4th sept:

**Degree of a vertex**: No of edges attached to it

An edge contributes 1 to the degree of each of its vertex. An edge contributes 2 in degree counting.

A loop at a vertex $v$ contributes 2 to the degree of the vertex
let $G=(V,E)$ be a graph with $n$ vertices. let $d_1 \dots d_n$ be degree of the vertice then:

$$\sum_{i=1}^{n} d_i = 2 \times e \quad \leftarrow \text{no. of edges}$$

no of vertices with odd degree is even.

$$\underbrace{\sum_{i=1}^{n} d_i}_{\text{even}} = \sum_{i:d_i \text{ is odd}} d_i + \sum_{i:d_i \text{ is even}} d_i$$

$$\Rightarrow \sum_{i:d_i \text{ is odd}} d_i = \text{even} \quad \text{i.e } |\{i : d_i \text{ is odd}\}| = \text{even}$$

as if not then $\sum_{i:d_i \text{ is odd}} d_i \neq \text{even}$

**Defn**: (multiplicity) It is for an edge in a graph $G=(V,E)$. It is the no of times it appears in E.

**Defn**: (Walk) It is a sequence of edges of the form $\{x_1,x_2\},\{x_2,x_3\},\dots\{x_n,x_{n+1}\}$ It is called a walk from $x_1$ to $x_{n+1}$, we denote it by

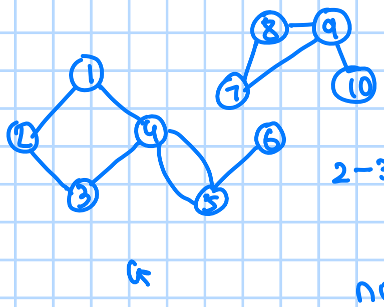
$$x_1 - x_2 - \dots - x_{n+1}$$

**Note**: (i) There could be repeated edges i.e $\{x_i, x_{i+1}\} = \{x_j, x_{j+1}\}$ for $i \neq j$
(ii) A vertex can repeat in a walk
(iii) closed walk: $x_1 = x_{n+1}$
open walk: $x_1 \neq x_{n+1}$
(iv) If $G=(V,E)$ is a graph, then an edge "e" in a walk is considered repeated if no of appearance of "e" in the walk crosses (>) multiplicity of "e"
(v) If G is a basic graph then point (iv) does not arise

**Defn**: (Isolated vertex) vertex of degree 0 is isolated.

**Defn**: (trail) It is a walk where no edge repeats.

Eg:




2-3-4-5-6-5-4-1 ← Repeated
Not repeated as multiplicity=2
not-a-trail

2-3-4-5-4-1 → trail
2-3-4-1 → trail

**Note**: vertex may repeat in a trail

**Defn**: (Path) It is a trail where no vertex repeat (except start & end if it is a closed path)

Eg: 2-3-4-1 is a path

**Defn**: (cycle) A closed path is called a cycle.

**Note:** Walk ⊇ Trails ⊇ Paths ⊇ cycle

**Defn:** (length) length of walk / trail / Path is no of edges (total) present

**Note:** proofs with graphs will be induction based or construction based

**lemma:** A walk from $u$ to $v$ contains a path from $u$ to $v$.

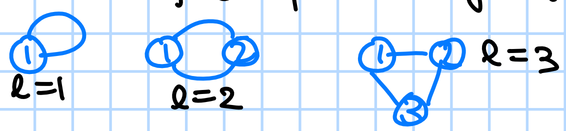
**proof:** (Induction on length of walk)
$u \neq v$, $l \rightarrow$ denote the length of the walk
suppose $l=1$, then there is an edge between $u$ & $v$
so path itself is a walk

assume result true for all $l < k$ then for $l=k$:
If no vertex is repeating then the walk is a path
otherwise if one vertex say $w$ is repeating more than one in a walk

Remove all vertices and edges between two $w$ (keeping one $w$)
by this operation we get a walk from $u$ to $v$, whose length is strictly less than $k$. By induction hypothesis, $\exists$ a path from $u$ to $v$.

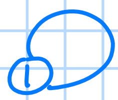
**Exe:** For above lemma, prove again using a constructive argument → done down

**Note:** In a basic graph min length of a cycle is 3, if loop allowed then 1, if only multiedge then 2.



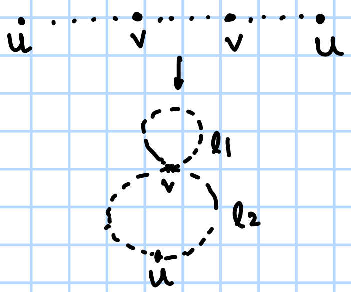
**lemma:** Every closed odd walk 'W' contains an odd cycle.
1-2-1-2-1 is a closed even walk but no cycle, so not true for even

**proof:** (Induction) here for $l=1 \rightarrow$ odd we have the below graph with a closed walk, we see that it's a cycle of length 1

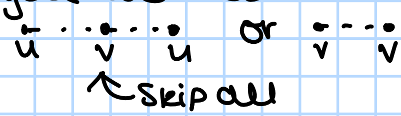


so, result is true for $l=1$
Suppose result is true for all odd $l < 2k+1$, then for $l = 2k+1$:
**case I:** W: $u - u_1 - \cdots - u_n - u$ where $\forall i \neq j$; $u_i \neq u_j$, then $u - u_1 - u_2 - \cdots - u_n - u$ is a cycle by defn of odd length

**case II:** Some vertex have repeated, let $v$ be one such vertex



then length of $\underset{v}{\cdot} \cdots \underset{v}{\cdot}$ be $l_1$ and $l_2 = 2k+1 - l_1$
now $\Rightarrow l_2 + l_1 = 2k+1$
so $\exists i$ s.t $l_i$ is odd
as $l_i < 2k+1$, $\exists$ an odd cycle and so
$\exists$ an odd cycle either from



part of walk.

Exe: For above lemma, prove again using a constructive argument

Ans: let $u - x_1 - x_2 \cdots x_n - v$

be the given walk, then

let $x_i$ be the first repeating vertex, then we remove everything in between

i.e $- x_i - \cdots - x_i - \Rightarrow - x_i -$ , same for all other repeating to get a path from $u$ to $v$

$11^{th}$ sept:

walk: $x_0 - x_1 \cdots - x_m$ (Edges and vertices may repeat)

trail: A walk when no edges repeats

Path: A trail when no vertex repeats

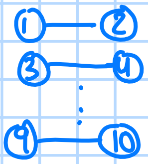
cycle: closed path

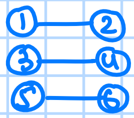
Defn: (subgraph) Let $G=(V,E)$ be a graph. $H=(U,F)$ is called a subgraph of $G$ if $U \subseteq V$, $F \subseteq E$

Eg:

$V=[10]$
$E=\{\{2n-1, 2n\} \mid n \in [5]\}$

$U=[6]$
$F=\{\{2n-1, 2n\} \mid n \in [3]\}$





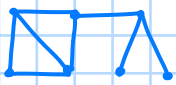
but $U=\{1,2,3\}$
$F=\{\{1,2\},\{3,5\}\}$ $(U,F)$ not a graph

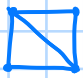
Defn: (Induced subgraph) A subgraph $H=(U,F)$ of $G=(V,E)$ is called induced subgraph if $F$ contains all the edges of $G$ whose end vertices are in $U$

Note: If $H$ is induced subgraph then $H=G_U$

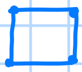
Defn: (spanning subgraph) A subgraph $H$ of $G$ is called spanning if $U=V$ and $F \subseteq E$

Eg:

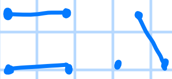




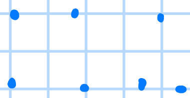
Induced subgraph



Subgraph but not induced



spanning subgraph



spanning subgraph

Defn: (connectedness) Two vertices $u,v$ in $G$ are said to be connected if $\exists$ a path from $u$ to $v$

Note: A vertex is connected to itself by a path of size 0

Note: (i) $u \sim u$ ($\sim$ is connected)
(ii) $u \sim v$ then $v \sim u$
(iii) $u \sim v$ and $v \sim w$ then $u \sim w$

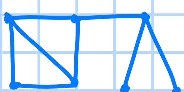
So from above we see, connection is equivalence relation on set of vertices of $G$
This equivalence relation partitions $V$ into nonempty block.

$u,v$ connected $\Leftrightarrow$ they are in same block

If $V = V_1 \cup V_2 \cdots \cup V_k$, then if we look inside subgraphs, $G_{V_1}, G_{V_2}, \ldots G_{V_k}$
$G_{V_i}$ for $i=1,2\ldots k$ are called connected components or components of $G$

Note: If $x,y \in G_{V_i}$, then $\exists$ a path from $x$ to $y$. If $x \in G_{V_i}$ and $y \in G_{V_j}$ when $i \neq j$ then there is no path from $x$ to $y$

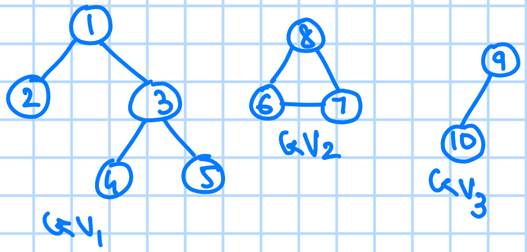
Eg:



only one connected component

Defn: (connected) A graph is called connected graph if $\exists$ a path between any two vertices
In other words, $G$ is called connected if it has only one connected component

Eg:




$V=[10]$
$V_1=[5]$
$V_2=\{6,7,8\}$
$V_3=\{9,10\}$

$G_{V_1}, G_{V_2}, G_{V_3}$ are connected components

**Note:** If we add an edge to a graph G, then the number of connected component of G decreases by atmost 1. As if new edge in same $G_{V_i}$ then same, otherwise if b/w $G_{V_i}, G_{V_j}$ $i \neq j$ then $G_{V_i}$ and $G_{V_j}$ becomes $G_{V}$.

**Note:** If we remove an edge from G, then number of connected components increases by atmost 1.

**Exe:** Every graph with n-vertices and k-edges has atleast $(n-k)$ connected components

**Ans:** Start with n vertices and 0 edges, now if we add an edge

No of componets $=n$ (0 edges case)

$\Rightarrow$ No of componets $\geq n-1$ (as $\underset{\substack{\uparrow \\ \text{loop}}}{n}$ or $\underset{\substack{\uparrow \\ \text{connecting two vertices}}}{n-1}$, when edge added)

for $k=1$, true, now if true for $k=r-1$ then

for $(n,k)=(n,r-1) \Rightarrow$ atleast $n-r+1$

for $k=r$: $(n,k)=(n,r)$

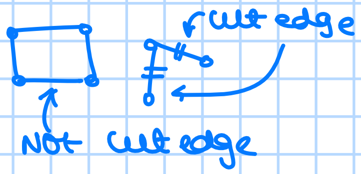
and no of vertices $= \underbrace{n-r+1}_{\text{edge in } G_{V_i}}$ or $\underbrace{n-r}_{\text{edge connecting two conneted componets}}$

$\Rightarrow$ atleast $n-r$ componets

by induction we get for $(n,k)$ atleast $n-k$ conneted componets

**Defn:** (cut edge) An edge is called cut edge if removal of that edge increases the number of components

Eg:




**Exe:** An edge is a cut edge iff it does not belong to any cycle

**Ans:**

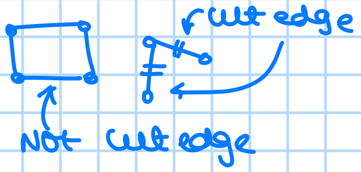
($\Rightarrow$) If an edge is cut edge then, if it did belong to a cycle then removing it will not inc total componets as it's in a cycle so there is one more path connecting $u,v$ if cut edge between u and v

($\Leftarrow$) If an edge not part of any cycle say $u \longrightarrow v$, then removing u and v makes $u,v$ of differet conneted sets, so total conneted sets increases and so $u \longrightarrow v$ is a cut edge

11th sept:

**Defn:** (cut edge) An edge is called cut edge if removal of that edge increases the number of components

eg:




**Thm:** An edge is a cut edge iff it does not belong to any cycle

Ans:

($\Rightarrow$) Suppose the cut edge 'e' belongs to a cycle, consider this component $G_1$, let $x, y \in G_1$ then $\exists$ path b/w $x$ and $y$

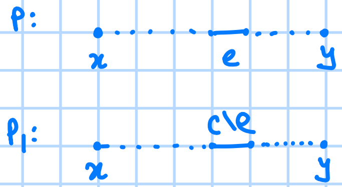
if P does not contain e, then P is a path in $G_1 \setminus e$

if P contains edge e, then P is not a path in $G_1 \setminus e$ connecting $x$ and $y$ but $\exists$ an alternate path

$$P_1 = (P \setminus e) + (C \setminus e)$$

so, $\forall x, y \in G_1 \setminus e$, $\exists$ path in $G_1 \setminus e$

so $G_1 \setminus e$ is connected




so e is not cutedge (contradiction)

($\Leftarrow$) if an edge is not cut edge and it does not belong to any cycle then if $G_1 \ni e$ and $G_1$ is a connected component then $\forall x, y \in G_1$ $\exists$ path P

now for $G_1 \setminus e$ wlog e is edge between $x$ & $y$

then as e is not part of any cycle, only connection b/w $x$ & $y$ is e, removing that we get 2 connected components

this is a contradiction

## Graph Isomorphism:

Two graph $G = (V, E)$ and $G' = (V', E')$ are isomorphic if $\exists$ a bijection

isomorphism map b/w G and G'

$\theta : V \longrightarrow V'$ s.t for any $x, y \in V$

no of edges b/w $x$ and $y$ is same as no of edges between $\theta(x)$ & $\theta(y)$

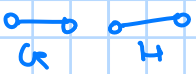
**Note:** isomorphism is an equivalence relation on the set of graphs

**Note:** Two graphs G and H belong to equivalence class iff $G \cong H$

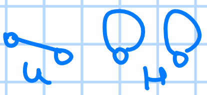
we see that if $G \cong H$ then G and H have same order and same no of edges

**Note:** converse is not true

eg:



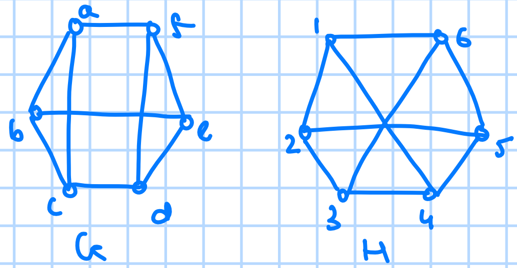

$G \cong H$




$G \ncong H$

now if $G \cong H$ then $\deg(x) = \deg(\theta(x))\ \forall x \in G$, converse is not true, previous example works here too.

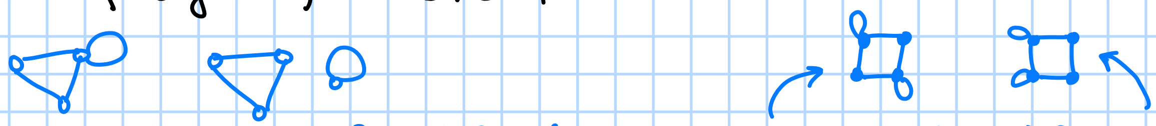
eg:




we see that G is not biparatate → odd length cycle
H is bipartate → No odd length cycle

so we see that $G \cong H$ if G has a cycle of length k, then H has a cycle of length k, converse not true

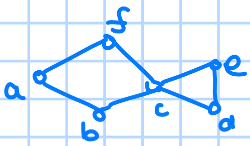
eg:



even if vertice no of cycles of k same, even no of vertices same

**Defn:** (Eulerian trail) A trail in G is called Eulerian if it contains every edge of G

eg:




a–b–c–d–e–c–f–a is Eulerian trail

**Theorm:** let G be a graph where degree of each vertex is even, then each edge belongs to a closed trail and hence belongs to a cycle

**proof:** let e be an edge between x and y of G. As every vertex has degree even x has one more edge connected to it other than self loop as self loop adds 2 in degree. Now, if this edge connects to y we are done, otherwise let $\exists v_1$ s.t x connects to $v_1$, then same thing happens on $v_1$ i.e it connects to y or $v_2$ let this happen for all vertices, then last unique vertex has odd degree and y has degree 1 so last vertex must connect to y, so in all cases $\exists$ a closed trail from x to y and as shown $\exists$ cycle as closed trail is made up of cycles.

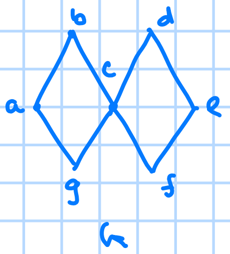
**Theorm:** G is connected, G has closed eulerian trail iff degree of each vertex is even

22nd sept:

so for connectedness, cut edge, edge is cut iff it is not part of cycle, graph isomorphism eulerian trail

Defn: (Eulerian trail) A trail which covers all the edge of the graph

Eg:




$a-b-c-d-e-f-g-a$ is a closed trail, moreover eulerian trail

$a-b-c-g-a$ is closed but not a eulerian trail

Note: There can also be open eulerian trails

Theorem: G be a finite connected graph, G has a closed eulerian trail iff degree of each vertex is even.

Proof: $(\Rightarrow)$ let $G=(V,E)$

$V=\{v_1, v_2, \ldots v_n\}$, suppose G has a closed eulerian trail C and suppose C starts from $v_1$ and end at $v_1$

suppose C visits vertex $v_i$, $k_i$ number of times $(i=1,2\ldots n)$

as C is a trail, it enters $v_i$ via $k_i$ different edges and leaves $v_i$ via $k_i$ different edges. There are atleast $2k_i$ many edges adjecent to $v_i$

since, C is eulerian i.e covers all the edges of G, there are exactly $2k_i$ many edges adjecent to $v_i$.

So, degree of $v_i = 2k_i$, $k_i > 0$ as G is connected so every edge has a path to other edge

Lemma: Let G be a finite graph and degree of each vertex is even, then each edge of G is part of a closed trail.

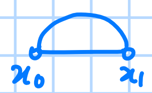
Proof: Let $G=(V,E)$ and let $e=\{x_0,x_1\}\in E$ then we have to show $e\in$ a closed trail, following the algorithm we shall construct a trail

we will construct a subgraph, say $(W,F)$ of G

Step 1: $W=\{x_0,x_1\}$ $F=\{\{x_0,x_1\}\}$

Step 2: find a vertex $x_2$ in G s.t $\{x_1,x_2\}\in E\setminus F$
such $\{x_1,x_2\}$ edge exist as degree of $x_1$ is even

If $x_2=x_0$ then update $W=\{x_0,x_1\}$ $F=\{\{x_0,x_1\},\{x_1,x_0\}\}$




$x_0-x_1-x_0$ is a closed trail

if $x_0\neq x_2$ then update $W=\{x_0,x_1,x_2\}$ $F=\{\{x_0,x_1\},\{x_1,x_2\}\}$

⋮

Step k: $W=\{x_0,x_1\ldots x_{k-1}\}$ $F=\{\{x_0,x_1\},\{x_1,x_2\},\ldots\{x_{k-2},x_{k-1}\}\}$
look for an vertex $x_k$ s.t
$\exists\{x_{k-1},x_k\}\in E\setminus F$
as degree of $x_{k-1}$ is even, such an edge exist

In this process, we visited $x_{k-1}$ m many times, we entered $x_{k-1}$, m many times and left m-1 many times, and at present at $x_{k-1}$, we have used $(2m-1)$ different edges adjecent to $x_{k-1}$. As degree is even, $\exists$ atleast one move edge adjacent to $x_{k+1}$.

If $x_k = x_0$ then we get our closed trail s.t $x_0 - x_1 \cdots - x_{k+1} - x_0$

if $x_k \neq x_0$ then update $W = \{x_0, \ldots x_k\}$
$F = \{\{x_0, x_1\}, \ldots \{x_{k-1}, x_k\}\}$

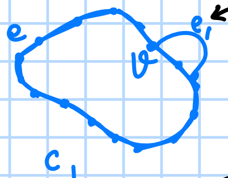
this process will terminate and repeat the algorithm.
This process will terminate, ie it will hit $x_0$ as G is a finite graph

**Theorem:** G be a finite connected graph, G has a closed eulerian trail iff degree of each vertex is even.

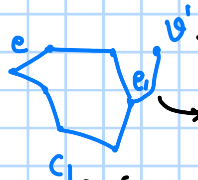
proof: ($\Leftarrow$) If we assume G is connected and degree of each vertex is even. Now if we start at edge 'e', applying the previous lemma then e is part of a closed trail $C_1$

if $C_1 = G$ then we are done as $C_1$ is a closed eulerian trail

now if $C_1 \neq G$ then case I: $C_1$ contains all the vertices of G but not all the edges




case II: $C_1$ does not contain all the vertices of G




$v' \notin C_1$, $v' \in V$, but as G is connected, $v'$ is connected to $C_1$. (any vertex on $C_1$ has a path from $v'$ to vertex)

$\exists$ path p from $v'$ to $C_1$, and p hits $C_1$ at a vertex $v$, say.
let $e_1$ be the edge on P, adjacent to $v$.

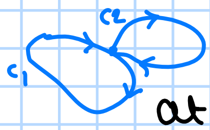
In any situation, we have a vertex V on $C_1$ s.t an edge $e_1$ adjacent to $v$ is not in $C_1$

Now remove all the edges of $C_1$ from G, due to this deletion process, if degree of vertex is zero, remove that vertex

we get a new graph G', we observe that

① degree of each vertex in G' is even (as removing one closed trail, removes even no of edges)

② consider components of G' containing $e_1$
call that component say $G'_1$, in $G'_1$ all vertices have degree even
so we apply lemma again on $e_1$, with $G'_1$ we will get a closed trail $C_2$. consider $C_1 \cup C_2$ = start from $u$, follow $C_1$, reach $v$, follow $C_2$, again $v$, then follow $C_1$.




If $C_1 \cup C_2 = G$, we are done, if not then repeat the argument, at one point we will cover G as G is finite graph

$C_1 \cup \cdots \cup C_k = G$ for some finite k iterations

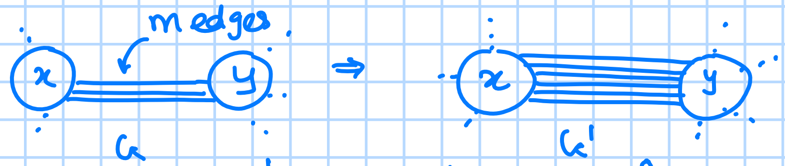
**Exe:** show if G is finite connected, G has eulerian trail from x to y $\Leftrightarrow$ G has all degree even except x & y which is odd

**Ans:** connect x and y, then we get above theorem when degree of all is even, G is finite connected $\Leftrightarrow$ G has a closed eulerian trail, then removing the edge b/w x and y we get above theorem

25th sept:

Exe: let G be a connected graph with k edges. Show that ∃ a closed walk of length 2k in which each edge is used twice of its multiplicity

Ans: If we construct a new graph G' where we add same number of edges b/w between two adjacent vertices present in G. So if there are m edges b/w two adjacent vertices x & y, we add m more b/w x and y



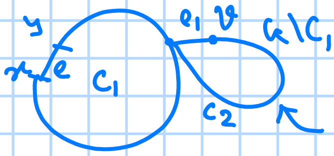

So, no of edges in G' = 2k and so by this construction degree of each vertex in G' is even, by last theorm ∃ a closed eulerian trail in G'. So, we have a closed walk that covers all edges twice

This closed eulerian trail has length 2k, and this is a closed walk in G, and each edge is used twice of its multiplicity

lemma: degree of each vertex is even, then each edge is a part of a closed trail

Theorm: G is conneted finite, G has an eulerian trail ⇔ degree of each vertex is even

proof:




we do $C_1 \cup C_2 = G$ or repeate if not this case, will end as finite G.

this argument works as G is finite connected

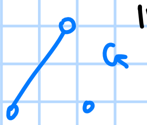
Hamiltonion Path:

Defn: A path that covers all the vertices of the graph

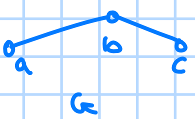
Defn: (Hamiltonion cycle) A cycle which covers all the vertices of the graph

Note: For existance of Hamiltonian path or cycle G has to be conneted

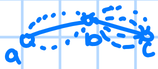
in this graph, no Hamiltonian path will exist




Note: For existance of Hamiltonian path, higher multiplicity (≥2) of edges does not play any role as since its a path, we cannot use same vertex

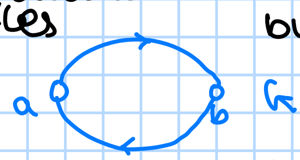



a-b-c is an hamiltonian path




Higher multiplicity does not help as path

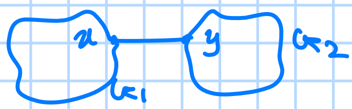
Same conclusion holds for hamiltonian cycle except for a graph with two vertices




but when G has more then 2 vertices, then multiplicity ≥2 of edge not needed

Exe: G is a connected graph with n vertices, n≥3 suppose G has a Hamiltonian cycle, then G cannot have a cut edge

Ans: let's suppose ∃ e s.t e is cut edge b/w vertex x and y



$G \setminus e$ will have two components $G_1$ & $G_2$. Now if we start from $x$, to construct a Hamiltonian cycle we have to go to $y$, i.e we have to pass $e$, so $e$ is part of the Hamiltonian cycle.

Now as we go from $x$ to $y$ and from $y$ to $x$, so we are going $x$ twice so, this is not a cycle, this is a contradiction and so, $e$ is not a cut edge

**Exe**: Show that removing a cut edge in a connected graph makes total components 2, i.e there will be two connected components

**Ans**:

let $G$ be a graph with connected component $G\langle V_1\rangle$, now if $e \in G\langle V_1\rangle$ is a cut edge then removing it will make $G\langle V_1\rangle$ into $G\langle V_{11}\rangle, G\langle V_{12}\rangle, G\langle V_{13}\rangle, \ldots G\langle V_{1n}\rangle$

now, as $e$ connects 2 vertices $x, y$ (wlog) $x \neq y$ we have $\exists i, j \in [n]$ s.t $x \in G\langle V_{1i}\rangle$, $y \in G\langle V_{1j}\rangle$ $i \neq j$ and this is by defn of connected components, as $e$ does not connect anything else $n=2$ where $G\langle V_{11}\rangle \ni x$ and $G\langle V_{12}\rangle \ni y$

29th Sept:

Exe: G is connected graph, suppose we remove a cut edge then there are exactly two connected components

Ans: Suppose $x-y$ is a cut edge. we remove it from G, then there is no path between x and y, There is at least two connected components (one containing x, another y) Suppose no of connected component is greater than 2. $G_1, G_2, \ldots G_k$ are connected components

where $k > 2$ and $x \in G_1$, $y \in G_2$

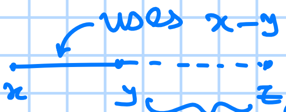
let z be a vertex in $G_3$

$\bigcup_{i=1}^{k} G_i = G' = G \setminus \{x, y\}$ → only this edge missing

There is no path from z to x in $G'$

but as ∃ path from z to x in G, a path from z to x has $\{x, y\}$ as one of the edges

So, this means there is a path from z to y




path from z to y in $G'$

So, $z \in G_2$ or $G_2 = G_3$, this is a contradiction

So, $K = 2$

**Hamiltonian path and cycle:**

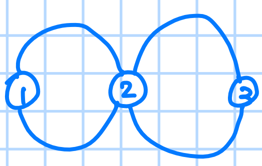
Hamiltonian path is a path which includes all vertices of G

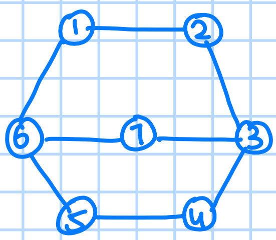
Hamiltonian cycle is a cycle which includes all the vertices of G

**Note:** we have seen that if G has Hamiltonian path/cycle then G is connected, we have also seen that if G has a Hamiltonian cycle, then G cannot have a cut edge / bridge } not true other way

as Hamiltonian cycle ⇒ no cut edge, we get
if ∃ cut edge ⇒ G cannot have Hamiltonian cycle

**Note:** converse is not true, i.e if G is connected and no cut edge, then also Hamiltonian cycle not guarantee




Above are two G, which are connected with no cut edge, G does not have a Hamiltonian cycle → no cut edge ⇏ Hamiltonian cycle

**Sufficient condition:**

we have a sufficient condition for existance of Hamiltonian cycle

**Ore property:** For any pair of non-adjacent vertice x and y, if

$$\deg(x) + \deg(y) \geq \text{order of } G$$

Theorem: if $G$ (Basic) satisfies ore property then $G$ has a Hamiltonian cycle

proof: observe that $G$ is connected as for any $x$ non-adjacent to $y$

$\deg(x)=k$

$\deg(y)=l$

as $k+l \geq n$

we will have a path of length 2

Total no of available vertices which are adjacent to $x$ or $y$ is $n-2$

suppose there is no vertex $z$, which is adjacent to both $x$ and $y$

Then $k+l \leq n-2$, this is a contradiction

so, $\exists$ vertex $z$ s.t

$$\begin{array}{l} z - x \\ | \\ y \end{array}$$

so, $x$ and $y$ are connected and hence $G$ is connected

to construct a Hamiltonian cycle,

step 1: start with vertex $v \in G$

attach adjacent vertices on both sides of $v$ and construct a path as large as possible

$\cdots v_2 - v - v_1 - v_3 \cdots$

Denote path by $\gamma$: call it $y_1 - y_2 - \cdots - y_m$

Step 2: we check if $y_1 - y_m$ or not (adjacent or not)

(i) if $y_1$ and $y_m$ are adjacent and $m=n$, we are done

$y_1 - y_2 - \cdots - y_m$ is Hamiltonian cycle

$k$ strictly from construction of $\gamma$

if $m<n$, then $\exists 1<k<m$ and vertex $z \notin \gamma$ s.t $y_k$ & $z$ are adjacent

similar argument from eulerian trail

$y_1 - y_2 - \cdots - y_k - \cdots - y_{m-1} - y_m$

$z$

$x \leftarrow$ for $x$ $\exists z$ so, $\exists z$ adjacent to $y_k$

now consider a longer path from $\gamma$ and $z$:

$\gamma_1$: $z - y_k - y_{k-1} - \cdots - y_1 - y_m - y_{m-1} - \cdots - y_{k+1}$, Rename $\gamma_1 : y_1' - y_2' - \cdots - y_{m+1}'$

now we go back to step 2

(ii) If $y_1$ and $y_m$ are not adjacent, then $\exists 1<k<m$ s.t

$y_1$ is adjacent to $y_k$ and

$y_m$ is adjacent to $y_{k-1}$

by ore property $\deg(y_1)=s$

$\deg(y_m)=t$

$s+t \geq n$, by ore property

this is from construction of $\gamma$

we observe that all the adjacent vertices of $y_1$ are on $\gamma$

similarly all the adjacent vertices of $y_m$ are on $\gamma$ — ①

suppose the adjacent vertices of $y_1$ are $y_{i_1}, y_{i_2}, y_{i_3}, \ldots y_{i_s}$ on $\gamma$:

$y_{i_1} < y_{i_2} < \cdots < y_{i_s}$, as $y_1$ and $y_m$ are not adjacent $y_{i_s} \neq y_m$

If our claim is not true, then $y_{i_1-1}, y_{i_2-1}, \ldots y_{i_s-1}$ are not adjacent to $y_m$ — ②

From ① and ②, the adjacent vertices of $y_m \in \{y_1,\dots,y_{m-1}\} \setminus \{y_{i_1},\dots,y_{i_s-1}\} = S$

then $|S| = m-1-s$

now $m-1-s$, as $\deg(y_m)=t$
from ① $|S| \geq t$
$\Rightarrow m-1-s \geq t$

as $m-1-s \geq t$
$\Rightarrow m-1 \geq s+t$
$\Rightarrow m-1 \geq n$

but $m \leq n$
$\Rightarrow n-1 \geq n$
$\Rightarrow 0 \geq 1$ this is a contradiction

so, claim is true, now we construct new $\gamma$ in following way

$$y_{k+1}-y_m-y_{m-1}-\cdots-y_k-y_1-y_2-\cdots-y_{k-2}$$

rename, $y_1'-y_2'-\cdots-y_m'$, go back to step 2 (i) to get Hamiltonian cycle

Exe: Try to find a similar sufficient condition weaker then one for the existence of Hamiltonian path in G

Ans: G s.t $\exists\, x,y \in V$ s.t

$G' \rightarrow$ induced by $V\setminus\{x,y\}$ is s.t $\forall\, u,v \in V$ $\underbrace{\deg(u)+\deg(v) \geq n-2}_{u,v \text{ not adjacent}}$

and $x,y$ s.t $x$ and $y$ are adjacent and $\deg x \geq 2$

then $G'$ will have a Hamiltonian cycle

$$v_1 - v_2 - \cdots - v_{n-2}$$

as $\deg x \geq 2 \rightarrow \exists\, v_\gamma$ s.t $x - v_\gamma$

so,
$$\begin{tikzcd}
v_1 \arrow[r, dash] & v_2 \arrow[r, dash] & \cdots \arrow[r, dash] & v_\gamma \arrow[r, dash] \arrow[d, dash] & \cdots & v_{n-2} \\
 & & & x \arrow[d, dash] & & \\
 & & & y & &
\end{tikzcd}$$

$$\Rightarrow v_{\gamma+1} - v_{\gamma+2} - \cdots - v_{n-2} - v_1 - \cdots v_\gamma - x - y$$

this is an Hamiltonian path

more weaker: G s.t $\exists\, x,y$ adjacent where $\deg x \geq 2$ and $G\setminus\{x,y\}$ s.t $\forall\, u,v$ not adjacent $\deg(u)+\deg(v) \geq n-2$

then $\exists$ Hamiltonian path

6th Oct :

## Hamiltonian path / cycles :

A path which covers all vertices, similarly hamiltonian cycle. We have seen sufficient condition for Hamiltonian cycle (Ore condition). We also saw we need graph to be connected, basic works

**Theorm:** $G$ is basic graph, $|V|=n$, For any two non-adjacent vertices $x, y$ if $\deg x + \deg y \geq n$ then $G$ has hamiltonian cycle

Eg: $G$ is a graph, $\deg x + \deg y \geq n-2$ for any two non adjacent vertex $x$ and $y$ and $\exists$ edge $\{u, v\}$ s.t $\deg u = 1$, $\deg v \geq 2$

this does not give us Hamiltonian path as removing $v$ may not get $\deg x + \deg y \geq n-2$

**Theorm:** $G$ is a basic graph, $|V|=n$, for any two non-adjacent vertices $x, y$ if $\deg x + \deg y \geq n-1$, then $G$ has a Hamiltonian path

proof : we observe $G$ is connected (similar to previous theorm)

Step 1 : Start with vertex $v$ and attach adjacent vertices to both sides, construct a path as long as possible

$$P: v_1 - v_2 - \cdots - v_m$$

if $m=n$, then $P$ is hamiltonian path

if $m \neq n$, then

Step 2 :
case (i): $v_1$ adjacent to $v_m$
case (ii): $v_1$ not adjacent to $v_m$

case (i) : $\exists$ vertex $z$ s.t $z$ is adjacent to some $v_k$ for $1<k<m$ (from previous theorm)
new path

$$P': z - v_k - v_{k+1} - \cdots - v_m - v_1 - v_2 \cdots - v_{k-1}$$

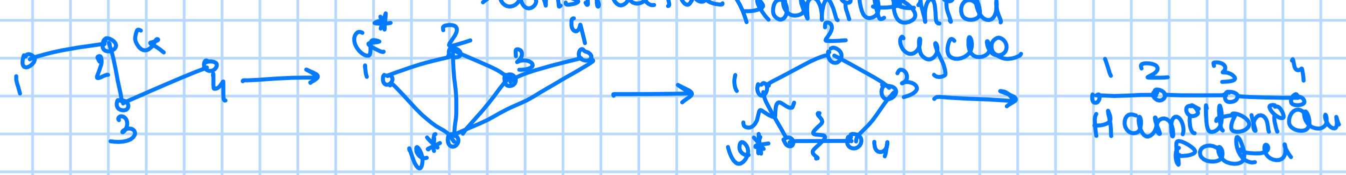
then we go back to step I

case (ii) : $\exists k$ s.t $1<k<m$ s.t $v_1$ is adjacent to $v_k$ and $v_{k-1}$ is adjacent to $v_m$ (same proof as previous theorm but $m<n$)

$\tilde{P}: v_1 - v_2 \cdots - v_{k-1} - v_m - \cdots - v_k$, so we get $v_1$ and $v_k$ adjacent and we can go back to case (i), so we will get hamiltonian path

**Note :** On the above graph, if we add a new vertex $v^*$ s.t it connects to all other vertex, then as we add 1 to all degree, we get sum of non adjacent vertex degree $\geq n-1+2 = n+1$, so $|V^*| = n+1$ and we get a Hamiltonian cycle, then we remove $v^*$ to get Hamiltonian path → constructive

Eg :



The above happend as started with $G$, added new vertex $z$ and added edge between $z$ and $v \in G$ $\forall v \in G$.

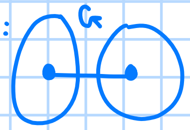
This $G'$ is s.t $\deg x + \deg y$ in $G'$
$= \deg x|_G + \deg y|_G + 1 + 1$
$\geq n-1+2 = n+1$, so by ore's property we will get $G'$ to have Hamiltonian cycle

$\gamma: z - v_1 - v_2 \ldots - v_n - z$
$\downarrow$
$P: v_1 - v_2 - \ldots - v_n$ this will be the Hamiltonian path in $G$

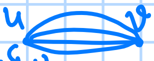
## Bipartate graph:

**Defn:** If $G=(V,E)$ s.t $V = X \cup Y$ s.t each edge in $G$ has one vertex in $X$ and another in $Y$

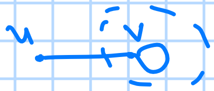
Eg: $G$



X Y



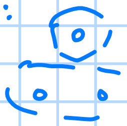
$X=\{u\}$ $Y=\{v\}$ but no loop



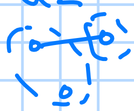
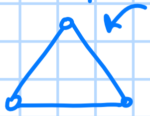
$\rightarrow$ Not bipartate

**Note:** Any graph with $|V|=2$ and no loops will be biparetate ($|V|=2$ is order 2)

Eg: $G_1$



, trivially bipatate

$G_2$  $G_3$  $G_4$  $\leftarrow$ Not Bipartate

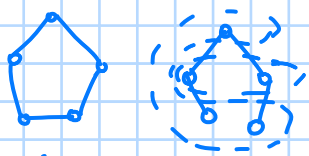
$G_1, G_2, G_3$ is bipartate, $G_4$ is not bipartate

Eg:

$G_1$ $G_2$ $G_3$ $G_4$ $G_5$ $G_6$ $G_7$

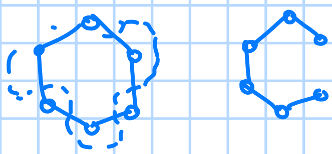
all except $G_6$ are bipartate

Eg:



$G_1$ $G_2$

Eg:

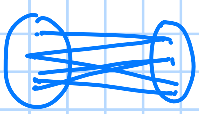


$G_1$ $G_2$

**Note:** From examples, we see that if graph has cycle of odd length then $G$ is not bipatate

**Theorm:** $G$ is basic, $G$ is bipartate iff $G$ does not have a cycle of odd length

proof: $(\Rightarrow)$ $G=(V,E)$ bipartate then $V = X \cup Y$ where $X$ and $Y$ are bipartate sets



X Y

then wlog if cycle starts at $X$, then ends at $X$, i.e even times edges used, so cycle has to be of even length

($\Leftarrow$) Assume $G$ is connected as if not then we do this argument for every connected component and union bipartate sets

now if $G$ does not have an odd cycle, we start $v \in V$ with $X = \{u \in V \mid dist(u,v) \text{ is odd}\}$

$Y = \{u \in V \mid dist(u,v) \text{ is even}\}$

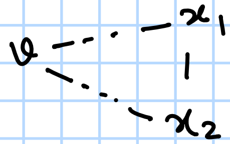
here $v \in Y$, $X \cup Y = V$ (this is trivial)

to show there is no edge b/w any two vertices of $X$ and there is no edge b/w any two vertices of $Y$

suppose $x_1, x_2 \in X$, and $x_1 - x_2$ is an edge, then

$$\left.\begin{array}{l}\alpha : v - \cdots - x_1 \\ \beta : v - \cdots - x_2\end{array}\right\}$$ shortest paths from $v$ to $x_1$ and $v$ to $x_2$

if there is no common vertex in $\alpha$ and $\beta$ other than $v$ then we get cycle of odd length which is a contradiction



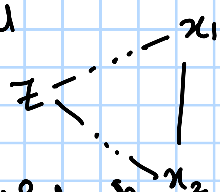

suppose there are common vertices in $\alpha$ and $\beta$, other than $v$

$$\alpha : \underbrace{v - \cdots - z}_{\alpha_1} \underbrace{- \cdots - x_1}_{\alpha_2}$$
$$\beta : \underbrace{v - \cdots - z}_{\beta_1} \underbrace{- \cdots - x_2}_{\beta_2}$$

suppose $z$ is last common vertex in $\alpha$ and $\beta$, then length of $\alpha_1$ and $\beta_1$ are same i.e

$|\alpha_1| = |\beta_1|$ as if not (wlog $|\alpha_1| < |\beta_1|$) then we get $\alpha_1 \cup \beta_2$ as new shorter path $v - \cdots - x_2$ which is a contradiction, similarly for $|\alpha_1| > |\beta_1|$

so, $|\alpha_2|$ and $|\beta_2|$ are both odd or both even, so $|\alpha_2| + |\beta_2|$ is even




cycle of odd length, which is a contradiction

so, $X$ is valid bipartate graph, same goes for $Y$, so $G$ is bipartate

$9^{th}$ oct:

## Bipartite graph:

$G=(V,E)$, $V = X \cup Y$ s.t if $e \in E$, then one vertex of $e$ is in $X$ and other one is in $Y$

**Theorem:** $G$ is bipartite iff $G$ does not have a cycle of odd length

**proof:** $(\Rightarrow)$ Trivial

$(\Leftarrow)$ let $v \in V$, $X = \{u \in V \mid dist(u,v) \text{ is odd}\}$

$$Y = \{u \in V \mid dist(u,v) \text{ is even}\}$$

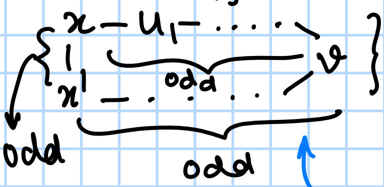
$x, x' \in X$

claim: there is no edge b/w $x$ and $x'$, as if there is then:

let $\alpha: x - u_1 - \cdots - v \to$ shortest path

$\beta: x' - \cdots - v \to$ shortest path

if $v$ is only common b/w $\alpha$ and $\beta$




} odd length cycle, which is a contradiction

cycle of odd length

suppose there are some common vertex other then $v$

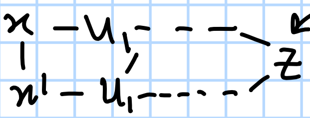
$$\alpha: x - \overbrace{u_1 - \cdots - z}^{\alpha_1} - \overbrace{\cdots - v}^{\alpha_2}$$

$$\beta: x' - \underbrace{u_1' - \cdots - z}_{\beta_1} - \underbrace{\cdots - v}_{\beta_2}$$

$z$ - last common vertex in $\alpha$ and $\beta$ (assuming $v$ is first) s.t $z \neq v$

claim: $|\alpha_2| = |\beta_2|$ as if not then wlog $|\alpha_2| < |\beta_2|$, then $\beta_1 \cup \alpha_2$ is shorter from $x'$ to $v$. this is a contradiction

so, $|\alpha_1|$ & $|\beta_1|$ have same parity




odd cycle

**Exe:** Suppose $G$ is a bipartite graph with bipartition $X$ and $Y$, then prove or disprove the following

(i) if $|X| \neq |Y|$, then $G$ does not have Hamiltonian cycle

(ii) if $|X| = |Y|$, then $G$ does not have a Hamiltonian path which starts at $Y$ and ends at $Y$

(iii) $|X|, |Y|$ differs by 2 then $G$ does not have Hamiltonian path $\to$ done down

**Exe:** Find the maximum number of edges in a bipartite graph with $2n$ vertices

**Ans:** if $V = X \cup Y$

then $|X| = x$, $|Y| = y$

$x + y = 2n$ and $xy$ = max vertices

$$xy \leq \left(\frac{x+y}{2}\right)^2 \quad (\because AM \geq GM)$$
$$\Rightarrow xy \leq (n)^2$$

**Exe:** If G is basic with 2n vertices and $n^2+1$ edges, then does G have a cycle of odd length? ($n \geq 2$)

**Ans:** from previous part, G is not Bipartate as maximum possible edges in bipartate is $(n)^2$

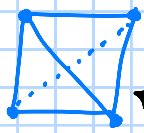
So, G is bipatate $\Leftrightarrow$ G has no cycle of len o

$\Rightarrow$ G is not bipatate $\Leftrightarrow$ $\exists$ atleast one cycle of len odd

**Theorm:** G is basic with 2n vertices, atleast $n^2+1$ edges, where $n \geq 2$ then G has atleast a triangle (cycle of length 3)

**proof:** for atleast one triangle:

for $n=2$: $|V|=4$, $|E|=5$




so true for $n=2$, suppose result is true for $n-1$ i.e $2n-2$ vertices and $(n-1)^2+1$ edges, then

we will show for n, 2n vertices and $(n)^2+1$ edges

let $x, y$ be adjecent vertices in G, suppose there is a vertex z s.t z is adjcent to both x and y, so we have a triangle

If such z does not exist, then:

max edges attached to x and y is $\underbrace{2n-2}_{\substack{x \text{ or } y \text{ conneted} \\ \text{to remining} \\ 2n-2 \text{ vertices}}} + \underbrace{1}_{\{x-y\}} = 2n-1$

total edges $= n^2+1$

if we remove x and y and all edges conneted to x and y

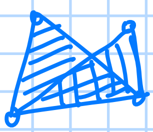
$\Rightarrow 2n-2$ vertices

and total edges $\geq n^2+1-(2n-1)$
$= (n-1)^2+1$

so from induction

we get $\exists$ a triagle and so this is true $\forall n$ from induction

**Note:** The above theorm is weaker, stronger case is when we have atleast n triagles, but here we use $(n)(n+\delta)$ $\delta \in \{0,1\}$ and then induction



$n=2$
$n^2+1=5$, n triagles

Exe: Suppose G is a bipartite graph with bipartition X and Y, then prove or disprove the following

(i) if $|X| \neq |Y|$, then G does not have Hamiltonian cycle

(ii) if $|X| = |Y|$, then G does not have a Hamiltonian path which starts at Y and ends at Y

(iii) $|X|, |Y|$ differs by 2, then G does not have Hamiltonian path

Ans: (i) G is bipartite and so if we start with $x_1$, we will have Hamiltonian cycle construction as

$$x_1 - y_1 - x_2 - y_2 \cdots - x_r - y_r$$

if $|X| = r$ then from $y_r$ we cannot come to X as all $x_i$'s used in construction

so, G does not have Hamiltonian cycle

(ii) This follows from i, as if we start from $y_1$, we will end at $x_r$ for $r = |X|$

(iii) $|X| = r$
$|Y| = m$ s.t $m - r \geq 2$
if we try to construct a path, then

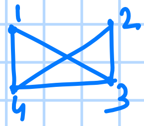
$$x_1 - y_1 - x_2 - y_2 \cdots - x_r - y_r$$

as now not back to X and $r < m \Rightarrow$ no such path exist

$13^{th}$ oct:

**Theorm**: suppose $G$ is a basic graph with $2n$ vertices ($n \geq 2$) and atleast $n^2+1$ edges then $G$ has atleast $n$ many triangles

proof: for $n=2$, this is trivial as we get




we get 1-3-4-1 and 2-3-4-2 as 2 triangles

Now, if we assume that result is true for a basic graph with $2k$ vertices, and atleast $k^2+1$ many edges where $2<k<n$

if we show this for $k=n$, we are done

$G$ is a basic graph, $2n$ vertices and atleast $n^2+1$ many edges from previous theorm we know that there is a triangle in $G$

$G=(V,E)$
let $R = V \setminus \{x,y,z\}$
$|R| = 2n-3$ vertices

case 1: There is no edge b/w vertices of $\{x,y,z\}$ and $R$ then induced sub graph on $R$ has atleast $n^2+1-3$ many vertices

induced subgraph $G(R) = (R, E(R))$

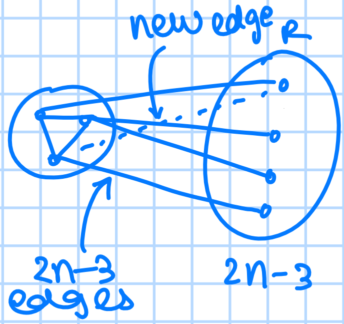
$$|E(R)| \geq n^2+1-3 = n^2-2 \geq (n-1)^2+1$$
as $n \geq 2$

so, $H = (\tilde{V}, \tilde{E})$ s.t $\tilde{V} = R \cup \{x\}$ $\tilde{E}$ = Edge set of $G(R)$
then $|\tilde{V}| = 2n-2$
$|\tilde{E}| \geq (n-1)^2+1$ so, $H$ has atleast $n-1$ many triangles (By induction hypothesis)

and so, $G$ has atleast $n$ many triangles

case 2: There are edges between $\{x,y,z\}$ and $R$

subcase (i): No of edges b/w $\{x,y,z\}$ and $R$ is $2n-3+k$ where $k \geq n-1$




after $2n-3$ vertices, whenever we add a new edge it will form one extra triangle in $G$ so, from PHP when $k \geq n-1$, we get $G$ has $n$ many triangles, and so, we are done in this case

as by PHP we got that there will be atleast $(n-1)$ triangles where two vertices are in $\{x,y,z\}$ and one in $R$

Total $n$ many triangles

Subcase (ii): No of edges b/w $\{x, y, z\}$ and $R$ is $K$ where $1 \leq K \leq 2n-3$

induced subgraph $G(R)$ has atleast $n^2+1-3-(2n-3)$ $=(n-1)^2$ many edges

add one edge, whose one vertex is from $\{x,y,z\}$ and other in $R$ in the induced subgraph $G(R)$ $(\because K \geq 1)$
wlog vertex is $x$ that this edge corresponds to

$$H = (\tilde{V}, \tilde{E}) \text{ s.t}$$

$$\tilde{V} = R \cup \{x\}$$

$$\tilde{E} = G(R) \cup \{e\}$$

$|\tilde{V}| = 2(n-1)$ and $|\tilde{E}| \geq (n-1)^2+1$, so by induction hypothesis $H$ has atleast $n-1$ many triangles, so $G$ has atleast $n$ triangles

Subcase (iii): No of edges b/w $\{x,y,z\}$ and $R$ is $2n-3+K$ where $1 \leq K < n-1$
i.e $1 \leq K \leq n-2$

now, $G(R)$ has atleast $n^2+1-3-(3n-5)$ many edges
$\Rightarrow n^2-3n+3$ many edges

now remove a vertex from $G(R)$ which has lowest number of edges i.e lowest degree in $G(R)$

$$0 \leq \text{lowest degree} \leq \frac{2(n^2-3n+3)}{2n-3}$$

so, remaing graph has atleast

$$(n^2-3n+3) - \frac{2(n^2-3n+3)}{2n-3} \times \frac{1}{2}$$

to get reming edges

$$= (n^2-3n+3)\left[1 - \frac{1}{2n-3}\right]$$

$$= (n^2-3n+3)\left[\frac{2n-3-1}{2n-3}\right]$$

$$= (n^2-3n+3)\left[\frac{2n-4}{2n-3}\right]$$

$$= ((n-2)(n-1)+1)\left(\frac{2n-4}{2n-3}\right)$$

$$> (n-2)(n-1)\left(\frac{2n-4}{2n-3}\right)$$

$$> (n-2)^2\frac{(2n-2)}{(2n-3)}$$

$$> (n-2)^2$$

$$\geq (n-2)^2+1 \quad \text{as } n \geq 3 \text{ ( } n=2 \text{ case done)}$$

so, new graph H has $2n-4$ vertices and atleast $(n-2)^2+1$ many edges

induction Hypothesis

so, atleast $n-2$ triangles and 2 triangles already in G as $k \geq 1$

$x-y-z$ and as $k \geq 1$ from PHP

so, G has atleast $n$ many triangles

16th oct :

**Theorm:** A basic graph with $2m$ vertices and atleast $(m^2+1)$ many edges has atleast $m$ triagles

proof : The last case :

as we know from previous theorm, $\exists$ a triagle $x\,y\,z$

$R = V \setminus \{x, y, z\}$

No of edges b/w $\{x, y, z\}$ and $R$ is $2m-3+p$ where

$1 \leq p \leq m-2$

$G(V,E)$, $|E| = m^2+1$, if $p = m-2$ then $G(R)$ has

$m^2+1-3-(m-2)-(2m-3)$ many edges

$= m^2-3m+3$

now, maximum value of minimum degree $= \dfrac{2(m^2-3m+3)}{2m-3}$ ← $2|E|/|V|$

if we remove vertex with mimium degree no of edges in $G(R)$ is atleast

$$(m^2-3m+3) - 2\frac{(m^2-3m+3)}{2m-3}$$

$$= (m^2-3m+3)\left(\frac{2m-5}{2m-3}\right)$$

$$> (m-2)^2 \quad \text{if } m > 3$$

$$\geq (m-2)^2+1 \quad \text{if } m > 3$$

In reduced graph H, no of vertices is $2(m-2)$ and has at least $(m-2)^2+1$ many edges

By induction hypothesis, so By induction Hypotesis, H has atleast $(m-2)$ triagles

we alrady have two triagles

we alredy have two triagles in G

① $\Delta xyz$

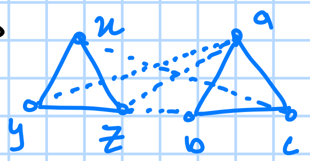
② A triagle whose two vertices are in $\{x, y, z\}$ and third vertex in $R$

Total G has atleast $m$-many triagles

If G has more than $(m^2+1)$ many edges or $1 \leq p < m-2$, then it has atleast $m^2-3m+3$ edges

By previous argument, $G(R)$ will have atleast $(m-2)$ triagles and so G will have atleast $m$-triagles

for $m=3$




$1 \leq p \leq 1 \Rightarrow p = 1$

we have to add 3 more edges in $G(R)$

so min 3 triagles

**Rodemacher:** If $G$ is basic graph with $n$ vertices and atleast $\lfloor \frac{n^2}{4} \rfloor + 1$ many edges then $G$ has atleast $\lfloor \frac{n}{2} \rfloor$ triangles

**Erdos:** First available proof was by him
For $r < \lfloor \frac{n}{2} \rfloor$ and $G$ has $n$ vertices and no of edges is atleast $\lfloor \frac{n^2}{4} \rfloor + r$ then $G$ has atleast $r \lfloor \frac{n}{2} \rfloor$ many triangles

**Defn:** A connected graph $G$ is called minimally connected if we remove any edge from $G$ it becomes disconnected

**Note:** From defn above, for a minimally connected graph, every edge will be a cut edge or a bridge

**Defn:** (Tree) A minimally connected graph is called a tree

Also, from defn we see minimally connected graph cannot have multiple edges and loops

**lemma:** A graph is minimally connected iff it has no cycle

**proof:**
($\Rightarrow$) let $G$ is minimally connected, suppose $G$ has a cycle $C$ then if $C$ contains $e$, $e \in E$, then $E$ is not a cut edge ($\because$ if $e \in C$ then $e$ is not a cut edge)

If we remove $e$ from $G$, it remains connected this is a contradiction, so $G$ has no cycle

($\Leftarrow$) $G$ is connected and has no cycle, any edge $e$

$x \overset{e}{\text{—}} y$ ← not part of any cycle

If we remove $e$ from $G$, then there is no path from $x$ to $y$
as if $\exists$ some path from $x$ to $y$ other then $x — y$ then PUe is a cycle which is a contradiction

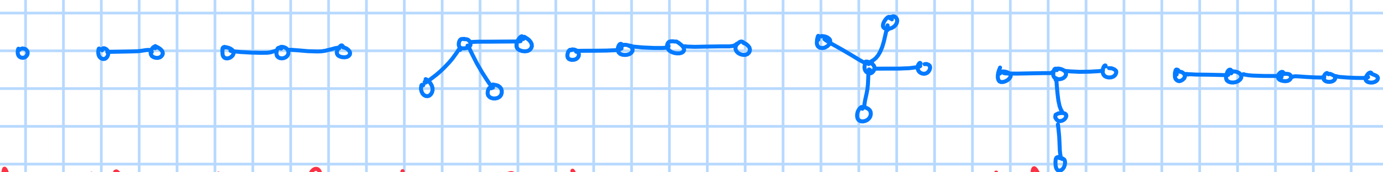
So, $e$ is cut edge, $\forall e \in E$

If we remove $e$ from $G$ then $G$ becomes disconnected so, $e$ is a cut edge

$\Rightarrow G$ is minimally connected

**Note:** A connected graph without a cycle is called a tree

**Eg:**



**Defn:** (leaves) vertices of a tree of degree one are called leaves

**lemma:** A tree with $n$ vertices ($n \geq 2$) has atleast two leaves

**proof:** Induction: for $n=2$, trivial. If true for $n-1$, then we can connect new vertex anywhere still leaves $\geq 2$ (2 cases)

Constructive:

If we consider longest possible path in T, then the vertices in both ends have degree 1

as suppose

$\circ\!-\!\circ \cdots \circ\!-\!\circ_y$ ($P$) if $\deg x > 1$ then we can add one more edge, which will be a contradiction

**Lemma:** A connected graph $G$ is a tree iff there is a unique path between any two pairs of vertices in $G$

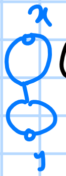
proof:

($\Leftarrow$) Let $e=\{x,y\}$ be an edge. $e$ is the unique path b/w $x$ and $y$. If we remove $e$, then there is no path from $x$ to $y$

$\Rightarrow e$ is a cut edge $\forall e \in E$

$\Rightarrow G$ is minimally connected

$\Rightarrow G$ is a tree

($\Rightarrow$) 



If $\exists\, x,y \in V$ $x \neq y$ st $\exists\, P_1 \neq P_2$ path from $x$ to $y$

$\underbrace{(P_1 \cup P_2) \setminus (P_1 \cap P_2)}_{\text{not end points}}$ ← union of cycles which is not possible

so, this is a contradiction

**Theorem:** A tree with $n$ vertices has $(n-1)$ edges

proof: If we assume true for $n-1$

then now if $T$ has $n$ vertices

$T$ has atleast 2 leaves

$x, y$

Remove $x$ and adjacent edge to get $G$ with $n-1$ vertices

as $T$ has no cycle, remaining we will get $G$ has no cycle

$\Rightarrow G$ is a tree

$\Rightarrow G$ has $n-2$ many edges

So $T$ has $n-1$ many edges

23rd oct :

Quiz-2: 2pm to 3:30 pm 30/10/25, only graph theory part, LC302
30/10/25: class at 9:30 in 114
No class: 27/10/25

**Tree**: minimally connected graph

**Theorem**: A connected graph is a tree iff it has no cycle

**Theorem**: $G$ is a connected graph with $n$ vertices

$G$ is a tree iff $G$ has $(n-1)$ edges

proof :
($\Rightarrow$) Result is true for $n=1$
suppose true for a tree with $n$ vertices
i.e it has $n-1$ edges

If $G$ is a tree with $(n+1)$ vertices then it will have atleast 2 leaves

Remove one leaf and the edge connected to it we get $G'$ a graph with $n$ vertices

$G'$ is connected and has no cycles as $G$ has no cycles
$\Rightarrow G'$ is a tree with $n$ vertices

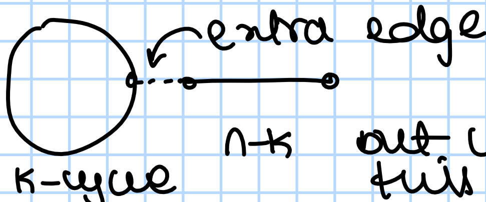
By Hypothesis $G'$ has $(n-1)$ edges, so $G$ has $n$ edges

($\Leftarrow$) $G$ is connected, $G$ has $n$ vertices, $n-1$ edges
induction: $n+1$ vertices, $n$ edges $\Rightarrow \sum d_i = 2n$
as $G$ is connected $d_i \geq 1 \; \forall i \in [n+1]$, now as $\sum d_i = 2n$
$d_i \geq 1$, by PHP, $\exists i \in [n+1]$ s.t $d_i = 1$
so, wlog $i = n+1$, then removing it makes $|V| = n$
$|E| = n-1$ so $G'$ becomes a tree by induction
Hypothesis (as trivial for $n=2$, $n-1=1$ case)

so, then $G' = G \setminus (n+1)$ is a tree, adding the one edge and vertex, we will get $G$ as tree

contradiction: If $G$ has atleast one cycle of len $k$, then the cycle uses $k$ vertex and $k$ edges, so we will have $n-1-k$ edges left for $n-k$ many vertex,

minimum edges needed to connect all $n-k$ many vertex is $n-k-1$, but as $G$ is connected, we need one more edge to connect the two components




so total minimum $n-k$ edges needed
but we only have $n-k-1$
this is a contradiction, and so $G$ has no cycles and connected
$\Rightarrow G$ is a tree

**Defn**: A graph without a cycle is called acyclic graph

**Theorm**: Suppose G is an acyclic graph with n vertices, then G is a tree iff G has (n-1) edges

**proof**:
(⇒) this is side is trivial from above theorm

(⇐) G is acyclic, with n vertices and (n-1) edges, suppose G is not connected and G has K-connected components

$K > 1$, suppose no. of vertices in $i^{th}$ component $G_i$ is $n_i$ $1 \leq i \leq K$

then, $G_i$ is connected component and it has $n_i$ many vertices and is acyclic as G is acyclic, so $G_i$ is a tree

$$\sum_{i=1}^{K} n_i = n,$$ and as we observed $G_i$ is a connected graph without a cycle, so $G_i$ is a tree, so no of edges in $G_i$ is $n_i - 1$, so total no of edges in G

$$\sum_{i=1}^{K} (n_i - 1) = n - K$$

as $K > 1 \Rightarrow n - K < n - 1$ (contradiction)

so, G is connected and so, G is a tree

**Spanning subgraph**:

$G = (V, E)$

$H = (U, F)$

where $U = V$, $F \subseteq E$

**Defn**: (Spanning tree) If a spanning subgraph of a connected graph is a tree, we call it spanning tree

G is connected graph, we want to construct a spanning tree of G. If we
① remove loops
② Remove multiple edges (keep only one edge)

we get a basic subgraph H of G, also as G is connected H will be connected

check an edge of H, weather it is cut edge or not
if not a cut edge of H, then remove it from H

continue this operation until all remaining edges are cut edges

**Lemma**: If T is a spanning tree of G, let $\alpha = \{a, b\}$, $(a \neq b)$ be an edge in G, but not an edge in T, then $\exists$ edge $\beta$ of T s.t
$T' = (T \cup \alpha) \setminus \beta$ is a spanning tree of G

30th oct:

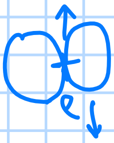
**Spanning tree**: If a spanning subgraph of $G$ is a tree, we call it spanning tree

**Lemma**: If $T$ is a spanning tree of $G$, let $\alpha = \{a, b\}$, $(a \neq b)$ be an edge in $G$, but not an edge in $T$, then $\exists$ edge $\beta$ of $T$ s.t
$$T' = (T \cup \alpha) \setminus \beta \text{ is a spanning tree of } G$$

**proof**: we add an edge to a tree, then a cycle will be formed as its an extra edge among two existing vertices, $T' = (T \cup \alpha)$ will have cycles

$\alpha$ is not a loop

$T'$ will have exactly one cycle as



still a cycle or if 2 cycles → more than 1 external path

Removing $\beta \neq \alpha$ from this one cycle will get us

$$T'' = (T \cup \alpha) \setminus \underset{\substack{\uparrow \\ \text{any edge in loop}}}{\beta}$$

only one cycle edge removed

$T''$ has $n$ vertices, $n-1$ edges and is acyclic so $T''$ is a tree

**Theorem**: suppose $T_1, T_2$ are two spanning trees of $G$. Let $\alpha$ be an edge in $T_1$. Show that $\exists$ an edge $\beta$ in $T_2$ s.t
$$(T_1 \cup \beta) \setminus \alpha \text{ is also spanning tree}$$

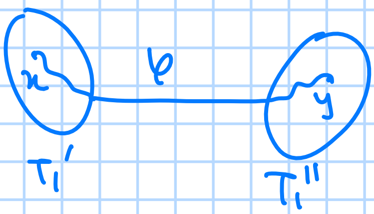
**proof**:

let $T_1 \setminus \alpha = T_1' \cup T_1''$

connected components as $\alpha \in$ edge in $T_1$ so $\alpha$ is cut edge of $T_1$

Claim is that $\exists$ $\beta \in$ edge of $T_2$ s.t it connects $T_1'$ and $T_1''$

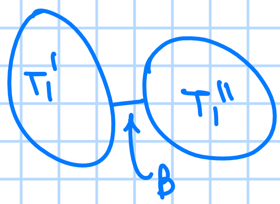
let $x \in T_1'$, $y \in T_1''$ then as $T_2$ is a tree, there exist be a unique path $\underbrace{x - \cdots - y}_{P}$ in $T_2$




so, $\exists$ $\beta \in$ edge of $T_2$ on that path $P$ s.t one vertex of $\beta$ is in $T_1'$ and other vertex of $\beta$ in $T_1''$

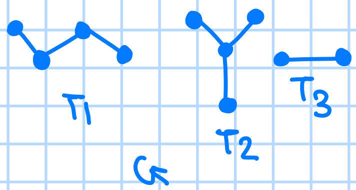
This $\beta$ will work, $(T_1 \cup \beta) \setminus \alpha$ is now connected and has $n$ vertices and $n-1$ edges, then $(T_1 \cup \beta) \setminus \alpha$ is a tree

$$(T_1 \cup \beta) \setminus \alpha = \underbrace{(T_1' \cup T_1'')}_{\text{connected}} \cup \beta$$



**Defn:** (Forest) A graph where each connected component is a tree

Eg:



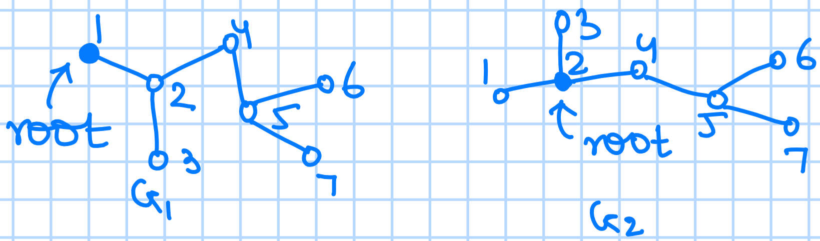

$G = T_1 \cup T_2 \cup T_3$
each component is a tree

**Note:** A forest with $n$ vertices and $k$ connect component will have $n-k$ many edges as if every connected component was $n_1, n_2, \ldots n_k$ trees

$$\text{edges} = \Sigma(n_i - 1) = n - \sum_{i=1}^{k} 1 = n-k$$

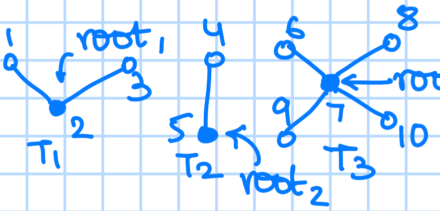
**Defn:** (Rooted tree) A tree where a vertex is declared as a root

Eg:



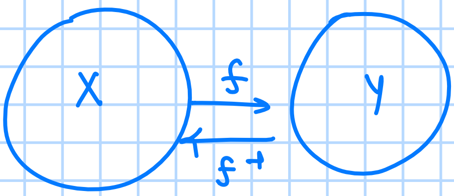

**Defn:** (Rooted forest) A graph where each connected component is a rooted tree

Eg:

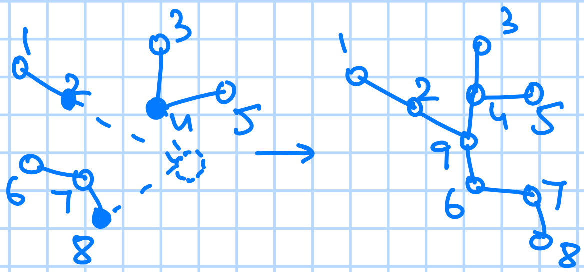



**Theorem:** Show that there is a bijection between the set of rooted forest with vertex set $[n]$ and the set of unrooted tree with vertex set $[n+1]$

proof:




X = set of rooted forest with vertex set $[n]$
Y = Set of rooted tree with vertex set $[n+1]$




$f(F) = T$
add edge between each root of F and vertex $(n+1)$

Then T is a connected graph with $(n+1)$ vertices and $n$ edges

f is a bijection as

① f is one-one, for every tree in Y, if $f(x_1) = f(x_2)$ then by removing $n+1$ vertex and making all conneted vertex to $n+1$ as roots, we will get $x_1$ trivial to see $x_1 = x_2$

so, $f(x_1) = f(x_2)$
$\Rightarrow x_1 = x_2$

② $f$ is onto, as for evey tree in $Y$, by removing $n+1$ vertex and making covered vertex to $n+1$ as roots we get a tree in $X$ say $x$ s.t

$f(x) = y$, so $\forall$ $y \in Y$, $\exists$ $x \in X$ s.t $f(x) = y$

so $f$ is onto

from ①, ② $\Rightarrow$ $f$ is a bijection and

$|X| = |Y|$

3rd Nov :

**Forest** : A graph where each connected component is a tree

we have seen that, F is a forest with $n$ vertices, $k$- connected components then the no of edges in F is $n-k$

so, if we know no of vertices $=n$, and no. of edges $=e$, then the no of components in this forest $=n-e$

**Exe**: suppose F, F' are two forest with n vertices. Suppose that no. of edges in F' is more than no. of edges in F show that $\exists$ edge e in F' s.t $F \cup e$ is also a forest

**Ans**: F' has less components than F

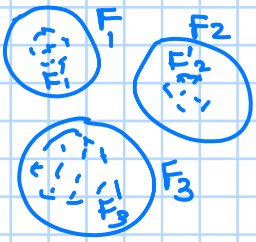
Now, if no such edge exist in F', then adding an edge of F' to F will form a cycle

That cycle will be formed within a connected component of F

Each connected component of F' is sitting inside some connected component of F, this is a contradiction and so $\exists e$
(no of comp of F' $\geq$ no of comp of F)

moreover, if there is an edge in F' whose vertex are in different components of F

That edge will serve the purpose



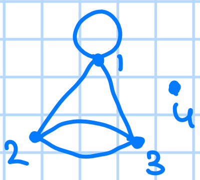

**Adjency matrix**:

**Defn**: (Adjency matrix) let $G=(V,E)$ be a graph where $V=\{v_1, v_2, \dots v_n\}$ The adjency matrix of G is

$A=((a_{ij}))_{n\times n}$ $\quad a_{ij}=$ no of edges b/w $v_i$ and $v_j$ $\;\; i\neq j$

$a_{ii}=$ no of loops of $v_i$

Eg:




$V=\{1,2,3,4\}$

$$A=\begin{array}{c|c} \begin{matrix}1&2&3&4\end{matrix} & \\ \begin{bmatrix}1&1&1&0\\1&0&2&0\\1&2&0&0\\0&0&0&0\end{bmatrix} & \begin{matrix}1\\2\\3\\4\end{matrix}\end{array}$$

we observe that :

① A is symmetric matrix, $a_{ij}=a_{ji}$

② Deg of $v_i = 2a_{ii} + \sum\limits_{j\neq i} a_{ij}$

**Note**: If G is a basic graph then $a_{ij}=0$ or $1$ $\forall i\neq j$ ($\because$ no multiple edge)

$a_{ii}=0 \;\forall i$ ($\because$ no loop)

**Theorem**: let $G$ be a graph with $n$ vertices and $A$ be the adjacency matrix of $G$. Then $(A^K)_{ij}$ denotes the no of walk from $v_i$ to $v_j$ of length $K$.

**proof**:

$\underline{K=1}$: $(A)_{ij} = a_{ij}$ = No of edges b/w $v_i$ and $v_j$
= no of walk of length 1

suppose true for $K$, then for

$\underline{K+1}$: $(A^{K+1})_{ij} = (A^K \cdot A)_{ij}$

$= (BA)_{ij}$

$= \sum_{l=1}^{n} b_{il} a_{lj}$ where $B = A^K$

$a_{lj}$: no of walks of $v_l$ to $v_j$ of len 1

$b_{il}$: no of walks from $v_i$ to $v_l$ of len $K$

$b_{il} a_{lj}$ = no of walks from $v_i$ to $v_j$ of length $K+1$ whose second last vertex is $v_l$

$\sum_{l=1} b_{il} a_{lj}$ = no of walks from $v_i$ to $v_j$ of length $K+1$

So, true for $K+1$, and so by induction, true for all $K$

**Theorem**: let $G$ be a basic graph with $n$ vertices and let $A$ be the adjacency matrix of $G$. Then $G$ is connected iff $(I+G)^{n-1}$ consist of strictly positive entries

**proof**: $(\Rightarrow)$

$(I+A)^{n-1} = \sum_{K=0}^{n-1} \binom{n-1}{K} A^K$ suppose $G$ is connected
$= B$

for $v_i, v_j \in V$, $\exists$ a path of $P_{ij}$ between $v_i$ and $v_j$ s.t length of $P_{ij} \leq n-1$ as no of vertices is $n$

Suppose length of $P_{ij}$ is $K$

$\Rightarrow (A)^K_{ij} \geq 1$, one of the term is positive in $B$

$\Rightarrow (I+A)^{n-1}_{ij} \geq 1$

as $B_{ij} = \sum_{l=0}^{n-1} \binom{n-1}{l} (A^l)_{ij} \geq \binom{n-1}{K} (A^K)_{ij} \geq \binom{n-1}{K} \geq 1$

as $\forall v_i, v_j \Rightarrow B_{ij} \geq 1 \ \forall i,j$

$\Rightarrow B$ has strictly positive

$(\Leftarrow)$ $B_{ij} = \sum_{K=0}^{n-1} \binom{n-1}{K} (A^K)_{ij} > 0 \Rightarrow \exists K \in \{0,1,\dots n-1\}$ s.t $(A^K)_{ij} > 0$

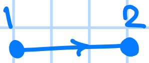
$\Rightarrow \exists$ a path between $v_i, v_j$

So, $\forall v_i, v_j$, $\exists$ a path
$\Rightarrow G$ is connected

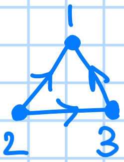
## Directed graph / digraph:

**Defn**: (Directed graph) Graph where each edge has a direction

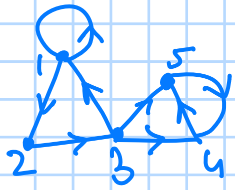
Eg:




$G_1$




$G_2$




**Defn**: (Adjacency matrix of digraph) Suppose $G$ is a digraph with $n$ vertices $v_1, v_2, \ldots v_n$

$$(A)_{ij} = \text{No of edges from } v_i \text{ to } v_j$$

$$(A)_{ii} = \text{no of loops on } v_i$$

Eg: $A_{G_1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$A_{G_2} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

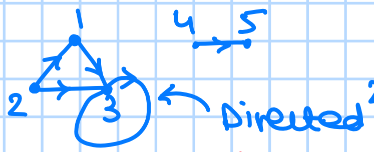
$$A_{G_3} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

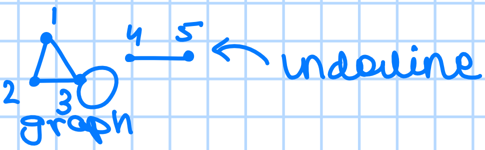
$6^{th}$ Nov :

Directed graph / Digraph :

A graph where each edge has a direction

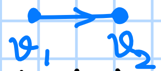
Eg:




Defn: (underline / associated) Given a digraph, a graph whose edges don't have direction is the underlying / associated

Consider an edge $\{v_1, v_2\}$ in a digraph




Here $v_1$ is called tail of the edge $\{v_1, v_2\}$, $v_2$ is called the head of edge $\{v_1, v_2\}$

Defn: (Adjency matrix) G digraph with vertices $\{v_1, \dots v_n\}$

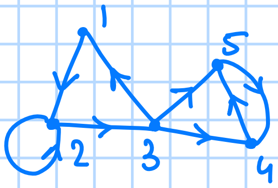
$(A)_{ij}$ = No of edges from $v_i$ to $v_j$

$(A)_{ii}$ = No of loops at $v_i$

Defn: (In degree of a vertex) No of edges ending at that vertex

Defn: (out degree of a vertex) No of edges starting at that vertex

Eg:




In deg (3) = 1
out deg (3) = 3

Defn: (Balanced diagraph) If indegree of $v$ = out degree of $v$ for all vertices $v$

Exe: Symmetric adjency matrix ⇒ Balanced diagraph → dose down

In a graph, a walk from $x$ to $y$ is collection $\{x, v_1\}, \dots \{v_k, y\}$ of edges in G

$$x - v_1 - v_2 - \cdots - v_k - y$$

Defn: (directed walk) A directed walk from vertex $x$ to $y$ of digraph G is

$$x \to v_{i_1} \to v_{i_2} \to v_{i_3} \cdots\cdots \to y$$

Eg: In above Example there is no directed walk from 4 to 3

but :

$2 \to 3 \to 1$ is a valid directed walk from 2 to 1

also a directed trail / path

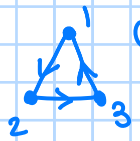
Defn: (Directed trail) Directed walk whose edge does not repeat

Defn : (Directed path) Directed trail whose vertices do not repeat

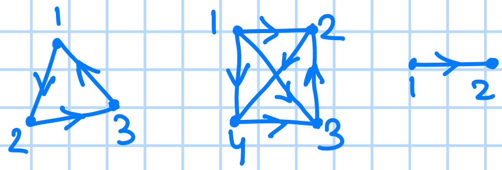
Defn : (Directed cycle) closed directed path

Defn : (Strongly connected) A digraph $G = (V, E)$ is called strongly connected if ∃ a directed path from $v_i$ to $v_j$ ∀ $v_i, v_j \in V$

Eg:

$G_1$:  $G_1$ is strongly connected

**Defn**: (Tournament) It is a complete graph with n vertices where each edge has a direction

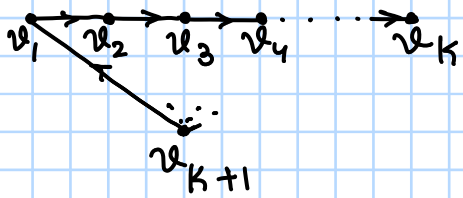
Eg:  Roundrobin tournament representation

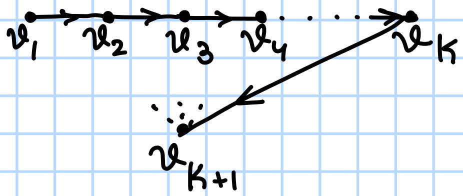
**Defn**: (Directed Hamiltonian path) A directed path in a diagraph which covers all the vertices

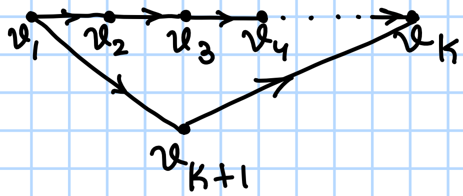
**Theorem**: A tournament always has a Hamiltonian path

**Proof**: $n=2$, trivial it is true

if true for $n=k$ then for $k+1$:

Case I: 

Case II: 

Case III: 

in this case if $v_{k+1} \to v_2$ then we are done as

$$v_1 \to v_{k+1} \to v_2 \to v_3 \to v_4 \cdots \to v_k$$

otherwise $v_{k+1} \leftarrow v_2$, then we check the next vertex

if ∃ vertex $v_i^{k+1}$ s.t $2 \le i \le k-1$ and

$$v_{k+1} \to v_i$$

then $v_1 \to \cdots \to v_{i-1} \to v_{k+1} \to v_i \to v_{i+1} \cdots \to v_k$ and we are done

otherwise $v_i \to v_{k+1}$ $\forall i \in [k-1]$, in this case

$$v_1 \to v_2 \cdots \to v_{k-1} \to v_{k+1} \to v_k$$

so, ∃ a hamiltonian path for $k+1$, then by induction true for all tournaments

**Note:** In undirected connected graph, closed E-trail exist iff degree of each vertex is even

**Note:** G is directed graph, For existence of closed E-trail in G we need the following

(i) Indegree = out degree ∀ vertex

(ii) Strongly connected

**Theorem:** Let G be a directed graph. G has a closed Eulerian trail iff G is strongly connected and indegree is same as outdegree for each vertex (Balanced)

**proof:** (⇒) Trivial

(⇐) Step 1: Each edge is part of a closed directed trail

Step 2: Using step 1 we show that ∃ a closed directed E-trail this follows similar to argument in undirected case

**Exe:** Symmetric adjecency matrix ⇒ Balanced diagraph

**Ans:**

$A = [(a_{ij})]$ $a_{ij}$ = no of edges from $i$ to $j$

as $A = A^T \Rightarrow a_{ij} = a_{ji}$ $\forall i, j \in [n]$

$$\Rightarrow \sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ji} \ \forall i \in [n]$$

$$\Rightarrow \text{indegree}(v_i) = \text{out degree}(v_i) \ \forall i \in [n]$$

So Digraph is balanced