2}{2!} + \cdots$$

$f(0) = 1^n = 1$

$f'(0) = n$

$f''(0) = (n)(n-1)$

$\vdots$

so, $f(x) = 1 + nx + n(n-1)\frac{x^2}{2} + \cdots$

4th sept:

Graphs:

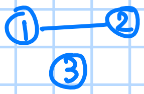
$G=(V,E)$, V = vertex set $\quad V=\{v_1,v_2 \dots v_n\}$
E = edge set $\quad E=\{\{v_1,v_2\},\dots\}$

basic graph: E is a set (not a multiset) and no $\{x,x\}\in E$ (no loops)

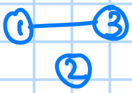
graph: E is a multiset and $\{x,x\}$ can be in E (loops & multiple edges exist)

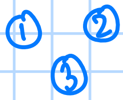
Defn: (loop) An edge with same start and end point, i.e $\{v,v\}$

Defn: (Adjacent vertex) Two vertices are called adjacent vertices if there is an edge between them.

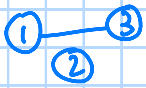
Eg:  here as $\{1,2\}$ is an edge, 1,2 are adjacent

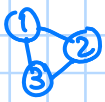
Defn: (independent set) It is a collection of pairwise non-adjacent vertices.

Eg:  $\{1,2\}$, $\{2,3\}$ are independent sets

 $\{1,2,3\}$ is an independent set as all pairwise non-adjacent

Defn: (Clique) A set of pairwise adjacent vertices

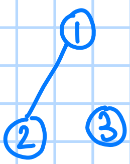
Eg: $\{1,3\}$ is clique of  $\{1,3\}$ is clique of size 1

$\{1,2,3\}$ is a clique of  $\{1,2,3\}$ is clique of size 2

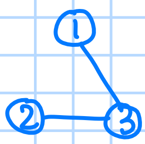
Note: By convention a point is considered a clique, and is called a clique of size 0

Defn: (complement graph) Complement of a basic graph $G=(V,E)$ is a basic graph $\bar{G}=(V,\bar{E})$ s.t $\{x,y\}\in\bar{E} \Leftrightarrow \{x,y\}\notin E$

Eg:

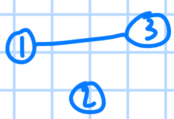


G



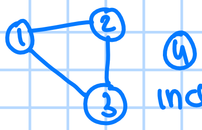
$\bar{G}$

Defn: (Bipartite graph) G is called bipartite if V, vertex set, can be written as union of two disjoint independent sets

Eg:  independent sets: $\{1,2\}$, $\{2,3\}$

$V=\{1,2\}\cup\{3\}$

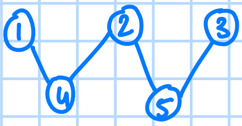
↑ trivially independent



independent sets $=\{1,4\}$, $\{1,2\}$, $\{1,3\}$

$V\neq\{1,4\}\cup\underbrace{-\cdot}$

so G is not bipartite

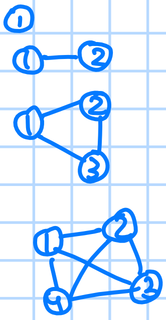


G

then $V = \underbrace{\{1,2,3\}}_{\text{independent set}} \cup \underbrace{\{4,5\}}_{\text{independent set}}$

Defn: (complete graph) It is a basic graph where each pair of vertices forms an edge

Eg: K_1 : K_2 : K_3 : K_4 :



Now, the number of edges in a complete graph with n-vertices in $\binom{n}{2}$ as if $V=[n]$ then $\binom{n}{2}$ is max possible edges

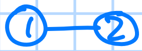
So, total no of basic graphs for $V=[n]$ is same as total no of subsets of edge set of comp graph of V

as for $G=(V,E)$
$E \subseteq$ Edges of complete graph with V

So, total $= 2^{\binom{n}{2}}$

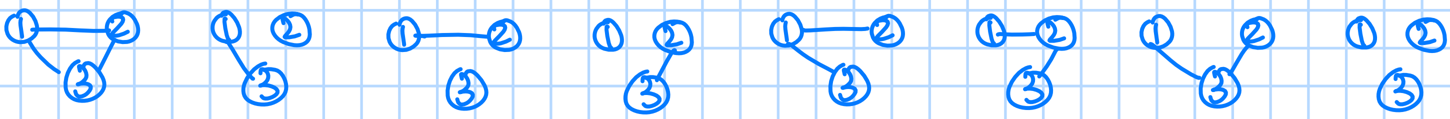
Eg: for $V=[2]$ $\binom{2}{2}=1$ & so $2^{\binom{2}{2}}=2$

G_1



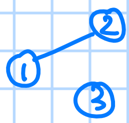
G_2

$V=[3]$, $\binom{3}{2} = \frac{3!}{2!\,1!} = 3$ so $2^3=8$

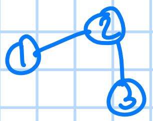


Defn: (Complete Bipartite graph) It is a basic bipartite graph s.t two vertices are adjacent iff they are in different independent sets.

Eg:

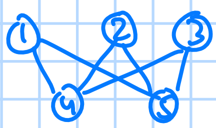


This is Bipartite as $V=\{1,3\}\cup\{2\}$, not complete as there should be an edge between 2 & 3



$\{1,3\} \cup \{2\} = V$

edges exist so complete bipartite graph



complete bi-partite

Note: These independent sets are called partite sets also (interchangable)

Defn: (Order of a graph) Number of vertices in the graph

Defn: (Degree of a vertex) No of edges connected to it.

Note: loop will contribute to 2 in degree of counting

Now, degree of a vertex in K_n(complete graph with n-vertices) is s.t degree $\forall v \in [n]$ is $(n-1)$

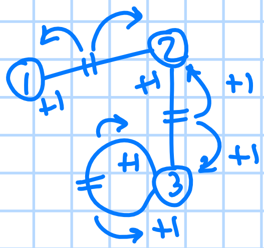
Note: we can observe that if G is a graph with n vertices and suppose d_i, $i=1,2,\dots,n$ are the degree of the vertices. Then

$$\sum_{i=1}^{n} d_i = 2e$$

as each edge contributes to 2 degree

No. of edges

Eg:



so each edge contributes 2

Note: Again from observation, total no of vertices with odd degree is even

$$\sum_{i=1}^{n} d_i = \text{even}$$

$$\sum_{i \text{ st } d_i \text{ is odd}} d_i + \overbrace{\sum_{i \text{ st } d_i \text{ is even}} d_i}^{\text{even}} = \text{even}$$

$$\Rightarrow \sum_{d_i \text{ is odd}} d_i = \text{even}$$