MA406 - Tuts

Tutorial-1 :

metric space has a distance function, given $\mathbb{R}$ the following topology

$S \in \tau \rightarrow$ collection of sets

$\forall x \in S, \ \exists \delta > 0$ s.t $(x-\varepsilon, x+\varepsilon) \in S$

① $\phi, \mathbb{R} \in \tau$ as if $S = \mathbb{R}$, $\forall x \in \mathbb{R}$, $\exists\, 1 > 0$ s.t $(x-1, x+1) \subseteq S$

if $S = \phi$, then as no x in ϕ, $\forall x \in \phi$ $\exists \varepsilon > 0$ s.t $(x-\varepsilon, x+\varepsilon) \subseteq \phi$

② if $A_i \in \tau$ for $i = 1, 2 \ldots n$ then

$\forall x \in A_i \in \tau$ then
$\exists \varepsilon_i > 0$ s.t

$$B_{\varepsilon_i}(x) \subseteq A_i$$

now if $\varepsilon = \min \{\varepsilon_i\}$ then

$$B_\varepsilon(x) \subseteq B_{\varepsilon_i}(x) \subseteq A_i \quad \forall i$$

$$\Rightarrow B_\varepsilon(x) \subseteq \bigcap_{i=1}^{n} A_i$$

$$\Rightarrow \bigcap_{i=1}^{n} A_i \in \tau$$

③ Infinite union of open set is open

1. $\mathcal{A}$ = collection of subsets of X

$\mathcal{A}'$ = collection of intersections of members of $\mathcal{A} \cup \{\phi\} \cup \{X\}$

B = union of member of $\mathcal{A}'$

Claim: B is closed under union

suppose $x \in B$ and $y \in B$

$$X = \bigcup_{i \in I} \left(\bigcap_{j=1}^{k} A_{ij} \right)$$

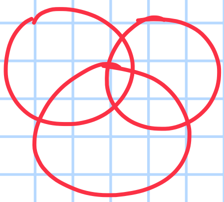
$x \in X$ then $\exists i$ s.t

$$Y = \bigcup_{m \in I} \left(\bigcap_{n=1}^{l} B_{mn} \right)$$

$x \in \bigcap_{j=1}^{k} A_{ij}$

or

$x \in A_{i_1}, A_{i_2} \ldots A_{i_k}$



$$X \cap Y = \bigcup_{i,m} \left(\bigcap_{j,n} (A_{ij} \cap B_{mn}) \right)$$

also $x \in Y$ then

$x \in B_{m_1}, B_{m_2}, \ldots B_{m_l}$

if $x \in A \cap B$ then

$x \in A_{i_1}, A_{i_2}, \ldots A_{i_k}$
and
$B_{m_1}, B_{m_2} \ldots B_{m_l}$

$\Rightarrow x \in \bigcap (A_{ij} \cap B_{mn})$

$\forall x \in X \cap Y \Rightarrow x \in \bigcup_{i,m} \left(\bigcap_{j,n} (A_{ij} \cap B_{mn}) \right)$

now, $x \in \bigcup_{i,m}\left(\bigcap_{j,n}(A_{ij} \cap B_{mn})\right)$

then $x \in \bigcap_{j,n}(A_{ij} \cap B_{mn})$

$\Rightarrow x \in X$ & $x \in Y$

$\Rightarrow x \in X \cap Y$

so $X \cap Y = \bigcup(\bigcap A_{ij} \cap B_{mn})$

1. $X \neq \phi$

$\mathcal{A} \subseteq 2^X$

$$\mathcal{A}' = \left\{ \bigcap_{A \in I}^{N} A \;\middle|\; \underset{N \geq 1}{I = \mathcal{A} \cup \{\phi\} \cup \{X\}} \right\}$$

$$\mathcal{B} = \left\{ \bigcup_{A \in I} A \;\middle|\; I \subseteq \mathcal{A}' \right\}$$

To prove : $\mathcal{B}$ is a topology

proof : as $\mathcal{B} = \left\{ \bigcup_{A \in I} A \;\middle|\; I \subseteq \mathcal{A}' \right\}$

① as $X \in \mathcal{A}' \Rightarrow X \in \mathcal{B}$

now, sim $\phi \in \mathcal{A}' \Rightarrow \phi \in \mathcal{B}$

② $S_1 \in \mathcal{B}$ and $S_2 \in \mathcal{B}$ then

$S_1 = \bigcup_{A \in I_1} A = \bigcup_j \bigcap_i A_{ij}$ $\quad A_{ij} \in \mathcal{A}$

$S_2 = \bigcup_{A \in I_2} A = \bigcup_{j'} \bigcap_{i'} B_{i'j'}$ $\quad B_{i'j'} \in \mathcal{A}$

now $S_1 \cap S_2 = \bigcup_j \left[\bigcap_i A_{ij} \cap B_{i'j'} \right]$ as $\bigcap A_{ij} \cap B_{ij} \in \mathcal{A}'$

$\Rightarrow \bigcup[\bigcap A_{ij} \cap B_{ij}] \in \mathcal{B}$

$\Rightarrow S_1 \cap S_2 \in \mathcal{B}$

③ $S_1 \cup S_2 = \underbrace{\Big[\bigcup \underbrace{\bigcap A_{ij}}_{\in \mathcal{A}'}\Big]}_{\in \mathcal{B}} \cup \underbrace{\Big[\bigcup \underbrace{\bigcap B_{i'j'}}_{\in \mathcal{A}'}\Big]}_{\in \mathcal{B}} \in \mathcal{B}$ is trivial

as $\mathcal{B} = \{\bigcup A\}$

By ①, ② and ③ $\mathcal{B}$ is a topology

2. (A) $X = \mathbb{R}$

$\mathcal{A} = \{[0,1]\}$

$\mathcal{A}' = \left\{ \bigcap_{A \in I}^{N} A \;\middle|\; I \subseteq [0,1] \cup \{\phi\} \cup \mathbb{R} \right\}$

$\mathcal{A}' = \{ [0,1], \phi, \mathbb{R} \}$

$B = \left\{ \bigcup_{A \in I'} A \;\middle|\; I' \subseteq \mathcal{A}' \right\}$

$B = \{ \mathbb{R}, \phi, [0,1] \}$

(B) $X = \mathbb{R}$

$\mathcal{A} = \{ [0,1], [3,4], [5,6) \}$

$\mathcal{A}' = \{ [0,1], [3,4], [5,6), \phi, \mathbb{R} \}$

$B = \{ \phi, \mathbb{R}, [0,1], [3,4], [5,6), [0,1] \cup [3,4],$
$[3,4] \cup [5,6), [0,1] \cup [5,6),$
$[0,1] \cup [3,4] \cup [5,6) \}$

(C) $X = \mathbb{R}^2$

$\mathcal{A} = \{ B((0,0), 1/n) \mid n \geq 1 \}$

$\mathcal{A}' = \{ \phi, \mathbb{R}, B((0,1), 1/n) \mid n \geq 1 \}$

$B = \{ \phi, \mathbb{R}, B((0,1), 1/n) \mid n \geq 1 \}$

(D) $X = \mathbb{R}$

$\mathcal{A} = \{$ set of all finite subsets of $\mathbb{R}^2 \}$

$\mathcal{A}' = \{ \phi, \mathbb{R}, \underset{\in \mathbb{R}}{\{\alpha\}}, \underset{\in \mathbb{R}}{\{\alpha, \beta\}}, \ldots \}$

$B = \{ \phi, \mathbb{R}, \{x_i\} \}$

$\{x_i\} = \{ \underset{\in \mathbb{R}}{x_1}, \ldots, \underset{\in \mathbb{R}}{x_i} \}$

for all $i \in \mathbb{N}$

(E) $X = \mathbb{R}^2$

$\mathcal{A} = \{$ set of all infinite subsets $\}$

$\mathcal{A}' = \{ \phi, \mathbb{R}, \underset{\forall \alpha \in \mathbb{R}}{\{\alpha\}},$ all inf subsets $\}$

$B = \{ \phi, \mathbb{R},$ all subsets $\} = 2^{\mathbb{R}}$

(F) $X = \mathbb{R}^2$, $\mathcal{A} = \{A \subset \mathbb{R}^2 \mid |\mathbb{R}^2 \setminus A| \leq 10\}$

$\mathcal{A}' = \{\phi, \mathbb{R}^2, A \subset \mathbb{R}^2 \mid |\mathbb{R}^2 \setminus A| \leq 10\}$

(G) $X = \mathbb{R}^2$, $A = \{A \subset \mathbb{R}^2 \mid |\mathbb{R}^2 \setminus A| < \infty\}$

$\mathcal{A}' = \{\phi, \mathbb{R}^2, A \subset \mathbb{R}^2 \mid |\mathbb{R}^2 \setminus A| < \infty\}$

(H) $\mathcal{A}' = \{\phi, \mathbb{R}^2, A \subset \mathbb{R}^2 \mid \mathbb{R}^2 \setminus A$ is countable $\}$

(3) $\tau = \{$open sets in $\mathbb{R}^n\}$

① as ϕ, $\mathbb{R}^n$ is open $\Rightarrow \phi, \mathbb{R}^n \in \tau$

② If $\bigcup A_i$ is also open as

$\forall x \in \bigcup A_i$, $\exists i$ s.t

$x \in A_i$

$\Rightarrow \exists \delta > 0$ s.t

$B_\delta(x) \subseteq A_i$

$\Rightarrow B_\delta(x) \subseteq \bigcup A_i$

③ $A \in \tau$

$B \in \tau$ then

$x \in A \cap B \Rightarrow x \in A$ and $x \in B$

$\Rightarrow \exists \delta_1$ s.t , $\exists \delta_2$ s.t

$B_{\delta_1}(x) \subseteq A$ $\quad B_{\delta_2}(x) \subseteq A$

$\Rightarrow \delta = \min\{\delta_1, \delta_2\}$

$B_\delta(x) \subseteq A$ and $B_\delta(x) \subseteq B$

$\Rightarrow B_\delta(x) \subseteq A \cap B$

(4) The above for n is general, and triangle inequality is not needed.

Tutorial-2 :

(1) (i) (X,d) $\qquad B_2=\{B(x,d), d\in\mathbb{Q}, x\in X\}$ $\qquad$ ① condition-1

$\tau \leftarrow$ metric $\qquad$ By density of $\mathbb{Q}$ inside $\mathbb{R}$ $\qquad X=\bigcup_{x\in X, r\in\mathbb{R}} B(x,r)$

$$B=\{B(p,r) \mid p\in X, r>0\}$$

$B_3=\{B(x,1/n)\}$ $\qquad \uparrow$ open Balls $\qquad$ ② $x\in B_2\cap B_3$ then $\underset{\in B}{B_1}\subseteq B_2\cap B_3$

as open balls consist open sets by defination

(iii) $\tau :=$ all subsets of $\mathbb{R}$ with countible complement

$\overset{\alpha}{\longleftrightarrow}$ $\forall\alpha\in\mathbb{R}$ $\in\tau$

$\{\mathbb{R}\setminus\{\alpha_1,\alpha_2,\ldots\}\} = $ Basis $\quad \forall \alpha_i\in\mathbb{R}$

as all sets will be open, and s.t its compliment has elements of cardinality of $\mathbb{N}$, so those sets $\in\tau$.

Also union of sets like this are

$\underset{\in\tau}{\mathbb{R}\setminus\{\underbrace{\ldots\ldots}_{\text{card } \mathbb{N}}\}}$ and $\mathbb{R}\setminus\{\underbrace{\ldots\ldots}_{n}\}$

(ii) τ; $\mathbb{R}$ with co-finite topology $\quad \mathcal{B}=\tau\setminus\{\mathbb{R}\setminus x_0\}$

$B_1, B_2\in\mathcal{B}$

$x\in B_1\cap B_2$

B_1, B_2 contain x_0 $\Rightarrow B_1\cap B_2$ also contain x_0

$x_0\notin B$ $2\neq\mathbb{R}\setminus\{x_0\}$ $\Rightarrow x_0\notin B_1\cap B_2$ $\exists x_1$ s.t $x_1\notin B_2$ so $x_0, x_1\notin B_1\cap B_2$ so $B_1\cap B_2\in\mathcal{B}$

$\therefore$ working

(iv) $(\mathbb{R},\tau)$

$$d(x,y)=\begin{cases}1; & x\neq y\\ 0; & x=y\end{cases}$$

same open balls

here open Ball

$$B(p,r)=\{r\in\mathbb{R} \mid |r-p|\leq r\}$$

for $r<1$

$$B(p,r)=\{p\}$$

here $\{p\}$ is open set so

$$\text{Basis}=\{\{p\} \mid \forall p\in\mathbb{R}\}$$

(2) for $1\leq p<\infty$

$\left(|x_1-y_1|^p+|x_2-y_2|^p\right)^{\frac{1}{p}}$ is a norm

① $|\alpha x|=|\alpha||x|$

② triagle inequality

③ positiviness

same for $\max(|x_1-y_1|, |x_2-y_2|)$ $\quad p=\infty$

property of norm:

$c_1\| \|_2 < \| \|_1 < c_2\| \|$

here this mean that say

τ_2 $d_2(x, y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$

τ_p here for τ_2 the basis is basis of open balls

now for τ_p

any open set in τ_p is s.t

$\forall x \in S \Rightarrow \exists r > 0$ s.t

$B_r(x) \subseteq S$

wrt d_p distance

but now if

$B_r(x)$ is s.t

$\exists r_1$ and r_2

$B_{r_1}(x) \subset B_r(x) \subseteq B_{r_2}(x)$

wrt d_2

$\Rightarrow B_{r_1}(x) \subseteq S$

$\Rightarrow$ every open set in τ_p can be written as union of open balls of τ_2

$\Rightarrow$ Basis = open balls wrt d_2

$\Rightarrow$ Basis τ_p = Basis τ_2

$\Rightarrow$ Basis of all topologies are same

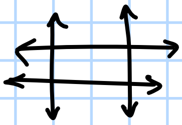
$\Rightarrow$ all topologies are same

Tutorial-3:

1. Problem 1 - tut-2

(i) (X,d) τ is a metric topology containg d.

$X = \{0,1\}$ with discrete $\{\phi, \{0\}, \{0,1\}, \{1\}\} = \tau$



Basis = {open retangles}

subbasis = {Rectangles with one pair of sides infinity}

$\}$ Basis $= \{\{0\},\{1\}\}$

sub-Basis = Basis

C = {Rectangles with one pair of sides infinity}

now $B = \{ \bigcap_N c \mid \forall c \}$

= {open retangles}

$\therefore$ B is basis

Def: let X be a set

B is a basis for some topology on X if

① $\forall x \in X, \exists U \in B$ s.t $x \in U$

② $x \in U_1 (\in B)$ & $x \in U_2 \in (B)$ then $\exists U_3 \in (B)$ s.t $x \in U_3 \subseteq U_1 \cap U_2$

(ii) $\tau :=$ all subsets of $\mathbb{R}$ with finite copliment

$C = \{ A \subseteq \mathbb{R} \mid |\mathbb{R} \setminus A| = 1 \}$

$\to$ A^c is singleton

$B = \{ \bigcap_N c \}$

$\to$ this is also finite

Def: Let X be set and $C \subseteq P(X)$

C is a subbasis for τ on X if union of all is X, topology = arbitrary union of finite intersions of C

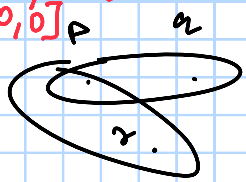
(iii) $\mathbb{R}$ co-countable topology

$C = \{ A \in \mathbb{R} \mid |A^c| = |\mathbb{N}| \}$ τ: all subsets of $\mathbb{R}$, coutable copliment $\leftarrow$ doubt

$A^c \subseteq (0,\infty)$ or $A^c \subseteq (-\infty, 0]$

$B = \{ x \subseteq \mathbb{R} \mid |\mathbb{R} \setminus x| = \infty \}$? Basis

(iv)



$\to$ set of all subsets s.t $|A| = 2$

$C = \{ A \subseteq \mathbb{R} \mid |A| = 2 \}$

as $\text{Basis} = \{ \cap A \}$

$\to$ singolton or $|A| = 2$

Tutorial-4 :

1. $A \subset X$
 ↳ subset of X

To find : A°, $\bar{A}$

A° = largest open set which lies inside A
$\in \tau$

$\bar{A}$ = smallest closed set containing A = A + all limit points of A
$(X \setminus \bar{A} \in \tau)$ (shown in class)

Limit points : if $p \in X$ is a limit point of $A \subset X$ then
$\forall V \in \tau$ (open)
s.t $p \in V \Rightarrow V \cap (A \setminus \{p\}) \neq \phi$

(A) $X = \mathbb{R}$, $(\mathbb{R}, \tau)$
↳ Normal euclidian dist

$A = \mathbb{Q}$ then

A° = largest open set lying inside $A = \phi$

as say $\exists x$ s.t $x \in A^\circ$
then
$\exists \delta > 0$ s.t
$B(x, \delta) \subseteq A$ as A° is open

but as $A^\circ \subseteq A \Rightarrow B(x, \delta) \subseteq A$

$\Rightarrow$ But ball will have points in $\mathbb{R} \setminus \mathbb{Q}$
so
$B(x, \delta) \subseteq A$ is a contradiction

$\therefore$ No such x exist

$\Rightarrow A^\circ = \phi$

now $\bar{A}$ = closure of A
$= \mathbb{R}$

as if $x \notin \bar{A}$ then

$\Rightarrow \mathbb{R} \setminus \bar{A}$ is open
so
$\exists \delta > 0$ s.t

$B(x, \delta) \subseteq \mathbb{R} \setminus \bar{A}$
but
as $B(x, \delta)$ contains point in $\mathbb{Q}$
and $\mathbb{Q}$ not in $\mathbb{R} \setminus \bar{A}$
this is a contradiction

$\Rightarrow \mathbb{R} \setminus \bar{A} = \phi$
$\Rightarrow \mathbb{R} = \bar{A}$

(B) $X = \mathbb{R}$
$\tau = \{\phi, \mathbb{Q}, \mathbb{R}\}$
$A = [0,1] \cap \mathbb{Q}$

A° = largest open set conteind in A

as $\phi \subseteq [0,1] \cap \mathbb{Q}$
but
$\mathbb{Q} \not\subseteq [0,1] \cap \mathbb{Q}$
& $\mathbb{R} \not\subseteq [0,1] \cap \mathbb{Q}$

$\therefore$ only $\phi \in \tau$ (open) in A
$\Rightarrow A^\circ = \phi$

now $\bar{A}$ = smallest closed set contaig A

as $\phi \in \tau \Rightarrow \phi$ is open
$\Rightarrow \mathbb{R} \setminus \phi$ is closed
$\Rightarrow \mathbb{R}$ is closed

now close sets in τ are

$$\{\mathbb{R} \setminus S \mid S \in \tau\} = \{\mathbb{R}\setminus\phi, \mathbb{R}\setminus\mathbb{R}, \mathbb{R}\setminus\mathbb{Q}\}$$
$$= \{\mathbb{R}, \phi, \mathbb{R}\setminus\mathbb{Q}\}$$

now $[0,1] \cap \mathbb{Q} \subseteq \mathbb{R}$
$[0,1] \cap \mathbb{Q} \not\subseteq \mathbb{R}\setminus\mathbb{Q}$ (trivial)
$[0,1] \cap \mathbb{Q} \not\subseteq \phi$ (trivial)

so $\bar{A} = \mathbb{R}$

(C) $X = M_2(\mathbb{R})$
$\longrightarrow$ set of all 2×2 matrices with entries in $\mathbb{R}$

$d : M_2 \times M_2 \longrightarrow [0, \infty]$

$$d(A,B) = \sup_{x \in \mathbb{R}^2,\ \|x\|=1} \|Ax - Bx\|$$

where $\|y\| = (y_1^2 + y_2^2)^{\frac{1}{2}}$

(i) <u>To prove</u>: (M_2, d) is a metric space

<u>proof</u>: ① $\underline{d(p,q) > 0}$, $\underline{p \neq q}$, $\forall p, q \in X$
② $d(p,p) = 0$, $\forall p \in X$
③ $d(p,q) = d(q,p)$
④ $d(p,q) \leq d(p,r) + d(r,q)$ $\forall p,q,r \in X$

we have to show this

①: $d(A,B)$ for $A \neq B$

$$d(A,B) = \sup_{x \in \mathbb{R}^2, \|x\|=1} \|Ax - Bx\|$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$Ax - Bx = (A-B)x$$

$$= \begin{bmatrix} a_{11}-b_{11} & a_{12}-b_{12} \\ a_{21}-b_{21} & a_{22}-b_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\neq 0$

as $A \neq B$ wlog $a_{11}-b_{11} \neq 0$

say $\underbrace{a_{11}-b_{11}}_{} = h_{11} \neq 0$

then $\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} h_{11}x_1 + h_{12}x_2 \\ h_{21}x_1 + h_{22}x_2 \end{bmatrix}$

$\|x\| = 1$ then

$x_1^2 + x_2^2 = 1$

now $\|(A-B)x\| = \left[(h_{11}x_1 + h_{12}x_2)^2 + (h_{21}x_1 + h_{22}x_2)^2\right]^{\frac{1}{2}}$

sup of the above

$\geq [(h_{11}(1) + h_{12}(0))^2 + (h_{21}(1) + h_{22}(0))^2]^{\frac{1}{2}}$

$\geq |h_{11}| > 0$

$\Rightarrow d(A,B) > 0$

② if $A = B$

$\Rightarrow \|Ax - Bx\| = 0$

$\Rightarrow \sup_{x \in \mathbb{R}^2, \|x\|=1} \|Ax - Bx\| = 0$

$\Rightarrow d(A,A) = 0$

③ $d(A,B) = d(B,A)$

as

$\|(A-B)x\| = \|-(A-B)x\|$

$= \|(B-A)x\|$

$\Rightarrow \sup_{x \in \mathbb{R}^2, \|x\|=1} \|(A-B)x\| = \sup_{x \in \mathbb{R}^2, \|x\|=1} \|(B-A)x\|$

④ $d(A,B) \leq d(A,C) + d(C,B)$

Proof:

as $\|Ax - Bx\| = \|Ax - Cx + Cx - Bx\|$

and $\|x+y\| \leq \|x\| + \|y\|$ (By triagle inequality)

$\Rightarrow \|(Ax - Cx) + (Cx - Bx)\|$

$\leq \|Ax - Cx\| + \|Cx - Bx\|$

$$\Rightarrow \sup_{\substack{x \in \mathbb{R}^2 \\ \|x\|=1}} \|Ax - Bx\| \leq \sup_{\substack{x \in \mathbb{R}^2 \\ \|x\|=1}} \|Ax - Cx\| + \sup_{\substack{x \in \mathbb{R}^2 \\ \|x\|=1}} \|Cx - Bx\|$$

so, (M_2, d) is a metric space by ①, ②, ③ and ④

(ii) first lets see how τ on (X, d) looks like:

open balls: $B_r(A) = \{B \in X \mid d(A,B) < r\}$

all open sets will be union of open balls for τ.
(By defination)

(A) $A = \{B \in M_2 \mid d(B, 0_{2\times 2}) \leq 1\}$

now

$$d(B, 0_{2\times2}) = \sup_{\substack{x \in \mathbb{R}^2 \\ \|x\|=1}} \|Bx\| \quad \text{as } 0_{2\times2}x = 0 \in \mathbb{R}^2$$

$$B_1(0_{2\times2}) = \{B \in M_2 \mid d(B, 0_{2\times2}) < 1\}$$

$\subseteq A$

and as this is a an open ball

$B_1(0_{2\times2}) \in \tau$

now to show that $A^\circ = B_1(0_{2\times2})$
we have to
show that
A° is biggest sum which is open

if $\forall x \in A^\circ$ s.t $x \notin B_1(0_{2\times2})$
then
as $A^\circ \subseteq A = \{B \mid d(B, 0_{2\times2}) \leq 1\}$

and $x \notin B_1(0_{2\times2})$

$\Rightarrow x \in \{B \mid d(B, 0_{2\times2}) = 1\}$

but if $x \in \{B \mid d(B, 0_{2\times2}) = 1\}$
then

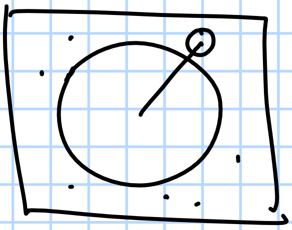
$\Rightarrow A^\circ = A$

but as A is closed, this is
not possible

so, $A^\circ = B_1(0_{2\times2})$

proof of A is closed:

$\forall\, B \in X \setminus A \Rightarrow d(B, 0_{2\times 2}) > 1$



now as $d(B, 0_{2\times2}) > 1$

$\delta_B = \dfrac{d(B, 0_{2\times 2})}{2}$

then $B_{\delta_B}(B) \subseteq X \setminus A$

this is because

$\forall\, B' \in B_{\delta_B}(B)$

$\Rightarrow d(B', B) < \delta_B = \dfrac{d(B, 0_{2\times2})}{2} > 0$

$\Rightarrow d(B', 0_{2\times2}) > d(B', B) + d(B, 0_{2\times 2})$

> 1

as $d(B', B) > 0$

$\Rightarrow B' \in X \setminus A$

$\therefore \forall\, B' \in B_{\delta_B}(B) \Rightarrow B' \in X \setminus A$

$\Rightarrow B_{\delta_B}(B) \subseteq X \setminus A$

$\Rightarrow A$ is closed

as A is closed $\Rightarrow \bar{A} = A$

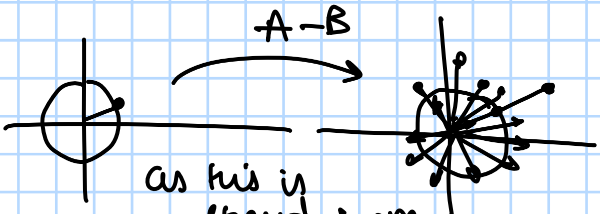
(B) $A = \{B \in M_2 \mid d(B, 0_{2\times2}) \leq 10\}$

same to above

$A^0 = \{B \in M_2 \mid d(B, 0_{2\times 2}) < 10\}$

$= B_{10}(0_{2\times2})$

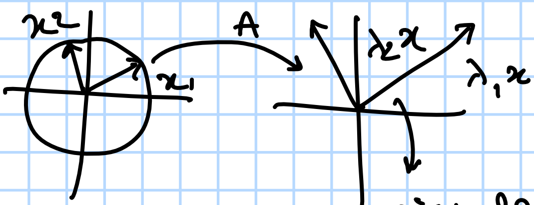
and $\bar{A} = A$ as A is closed



as this is shewed from its eigenvalues Basically

(C) $A = \{B \in M_2 \mid \det(B) > 0\}$

$d(B, 0_{2\times2}) = \sup\limits_{\theta \in [0, 2\pi)} \left\| \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\|$



$= \sup\limits_{\theta \in [0, 2\pi)} \left\| \begin{pmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \end{pmatrix} \right\|$

$= \sup\limits_{\theta \in [0, 2\pi)} \left(\lambda_1^2 \cos^2\theta + \lambda_2^2 \sin^2\theta \right)^{\frac{1}{2}}$

$= \|\lambda\| = \max\{\|\lambda_1\|, \|\lambda_2\|\} > 0$

eigvalue

$\det(B) > 0$

$$\Rightarrow \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} > 0$$

$$\Rightarrow b_{11}b_{22} - b_{12}b_{21} > 0$$

$$\Rightarrow \underbrace{P(b_{11}, b_{22}, b_{12}, b_{21})}_{\text{some polynomial with } b_{11}, b_{22}, b_{12}, b_{21} \text{ variables}} > 0$$

as polynomials are continious
this mean $\exists$ a ball around $(b_{11}, b_{22}, b_{12}, b_{21})$ s.t
all points inside the ball
make $\det(A) > 0$
and all these points are very close
so, $d(A,B) < \varepsilon$ this can be done for
any ε.

$\therefore \forall B \in A$, $\exists$ $\varepsilon > 0$ s.t

$B_\varepsilon(B) \subseteq A$

$\Rightarrow A$ is open $\Rightarrow A^\circ = A$, $\bar{A} = \{B \mid \det(B) \geq 0\}$

$A = \{B \in M_2 \mid tr(B) = 0\}$
$int\ A = \phi$
as if $B \in int A$
then
$tr(B) = 0$
also $B + \frac{1}{n} I \leftarrow$ dist from
B is $< \varepsilon$
for any epsilon $tr(B + \frac{1}{n}) \neq 0$
$d(B, B + \frac{1}{n} I) = 1/n \rightarrow B\ (B, 2/n) \rightarrow$ for all n not possible

(D) using (C) calculation, (D) is closed. $\bar{A} = A$, $A^\circ = \phi$

(E) $tr(B) = 0 \Rightarrow$ for $tr(B) \neq 0$
we have A

$$P'(a_{11}, a_{12}, a_{21}, a_{22}) = 0$$

some polynomial

$\Rightarrow A$ is closed
$\Rightarrow \bar{A} = A$ and
$A^\circ = \phi$

(F) $BB^T = B^TB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

the $b_{11}, b_{12}, b_{21}, b_{22}$
follow
rules

$\Rightarrow P(b_{11}, b_{12}, b_{21}, b_{22}) = 0$
$\Rightarrow A$ is closed

$\Rightarrow \bar{A} = A$, $A^\circ = \phi$ Boundry

(G) $A = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \middle| \underbrace{a_{11}^2 + a_{12}^2 + a_{21}^2}_{\text{Polynomial}} = a_{22} \right\}$

so $\bar{A} = A$, $A^\circ = \phi$

(2) $B_n = \begin{cases} X \setminus B_{n-1} ; & n \text{ is odd} \\ \overline{B_{n-1}} ; & n \text{ is even} \end{cases}$

c = compliment
K = closure

A	n
ϕ	2
Z	8
KcZ	6
KcKcZ	4

our 8 giving example

$B_0,\ B_1 = X\setminus B_0,\ B_2 = \overline{B_1} = \overline{X\setminus B_0},\ B_3 = X\setminus \overline{B_2}$ .....

now if $n = 2, 4, 6, \ldots$ then B_n is always closed

also $n = 3, 5, 7, \ldots$ ← B_n is always open

for $n=0$, $n=1$ → anything

B_0 (0)	$X\setminus B_0$ (1)	$\overline{X\setminus B_0}$ (2)	$X\setminus\overline{X\setminus B_0}$ (3)	$\overline{X\setminus\overline{X\setminus B_0}}$ (4)	$X\setminus(\overline{X\setminus\overline{X\setminus B_0}})$ .... (5)	6
$\{\alpha\}$	$\mathbb{R}\setminus\{\alpha\}$	$\mathbb{R}$	ϕ	ϕ	$\mathbb{R}$	$\mathbb{R}$
$(0,1)$	$\mathbb{R}\setminus(0,1)$	$\mathbb{R}\setminus(0,1)$	$(0,1)$	$[0,1]$	$\mathbb{R}\setminus[0,1]$	$\mathbb{R}\setminus(0,1)$
$[0,1]$	$\mathbb{R}\setminus[0,1]$	$\mathbb{R}\setminus(0,1)$	$(0,1)$	$[0,1]$	$\mathbb{R}\setminus[0,1]$	$\mathbb{R}\setminus(0,1)$
$(0,1]$	$\mathbb{R}\setminus(0,1]$	$\mathbb{R}\setminus(0,1)$	$(0,1)$	$[0,1]$	$\mathbb{R}\setminus[0,1]$	$\mathbb{R}\setminus(0,1)$
$[0,1)$	$\mathbb{R}\setminus[0,1)$	$\mathbb{R}\setminus(0,1)$	$(0,1)$	$[0,1]$	$\mathbb{R}\setminus[0,1]$	$\mathbb{R}\setminus(0,1)$
$(-\infty,0]$	$\mathbb{R}\setminus(-\infty,0]$	$\mathbb{R}\setminus(-\infty,0)$	$(-\infty,0)$	$(-\infty,0]$	$\mathbb{R}\setminus(-\infty,0]$	$\mathbb{R}\setminus(-\infty,0)$
$(-\infty,0)$	$\mathbb{R}\setminus(-\infty,0)$	$\mathbb{R}\setminus(-\infty,0)$	$(-\infty,0)$	$(-\infty,0]$	.	.

__claim__ : $B_3 = (B_0^\circ)$ → biggest open set contained in B_0

__proof__ : if B_3 is __not__ the biggest open set contained in B_0

$B_3 = X\setminus(\overline{X\setminus B_0})$

then $\exists\, x \in B_0\setminus B_3$ s.t $\exists\, \delta > 0$
$B(x,\delta) \subseteq B_0$

now as $x \in B_0\setminus B_3$ $\quad B_3 = X\setminus(\overline{X\setminus B_0})$

$x \in B_0$ and as $B(x,\delta)\subseteq B_0$
x is an interior point

$\Rightarrow x \notin X\setminus B_0$

$\Rightarrow x \notin \overline{X\setminus B_0}$

$\Rightarrow x \in X\setminus(\overline{X\setminus B_0})$

$\Rightarrow x \in B_3$ this is a contradiction

so $(B_0)^\circ = B_3$

5 or 7 not possible as if 5 then:
if 5 then

$c(KcKcA) = \{A, cA, KcA, cKcA, KcKcA\}$

Not possible

$\Rightarrow cKcKcA \in \{A, cA, KcA, cKcA\}$

$\downarrow c$

$ccKcKcA = KcKcA \in \{cA, A, cKcA, KcA\}$

but $KcKcA \notin \{A, cA, KcA, cKcA\}$ → Not possible claim : $B_4 = \overline{A^o}$, $B_7 = (\overline{A^o})^o$, $B_8 = \overline{(\overline{A^o})^o}$, $B_{11} = (\overline{(\overline{A^o})^o})^o \dots$

proof : as $B_3 = A^o$

$B_4 = \overline{A^o}$ — $0 = A_1 = \overline{A^o}$

$B_5 = X \setminus \overline{A^o}$ — 1 .

$B_6 = \overline{X \setminus \overline{A^o}}$ — 2 .

$B_7 = X \setminus \overline{X \setminus \overline{A^o}}$ — $3 = A_1^o$ (from above)

$= (\overline{A^o})^o$ claim

claim : $(\overline{A^o}) = (\overline{(\overline{A^o})^o})$ for $A \subset \mathbb{R}$

proof : $(\overline{A^o}) \supseteq (\overline{A^o})^o$

$\Rightarrow (\overline{\overline{A^o}}) \supseteq \overline{(\overline{A^o})^o}$

$\Rightarrow \overline{(\overline{A^o})^o} \subseteq (\overline{A^o})$ —— ①

and now $\forall x \in \overline{A^o}$

$B \subseteq C \Rightarrow B^o \subseteq C^o$
$\overline{B} \subseteq \overline{C}$

now $\overline{A^o} = \overline{(A^o)^o} \subseteq \overline{(\overline{A^o})^o}$

as $B = A^o$
then $B^o \subseteq B \subseteq \overline{B}$

$\Rightarrow (A^o)^o \subseteq A^o \subseteq \overline{A^o}$

$\Rightarrow A^o \subseteq \overline{A^o}$

$\Rightarrow A^o \subseteq (\overline{A^o})^o$

$\Rightarrow \overline{A^o} \subseteq \overline{(\overline{A^o})^o}$

so $\overline{A^o} = \overline{(\overline{A^o})^o}$

now $B_4, B_5, B_6, B_7, \underset{\underset{B_4}{\parallel}}{B_8}, \dots\dots$

Repeats

$\therefore$ This can atmost have 8, $\{B_0, \dots B_8\}$

show : $kckckckc = kckc$
$\text{int}(A) = A^o$

$kc = ci$

as $\overline{A^c} = (A^o)^c$ ← using set theory

then $kckc = kc(ci)$
$kckc = ki$
$(kckc)(kckc) = (ki)(ki)$

to show : $(ki)(ki) = (ki)$

To show: $(\overline{A^o}) = (\overline{(\overline{A^o})^o})$

$\overline{(A^o)^o} \subseteq \overline{A^o}$

$\Rightarrow \overline{(\overline{A^o})^o} \subseteq \overline{A^o}$

also

$A^o \subseteq (\overline{A^o})$
$\Rightarrow A^o \subseteq (\overline{A^o})^o$
$\Rightarrow \overline{A^o} \subseteq (\overline{(\overline{A^o})^o})$

so $\overline{A^o} = (\overline{(\overline{A^o})^o})$

$\Rightarrow (ki)(ki) = (ki)$

$\{A, cA, kcA, ckcA, kckcA, ckckcA, kckckcA, ckckckcA\}$ distinct then repeats

(A) $n=1$ not possible as for this

$B_0 = B_1 = X \setminus B_0$

$B_0 = X \setminus B_0$ ✱

$n=2$ possible for $B_0 = \phi$
$B_1 = \mathbb{R}$
$B_2 = \mathbb{R}$
$B_3 = \phi$
$B_4 = \phi$
$\vdots$

so $n=2$ $\{\mathbb{R}, \phi\}$

$n=3$ not possible

as $B_0, X\setminus B_0, \overline{X\setminus B_0}, X\setminus\overline{X\setminus B_0}, \overline{X\setminus\overline{X\setminus B_0}}$

$0 \quad 1 \quad 2 \quad B_0^\circ \quad \overline{B_0^\circ}$

$3 \quad 4$

$B_0, X\setminus B_0, \overline{X\setminus B_0}, B_0^\circ, \ldots$

claim : $|B|$ can never be odd

proof : if $|B|$ is odd:
say $B_0, B_1 \ldots \ldots B_{2n}$

$\{B_0, B_1, B_2, B_3, B_4, B_5, B_6, B_7\}$

$B \; X\setminus B \; (\overline{X\setminus B}) \; B^\circ \; \overline{B^\circ} \; (\overline{X\setminus B})^\circ \; \cancel{(\overline{X\setminus B})} (\overline{B^\circ})^\circ$

if this is same
$X\setminus B$ is closed $\Rightarrow B$ is open

if we want to remove B_7 then $(\overline{B^\circ})^\circ = B^\circ$

$\overline{A} \qquad (\overline{A})^\circ$

$\overline{A} = \overline{(\overline{A})^\circ}$

$A \subseteq \overline{A} \Rightarrow (\overline{A})^\circ \subseteq (\overline{A})$

$\Rightarrow \overline{((\overline{A})^\circ)} \subseteq (\overline{A})$

now $\overline{A} \subseteq \overline{((\overline{A})^\circ)}$ in $\mathbb{R}$ for this

$A^\circ \subseteq \overline{A} \Rightarrow A^\circ \subseteq (\overline{A})^\circ$

$\Rightarrow A^\circ \subseteq (\overline{\overline{A}})^\circ$

now in $\mathbb{R}$, if x is a limit point of A then $x \in \overline{A} \Rightarrow x$ lies on body of $(\overline{A})^\circ$
$\Rightarrow x \in \overline{(\overline{A})^\circ}$

for $\mathbb{R}$ $\quad \overline{A} = (\overline{(\overline{A})^\circ})$

now,

B_0	B_1	$\cancel{B_2}$	$\cancel{B_3}$	$\cancel{B_4}$	$\cancel{B_5}$	$\cancel{B_6}$	$\cancel{B_7}$	B_8	B_9
B	$X\setminus B$	$\overline{X\setminus B}$	B°	$\overline{B^\circ}$	$X\setminus\overline{B^\circ}$	$\overline{X\setminus\overline{B^\circ}}$	$(\overline{B^\circ})^\circ$	$\overline{B^\circ}$	$\dots$
B	$X\setminus B$	$\overline{X\setminus B}$	B°	$\overline{B^\circ}$	$(\overline{X\setminus B})^\circ$				

Rep

$(-1,2$

$(-\infty,-1]\cup[2,\infty)$

$[-\infty,-1]\cup\{1\}\cup[2,\infty)$ $\quad (-1,1)\cup(1,2)$

$[-1,1)\cup(1,2)$ $\quad (-\infty,-1)\cup(2,\infty)$

$[-1,2]$

$(-\infty,-1)\cup\{1\}\cup[2,\infty)$

8 ✓
4 ✓
6 ✓
2 ✓
0 ✓ $\quad$ 1 ✗ $\quad$ now 3, 5, 7 — 3 cases left

if $B_7 = B_3$

$B^\circ = (\overline{B^\circ})^\circ$

then

$\underbrace{X\setminus B^\circ}_{\text{open}} = \underbrace{\overline{X\setminus B}}_{B_2} = (X\setminus\overline{B^\circ}) = \underbrace{\overline{X\setminus\overline{B^\circ}}}_{B_6}$

closed $\quad$ closed

if B_5 repeates : $(\overline{X\setminus B})^\circ = (X\setminus B)$

$\Rightarrow$ $X\setminus B$ is open

To show: $\quad B$ is closed

$B = \overline{B^\circ}$

$B^\circ \subseteq B \Rightarrow \overline{B^\circ} \subseteq B$ —— ①

$\text{int } B = B^\circ$

int B + limit point of $B = \overline{B^\circ} = B$

so if $B_5 = B_1$

$\Rightarrow B_4 = B_0$

if $B_3 = B_0$

$\Rightarrow B^\circ = B$

$\Rightarrow B$ is open

$\Rightarrow \overline{X\setminus B} = X\setminus B$

$\Rightarrow B_1 = B_2$

$\therefore$ 3, 5, 7 not possible

$\therefore$ 0 ✓, 1 ✗, 2 ✓, 3 ✗, 4 ✓, 5 ✗, 6 ✓, 7 ✗, 8 ✓, 9 ✗, 10 ✗, ...

(B) NO, proof is above

Tutorial-5 :

17.1 To prove : $\mathcal{C} \rightarrow$ arbitrary subsets in X

① $\phi \in \mathcal{C}$, $X \in \mathcal{C}$

② $\bigcup_{\alpha \in I} C_\alpha \in \mathcal{C}$ $\forall C_\alpha \in \mathcal{C}$

③ $\bigcap_{i=1}^{n} C_i \in \mathcal{C}$ $\forall C_i \in \mathcal{C}$

① finite union

② arbitrary intersection

$\tau = \{X - c \mid \forall c \in \mathcal{C}\}$ is a topology

proof: as $\phi \in \mathcal{C}$ for $c = \phi$

$\Rightarrow X \in \tau$

and $X \in \mathcal{C}$ for $c = X$

$\Rightarrow \phi \in \tau$

so ① $\phi, X \in \tau$ is true

for $A_\alpha \in \tau$ $\forall \alpha \in I \leftarrow$ arbitary collection

$\Rightarrow (A_\alpha)^c \in \mathcal{C}$ $\forall \alpha \in I$

$\Rightarrow \bigcap_{\alpha \in I} (A_\alpha)^c \in \mathcal{C}$ [By defination]

$\Rightarrow \left(\bigcup_{\alpha \in I} A_\alpha\right)^c \in \mathcal{C}$ [By demorgan's rule]

$\Rightarrow \bigcup_{\alpha \in I} A_\alpha \in \tau$ —— ②

now, $A_i \in \tau$ for some $N \in \mathbb{N}$ s.t $i \in \{1, 2 \ldots N\}$

then

$\Rightarrow (A_i)^c \in \mathcal{C}$ $\forall i \in \{1, 2 \ldots N\}$

$\Rightarrow \bigcup_{i=1}^{N} (A_i)^c \in \mathcal{C}$ [By defination]

$\Rightarrow \left(\bigcap_{i=1}^{N} A_i\right)^c \in \mathcal{C}$ [By demorgan's law]

$\Rightarrow \bigcap_{i=1}^{N} A_i \in \tau$ —— ③

from ①, ② and ③ $\Rightarrow \tau$ is a topology

17.2 To prove : A is closed in Y and Y is closed in X then A is closed in X.

proof: A being closed in Y means that

$\exists B \leftarrow$ closed in topology on X s.t $A = B \cap Y$

and Y is closed in X means

$X \setminus Y \in \tau$

now as $A = B \cap Y$
and $X \setminus Y \in \tau$

$$X - A = X - B \cap Y$$
$$= (B \cap Y)^c$$
$$= \underset{\in \tau}{(B^c)} \cup \underset{\in \tau}{(Y)^c} \Rightarrow X \setminus A \in \tau$$
$\Rightarrow A$ is closed in X

17.3 given: A is closed in X $(A^c \in \tau_X)$
B is closed in Y $(B^c \in \tau_Y)$

To prove: $A \times B$ is closed in $X \times Y$

proof: If we can show
$X \times Y - A \times B \in \tau$ → product topology
then we are done

$\forall (x, y) \in (X \times Y - A \times B)$
$\Leftrightarrow (x, y) \in X \times Y$ and
$(x, y) \notin A \times B$
$\Leftrightarrow (x \in X \wedge y \in Y) \wedge \sim (x \in A \wedge y \in B)$
$\Leftrightarrow (x \in X \wedge y \in Y) \wedge (x \notin A \vee y \notin B)$
$\Leftrightarrow (x \in X \wedge y \in Y \wedge x \notin A) \vee$
$(x \in X \wedge y \in Y \wedge y \notin B)$
$\Leftrightarrow (x \in X \setminus A \wedge y \in Y) \vee (x \in X \wedge y \in Y \setminus B)$
$\Leftrightarrow \forall (x, y) \in (X \setminus A \times Y) \cup (X \times Y \setminus B)$

so $X \times Y - A \times B = (X \setminus A \times Y) \cup X \times Y \setminus B$

now as $X \setminus A \in \tau_X$
$Y \in \tau_Y$ (trivial)
$\Rightarrow X \setminus A \times Y \in \tau$ —— ①

and $X \in \tau_X$
$Y \setminus B \in \tau_Y \Rightarrow X \times Y \setminus B \in \tau$ —— ②

from ①, ② $\Rightarrow (X \setminus A \times Y) \cup (X \times Y \setminus B) \in \tau$

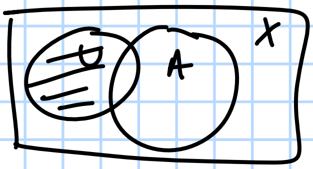
17.4 <u>given</u> : $U \in \tau_X$

$X \setminus A \in \tau_X$

<u>To prove</u> : $U - A \in \tau_X$

$X \setminus (A - U) \in \tau_X$

<u>proof</u> : as $U \in \tau_X$ $\quad X \setminus A \in \tau_X$



$U \cap (X \setminus A) = U - A$

as

$\forall x \in U - A, \; x \in U \wedge x \notin A$

$\Leftrightarrow x \in U \wedge x \in X \setminus A$

$\Leftrightarrow \forall x \in U \cap (X \setminus A)$

as τ_X is topology where

$U \in \tau_X, \; X \setminus A \in \tau_X$

$\Rightarrow U \cap (X \setminus A) \in \tau_X$

now $\quad A - U = A \cap (X \setminus U)$

now if we can show

intersection of two closed sets is closed in a topology

then we are done

A, B closed

$\Rightarrow A^c \in \tau, \; B^c \in \tau$

$\Rightarrow A^c \cup B^c \in \tau$

$\Rightarrow (A \cap B)^c \in \tau$

$\Rightarrow A \cap B$ is closed

so $A \cap (X \setminus U)$ is closed

one more useful approach :

<u>To prove</u> : $A - (B - C) = (A - B) \cup (A \cap C)$

<u>proof</u> : $\forall x \in A - (B - C)$

$\Leftrightarrow x \in A \wedge x \notin (B - C)$

$\Leftrightarrow x \in A \wedge \sim(x \in B \wedge x \notin C)$

$\Leftrightarrow x \in A \wedge (x \notin B \vee x \in C)$

$\Leftrightarrow (x \in A \wedge x \notin B) \vee (x \in A \wedge x \in C)$

$\Leftrightarrow \forall x \in (A - B) \cup (A \cap C)$

so, $U - A$

$\rightarrow$ $X - A$ is open say B

$X - A = B \Rightarrow A = X - B$

$U - A = U - (X - B) = (U - X) \cup (U \cap B) = \phi \cup (U \cap B) = U \cap B$

as $U \subseteq X$

$X-(A-U)=(X-A)\cup(X\cap U)$
$=(X-A)\cup(U)$
$=(B)\cup(U)\in\mathcal{T}$

17.6 (a) <u>To prove</u> : $A\subset B$ then $\bar{A}\subset\bar{B}$

<u>proof</u> : $\bar{A}=A\cup A'$

$\hookrightarrow$ all the limit points of A

$A\subseteq B$

$\bar{B}=B\cup B'\Rightarrow A\subseteq B\cup B'$

$A\subseteq\bar{B}$ — ①

To show now $A'\subseteq\bar{B}$

$\forall x\in A'$, we have $\exists U\in\mathcal{T}$ s.t

$U\cap(A-\{x\})\neq\phi$

$\Rightarrow\exists y$ s.t

$y\in A$ and $y\in U$

$\Rightarrow y\in A\subseteq B$ and $y\in U$

$\Rightarrow y\in B\cap U$

$\Rightarrow\exists U\in\mathcal{T}$ s.t $U\cap(B-\{x\})\neq\phi$

$\Rightarrow x\in B'$

$\Rightarrow x\in B\cup B'$

$\Rightarrow x\in\bar{B}$

so $A\subseteq\bar{B}$ and $A'\subseteq\bar{B}$

$\Rightarrow\bar{A}\subseteq\bar{B}$

(b) <u>To prove</u>: $\overline{A\cup B}=\bar{A}\cup\bar{B}$

<u>proof</u> : $\forall x\in\overline{A\cup B}$

$\Rightarrow x\in A\cup B$ or

$x\in(A\cup B)'$

if $x\in A\cup B$ then

as $A\subseteq\bar{A}$

$\Rightarrow A\cup B\subseteq\bar{A}\cup\bar{B}$

$\Rightarrow x\in\bar{A}\cup\bar{B}$

if $x\in(A\cup B)'$ then

$\exists U\in\mathcal{T}_x$ s.t

$U\cap(A\cup B-\{x\})\neq\phi$

$\exists y\neq x$

s.t $y\in U\cap(A\cup B)$

$\Rightarrow y\in U$ and $y\in A$ or B

$\Rightarrow x \in A'$ or $x \in B'$

$\Rightarrow x \in \bar{A} \cup \bar{B}$

so $\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$ —— ①

now $\forall\, x \in \bar{A} \cup \bar{B}$

$\Rightarrow x \in \bar{A}$ or $x \in \bar{B}$

$\Rightarrow x \in A$ or A', B or B'

if $x \in A \Rightarrow x \in \underline{A \cup B}$

$\Rightarrow x \in \overline{A \cup B}$

same for B

if $x \in A' \Rightarrow \exists\, V \in \tau_x$ s.t

$V \cap [A - \{x\}] \neq \phi$

$\Rightarrow y \in V \cap A \quad y \neq x$

$\Rightarrow y \in V$ and $y \in A$

$\Rightarrow y \in V$ and $y \in A \cup B$

$\Rightarrow x \in (A \cup B)'$

$\Rightarrow x \in (\overline{A \cup B})$

Same for B'

$\bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$ —— ②

from ①, ② $\Rightarrow \bar{A} \cup \bar{B} = \overline{A \cup B}$

(c) To prove: $\cup \bar{A}_\alpha \subseteq \overline{\cup A_\alpha}$

proof : $\forall\, x \in \cup \bar{A}_\alpha$

$\Rightarrow x \in \bar{A}_\beta$ for some β

$\Rightarrow x \in A_\beta \cup (A_\beta)'$

limit points of A_β

$\Rightarrow (x \in A_\beta) \vee (x \in A'_\beta)$

now if $x \in A_\beta$ then

$x \in \cup A_\alpha \Rightarrow x \in \overline{\cup A_\alpha}$

if $x \in A'_\beta$ then, $\exists\, V \in \tau_x$ s.t

$V \cap (A_\beta - \{x\}) \neq \phi$

$\Rightarrow \exists\, y \neq x$ s.t $y \in V \cap A_\beta$

$\Rightarrow y \in U \cap [UA_B]$

$\Rightarrow x \in (UA_B)'$

$\Rightarrow x \in \overline{UA_B}$

$\therefore \forall x \in U\overline{A_\alpha} \Rightarrow x \in \overline{UA_\alpha}$

so $U\overline{A_\alpha} \subseteq \overline{UA_\alpha}$

let $A_n = (\frac{1}{n}, 2]$

$n \in \mathbb{N}$ then

$A_1 = (1,2]$ $A_2 = (\frac{1}{2}, 2]$ ....

$\bigcup_{\alpha \in \mathbb{N}} A_\alpha = (0,2]$

$\overline{\bigcup_{\alpha \in \mathbb{N}} A_\alpha} = [0,2]$

now $\overline{A_n} = [\frac{1}{n}, 2]$

$\bigcup_{n \in \mathbb{N}} \overline{A_n} = [1,2] \cup [\frac{1}{2},2] \cup [\frac{1}{3},2] \cdots$

$= (0,2]$

as $(0,2] \neq [0,2]$

here equality is <u>not</u> holding

17.7 $\overline{UA_\alpha} \subseteq U\overline{A_\alpha}$
is <u>not</u> true as

$\{A_\alpha\}$ is a collection

$x \in \overline{UA_\alpha}$ then every neighbourhood of x intersects UA_α

$\Rightarrow U \cap (UA_\alpha) \neq \phi$

$\Rightarrow U \cap (A_\alpha) \neq \phi$

$\uparrow$ some A_α

$\Rightarrow x \in \overline{A_\alpha}$

the issue is that just because some U is s.t $U \cap (A_\alpha) \neq \phi$ this does not garantee that other neigbourhood $U \cap (A_\alpha) \neq \phi$

so not true for all U that they will intersect same A_α

so, this is <u>not</u> true

17.8

(a) $\overline{A \cap B} = \bar{A} \cap \bar{B}$ or $A \cap B \subseteq A \Rightarrow \overline{A \cap B} \subseteq \bar{A}$

$A \cap B \subseteq B \Rightarrow \overline{A \cap B} \subseteq \bar{B}$

so

$\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$

$\forall x \in \overline{A \cap B}$

$\Rightarrow x \in (A \cap B) \cup (A \cap B)'$

$\Rightarrow x \in (A \cap B)$ or $x \in (A \cap B)'$

$\Rightarrow x \in \bar{A} \cap \bar{B}$ or $x \in (A \cap B)'$

for $x \in (A \cap B)' \Rightarrow \exists U \in \tau_x$ s.t

$U \cap (A \cap B - \{x\}) \neq \phi$

$\exists y \neq x$ s.t

$y \in U \cap (A \cap B)$

$\Rightarrow y \in U \cap A$ and $y \in U \cap B$

$\Rightarrow x \in (A)'$ and $x \in (B)'$

$\Rightarrow x \in \bar{A}$ and $x \in \bar{B}$

$\Rightarrow x \in \bar{A} \cap \bar{B}$

so $x \in \overline{A \cap B} \Rightarrow x \in \bar{A} \cap \bar{B}$

$\Rightarrow \overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$

now $\overline{A \cap B} = \bar{A} \cap \bar{B}$

<u>does</u> <u>not</u> hold always

$A = (0,1)$

$B = (1,2)$

$A \cap B = \phi \Rightarrow \overline{A \cap B} = \phi$

$\bar{A} \cap \bar{B} = [0,1] \cap [1,2] = \{1\}$

as $\phi \neq \{1\}$ equality

<u>does</u> not hold always

(b) $\overline{\cap A_\alpha} = \cap \bar{A}_\alpha$ Same as previous

equality does not hold from prev example

$\forall x \in \overline{\cap A_\alpha} \Rightarrow x \in (\cap A_\alpha) \cup (\cap A_\alpha)'$

if $x \in \cap A_\alpha$ then

$\Rightarrow x \in \cap \bar{A}_\alpha \left(\forall \alpha, x \in A_\alpha \subseteq \bar{A}_\alpha \right)$

if $x \in (\cap A_\alpha)'$

then $\forall U \in \tau_x$ s.t $x \in U$ we have

$$U \cap (\cap A_\alpha - \{x\}) \neq \phi$$

$\exists\ y \in U \cap (\cap A_\alpha)$ s.t $y \neq x$

$\Rightarrow y \in U \cap A_\alpha \ \forall \alpha \in I$

$\Rightarrow x \in (A_\alpha)' \ \forall \ \alpha \in I$

$\Rightarrow x \in \overline{A_\alpha} \ \ \forall \ \alpha \in I$

$\Rightarrow x \in \cap \overline{A_\alpha}$

so $\overline{\cap A_\alpha} \subseteq \cap \overline{A_\alpha}$

(C) $\overline{A - B} = \overline{A} - \overline{B}$

here equality is <u>not</u> true as

$A = (0, 1]$

$B = (0, 1)$

$A - B = \{1\} \Rightarrow \overline{A - B} = \{1\}$

$\overline{A} = [0, 1]$

$\overline{B} = [0, 1]$ & $\overline{A} - \overline{B} = \phi$

so equality not true

but, $\overline{A} - \overline{B} \subseteq \overline{A - B}$

$\forall x \in \overline{A} - \overline{B}$, then

$\Rightarrow x \in \overline{A} \ \wedge x \notin \overline{B}$

$\Rightarrow x \in \overline{A}, \ x \notin B, x \notin B'$

$\Rightarrow x \in A$ or A', $x \notin B$, $x \notin B'$

if $x \in A \Rightarrow x \in A - B \Rightarrow x \in \overline{A - B}$ —— ①

now if $x \in A'$ we have to show $x \in \overline{A - B}$

$x \in A'$, $x \notin B$, $x \notin B'$

as $\forall \ U \in \tau$ s.t $x \in U$, we have

$$U \cap (A - \{x\}) \neq \phi$$

as $x \notin B$ and $x \notin B'$ for some U, $U \cap (B - \{x\}) = \phi$

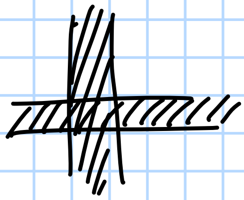
$$U \cap (A - \{x\}) \neq \phi$$
$$U \cap (B) = \phi$$

then

$$U \cap (A - \{x\} - B) \neq \phi$$
$$\Rightarrow x \in (A - B)'$$
$$\Rightarrow x \in \overline{A - B}$$
$$\Rightarrow \bar{A} - \bar{B} \subseteq \overline{A - B}$$

17.9 $A \subseteq X$, $B \subseteq Y$

<u>To prove</u> : $\overline{A \times B} = \bar{A} \times \bar{B}$



<u>proof</u> :

$\forall (x, y) \in \overline{A \times B}$, we have

$\forall\ W \times V$ s.t $x \in W$, $y \in V$

$\exists (w, v) \in W \times V$

and $(w, v) \in A \times B$ where $(w, v) \neq (x, y)$

as $w \in W$ and $w \in A$

$\Rightarrow x \in \bar{A}$ as this is for any arbitrary W

similarly $y \in \bar{B}$ as this is for any arbitrary V

$\Rightarrow (x, y) \in \bar{A} \times \bar{B}$

so $\overline{A \times B} \subseteq \bar{A} \times \bar{B}$

now $\forall (x, y) \in \bar{A} \times \bar{B}$

$\Rightarrow x \in \bar{A}$ and $y \in \bar{B}$

so $\forall\ W \in \tau_x$ s.t $x \in W$

$\Rightarrow W \cap (A - \{x\}) \neq \phi$

so $\exists\ x'$ s.t $x' \in W \cap A$ and $x' \neq x$

similarly $\forall\ V$ we have $y' \in V \cap B$ and $y' \neq y$

now, this means that all open sets in $X \times Y$ s.t $(x, y) \in W \times V$ we have (x', y') s.t $(x', y') \neq (x, y)$

$(x', y') \in$ intesection of $W \times V$ and $A \times B$

$\Rightarrow (x, y) \in \overline{A \times B}$

$\Rightarrow \bar{A} \times \bar{B} \subseteq \overline{A \times B}$

17.11 let X, Y be Hausdoff

for $x_1, x_2 \in X$ s.t $x_1 \neq x_2$

$\exists\, U_1, U_2$ neighborhood of x_1, x_2 s.t

$U_1 \cap U_2 = \phi$

now let $(x_1, y_1) \in X \times Y$
$(x_2, y_2) \in X \times Y$

if $(x_1, y_1) \neq (x_2, y_2)$ then

$x_1 \neq x_2 \vee y_1 \neq y_2$

$\Rightarrow \exists\, U_1, U_2$ s.t $U_1 \cap U_2 = \phi$ $\vee \exists\, V_1, V_2$ s.t $V_1 \cap V_2 = \phi$

$\Rightarrow U_1 \times Y$ and $U_2 \times Y$ is s.t $(x_1, y_1) \in U_1 \times Y$
$(x_2, y_2) \in U_2 \times Y$
and as $x_1 \in U_1, x_2 \notin U_2$

$(U_1 \times Y)$ and $(U_2 \times Y)$ are disjoint

similarly for y_1 and y_2 same

so, for $(x_1, y_1) \neq (x_2, y_2) \in X \times Y$

$\exists\, U_1 \times U_2$ and $V_1 \times V_2$ disjoint

$\Rightarrow X \times Y$ is Hausdoff

17.12 <u>To prove</u>: subspace of Hausdorff is Hausdorff

<u>proof</u>: $X =$ Hausdoff and $Y \subseteq X$ $\to$ subspace of X

$\forall\, x_1, x_2 \in Y$ s.t

$x_1 \neq x_2$ as $x_1, x_2 \in X$

$\exists\, U_1, U_2$ s.t

$x_1 \in U_1$ and $x_2 \in U_2$
and
$U_1 \cap U_2 = \phi$

now as $x_1 \in Y \Rightarrow x_1 \in U_1 \cap Y$
$x_2 \in Y \Rightarrow x_2 \in U_2 \cap Y$

and $U_1 \cap Y$, $U_2 \cap Y$ is open in Y

$U_2 \cap Y$ is open

for $x_1 \in U_1 \cap Y$
$x_2 \in U_2 \cap Y$
as
$U_1 \cap U_2 = \phi$
$\Rightarrow (U_1 \cap Y) \cap (U_2 \cap Y) = \phi$

so, $\forall\, x_1, x_2 \in Y$ s.t
$x_1 \neq x_2$

$\exists\, V_1 = U_1 \cap Y$
$V_2 = U_2 \cap Y$
open in Y s.t

$x_1 \in V_1$, $x_2 \in V_2$ and

$V_1 \cap V_2 = \phi$

so Y is Hausdorff

17.13 To prove: X is Hausdorff $\Leftrightarrow \Delta = \{x \times x \mid \forall x \in X\}$ is closed in $X \times X$
diagonal

proof: $(\Rightarrow)$ if X is Hausdorff then for $x \times y \in X \times X$
where $x \times y \notin \Delta$

then $x \neq y$
so, $\exists\, U$ and V s.t
$x \in U$
$y \in V$
and $U, V \in \mathcal{T}_X$
where
$U \cap V = \phi$

now $U \times V$ is a basis element for $X \times X$
so it is open.

$\forall\, (x,y) \notin \Delta$, $\exists\, U, V$ s.t

$(x,y) \in U \times V$, $U \cap V = \phi$ and $U \times V \in \mathcal{T}_{X \times X}$

now, $\forall\, (u,v) \in U \times V$, as $U \cap V = \phi$

no $u = v \Rightarrow U \times V$ does not
have any elements
from Δ

$\Rightarrow U \times V \subseteq X \times X - \Delta$

$\therefore \forall (x,y) \in X \times X - \Delta$

$\exists\, U, V$ s.t $U \times V \subseteq X \times X - \Delta$ and $U \times V$ open

$\Rightarrow (\Delta)^c$ is open

$\Rightarrow \Delta$ is closed

($\Leftarrow$) if Δ is closed then

$\Delta = \{x \times x \mid \forall x \in X\}$

$\Rightarrow X \times X - \Delta$ is open

or

$\forall (x,y) \in X \times X - \Delta$ $\exists\, T \subseteq X \times X - \Delta$ open s.t $(x,y) \in T$

now, as $T = \bigcup_{B \in \mathcal{B}} B$

and basis $= \{U \times V \mid U, V \in \mathcal{T}_X\}$

$\exists\, U \times V$ s.t $U \in \mathcal{T}_X$, $V \in \mathcal{T}_X$

and $(x,y) \in U \times V \subseteq T \subseteq X \times X - \Delta$

$\Rightarrow (x,y) \in U \times V \subseteq X \times X - \Delta$

$\forall (u,v) \in U \times V$

if $u = v \Rightarrow$ then $(u,v) \in \Delta$ $\ast$

so

no $u = v \Rightarrow U \cap V = \phi$

$\Rightarrow$ for $x, y \in X$ s.t $x \neq y$

$\exists\, U, V$ s.t

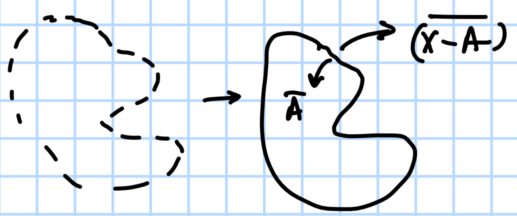
$x \in U$, $y \in V$, $U, V \in \mathcal{T}_X$

and $U \cap V = \phi$

$\therefore X$ is Hausdorff

17.19 $A \subseteq X$

Boundary of A = Bd $A = \overline{A} \cap \overline{(X - A)}$



(a) <u>To prove</u>: $A^\circ \cap \text{Bd } A = \phi$

$\overline{A} = A^\circ \cup \text{Bd } A$

<u>proof</u>:

$\forall x \in A^\circ$, $\exists\, U \in \mathcal{T}_X$ s.t $x \in U$ and $U \subseteq A$

now, $\forall y \in U$, as $y \in A$

$\Rightarrow y \notin X - A$

or

$U \cap (X-A) = \phi$

$\Rightarrow x$ is not a limit point of $X-A$

$\Rightarrow x \notin (\overline{X-A})$

$\Rightarrow x \notin \overline{A} \cap (\overline{X-A})$

$\forall x \in A^\circ \Rightarrow x \notin \overline{A} \cap (\overline{X-A}) \Rightarrow x \notin Bd(A)$

$\Rightarrow A^\circ \cap Bd(A) = \phi$

$\overline{A} = A^\circ \cup Bd(A)$

$\forall x \in A^\circ \cup Bd(A)$

if $x \in A^\circ$ then

$\Rightarrow x \in A \Rightarrow x \in \overline{A}$

if $x \in Bd(A)$

$\Rightarrow x \in \overline{A} \cap (\overline{X-A})$

$\Rightarrow x \in \overline{A}$

$\therefore \forall x \in A^\circ \cup Bd(A)$

$\Rightarrow x \in \overline{A}$

so, $A^\circ \cup Bd(A) \subseteq \overline{A}$

now, $\forall x \in \overline{A} = A \cup$ limit points of A

if $x \in A^\circ$

we are done

else

$x \notin A^\circ$

but as $x \in \overline{A}$

let U be any openset containg x

if $U \subset A$ then $\Rightarrow x \in U$

and so $x \in A^\circ$

this is not true

so $\forall$ such U, $\exists y \in U$ s.t

$y \notin A \Rightarrow y \in X - A$

this means that

$U \cap (X - A - \{x\}) \neq \phi \leftarrow$ as $\exists y$

$\Rightarrow x \in (\overline{X-A})$

and as $x \in \overline{A}$

$\Rightarrow x \in \overline{A} \cap (\overline{X-A}) = Bd(A)$

$\Rightarrow x \in A^\circ \cup Bd(A)$

$\therefore$ in both cases $\overline{A} \subseteq A^\circ \cup Bd(A)$

$\Rightarrow \overline{A} = A^\circ \cup Bd(A)$

(b) To prove: $Bd\, A = \phi \Leftrightarrow A$ is both open and close

proof: $(\Rightarrow)$ as $\bar{A} = A^{o} \cup Bd(A)$

if $Bd(A) = \phi$

$\Rightarrow \bar{A} = A^{o}$

or $A \subseteq \bar{A} = A^{o}$

$\Rightarrow A \subseteq A^{o} \Rightarrow A = A^{o}$

$\therefore$ A is open

as $A = A^{o}$

also $\bar{A} \subseteq A^{o} \subseteq A$

$\Rightarrow A \subseteq \bar{A} \subseteq A$

$\Rightarrow A = \bar{A}$

$\therefore$ A is closed

as $A = \bar{A}$

$(\Leftarrow)$ if A is both open and closed

as

$\bar{A} = A,\ A^{o} = A$

$\Rightarrow \bar{A} = A^{o}$

as $A^{o} \cap Bd(A) = \phi$

$\Rightarrow \bar{A} \cap Bd(A) = \phi$

$\Rightarrow \bar{A} \cap (\bar{A} \cap \overline{X-A}) = \phi$

$\Rightarrow \bar{A} \cap (\overline{X-A}) = \phi$

$\Rightarrow Bd(A) = \phi$

(c) To prove: U is open $\Leftrightarrow Bd\, U = \bar{U} - U$

proof: $(\Rightarrow)$ U is open, then $X-U$ is closed

$\Rightarrow \overline{X-U} = X-U$

and for $x \notin U \Rightarrow x \in \overline{X-U}$

now for $x \notin U \Rightarrow x \in \overline{X-U}$

and

$x \in \bar{U} \Rightarrow x \in \bar{U}$

then for $x \in \bar{U} - U \Rightarrow x \in \bar{U}$ and

$x \in \overline{X-U}$

$\Rightarrow x \in Bd(U)$

for $x \in Bd(U) \Rightarrow x \in \bar{U}$ and $x \in \overline{X-U}$

$\Rightarrow x \in \bar{U}$ and $x \in X-U$

$\Rightarrow x \in \bar{U}$ and $x \notin U$

$\Rightarrow x \in \bar{U} - U \Rightarrow Bd(U) \subseteq \bar{U} - U$

so $Bd(U) = \bar{U} - U$

$(\Leftarrow)$ $Bd(U) = \bar{U} - U$

let $x \in U$ then

$x \notin \bar{U} - U \Rightarrow x \notin Bd(U)$

as $x \in U \Rightarrow x \in \bar{U}$

as $x \notin \bar{U} \cap (\overline{X - U})$

$\Rightarrow x \notin \overline{X - U}$

$\exists V \subseteq X \rightarrow$ open s.t $x \in V$

$V \cap (X - U - \{x\}) = \phi$

as

$x \notin X - U \Rightarrow V \cap (X - U) = \phi$

$\therefore V \subseteq U$ s.t V is open

now

$x \in V$, $V \subseteq U$ and U is open

so $x \in U^\circ$

$\Rightarrow U \subseteq U^\circ \Rightarrow U = U^\circ$

$\Rightarrow U$ is open

(d) U is open

$U = \text{int}(\bar{U})$ ← this is not always true

for $U = (0,1) \cup (1,2)$

$\bar{U} = [0,2]$

$\text{int}(\bar{U}) = (0,2)$, clearly $(0,1) \cup (1,2) \neq (0,2)$

17.20 (a) $A = \{x \times y \mid y = 0\}$

$Bd(A) = \bar{A} \cap (\overline{X - A})$

now for $\forall (x,y) \in A$

← Ball around (x,y) intersects A and also $X - A$

$\Rightarrow (x,y) \in (\overline{X - A}) = \bar{A}$

$\Rightarrow Bd(A) = \bar{A} \cap (\overline{X - A})$

$= \bar{A} = A$

now $Bd(A) = A$

$A^\circ \cup Bd(A) = \bar{A} = A$

$\Rightarrow A^\circ = \phi$ if not then $y = 0$ *

(b) $B = \{x \times y \mid x > 0,\ y \neq 0\}$

$\text{int } B = B \longrightarrow B$ is open

$\text{Bd } B = \bar{B} - B = \{x \times y \mid x \geq 0\} - \{x \times y \mid x > 0, y \neq 0\}$

$= \{x \times y \mid x = 0\} \cup \{x \times y \mid x > 0, y = 0\}$

(c) $C = A \cup B$

$= \{x \times y \mid y = 0\} \cup \{x \times y \mid x > 0\}$

$\bar{C} = \{x \times y \mid y = 0\} \cup \{x \times y \mid x \geq 0\}$

$\mathbb{R}^2 - C = \{x \times y \mid x \leq 0 \text{ and } y \neq 0\}$

$\overline{\mathbb{R}^2 - C} = \{x \times y \mid x \leq 0\}$

$\text{Bd } C = \bar{C} \cap (\overline{\mathbb{R}^2 - C}) = \left(\{y = 0\} \vee \{x \geq 0\}\right) \wedge \left(\{x \leq 0\}\right)$

$= \left(\{y = 0, x < 0\} \vee \{x = 0\}\right)$

$\text{int}(C) = \{x \times y \mid x > 0\}$

(D) $D = \{x \times y \mid x \in \mathbb{Q}\}$

$\text{int } D = \phi$ no such x

$\bar{D} = \{x \times y\}$

$X - D = \{x \times y \mid x \in \mathbb{R}/\mathbb{Q}\}$

$\overline{X - D} = \{x \times y\}$

$\Rightarrow \text{Bd}(D) = \mathbb{R}^2$

(e) $E = \{x \times y \mid 0 < x^2 - y^2 \leq 1\}$

$\text{Bd } E = \{|x| = |y|\} \vee \{x^2 - y^2 = 1\}$

$\text{int } E = \{x \times y \mid 0 < x^2 - y^2 < 1\}$

(f) $F = \{x \times y \mid x \neq 0 \quad y \leq \frac{1}{x}\}$

$\text{int } F = \{x \times y \mid x \neq 0,\ y < \frac{1}{x}\}$

$\bar{F} = \{x \times y \mid x = 0\} \cup \{x \times y \mid x \neq 0,\ y \leq \frac{1}{x}\}$

$\text{Bd } F = \bar{F} - \text{int } F$

$= \{x = 0\} \cup \{x \neq 0,\ y = \frac{1}{x}\}$

(2) Here if $\{x_n\}_{n\geq 1}$ we say $x_n \to p$ if $x_n \in X$ and $p \in X$

$\forall\, U \leftarrow$ open s.t $U \in \tau_X$ and $p \in U$

$\exists N \in \mathbb{N}$ s.t

$\forall\, n \geq N \Rightarrow x_n \in U$

now τ on $[0,1]$ s.t $\{\frac{1}{n}\}$ is only conv seq

$\{x_n\}$ conv $\Leftrightarrow$ ① after some N it is const

or

② almost a subseq of $\{\frac{1}{n}\}$

$\{x_N, x_{N+1}, \dots\} \subseteq \{0, 1, \frac{1}{2}, \dots\}$

for some N and

$\{n \mid x_n = \frac{1}{m}\}$ is finite for all m

$A_N = \{0\} \cup \{1/n \mid n \geq N\}$

τ generated by (smallest τ generated by using below sets)

$\{A \subseteq [0,1]\} \cup \{A_N \mid \forall N\}$

now all $\{p\} \subseteq [0,1]$ s.t $p \neq 0$

then $\{p\}$ is open

for any seq to conv to p we need it s.t after some N, $x_n = p \;\; \forall\, n \geq N$

from ①

now if we remove all the countable sets from $[0,1]$ then no seq will ever converge

as for any open set s.t

$\underbrace{\{x_N, x_{N+1} \dots\}}_{\text{this is countable}} \subseteq$ open set

we have already removed that set

so τ = generated by $\{A \subseteq [0,1] \setminus A$ is countable$\} \cup \{A_N \mid \forall N\}$

so nothing converges on $[0,1]$

now $0 \in A_n$

and every A_n is open, so $\quad A_N = \{0\} \cup \{\frac{1}{n} \mid n \geq N\}$

(i) $\forall$ conv seq, $\exists N$ s.t

$\{x_N, x_{N+1} \dots\} \subseteq \{0\} \cup \underbrace{\{1, \frac{1}{2}, \frac{1}{3} \dots\}}_{A_1}$

(ii) $\{n \mid x_n = \frac{1}{m}\}$ should be finite for all m

now this is true as it was not finite then

$x_n = \frac{1}{m}$ happens infinitely many times

for $A_{m+1} = \{0\} \cup \{\frac{1}{m+1}, \frac{1}{m+2}, \dots\}$

is an open set which will not have the seq as $\frac{1}{m}$ is repeated infintely many times

so will be pesent after every N

$\frac{1}{m} \in \{x_N, x_{N+1} \dots\} \; \forall N$

$\therefore$ any seq conv follow (ii) also

$\therefore$ for τ = generated by $\{A \subseteq (0,1] \setminus A \text{ is countable}\} \cup \{A_N\}$

$A_N = \{0\} \cup \{\frac{1}{n} \mid n \geq N\}$

only conv seq τ has will be st $\frac{1}{n} \to 0$

we can just add a finite countable set back from $A \to A$ is countable to generate ∞ many topologies.

$\tau = \{A \subseteq (0,1] \setminus A \text{ is countable}\} \cup \{\{0\} \cup \{\frac{1}{n} \mid n \geq N\} \mid N \in \mathbb{N}\}$

Tutorial - 6 :

18.2 No as if $f(x) = y_0$
for $f: \mathbb{R} \to \mathbb{R}$
we have
$U \leftarrow$ open if it contain y_0
$\Rightarrow f^{-1}(U) = \mathbb{R}$
$\downarrow$
open

$U \leftarrow$ does not contain y_0
$\Rightarrow f^{-1}(U) = \phi \leftarrow$ open

now as $x_n \to x_0$ has a limit point
then
$f(x_n) = y_0$
so
$\{f(x_n)\} = \{y_0\}$

now as y_0 is the only point
$U \cap (\{y_0\} - \{y_0\}) = \phi$
for any open containing y_0
So, y_0 is not a limit point

A = any subset of $\mathbb{R}$ s.t x is a limit point
$f(A) = \{y_0\}$
$f(x) = y_0 \leftarrow$ not a limit point of $f(A)$

18.3 $i : X \to X$
$U \mapsto U$
i.e $i(U) = U$ and $U = i^{-1}(U)$
$\uparrow$
identity function

To prove : $i : X' \to X$ where
$X' \in \tau'$
$X \in \tau$

i is cont $\Leftrightarrow \tau'$ is finer than τ

proof : $(\Rightarrow)$ i is cont, then $i^{-1}(U)$ is open in τ'
$\uparrow$
open in τ

now $i^{-1}(X) = X \leftarrow$ open in τ'
$\uparrow$
open in τ

So every open set in $\tau \in \tau'$

$\Rightarrow \tau \subseteq \tau'$ or τ' is finer than τ

($\Leftarrow$) τ' is finer than τ then

$\forall$ open sets in $\tau \in \tau'$

as $i^{-1}(U) = U \in \tau \subseteq \tau' \Rightarrow U \in \tau'$

open in τ $\quad$ open in τ'

$\Rightarrow i^{-1}(\text{open in } \tau) = \text{open in } \tau'$

To prove: i is Homeomorphic $\Leftrightarrow \tau' = \tau$

i is Homeomorphic $\Leftrightarrow i$ and i^{-1} are cont

$\Leftrightarrow \tau' \subseteq \tau$ and $\tau \subseteq \tau'$

$\Leftrightarrow \tau' = \tau$

Here bijectivity is trivial as domain and image are same sets

18.5 $\quad f:\underset{X}{(a,b)} \longrightarrow \underset{Y}{(0,1)}$

$f(x) = \dfrac{x-a}{b-a}$

now for $\dfrac{x-a}{b-a}$, as $x \in (a,b)$

① f is cont $\leftarrow$ trivial

$y = \dfrac{x-a}{b-a} \Rightarrow x = a + (b-a)y$

$\Rightarrow f^{-1}(x) = a + (b-a)x \leftarrow$ cont is trivial

② f is bijective as:

(i) $\dfrac{x_1 - a}{\cancel{b-a}} = \dfrac{x_2 - a}{\cancel{b-a}} \Rightarrow x_1 = x_2$

(ii) $\forall\ y \in (0,1),\ \exists\ x = a + (b-a)y \in (a,b)$

s.t

$f(\underbrace{a+(b-a)y}_{x}) = y$

$\therefore$ onto

so $(a,b) \cong (0,1)$

now for $[a,b]$ and $[0,1]$

if $a \neq b$ then

same as above

if $a = b \Rightarrow \{a\}$ is singleton while $[0,1]$ is not

$f(x) = \dfrac{x-a}{b-a}$ is cont as

$U \to$ open in $Y \Rightarrow f^{-1}(U)$ open in X

U is open in Y

then

$U = \bigcup B_\delta(y)$

now $f^{-1}(B_\delta(y)) = f^{-1}((\alpha,\beta)) = (a+(b-a)\alpha, a+(b-a)\beta)$

open ball $\in (0,1)$ is of form (α,β)

$\alpha \geq 0, \beta \leq 1$

now as $\alpha \geq 0 \Rightarrow a+(b-a)\alpha \geq a$

$\beta \leq 1 \Rightarrow a+(b-a)\beta \leq b$

so $f^{-1}(B_\delta(y)) \subseteq X$

and is open

so f is cont

or: f is cont by ε-δ $\Rightarrow f^{-1}(B(y,\varepsilon))$ is open

$\Rightarrow f^{-1}(\bigcup B(y,\varepsilon))$ is open

$\Rightarrow f^{-1}(U)$ is open for U open

$\Rightarrow f$ is cont

18.6 Function cont at only one point:

$$f(x) = \begin{cases} x \; ; & x \in \mathbb{R}\setminus\mathbb{Q} \\ 0 \; ; & x \in \mathbb{Q} \end{cases}$$

$f: \mathbb{R} \to \mathbb{R}$

now to show f is cont at only one point:

let's use ε-δ defination

for f to be cont at 0:

$\forall \varepsilon > 0, \exists \delta > 0$ s.t

$|f(x) - f(0)| < \varepsilon \;\; \forall |x-0| < \delta$

now

$f(0) = 0$ and $f(x) = x$ or 0

if $x \in \mathbb{R}\setminus\mathbb{Q}$

$|f(x) - 0| = |x-0|$

putting $\delta = \varepsilon$

we get

$|f(x) - f(0)| < \varepsilon$

now, f is not cont elsewhere as

$\forall x \neq 0$ ① $x \in \mathbb{Q}$

② $x \notin \mathbb{Q}$

for

$|f(x_0) - f(x)| = |0 - f(x)|$

$= |f(x)|$

now as $|x - x_0| < \delta$

for $x \in \mathbb{R} \setminus \mathbb{Q}$ s.t

$|x - x_0| > \delta/2$

$\Rightarrow |x| > \delta/2 + |x_0|$

$|f(x)| = |x| > \delta/2 + |x_0|$

$\therefore$ not for this ε

for $x_0 \in \mathbb{R} \setminus \mathbb{Q}$

let $x \in \mathbb{Q}$

then

$|f(x) - f(x_0)| = |x_0|$

putting $\varepsilon > |x_0|$

never possible

$\therefore f$ is cont by ε-δ definition on 0 only

$\Rightarrow f$ is cont on one-point

18.9 $\{A_\alpha\}$ s.t $A_\alpha \subseteq X \ \forall \alpha \in I$

$$X = \bigcup_{\alpha \in I} A_\alpha$$

$f : X \longrightarrow Y$

$f|_{A_\alpha}$ is cont $\forall \alpha \in I$

(a) <u>To prove</u>: If $\{A_\alpha\}$ is finite and each A_α is closed then f is cont on X

<u>proof</u>: lets use induction

for $\alpha \in \{1\}$

$A_1 = X$ and as $f|_{A_1} = f|_X = f$

or f is cont on X

now if true for $\alpha \in \{1, 2, \ldots n-1\}$

now, for $\alpha \in \{1, \ldots n\}$

i.e $\{A_\alpha\}$ s.t A_α is closed

$A_\alpha \subseteq X$

and $\bigcup_{\alpha \in \{1 \ldots n\}} A_\alpha = X$

we $A = \bigcup_{\alpha \in \{1 \ldots n-1\}} A_\alpha$

$B = A_n$

s.t $A \cup B = X$

and as A is union of $n-1$ closed sets (finite union of closed sets is closed)

from induction

$f|_A$ is cont also $f|_B$ is cont

now if $x \in A \cap B$ then
as $f|_A$ and $f|_B$ is cont

$\Rightarrow f|_A = f|_B$ for $A \cap B$

now let $f = \begin{cases} f|_A & x \in A \\ f|_B & x \in B \cap A^c \end{cases}$

then as $(B \cap A^c) \cup (A) = X$
if we show f is cont we are done

as $X = A \underset{\uparrow}{\sqcup} (B \cap A^c)$
disjoint union

$f^{-1}(V) = [f^{-1}(V) \cap A] \cup [f^{-1}(V) \cap (B \cap A^c)]$

closed in Y

closed as cont in A

closed in X

closed as cont in $B \cap A^c$

closed in X

$\Rightarrow f^{-1}(V)$ is closed

$\therefore \forall V$ (closed in Y) $\Rightarrow f^{-1}(V)$ is closed

$\Rightarrow f$ is cont

(b) $\{A_\alpha\}$ contable
each A_α is closed
but f is not cont

$\bigcup [1, \frac{1}{n}] = [1, 0)$

$[1, 0) \cup \{0\} = [0, 1]$

let $\underset{\uparrow}{\{A_\alpha\}} = \{[1, \frac{1}{n}]\}_{n \in \mathbb{N}} \cup \{\{0\}\}$
coutable

let $f_{[1, \frac{1}{n}]} = 1 \leftarrow$ cont on $[1, \frac{1}{n}]$
and
$f_{\{0\}} = 0 \leftarrow$ cont as single point (trivial)

but $f = \begin{cases} 1 \; ; & x \in [1, 0) \\ 0 \; ; & x = 0 \end{cases}$
is not cont on $[0, 1]$ as not cont on 0 (trivial)

(c) $\{A_\alpha\}$ index family is locally finite if each point x of X has neighbourhood that intersects A_α for finite many values of α.

for $U' \leftarrow$ neighbourhood of some $x \in X$
we have

U' intersects $\{A_k\}_{k=1}^{n}$

$\Rightarrow U' \subseteq \bigcup_{k=1}^{n} A_k = A$

now as this is finite union
$\Rightarrow A$ is closed
and $f|_A$ is cont

now, $f|_A$ is cont
then
any neighbourhood of $f(x)$ (open) say $V \subseteq Y$
then as $x \in A \Rightarrow f(x)$ is in image of $f|_A$
so $\exists U_A \subseteq A \leftarrow$ subset topology

$f|_A(U_A) \subseteq V$
open wrt A or $\exists U_X \subseteq X$ open in X — By theorm below

s.t $U_A = A \cap U_X$

now $U = U_X \cap U'$
this is open in X
as U_X, U' open in X

and as $x \in U'$ and also $x \in U_A \Rightarrow x \in U_X$

$\Rightarrow x \in U$
$\Rightarrow U$ is a neighbourhood of x

if z is element of $f(U)$ s.t $z = f(y)$ for $y \in U$
then
$y \in U'$ and $y \in U_X$
as $U' \subseteq A$
$\Rightarrow y \in A$ and $y \in U_X$

$\Rightarrow y \in A \cap U_X = U_A$

$\Rightarrow z = f(y) = f|_A(y) \in f|_A(U_A) \subseteq V$

as this is true for all $z \in f(U)$

$\Rightarrow f(U) \subseteq V$

here every $x \in X$, and each neighborhood V of $f(x)$
$\exists U \leftarrow$ Neighborhood of x s.t
$f(U) \subseteq V$ then f is cont

Statement: f is cont $\Leftrightarrow \forall x \in X$, for every neighbourhood of $f(x)$ in Y say V

$\exists\, U \subseteq X$

$\hookrightarrow$ neighbourhood of x in X

s.t

$f(U) \subseteq V$

($\Rightarrow$) for f cont, $x \in X$, V be a neighbourhood of $f(x)$

then

$U = f^{-1}(V)$

$\uparrow$

open in X is s.t

$x \in U$

and

$f(U) \subseteq V$

($\Leftarrow$) let V be open set of Y, now for $x \in f^{-1}(V)$

$\Rightarrow V$ is neighbourhood of $f(x)$

so

$\exists\, U_x \leftarrow$ neighbourhood of x s.t

$f(U_x) \subseteq V$

$\Rightarrow U_x \subseteq f^{-1}(V)$

$\uparrow$

open

$\Rightarrow \quad f^{-1}(V) = \bigcup_{\alpha \in I} (U_\alpha)$

$\Rightarrow f^{-1}(V)$ is open

$\Rightarrow f$ is cont

18.13 $\quad A \subseteq X$

$f : A \to Y$

cont $\quad \searrow$ Hausdorff

To prove: if $g : \overline{A} \to Y$ then g is uniquely determi d by f

$\downarrow$

extension of f

proof: $\quad g|_A = f$

$g|_{\overline{A}-A}$ = some funtion cont

now if this is true for g_1, g_2

then $g_1(x) = g_2(x) = f(x) \ \forall x \in A$

$\exists x \in \bar{A} - A$ s.t

$g_1(x) \neq g_2(x)$

now, Y is Hausdorff and

$g_1(x) \neq g_2(x)$ then

$\exists V_1, V_2 \leftarrow$ Neigubourhoods s.t

$g_1(x) \in V_1 \subseteq Y$

$g_2(x) \in V_2 \subseteq Y$

and $V_1 \cap V_2 = \phi$

as g_1, g_2 are cont $\exists U_1, U_2 \leftarrow$ Neiguoorhoods of x

s.t

$g_1(U_1) \subseteq V_1$

$g_2(U_2) \subseteq V_2$

let $U = U_1 \cap U_2$

And $x \in U$

as $x \in \bar{A}$ And $U \cap A$, $\exists y \in U$ s.t

$y \in A$ and $y \in U$

limit point $\Rightarrow g_1(y) = g_2(y)$ as $y \in A$

and $y \in U_1$ and U_2

$\Rightarrow g_1(y) \in V_1$

& $g_2(y) \in V_2$

$\Rightarrow g_1(y) \in V_1 \cap V_2$

$\Rightarrow V_1 \cap V_2 \neq \phi$ $\#$

$\Rightarrow g_1 \neq g_2$ is $*$

$\Rightarrow g_1 = g_2$

(2) (i) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ (Std topology)

$(x,y) \longmapsto x - y$

now, $\underbrace{f^{-1}(U)}_{\text{open}} \subseteq \mathbb{R}^2$

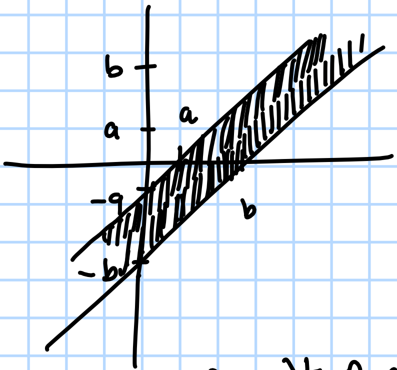
open for $U \leftarrow$ open in $\mathbb{R}$

if $U = (a,b) \leftarrow$ open ball

then

$a \leqslant x - y \leqslant b$

$x - b \leq y$
$x - a \geq y$



so $f^{-1}(B_r(t)) = \{(x,y) \mid x-(t+r) \leq y$ and $x-(t-r) \geq y\}$

$(t-r, t+r)$

this strip

as $\forall B \in \mathcal{B} \Rightarrow f^{-1}(B)$ is open

basis for $\mathbb{R}^2$ can be diagonal strips:

$$\mathcal{B}' = \{(x,y) \mid a \leq x-y \leq b \quad \forall a,b \in \mathbb{R}\}$$

then this basis is <u>non</u> trivial and makes f cont

as $\forall$ $U \to$ open in $\mathbb{R}$

$\Rightarrow f^{-1}(U)$ is open

(ii) $f : \mathbb{R} \longrightarrow \mathbb{R}$ (std topology)

$x \longmapsto x^2$

now for $U \to$ open in $\mathbb{R}$

for a ball (a,b) s.t $a,b > 0$

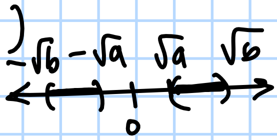
$$f^{-1}((a,b)) = (-\sqrt{b}, -\sqrt{a}) \cup (\sqrt{a}, \sqrt{b})$$

for ball (a,b) s.t $a,b < 0$

$$f^{-1}((a,b)) = \phi \leftarrow \text{always open}$$

for (a,b) st $a < 0$ $b > 0$

$$f^{-1}((a,b)) = (-\sqrt{b}, \sqrt{b})$$



so now for every $B \in \mathcal{B}$

B ↑ Ball

$\mathcal{B}$ ↖ Basis in $\mathbb{R}$ (Balls)

$f^{-1}(B_r(x)) =$ one of three cases

putting $\mathcal{B}' = \{(-\sqrt{b}, -\sqrt{a}) \cup (\sqrt{a}, \sqrt{b}) \mid \forall a,b \geq 0,\ b > a\}$

↖ ball

we get $f^{-1}(B) \in \mathcal{B}'$

or $f^{-1}(B)$ is basis for $B \in \mathcal{B}$

$\therefore$ this is a new basis which is <u>not</u> trivial

(iii) $f : X \longrightarrow \mathbb{R}$ (std)

(X,d) metric space

$d(x, x_0)$ fixed point $\in X$ $\in X$

$f : x \longmapsto d(x, x_0)$

now open balls in $\mathbb{R}$ s.t (a,b) $a,b>0$

we have

$f^{-1}((a,b)) = \{x \in X \mid a < d(x,x_0) < b\}$

dist b/w a and b

now if $a<0$, $b>0$ then

$f^{-1}((a,b)) = \{x \in X \mid d(x,x_0) < b\}$

if $a,b<0 \Rightarrow f^{-1}((a,b)) = \phi$

so for $\mathcal{B}' = \left\{ \{x \in X \mid a < d(x,x_0) < b \text{ s.t } a,b \geq 0, b>a \} \right\}$

for $B \in \mathcal{B} \rightarrow$ Basis of $\mathbb{R}$

as $f^{-1}(B) \in \mathcal{B}'$

we have $f^{-1}(B)$ as open and

$\mathcal{B}'$ is <u>non</u> trivial

(3) $\tau = \{\phi, A_1, \ldots A_N, X\}$ topology on X with $N>0$

<u>To prove</u> : $\exists f : X \longrightarrow \mathbb{R}$(std) s.t

$\tau_f = \tau$ iff $\exists$ sub-collection of mutually disjoint $\{A_1, \ldots A_m\}$ $N = 2^m - 2$

$\{A_{m+1}, \ldots A_N, X\}$ = all possible unions (of size >1)

<u>proof</u> : $(\Rightarrow)$ $\tau_f = \tau$ then every open set in $\mathbb{R}$ is s.t

$f^{-1}(U)$ is open in τ

now $f^{-1}(U) = \bigcup_{\alpha \in I} A_\alpha$ as $\{\phi, A_\alpha, X\}_{\alpha \in \{1 \ldots N\}}$ only open in τ

now, τ_f is s.t $f^{-1}(B) \in \mathcal{B}_{\tau_f}$ for $B \in \mathcal{B}_{\mathbb{R}}$

now $\mathcal{B}_{\tau_f} = \{ f^{-1}(B) \mid B \in \mathcal{B}_{\mathbb{R}} \}$

as $\tau = \tau_f \Rightarrow \mathcal{B}_{\tau} = \mathcal{B}_{\tau_f} = \{ f^{-1}(B) \mid B \in \mathcal{B}_{\mathbb{R}} \}$

now $\mathcal{B}_{\tau}$ = Basis of τ

if we can make a basis for

$\{ \phi, A_1, A_2 \dots A_N, X \}$ we are done.

now let $\mathcal{B}_{\tau} = \{ A_1, A_2 \dots A_m \}$

for some $m < N$ s.t

doubt

all $\phi, A_1, \dots A_N, X$ can be made using them

(definition of basis)

$A_i \cap A_j = \phi$

then 2^m combinations of $\mathcal{B}_{\tau}$

$N = 2^m - 1 - 1$ → for X

→ for ϕ

$\Rightarrow N = 2^m - 2$ then $\{ A_1 \dots A_m \}$ s.t

union of all is in τ

X can be constructed similarly

($\Leftarrow$) $\exists$ subcollection s.t

$\{ A_1 \dots A_N \}$

$\bigcup_{\alpha \in I} A_i = A \quad \forall A \in \tau = \{ \phi, A_1 \dots A_N, X \}$

when $N = 2^m - 2$

now as all are disjoint, let $\mathcal{B}_{\tau} = \{ A_1 \dots A_m \}$

then 2^m combinations in τ construction

also, $f^{-1}(B) \in \mathcal{B}_{\tau_f}$ now for $B \in$ Basis elet of $\mathbb{R}$

→ Basis elemt of τ_f

this is only possible if $\exists r$ s.t

$f^{-1}(B) = A_r$

so $f^{-1}(U)$ can be open

for $U = \bigcup_{\alpha} B_{\alpha} \Rightarrow f^{-1}(\bigcup_{\alpha} B) = \underbrace{\bigcup_{\alpha} f^{-1}(B_{\alpha})}_{\text{open}}$

so $f^{-1}(B_\alpha) = A_\alpha$

for $\alpha = 1, 2 \dots m$

$$\therefore \mathcal{B}_{\tau_f} = \{A_1, \ldots A_m\} = \mathcal{B}_{\tau}$$

as $\mathcal{B}_{\tau} = \mathcal{B}_{\tau_f}$

$\Rightarrow \tau = \tau_f$

Tutorial-7 :

1. (A) $X = (0,1)$
$Y = (-1, \infty)$

They are homeomorphic as $F : (0,1) \longrightarrow (-1, \infty)$

$x \longmapsto \frac{1}{x} - 2$

then ① F is onto :

$\forall\, y \in (-1, \infty)$

$y = \frac{1}{x} - 2$

$\Rightarrow y + 2 = \frac{1}{x}$

$\Rightarrow x = \frac{1}{y+2}$

$\exists\, x \in (0,1)$ s.t $f(x) = y$

② F is one-one

if $f(x_1) = f(x_2)$

$\frac{1}{x_1} - 2 = \frac{1}{x_2} - 2$

$\Rightarrow \frac{1}{x_1} = \frac{1}{x_2}$

$\Rightarrow x_1 = x_2$

③ F is cont :

from the argument of $\varepsilon - \delta$

as x is cont

$\Rightarrow \frac{1}{x}$ $(x \neq 0)$ is cont

$\Rightarrow \frac{1}{x} - 2$ is cont

$\Rightarrow F(x)$ is cont

④ F^{-1} is $y = \frac{1}{x} - 2$

$\Rightarrow x = \frac{1}{y+2}$

$\Rightarrow F^{-1}(x) = \frac{1}{x+2}$

as $x + 2$ cont by $\varepsilon - \delta$

$\Rightarrow \frac{1}{x+2}$ cont by $\varepsilon - \delta$

$\Rightarrow F^{-1}(x)$ cont

from ①, ②, ③ and ④ $(0,1) \cong (-1, \infty)$

(B) $(0,1)$ is not homeomorphic with $(-1, 1]$

as :

if $(0,1) \cong (-1, 1]$

then $\exists\, F$ s.t

$F : (0,1) \rightarrow (-1, 1]$ is bijective

and F, F^{-1} are continuous

now, if F is bijective

then on $(0,1)$

Say $c \in (0,1)$
$F(c) = 1$

as F is bijective $\Rightarrow$ F is monotonic over $(0,1)$ (done in class)

$F(c) = 1$

then for $\forall x \in (c,1)$
$F(x) > 1 \longrightarrow$ this is a contradiction

$\therefore$ not possible

(C) $X = (0,1)$ $Y = [-1,1]$

same argument as (B)

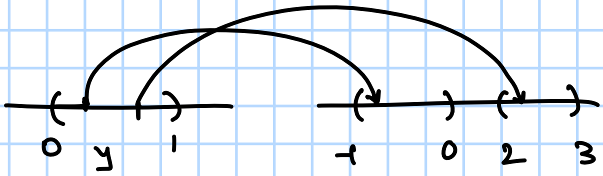
as if possible then

$F(c) = 1$

for some $c \in (0,1)$

$\Rightarrow \forall x \in (c,1)$
$F(x) > 1$ $*$

(D) $X = (0,1)$ $Y = (-1,0) \cup (2,3)$



if F is bijective and continious
wlog montonically inc
then

if $x = \sup_{y>0} F^{-1}(-1,0)$ then

$\forall y < x \Rightarrow F(y) \in (-1,0)$

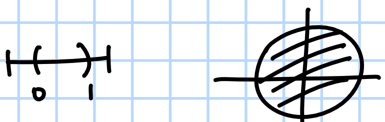
$\forall \underset{y<1}{y > x} \Rightarrow F(y) \in (2,3)$

now then $|F(x - \varepsilon/2) - F(x + \varepsilon/2)| > 2$

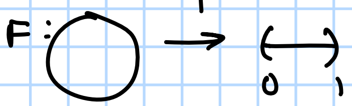
$\Rightarrow$ F is not cont at $F(x)$

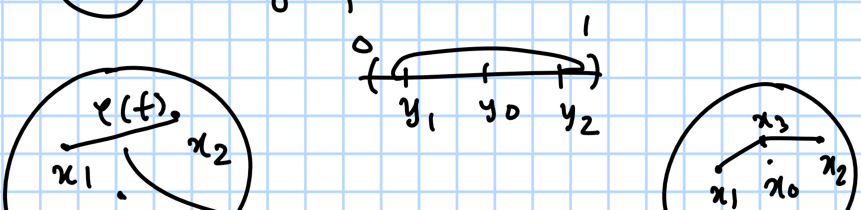
$\Rightarrow$ This is a contradiction

(E) $X = (0,1)$ $Y = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$



Not homeomorphic





Should not pas x_0

$\varphi(t)$ is cont, $h(\varphi(t)) \in (0,1)$

s.t $h(\varphi(0)) = y_1$

$h(\varphi(1)) = y_2$

$\exists\, t_0$ s.t

$h(\varphi(t_0)) = y_0$

but for that point t_0

$F(\varphi(t_0)) \neq y_0$ *

formally : let F ① bijective

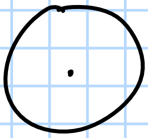
② one-one onto

then

$F: Y \longrightarrow X$

let $\tilde{x}_0 = F^{-1}(0,0)$

now



for $x_0 \in X$

s.t $x_0 \neq \tilde{x}_0$

$x_1 \in (0, x_0)$ and $x_1 \neq \tilde{x}_0$

$x_2 \in (x_0, 1)$ and $x_2 \neq \tilde{x}_0$

we have $y_1 = F^{-1}(x_1)$

$y_2 = F^{-1}(x_2)$

$y_0 = F^{-1}(x_0)$

and as $x_i \neq \tilde{x}_0$ for $i = 0,1,2$

$\Rightarrow y_i \neq (0,0)$ as F is bijective

now, let $\varphi(t) = t\, y_1 + (1-t)\, y_2 \quad 0 \leq t \leq 1$

if y_0 on this line

then

$$\varphi(t) = \begin{cases} (1-2t)\, y_2 \ ; & 0 \leq t < 1/2 \\ (2t-1)\, y_1 \ ; & \frac{1}{2} \leq t \leq 1 \end{cases}$$

$\varphi(t)$ is cont as for $t = 1/2$

$\varphi(1/2) = 0$

$\lim_{t \to 1/2^-} \varphi(t) = 0$

now, $\varphi : [0,1] \longrightarrow Y$

$\hookrightarrow$ domain

then $h = F \circ \varphi$

is s.t

$h : [0,1] \longrightarrow X$

where

$F(\varphi(t)) = h(t)$

where

$h(0) = x_1$

$h(1) = x_2$

then by IVT $\exists\, t_0$

s.t $F(\varphi(t_0)) = x_0$

$\Rightarrow \varphi(t_0) = F^{-1}(x_0) = y_0$

but y_0 not on $\varphi(t)$ by construction

(F)

$X = (0,1)$ $\quad Y = \mathbb{R}\setminus(0,1)$

if $F \to$ s.t F is bijective
F is cont
F^{-1} is cont

then F is monotonic

let $x_0 = F^{-1}(0)$

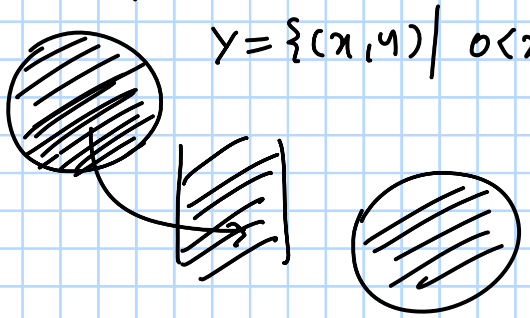
then wlog monotonically inc

then

$F(x_0 + \epsilon) \geq 1 \to$ this means not cont at x_0

(K) $X = \{(x,y) \mid x^2+y^2<1\}$

$Y = \{(x,y) \mid 0<x<1\}$



$(x,y) \in \mathbb{R}^2$

then

$$\frac{(x,y)}{1+\sqrt{x^2+y^2}} \in B_1(0,0)$$

$$F : \mathbb{R}^2 \to B_1(0,0)$$

$$(x,y) \mapsto \frac{x}{1+\sqrt{x^2+y^2}}, \frac{y}{1+\sqrt{x^2+y^2}}$$

$(0,1)\times\mathbb{R} \to \mathbb{R}^2$

$\hookrightarrow (x,y) \to (2x-1, y)$

$\downarrow$

$$\left(\frac{\pi}{2}(2x-1), y\right)$$

$\downarrow$

$$F : (0,1)\times\mathbb{R} \to \left(\tan^{-1}\left(\frac{\pi}{2}(2x-1)\right), y\right)$$

$F : Y \to X$ is

$$(x,y) \mapsto \left(\frac{\tan^{-1}\left(\pi x - \frac{\pi}{2}\right)}{1+\sqrt{\left(\tan^{-1}\left(\pi x - \frac{\pi}{2}\right)\right)^2+1}}, \frac{y}{1+\sqrt{x^2+y^2}} \right)$$

F is Bijective by construction

F is cont by cont

$$(0,1)\times\mathbb{R} \xrightarrow{f_1} \mathbb{R}\times\mathbb{R} \xrightarrow{f_2} B_1(0,0)$$

$$\leftarrow f_1^{-1} \leftarrow f_2^{-1} \leftarrow$$

$$f_2(z) = \frac{z}{1+|z|}$$

$$|z'| = \frac{|z|}{1+|z|}$$

$$|z'| + |z||z'| = |z|$$

$$\frac{|z'|}{1-|z'|} = |z|$$

$f_1^{-1}(z) = \dfrac{z'}{1-|z'|}$

f_2^{-1} is similar

(H) $X = \{(x,y) \mid x^2+y^2<1\}$

$Y = \{(x,y,z) \mid x^2+y^2+z^2=1\}$

if $X \cong Y \Rightarrow \exists F$ s.t

$F: Y \longrightarrow X$ is cont

$\Rightarrow F(Y) = X$ also F is bijection

but $Y = S^2$ is compact (closed, bounded in $\mathbb{R}^3$)

X is not compact (open, in $\mathbb{R}^2$)

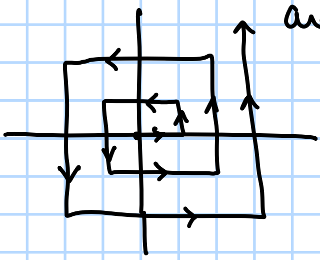
(I) $X = \mathbb{Z}$

$Y = \mathbb{Z}\times\mathbb{Z} \subseteq \mathbb{R}^2$

$$f(r) = \begin{cases} \dfrac{4r-1}{2\operatorname{sgn}(r)} - \dfrac{1}{2} & ; r \neq 0 \\ 0 & ; r = 0 \end{cases}$$

$f: \mathbb{Z} \longrightarrow \mathbb{N}$ is an homeomorphic map

$\mathbb{N} \cong \mathbb{Z}\times\mathbb{Z}$ by



and as F, F^{-1} are always cont

$\rightarrow$ every singleton is open

(J) $X = \mathbb{Z}$ $Y = \mathbb{Q}$

Not if so then

$f: Y \longrightarrow X$
cont

$\{q\} \subseteq \mathbb{Q}$ is singleton

then $f(\{q\}) =$ also singelton open

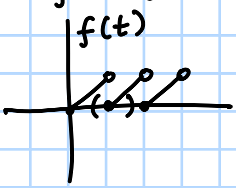
but $\{q\}$ is closed in $\mathbb{Q}$

(K) $\tau_f \leftarrow$ topology on $\mathbb{R}$

$f(t) = t - \lfloor t \rfloor \longrightarrow \mathbb{R}$

s.t $f^{-1}(U)$ is open for U open

$t - \lfloor t \rfloor$

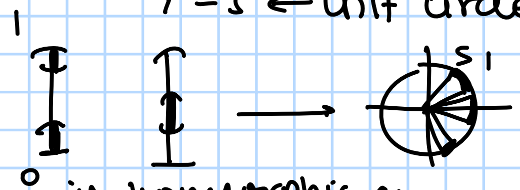


$\mathcal{B}_{\tau_f} = \{ f^{-1}(B) \mid B \in \mathcal{B}_{\tau} \}$

$$\mathcal{B}_{\tau_f} = \{(a,b) \mid a>0,\ b<1\} \cup \{[0,\varepsilon) \cup (1-\varepsilon,1) \mid \forall \varepsilon < \tfrac{1}{3}\}$$

$\tau = \tau_f$ where $[0,1)$ is open

$Y = S^1 \leftarrow$ unit circle in $\mathbb{R}^2$

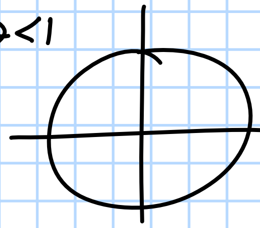


is homeorphic as

$F(\theta) = e^{2\pi i \theta}$

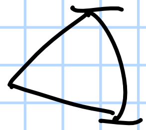
$0 \leq \theta < 1$

then



① Bijective by defination

②  → open

③  open

2. $A \subseteq \mathbb{R}$ homeomorphic to $(0,1)$

$(a,b) \cong (0,1)$ (Provable using normal F)

if more than one ball (disjoint) then not cont

← Not $[a,b)$ due to IVP

$(a,b]$ same

$[a,b]$ same

Tutorial-8 :

22.2 (a) $P: X \to Y$ cont map $\quad$ U open in $Y \Rightarrow P^{-1}(U)$ open in X
$f: Y \to X$ s.t $\quad$ $U \to$ open in $X \Rightarrow f^{-1}(U)$ open in Y
$P \circ f =$ Identity
then P is a quotient map

$P \circ f: Y \to Y$

$P \circ f(U) = U$

now ① $U \leftarrow$ open in Y
then
$\Rightarrow P^{-1}(U)$ open in X
now if $P^{-1}(U)$ is open in X say V
then $f^{-1}(P^{-1}(U)) = U \Rightarrow V$ is open
$\Rightarrow f^{-1}(V)$ is open $\Rightarrow U$ is open

② $P: X \to Y$
$\forall y \in Y$, as $P \circ f: Y \to Y$ is Id
$P \circ f = Id$
$P \circ f(y) = y$
or $\exists f(y) \in X$ s.t
$P(f(y)) = y$
$\therefore P$ is surjective
from ①,② P is a quotient map

(b) $A \subset X$ retraction of X onto A
$r: X \to A$ (cont)
s.t $r(a) = a \quad \forall a \in A$
To prove : r is a quotient map

proof : for $B_A \leftarrow$ open wrt A i.e $B_A = A \cap B_X$ $\to$ some open in X

given r is cont
then
$r^{-1}(B_A)$ is open in X
$\Rightarrow A \cap B_X$ open in X
$\Rightarrow A$ is open in X
as for $B_X = X$
we have
$A \cap X = A$
or $r^{-1}(A)$ is open

now, if $f^{-1}(B)$ is open in X then:

$f^{-1}(B) = B_X$
now let $A \cap f^{-1}(B) = A \cap B_X = B_A$
open in X
then $A \cap B_X$ is also open as A is open
and $f^{-1}(A \cap B_X) = A \cap B_X$
so $f^{-1}(B) \subseteq A$

$\Rightarrow f^{-1}(B) = B_A$
$\Rightarrow f^{-1}(A \cap B_X) = A \cap B_X$ $\to$ open

or $A \cap f^{-1}(B) = f^{-1}(B) = A \cap B \times$
open
open

$\Rightarrow B$ is open in A

so $f^{-1}(B)$ open $\Leftrightarrow B$ is open

now, surjectivity is trivial

22.3 $\pi_1 : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$

$\pi_1(x,y) = x$

$A = \{(x,y) \subseteq \mathbb{R} \times \mathbb{R} \mid x \geq 0\} \cup \{(x,y) \subseteq \mathbb{R} \times \mathbb{R} \mid y = 0\}$

$q : A \longrightarrow \mathbb{R}$

$q : A \longrightarrow \mathbb{R}$
is
$\pi_1|_A : A \longrightarrow \mathbb{R}$
q

now $f : \mathbb{R} \longrightarrow A$
$(x) \longrightarrow (x,0)$

then
$f(x) = (x,0)$

$f^{-1}(V \times U)$
open open
(Basis

if V or $U = \phi$ then
$V \times U = \phi \Rightarrow f^{-1}(V \times U) = \phi$ open in $\mathbb{R}$

if $0 \notin V$ then $f^{-1}(U \times V) = \phi \rightarrow$ open in $\mathbb{R}$

if $0 \in V$ then $f^{-1}(U \times V) = U$
as
$\forall x \in f^{-1}(U \times V) \Leftrightarrow f(x) \in U \times V$
$\Leftrightarrow x \times 0 \in U \times V$
$\Leftrightarrow \forall x \in U$

$\therefore U = f^{-1}(U \times V)$

as U is open

$f^{-1}(U \times V)$ is open $\forall\, U \times V$ open

now
$\Rightarrow f$ is cont

now $q : A \rightarrow \mathbb{R}$
for q to be cont
$q^{-1}(U)$ cont in A

now $U \leftarrow$ Ball in $\mathbb{R}$ say (a,b) s.t
$0 \notin (a,b)$

then $q^{-1}(U) = A \cap [(a,b) \times \mathbb{R}]$ open wrt subspace topology if $0 \in (a,b)$ then also

$$q^{-1}(U) = A \cap \underbrace{[(a,b) \times \mathbb{R}]}_{\text{open}}$$

$\Rightarrow q^{-1}(U)$ open

$\Rightarrow q$ is continious

now

q is cont

$q : A \longrightarrow \mathbb{R}$
$f : \mathbb{R} \longrightarrow A$
then
$q \circ f : \mathbb{R} \longrightarrow \mathbb{R}$

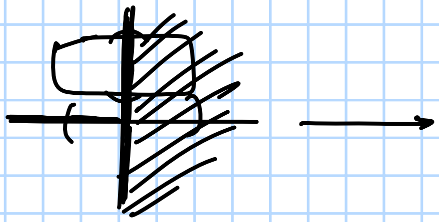
$f(x) = (x, 0)$

$q(x,0) = x \quad \Rightarrow \quad q \circ f(x) = x \quad \forall x \in \mathbb{R}$

$\Rightarrow$ then as q is cont
f is cont
from prev q
$\Rightarrow q$ is Quotient map

now, to show q is not close/not open map:

A:



$\mathbb{R}$

$\underset{\text{open}}{q(U)}$ then $q(U)$ open $\forall$ such U for q to be open map

let $U = A \cap [(-1,1) \times (1,2)]$

then $\underset{A = \{(x,y) \mid x \geq 0\} \cup \{(x,y) \mid y = 0\}}{A} \cap [(-1,1) \times (1,2)] = [0,1) \times (1,2)$ open in A

now, $\pi_1([0,1) \times (1,2)) = [0,1)$
this is not open

for closed:

$C = \{(x, \frac{1}{x}) \mid x > 0\}$

this is just a  Boundary there

$\underset{\text{closed}}{\pi_1(C)} = \underset{\text{open}}{(0,\infty)}$

$C = \{(x, \frac{1}{x}) \mid \forall x > 0\}$ is closed as

$$A - C = \left\{ \begin{matrix} (xy) \neq 1 \\ \left. \begin{matrix} x > 0 \\ y > 0 \end{matrix} \right\} \end{matrix} \right\}$$

this is open by $\underbrace{(xy-1)}_{\text{cont polynomial}} \neq 0$

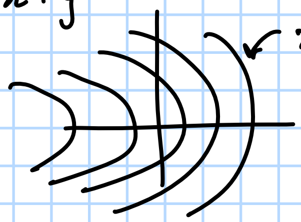
22.4 (a) $X = \mathbb{R}^2$

$x_0 \times y_0 \sim x_1 \times y_1$

$x_0 + y_0^2 = x_1 + y_1^2$

$X^* =$ Quotient space

$g(x \times y) = x + y^2$



let $X^* =$ Quotient space of $g(x \times y) = x + y^2$
i.e

$G : \mathbb{R}^2 \to X^*$ ← space of all values

let $[c] = \{(x,y) \mid c = x + y^2\}$

then let

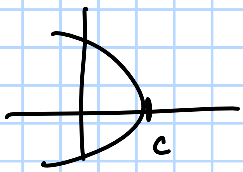
$G : \mathbb{R} \to X^*$ ← eq class

$(x,y) \to [x + y^2]$

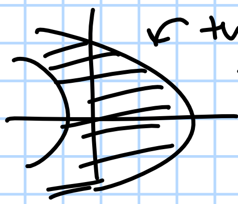
now, $\tau_{x^*} = \{ U \mid g^{-1}(U) \text{ is open} \}$

now

$g^{-1}[c] =$ is just $x + y^2 = c$ function



$g^{-1}(c_1 < x + y^2 < c_2)$



this is open as:

$\forall \ (x,y) \in \{(x,y) \mid c_1 < x + y^2 < c_2\}$

$\Rightarrow \begin{matrix} x + y^2 < c_2 \\ x + y^2 > c_1 \end{matrix}$ by continuity of polynomial

$\exists$ Ball s.t
$\forall (x,y) \in$ Ball

$x+y^2 < c_2 \quad x+y^2 > c_1$

then $\mathcal{B}_{\tau_{X^*}} = \{ (c_1, c_2) \mid \forall c \in (c_1, c_2)$ $\hookrightarrow [c]$ is the corresponding equation $\}$

now $X^* \cong \mathbb{R}$ as:

$F : X^* \longrightarrow \mathbb{R}$
$[c] \longrightarrow c$

then

① F is onto and one-one (trivial)

② F is cont as
$\forall (c_1, c_2) \in \mathbb{R}$
$F^{-1}((c_1, c_2))$ is by defination basis

③ F^{-1} is cont as

$F(c_1, c_2) = (c_1, c_2) \subseteq \mathbb{R}$ is open

(b) $x_0 \times y_0 \sim x_1 \times y_1$
$x_0^2 + y_0^2 = x_1^2 + y_1^2$

let $g(x,y) = x^2 + y^2$
then

let
$g : \mathbb{R}^2 \longrightarrow X^*$
s.t

$[c] \leftarrow$ equivalence cos

$[c] = \{ (x,y) \mid x^2 + y^2 = c \}$

then $c \geqslant 0$ (By $x^2 + y^2 \geqslant 0$)

and we have:

$g : \mathbb{R}^2 \longrightarrow X^*$

let $\tau_{X^*} = \{ U \mid g^{-1}(U)$ is open in $\mathbb{R}^2 \}$

then

$0 \leqslant c < \infty$ we have

$g^{-1}(U)$ open if

$0 \leqslant c_1 < x^2 + y^2 < c_2$

then

we have the corresponding set open (trivial)

$\Rightarrow \mathcal{B}_{\tau_{X^*}} = \{ \{ [c] \mid c_1 < c < c_2 \} \}$
$\ni c_1 \geqslant 0$

then
now
$X^* \cong \mathbb{R}_{\geq 0}$

as $F: X^* \longrightarrow \mathbb{R}_{\geq 0}$
$[c] \longmapsto c$

then

① F is one-one and onto (trivial)
② F is cont as:
$F^{-1}((c_1, c_2)) = \{[c] \mid c_1 < c < c_2\}$
$\in \mathcal{B}_{\tau_{X^*}}$

③ F^{-1} is cont as:

$F(\{[c] \mid c_1 < c < c_2\}) = (c_1, c_2) \in \mathcal{B}_{\mathbb{R}_{\geq 0}}$
for $c_2 > c_1 \geq 0$

22.5 $p: X \longrightarrow Y$
↳ open map

$\forall\, U \subseteq X$ s.t U is open in X

$P(U) \subseteq Y$ is open in Y

$A \leftarrow$ open in X
$q: A \longrightarrow P(A)$ is an open map

Proof: as $q: A \longrightarrow P(A)$
if U_A is open in A
then
$U_A = A \cap U_X$ for some
U_X open in X
now
$q(U_A) = P(A \cap U_X)$
as A is open, U_X is open

$\Rightarrow A \cap U_X$ is open in X
$\Rightarrow P(A \cap U_X)$ is open in Y

and as $P(A \cap U_X) \subseteq P(A)$

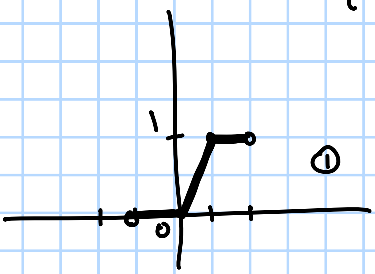
then $P(A) \cap P(A \cap U_X) = P(A \cap U_X)$
so $P(U_A)$ is open in $P(A)$
then $q: A \longrightarrow P(A)$
is an open map

2. (A) $\pi: X \longrightarrow Y$
s.t $Y \cong Z$
now
$F: X \longrightarrow Z$ (quotient map)
then
$F: Y \longrightarrow Z$
↳ is the required homeomorphism

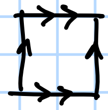
$X = (-1,1) \qquad Z = [0,1]$

$F : X \longrightarrow Z$
$\quad x \longrightarrow 2x$

$$F(x) = \begin{cases} 2x \ ; & 0 \leqslant x \leqslant \frac{1}{2} \\ 1 \ ; & x > 1/2 \\ 0 \ ; & x < 0 \end{cases}$$



① now F is surjective as $\forall\ y \in [0,1]$
$$F(y/2) = y$$



$S^1 \times S^1$
$e^{2\pi i t} \times e^{2\pi i s}$

② $F^{-1}(U)$ is open $\Leftrightarrow U$ is open
as $(\Leftarrow)$ U is open then

$$\mathcal{B}_Z = \left\{ (a,b) \cap [0,1] \;\middle|\; \substack{a,b \in \mathbb{R} \\ a<b} \right\}$$

now if $(a,b) \cap [0,1] \in \mathcal{B}_Z$ s.t

$$0 < a < b < 1$$

$$\Rightarrow (a,b) \cap [0,1] = (a,b)$$

$$F^{-1}(a,b) = \left(\frac{a}{2}, \frac{b}{2}\right) \in \mathcal{B}_X$$

if (a,b) s.t $0 \in (a,b)$ then

$$F^{-1}([0,b)) = \left(-1, \frac{b}{2}\right) \in \mathcal{B}_X$$

$$F^{-1}((a,1]) = \left(\frac{a}{2}, 1\right) \in \mathcal{B}_X$$

So $F^{-1}(U)$ open

$(\Rightarrow)$ $F^{-1}(U)$ is open then

$$\mathcal{B}_X = \left\{ (-1,1) \cap (a,b) \;\middle|\; a<b,\ a,b \in \mathbb{R} \right\}$$

if (a,b) s.t $(a,b) \subset (0,\frac{1}{2})$ then

if $F^{-1}(U) = (a,b)$

$$\Rightarrow U = (2a, 2b)$$

else $\quad \left.\begin{array}{l} (U) = [0, 2b) \\ (U) = (2a, 1] \end{array}\right\}$ other cases

$\Rightarrow U$ is open

So F is a Quotient map

now, $\pi : X \longrightarrow Y$ should be s.t

for $\alpha \in [0,1]$

$$F^{-1}(\alpha) = \begin{cases} 2\alpha & ; \alpha \in (0,1) \\ (-1,0] & ; \alpha = 0 \\ [\frac{1}{2}, 1) & ; \alpha = 1 \end{cases}$$

let $\pi : X \longrightarrow Y$

$$\pi(x) = \begin{cases} x ; & x \in (0, 1/2) \\ 0 ; & x \in (-1, 0] \\ \frac{1}{2} ; & x \in [\frac{1}{2}, 1) \end{cases} \quad \Bigg\} \ \pi \text{ is surjective \& trivial}$$

then $Y = [0, \frac{1}{2}]$ is s.t

$\pi : X \longrightarrow Y$

$$\mathcal{B}_X = \{ (a,b) \cap (-1,1) \}$$

$$\tau_Y = \{ U \mid \pi^{-1}(U) \text{ open in } X \}$$
$$= \{ (a,b) \mid a > 0, \ b < \tfrac{1}{2} \} \cup \{ [0,b) \mid b < \tfrac{1}{2} \} \cup \{ (a, \tfrac{1}{2}] \mid a > 0 \}$$

Basis making τ_Y (Quotient)

now $F : Y \longrightarrow Z$

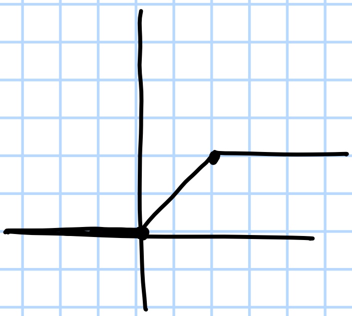
wrt τ_Y

we have by the theorm $Y \cong Z$

(B) $X = \mathbb{R}$ $\qquad Z = [0,1]$

$$F : X \longrightarrow Z$$
$$x \longrightarrow x$$

$$F(x) = \begin{cases} x & ; x \in (0,1) \\ 1 & ; x \in [1, \infty) \\ 0 & ; x \in (-\infty, 0] \end{cases}$$



① now F is surjective as $\forall \ y \in [0,1]$

$\exists \ F(y) = y$

② $F^{-1}(U)$ is open $\Leftrightarrow U$ is open

as $(\Leftarrow)$ U is open then

$$\mathcal{B}_Z = \{ (a,b) \cap [0,1] \mid a, b \in \mathbb{R}, \ a < b \}$$

now if $(a,b)\cap[0,1]\in \mathcal{B}_Z$ s.t

$0<a<b<1$

$\Rightarrow (a,b)\cap[0,1] = (a,b)$

$F^{-1}(a,b) = (a,b) \in \mathcal{B}_X$

if (a,b) s.t $0\in(a,b)$ then

$F^{-1}([0,b)) = (-\infty,b) \in \mathcal{B}_X$

$F^{-1}((a,1]) = (a,\infty) \in \mathcal{B}_X$

so $F^{-1}(U)$ open

$(\Rightarrow)$ $F^{-1}(U)$ is open then

$\mathcal{B}_X = \{ \mathbb{R} \cap (a,b) \mid a<b, a,b\in\mathbb{R} \}$

if (a,b) s.t $(a,b)\subset(0,1)$ then

if $F^{-1}(U) = (a,b)$

$\Rightarrow U = (a,b)$

else $\left.\begin{array}{l}(U) = [0,b) \\ (U) = (a,1]\end{array}\right\}$ other cases

$\Rightarrow U$ is open

so F is a Quotient map

now, $\pi : X \longrightarrow Y$ should be s.t

for $\alpha\in[0,1]$

$$F^{-1}(\alpha) = \begin{cases} \alpha & ; \alpha\in(0,1) \\ (-\infty,0] & ; \alpha=0 \\ [1,\infty) & ; \alpha = 1 \end{cases}$$

let $\pi : X \longrightarrow Y$

$$\pi(x) = \begin{cases} x ; & x\in(0,1) \\ 0 ; & x\in(-\infty,0] \\ 1 ; & x\in[1,\infty) \end{cases}$$

π is surjective is trivial

then $Y = [0,1]$ is s.t

$\pi : X \longrightarrow Y$

$\mathcal{B}_X = \{ (a,b)\cap \mathbb{R} \}$

$\tau_Y = \{U \mid \pi^{-1}(U) \text{ open in } X\}$

$= \{(a,b) \mid a>0,\ b<1\} \cup \{[0,b) \mid b<1\} \cup \{(a,1] \mid a>0\}$

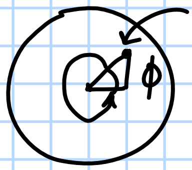
Basis making τ_Y (Quotient)

now $F : Y \longrightarrow Z$

wrt τ_Y

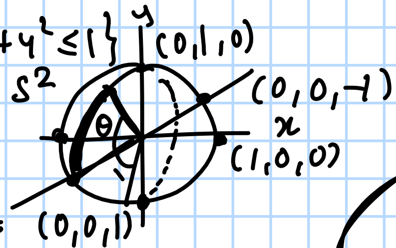
we have by the theorm $Y \cong Z$

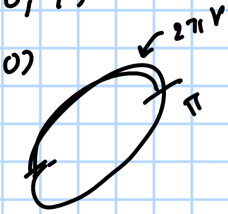
(c) $x^2+y^2 \leq 1$



$z \in \{x^2+y^2 \leq 1\}$

$|z| \leq 1$





$F : X \longrightarrow Z$

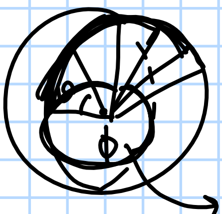
s.t if $|z| = 1$ then

$z \longrightarrow (0,0,-1)$

now $|z| = 1 \longrightarrow \pi$

i.e πz

$|\pi z| = \pi$ then



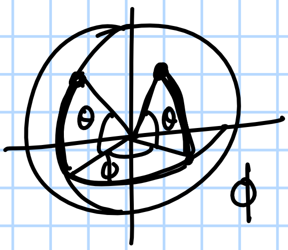
ϕ depends on $\tan^{-1}\left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right)$

$z = |z| e^{i\theta}$

this $\theta \in [0, 2\pi]$

$\frac{z}{|z|} = e^{i\theta}$

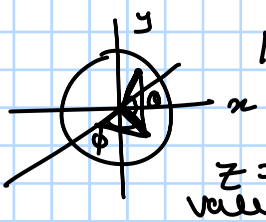
this $\cos\theta + i\sin\theta = \frac{z}{|z|}$



$(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$

$\theta \in [0, 2\pi]$

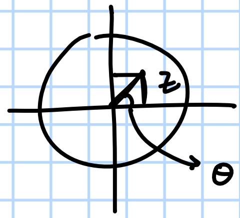
$\phi \in [0, \pi]$



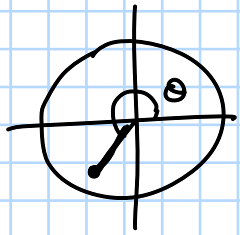
$|\pi z| \leq \pi$

ϕ axis on z

$z = \cos(|\pi z|)$ value



$\theta = \tan^{-1}\left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right)$



$\cos(\pi|z|) = \cos(\pi(\sqrt{x^2+y^2}))$

$(\cos\theta \sin\phi, \sin\theta\sin\phi, \cos\phi)$

$= (\cos\theta \sin(\pi\sqrt{x^2+y^2}), \sin\theta \sin(\pi\sqrt{x^2+y^2}), \cos(\pi\sqrt{x^2+y^2}))$

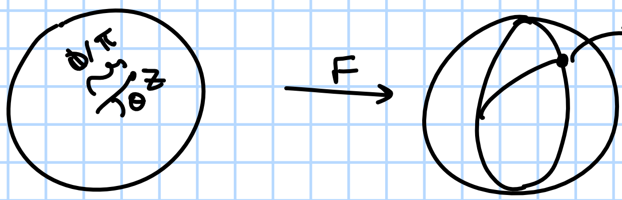
$$\theta = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) & ; y \geq 0 \\ \tan^{-1}\left(\frac{y}{x}\right) + \pi & ; y < 0 \end{cases}$$

$\theta \in [0, 2\pi]$

Now this $x^2+y^2 \leq 1 \longrightarrow S^2$

s.t $|z| = 1 \rightarrow$ goes to last point

$|z| < 1 \longrightarrow$ then some unique point



$\rightarrow (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$

for $\phi = \pi$
or $|z| = 1$

$\cos(\pi) = -1$
$\sin(\pi) = 0 \Rightarrow (0,0,-1)$

and $0 \rightarrow \cos(0) = 1$
$\sin(0) = 0 \Rightarrow (0,0,1)$

now, F is a Quotient map as:

① $\forall (x,y,z) \in S^2$

$z = \cos\phi$

$\phi = \cos^{-1}(z)$
$\in [0,\pi]$

$\frac{x}{\sin\phi} = \cos\theta$

$\frac{y}{\sin\phi} = \sin\theta$

and we can fin $\theta \Rightarrow z \in x^2+y^2 \leq 1$ s.t

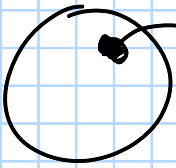
$\phi = \pi|z| \quad \theta = \tan^{-1}(y/x)$

if $|z|=1$ then choose $(1,0)$

now, $F^{-1}(U)$ is open $\Leftrightarrow$ U is open

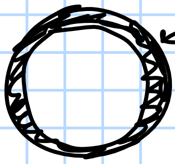
($\Leftarrow$) U is open on S^2 and $(0,0,-1) \notin U$ then

if $U \in \mathcal{B}_{S^2}$(Ball) then



Distorted open $\Rightarrow$ open

if $(0,0,-1) \in U$ then

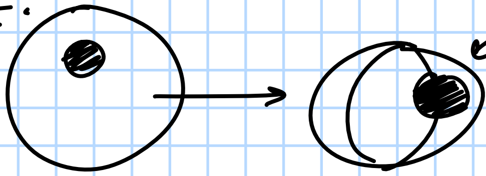


open set around whole boundary (open)

so F is a Quotient map

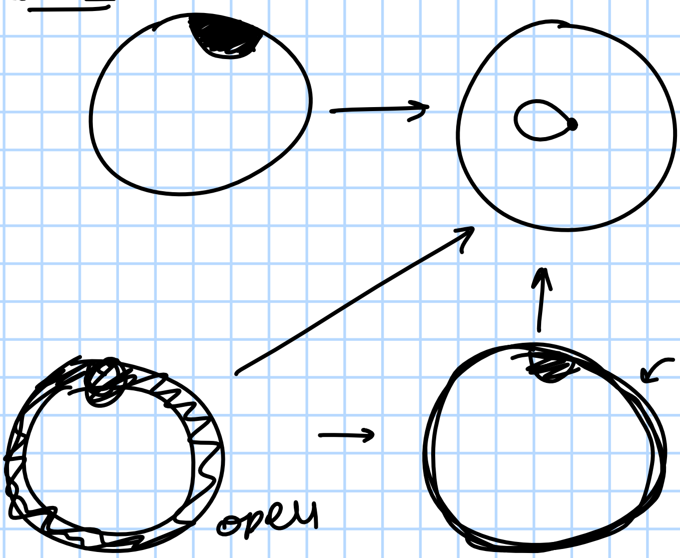
now, ($\Rightarrow$) if $F^{-1}(U)$ is open

case I:



open

case II:



but if 0 put then

not open

open

now, $F^{-1}(\alpha)$ for $\alpha \in S^2$ is unique if not $(0,0,-1)$ then

$\pi : X \longrightarrow Y$ is Id otw then boundary

$z \longmapsto z$ if $|z| \neq 1$

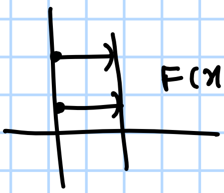
if $|z|=1$ then let it be $(1,0)$

then Y:

$$\mathcal{B}_{\tilde{\tau}_\pi} = \{ U \mid \pi^{-1}(U) \in \mathcal{B}_{\tau_X} \}$$
$$= \{\{ B_\delta(x,y) \mid (1,0) \notin B_\delta(x,y) \} \cup \{ Y \cap (\quad x^2+y^2 > 1-\varepsilon \}\}$$

now, $Y \cong Z$

(D) $X = \mathbb{R}^2$ $\quad Z = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2=1\}$



$$F(x,y) = \begin{cases} (e^{i2\pi x}, y) & ; x \in [0,1) \\ (1,0,y) & ; x \in (-\infty,0) \cup [1,\infty) \end{cases}$$

now $F: X \longrightarrow Z$

① onto: as $\forall\ x,y,z \in Z$ s.t
$x^2+y^2=1$
$z \leftarrow$ any

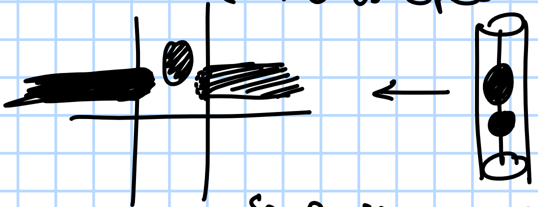
$\exists \quad \theta = \tan^{-1}\left(\frac{y}{x}\right) \to [0,2\pi)$

s.t $\frac{\theta}{2\pi} \Rightarrow [0,1)$

$F^{-1}\left(\frac{\theta}{2\pi}, z\right) = (x,y,z)$

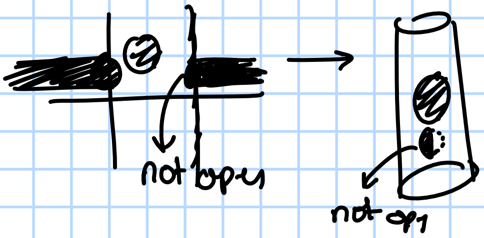
② $F^{-1}(U)$ is open $\Leftrightarrow$ U is open

($\Leftarrow$) U is open then



So Basis are open $\Rightarrow$ $F^{-1}(U)$ is open

($\Rightarrow$) $F^{-1}(U)$ is open then:



So $F^{-1}(U)$ open $\Rightarrow$ U is open now, $\pi: X \longrightarrow Y$

s.t $F^{-1}(\alpha)$ $\to$ $\alpha \in \mathbb{Z}$

if $\alpha \neq$ where $x=0$ then

it's one-one so

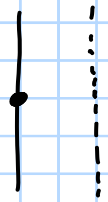
and if $\alpha =$ has $x=0$

then

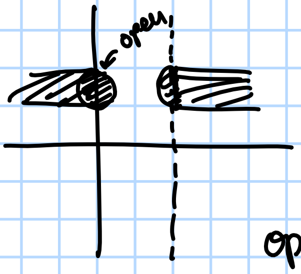
$F^{-1}(\alpha)$ $\to$ infinite many then

fix to $x=0$

then

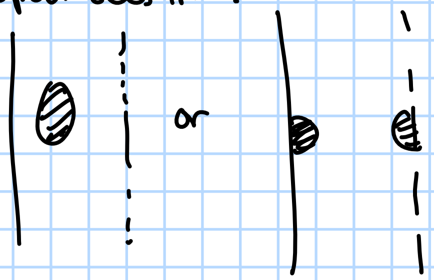


Y



$$\tilde{\tau}_\pi = \{ U \mid F^{-1}(U) \text{ is open in } X \}$$

open sets in Y:



$Y \cong \mathbb{Z}$

(E) $X = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x, y \leq 1\}$ $Y =$ Torus $S^1 \times S^1$

$F: X \longrightarrow S^1 \times S^1$

$(x,y) \longrightarrow (e^{2\pi i x}, e^{2\pi i y})$

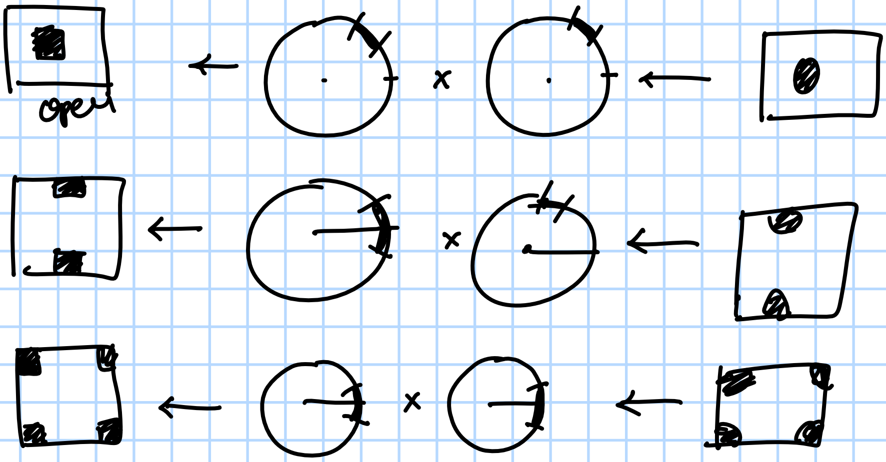
now

F is surjective (trivial)

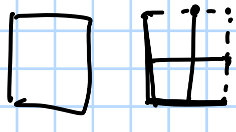
$F^{-1}(U)$ is open $\Leftrightarrow$ U is open:

($\Leftarrow$) U is open then

($\Rightarrow$)



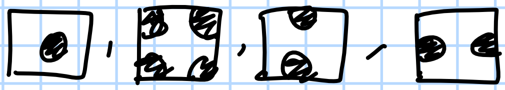
now, $\pi: X \longrightarrow Y$



all points to all

if boundary then pnd

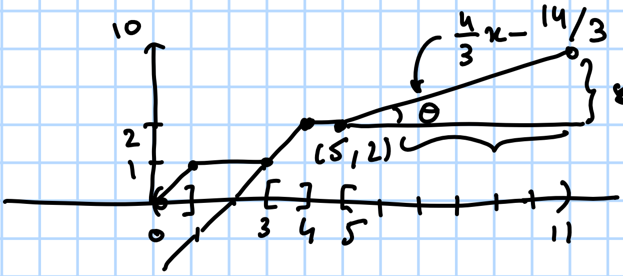
if conv fixes $(0,0)$

$\tilde{\zeta}_{\pi} = \{$  $\}$

$Y \cong Z$

Tutorial-9:

1. (A) $X = (0,1] \cup [3,4] \cup [5,11)$ $\qquad Z = (0,10)$



$\frac{20}{3} + c = 2$

$c = 2 - \frac{20}{3}$

$= \frac{6-20}{3} = \frac{-14}{3}$

$F : X \longrightarrow Z$

$$F(x) = \begin{cases} x & ; x \in (0,1] \\ x-2 & ; x \in [3,4] \\ \frac{4}{3}x - \frac{14}{3} & ; x \in [5,11) \end{cases}$$

$X/\sim \cong Y$

Quotient topology:

$f : X \to Y$

$\to$ f is Quotient map

then $\tau_f = \{ U \mid f^{-1}(U) \text{ is open} \}$

① F is onto as $\forall\, y \in (0,10)$

then
$y \in (0,1)$ $\quad F(y) = y$
$y = 1$ $\quad F(y) = y$
$y \in (1,2)$ $\quad F(y+2) = y$
$y = 2$ $\quad F(y+2) = y$
$y \in (2,10)$ $\quad F\left(\frac{3y+14}{4}\right) = y$

$\frac{4}{3}x - \frac{14}{3} = y$

$X \leftarrow$ topology, $\sim$ eq relation

$X/\sim$ equivalence classes
s.t $U \subseteq$ open in X
$\Leftrightarrow q^{-1}(U)$ open in $X/\sim$

② $F^{-1}(U)$ is open $\Leftrightarrow$ U is open

$(\Leftarrow)$ U is open then

$$\mathcal{B}_Z = \{ \{ (a,b) \mid 0 < a < b < 10 \} \}$$

$q : X \longrightarrow \frac{X}{\sim}$

$x \longmapsto [x]$

Basis of the following type:

$\tau_{X/\sim} = \{ A \subseteq X/\sim \mid \phi^{-1}(A) \subseteq \text{open } X \}$

1	0	2
×		×
✓		×
×		✓
✓		✓

case I: $F^{-1}(U) = (a,b)$ if $a,b \in (0,1)$
$F^{-1}(U) = (a+2, b+2)$ if $a,b \in (1,2)$
$F^{-1}(U) = \left(\frac{3a+14}{4}, \frac{3b+14}{4} \right)$ if $a,b \in (2,10)$

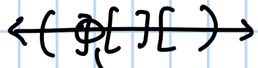
Case II: $F^{-1}(U) = (a, 1] \cup [3, b+2)$ if
$0 < a < 1 < b < 2$

Case III: $F^{-1}(U) = (a+2, 4] \cup \left[5, \frac{3b+14}{4}\right)$
$1 < a < 2 < b < 10$

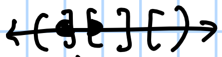
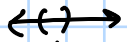
$(\Rightarrow)$ $F^{-1}(U)$ is open
then

$\longleftarrow (( )] \; [\;] \; [\longrightarrow$

trivial case if not touching boundary



if boundary touched then U becomes ( ] and now we have to add the point on right ⇒ the whole set add

 ⇒ 

open U

new open $f^{-1}(U)$

now $\pi : X \longrightarrow Y$

$$\pi(x) = \begin{cases} x \; ; & x \in (0,1) \cup (3,4) \cup (5,11) \\ 3 \; ; & x = 1 \\ 5 \; ; & x = 4 \end{cases}$$

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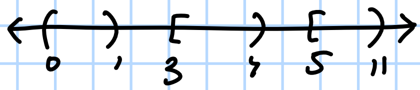
then $\pi : X \longrightarrow Y$ is a quotient map (trivial)

Y :



$$\mathcal{B}_Y = \{ U \mid \pi^{-1}(U) \text{ is open} \}$$

$$= \{ (a,b) \mid (a,b) \subseteq (0,1) \} \cup \{ (a,b) \mid (a,b) \subseteq (3,4) \}$$

$$\cup \{ (a,b) \mid (a,b) \subseteq (5,11) \}$$

$$\cup \{ (1-\varepsilon, 1) \cup [3, 3+\varepsilon) \mid 0 < \varepsilon < \tfrac{1}{2} \}$$

$$\cup \{ (4-\varepsilon, 4) \cup [5, 5+\varepsilon) \mid 0 < \varepsilon < \tfrac{1}{2} \}$$

$X/\sim$

$= \{ \{t\} ; 0 < t < 1$ or $3 < t < 4$ or $5 < t < 11 \}$

$\cup \{ \{1,3\} \} \cup \{ \{4,5\} \}$

and $Y \cong Z$ wrt τ_Y made by $\mathcal{B}_Y$

(B) $X = [0,1] \times [0,1]$

$Z = [0,1]$

$F : X \longrightarrow Z$

$(x,y) \longmapsto x$

$F(x,y) = x$

now, F is ① onto as $\forall \; x \in [0,1]$

$F(x,0) = x$

② $F^{-1}(U)$ is open $\Leftrightarrow$ U is open

($\Leftarrow$) [redacted]

$$F^{-1}\big((a,b)\cap[0,1]\big) = \underbrace{\big((a,b)\cap[0,1]\big)}_{\text{this is open}} \times [0,1]$$

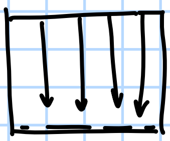
($\Rightarrow$) if $F^{-1}(U)$ is open
then if some $x \in F^{-1}(U)$
then the whole

$\{(x,y) \mid y \in [0,1]\}$

should be in $F^{-1}(U)$

$U = (x_1, x_2)$ open in $[0,1]$

now $\pi : X \longrightarrow Y$



$\pi(x,y) = (x,0)$

Y:

then π is onto as $\forall\ (x,0) \in Y$

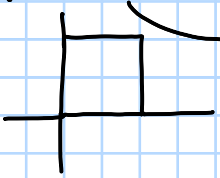
$F(x,0) = (x,0)$

also $\mathcal{B}_{\tilde{\tau}_\pi} = \big\{\big((a,b)\cap[0,1]\big)\times(\{0\})\big\}$

and $Y \cong Z$

(c) $X = [0,1]\times[0,1]$ $\longrightarrow$ maps to 0

$Y = S^1$ $\longrightarrow$ $S^1 \equiv (\cos\theta, \sin\theta)$ $\longrightarrow$ $\theta \in [0, 2\pi]$



$F : X \longrightarrow Z$

$F(x,y) = e^{2\pi i x}$

then

$\forall\ y \in S^1$

$y = e^{2\pi i t}$

$t \in [0,1)$

$F(t,0) = e^{2\pi i t}$

$\therefore F$ is onto

also $F^{-1}(U)$ is open $\Leftrightarrow$ U is open

$X \cong Y$ then
$X \setminus p \cong Y \setminus f(p)$ by restriction topology

$\mathbb{R} \not\cong \mathbb{R}^2$ as

$\mathbb{R}\setminus\{0\}$ not connected

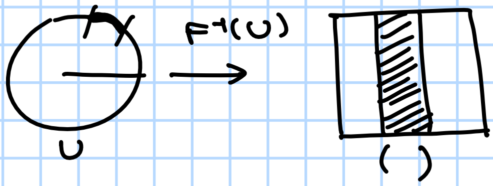
$\mathbb{R}^2\setminus\{F(0)\}$ is connected

so $\mathbb{R}\setminus\{0\} \not\cong \mathbb{R}^2\setminus\{F(0)\}$

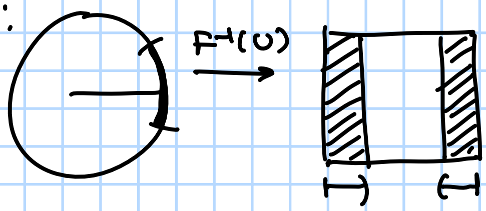
$\Rightarrow \mathbb{R} \not\cong \mathbb{R}^2$

$(\Leftarrow)$ U is open

Case I:



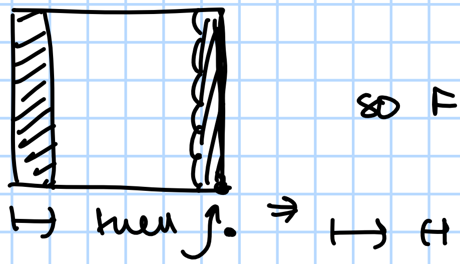
Case II:



$(\Rightarrow)$

but if $x \in F^{-1}(U)$

then $\{(x,y) \mid \forall y \in [0,1]\} \in F^{-1}(U)$



so $F^{-1}(U)$ is open $\Rightarrow$ U is open

$\pi : X \longrightarrow Y$

$$\pi(x,y) = \begin{cases} (x,0) & ; x \in [0,1) \\ (0,0) & ; x = 1 \end{cases}$$

then $\mathcal{B}_{\tilde{\tau}_\pi} = \{U \mid \pi^{-1}(U) \text{ is open}\}$

$$= \left\{ (a,b) \times \{0\} \;\middle|\; 0 < a < b < 1 \right\}$$

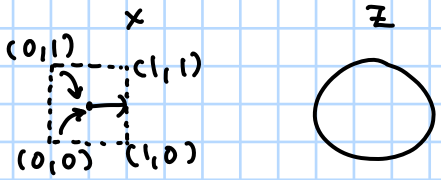
$$\cup \left\{ [0,\varepsilon) \cup (1-\varepsilon,1) \times \{0\} \;\middle|\; \forall \varepsilon < \tfrac{1}{2}, \varepsilon > 0 \right\}$$

$Y \cong Z$

(D) $X = (0,1) \times (0,1)$

$Z = S^1$

$F : \underset{X}{(0,1) \times (0,1)} \longrightarrow \underset{Z}{S^1}$



$F : X \longrightarrow Z$

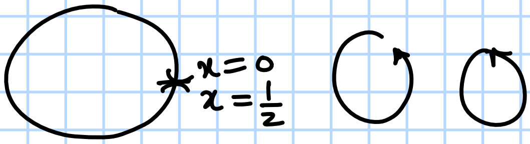
$F(x,y) = e^{2\pi i(2x-1)}$

then

$x = \frac{1}{2}$ $\quad e^{2\pi i(1-1)} = (1,0)$

$x = 1$ $\quad e^{2\pi i(2-1)} = (1,0)$

$x = 0$ $\quad e^{2\pi i(-1)} = (1,0)$



now for $\forall y \in S^1$ i.e

$y = e^{2\pi i t}$

for $t \in [0,1)$

let $2x-1 = t$

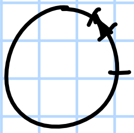
$x = \frac{1+t}{2} \in [\frac{1}{2}, 1)$

so $F(x) = y \quad \forall y \in S'$

$\therefore F$ is surjective

now $F^{-1}(U)$ is open $\Leftrightarrow$ U is open:

($\Leftarrow$) U is open in S' then:



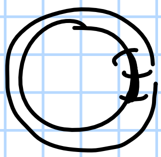
$\mathcal{B}_{S^1} = \left\{ \left\{ e^{2\pi i t} \,\middle|\, 0 < t < 1 \right\} \right\}$

$\cup \left\{ e^{2\pi i t} \,\middle|\, t \in (1-\varepsilon, 1) \cup [0, \varepsilon) \right\}$

now

$F^{-1}\left(\left\{ e^{2\pi i t} \,\middle|\, \begin{matrix} \forall t \in (t_1, t_2) \\ 0 < t_1 < t_2 < 1 \end{matrix} \right\} \right) = \left(\frac{t_1}{2}, \frac{t_2}{2} \right) \cup \left(\frac{1+t_1}{2}, \frac{1+t_2}{2} \right)$

$F^{-1}\left(\left\{ e^{2\pi i t} \,\middle|\, \begin{matrix} t \in (1-\varepsilon, 1) \cup [0, \varepsilon) \\ \varepsilon < 1 \end{matrix} \right\} \right)$



$(0, \frac{\varepsilon}{2}) \cup (\frac{1-\varepsilon}{2}, \frac{1}{2}] \cup [\frac{1}{2}, \frac{1+\varepsilon}{2})$

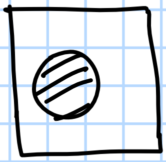
$= (0, \frac{\varepsilon}{2}) \cup (\frac{1-\varepsilon}{2}, \frac{1+\varepsilon}{2})$

now as $\forall B \in \mathcal{B}_{S^1} \Rightarrow F^{-1}(B)$ is open

$\forall U$ open in S^1

$U = \bigcup_{\alpha \in I} B_\alpha \Rightarrow F^{-1}(U) = \bigcup_{\alpha \in I} \overset{\text{open}}{F^{-1}(B_\alpha)}$

($\Rightarrow$) Now if $F^{-1}(U)$ is open:



$\forall x \in F^{-1}(U)$

as $F(x,y) = u$
for some x

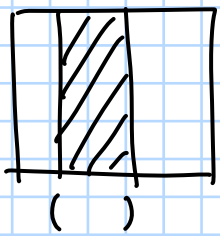
$\forall y \in (0,1) \Rightarrow F(x,y) = u$

$\Rightarrow \forall y \in (0,1)$
$y \in F^{-1}(U)$

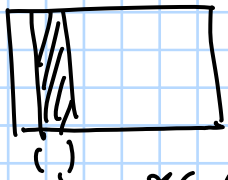
so if $x \in F^{-1}(U)$ then

$\{(x,y) \mid \forall y \in (0,1)\} \subseteq F^{-1}(U)$

so $F^{-1}(U)$ looks like:



case I: $F^{-1}(U)$:



$x \in (a,b)$
s.t $0 < a < b < \frac{1}{2}$

or $F^{-1}(U) = (a,b) \times (0,1)$

now
$U = \{e^{2\pi i t} \mid t \in (2a-1, 2b-1)\}$
open

for $\frac{1}{2} < a < b < 1 \Rightarrow U = \{e^{2\pi i t} \mid t \in (2a-1, 2b-1)\}$
open

for $0 < a < \frac{1}{2} < b < 1$

we have $U = \{e^{2\pi i t} \mid t \in (2a-1, \frac{1}{2}] \cup [\frac{1}{2}, 2b-1)\}$

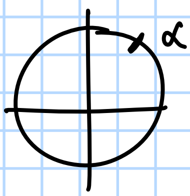
$\therefore U$ is open

so F is a quotient map

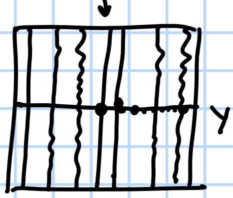
$F: X \longrightarrow Z$
is a quotient map

now
$\pi: X \longrightarrow Y$
be s.t
for $\alpha \in Z$
$F^{-1}(\alpha) =$ some set
then
$\pi:$ some set $\longrightarrow$ point in Y

then



$F^{-1}(\frac{1}{2}) = \{(\frac{1}{2})\} \times (0,1)$



$\pi : X \longrightarrow Y$

$$\pi(x,y) = \begin{cases} (x, \frac{1}{2}) & x \in [\frac{1}{2}, 1) \\ (\frac{1}{2}, \frac{1}{2}) & x \in (0, \frac{1}{2}) \end{cases}$$

now $\mathcal{B}_{\tilde{\tau}_\pi} = \{ (a,b) \mid \frac{1}{2} < a < b < 1 \} \cup$
$\{ [\frac{1}{2}, \frac{1}{2}+\varepsilon) \cup (1-\varepsilon, 1) \mid 0 < \varepsilon < \frac{1}{3} \}$

this from mapping π
to keep
$\pi^{-1}(U)$ open
we have to include
some of $(1-\varepsilon, 1)$

$Z \cong Y$

(E) $X = S^1 \times (0,1)$
$Z = S^1$

now $F : X \longrightarrow Z$

s.t $F(e^{2\pi i t}, y) = e^{2\pi i t}$

this is ① onto (trivial)
for $F(e^{2\pi i t}, 0) = e^{2\pi i t}$

② $F^{-1}(U)$ is open $\Longleftrightarrow$ U is open
this is as
U is open $\Longleftrightarrow$ $U \times (0,1)$ is open
$\Updownarrow$
$F^{-1}(U) = U \times (0,1)$ is open

now, $\pi : X \longrightarrow Y$

$$\pi(e^{2\pi i t}, y) = (e^{2\pi i t}, \tfrac{1}{2})$$

then

π is onto (trivial

now $\mathcal{B}_{\tilde{\tau}_\pi} = \{ \text{Basis of } S^1 \times \{\frac{1}{2}\} \}$

and $Z \cong Y$ by theorm

(F) $S^1 \times [0,1]$ $\quad S^2$

$X = S^1 \times [0,1]$
$Z = S^2$

$F : X \longrightarrow Z$
$F(e^{2\pi i t}, y) = (\cos(2\pi t)\sin(\pi y), \sin(2\pi t)\sin(\pi y), \cos(\pi y))$

then $\forall (x,y,z) \in S^2$

$\exists\, t$ and y

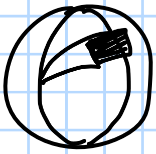
only at $y=0$ or 1 not unique

else unique

F is onto

also $F^{-1}(U)$ is open $\Leftrightarrow$ U is open

($\Leftarrow$) if U is open in S^2 then:



$0 < 2\pi t < 2\pi \Rightarrow 0 < t < 1$

$0 < \pi y < \pi \Rightarrow 0 < y < 1$

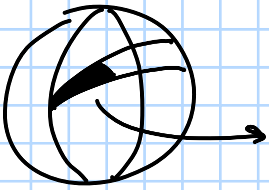
then

open in S^1

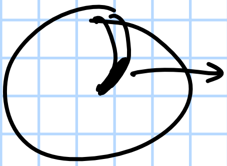
$F^{-1}(U) = \{\{e^{2\pi i t} \mid 0 < t < 1\}\}$

$\times \{(0,1)\}$

open in $[0,1]$

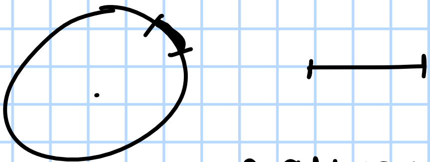


$0 < y < 1$ still open



open

($\Rightarrow$) if $F^{-1}(U)$ is open



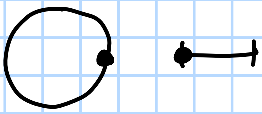
in all cases it is trivial

S^2 is open

so F is a quotient map

now $\pi : X \longrightarrow Y$

F is one-one except for



$\pi : X \longrightarrow Y$ $y=0$ and $y=1$

let $t=0$ $t=0$

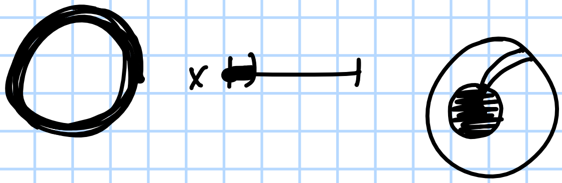
in both cases

$\pi : X \longrightarrow Y$

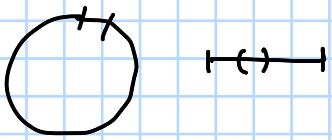
$$\pi(e^{2\pi i t}, y) = \begin{cases} (e^{2\pi i t}, y) & ; \; y \neq 0,1 \\ (e^{2\pi i (0)}, y) & ; \; y = 0 \text{ or } 1 \end{cases}$$

then π is also is trivial

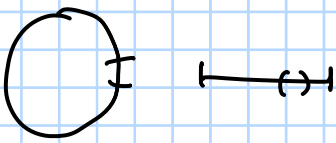
now $\tau_Y = \{U \mid \pi^{-1}(U) \text{ open in } X\}$



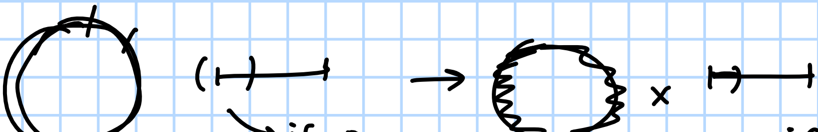
$\pi^{-1}(U)$ cases:



open in Y

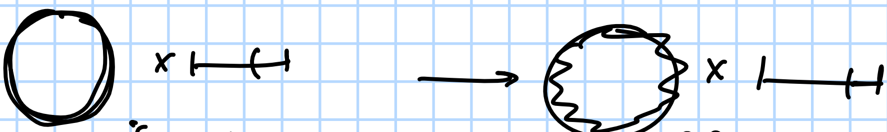


open in Y



if 0 then whole circle should be taken for open

open in Y



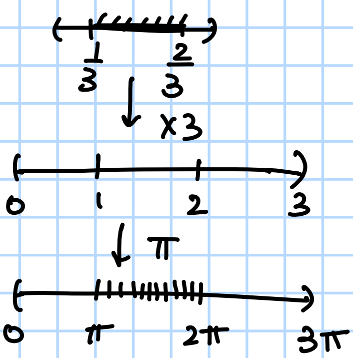
if 1 in Y then whole S^1 should be taken by defination

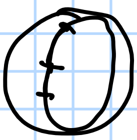
open in Y

then $Y \cong Z$

(a) $X = S^1 \times (0,1)$

$Z = S^2$





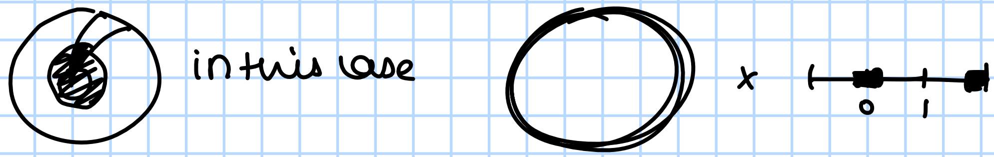
$F: X \longrightarrow Z$

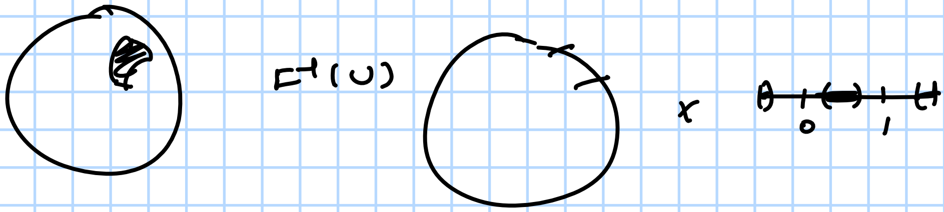
$$F(e^{2\pi i t}, y) = (\cos(2\pi t)\sin(3\pi y - \pi),\ \sin(2\pi t)\sin(3\pi y - \pi),\ \cos(3\pi y - \pi))$$

F is onto (trivial) as any point from $3\pi y - \pi \in [0, \pi]$

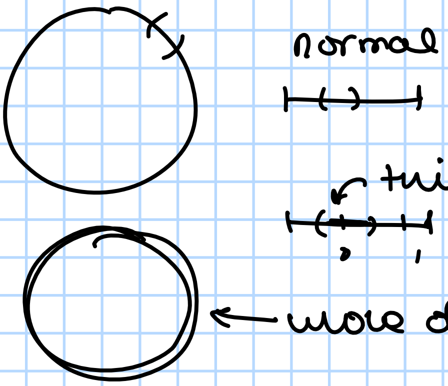
now $F^{-1}(U)$ is open $\Leftrightarrow$ U is open

($\Leftarrow$) U is open then:





($\Rightarrow$) $F^{-1}(U)$ is open

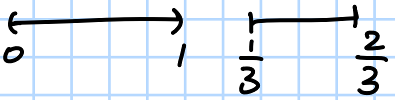


← whole of S^1 $\Rightarrow$ open U

so F is quotient map

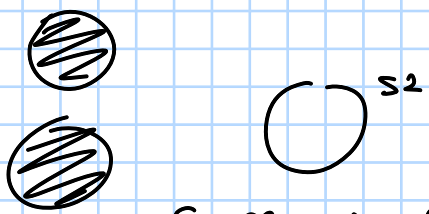
now, $\pi: X \longrightarrow Y$

$S^1\ Y$ $\quad$ $S^1\ X$



same calculation as previous question with added constraint

(H)



same as $S^1 \to S^2$ but we map both S^1 and the upper circle together so if something is open in S^1 its image/interpolation should be open on above circle

(2) To prove : Homeomorphism $F : X \to Y$ is a Quotient map

proof: as $F: X \to Y$ is Bijective
$\Rightarrow F : X \to Y$ is onto — ①

now as F is cont
$\forall\, U$ open in $Y \Rightarrow F^{-1}(U)$ is open in X — ②

also as F^{-1} is cont
$\forall\, U$ open in $X \Rightarrow F(U)$ open in Y
now as F is bijective
let
U_X open in $X \Rightarrow F(U)$ open in Y
$U_X = F^{-1}(U) \quad \Rightarrow F(F^{-1}(U))$ open in Y
for some U
as F is bijective

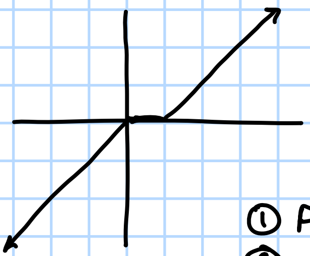
so $F^{-1}(U) \Rightarrow V$ is open in Y — ③

$\therefore$ from ①, ②, ③:
① F is onto
② $F^{-1}(U)$ open $\Leftrightarrow V$ is open

we have F to be Quotient map

converse is not true as:

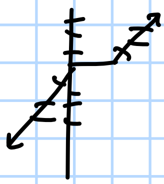
$F: \mathbb{R} \to \mathbb{R}$
be a quotient
map st:



$$F(x) = \begin{cases} x & ; \ x \in (-\infty, 0) \\ 0 & ; \ x \in [0, 1) \\ x-1 & ; \ x \in [1, \infty) \end{cases}$$

① F is onto (trivial)
② $F^{-1}(U)$ is open $\Leftrightarrow U$ is open

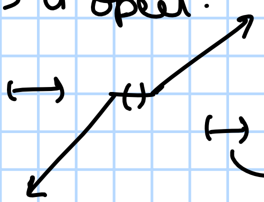
($\Leftarrow$) U is open



0
✗ then it is open
✓ then $(-\varepsilon, 0) \cup [0, 1] \cup [1, 1+\varepsilon)$
if $U = (-\varepsilon, \varepsilon)$ $\qquad = (-\varepsilon, 1+\varepsilon)$

$\therefore$ open

now if $F^{-1}(U)$ is open:



trivial

if $x \in (0,1)$
then whole $[0,1]$ in $F^{-1}(U)$
by construction
and so $F^{-1}(U)$ if $\supseteq [0,1)$ then
$F^{-1}(U) = (-\varepsilon, 1+\varepsilon)$

$\Rightarrow U = (-\varepsilon, \varepsilon)$

$\therefore$ open

so, F is a quotient map, but F is not bijective

($F(0.1) = F(0.2)$)
not one-one

Tutorial-10:

$M_2(\mathbb{R}) \cong \mathbb{R}^4$

as $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \longrightarrow (a_{11}, a_{12}, a_{21}, a_{22})$

$Gl_2(\mathbb{R})$: $f : X \longrightarrow Y$ cts

X connected $\Rightarrow$ Y is connected

$X = Gl_2(\mathbb{R})$

$f = \det$

then

$f : X \longrightarrow Y$

$M_{2\times 2} \longmapsto \det(M_{2\times 2})$

$= \mathbb{R} \setminus 0$

not conncted

$\Rightarrow$ X is not conneted

$X = GL_2^+(\mathbb{R}) \sqcup GL_2^-(\mathbb{R})$

$\overset{\shortparallel}{\{A \in X \mid \det A > 0\}}$

<u>Theorm</u> : (pg 152, Q11) $P : X \longrightarrow Y$ is a Quotient map, if $P^{-1}(y)$ is conneted $\forall y$, Y is also conned $\Rightarrow$ X is also conneted

If X is not conneted, then $X = U \sqcup V$

then $Y = P(U) \cup P(V)$

forms a seperation

as $P : X \longrightarrow Y$ is sujetie

$P(Y) = P(U) \cup P(V)$

now as U is open, V is open

we have to show

① $P(U)$, $P(V)$ is open

② $P(U) \cap P(V) = \phi$

③ $P(U), P(V) \neq \phi$ (trivial)

If $P(U) \cap P(V) \neq \phi$

then $y \in P(U) \cap P(V)$

$\Rightarrow P^{-1}(y) \in U$, $P^{-1}(y) \in V$

$\Rightarrow P^{-1}(y) \in U \cap V = \phi$ $*$

as $P^{-1}(y)$ (fiber) is conneted

② is tree, now

$P(U)$ is open as:

$U \subseteq P^{-1}(P(U))$

now to show $U \supseteq P^{-1}(P(U))$

let $x \in P^{-1}(P(U))$

$\Rightarrow P(x) \in P(U)$

$\Rightarrow \underbrace{P^{-1}(P(x))}_{\text{conneted}} \subseteq U$

$\Rightarrow P^{-1}(P(U)) \subseteq U$

$\Rightarrow P^{-1}(P(U)) = U$

as U is open $\Rightarrow$ $P^{-1}(P(U))$ is open $\Rightarrow$ $P(U)$ is open

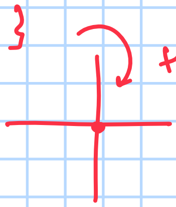
$\forall\ p^{-1}(y)$ connected
X is not connected $\Rightarrow$ y is not connected

$p^{-1}(y)$ is connected
& Y is connected $\Rightarrow$ X is connected

$g : GL_2^+(\mathbb{R}) \longrightarrow \mathbb{R}^2 \setminus \{0,0\}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a \\ c \end{pmatrix}$

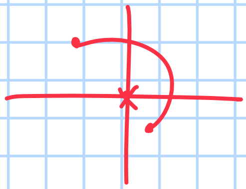
this is connected



open, cont + surjective

$\Rightarrow$ Quotient map

range is connected



$p^{-1}(y)$ is connected for all $y \in \mathbb{R}^2 \setminus \{(0,0)\}$
and Y is connected
$\Rightarrow X = GL_2^+(\mathbb{R})$ is connected

also $GL_2^+(\mathbb{R}) \cong GL_2^-(\mathbb{R})$ (homeomorphism)
$\Rightarrow GL_2^-(\mathbb{R})$ is connected

$f : GL_2^+(\mathbb{R}) \longrightarrow GL_2^-(\mathbb{R})$
$M \longmapsto A_0 M$
$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$SO_2(\mathbb{R}) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = A_\theta \right\}_{\theta \in [0, 2\pi)}$

$I \longrightarrow SO_2(\mathbb{R})$
$\varphi(t) = \begin{pmatrix} \cos t\theta & -\sin t\theta \\ \sin t\theta & \cos t\theta \end{pmatrix}$

$\varphi(0) = I$
$\varphi(1) = A_\theta$
and $\varphi(t) \in SO_2(\mathbb{R})$

$X = \{ A \mid \det A = 1 \}$

$GL_2^+(\mathbb{R})$ is connected

$GL_2^+(\mathbb{R}) \longrightarrow X$
$A \longmapsto \frac{A}{|A|}$ image of connected $\Rightarrow$ connected
$\Rightarrow X$ is connected

tra $A = a_{11} + a_{22} = 2$

now

tra $I = 2$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$0 \qquad 1$

$0 \longrightarrow a_{12}$

$0 \longrightarrow a_{21}$

$1 \longrightarrow a_{11}$

$1 \longrightarrow a_{22}$

s.t $\quad 1+1 = a_{11} + a_{22}$

$(a_{11} - 1)(t) + 1$

$(a_{22} - 1)(t) + 1$

adding both $= 2$

$$\gamma(t) = \begin{pmatrix} (a_{11}-1)t+1 & t a_{12} \\ t a_{21} & (a_{22}-1)t+1 \end{pmatrix}$$

or $\gamma(t) = tA + (1-t)I$

$X = \{A \mid a_{11}a_{22} > 0\}$

$f : X \longrightarrow \mathbb{R}^2$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \longmapsto (a_{11}, a_{22})$$

image is disconnected

$\Rightarrow X$ is disconnected

Pg152, Q4

(X, τ) $\quad \tau =$ cofinite topology

$\hookrightarrow$ infinite

if $X = U \sqcup V$

$= U^c \sqcup V^c$

$U^c =$ finite

$V^c =$ finite

$\Rightarrow X$ is finite $\quad *$

Pg158 Q2: $f : S^1 \to \mathbb{R}$ cont

to show: $\exists\, x \in S^1$

$f(x) = f(-x)$

let $\quad g(x) = f(x) - f(-x)$

$x_0 \in S^1$

$g(x_0) = 0 \quad$ we are done

If $\quad g(x_0) > 0 \quad$ then

$g(-x_0) < 0 \quad$ (By defination)

$\Rightarrow \exists\, x_1 \in \quad$ b/w x_0 and $-x_0$

IVP $\quad g(x_1) = 0$

$$g(x_0, -x_0) \leq (g(x_0), g(-x_0))$$

$f : X \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}$ then

connected

for $f(x_0), f(x_1) \in \mathbb{R}$
$f(x_0) < f(x_1)$

$\exists\ c \in X$ s.t
$f(c) = y \quad f(x_0) < y < f(x_1)$

case III:
$g(x_0) < 0$
$g(-x_0) > 0$

$0 \in (g(x_0), g(-x_0))$
$\exists\ c \in$ b/w x_0 and $-x_0$
s.t $\quad g(c) = 0$

Pg 158 Q3:

$[0,1] \longrightarrow [0,1]$
now, $\exists\ x \in [0,1]$ s.t $f(x) = x$

$g(x) = f(x) - x$

$\mathbb{D}^2 \cong$ 

Disk $\quad (0,1) \times (0,1) \quad \mathbb{R} \times (0,1)$

23.1 $\quad \tau, \tau'$ two topologies on X
$\tau \subseteq \tau'$
if (X, τ') is connected
and if (X, τ) is not connected
then
$X = U \sqcup V \quad$ for $U, V \in \tau$
$U \cap V = \phi$
$U, V \neq \phi$
as $U \in \tau \subseteq \tau'$
$\Rightarrow U \in \tau'$ and
similary $V \in \tau'$
but as X is connected on τ' $\ast$

$\therefore (X, \tau')$ connected $\Rightarrow (X, \tau)$ is connected

the converse is not true

$$\tau = \{\phi, X\}$$

$$\tau' = \{\phi, U, V, X\}$$

s.t $\quad U^c = V$
$V^c = U$
$U \sqcup V = X$

$\tau \subseteq \tau'$
(X, τ) is connected $\not\Rightarrow (X, \tau')$ is connected

23.2 $\{A_n\}$ sequence of connected subspaces of X, s.t $A_n \cap A_{n+1} \neq \phi$ $\forall n$

To prove : $\cup A_n$ is connected

proof : now let pick n_0

then

$\exists p \in A_{n_0} \cap A_{n_0+1}$

if $\cup A_n$ is not connected then

$\cup A_n = U \sqcup V$

wlog $p \in U \Rightarrow A_{n_0} \subseteq U$

as A_{n_0} is connected

now $p_n \in A_{n_0} \cap A_n$

$\Rightarrow p_n \in C \Rightarrow A_n \subseteq C$ $\forall n$

$\Rightarrow \cup A_n \subseteq C$ $*$

$\therefore \cup A_n$ is connected

23.3 $\{A\alpha\}$ is a collection of connected subspaces of X

A is connected

$\exists p_\alpha \in A \cap A_\alpha$

then if $A \cup (\cup A_\alpha)$ not connected

$A \cup (\cup A_\alpha) = U \sqcup V$

$p_\alpha \in U$

$\Rightarrow A \subseteq U$, $A_\alpha \subseteq U$

$\Rightarrow A \cup (\cup A_\alpha) \subseteq U$

$\Rightarrow A \cup (\cup A_\alpha)$ is not connected $*$

23.4 To prove : If X is an infinite set, then it is connected in cofinite topology

proof : If X is not connected, then

$X = U \sqcup V$ s.t

$U, V \in \tau$

$U \cap V = \phi$

$U, V \neq \phi$

now as $U \in \tau$ $V \in \tau$

$U^c = V \Rightarrow V$ is finite

similarly U is finite

$\Rightarrow U \sqcup V$ is finite $*$

$\therefore$ X is connected

23.5 Totally disconnected : space if only connected subspaces are one point sets

To prove : $(X, 2^X)$ then X is totally disconnected

proof : let $A \subseteq X$ be s.t $A \in 2^X$

now if $|A| > 1$ then

$\exists$ $p_A \in A$ s.t

$|A \setminus \{p_A\}| > 0$

now let

$A = \{p_A\} \cup (A \setminus \{p_A\})$

then as $\tau = 2^X$, $\{p_A\} \in \tau$, $A \setminus \{p_A\} \in \tau$

and $\{p_A\} \neq \phi$, $A \setminus \{p_A\} \neq \phi$

$\therefore$ A is not conneted

if $|A|=1 \Rightarrow A = \{p_A\}$ and is conneted

as if $A = U \sqcup V$

then $|U|+|V|=1$

wlog $|U|=0$

$\Rightarrow U = \phi$

$X = \mathbb{Q}$

$A \subseteq \mathbb{Q}$

then if $x_0, x_1 \in A$

then

$x_0, x_1 \in \mathbb{Q}$

now $\left(\frac{x_0+x_1}{2}\right)\sqrt{2} \in (x_0, x_1)$

but $\left(\frac{x_0+x_1}{2}\right)\sqrt{2} \in \mathbb{R}\setminus\mathbb{Q}$

so $\exists\, p$ s.t

$x_0 < p < x_1$

but $p \notin A$

so $A = U \sqcup V$

st $U = \{x \in A \mid x < p\}$

$V = \{x \in A \mid x > p\}$

then

$A = U \sqcup V$

$U \cap V = \phi$

$U, V \neq \phi$ as $x_0 \in U$

$x_1 \in V$

$\therefore$ if $|A| \geqslant 2 \Rightarrow$ A is not conneted

if $|A|=1 \Rightarrow$ A is conneted (trivial)

$\therefore (\mathbb{Q}, \tau)$ is totally disconnected

23.11 $p: X \longrightarrow Y$

is a Quotient map

To prove : $p^{-1}(\{y\})$ is conneted and Y is conneted $\Rightarrow X$ is conneted

proof:

If X is not conneted

then

$X = U \sqcup V$

$\Rightarrow p(X) = Y = p(U) \cup p(V)$

as p is onto

now if $y \in p(U) \cup p(V)$

$\Rightarrow p^{-1}(\{y\}) \subseteq U$

and $p^{-1}(\{y\}) \subseteq V$ $\nrightarrow$

so $P(U) \cap P(V) = \phi$

now $Y = P(U) \sqcup P(V)$
$P(U) \neq \phi$
$P(V) \neq \phi$
as $U, V \neq \phi$

now, $U \subseteq p^{-1} P(U)$
$\forall\ x \in U$
$\Rightarrow P(x) \in P(U)$
now $y = P(x) \in P(U)$
then
$p^{-1}(\{y\}) \subseteq U$
as $x \in p^{-1}\{y\}$
and $x \in U$
$\Rightarrow p^{-1}\{y\} \subseteq U$
as $p^{-1}\{y\}$ is conneted

$\Rightarrow p^{-1} P(x) \subseteq U$
$\forall x \in U$

$\Rightarrow p^{-1} P(U) \subseteq U$

now, $p^{-1} P(U) = U$
as U is open $\Rightarrow p^{-1}(P(U))$ is open
$\Rightarrow P(U)$ is open
as P is quotient map

similary $P(V)$ is open
$\Rightarrow Y = P(U) \sqcup P(V)$ $\ast$

$\Rightarrow X$ is conneted

23.12 $Y \subseteq X$
X, Y be connected

To prove: If $A \sqcup B = X - Y$ then
$Y \cup A$, $Y \cup B$ are conneted

proof: If $Y \cup A$ is not conneted
then
$Y \cup A = D \sqcup E$
where $D, E \neq \phi$
$D \cap E = \phi$
and D, E open and closed in $Y \cup A$

now as Y is conneted
wlog $Y \subseteq D$

$\Rightarrow E \subseteq A$
now, as A is open and closed in $X \setminus Y$
$\exists\ U \subseteq X$ open
$V \subseteq X$ closed s.t

$A = (X \setminus Y) \cap U$
$A = (X \setminus Y) \cap V$

now $A = (X \cap Y^c) \cap U$

$= (U \cap Y^c) = (U \setminus Y)$

open in X

$A = (U \setminus Y)$

$A = (V \setminus Y)$

closed in X

now $E \subseteq A = (U \setminus Y)$

we have

$E \subseteq U \setminus Y \subseteq U$

$\Rightarrow E \subseteq U$ or E is open in U

$\Rightarrow E$ is open in X

now $E \subseteq (V \setminus Y) \subseteq V$

$\Rightarrow E \subseteq V$ or E is closed in V

$\Rightarrow E$ is closed in X

E is both open and closed $\times$

$\Rightarrow Y \cup A$ is connected

similarly, $Y \cup B$ is connected

24.1 (a) $(0,1)$

$(0,1]$

$[0,1]$ are not homeomorphic

$(0,1] \setminus \{1\} \not\cong (0,1) \setminus \{f(1)\}$

connected   not connected

now $[0,1] \setminus \{0,1\} \not\cong (0,1) \setminus \{f(0), f(1)\}$

$\not\cong [0,1] \setminus \{f(0), f(1)\}$

$\therefore (0,1) \not\cong [0,1] \not\cong (0,1]$

To show : $(0,1) \not\cong (0,1]$

proof : if $(0,1) \cong (0,1]$ then

$\exists F : (0,1] \longrightarrow (0,1)$

s.t

① F is bijective

② F is cont and F^{-1} is cont

then $F : (0,1] \setminus \{1\} \longrightarrow (0,1) \setminus \{F(1)\}$

is also a bijection

but $(0,1)\setminus\{p\}$ is s.t

$$(0,1)\setminus\{p\} = (0,p)\cup(p,1)$$

or this is not connected

$\therefore$ $(0,1)\setminus\{p\}$ not connected but

$(0,1]\setminus\{1\} = (0,1)$ is connected

($\because$ it is path connected)

$\therefore$ $(0,1] \not\cong (0,1)$

(c) To prove : $\mathbb{R}^n \not\cong \mathbb{R}$ for $n>1$

proof : if $\mathbb{R}^n \cong \mathbb{R}$ then, $\exists F$ s.t

$$f : \mathbb{R} \longrightarrow \mathbb{R}^n$$

① F is bijective

② F is cont, F^{-1} is cont

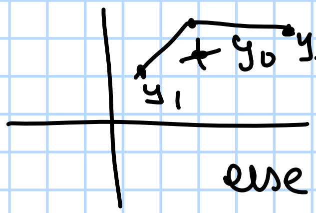
now then $F : \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}^n\setminus\{f(0)\}$

should be s.t

$$\mathbb{R}\setminus\{0\} \cong \mathbb{R}^n\setminus\{f(0)\}$$

but $\mathbb{R}\setminus\{0\}$ is not connected

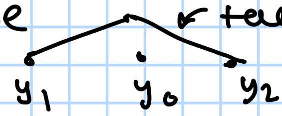
while: $\mathbb{R}^n\setminus\{\underbrace{f(0)}_{y_0}\}$ $\forall\ y_1, y_2 \in \mathbb{R}^n\setminus\{y_0\}$



$\varphi(t) = t y_1 + (1-t) y_2$

if $y_0 \notin \varphi(t)$ $\forall\ t \in [0,1]$

else take path like this



(φ is cont)

24.2 $f : S^1 \longrightarrow \mathbb{R}$

continuous map

To prove : $\exists\ x \in S^1$ s.t $f(x) = f(-x)$

proof : let $g(x) = f(x) - f(-x)$

now if for some $x_0 \in S^1$

Case I : $g(x_0) = 0$ then

we are done

case II : $g(x_0) > 0$

then

$g(-x_0) < 0$ (By construction)

$\Rightarrow \exists\ c \in$ B/W x_0 and $-x_0$
s.t $g(c) = 0$

By IVP

Case III > similar to II

$\therefore \exists\ x \in S^1$ s.t
$f(x) = f(-x)$

24.3 $f : X \longrightarrow X$ is cont
$X = [0,1]$ $g(x) = f(x) - x$
then
Case I : $f(1) = 1$ or
$f(0) = 0$
then done

Case II : If $0 < f(0) \leq f(1) < 1$
then
$g(1) = f(1) - 1 < 0$
$g(0) > 0 \Rightarrow \exists\ c \in (0,1)$ s.t
$f(c) = c$
IVP

Case III > $0 < f(1) \leq f(0) < 1$
same

for $X = [0,1)$ not true as

$f(x) = x/2 + 1/2$ then
$x/2 + 1/2 = x$
$\Rightarrow 1/2 = x/2$
$\Rightarrow x = 1$ $\ast$

for $X = (0,1)$

$f(x) = x/2$
then

$f(x) = x \Rightarrow x/2 = x$
$\Rightarrow x = 0$ $\ast$

(2)

(A) $X = M_2(\mathbb{R})$ (2x2 metric space)

now then

$$M_2(\mathbb{R}) \overset{?}{\cong} \mathbb{R}^4$$

$$f : M_2(\mathbb{R}) \longrightarrow \mathbb{R}^4$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto (a, b, c, d)$$

f is bijective (trivial)

① f is cont:

$f^{-1}(U)$ is open for U open in $\mathbb{R}^4$

let $U = B_\delta(a_0, b_0, c_0, d_0)$

then
$$U = \left\{ (a, b, c, d) \;\middle|\; (a-a_0)^2 + (b-b_0)^2 + (c-c_0)^2 + (d-d_0)^2 < \delta^2 \right\}$$

now
$$f^{-1}(U) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; (a-a_0)^2 + (b-b_0)^2 + (c-c_0)^2 + (d-d_0)^2 < \delta^2 \right\}$$

is open as $(a-a_0)^2 + (b-b_0)^2 + (c-c_0)^2 + (d-d_0)^2$
= polynomial
$\Rightarrow$ cont
$\Rightarrow$ Ball is open

so $f^{-1}(U)$ is open $\forall$ U open in $\mathbb{R}^4$

② now $f(U)$ is open $\forall$ U open in $M_{2\times 2}(\mathbb{R})$

let $U = B_\delta(M_0) = \left\{ M \in M_{2\times 2}(\mathbb{R}) \;\middle|\; d(M, M_0) < \delta \right\}$

now $f(B_\delta(M_0)) = \left\{ (a\ b\ c\ d) \in \mathbb{R}^4 \;\middle|\; d(M, M_0) < \delta \right\}$

this is a norm (Proved)

$$c_2\, d_0(M, M_0) < d(M, M_0) < c_1\, d_0(M, M_0)$$

euclidian distance   euclidian distance

$\therefore$ open as $f(U)$ is just the ball wrt d norm

$\therefore$ $M_{2\times 2}(\mathbb{R}) \overset{?}{\cong} \mathbb{R}^4$

as $\mathbb{R}^4$ is connected $\Rightarrow M_{2\times 2}(\mathbb{R})$ is connected

$\Rightarrow$ Required connected subspace $= M_{2\times2}(\mathbb{R})$

(B) $X = GL_2(\mathbb{R})$

$\hookrightarrow \det(A) \neq 0$ for $A \in GL_2(\mathbb{R})$

now, $\det(A) > 0$ or < 0

$$f : GL_2(\mathbb{R}) \longrightarrow \mathbb{R}\setminus\{0\}$$
$$A \longmapsto \det(A)$$

now $f(A) = \det(A)$

$= \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$= ad - bc$

① f is cont as

$$\mathcal{B}_{\mathbb{R}\setminus\{0\}} = \left\{ (a,b) \;\middle|\; \begin{array}{l} 0<a<b \\ \text{or } a<b<0 \end{array} \right\}$$

now $B \in \mathcal{B}_{\mathbb{R}\setminus\{0\}}$

wlog $B \subseteq (0,\infty)$

then $f^{-1}(B) = \{ M \mid a < \det M < b \}$

$= \{ M \mid 0 < a < \underbrace{ad - bc}_{\substack{\text{polynomial} \\ \Rightarrow \text{cont} \\ \Rightarrow \text{open}}} < b \}$

$\Rightarrow f^{-1}(B)$ is open

② $f(GL_2(\mathbb{R})) = \mathbb{R}\setminus\{0\}$ is trivial

as for $\alpha \in \mathbb{R}\setminus\{0\}$ then

$\det I = 1$

$\det\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} = \alpha$

$\therefore$ from ①, ② if $\mathbb{R}\setminus\{0\}$ is not connected $\Rightarrow GL_2(\mathbb{R})$ is not connected now, then $GL_2(\mathbb{R}) = GL_2^+(\mathbb{R}) \sqcup GL_2^-(\mathbb{R})$

$$g : GL_2^+(\mathbb{R}) \longrightarrow \mathbb{R}^2 \setminus \{(0,0)\}$$

s.t $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a \\ c \end{pmatrix}$

then ① g is onto as:

if $a>0$ then

$$g\begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$$

$a<0$

$1 \to -1$

$a=0$

$1 \to 0$

$0 \to 1$

c changed accordingly

② $g^{-1}(U)$ open $\Leftrightarrow$ U is open

$(\Leftarrow)$ U is open $\Rightarrow$ $U = B_\delta(a_0, c_0)$

$$g^{-1}(U) = \{ M \mid \underbrace{|a-a_0|^2 + |c-c_0|^2}_{\text{cont}} < \delta^2 \}$$

$\Rightarrow g^{-1}(U)$ is open

$(\Rightarrow)$ $g^{-1}(U)$ is open then $\begin{pmatrix} * & b \\ * & d \end{pmatrix} \in g^{-1}(U)$ then for all b d $\begin{pmatrix} * & * \\ * & * \end{pmatrix} \in g^{-1}(U)$

$$g^{-1}(U) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; 0 < a_0 < \det M < b_0 \right\}$$

$a_0 < ad - bc < b_0$
for all such b, d

$$U = \left\{ \begin{pmatrix} a \\ c \end{pmatrix} \middle| \; a_0 < ad - bc < b_0 \text{ for all such } b, d \right\}$$

polynomial

$\Rightarrow$ cont

$\Rightarrow$ U is open

$\therefore$ g is Quotient map

$$g^{-1}\left(\left\{\begin{pmatrix} a \\ c \end{pmatrix}\right\}\right) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; \begin{matrix} ad - bc > 0 \\ d, b \text{ variable} \end{matrix} \right\}$$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; ad > bc \right\}$$

now $g^{-1}\left\{\begin{pmatrix} a \\ c \end{pmatrix}\right\}$ is path connected as

$$\begin{pmatrix} a & b_0 \\ c & d_0 \end{pmatrix} \longrightarrow \begin{pmatrix} a & b_1 \\ c & d_1 \end{pmatrix}$$

$d_0 > \left(\frac{c}{a}\right) b_0 \qquad d_1 > \left(\frac{c}{a}\right) b_1$

$$\varphi(t) = \begin{pmatrix} a & (1-t)b_0 + tb_1 \\ c & (1-t)d_0 + td_1 \end{pmatrix}$$

$$\det(\varphi(t)) = (1-t)d_0 a + t d_1 a - (1-t)b_0 d - t b_1 d$$

$>0 \qquad >0$

$\det(\varphi(t)) > 0 \quad \forall\ t \in [0,1]$

$\varphi(t)$ is cont as linear

$$\varphi(t) = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + [(1-t)b_0 + tb_1]\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + [(1-t)d_0 + td_1]\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \varphi(0) = \begin{pmatrix} a & b_0 \\ c & d_0 \end{pmatrix}$$

$$\varphi(1) = \begin{pmatrix} a & b_1 \\ c & d_1 \end{pmatrix}$$

$\therefore g^{-1}(\{(a,c)\})$ is path connected

$\Rightarrow GL_2^+(\mathbb{R})$ is connected

similarly $GL_2^-(\mathbb{R})$ is connected

as $GL_2(\mathbb{R})$ is not connected

$GL_2(\mathbb{R}) = GL_2^+(\mathbb{R}) \cup GL_2^-(\mathbb{R})$

$\Rightarrow GL_2^+(\mathbb{R})$ biggest connected cont I

(c) $X = O(\mathbb{R})$ 2×2 orthogonal matrices

then $O(\mathbb{R}) = \{ M \mid MM^T = M^TM = I \}$

$f : O(\mathbb{R}) \longrightarrow \{+1,-1\}$

$M \longmapsto \det(M)$

now, $O(\mathbb{R}) = SO_2(\mathbb{R}) \cup [O(\mathbb{R}) \setminus SO_2(\mathbb{R})]$

now $SO_2(\mathbb{R})$ is open in $O(\mathbb{R})$

$SO_2(\mathbb{R}) = \underbrace{\{ M \mid \det M > 0 \}}_{\text{open}} \cap O(\mathbb{R})$

sim $O(\mathbb{R}) \setminus SO_2(\mathbb{R})$ is open

$f : O(\mathbb{R}) \longrightarrow \{+1,-1\}$

$f^{-1}(1) = SO_2(\mathbb{R}) \longrightarrow$ open

$f^{-1}(-1) = O(\mathbb{R}) \setminus SO_2(\mathbb{R}) \longrightarrow$ open

① f is cont
② $f(O(\mathbb{R})) = \{+1, -1\}$

$\therefore$ $O(\mathbb{R})$ is disconneted
and

$$O(\mathbb{R}) = SO_2(\mathbb{R}) \sqcup \left(O(\mathbb{R}) \setminus SO_2(\mathbb{R})\right)$$

now $SO_2(\mathbb{R})$ is conneted as:

$$\varphi(t) = \begin{bmatrix} \cos(t\theta) & -\sin(t\theta) \\ \sin(t\theta) & \cos(t\theta) \end{bmatrix}$$

$\varphi(t)$ is cont as $\varphi(t)$ is the rotation matrix

$\varphi(0) = I$
$\varphi(1) = A_\theta$

$\Rightarrow SO_2(\mathbb{R})$ is path conneted

(D) $X = \{A \in M_2(\mathbb{R}) \mid \text{tra}(A) = 2\}$

let $\varphi(t) = (1-t)I + tA$
then φ is cont as line
φ is s.t
$\varphi(0) = I$
$\varphi(1) = A$
$\varphi(t) \in X$ given

$\Rightarrow X$ is conneted
$\Rightarrow I \in X$
$\hookrightarrow$ largest such set

(E) $X = \{A \in M_2(\mathbb{R}) \mid \det A = 1\}$

as $GL_2^+(\mathbb{R})$ is conneted

$$f : GL_2^+(\mathbb{R}) \longrightarrow X$$
$$M \longmapsto \frac{M}{|M|}$$

then ① f is onto as
$\forall M \in X$
$f(M) = M \in X$

② f is cont as:

$f^{-1}(U)$ is open for U open
U is open then
$U \subseteq X$
s.t U is open
then $U = B_\delta(M) \cap X$ s.t $B_\delta(M) \subseteq GL_2^+(\mathbb{R})$

$f^{-1}(U) = \{B_\delta(M)\}$
$\hookrightarrow$ open

$\therefore X$ is conneted

(F) $X = \{ A \in M_2(\mathbb{R}) \mid a_{11} a_{22} > 0 \}$

$$f : X \longrightarrow \{1, -1\}$$
$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \longmapsto \operatorname{sgn}(a_{11})$$

then ① f is onto as
$f(X) = \{1, -1\}$ is trivial

② f is cont as:

$$f^{-1}(-1) = X^- = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in X \;\middle|\; a_{11} < 0 \right\}$$

$$f^{-1}(+1) = X^+ = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in X \;\middle|\; a_{22} > 0 \right\}$$

$X = X^+ \sqcup X^-$
$\therefore X$ is disconnected

now $$X^+ = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \;\middle|\; a_{11} > 0, a_{22} > 0 \right\}$$

now $I \in X^+$
if $A \in X^+$ then

$$\varphi(t) = (1-t) I + t A$$
$$= (1-t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

as $(1-t) + t a_{11} > 0$
for $t \in (0,1)$
as
$1 - t > 0$, $t a_{11} > 0$

and $t = 0, 1$ trivial

$\Rightarrow \varphi(t) \in X^+ \;\forall\; t \in [0,1]$
$\varphi(t)$ is cont (trivial)
$\varphi(0) = I$
$\varphi(1) = A$
$\Rightarrow X^+$ is conneted

Tutorial-11:

25.4 To prove: If X is locally path connected then every connected open set in X is path connected

proof: Let U be connected open set in X

i.e. $U \in \tau_X$

and U is connected

now $\forall x \in U$ as U is open

$\exists V \subseteq U$ s.t

$V \in \tau_X$

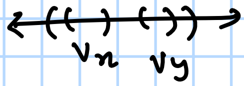
and $x \in V \subseteq U$

V is path-connected

now, for $x, y \in U$

$\exists V_x \subseteq U, V_y \subseteq U$

$\cup$ $x \in V_x, y \in V_y$



if $V_x \cap V_y \neq \phi$ then we are done

if $V_x \cap V_y = \phi$ then

as

$V_x \subseteq U$

$V_y \subseteq U$

U is connected

$U \neq V_x \sqcup V_y$

$\exists z \in U$ s.t

$z \notin V_x, V_y$

and so $\exists V_z \subseteq U$

now

$V_x \cap V_z \neq \phi$

and $V_z \cap V_y \neq \phi$

then we are done

we do this for all z_i

s.t

$x, z_1, z_2, \dots z_n, y$

s.t

$V_x \cap V_{z_1} \neq \phi$

$V_{z_i} \cap V_{z_{i+1}} \neq \phi$

$V_n \cap y \neq \phi$

then $\exists \varphi_1(t)$ s.t

$\varphi_1(0) = x$

$\varphi_1(1) = z_1$

now

$$\varphi(t) = \begin{cases} \varphi_1(nt) & ; 0 \leq t \leq 1/n \\ \varphi_2(nt-1) & ; \frac{1}{n} \leq t \leq \frac{2}{n} \\ \vdots & \\ \varphi_n(nt-n+1) & ; \frac{n-1}{n} \leq t \leq 1 \end{cases}$$

so U is path connect

generally if $\{V_{x_\alpha} | \alpha \in I\}$ are such sets

or use the theorm to say if X is locally path connected then if U is open then path $C \rightarrow$ component of U is open

$\Rightarrow U = \bigcup C$

$\Rightarrow U = C$ as U is connected

$\Rightarrow U$ is path connected

then $\bigcup_{\alpha \in I} V_{x\alpha} \supseteq U$

$\downarrow$

union of path conneted sets s.t
Union is conneted $\Rightarrow U$ is path conneted

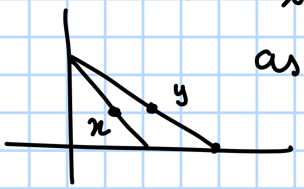
25.5 $([0,1] \times \{0\}) \cap (\mathbb{Q} \times \mathbb{Q}) = X$

$T = \bigcup_{x \in X} l_x$

l_x = line segmet conneting $(0,1)$ to x.

(a) To show : T is path conneted

let $x, y \in T$ s.t $\exists \tilde{x}, \tilde{y}$
$x \in l_{\tilde{x}}, y \in l_{\tilde{y}}$

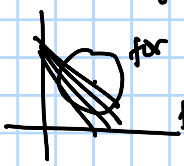


as $T = \bigcup_{x \in I} l_x$ where l_x is path conneted

$\bigcap_{x \in I} l_x \neq \phi$

$\Rightarrow \bigcup_{x \in I} l_x$ is path conneted
$\Rightarrow T$ is path conneted

locally conneted only at P
as



for $x \in T$
$x \neq P$ then
for $\delta < |x - P|$
$B_\delta(x) \cap T$ is open
and no line

$\tilde{l}_x = l_x \cap B_\delta(x)$

$\Rightarrow \bigcap \tilde{l}_x = \{P\} \cap B_\delta(x) = \phi$

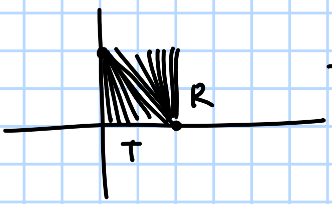
$\Rightarrow T \cap B_\delta(x) = \bigsqcup_{x \in I} \tilde{l}_x$

$\Rightarrow T \cap B_\delta(x)$ is disconneted
$\Rightarrow T \cap B_\delta(x)$ is not path conneted

now $B_\delta(P) \cap T$ is s.t

$\bigcap_{x \in I} \tilde{l}_x \neq \phi \Rightarrow B_\delta(P) \cap T$ is path conneted

(b)



$T \cup R$
$\rightarrow$ path conneted
not locally two

25.6 X ← weakly locally connected at x

$\forall U$ s.t
$U \in \tau_X$, $x \in U$, $U \subseteq X$

$\exists A \subseteq X$ s.t
$A \subseteq U$ s.t A is connected
s.t $\exists V \in \tau_X$ s.t
$V \subseteq X$
$x \in V \subseteq A \subseteq U \subseteq X$
↓
connected

or U be open in X and C be it's component
$\exists A_x$ s.t A_x is connected and
$V_x \subseteq A_x \subseteq U$
↓
open nbd of x

if $x \in C$ then $A_x \subseteq C$ of U
then $V_x \subseteq A_x \subseteq C$ and so

$C = \bigcup_{x \in C} V_x \Rightarrow C$ is open in X
$\Rightarrow$ each component of U is open in X
$\Rightarrow X$ is locally connected

To prove: $\forall x \in X$, x is weakly locally connected $\Rightarrow X$ is locally connected

proof: now for $\forall x \in X$
for $\forall U_x \ni x$ s.t V_x is nbd of x
$U_x \subseteq X$
$\Rightarrow \exists A_x$ connected s.t
$A_x \subseteq U_x$
and $\exists V_x \ni x$
s.t $x \in V_x \subseteq A_x \subseteq U_x$
as A_x is connected
$\Rightarrow U_x$ is connected
$\therefore \forall x \in X$, $\forall$ nbd of x
$\exists V \subseteq$ nbd of x
s.t V is also nbd and connected

25.8 $p: X \to Y$ is a quotient map

To prove: X is locally connected $\Rightarrow Y$ is locally connected

proof: we will use that for a space X to be locally connected
$\Updownarrow$
$\forall U$ open $\subseteq X$, every component $(x \sim y)$ of U is open in X

let U be open in Y, let C be a component of U
if D is a component of $p^{-1}(U)$
s.t it intersects $p^{-1}(C)$ then
D is connected as open in X
and $p(D)$ is also connected and intersects C
$\Rightarrow p(D) \subseteq C$
$\Rightarrow D \subseteq p^{-1}(C)$
so if D is a component of $p^{-1}(U)$
then wlog $D \subseteq p^{-1}(C)$
$\Rightarrow p^{-1}(U) = \bigcup p^{-1}(C)$
now as X is locally connected
as $p^{-1}(U)$ is open every component of $p^{-1}(U)$ is open
$\Rightarrow p^{-1}(C)$ is open
$\Rightarrow C$ is open in Y
so C being a component of U open in Y
$\Rightarrow C$ is open in Y
this tells us that Y is a locally connected space.

Theorm: A space is locally conneted $\Leftrightarrow$ every open set U of it, each componet of U is open

Theorm: A space is locally path conneted $\Leftrightarrow$ every open set U of it, each path componet of U is open.

componets are just $C_1, C_2 \ldots C_\alpha$ $\alpha \in I$

2. $d \to$ metric

(A) $X = M_2(\mathbb{R})$, space of 2×2 matrices
$\downarrow$
conneted as $M_2(\mathbb{R}) \cong \mathbb{R}^4$

(B) $X = GL_2(\mathbb{R}) = GL_2^+(\mathbb{R}) \sqcup GL_2^-(\mathbb{R})$
$\downarrow$ conneted $\downarrow$ conneted
as
$g : \underset{X}{GL_2^+(\mathbb{R})} \longrightarrow \underset{Y}{\mathbb{R}^2 \setminus \{(0,0)\}}$
is a quotient map we
Y is path conneted
and so are fibers

(c) $X = O_2(\mathbb{R})$ space of all 2×2 ortogonal matrices

$$O_2(\mathbb{R}) = \underset{\text{path conneted}}{SO_2(\mathbb{R})} \sqcup \left(O_2(\mathbb{R}) \setminus SO_2(\mathbb{R})\right)$$

(d) $X = \{A \in M_2(\mathbb{R}) \mid tr(A) = 2\} \to$ path conneted

(e) $X = \{A \in M_2(\mathbb{R}) \mid \det(A) > 1\} \to GL_2^+(\mathbb{R})$ with $\det > 1$

$$g : X \longrightarrow \mathbb{R}^2 \setminus \{(0,0)\}$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a \\ c \end{pmatrix}$$
then $ad - bc > 1$

$$g^{-1}\left(\begin{pmatrix} a_0 \\ c_0 \end{pmatrix}\right) = \left\{ \begin{bmatrix} a_0 & b \\ c_0 & d \end{bmatrix} \middle| \ a_0 d - b c_0 > 1 \right\}$$

$$\begin{bmatrix} a_0 & b_1 \\ c_0 & d_1 \end{bmatrix} \longrightarrow \begin{bmatrix} a_0 & b_2 \\ c_0 & d_2 \end{bmatrix}$$

$$\det \begin{bmatrix} a_0 & (1-t)b_1 + tb_2 \\ c_0 & (1-t)d_1 + td_2 \end{bmatrix} > 1$$ fibre is conneted

so path conneted, $\therefore$ conneted (from quotient map theorm)

(f) $X = \{A \in M_2(\mathbb{R}) \mid \underset{\substack{+ \\ -}}{a_{11}}\, \underset{\substack{+ \\ -}}{a_{22}} > \frac{1}{2}\}$

$f : X \longrightarrow \{+, -\}$
$\hookrightarrow$ sgn of a_{11}
$\downarrow$
can be made continious, so not conneted

now $X^+ = \{A \mid a_{11}a_{22} > \frac{1}{2},\ a_{11} > 0\}$

now X^+ is connected as:

$(1-t)A^{\in X^+} + tB^{\in X^+}$

now

$(1-t)A + tB$

$$a'_{11} = (1-t)a_{11} + t\,b_{11}$$
$$a'_{22} = (1-t)a_{22} + t\,b_{22}$$
$$a'_{11}a'_{22} = (1-t)^2 a_{11}a_{22} + (1-t)(t)[a_{11}b_{22} + b_{11}a_{22}] + t^2[b_{11}b_{22}]$$
$$> (1-t)^2\frac{1}{2} + t^2\frac{1}{2} + (1-t)(t)(a_{11}b_{22} + b_{11}a_{22})$$

now $\dfrac{a_{11}b_{22} + b_{11}a_{22}}{2} > (a_{11}b_{22}b_{11}a_{22})^{1/2}$

$\Rightarrow a_{11}b_{22} + b_{11}a_{22} > \cancel{2}\left(\frac{1}{\cancel{2}}\frac{1}{\cancel{2}}\right)^{\frac{1}{2}}$

$\Rightarrow a_{11}b_{22} + b_{11}a_{22} > 1$

so $a'_{11}a'_{22} > (1-t)^2\frac{1}{2} + t^2\frac{1}{2} + (t)(1-t)$

$$> \frac{(1-t+t)^2}{2} = \frac{1}{2}$$

so X^+ is connected
so is X^-

Tutorial-12 :

lemma : Y be subspace of X, Y is compact iff every cover of Y by open sets has a finite subcover

theorm : every closed subspace of compact space is compact

theorm : every compact space of Hausdorff is closed

proof :

let Y be compact subspace of Hausdorff X.

$X-Y$ is open then we are done

for $x_0 \in X-Y$

$\forall y_0 \in Y$ we have

$\exists U_{x_0}, U_{y_0}$ s.t

$x_0 \in U_{x_0}$, $y_0 \in U_{y_0}$

$U_{x_0} \cap U_{y_0} = \phi$

and also $\{U_y \mid y \in Y\}$ cover Y

$\Rightarrow \{U_{y_0}, U_{y_1} \ldots . U_{y_n}\}$ cover Y

let $\underset{\downarrow \text{open}}{U_x} = U_{x_0} \cap U_{x_1} \cap \cdots \cap U_{x_n}$

open and as $U_x \cap U_{y_i} = \phi$

$\Rightarrow U_x \subseteq X-Y$

so $\forall x_0 \in X-Y$ $\exists U_x \subseteq X-Y$, $x_0 \in U_x$

so $X-Y$ is open

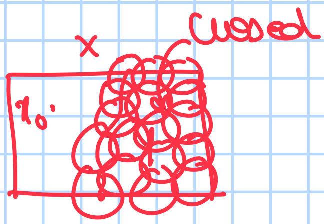
$\Rightarrow Y$ is closed

lemma : Y is compact subspace of Hausdorff X

x_0 is not in Y then $\exists$ U and V of X

s.t $x_0 \in U$, $Y \subseteq V$

$\underbrace{U \cap V}_{\text{open}} = \phi$



theorm : $f : X \to Y$ is cont & X is compact

$\Rightarrow f(X)$ is compact

theorm : $f : X \to Y$ is bijective cont funtion, X compact, Y Hausdorff then $X \cong Y$

theorm : product of finite compact spaces is compact.

26.1 (a) τ, τ' topologies on X

$\tau \subseteq \tau'$

$\downarrow$

finer than τ

if X is compact in τ'

then any cover of X in τ

is also cover in τ'

$\Rightarrow$ is finite $\Rightarrow$ X is compact in τ

converse is not true in general

(b) X is compact Hausdorff under τ, τ'
then either $\tau = \tau'$ or not comparable

if $\tau \subseteq \tau'$ $A \in \tau'$ $\Rightarrow X - A$ is closed
$\downarrow$ $\downarrow$
open in τ' X is closed

$\Rightarrow X - A$ is compact
as $\tau \subseteq \tau'$
$X \setminus A$ compact in τ'
$\Rightarrow X \setminus A$ compact in τ
$\Rightarrow X \setminus A$ closed in τ
$\Rightarrow A \in \tau$
so if $\tau \subseteq \tau'$ then
$\tau = \tau'$

2.(a) finite complement topology on $\mathbb{R}$
every subspace is compact
$$\tau = \{ A \mid |A^c| < \infty,\ A \subseteq \mathbb{R} \} \cup \{ \phi, \mathbb{R} \}$$

now $A \subseteq \mathbb{R}$ s.t $|A^c| < \infty$
i.e let
$$A = \mathbb{R} \setminus \{ x_0, x_1 \dots x_n \}$$
now cover of A : C
let $B \in C$ s.t
$$B = \mathbb{R} \setminus \{ x_{B_1}, x_{B_2}, \dots x_{B_r} \}$$
then
now $\mathbb{R} \setminus A = \{ x_0, x_1 \dots x_n \}$
$\mathbb{R} \setminus B = \{ x_{B_1}, x_{B_2} \dots x_{B_r} \}$

then for every x_i of $\mathbb{R} \setminus A$
$\exists B_i$ s.t $x_i \in \mathbb{R} \setminus B_i$
$B_i \in C$
then $C' = \{ B_0, B_1 \dots B_n \}$ finite subcover of A

so A is compact

(b) $\mathbb{R}$, $\tau = \{ A \subseteq \mathbb{R} \mid \mathbb{R} - A$ is countable or all of $\mathbb{R} \}$

now $[0,1]$ is s.t the open sets are
$\tau = \{ A \subseteq \mathbb{R} \mid \mathbb{R} - A$ is countable or all of $\mathbb{R} \}$

now let $B_n = \{ 1/k \mid k \geq n \}$
now as B_n is countable
$\Rightarrow \mathbb{R} \setminus B_n \in \tau$
let $\underbrace{A_n = [0,1] \cap \mathbb{R} \setminus B_n}_{\text{open in subspace topology of } [0,1]}$

now, $B_n = \{ \frac{1}{n}, \frac{1}{n+1}, \dots \}$

$B_{n+1} = \{ \frac{1}{n+1}, \dots \}$

$B_{n+1} \subseteq B_n$
$\Rightarrow \mathbb{R} \setminus B_n \subseteq \mathbb{R} \setminus B_{n+1}$
$\Rightarrow A_n \subseteq A_{n+1}$

also $\bigcup_{i=1}^{\infty} A_i = [0,1] \cap [\mathbb{R} \setminus B_1 \cup \mathbb{R} \setminus B_2 \cup \dots]$
$= [0,1]$

so $\{ A_i \mid i \in \mathbb{N} \}$ is an open cover of $[0,1]$

but as $A_n \subseteq A_{n+1}$ if $\exists$ a finite subcover
then by wop let n_0 be max index of A_i's
then as $A_{n_0} \subseteq A_{n_0+1}$
$\exists x \in A_{n_0+1}$ not in A_{n_0}
so not possible

3. finite union of compact subspaces of X is compact

now let $Y_1, Y_2, \dots Y_n$ be compact subspaces of X
any open cover O of Y_i is s.t
$\exists \tilde{O} = \{ C_{i1}, C_{i2}, \dots C_{ir} \}$ finite subcor of Y_i

then for $O \rightarrow$ cover of $Y_1 \cup Y_2 \dots Y_n$
O is also a cover of Y_i
$\Rightarrow \exists O_i \rightarrow$ finite subcor of Y_i
then
$O_1 \cup O_2 \cup \dots \cup O_n \rightarrow$ also finite
and as $O_i \subseteq O$
$\Rightarrow \bigcup O_i \subseteq O$
$\Rightarrow$ finite subcor of O
s.t $\bigcup O_i$ is finite subcor of $Y_1 \cup Y_2 \dots \cup Y_n$

$\Rightarrow Y_1 \cup \dots Y_n$ is compact

4. every compact subspace of metric space is bounded and closed in that metric space

now let X = metric space
$Y \subseteq X$ — compact subspace
X is metric space say with d
$\Rightarrow$ X is hausdorff
as X is hausdorff
Y is compact
$\Rightarrow$ Y is closed (By theorm)

now to show Y is bounded in X
$\forall y \in Y$ let $B_n := B_d(y, n)$ be the open ball with d, $n \in \mathbb{N}$
then $\{ B_n \mid n \in \mathbb{N} \}$ is a cover for Y
$\Rightarrow \exists$ finite subcover

as $B_n \subseteq B_{n+1}$
we have
finite subcover $= \{B_{n_1}, B_{n_2} \dots B_{n_r}\}$

then B_{n_r} largest index by wop

is s.t
$Y \subseteq B_{n_r}$ as $B_{n_r} \supseteq B_n \ \forall\, n \leq n_r$

$\Rightarrow Y \subseteq \underset{\text{open}}{B_{n_r}}$
call it B_y

then $\forall\, y \in Y$, $\exists\, B_y$ s.t
$Y \subseteq B_y$
$\Rightarrow Y$ is bounded

now, Y is closed & bounded but not compact in X. d = discrete metric
$\mathbb{R} = X$
X is closed and open both
also $X \subseteq B_d(x, 2)\ \forall\, x \in X$
so X is bounded
but X is not compact
as $\bigcup_{x \in X} \{x\} = X$ but no finite subcover

5. A, B be disjoint compact spaces of Hausdorff X

$A \cap B = \phi$
A, B are closed

now for $x \in A$, and $\forall\, y \in B$, $\exists\, U_x, V_y$ s.t

$x \in U_x$
$y \in V_y$, $U_x \cap V_y = \phi$
and so $\forall\, x \in A, \forall\, y \in Y$
$U_x \cap V_y = \phi$

now let $\{U_x \mid x \in A\}$ be open cover of A

$\{V_y \mid y \in B\}$ be open cover of B

now $\{U_{x_1}, U_{x_2} \dots U_{x_n}\}$ finite subcover of A

$\{V_{y_1}, V_{y_2}, \dots V_{y_r}\}$ finite subcover of B

now $\forall\, \tilde{U} \in \{U_{x_1}, \dots U_{x_n}\}$
$\forall\, \tilde{V} \in \{V_{y_1}, \dots V_{y_r}\}$
$\tilde{U} \cap \tilde{V} = \phi$

now, let $U = \bigcup U_{x_i}$
$V = \bigcup V_{y_i}$
then $U \cap V = \phi$
and $A \subseteq U$, $B \subseteq V$

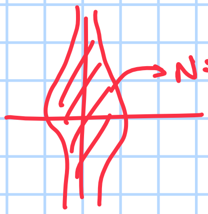
6. $f : X \to Y$ is cts
X compact, Y Hausdorff

we have to show f is a closed map
i.e U closed in X
$\Rightarrow f(U)$ closed in Y

now X is compact
C is closed in X
$\Rightarrow C$ is compact
$\Rightarrow f(C)$ is compact
as f is cts
$\Rightarrow f(C)$ is closed
as $f(C)$ is
compact in Hausdorff space

$\Rightarrow f(C)$ is closed

lemma: (Tube lemma) $X \times Y$, Y is compact. If N is open set of $X \times Y$ contain $x_0 \times Y$ slice of $X \times Y$ then N contain some tube $W \times Y$ about $x_0 \times Y$
W: Nbd of x_0



$N = \{x \times y \mid |x| < \frac{1}{y^2+1}\}$

$0 \times \mathbb{R} \subseteq N$ but
no such tube exist
as $\mathbb{R}$ is not compact
$\mathbb{R} = Y$

7. Y is compact $\pi_1 : X \times Y \to X$
π_1: projection map

topology on $X \times Y$ is s.t π_1 is cts
by defination of product topology

now, then for C closed in $X \times Y$
$\pi_1(C)$ is s.t
for $x_0 \in X \setminus \pi_1(C)$
$x_0 \times Y \subseteq X \times Y \setminus C$ (open)
now as $X \times Y \setminus C$ is open we can use tube lemma
and $x_0 \times Y \subseteq X \times Y \setminus C$ as Y is compact
$\exists\, W$ open in X
s.t $x_0 \in W$
and $W \times Y \subseteq X \times Y \setminus C$
now for $x \in W$ & $y \in Y$
as $x \times y \in X \times Y \setminus C$
$\Rightarrow x \times y \notin C$
$x \in W \Rightarrow x \notin \pi_1(C)$
$\Rightarrow W \subseteq X \setminus \pi_1(C)$
so $\forall\, x_0 \in X \setminus \pi_1(C)$, $\exists\, W$ open
s.t
$W \subseteq X \setminus \pi_1(C)$
or $X \setminus \pi_1(C)$ is open
$\Rightarrow \pi_1(C)$ is closed
C, closed in X $\Rightarrow \pi_1(C)$ closed

8. <u>To prove</u>: let $f: X \to Y$, Y is compact and Hausdorff.

f is cont $\Leftrightarrow$ $G_f = \{x \times f(x) \mid x \in X\}$ graph is closed in $X \times Y$.

<u>proof</u>: ($\Rightarrow$) f is continious, then $f^{-1}(U)$ is open for U open in Y

now, $X \times Y \setminus G_f$ is s.t
$x_0 \times y_0 \in X \times Y \setminus G_f$
then
$x_0 \in X$
$y_0 \in Y$
s.t $f(x_0) \neq y_0$
now, then as Y is hausdorff
$\exists$ U, V s.t
$f(x_0) \in U$
$y_0 \in V$
and $U \cap V = \phi$
and $x_0 \in f^{-1}(U)$
$y_0 \in V$

$$\Rightarrow \underbrace{\underset{\text{open}}{f^{-1}(U)} \times \underset{\text{open}}{V}}_{\text{open in } X\times Y} \ni x_0 \times y_0$$

open in $X \times Y$ is a nbd of $x_0 \times y_0$

and if $G_f \cap (f^{-1}(U) \times V) \neq \phi$
then
$\exists$ $x \times y \in f^{-1}(U) \times V$ s.t
$f(x) \in U$
$y \in V$
and $f(x) = y$
but $U \cap V = \phi$ *

so $f^{-1}(U) \times V \cap G_f = \phi$

$\Rightarrow f^{-1}(U) \times V \subseteq X \times Y \setminus G_f$

or $\forall$ $x_0 \times y_0 \in X \times Y \setminus G_f$
$\exists$ $f^{-1}(U) \times V$ open
s.t
$f^{-1}(U) \times V \subseteq X \times Y \setminus G_f$

so, $X \times Y \setminus G_f$ is open
$\Rightarrow G_f$ is closed

($\Leftarrow$) Now if G_f is closed

(we will use f is cts $\Leftrightarrow$ $\forall x \in X$, nbd of $f(x) = V$ is s.t $\exists$ U nbd of x s.t $f(U) \subseteq V$)

let V be a nbd of $f(x_0)$ for $x_0 \in X$

$$\text{now } C = \underbrace{G_f}_{\text{closed}} \cap \underbrace{(X \times (Y \setminus V))}_{\text{closed}}$$

$\Rightarrow C =$ closed in $X \times Y$

and as Y is compact (By last question)
$\pi_1(C)$ is closed in X

let $U = X \setminus \pi_1(C)$ then
if $x \in U$, then $f(x) \in V$, if not then $f(x) \notin V$

then $x \times f(x) \in G_f$
$x \times f(x) \in X \times Y \setminus V$

$\Rightarrow x \times f(x) \in G_f \cap (X \times Y \setminus V)$
$\Rightarrow x \times f(x) \in C$

$\Rightarrow \pi_1(x \times f(x)) = x \in \pi_1(C)$
but as $x \in U = X \setminus \pi_1(C)$
this is a contradiction
or $f(x) \notin V$ is a contradiction *

so for $x \in U$, $f(x) \in V$

$\Rightarrow f(U) \subseteq V$

$\Rightarrow f$ is continuous

9. To prove: let A, B be subspaces of X, Y. N is an open set containing $A \times B$
$A \times B \subseteq N \subseteq X \times Y$
if A, B are compact then $\exists\, U \subseteq X, V \subseteq Y$ open s.t. $A \times B \subseteq U \times V \subseteq N$

proof: $a \in A$, consider
$a \times B$ slice of $A \times B$
then $a \times B \subseteq N$
now as $a \times B \subseteq N$
and N is an open set in $X \times Y$
$N = \{\bigcup(U \times V),\ U \text{ open in } X,\ V \text{ open in } Y\}$
Basis elements
let these elements be
$\{W_1 \times Z_1, W_2 \times Z_2, \ldots\}$
cover $N \Rightarrow$ cover $a \times B$
and as B is compact
and a is just an element
$\Rightarrow a \times B$ is compact
and so $\exists\ \{W_1 \times Z_1, W_2 \times Z_2, \ldots W_n \times Z_n\}$ s.t. it covers $a \times B$
now $U_a = \bigcap_{i=1}^{n} W_i$

$V_a = \bigcup_{i=1}^{n} Z_i$

then $U_a \ni a$ and is open
V_a is open in Y and contains B
then $a \times B \subseteq U_a \times V_a$
and also $U_a \times V_a \subseteq N$
now as $U_a \times U_a \subseteq N$ and $U_a \times V_a$ is open
$\{U_a \times V_a \mid a \in A\}$ is s.t. it is a open collection of $A \times B$
which is compact
$\Rightarrow \{U_{a_1} \times V_{a_1}, U_{a_2} \times V_{a_2}, \ldots U_{a_k} \times V_{a_k}\}$ is a finite subcover
s.t. $U_{a_i} \times V_{a_i} \subseteq N$

as $\forall a_i,\ B \subseteq V_{a_i}$

$$\Rightarrow B \subseteq \bigcap_{i=1}^{k} V_{a_i} = V$$

$\downarrow$ open in Y

and $\bigcup_{i=1}^{k} U_{a_i}$ covers A by construction

$$\Rightarrow U = \bigcup_{i=1}^{k} U_{a_i}$$

s.t $A \subseteq U$

so $A \times B \subseteq U \times V$

now $\forall\ a \times b \in U \times V$

$\exists\, i$ s.t

$b \in V_{a_i}$

so $a \in U_{a_i}$

and $a \times b \in U_{a_i} \times V_{a_i} \subseteq N$

so $A \times B \subseteq U \times V \subseteq N$

10. (a) <u>To prove</u>: let $f_n : X \to \mathbb{R}$ be a seq of cts fuctions $f_n(x) \to f(x)\ \forall x \in X$

now if f is cts and f_n is monotone increasing.

If X is compact then the conv is uniform

<u>proof</u>: for any $x \in X$, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t

$$|f(x) - f_n(x)| < \varepsilon \quad \forall n \geq N$$

now as f_n is mon inc

$$\Rightarrow f(x) - f_n(x) < \varepsilon \quad \forall n \geq N$$

as f and f_n are cts, $F_n(x) = f(x) - f_n(x)$ is cts & $< \varepsilon$

$\exists\, U_x \to$ open cover s.t

$$\forall z \in U_x \quad F_n(z) < \varepsilon$$

$$\Rightarrow f(z) - f_n(z) < \varepsilon \quad \forall z \in U_x$$

by property of continuity

and so $\{U_x \mid x \in X\}$ is an open cover

$\exists$ subcover finite cong X

$$\Rightarrow \{U_{x_1}, U_{x_2}, \ldots U_{x_n}\}$$

let $N = \max\{N_{x_i}\}$

$\to N$ from how we defined it

then $\forall x \in X$, $\exists\, i \in \{1, \ldots n\}$ s.t $x \in U_{x_i}$

$\Rightarrow$ as $N \geq N_{x_i}$

$$f(x) - f_n(x) < \varepsilon \quad \forall n \geq N$$

so, $\forall x \in X$, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t

$$f(x) - f_n(x) < \varepsilon \quad \forall n \geq N$$

$\therefore$ uniform

(b) If X is not compact then

$f_n : \mathbb{R} \to \mathbb{R}$
$\overset{||}{X}$ (not compact)

s.t $f_n(x) = \dfrac{1}{n^3[x - \frac{1}{n}]^2 + 1}$ is cts (trivial)

& now as $n \to \infty$

$f_n(x) \to 0 \quad \forall x$
$\downarrow$
cts

but not uniformly as

for $x_n = \frac{1}{n}$

$f_n(x_n) = 1 \neq 0 \ \forall n$

so the function does not conv to 0 wrt this x_n

now for $X = [0,1]$
we have
f_n not monotone
as

for $f_1(x) = \dfrac{1}{(x-1)^2+1}$

$f_2(x) = \dfrac{1}{8(x-\frac{1}{2})^2+1}$

$f_1(1) = 1$

$f_2(1) = \dfrac{1}{\cancel{2}8(\frac{1}{4})+1} = \frac{1}{3} < f_1(1)$

so the function are not monotone

$f_n : X = [0,1] \to \mathbb{R}$ s.t
f_n, f cts but
not monotone
so not uniform conv example.

for $X = \mathbb{R}$

$f_n : \mathbb{R} \to \mathbb{R}$

$f_n(x) = \tan^{-1}(x+n)$

as $n \to \infty$

$f_n \to \pi/2$ const

while

$f_n(x) < f_{n+1}(x)$

from property of $\tan^{-1}$

$\therefore$ monotone but

$x_n = -n$ is s.t

$f_n(x_n) = 0 \quad \forall n \in \mathbb{N}$

so not uniform

so one more counterexample

12. $p: X \to Y$ closed cont surjective map $p^{-1}(\{y\})$ is compact for each $y \in Y$ (perfect map)

now ① $p(X) = Y$
② p is cont
③ p is closed map
④ $p^{-1}(\{y\})$ is compact for $y \in Y$

now

Y is compact then

if U is open set containing $p^{-1}(\{y\})$
then $X - U$ is closed.
$\Rightarrow p(X-U)$ is closed in Y
and as $p^{-1}(\{y\}) \subseteq U$
$\Rightarrow y \notin p(X-U) \Rightarrow y \in p(X-U)^c$
then $Y \setminus p(X-U)$ is open s.t
$y \in Y \setminus p(X-U)$
let $W = Y \setminus p(X-U)$
now
$$p^{-1}(W) = p^{-1}(Y \setminus p(X-U))$$
$$= p^{-1}(Y) \setminus p^{-1}p(X-U)$$
$$= X \setminus p^{-1}p(X-U)$$
$$\subseteq X \setminus (X-U) = U$$
as $(X-U) \subseteq p^{-1}p(X-U)$

so $p^{-1}(W) \subseteq U$

so for U s.t $p^{-1}(\{y\}) \subseteq U$ open
is s.t $\exists W \subseteq Y$
$y \in W$
and $p^{-1}(W) \subseteq U$

now let C = open cover of X
$\forall y \in Y$ let C_y be subcollection of C s.t
$$p^{-1}(\{y\}) \subseteq \bigcup_{A \in C_y} A$$
as $p^{-1}(\{y\})$ is compact, $\exists \tilde{C}_y$ finite subcollection
let $\tilde{C}_y = \{A_{y_1}, A_{y_2}, \dots A_{y_n}\}$
then $\bigcup_{i=1}^{n} A_{y_i} = U_y$ s.t $p^{-1}(\{y\}) \subseteq U_y$
open
so, $\exists W_y \ni y$ s.t $p^{-1}(W_y) \subseteq U_y$
open
now $\{W_y \mid \forall y \in Y\}$ is a cover for Y
$\Rightarrow \{W_{y_1}, W_{y_2}, \dots W_{y_k}\}$ is a subcover for Y
and $X = p^{-1}(Y) \subseteq p^{-1}(\bigcup_{j=1}^{k} W_{y_j}) = \bigcup p^{-1}(W_{y_j})$
$\subseteq \bigcup U_{y_j}$

so for a cover C
$\exists \tilde{C}_{y_1} \cup \tilde{C}_{y_2} \cup \dots \tilde{C}_{y_k}$ s.t it is a finite subcover for X. So X is compact

isolated point of X, for $x \in X$ $\{x\}$ is open

<u>Theorm</u>: X be non-empty, compact, Hausdorff
no isolated points $\Rightarrow X$ is uncountable

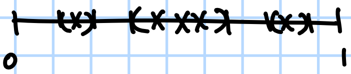
<u>proof</u>: for a non-empty open set U of X
as $U \neq \{x\}$ $\exists\, x, y \in U$
s.t as hausdorff, $\exists\, U_x$ nbd of x
U_y nbd of y
s.t $U_x \cap U_y = \phi$
let $V = U \cap U_y$
then V is open and as $x \in U \cap U_x$
open
$\Rightarrow x \notin (U \cap U_x)^c \subseteq \bar{V}$
$\Rightarrow x \notin \bar{V}$

any $f : \mathbb{N} \to X$ and show not surjective $(f(\mathbb{N}) \neq X)$
$x_n = f(n)$
now applying the first step to $U = X$
choose $V_1 \subseteq X$ s.t
$x_1 \notin \bar{V}_1$
then V_1 s.t $x_1 \notin \bar{V}_1$
V_2 s.t $x_2 \notin \bar{V}_2$ but $x_2 \in V_1$
$V_n \subseteq V_{n-1}$
& $\bar{V}_n \not\ni x_n$
now, $\bar{V}_1 \supseteq \bar{V}_2 \supseteq \dots\dots$
as X is compact
let $x_1 \in \bar{V}_1, x_2 \in \bar{V}_2, \dots$
then $\exists\, x = \cap \bar{V}_n$ s.t
$x_n \to x$
now, but $x_n \neq x$ as $x \in \bar{V}_n$ $\forall n$
but $x_n \notin \bar{V}_n$
contradiction
so X is not countable

4. connected metric space having more than one point is uncountable
① connected
② non-empty
③ Hausdorff
now $d : X \times X \to \mathbb{R}$ is cts
for $x \in X$, $d_x : X \to \mathbb{R}$
by $d_x(y) = d(x, y)$ is cts
now X is connected
$\Rightarrow d_x(X)$ is connected subspace of $\mathbb{R}$
as $x \in X$ more
$\Rightarrow d_x(x) = 0$ for $y \neq x$, $y \in X$ than 1 point
so $[0, \delta] \subseteq d_x(X)$
$\delta = d_x(y) > 0$
so $d_x(X)$ contains
$[0, \delta]$ and so is uncountable
$\Rightarrow X$ is uncountable
as d_x is continious

6. $A_0 \leftarrow$ closed $[0,1]$ in $\mathbb{R}$

$A_1 = A_0 - \left(\frac{1}{3}, \frac{2}{3}\right)$



$$A_n = A_{n-1} - \bigcup_{k=0}^{\infty} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n}\right)$$

$$C = \bigcap_{n \in} A_n$$

Cantor set $\subseteq [0,1]$

(a) If B is a connected component of Cantor set

if $x, y \in B$

s.t $x \neq y$, $x, y \in \mathbb{R}$

then $\exists n \in \mathbb{N}$ s.t

$1/3^n > |x-y|$

$\Rightarrow x$ and y in different closed Interval of A_n

$\Rightarrow B$ is not connected

so B to be connected

$x = y$

or $B = \{x\}$

$\therefore$ Cantor set is totally disconnected.

(b) As $A_n = A_{n-1} - \bigcup_{k=0}^{\infty} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n}\right)$

$$C = \underbrace{[0,1] \setminus \overbrace{\bigcup_{k=0}^{\infty} \bigcup_{n=1}^{\infty} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n}\right)}^{\text{this is countable union of open sets} \Rightarrow \text{open in } [0,1]}}_{\text{closed in } [0,1]}$$

so $C =$ Cantor set is closed in $[0,1]$

$\Rightarrow C$ is compact

(c) By induction $A_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

now at each step

closed intervals of $\frac{1}{3^n}$ of A_n

is divided into three closed interval of $\frac{1}{3^{n+1}}$

but interior of middle is removed

keeping 2 closed interval of $\frac{1}{3^{n+1}}$ in A_{n+1}

now, as interiors removed

$\Rightarrow$ end points lie in C

(d) let $x \in C$, then $x \in A_n \; \forall n \in \mathbb{N}$

so $\exists I_n \subseteq A_n$ s.t

$x \in I_n$

now choose endpoint x_n of I_n s.t $x_n \neq x$ (wlog)

then $x_n \to x$ as $n \to \infty$

$\Rightarrow x$ is not an isolated point

as this is true $\forall x \in C$

$\Rightarrow x$ is not an isolated point

$\Rightarrow C$ has <u>no</u> isolated point

(e) ① C has no isolated points

② C is compact

③ C is non-empty

④ C is a metric space so Hausdorff

$\Rightarrow C$ is uncountable

Tutorial - 13 :

Defn : (limit point compactness) A space X is said to be limit point compact if every infinite subset of X has a limit point.

Theorm : compactness ⇒ limit point compactness

if X is compact

then every cover has finite subcover

now if X is not limit point compact

then

let A be a subset of X

then if A has no limit point

when X is compact

⇒ A contains all its limit point

$\Rightarrow A = \bar{A}$ & $\forall a \in A, \exists U_a \ni a$ s.t

$U_a \cap A \setminus \{a\} = \phi$ (as no limit point of A)

now as A is closed ⇒ $X \setminus A$ is open

$\{X \setminus A, U_a \mid \forall a \in A\}$ cover X

∃ finite subcover

s.t $\{X \setminus A, U_{a_1}, \ldots U_{a_n}\}$ cover X

$\Rightarrow A \subseteq \{a_1, \ldots a_n\} \rightarrow$ finite

so X is compact then A has no limit point ⇒ A is finite

~(A is finite) ⇒ ~(A has no limit point)

A is infinite ⇒ A has limit point

so if X is compact then X is limit-point-compact

Defn : X is a topological space

(x_n) is a sequence of points of X

$n_1 < n_2 \ldots \; n_i < \ldots$ is an inc seq of poisitive int

then (y_i) s.t

$y_i = x_{n_i}$ is called a subsequence of (x_n)

X is said to be sequentially compact if every seq of points of X has a conv subseq.

Theorm : (X, d) is a metric space, then

X is compact ⇔ X is a limit point compact

⇕

X is sequentially compact

6. (X,d) is a metric space

$f: X \longrightarrow X$ satisfies

$d(f(x), f(y)) = d(x,y)$
$\forall\, x,y \in X$
then f is called isometry of X

To prove: f is an isometry of X, X is compact then f is bijective & hence homeomorphism

proof: now if $g: X \longrightarrow Y$
s.t X is compact
Y is Hausdorff
g is bijective and cont
then
g is homeomorphism

now $f: X \longrightarrow X$
$\downarrow$ compact, $\downarrow$ Hausdorff
we have

① f is 1-1 as
if $f(x_1) = f(x_2)$ for $x_1, x_2 \in X$
then
$d(f(x_1), f(x_2)) = d(x_1, x_2) = 0$
$\Rightarrow x_1 = x_2$

② f is cont as
for $d(x_1, x_2) < \delta$
$\Rightarrow d(f(x_1), f(x_2)) < \delta = \varepsilon$

$\therefore \forall \varepsilon > 0, \exists \delta = \varepsilon$ s.t
$\forall\, d(x,a) < \delta$
$\Rightarrow d(f(x), f(a)) < \varepsilon$

so f is continious

③ f is surjective as:
now if not then
if $\exists a \notin f(X)$
as X is compact $f(X)$ is also compact
$\Rightarrow f(X)$ is closed

so $\exists \varepsilon > 0$ s.t
$B_d(a, \varepsilon) \subseteq X \setminus f(X)$

now $x_1 = a$ and
$x_2 = f(x_1)$
$\vdots$
$x_{n+1} = f(x_n)$ $\forall n \in \mathbb{N}$
wlog $n > m$
now $d(x_n, x_m) = d(f(x_{n-1}), f(x_{m-1}))$
$= d(x_{n-1}, x_{m-1})$
$\vdots$
$= d(x_{n-m}, x_0)$

$d(x_n, x_m) = d(x_{n-m}, a)$

now $x_{n-m} = f(x_{n-m+1}) \in f(X)$

and $B(a,\varepsilon) \cap f(X) = \phi$

$\Rightarrow d(x_{n-m}, a) \geq \varepsilon$

$\Rightarrow d(x_n, x_m) \geq \varepsilon$

so (x_i) is a seq with no conv subsequence

$\Rightarrow X$ is not seq compact $*$

$\Rightarrow f$ is surjective

now ①, ②, ③ $\Rightarrow f$ is bijective and cont

$$\begin{array}{ccc} f: X & \longrightarrow & X \\ \downarrow & & \downarrow \\ \text{compact} & & \text{Hausdorff} \end{array}$$

$\Rightarrow f$ is homeomorphism

7. (X,d) metric space

$d(f(x), f(y)) < d(x,y)$
$\forall x, y \in X$
$x \neq y$

then f is a shrinking map

now if $\exists \alpha < 1$ s.t
$d(f(x), f(y)) \leq \alpha d(x,y)$
$\forall x, y \in X$
then f is called a contraction

A fixed point of f is point x s.t $f(x) = x$

(a) f is a contraction, X is compact
<u>To prove</u>: f has a unique fixed point

<u>proof</u>: let $f' = f$
now as f' is a contraction
$\exists \alpha < 1$ s.t
$d(f(x), f(y)) \leq \alpha d(x,y)$

now
$f^2 = f \circ f$

also
$A_n = f^n(X)$

then
$A_{i-1} = f^{i-1}(X)$
$A_i = f^i(X)$

X is a metric space & X is compact
$\Downarrow$
X is closed

so A_i is closed $\forall i$

now $A = \bigcap_{i \in \mathbb{N}} A_i \rightarrow$ is closed as A_i's are closed

also a $d(f(x), f(y)) \leq \alpha d(x,y)$

$\Rightarrow A_{n+1} \subseteq A_n$

now as $A = \bigcap_{i \in \mathbb{N}} A_i$ and $A_{n+1} \subseteq A_n \Rightarrow A$ is non-empty

let $y \in f(A)$ then $\exists x \in A$
s.t $f(x) = y$
as $x \in A = \bigcap_{i \in \mathbb{N}} A_i$, $\exists n \in \mathbb{N}$ s.t
$x \in A_n$
now if $n=1$ then $x \in A_1 = f(X)$
now $y = f(x) \in f(f(X))$
$\Rightarrow A_2$
$y \in A_2$
so for $x \in A_{n-1} \Rightarrow y \in A_n$
now this is true for $\forall n$
$\Rightarrow y \in \bigcap_{n \in \mathbb{N}} A_n$
$\Rightarrow y \in A$
so for $y \in f(A) \Rightarrow y \in A$
$\Rightarrow f(A) \subseteq A$

now, let $x \in A$ then $x \in f^n(X)$ $\forall n$
so $x \in f^{n+1}(X)$
$\exists x_n \in X$
s.t
$f^{n+1}(x_n) = x$ $\forall n \in \mathbb{N}$
now let $y_n = f^n(x_n)$
as X is a metric space
it is sequentially compact
so $\{y_n\}_{n \in \mathbb{N}}$ has a conv subseq
$\{z_n\}_{n \in \mathbb{N}}$
then for $a \in X$
$z_n \longrightarrow a$
& $z_n \in A$ $\forall n$
as X is a metric space
$a \in \overline{A} = A$
$\Downarrow$
A is closed

so $z_n \longrightarrow a$
now as f is cont
$\lim_{n \to \infty} f(z_n) = x$
$= f(a)$
$x = f(a) \Rightarrow x \in f(A)$
$\therefore$ $A \subseteq f(A)$
so $A = f(A)$

now, A has only one point as if more then one then as:
$x \neq y \in A$
we have $d(f(x), f(y)) \leq \alpha d(x,y)$

$\Rightarrow d(x,y) \leq \alpha d(x,y)$
$\Rightarrow d(x,y) = 0$ $*$
so only one point in A

(b) similar calculation

(c) $X = [0,1]$ $\quad f(x) = x - \frac{x^2}{2} : X \longrightarrow X$

To prove: f is a shrinkg map but not a contraction

proof : $\forall x \in [0,1]$

$$(x)(1-\frac{x}{2})$$

$$= \frac{(x)(2-x)}{2}$$

now $f(x) = \frac{1}{2}(x)(2-x)$

$f'(x) = 1-x = 0$

$f'(x) = 0$

for $x=1$

now $f''(x) = -1 < 0$

so maxima at $x=1$

ad $f'(x) \geq 0$ for $x \in [0,1]$

so inc

now $f(0) = 0$

$f(1) = \frac{1}{2}$ $\quad$ so $f(X) = [0,\frac{1}{2}] \subseteq X$

$\therefore f : X \longrightarrow X$

now to show it is a shrinking map, let

$x, y \in [0,1]$

wlog $x < y$

now

$d(x,y) = y - x$ and as $x, y \in [0,1]$

$\Rightarrow y - x \leq 1$

now $d(f(x), f(y))$

$= f(y) - f(x)$

as f is inc function

$$f(y) - f(x) = y - \frac{y^2}{2} - x + \frac{x^2}{2}$$

$$= y - x + \frac{1}{2}(x^2 - y^2)$$

$$= y - x + \frac{1}{2}(x-y)(x+y)$$

$$= y - x - (y-x)\left(\frac{x+y}{2}\right)$$

$$= (y-x)\left(1 - \left(\frac{x+y}{2}\right)\right)$$

now as $(y-x)\left(\frac{x+y}{2}\right) \geq 0$

$\Rightarrow -(y-x)\left(\frac{x+y}{2}\right) \leq 0$

$\Rightarrow (y-x) - (y-x)\left(\frac{x+y}{2}\right) \leq (y-x)$

$\Rightarrow f(y) - f(x) \leq y - x$

now equality when $x = y$

so, for $x \neq y$
$$d(f(x), f(y)) < d(x, y)$$

now, if $\exists \alpha$ s.t
$$d(f(x), f(y)) \leqslant \alpha\, d(x, y)$$
for $y = 0$
$$|f(x) - f(0)| = (x - 0)\left(1 - \frac{x}{2}\right)$$
now for $\alpha < 1 - \frac{x}{2}$

condition not satisfied

(d) for $x, y \in \mathbb{R}$ $x \neq y$
$$|f(x) - f(y)| = \frac{|x-y|}{2}\left| 1 + \frac{(x^2+1)^{\frac{1}{2}} - (y^2+1)^{\frac{1}{2}}}{(x-y)} \frac{((x^2+1)^{\frac{1}{2}} + (y^2+1)^{\frac{1}{2}})}{((x^2+1)^{\frac{1}{2}} + (y^2+1)^{\frac{1}{2}})} \right|$$
$$= \frac{|x-y|}{2}\left| 1 + \frac{(x^2+1) - (y^2+1)}{(x-y)\left[(x^2+1)^{\frac{1}{2}} + (y^2+1)^{\frac{1}{2}}\right]} \right|$$
$$= \frac{|x-y|}{2}\left| 1 + \frac{x+y}{(x^2+1)^{\frac{1}{2}} + (y^2+1)^{\frac{1}{2}}} \right|$$
$$< \frac{|x-y|}{2} + \frac{|x-y|}{2}\left| \frac{x+y}{(x^2+1)^{1/2} + (y^2+1)^{1/2}} \right|$$
$$\frac{|x| + |y|}{(|x|^2+1)^{\frac{1}{2}} + (|y|^2+1)^{\frac{1}{2}}} < 1$$
$$\Rightarrow < |x-y|$$

so $|f(x) - f(y)| < |x-y|$
$\therefore f$ is a shrinking map

now for $|f(x) - f(0)| = \left|f(x) - \frac{1}{2}\right|$

if $\exists\ \alpha < 1$ s.t
$$\left|f(x) - \frac{1}{2}\right| \leqslant \alpha |x|$$
as f is strictly inc for $x > 0$ we have
$$f(x) > \frac{1}{2} \text{ for } x > 0$$
now $f(x) - \frac{1}{2} \leqslant \alpha |x|$

but $f(x) - \frac{1}{2} = \frac{x}{2}\left(1 + \frac{x}{(x^2+1)^{1/2}}\right)$
$$= \frac{x}{2}\left(1 + \frac{1}{(1+\frac{1}{x^2})^{1/2}}\right)$$
so $\alpha > 1 + \frac{1}{(1+\frac{1}{x^2})^{1/2}} > 1$ *

$$f(x) = \frac{x + (x^2+1)^{1/2}}{2} > \frac{x + |x|}{2} > \frac{x+x}{2} = x$$

$\Rightarrow f(x) > x \;\forall x \in \mathbb{R}$

$\therefore$ No fixed point

6. f defined on E

$G = \{(x, f(x)) \mid x \in E\}$

E is compact

To prove: f is cont on $E \Leftrightarrow$ graph is compact

proof: $(\Rightarrow)$ let $Y = f(E)$

then

$E \times Y$ as (x, y), $x \in E, y \in Y$

and

$(E \times Y, \rho)$ s.t

$\rho((x_1, y_1), (x_2, y_2)) = d_E(x_1, x_2) + d_Y(y_1, y_2)$

$\varphi(x) = (x, f(x))$ is s.t

$\varphi : E \longrightarrow E \times Y$

now if f is cont then

$\forall \varepsilon > 0, \exists \delta > 0$ s.t

$d_Y(f(x), f(u)) < \varepsilon$

if $d_E(x, u) < \delta$

let $\delta' = \min(\delta, \varepsilon)$

then for $d_E(x, u) < \delta'$

$\Rightarrow d_Y(f(x), f(u)) < \varepsilon$

& $d_E(x, u) + d_Y(f(x), f(u)) < 2\varepsilon$

so $\forall \varepsilon' = 2\varepsilon > 0, \exists \delta' > 0$ s.t

$\rho(\varphi(x), \varphi(u)) < \varepsilon'$

$\forall d_E(x, u) < \delta'$

so $\varphi(x)$ is cont

now

$\varphi : E \longrightarrow E \times Y$

now $\varphi(E) = G$

$\downarrow$ compact

continious $\Rightarrow G$ is compact

$(\Leftarrow)$ Now if f is not continious then

then $\exists (x_n) \longrightarrow x$ s.t

$f(x_n) \not\rightarrow f(x)$

so $\{(x_n, f(x_n)\}_{n=1}^{\infty}$ no cvg subseq $\Rightarrow G$ not compact

if $\exists$ cvg subsequence $f(x_{n_k}) \rightarrow z$ but $z \neq f(x)$

so f fails to contain limit point (x, z)

$\therefore$ Not closed

20. $E \neq \phi$ $E \subseteq X$ (metric space) $\quad \rho_E(x) = \inf_{\substack{z \in E \\ x \in X}} d(x, z)$

(a) <u>To prove</u>: $\rho_E(x) = 0 \Leftrightarrow x \in \bar{E}$

<u>proof</u>: $(\Rightarrow)$ let $\rho_E(x) = 0$

now for $z_n \in E$

let

$$\underset{\text{infimum property}}{\rho_E(x)} \leq d(x, z_n) < \rho_E(x) + \frac{1}{n} \quad (\exists n \in \mathbb{N} \text{ s.t above occurs})$$

now $d(x, z_n) < \frac{1}{n}$

$\Rightarrow$ as $n \to \infty$

$d(x, z_n) \to \rho_E(x)$

so $(z_n) \to x \Rightarrow x \in \bar{E}$

$(\Leftarrow)$ Now if $x \in \bar{E}$ then $\exists \{z_n\} \subseteq E$ s.t

$z_n \longrightarrow x$

and so $d(z_n, x) \longrightarrow 0$

$\Rightarrow \rho_E(x) = 0$

(b) $\quad d(x, z) \leq d(x, y) + d(y, z)$

$\Rightarrow \rho_E(x) \leq d(x, y) + \rho_E(y)$

$\Rightarrow \rho_E(x) - \rho_E(y) \leq d(x, y)$

interchange x, y

so $|\rho_E(x) - \rho_E(y)| \leq d(x, y)$

$\forall x, y \in X$

so for $|x - y| < \delta$

$|\rho_E(x) - \rho_E(y)| < \delta$

$\forall \varepsilon > 0$, $\exists \delta = \varepsilon > 0$ s.t

the continuity is satisfied

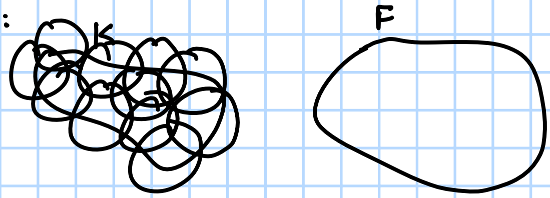
21. K, F $\quad K \cap F = \phi$

$K, F \subseteq X$

K is compact, F is closed

<u>To prove</u>: $\exists \delta > 0$ s.t $d(p, q) > \delta$ if $p \in K$, $q \in F$

<u>proof</u>:



$\forall p \in F^c$ (open), $\exists r > 0$ s.t

$B_r(p) \subseteq F^c$

now

$\forall p \in K$, $B_{r_p}(p) \subseteq F^c$

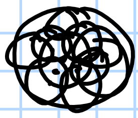
$\exists r_p$

now $K \subseteq \bigcup_{p \in K} B_{\frac{r_p}{2}}(p) \Rightarrow \exists \{p_1 \dots p_n\}$ s.t

$$\bigcup_{i=1}^{n} B_{\frac{r_{p_i}}{2}}(p_i) \supseteq K$$

now, $\forall z \in B_{\frac{r_p}{2}}(p)$

$B_{\frac{r_p}{2}}(z) \subseteq B_{r_p}(p)$



now, let $\delta = \min\left\{\frac{r_{p_i}}{2}\right\}$

then $\forall z \in K$

$\exists p_i$ s.t

$z \in B_{\frac{r_{p_i}}{2}}(p_i)$

then $B_\delta(z) \subseteq B_{r_{p_i}}(p_i)$

or

$d(z,q) > \delta \;\; \forall z \in K,\; q \in F$

25. $A \subseteq \mathbb{R}^k$
$B \subseteq \mathbb{R}^k$

$A + B =$ set of $x+y$, $x \in A$, $y \in B$

(a) K is compact, C is closed then $K+C$ is closed

for $z \notin K+C$ then let

$F = \{z - y \mid y \in C\}$
$= z - C$

now if $x \in K \cap F$

then $x \in K$

and $x \in z - C$ so $\exists y \in C$ s.t

$x = z - y$

$\Rightarrow z = x + y$

$x \in K \quad y \in C$

$\Rightarrow z \in K + C$ *

so $K \cap F = \phi$

now, K is compact, F is closed

so $\exists \delta > 0$ s.t

$|x - y| > \delta \;\; \forall x \in K,\; y \in F$

$\Rightarrow |x - (z - y)| > \delta \;\; \forall x \in K,\; y \in C,\; z \notin K+C$

$\Rightarrow |(x+y) - z| > \delta \;\; \forall (x+y) \in K+C,\; z \notin K+C$

$\Rightarrow |w - z| > \delta \;\; \forall w \in K+C,\; z \notin K+C$

so, $\forall z \notin K+C$, $\exists \delta > 0$ s.t

$B_\delta(z) \subseteq (K+C)^c \Rightarrow K+C$ is closed

(b) α is irrational $\quad C_1 = \mathbb{Z}$

$C_2 = \{\alpha n \mid n \in C_1\}$

To prove: C_1, C_2 are closed subsets of $\mathbb{R}^1$

$C_1 + C_2$ is not closed

proof: $C_1\ C_2$ does not have a limit point

so both $C_1 = \bar{C_1}$

$C_2 = \bar{C_2}$

now $\forall N \in \mathbb{N}$ consider $N \geq 2$

$\beta_1 = \alpha - [\alpha]$

$\beta_2 = 2\alpha - [2\alpha]$

$\vdots$

$\beta_N = N\alpha - [N\alpha]$

then $\left[\frac{k-1}{N-1}, \frac{k}{N-1}\right)$ $\quad k = 1, 2 \ldots N-1$

is s.t

$\exists\, k$ s.t

$\beta_i, \beta_j \in \left[\frac{k-1}{N-1}, \frac{k}{N-1}\right)$

now, if $\beta_i = \beta_j$ then as $i \neq j$

$\alpha = \frac{[i\alpha] - [j\alpha]}{i-j} \rightarrow$ rational $*$

so $\beta_i \neq \beta_j$

now $0 < (i\alpha - [i\alpha]) - (j\alpha - [j\alpha]) < \frac{1}{N-1}$

$\Rightarrow (i-j)\alpha + ([j\alpha] - [i\alpha]) \in \left(0, \frac{1}{N-1}\right)$

so, $\exists$ a point in $C_1 + C_2$ in $\left(0, \frac{1}{N-1}\right)$ for $\forall N \geq 2$

for any $k \in \mathbb{Z}$ fix $q \in \mathbb{Z}$

let $qn \leq k < (q+1)n$

$y \in C_1 + C_2$

s.t $0 < y < \frac{1}{n}$

then $x = ny \in C_1 + C_2$

$0 < x < 1$

$\exists\, p \in \mathbb{N}$ s.t

$k < px + qn < k+1$

$\Rightarrow \frac{k}{n} < py + q < \frac{k+1}{n}$

and $py + q \in C_1 + C_2$

so $\exists$ a point of $C_1 + C_2$ in $\left(\frac{k}{n}, \frac{k+1}{n}\right)$ $\forall k \in \mathbb{Z}$ $\forall n \geq 2$

O non empty of $\mathbb{R}$

then $(a, b) \subseteq O$

if $n > \frac{2}{b-a}$

then $\exists k$ s.t $\left(\frac{k}{n}, \frac{k+1}{n}\right) \subseteq (a, b)$

contains a point of $C_1 + C_2$

$\therefore O$ contains a point of $C_1 + C_2$

$\therefore C_1 + C_2$ dense in $\mathbb{R}$

26. X, Y, Z are metric space

Y is compact

$f : X \longrightarrow Y$

g be cont, one-one

$g : Y \rightarrow Z$

$h(x) = g(f(x))$

$h : X \longrightarrow Z$

To prove : f is unif cont if h is unif cont

proof : g^{-1} has compact domain $g(Y)$

$g^{-1} : g(Y) \longrightarrow Y$

$\downarrow$ cont, $\hookrightarrow$ compact

$\Rightarrow g^{-1}(h(x)) = f(x)$

so if $h(x)$ is uniform cont then f is uniform cont → doubt

Counterexample :

$X = [0,1]$

$Z = [0,1]$

$Y = \{0\} \cup [1, \infty)$

$f : X \rightarrow Y$

$$x \mapsto \begin{cases} \frac{1}{x} & 0 < x \leq 1 \\ 0 & x = 0 \end{cases}$$

$g : Y \rightarrow Z$

$$y \mapsto \begin{cases} \frac{1}{y} & 1 \leq y < \infty \\ 0 & y = 0 \end{cases}$$

$h(x) = g(f(x)) = x$

$\downarrow$

uniformly cont g is cont & one-one but f is not even cont

Tutorial-15 :

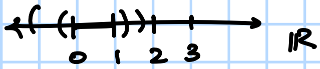
1. U_r in Urysohn lemma

$$S=\{U_r \mid 0\leq r\leq 1,\ U_r\subseteq X \text{ open},\ \overline{U_r}\subset U_{r+\varepsilon},\ A\subset U_0,\ B\subset X\setminus U_1\}$$

(A) $X=\mathbb{R}$

$A=[0,1]$

$B=[2,3]$



$U_0=(-0.5,\ 1.5)$

$U_1=(-1,\ 1.75)$

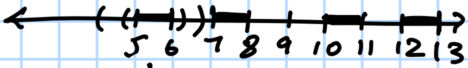
where $U_r=(-0.5(1-r)-1(r),\ 1.5(1-r)+1.75(r))$

$=(-0.5-0.5r,\ 1.5-1.5r+1.75r)$

$=(-0.5-0.5r,\ 1.5+0.25r)$

(B) $X=\mathbb{R}$, $A=\bigcup_{n=1}^{\infty}[5n,5n+1]$

$B=\bigcup_{n=1}^{\infty}[5n+2,5n+3]$

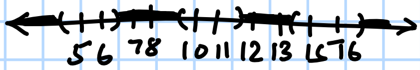


$U_0=\bigcup_{n=1}^{\infty}\left(5n-\frac{1}{3},\ 5n+1+\frac{1}{3}\right)$

$U_1=\bigcup_{n=1}^{\infty}\left(5n-\frac{2}{3},\ 5n+1+\frac{2}{3}\right)$

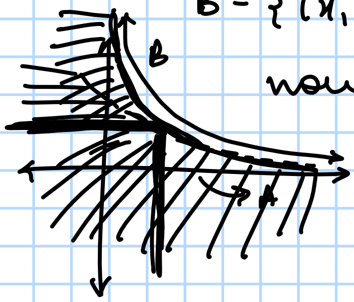
now $X\setminus U_1=\mathbb{R}\setminus\bigcup_{n=1}^{\infty}\left(5n-\frac{2}{3},\ 5n+1+\frac{2}{3}\right)$

then $B\subseteq X\setminus U$



(C) $X=\mathbb{R}^2$ $A=\{(x,0)\mid x\in\mathbb{R}\}$

$B=\{(x,\frac{1}{x})\mid x>0\}$



now 0 and $\frac{1}{x}$ as $x>0$ is s.t

$0<\frac{1}{x}$

let $r=\frac{1}{x}-0=\frac{1}{x}$

now for any $(x,0)$ and $(x,\frac{1}{x})$

let $s\in(0,1)$ be s.t

$(x(1-s),\ 0\times(1-s))+(x(s),\frac{1}{x}(s))$

$=(x,\frac{s}{x})$

now let s go from $\frac{1}{4}$ to $\frac{3}{4}$

i.e

$\frac{1}{4}(1-t)+\frac{3}{4}(t)=\frac{1}{4}+\frac{1}{2}t$

$S_t=\left\{\left(x,\frac{\frac{1}{4}+\frac{t}{2}}{x}\right)\in\mathbb{R}^2 \;\middle|\; x>0\right\}$

$f: (0,\infty) \times (0,\infty) \longrightarrow \mathbb{R}$

$(x,y) \longrightarrow \dfrac{\frac{1}{2}+\frac{r}{4}}{x} - y$

$f_r(x,y)$ is continious as $\dfrac{\frac{1}{2}+\frac{r}{4}}{x}$ and y are cont for $(0,\infty)$

and so $f_r^{-1}((-\infty,0])$ is closed

this region is closed

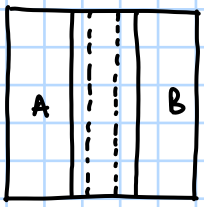
now $\mathbb{R}^2 \setminus f_r^{-1}(-\infty,0]$ is open

let $U_r = \mathbb{R}^2 \setminus f_r^{-1}(-\infty,0]$

then we get what we want

(D) $X=[0,1]^2$ $A=[0,\frac{1}{3}] \times [0,1]$

$B=[\frac{2}{3},1] \times [0,1]$



let $U_0 = [0, \frac{1}{3}+\frac{1}{10}) \times [0,1]$

$U_1 = [0, \frac{1}{3}+\frac{2}{10}) \times [0,1]$

then $U_r = [0, \frac{1}{3}+\frac{1}{10}+\frac{r}{10})$

(E) $X=[0,1]^2$

$A = \{(x,y) \in [0,1]^2 \mid x \leq y - \frac{1}{10}\}$

$B = \{(x,y) \in [0,1]^2 \mid x \geq y\}$

let $f: [0,1] \times [0,1] \longrightarrow [-1,1]$

$(x,y) \longmapsto y-x$

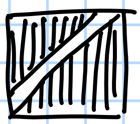
$f(x,y)=y-x$ is continious (trivial)

now $f(x,y) \leq 0$ is first

$f^{-1}[-1,0] \subseteq [0,1]^2$

closed in $[-1,1]$

$f^{-1}[-1,0]$ closed in $[0,1]^2 = B$



and now $f(x,y) \geq \frac{1}{10}$ is also closed in $[0,1]^2$ and A

now so $A = f^{-1}[\frac{1}{10}, 1]$
$B = f^{-1}[-1, 0]$

then $f^{-1}\left(\frac{1}{20}, 1\right]$ is open and

as $\left[\frac{1}{10}, 1\right] \supseteq \left(\frac{1}{20}, 1\right]$

$\Rightarrow f^{-1}\left[\frac{1}{10}, 1\right] \supseteq f^{-1}\left(\frac{1}{20}, 1\right)$

and let $U_r = f^{-1}\left(\frac{1}{20+r}, 1\right)$

then
$U_0 = f^{-1}\left(\frac{1}{20}, 1\right)$
$U_1 = f^{-1}\left(\frac{1}{21}, 1\right) \supseteq f^{-1}\left(\frac{1}{20}, 1\right) = U_0$

and so $f(x, y) > \frac{1}{20+r}$

$\Rightarrow y - x > \frac{1}{20+r}$ open

so, $U_r = \left\{ (x, y) \in [0,1]^2 \;\middle|\; y - x > \frac{1}{20+r} \right\}$

open ball

(F) $X = [0,1]^{\mathbb{N}}$
$A = \{ (x_n) \in X \mid x_1 \leq x_2 - \frac{1}{10} \}$
$B = \{ (x_n) \in X \mid x_1 \geq x_2 \}$

now let $f : [0,1]^{\mathbb{N}} \longrightarrow [-1, 1]$
$(x_n) \longmapsto x_2 - x_1$

then it is trivial to see that f is continious as by defination

$\pi_1 : [0,1]^{\mathbb{N}} \longrightarrow [0,1]$
$(x_n) \longmapsto x_1$

$\pi_2 : [0,1]^{\mathbb{N}} \longrightarrow [0,1]$
$(x_n) \longmapsto x_2$

π_1, π_2 is cont

$\Rightarrow \pi_2 - \pi_1$ is continious, we just have to change range

now $f = \pi_2 - \pi_1$, now

$f \leq 0 \quad f \geq \frac{1}{10}$

now $f^{-1}[-1, 0] = B$ $\quad [-1, 0]$ is closed $\Rightarrow f^{-1}[-1, 0]$ is closed
$f^{-1}\left[\frac{1}{10}, 1\right] = A$ $\quad [\frac{1}{10}, 1]$ is closed $\Rightarrow f^{-1}\left[\frac{1}{10}, 1\right]$ is closed

$0 \le r \le 1$

now $\left(\frac{1}{20+r}, 1\right) \subseteq \left[\frac{1}{10}, 1\right]$

$\Rightarrow f^{-1}\left(\frac{1}{20+r}, 1\right) \subseteq A$

$\quad \shortparallel$

U_r (U_r is open as $\left(\frac{1}{20+r}, 1\right)$ is open)

and $X \setminus U_r = f^{-1}\left[-1, \frac{1}{20+r}\right] \supseteq f^{-1}[-1, 0] \supseteq B$

$$\therefore U_r = f^{-1}\left(\frac{1}{20+r}, 1\right)$$

$$= \left\{ (x_n) \in [0,1]^{\mathbb{N}} \;\middle|\; f > \frac{1}{20+r} \right\}$$

$$= \left\{ (x_n) \in [0,1]^{\mathbb{N}} \;\middle|\; x_2 - x_1 > \frac{1}{20+r} \right\}$$

(G) $X = [0,1]^{\mathbb{N}}$ $\quad A = \left\{ (x_n) \in X \;\middle|\; \sum_n \frac{x_{n+1} - x_n}{n^2} \geq \frac{1}{10} \text{ and } x_n \le x_{n+1} \ \forall n \right\}$

$B = \left\{ (x_n) \in X \;\middle|\; x_n \geq x_{n+1} \ \forall n \right\}$

$f : x_n \longrightarrow x_{n+1}$

$\quad$ then $\quad x_n \geq x_{n+1}$

$\quad\quad\quad\quad x_n \le x_{n+1}$ $\quad$ both closed

continious

now $f(x_i) = \sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{n^2}$

then afr some N

$$\underbrace{\sum_{n=N}^{\infty} \frac{x_{n+1} - x_n}{n^2}}_{<\varepsilon} \quad \text{as reminder} \longrightarrow 0$$

so f is continious

now, $f \geq \frac{1}{10} \Rightarrow f^{-1}\left[\frac{1}{10}, \infty\right)$ $\longrightarrow$ closed in $\mathbb{R}$

now, $f_n(x_i) = x_{n+1} - x_n$

then $f_n \geq 0$ $\longrightarrow$ cont

so $f_n^{-1}[0, \infty) \Rightarrow$ closed

$\longrightarrow$ closed

so $\left(\bigcap_{n=1}^{\infty} f_n^{-1}[0, \infty) \right) \cap f^{-1}\left[\frac{1}{10}, \infty\right)$ is also closed

$= A$

arbitrary intersection of closed sets is closed

$\left(\bigcap_{n=1}^{\infty} f_n^{-1}(-\infty, 0] \right) = B$ $\longleftarrow$ closed (similarly) now let $U_\gamma = \left\{ (x_n) \in X \;\middle|\; f > \frac{1}{11+\gamma} \right\}$

(H) $X = [0,1]^{(0,1)}$ $\quad A = \left\{ (x_\alpha) \in X \;\middle|\; \sup_{n \geq 2} \inf_{\alpha < \frac{1}{n}} n^{-1} |x_\alpha - x_{\frac{1}{n}}| \geq \frac{1}{10} \text{ and } x_\alpha \leq x_\beta \text{ for } \alpha \leq \beta \right\}$

$B = \left\{ (x_\alpha) \in X \;\middle|\; x_\alpha \geq x_\beta \text{ for } \alpha \leq \beta \right\}$

$f_{\alpha,\beta} : X \longrightarrow \mathbb{R}$

$(x_\alpha) \longmapsto x_\beta - x_\alpha$

then similar to previos one B is closed

and

$f : X \longrightarrow \mathbb{R}$

$(x_\alpha) \longmapsto \sup_{n \geq 2} \inf_{\alpha < \frac{1}{2}} \frac{1}{n} |x_\alpha - x_{\frac{1}{n}}|$

for $n \longrightarrow \infty$

$\underbrace{\inf_{\alpha < \frac{1}{2}} \frac{1}{n} |x_\alpha - x_{\frac{1}{n}}|}_{\text{goes to } 0 \Rightarrow \text{ is controlable}}$

so f is continious

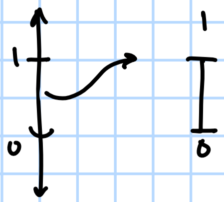
and so is A closed

now

$U_\gamma = \left\{ f > \frac{1}{20+\gamma} \right\}$

$0 \leq \gamma \leq 1$

2. (A) $X = (0,1]$



$\tau_Y = \{ V \text{ open in } X \} \cup \{ \{0\} \cup K^c \mid K \text{ is compact in } X \}$

then $\tau_Y = \{ (a,b); 0<a<b<1 \} \cup \{ [0,a) \mid 0<a<1 \}$

$\cup \{ (a,1]; 0<a<1 \} \cup \{ [0,1] \}$

$Y = [0,1]$

$[0,1]$ with euclidian metric space

$= \{ (a,b) \} \cup \{ [0,b) \} \cup \{ (a,1] \} \cup \{ [0,1] \}$

$= \tau_Y$ (trivial)

① Y is compact + Hausdorff ($\because$ metric space + closed)
② $X \subseteq Y$
③ $|Y - X| = 1$

(B) $[0, \frac{1}{2}) \cup (\frac{1}{2}, 1] = X$

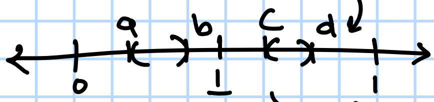
now $Y = [0,1]$ with euclidean metric

as $\tau_Y = \{ \underbrace{U \subseteq X}_{\text{open in } X} \} \cup \{ \{\frac{1}{2}\} \cup K^c \mid K \text{ is compact in } X \}$

$K = \left\{ [a,b] \;\middle|\; 0 \le a \le b < \frac{1}{2} \text{ or } \frac{1}{2} < a \le b \le 1 \right\}$

($\because$ closed and bounded)

$\left\{\frac{1}{2}\right\} \cup K^c = \left\{ [0,a) \cup (b,c) \cup (d,1] \right\}$



following this rules

then $\tau_Y = \left\{ (a,b) \cap [0,1] \right\}$

$= ([0,1], d)$

so Y is generated by euclidian metric

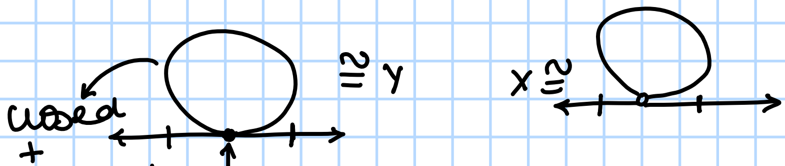
and so

① Y is compact + Hausdorff ($\because$ metric space)

② $|Y - X| = 1$

③ $X \subseteq Y$

(C) $[0,1) \cup (1,2) \cup (2,3]$



one point added

$\Rightarrow$ compact + Hausdorff

let $Y = X \cup \{4\}$

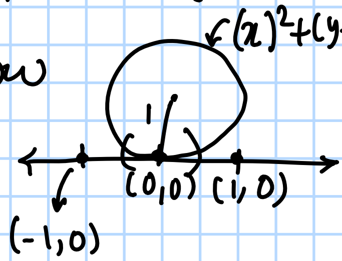
$\tau_Y = \{ \underset{\text{open in } X}{U \subseteq X} \} \cup \{ \{4\} \cup K^c \mid K \text{ is compact in } X \} \cup \{Y\}$

becomes open sets like this in $\mathcal{B}_Y$

$\mathcal{B}_Y = \mathcal{B}_X \cup \left\{ \{4\} \cup (1-\varepsilon, 1+\varepsilon) \cup (2-\varepsilon, 2+\varepsilon) \right\}$

$\varepsilon < \frac{1}{3}$

now



let $f : X \longrightarrow \mathbb{R}^2$

$$f(x) = \begin{cases} (1,0) + e^{2\pi i (x + \frac{3}{4})} ; & x \in (1,2) \\ (x-2, 0) ; & x \in (2,3] \\ (x-1, 0) ; & x \in [0,1) \end{cases}$$

for clock wise

start

then let Range be X_0 $f : X \longrightarrow X_0$

then $\bar{X}_0 \cong Y$ $\Rightarrow X \cong X_0$

$\bar{X}_0 \cong Y \Rightarrow Y$ is compact + Hausdorff

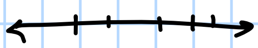
and $X \subseteq Y$

and $|Y - X| = 1$

so Y is one-point compactification of X

(D) same as (C) but shift $(5,6] \longrightarrow (2,3]$

(E) $\mathbb{N}$



$Y = \{ \underset{\text{open in } X}{U \subseteq X} \} \cup \{ \{\infty\} \cup K^c \mid K \text{ is compact in } X \}$

now for any bounded set in X it is compact

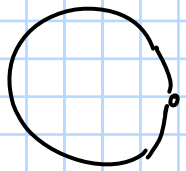
$$\mathcal{B}_Y = \mathcal{B}_X \cup \left\{ \{\infty\} \cup \left(\mathbb{N} \cap (-\infty, -r) \cap (r, \infty) \quad r > 10 \right) \right\}$$

then Y is compact by costution and is also hausdorff (trivial)

also $|Y - X| = 1$ and $X \subseteq Y$

(F) $S^1 \setminus (1,0) = X$

$Y = S^1 = X \cup (1,0)$ trivial to see closed + Bounded in $\mathbb{R}^2$ $\Rightarrow$ compact + Hausdorff

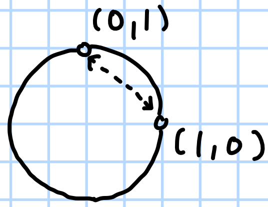


$Y = S^1$

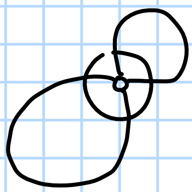
topology same as of S^1

(G) $S^1 \setminus \{(1,0), (0,1)\} = X$

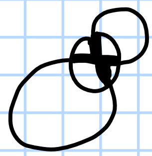
X :



$\cong$



$Y \cong$

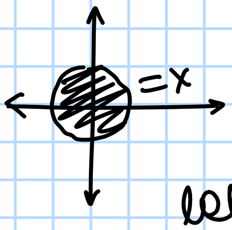


$$\tau_Y = \{ \text{open in } X \} \cup \{ \{(0,0)\} \cup K^c \mid K \text{ is compact in } X \}$$

$$\mathcal{B}_Y = \left\{ (e^{2\pi i t}) \mid 0 < t < \tfrac{1}{2} \text{ or } \tfrac{1}{2} < t < 1 \right\}$$

$$\cup \left\{ (0,0) \cup e^{2\pi i t} \;;\; t \in [0, \varepsilon) \cup (1-\varepsilon, 1) \cup \left(\tfrac{1}{2} - \varepsilon, \tfrac{1}{2} + \varepsilon\right) \text{ s.t } \varepsilon < \tfrac{1}{100} \right\}$$

(H) $B((0,0),1) \subseteq \mathbb{R}^2$



$Y \cong S^2$ (intuition)

let $Y = X \cup \{\infty\}$

then

$$\tau_Y = \{ \underbrace{U \subseteq X}_{\text{open in } X} \} \cup \{ \{\infty\} \cup K^c \mid K \text{ is compact} \}$$

$$\mathcal{B}_Y = \mathcal{B}_X \cup \left\{ \{\infty\} \cup (x,y) \mid 1 - \varepsilon < x^2 + y^2 < 1 \,;\, \text{for } \varepsilon < \tfrac{1}{3} \right\}$$

3. (X,d) is locally compact but not compact metric space

<u>To prove</u>: Y (the one point compactification of X) is a metric space

<u>proof</u>: Y is ① Hausdorff + compact

② $|Y - X| = 1$

③ $X \subseteq Y$

now, wlog $Y = X \cup \{\infty\}$
then

$$\tau_Y = \{ U \underset{\text{open}}{\subseteq} X \} \cup \{ \{\infty\} \cup K^c \mid K \text{ is compact in } X \} \cup \{Y\}$$

now, let $\mathcal{B}_Y = \mathcal{B}_X \cup \{ \{\infty\} \cup [p-\varepsilon, p+\varepsilon]^c \}$ $\varepsilon > \frac{1}{3}$

for some fixed p

now let $\hat{d}(x,y) = \min\left\{ \frac{1}{1+d(x,p)} + \frac{1}{1+d(y,p)}, d(x,y) \right\}$

given x or $y \neq \infty$

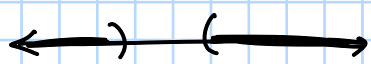
$\hat{d}(\infty,\infty) = 0$

$\hat{d}(\infty, x) = \frac{1}{1+d(x,p)}$

So now $\hat{B}_x(\gamma) = \{ y \in Y \text{ s.t } \hat{d}(x,y) < \gamma \}$

$\frac{1}{1+d(p,x)} + \frac{1}{1+d(p,y)} < \gamma$
and $d(x,y) < \gamma$

$\hat{B}_x(\gamma) = B_\gamma(x) \cap [\{\infty\} \cup K^c]$ → K is some compact set s.t $p \in K$



now $\hat{B}_\gamma(x) \in \mathcal{B}_X$ or

then if it contains $\{\infty\}$
$\hat{B}_\gamma(x) \in \{ \{\infty\} \cup [p-\varepsilon, p+\varepsilon]^c \}$

for some ε

$\Rightarrow \hat{B}_\gamma(x) \in \mathcal{B}_Y$

$\Rightarrow$ if $\hat{d}$ is a metric then Y is a metric space

now $\hat{d}$ is a metric as

① $\hat{d}(x,y) = 0 \Rightarrow d(x,y) = 0$ or $\frac{1}{1+d(x,p)} + \frac{1}{1+d(y,p)} = 0$

$\Rightarrow x = y$ or $x = y$

$\Rightarrow x = y$

② $\hat{d}(x,y) = \hat{d}(y,x)$ (trivial)

③ $\hat{d}(x,y) \geq 0$ (trivial)

④ $\hat{d}(x,y) + \hat{d}(y,z)$

$$= \min\left\{ d(x,y), \frac{1}{1+d(x,p)} + \frac{1}{1+d(y,p)} \right\}$$

$$+ \min\{ \quad \}$$

$$\geq d(x,y) + d(y,z) \geq d(x,z) \geq \hat{d}(x,z)$$

similarly for other cases

$$\hat{d}(x,y) + \hat{d}(y,z) \geq \hat{d}(x,z)$$

Tutorial-14 :

1. <u>To prove</u>: $\mathbb{Q}$ are not locally compact

<u>proof</u>:

let $\mathbb{Q}$ be locally compact, then

$\forall p \in \mathbb{Q}, \exists U_p \ni p$ open in $\mathbb{Q}$ s.t $\overline{U_p}$ is compact

now, for $0 \in \mathbb{Q}$, $\exists U_0 \ni 0$ open

$\exists r > 0$ s.t

$B_r(0) \cap \mathbb{Q} \subseteq U_0$

then as $B_r(0) \cap \mathbb{Q}$ is open in

$\Rightarrow (-r, r) \cap \mathbb{Q}$ is open in $\mathbb{Q}$

and

$(-r, r) \cap \mathbb{Q} \subseteq U_r$

now $\overline{(-r, r) \cap \mathbb{Q}} = [-r, r] \cap \mathbb{Q}$

let $\alpha \in \mathbb{Q}$ then $\{\alpha\}$ is singleton and open in $\mathbb{Q}$

$\Rightarrow \bigcup\limits_{\alpha \in [r,r] \cap \mathbb{Q}} \{\alpha\} = [-r, r] \cap \mathbb{Q}$

cover of $[-r, r] \cap \mathbb{Q}$

but no finite subcover as all are singletons

$\therefore \mathbb{Q}$ is <u>not</u> locally compact

2. $\{X_\alpha\}$ indexed family of non-empty sets

(a) <u>To prove</u>: $\prod X_\alpha$ is locally compact $\Rightarrow X_\alpha$ is locally compact and X_α is compact $\forall$ all α but finitely many

<u>proof</u>: $\pi_\alpha : \prod X_\alpha \longrightarrow X_\alpha$

$\pi_\alpha((x_\alpha)) = x_\alpha \in X_\alpha$

now π_α is a projection map so cont + open map + surjective

here $\pi_\alpha(\prod X_\alpha) = X_\alpha$ ($\because$ surjectivity)

also now, for $x_\alpha \in X_\alpha$

as X_α's are non empty, pick any element from it to get $(x)_\alpha$ s.t

$\pi_\alpha((x)_\alpha) = x_\alpha$

then $(x)_\alpha \in \prod X_\alpha$

as $\prod X_\alpha$ is locally compact, by defination

$\exists\, U \ni (x)_\alpha$ open in $\prod X_\alpha$ s.t
$\exists\, C \supseteq U$ compact in $\prod X_\alpha$

$\Rightarrow \pi_\alpha(U) = U_\alpha \leftarrow$ open in X_α
and $x_\alpha \in U_\alpha$

$\pi_\alpha(C) = C_\alpha \leftarrow$ compact in X_α
and
as $U \subseteq C$
$\Rightarrow \pi_\alpha(U) \subseteq \pi_\alpha(C)$
$\Rightarrow U_\alpha \subseteq C_\alpha$
$\downarrow$
compact

So, $\forall\, x_\alpha \in X_\alpha$, $\exists\, U_\alpha$ open in X_α s.t
$\exists\, C_\alpha$ compact in X_α

$U_\alpha \subseteq C_\alpha$

So X_α is locally compact

now, open sets in $\prod X_\alpha$ look like:

$$U = \prod_{i=1}^{N} U_i \times \underbrace{X_{i+1} \times \cdots\cdots}_{\text{after some } N, \text{ all are } X_\alpha}$$

So, $\pi_\alpha(U)$ after the finitely many N become X_α

$\Rightarrow \pi_\alpha(U) = X_\alpha \;\; \forall \alpha$ but finitely many

as $U \subseteq C \Rightarrow \pi_\alpha(C) = X_\alpha \;\forall \alpha$ but finitely many

$\Rightarrow X_\alpha$ is compact

$\therefore X_\alpha$ is compact $\forall \alpha$ but finitely many

(b) <u>To prove</u>: X_α is compact $\forall \alpha$ but finitely many then $\prod X_\alpha$ is locally compact

<u>proof</u>: let $\prod X_\alpha = X_1 \times X_2$, where X_1 is product of those finite locally compact spaces

now, finite product of locally compact spaces is locally compact (trivial)

and let X_2 be product of all but finite X_α which are compact

by Tychonoff theorem X_2 is compact

$\Rightarrow X_2$ is locally compact

$\Rightarrow X_1 \times X_2$ is locally compact
$\Rightarrow \prod X_\alpha$ is locally compact

3. X is locally compact

$f: X \to Y$ is cont
then $f(X)$ is locally compact

<u>Counterexample</u> : $S \subseteq \mathbb{R}^2$ be graph of $\sin(\frac{1}{x})$

i.e $f(x) = (x, \sin\frac{1}{x})$

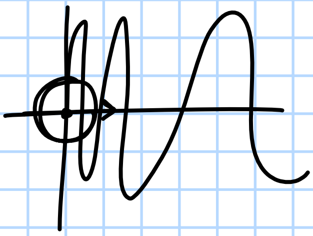
$f: (0,1] \longrightarrow S$

then let $f: \{-1\} \cup (0,1] \longrightarrow S \cup \{(0,0)\}$

s.t $f(x) = \begin{cases} (x, \sin\frac{1}{x}) ; & x \in (0,1] \\ (0,0) ; & x \in \{-1\} \end{cases}$

now $f(\{-1\} \cup (0,1]) = \{(0,0)\} \cup S$

also for $(0,0)$ all U contains $(0,0)$



any U cannot be in C compact as U contains infinite <u>disjoint open sets</u>

if f is cont + open map then trivial to see that it preserves compactness and openness
$\Rightarrow f(X)$ is locally-compact

5. $f: X_1 \longrightarrow X_2$ homeomorphism of locally compact Hausdorff spaces

$X_1 \cong X_2$
$\hookrightarrow$ locally compact + Hausdorff

then let Y_1 be s.t $Y_1 = X_1 \cup \{\infty\}$

$\tau_{Y_1} = \{ U \subseteq X_1 \text{ open in } X_1 \} \cup \{ Y_2 - K \mid K \text{ is compact in } X_1 \} \cup \{Y_2\}$

similarly for X_1

now, if $F: Y_1 \longrightarrow Y_2$
s.t $F|_{X_1} \equiv f$ s.t

$F: Y_1 \longrightarrow Y_2$

$F|_{X_1} \equiv f$ and let $F(\infty) = \infty$

then

① F is one-one
② F is onto } trivial

now $F^{-1}(U) = f^{-1}(U)$ is open in X_1 ($\because f$ is cont)

(open in X_2 above $f^{-1}(U)$; open in X_2 below U)

$F^{-1}(Y_1 - K) = F^{-1}(Y_1) - F^{-1}(K)$
$= Y_2 - F^{-1}(K)$ (compact in X_2)
$= Y_2 - K_2$ (open in Y_2 by construction)

$\Rightarrow$ F is continious

So, F is continious, bijective map s.t

$F: Y_1 \longrightarrow Y_2$ $\Rightarrow$ F is homeomorphism

(Y_1: compact; Y_2: Hausdorff)

($\because$ one point compatification is Y condition s.t Y is compact + Hausdorff)

6. $\mathbb{R}$ one-point compatification

$Y = \mathbb{R} \cup \{\infty\}$

$\tau_Y = \{\text{open in } \mathbb{R}\} \cup \{Y - K \mid K \text{ is compact in } \mathbb{R}\} \cup \{Y\}$

<u>To prove</u>: $Y \cong S^1$

<u>Proof</u>: let $F: Y \longrightarrow S^1$ be s.t

$$F(x) = \begin{cases} e^{2\pi i\left(\frac{1}{1+e^x}\right)} & ; x \in \mathbb{R} \\ e^{2\pi i} & ; x = \infty \end{cases}$$

as $e^x \subseteq (0,\infty)$
$\frac{1}{1+e^x} \subseteq (0,1)$

now ① F is bijective:

If $f(x_1) = f(x_2)$
then if $f(x_2) = (1,0)$
then $x_2 = \infty$
as no other x can get $(1,0)$
as $e^{2\pi i t}$ for $t \in (0,1) \neq (1,0)$
$\therefore f(x) = (1,0) \Rightarrow x = \infty$
$\Rightarrow x_1 = x_2$

if $f(x_2) = f(x_2) \neq (1,0)$

then $e^{2\pi i \left(\frac{1}{1+e^{x_1}}\right)} = e^{2\pi i \left(\frac{1}{1+e^{x_2}}\right)}$ $\rightarrow (0,1)$ $\rightarrow (0,1)$

$\Rightarrow \frac{1}{1+e^{x_1}} = \frac{1}{1+e^{x_2}}$

$\Rightarrow e^{x_1} = e^{x_2}$

$\Rightarrow x_1 = x_2$

Also $\forall\, t \in (0,1]$ $e^{2\pi i t}$ is s.t

if $t \in (0,1)$ then

$\exists\ t = \frac{1}{1+e^{x}}$

$\Rightarrow e^{x} + 1 = \frac{1}{t}$

$\Rightarrow e^{x} = \frac{1}{t} - 1$

$\Rightarrow x = \log_e\left(\frac{1}{t} - 1\right)$

if $t = 1 \Rightarrow x = \infty$

$\therefore \forall\ e^{2\pi i t} \in S^1, \exists\ x \in Y$ s.t

$f(x) = e^{2\pi i t}$

② F is continious :

if $U = \left\{ e^{2\pi i t} \;\middle|\; \begin{array}{l} 0 < a < b < 1 \\ \text{and } t \in (a,b) \end{array} \right\}$

$\rightarrow$ this type of open set of S^1

then as $(1,0) \notin U \Rightarrow F^{-1}(U) = \left(e^{2\pi i}\left(\frac{1}{1+e^{x}}\right)\right)^{-1}(U)$

$\rightarrow$ alway continious wrt X

$\Rightarrow$ open in $X = \mathbb{R}$

$\Rightarrow$ open in Y

if $U = \left\{ e^{2\pi i t} \;\middle|\; t \in (0,\varepsilon) \cup (1-\varepsilon, 1] \right\}$

then $U = \underbrace{\left\{ e^{2\pi i t} \;\middle|\; t \in (0,\varepsilon) \cup (1-\varepsilon,1) \right\}}_{U_1} \cup \underbrace{\left\{ (1,0) \right\}}_{U_2}$

now $F^{-1}(U_1) = f^{-1}(U_1)$

$= (-\infty, a) \cup (b, \infty)$

$= \mathbb{R} \setminus \underbrace{[a,b]}_{\text{compact}}$

$F^{-1}(U_2) = \{\infty\}$

$\Rightarrow F^{-1}(U) = Y \cup K^c \Rightarrow$ open in Y

$\Rightarrow F$ is continious

as Y is compact and S^1 is hausdorff ($\because$ metric space)

$\Rightarrow F : Y \longrightarrow S^1$ bijective continious map is homeomorphic

$\Rightarrow Y \cong S^1$

1. To prove : closed subspace of normal space is normal

proof: let A be a closed subspace of normal space, then $A = \bar{A}$

and for $U_A, V_A \subseteq A$ closed or

$\exists U_X$ closed s.t

$U_X \cap A = U_A \leftarrow$ closed in A

$\exists U_Y$ closed s.t

$V_X \cap A = V_A \leftarrow$ closed in A

now if $U_A \cap V_A = \phi$ then

as U_A, V_A are closed in X

$\exists U, V$ open in X s.t

$U_A \subseteq U$, $V_A \subseteq V$ and $U \cap V = \phi$

now $\underset{\substack{\parallel \\ U_A}}{U_A \cap A} \subseteq \underbrace{U \cap A}_{\text{open in } A}$

$\Rightarrow U_A \subseteq U \cap A$

$V_A \subseteq V \cap A$

and $\underset{\substack{\parallel \\ U_A'}}{(U \cap A)} \cap \underset{\substack{\parallel \\ V_A'}}{(V \cap A)} = \phi$

then $\forall U_A, V_A$ closed in A

$U_A \cap V_A = \phi$

$\exists U_A', V_A'$ open in A

s.t

$U_A \subseteq U_A'$

$V_A \subseteq V_A'$

$U_A' \cap V_A' = \phi$

$\therefore A$ is normal

2. ΠX_α is Hausdorff or regular or normal

ΠX_α is any of three then

$\pi_\alpha(\Pi X_\alpha) = X_\alpha$

$\hookrightarrow$ projection is cont + open map + surjective

then all these properties can be proved
by making sets in ΠX_α and then using
it's property and then projecting it over X_α by π_α

3. <u>To prove</u> : every locally compact Hausdorff space is regular

<u>proof</u> :

we have to show that for $x \in X$, $C \subseteq X$ closed

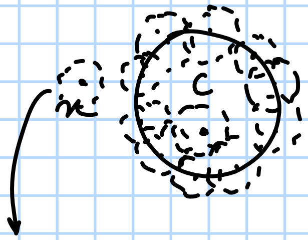
s.t $\{x\} \cap C = \phi$

$\exists\ U_x \ni x$ open

$V_C \supseteq C$ open s.t

$U_x \cap U_C = \phi$

now as X is hausdorff



as $x \in X$, $\exists\ U \ni x$ s.t U is open and
$C' \supseteq U$
compact subset

now $\forall y \in C$, as $x \neq y$
$\exists\ U_x \ni x$, $U_y \ni y$ s.t
both open

now then $U_x \cap U_y = \phi$
as C' is compact if

$\bigcup\limits_{y \in C} U_x$ covers some space
then $\exists$ a finite subcover

say $\{U_{x_1}, \dots U_{x_n}\}$

then $\bigcap\limits_{i=1}^{n} U_{x_i}$ is open and

$x \in \underbrace{\bigcap\limits_{i=1}^{n} U_{x_i}}_{V_x}$ and also

also $V_x \cap \left(\bigcup\limits_{y \in C} U_y\right) = \phi$

and also $x \in V_x$

$C \subseteq \bigcup\limits_{y \in C} U_y$

So, X is Regular

6. X is comp normal if every subspace of X is normal

<u>To prove</u> : X is comp normal $\Leftrightarrow$ every pair A, B of sep sets in X
$\bar{A} \cap B = \phi$, $A \cap \bar{B} = \phi$, $\exists$ disjoint
open sets containing them

proof : ($\Rightarrow$) let X be completely normal, then

$X-(\bar{A}\cap\bar{B})$ is also normal

now, $\bar{A}\cap B=\phi$
$A\cap\bar{B}=\phi$

then $A\subseteq(X-\bar{A}\cap\bar{B})$

as

$$X-\bar{A}\cap\bar{B} = X\cap(\bar{A}\cap\bar{B})^c$$
$$= X\cap\underbrace{((\bar{A})^c\cup(\bar{B})^c)}_{\substack{A\subseteq(\bar{B})^c \\ B\subseteq(\bar{A})^c}}$$

so $A, B\subseteq X\setminus\bar{A}\cap\bar{B}$

now as A, B open and in X

and $A\cap B=\phi$

now as $X-(\bar{A}\cap\bar{B})=(X-\bar{A})\cup(X-\bar{B})$

$A\subseteq X-(\bar{A}\cap\bar{B})$
closure of A is $=\bar{A}-\bar{B}$
and close of B is $\bar{B}-\bar{A}$

as $(\bar{A}-\bar{B})\cap(\bar{B}-\bar{A})=\phi$

and $\bar{A}-\bar{B}\subseteq X\setminus\bar{A}\cap\bar{B}$

so $\exists\, U_{\bar{A}-\bar{B}}$ and $U_{\bar{B}-\bar{A}}$ open and disjoint in $X\setminus\bar{A}\cap\bar{B}$

s.t $\bar{A}-\bar{B}\subseteq U_{\bar{A}-\bar{B}}$
$\Rightarrow A\subseteq U_{\bar{A}-\bar{B}}$
$B\subseteq U_{\bar{B}-\bar{A}}$

so, $\exists\, U, V$ open s.t
$A\subseteq U,\ B\subseteq V$

($\Leftarrow$) let Y be a subspace of X, let A, B be disjoint closed subspaces of Y
then
$\exists\, U, V$ open s.t

$U\supseteq A,\ V\supseteq B$ and $U\cap V=\phi$

$\Rightarrow X$ is normal

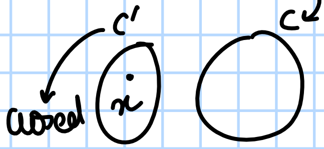
7. (a) Yes, as Y is subspace of X
then any subspace of Y is also subspace of X
$\Rightarrow Y$ is completely normal

(b) No as we take order topology
$X=\mathbb{Q}$
$Y=\mathbb{R}$
then $\mathbb{Q}\times\mathbb{R}$ then $\{1\}\times(a,b)$ is not normal $\Rightarrow$ Not comp. normal

(d) Yes, metrizable space $\Rightarrow$ Normal
and subspace of metric $\Rightarrow$ metric $\Rightarrow$ Normal

(e) $[0,1]\times[0,1]$ is compact + Hausdorff
but $\{0\}\times(a,b)$ is not normal

(f) X is regular space with countable basis
then $x \in X$ closed set in X



$\forall\, x \in C'$, $\exists\, U_x \ni x$ and $V_x \supseteq C$ s.t. $U_x \cap V_x = \phi$

now as there is countable basis, τ has same properties as that of a borel field as Borel σ-field contains all open and as countable all closed
$\Rightarrow$ intersection of countable intersection is open

$\Rightarrow \bigcap_{x\in C'} V_x \supseteq C \quad \bigcup_{x\in C'} U_x \supseteq C'$
two open sets s.t $(\bigcap V_x)\cap(\bigcup U_x)=\phi$
$\Rightarrow X$ is normal

now every subset of X will also have countable basis and will be regular
$\Rightarrow X$ is comp normal

2.(i) $X=[0,1]\times[0,1]\times\ldots\ldots$

$$d(a,b)=\sum_{n=1}^{\infty}\frac{|a(n)-b(n)|}{2^n}$$

after all n, the tail $\to 0$

now this is a metric as:
① $d(a,b)=d(b,a)$
② $d(a,b)+d(b,c) = \sum_{n=1}^{\infty}\frac{|a(n)-b(n)|}{2^n} + \sum_{n=1}^{\infty}\frac{|b(n)-c(n)|}{2^n}$

ab. conv so we can rearrange them

$$= \sum_{n=1}^{\infty}\frac{|a(n)-b(n)|+|b(n)-c(n)|}{2^n}$$

$$\geq \sum_{n=1}^{\infty}\frac{|a(n)-c(n)|}{2^n} = d(a,c)$$

③ $d(a,b) \geq 0$ is trivial
also
$d(a,b) = 0 \Rightarrow a = b$ is trivial

now open sets in $[0,1]^{\mathbb{N}}$ look like:

$$\underbrace{U_1 \times U_2 \times \cdots \; U_N}_{\text{some } N} \times [0,1] \times [0,1] \times \cdots\cdots$$

now, $B_a(r) = \left\{ b \in [0,1]^{\mathbb{N}} \,\middle|\, d(a,b) < r \right\}$

$d(a,b) < r$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{|a(n) - b(n)|}{2^n} < r \quad \text{or} \quad \sup_n \frac{|a_n - b_n|}{n} = d(a,b)$$

after some N, $\underbrace{\sum_{n=N+1}^{\infty} \frac{|a(n) - b(n)|}{2^n}}$ $\longrightarrow 0$

$$\text{so} \Rightarrow \sum_{n=1}^{N} \frac{|a(n) - b(n)|}{2^n} < r$$

now, as d is a cont function

$$\Rightarrow \sum_{n=1}^{N} \frac{|a(n) - b(n)|}{2^n} < r$$

to be open and so

$\Rightarrow U_1 \times U_2 \times \cdots U_N \times [0,1] \times [0,1] \times \cdots$

will be generated as a ball

$\Rightarrow$ open in $[0,1]^{\mathbb{N}}$

$\therefore$ Ball word d is open in $[0,1]^{\mathbb{N}}$
and open set in $[0,1]^{\mathbb{N}}$ can be mced of sum balls form d

$\Rightarrow [0,1]^{\mathbb{N}}$ is a metric space

(ii) $X = \{ (x_\alpha)_{\alpha \in (0,1)} ; x_\alpha \in [0,1] \}$

$X = [0,1]^{(0,1)}$

Firstly if J is uncountable then $\mathbb{R}^J$ is not normal

let $X = (\mathbb{Z}_+)^J$
if X is not normal then as X is closed
$\Rightarrow \mathbb{R}^J$ is not normal

(if $\mathbb{R}^J$ is normal then any closed set of $\mathbb{R}^J$ is also normal)

$x \in X$, B is finite
let $x : J \longrightarrow \mathbb{Z}_+$
then now

$$U(x,B) = \left\{ y \in X \,\middle|\, y_\alpha = x_\alpha ,\ \forall \alpha \in B \right\}$$

$\Rightarrow U(x,B)$ is basis for $\mathbb{R}^J$ (trivial) now let P_n be a subset s.t

$x \in P_n$ if $x : J \longrightarrow \mathbb{Z}+$ is s.t

$x : J - x^{-1}(n) \longrightarrow \mathbb{Z}+$ is one-one

i.e $x_\alpha = \alpha_{\mathbb{Z}+}$ $\quad \alpha \in J - x^{-1}(n)$, $\alpha_{\mathbb{Z}_+} \in \mathbb{Z}+$

$x_\alpha = x_\beta \Rightarrow \alpha = \beta$

finite subset of J

$$U(x, B) = \left\{ y \in X \;\middle|\; x_\alpha = y_\alpha ; \forall \alpha \in B \right\}$$

$$P_n = \left\{ x \in X \;\middle|\; x : J - x^{-1}(n) \longrightarrow \mathbb{Z}+ \text{ is one-one} \right\}$$

for every $x \in P_n$

$x : J - x^{-1}(n) \longrightarrow \mathbb{Z}+$ is one-one

i.e if $x_\alpha = x_\beta$ then $\alpha = \beta$

so $x \notin U(x, \{\alpha\})$

$\Rightarrow x \in \underbrace{U^c(x, \{\alpha\})}_{\text{closed}}$

intersection of all

$\Rightarrow P_1$ and P_2 closed

and

$P_1 \cap P_2 = \phi$

let U, V be open containing P_1 and P_2

given $\alpha_1, \alpha_2, \ldots$ of distinct elems of J

and $\{n_i\}_{i=0}^{\infty}$ s.t

$0 = n_0 < n_1 < \cdots$ and $n_i \in \mathbb{Z}_{\geq 0}$

let $\forall\, i \geq 1$, $B_i = \{\alpha_1, \ldots \alpha_{n_i}\}$

and $x_i \in X$ s.t

$x_i(\alpha_j) = j \qquad 1 \leq j \leq n_i - 1$

$x_i(\alpha) = 1 \qquad$ for other values

now $U(x_i, B_i) \subseteq U$

and similarly $y : J \longrightarrow \mathbb{Z}+$

$y(\alpha_j) = j \qquad \alpha_j \in \{\alpha_1, \ldots\}$

$y(\alpha) = 2 \qquad$ all else

$\Rightarrow U(y, B) \subseteq V$ [s.t $B \cap A \subset B_i$

but $U(x_{i+1}, B_{i+1}) \cap U(y, B) \neq \phi$

$\Rightarrow U \cap V \neq \phi$

$\Rightarrow \mathbb{R}^J$ not normal

$\Rightarrow (0,1)^{(0,1)}$ not normal

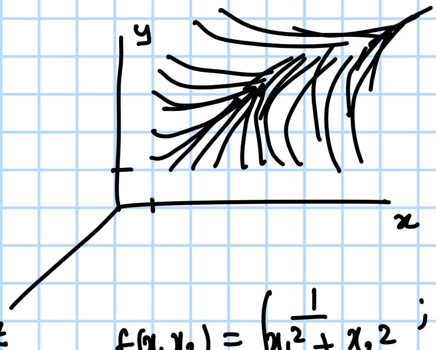
but $[0,1]^{(0,1)}$ is not completely normal
as $(0,1)^{(0,1)}$ is not normal

$\Rightarrow [0,1]^{(0,1)}$ is not a metric space

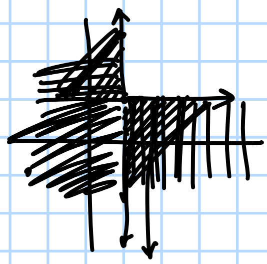
$(\because$ metric space $\Rightarrow$ comp normal $)$

3. (i) $A = \{x \in \mathbb{R}^3 \mid x_3 = 0\} = \{(x,y,0) \in \mathbb{R}^3 \mid x, y \in \mathbb{R}\}$

plane of $x-y$
as $z=0$
always



$$f(x_1,x_2) = \begin{cases} \dfrac{1}{x_1^2 + x_2^2} ; & x_1 > 1, \\ & x_2 > 1 \\ \dfrac{1}{1+x_1^2} ; & x_2 = 1 \\ & x_1 > 1 \\ \dfrac{1}{1+x_2^2} ; & x_1 = 1 \\ & x_2 > 1 \\ \dfrac{1}{2} ; & x_1 = 1, x_2 = 1 \end{cases}$$



now, lets extend this f:

$$f(x_1, x_2) = \begin{cases} \dfrac{1}{x_1^2 + x_2^2} ; & x_1 > 1, x_2 > 1 \\ \dfrac{1}{2} ; & x_1 \leq 1, x_2 \leq 1 \\ \dfrac{1}{1+x_1^2} ; & x_2 \leq 1, x_1 > 1 \\ \dfrac{1}{1+x_2^2} ; & x_2 > 1, x_2 \leq 1 \end{cases}$$

now lets make this 2/3 shift ie

$g(x_1, x_2) = \frac{2}{3} f(x_1, x_2)$

then for all x_3 s.t

$V = \{x \in \mathbb{R}^3 \mid x_1, x_2 \in \mathbb{R}, x_3 > \frac{2}{3} f(x_1, x_2)\}$

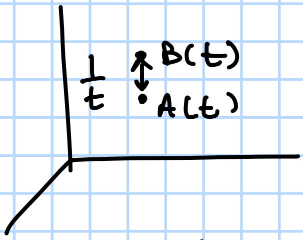
$U = \{x \in \mathbb{R}^3 \mid x_1, x_2 \in \mathbb{R}, x_3 < \frac{1}{3} f(x_1, x_2)\}$

U, V are open as f is cont (trivial)

and also $V \cap V = \phi$ as if $x \in U$ and $x \in V$
then $x_3 > \frac{2}{3} f(x_1, x_2)$
and $x_3 < \frac{1}{3} f(x_1, x_2)$
is not possible

(ii) $A = \{(\sin t, \cos t, t) \mid t > 1\}$

$B = \{(\sin t, \frac{1}{t} + \cos t, t) \mid t > 1\}$



A(t) and B(t) will always have $\frac{1}{t}$ gap
ie $\forall t \ |A(t) - B(t)| = \frac{1}{t}$

lets extend this for all t:
the second coordinate:

$$f(t) = \frac{1}{t} + \cos t \quad \text{as } t > 1$$

then

$$f(t) = \begin{cases} \frac{1}{t} + \cos t \ ; & t > 1 \\ 1 + \cos(1) \ ; & t \leq 1 \end{cases}$$

value of $f(1) = 1 + \cos(1)$

$\Rightarrow f(t)$ is continious
(By construction)

then now let
$$g_r(t) = \begin{cases} r \frac{1}{t} + \cos t \ ; & t > 1 \\ \frac{r}{t} + \cos(1) ; & t \leq 1 \end{cases}$$

then now

$$V = \{(\sin t, x_2, t) \mid \underbrace{g_{\frac{2}{3}}(t) < x_2 < g_{\frac{3}{2}}(t)}_{\text{open as } g_r \text{ is cont}}\}$$

$$U = \{(\sin t, x_2, t) \mid \underbrace{g_{-\frac{1}{3}}(t) < x_2 < g_{\frac{1}{3}}(t)}_{\text{open as } g_r \text{ is continious}}\}$$

and $U \cap V = \phi$ as $x \in U$ and $x \in V$
then
$x_2 < g_{\frac{1}{3}}(t)$ and $x_2 > g_{\frac{2}{3}}(t)$
not possible

4. $A \subseteq [0,1]^{\mathbb{N}}$

(i) $A = \{ (x_n) \in [0,1]^{\mathbb{N}} \mid \sup x_n \leqslant \frac{1}{10} \}$

if A is not closed then A is not compact
as $[0,1]^{\mathbb{N}}$ is also hausdorff + metric space

now for any conv seq of A, its subseq should also be in A

now,