SI 507 Tutorial

Tutorial-1:

Bisection method: $f \in C[a,b]$

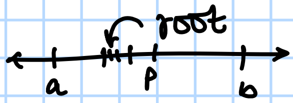
$f(a).f(b) < 0$

$\exists\, P \in (a,b)$ s.t

$f(P) = 0$

$P_1 = \frac{a+b}{2}$ $\quad f(a)f(P_1) < 0$ or $f(b)f(P_1) < 0$

$\vdots$



Eg: try to avoid $|P_{n+1} - P_n| < \varepsilon$ as if

$P_n = \sum_{i=1}^{n} \frac{1}{i}$

$\Rightarrow |P_{n+1} - P_n| = \frac{1}{n+1} < \varepsilon$ but $P_n \not\to$

Eg: try to avoid

$f(z) = 0 \quad |f(P_n)| < \varepsilon$

$f(x) = (1-x)^{10}$

$P_n = 1 + \frac{1}{n}$

$P = 1$ then $f(P_n) = \frac{1}{n^{10}} < \varepsilon$ very fast

$|P_n - P| = \left(1 - 1 + \frac{1}{n}\right) < 10^{-3}$

$\Rightarrow \frac{1}{n} < 10^{-3}$

$\Rightarrow n > 1000$

Eg: for $f(x) = \sin(\pi x)$

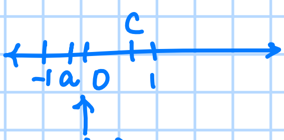
when $-1 < a < 0$, $2 < b < 3$ the bisection method converge to $\begin{cases} 0 & \text{if } a+b < 2 \\ 2 & \text{if } a+b > 2 \\ 1 & \text{if } a+b = 2 \end{cases}$

$1 < a + b < 3$

$\Rightarrow \frac{1}{2} < \frac{a+b}{2} < \frac{3}{2}$

$0.5 < c < 1.5$

now if $0.5 < c < 1$ then



this is the only zero in (a,c)

as $\sin \pi x < 0 \ \forall\ -1 < x < 0$

$\sin \pi x > 0 \ \forall\ 2 < x < 3$

$\sin(\pi a)\sin(\pi b) < 0$

now if $0.5 < c < 1$ then $f(c) > 0$

so $f(a)f(c) < 0$

if $1 < c < 1.5$

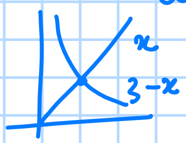
then $f(c) < 0 \quad f(b)f(c) < 0$

if $c=1$
then $f(c)=0 \Rightarrow$ converges to $c=1$

Fixed point theorem:

$f \in C[a,b]$
$f(x) \in [a,b] \quad \forall x \in [a,b]$
f is a differentiable function in (a,b)
s.t $\exists\, 0<k<1$
$|f'(z)| \leq k \quad \forall z \in (a,b)$

Eg: $f(x)=3-x$ converse in not true so, $\exists$ fixed point unique but above not satisfied



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Assignment -1

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1. $\{P_n\}$ given seq s.t

$$P_n = \sum_{k=1}^{n} \frac{1}{k}$$

To prove: $\lim_{n\to\infty} (P_n - P_{n-1}) = 0$ but $\{P_n\}$ diverges

proof: Now $P_n = \sum_{k=1}^{n} \frac{1}{k}$

$$P_{n-1} = \sum_{k=1}^{n-1} \frac{1}{k}$$

$$\Rightarrow P_n - P_{n-1} = \frac{1}{n} + \sum_{k=1}^{n-1} \frac{1}{k} - \sum_{k=1}^{n-1} \frac{1}{k}$$

$$= \frac{1}{n}$$

$$\Rightarrow P_n - P_{n-1} = \frac{1}{n}$$

now, $\forall \varepsilon > 0, \exists N = \lfloor \frac{1}{\varepsilon} \rfloor + 1$ s.t $\forall n > N$

$\Rightarrow n > \lfloor \frac{1}{\varepsilon} \rfloor + 1$ (from construction of N)

$\Rightarrow n > \frac{1}{\varepsilon}$

$\Rightarrow \frac{1}{n} < \varepsilon$

$\Rightarrow \left| \frac{1}{n} - 0 \right| < \varepsilon$

So, $\lim_{n\to\infty} \frac{1}{n} = 0$

$\Rightarrow \lim_{n\to\infty} P_n - P_{n-1} = 0$

now, to show $\{P_n\}$ diverges we have to prove that

$\forall \gamma \in \mathbb{R}, \exists \varepsilon > 0$ s.t $\forall N \in \mathbb{N}, \exists n > N$ s.t

$$|P_n - \gamma| \geq \varepsilon$$

for any N pick k s.t
$n = 2^k$ and $2^k > N$

now, $P_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots \frac{1}{2^k}$

$$= \left(\frac{1}{2^{-1}} + \frac{1}{2^1}\right) + \left(\frac{1}{2^1+1} + \dots \frac{1}{2^2}\right) + \left(\frac{1}{2^2+1} + \dots \frac{1}{2^3}\right) + \dots \left(\frac{1}{2^{k-1}+1} + \dots \frac{1}{2^k}\right)$$

$$> 2 \times \frac{1}{2} + 2 \times \frac{1}{2^2} + 2^2 \times \frac{1}{2^3} + \dots + 2^{k-1} \times \frac{1}{2^k}$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \dots \frac{1}{2}$$

$$= 1 + \frac{k-1}{2} = \frac{k}{2} + \frac{1}{2}$$

So, $P_n > \frac{k}{2} + \frac{1}{2}$ when $n = 2^k$

now, if $\gamma > 0$ then choose k s.t $2^k > N$
and $\frac{k}{2} > \gamma$

then $p_n - r > \frac{k}{2} - r + \frac{1}{2} > \frac{1}{2}$

$\Rightarrow |p_n - r| > \frac{1}{2} = \varepsilon$

if $r \leq 0$ then

$p_n - r > \frac{k}{2} - r + \frac{1}{2} > \frac{k}{2} + \frac{1}{2} \quad (\because -r > 0)$

$\Rightarrow |p_n - r| > \frac{1}{2} = \varepsilon$

so, p_n diverges

2. $f(x) = \sin(\pi x)$

To prove: $-1 < a < 0$, $2 < b < 3$ then bisection method converges to
(a) 0 if $a+b<2$
(b) 2 if $a+b>2$
(c) 1 if $a+b=2$

proof: now $f(a) = \sin(\pi a)$

as $-1 < a < 0$
$\Rightarrow -\pi < \pi a < 0$
$\Rightarrow \sin(\pi a) < 0$
as $\sin(x) < 0$ for $x \in (-\pi, 0)$

similarly $2 < b < 3$
$\Rightarrow 2\pi < \pi b < 3\pi$
$\Rightarrow \sin(\pi b) > 0$ as $\sin(x) > 0$ for $x \in (2\pi, 3\pi)$

so, $f(a) < 0$, $f(b) > 0$

$\Rightarrow f(a) f(b) < 0$

so, we can apply bisection method

now next step: $c = \frac{a+b}{2}$

case (a) $a+b<2$ then $c = \frac{a+b}{2} < \frac{2}{2} = 1$

$\Rightarrow c < 1$

as $c < 1$ and $c > a$
$\Rightarrow c > -1$

as $c \in (-1, 1)$ and we would have to repeate and find c' s.t
$f(c) f(a) < 0$ or $f(c) f(b) < 0$
but as the interval narrowed down to $(-1,1)$ and given in question zeros are integers

as only one integer 0 between $(-1,1)$ our bisection method will converge to 0

case (b) $a+b>2 \Rightarrow c = \frac{a+b}{2} > \frac{2}{2}$

$\Rightarrow c > 1$

then as $c < b < 3$ (given)
$\Rightarrow 1 < c < 3$
$c \in (1, 3)$

we will itterate to next step of bisection method depending on
$f(c)f(a)<0$ or $f(c)f(b)<0$
but as $c \in (1,3)$ and zeros only integers only 2 is the integer and so our method will converge to 2.

case(c) $a+b=2 \Rightarrow c = \frac{a+b}{2} = \frac{2}{2} = 1$

now $f(c) = \sin(\pi) = 0$
so our bisection method is terminated and converges to 1.

3. $I = [0,1]$ $f(x) = \sqrt{x} - \cos x$

now let $a_1 = 0$, $b_1 = 1$
then
$f(a_1) = \sqrt{0} - \cos(0) = -1$
$f(b_1) = 1 - \cos(1) > 0$ ($\because \cos(x) \leq 1 \ \forall x$ and $\cos(1) \neq 1$)
$\Rightarrow f(a_1)f(b_1) < 0$

so now by bisection method

$$p_1 = \frac{a_1+b_1}{2} = \frac{0+1}{2} = \frac{1}{2}$$

now $f(p_1) = \sqrt{\frac{1}{2}} - \cos\left(\frac{1}{2}\right)$

$\sqrt{\frac{1}{2}} \approx \frac{1}{1.414} = 0.709$ and $\cos\left(\frac{1}{2}\right) \approx \cos 30^\circ = \frac{\sqrt{3}}{2} = \frac{1.732}{2} = 0.866$

so $f(p_1) < 0$ as $0.709 - 0.866 < 0$

then $f(p_1)f(1) < 0$, then by our bisection method

$a_2 = p_1 = 1/2$
$b_2 = 1$
and $p_2 = \frac{1/2 + 1}{2} = 0.25 + 0.5 = 0.75$

$f(a_2) < 0$
$f(b_2) > 0$ and now

$$f(p_2) = f(0.75) = \sqrt{\frac{3}{4}} - \cos\left(\frac{3}{4}\right) = \frac{\sqrt{3}}{2} - \cos\left(\frac{3}{4}\right)$$

$\frac{\sqrt{3}}{2} \approx \frac{1.732}{2} = 0.866$

$\cos\left(\frac{3}{4}\right) \approx \cos(45^\circ) = \frac{1}{\sqrt{2}} = 0.709$

so $f(p_2) > 0$ as $\frac{\sqrt{3}}{2} - \cos\left(\frac{3}{4}\right) \approx 0.866 - 0.709 > 0$

then $f(a_2)f(p_2) < 0$, so
$a_3 = a_2$, $b_3 = p_2 \Rightarrow a_3 = 0.5$, $b_3 = 0.75$

then from bisection method

$$p_3 = \frac{a_3+b_3}{2} = \frac{\frac{1}{2}+\frac{3}{4}}{2} = \frac{\frac{5}{4}}{2} = \frac{1.25}{2} = 0.625$$

$p_3 = 0.625$

4. To prove: $x_n = \frac{1}{2}\left(x_{n-1} + \frac{2}{x_{n-1}}\right), n \geq 1$ converges to $\sqrt{2}$ when $x_0 > \sqrt{2}$

proof:

let $g(x) = \frac{1}{2}\left(x + \frac{2}{x}\right)$

$I = \left[\frac{2}{x_0}, x_0\right]$

as $x_0 > \sqrt{2}$

$\frac{1}{x_0} < \frac{1}{\sqrt{2}}$

$\Rightarrow \frac{2}{x_0} < \sqrt{2}$

so $\sqrt{2} \in I$

and now

$\frac{2}{x_0} \leq x \leq x_0$

$\Rightarrow \frac{1}{x_0} \leq \frac{1}{x} \leq \frac{x_0}{2}$

$\Rightarrow \frac{2}{x_0} \leq \frac{2}{x} \leq x_0$

$\Rightarrow \frac{2}{x_0} + \frac{2}{x_0} \leq x + \frac{2}{x} \leq 2x_0$

$\Rightarrow \frac{2}{x_0} \leq g(x) \leq x_0$

so for $I = \left[\frac{2}{x_0}, x_0\right]$ $\frac{2}{x_0} \leq g(x) \leq x_0$

now, it is trivial to see $g \in C^1\left(\left[\frac{2}{x_0}, x_0\right]\right)$ as $x, \frac{1}{x}$ C^1 on $(0,\infty)$

$g'(x) = \frac{1}{2} \times \left[1 - \frac{2}{x^2}\right]$

for $I = \left[\frac{2}{x_0}, x_0\right]$ as $x > 0$

$\frac{1}{x^2} > 0$

$\Rightarrow -\frac{1}{x^2} < 0$

so $g'(x) = \frac{1}{2} - \frac{1}{x^2} < \frac{1}{2}$

$\forall x \in I$, $|g'(x)| < \frac{1}{2} < 1$

so from theorem done in class $\left(g \in C^1[a,b],\ a \leq g(x) \leq b,\ |g'(x)| \leq k < 1 \ \forall x \in [a,b]\right)$

for any $p_0 \in \left[\frac{2}{x_0}, x_0\right]$ the

sequence $p_n = g(p_{n-1})$ converges to a unique fixed point $p \in \left[\frac{2}{x_0}, x_0\right]$

let $p_0 = x_0$ then

$p_n = g(p_{n-1})$ converges to a unique fixed point $p \in \left[\frac{2}{x_0}, x_0\right]$

$\Rightarrow x_n = \frac{1}{2}\left(x_{n-1} + \frac{2}{x_{n-1}}\right)$ conv to a unique $p \in \left[\frac{2}{x_0}, x_0\right]$

$g(x) = x$ when

$$\frac{1}{2}\left(x + \frac{2}{x}\right) = x$$

$$\Rightarrow \quad \frac{1}{x} = \frac{x}{2} \Rightarrow x^2 = 2$$

$\Rightarrow x = \pm\sqrt{2}$ so 2 fixed point but

$-\sqrt{2} \notin [\frac{2}{x_0}, x_0]$

the unique fixed point P where

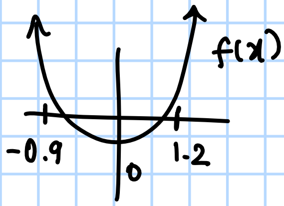
$x_n = f(x_{n-1})$ converges is $\sqrt{2}$

5. $f(x) = x^4 + 2x^2 - x - 3$
$f'(x) = 4x^3 + 4x - 1$
$f''(x) = 12x^2 + 4 > 0$ for all values
so f is a convex polynomial

$\Rightarrow f$ has 2 real roots
now $f(0) < 0$

$f(-0.9) = (-0.9)^4 + 2(-0.9)^2 - (-0.9) - 3 = 0.1761 > 0$

$f(1.2) = (1.2)^4 + 2(1.2)^2 - (1.2) - 3 = 0.7536 > 0$



so $\text{root}_1 \in (-0.9, 0)$ (only real root)
$\text{root}_2 \in (0, 1.2)$ (IVP)

(a) $g_1(x) = (3 + x - 2x^2)^{1/4}$ now

let $P_1(x) = 3 + x - 2x^2$

then $P_1'(x) = 1 - 4x$

$P_1''(x) = -4 < 0$
and so $P_1'(x) = 0 \Rightarrow x = \frac{1}{4}$ is maxima

and $P_1\left(\frac{1}{4}\right) = 3 + \frac{1}{4} - 2\left(\frac{1}{4}\right)^2 = 3.125 > 0$

now, $P_1(-0.9) = 3 - 0.9 - 2(-0.9)^2 = 0.48 > 0$
$P_1(1.2) = 3 + (1.2) - 2(1.2)^2 = 1.32 > 0$

so, $3 + x - 2x^2 > 0$ for $x \in (-0.9, 1.2)$ $P_1(x)$

now, $g_1(x) = x$ if $(3 + x - 2x^2)^{1/4} = x$
for $x \in (-0.9, 1.2)$

but as $(3 + x - 2x^2)^{1/4} > 0$ root_2 cannot be fixed point (IVP)

$\Rightarrow 3 + x - 2x^2 = x^4$
$\Rightarrow x^4 + 2x^2 - x - 3 = 0$

so $g_1(P) = P \Rightarrow P^4 + 2P^2 - P - 3 = 0$
and $P > 0$ as $(3 + P - 2P^2)^{1/4} > 0$
so $\text{root}_2 \in (0, 1.2)$ is solution to $g_1(P) = P$ and

also $f(P)=0$

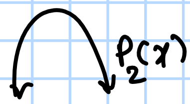
(b) $g_2(x) = \left(\frac{x+3-x^4}{2}\right)^{1/4}$

now

$P_2(x) = x+3-x^4$

$P_2'(x) = 1-4x^3$

$P_2''(x) = -12x^2 < 0$ so only 2 roots



now, $P_2(-0.9) = 1.4439 > 0$

$P_2(1.2) = 2.1264 > 0$

so, $P_2(x) > 0$ for $x \in (-0.9, 1.2)$

$\Rightarrow \left(\frac{P_2(x)}{2}\right)^{1/2} > 0$ for $x \in (-0.9, 1.2)$

now $g_2(P) = P$ is fixed point

$\Rightarrow \left(\frac{x+3-x^4}{2}\right)^{1/2} = x$

$\Rightarrow \left(\frac{x+3-x^4}{2}\right) = x^2$

$\Rightarrow 0 = x^4 + 2x^2 - x - 3$

so $g_2(P) = P \Rightarrow f(P) = 0$ but root$_1 < 0$ not a fixed point as (IVP)

$\left(\frac{P_2(P)}{2}\right)^{1/2} > 0$, so $P > 0$

so in both cases root$_2$ (positive real root) is the fixed point for $g_1(x)$ and $g_2(x)$

6. P is zero of mul m of f

f^m is cont on I s.t I is open & $P \in I$

<u>To prove</u>: $g(x) = x - \frac{mf(x)}{f'(x)}$ has $g'(P) = 0$

<u>proof</u>: as P is a zero of multiplicity m and $f^{(m)}$ is cont

$\Rightarrow f^{(m-1)}$ is diff & cont

so, $f^{(m-1)}$ is diff & cont and similarly

$f^{(m-2)}$ is diff & cont

$\vdots$

$\Rightarrow f$ is diff & cont, moreover $f \in C^m(I)$

now, as P is zero of mul $= m$

$f(x) = (x-p)^m h(x)$ where $h(p) \neq 0$ and $h \in C^m(I)$
(from defn of multiplicity and $f \in C^m(I)$)

now, $f'(x) = m(x-p)^{m-1}h(x) + (x-p)^m h'(x)$ as ($h \in C^m(I)$, $h'(x)$ exist & is cont)

$$m\frac{f(x)}{f'(x)} = \frac{m[x-p]^m h(x)}{\frac{m(x-p)^{m-1}h(x) + \cancel{(x-p)^m} h'(x)}{(x-p)^m}}$$

$$= \frac{m\, h(x)(x-p)}{m\, h(x) + (x-p)h'(x)}$$

also as $h(p) \neq 0$ $f'(p) \neq 0$ and also cont so $\exists I_p$ s.t $\forall x \in I_p \subseteq I$
$f'(p) \neq 0$ (property of continuity)

and so, $m\frac{f'(x)}{f(x)}$ exist for $\forall x \in I_p$

now, $g(x) = x - m\frac{f(x)}{f'(x)}$ $\forall x \in I_p$

$$\Rightarrow g(x) = x - \frac{m h(x)(x-p)}{m h(x) + (x-p)h'(x)}$$

$$\Rightarrow g'(x) = 1 - \frac{d}{dx}\left(\frac{m h(x)(x-p)}{m h(x) + (x-p)h'(x)}\right)$$

$$= 1 - \frac{m[h'(x)(x-p) + h(x)(1)]}{m h(x) + (x-p)h'(x)}$$

$$+ \frac{[m h(x)(x-p)][m h'(x) + h'(x) + (x-p)h''(x)]}{(m h(x) + (x-p)h'(x))^2}$$

exist as $h \in C^m(I)$

exist as denominator $\neq 0$ for $x \in I_p$

now, $g'(p) = 1 - \frac{m[0 + h(p)]}{m\, h(p)} + 0$

$= 1 - \frac{m h(p)}{m h(p)}$

$= 1 - 1$

$= 0$

so, $g'(p) = 0$

7. f has m cont derivaties

<u>To prove</u>: f has zero of multiplicity m at $p \Leftrightarrow 0 = f^i(p)$ $i = 0, 1 \ldots m-1$
$f^m(p) \neq 0$

<u>proof</u>: ($\Rightarrow$) $f(x) = (x-p)^m g(x)$ $g \in C^m$, $g(p) \neq 0$
by defn of multiplicity

then
$f(p) = 0$ (trivial)

$f'(x) = m(x-p)^{m-1}g(x) + (x-p)^m g'(x)$

then $f'(x) = \frac{m!}{(m-1)!}(x-p)^{m-1} g(x) + (x-p)^m r(x)$

$f'(p) = 0$ (trivial)

let $f^i(x) = \frac{m!}{(m-i)!}(x-p)^{m-i} g(x) + (x-p)^{m-i+1} r_i(x)$

true for $i=2$ $\quad i < m-1$

for $i+1$: $f^{i+1}(x) = \frac{m!}{(m-i)!}(m-i)(x-p)^{m-i-1} g(x)$ $\quad \left(\because \frac{d}{dx}(f^i(x))\right)$

$+ (x-p)^{m-i} r_{i+1}(x)$

$= \frac{m!}{(m-i-1)!}(x-p)^{m-i-1} g(x) + (x-p)^{m-i} r_{i+1}(x)$

true for $i+1$ (induction step)

$\therefore \forall\, i<m$ true, so

$f^i(p) = 0$ as $(x-p)$ common for $i<m$ from induction

now $\quad f^{m-1}(x) = m!(x-p)\, g(x) + (x-p)^2 r_{m-1}(x)$ from our induction result

and so $\frac{d}{dx} f^{m-1}(x) = f^m(x) = m!\, g(x) + m!(x-p) g'(x)$

$+ 2(x-p) r_{m-1}(x) + (x-p)^2 r'_{m-1}(x)$

$f^m(p) = m!\, g(p)$ as $g(p) \neq 0$

$\Rightarrow f^m(p) \neq 0$

$\therefore f^i(p) = 0 \ \forall i = 0,1..m-1$

$f^m(p) \neq 0$

$(\Leftarrow)$ $f^m(p) \neq 0$, $f^{(i)}(p) = 0 \ \forall i = 0,1\ldots m-1$

let the multiplicity be k, then

$f^0(x) = (x-p)^k g_0(x)$ s.t $g_0(p) \neq 0$

$= \frac{k!}{(k-0)!}(x-p)^k g_0(x)$

claim: $f^i(x) = \frac{k!}{(k-i)!}(x-p)^{k-i} g_0(x) + (x-p)^{k-i+1} r_i(x)$

for $i \leq k$

true for $i=0$, assume true for $i<k$ then for $i+1$:

$f^{i+1}(x) = \frac{d}{dx} f^i(x) = \frac{d}{dx}\left[\frac{k!}{(k-i)!}(x-p)^{k-i} g_0(x) + (x-p)^{k-i+1} r_i(x)\right]$

$= \frac{k!}{(k-i)!}(i)(x-p)^{k-i-1} g_0(x)$

$+ (x-p)^{k-i} \underbrace{\left[\frac{k!}{(k-i)!} g_0'(x) + (k-i+1)(x-p)^{k-i} r_i(x) + (x-p)^{k-i+1} r_i'(x)\right]}_{r_{i+1}(x)}$

$\Rightarrow$ true for $i+1$, $\therefore \forall i \leq k$ true

now as $f^{(m-1)}(p) = 0$ as $f^{m-1}(p) = 0$ for $i = m-1$

$k - i = k-(m-1) \geq 1$

$\Rightarrow K-m \geq 0$
$\Rightarrow K \geq m$

but $f^m(p) \neq 0$, so $K=m$ as smallest such $f^i(p) \neq 0 \Rightarrow i=m$

$\therefore$ $K=m$ and so

$$f(x) = (x-p)^m g_0(x) \quad \text{s.t } g_0(p) \neq 0$$

$\Rightarrow f$ has multiplicity m

8. To prove: $g(x) = x - \frac{f(x)}{f'(x)} - \frac{f''(x)}{2f'(x)} \times \left(\frac{f(x)}{f'(x)}\right)^2$ has $g'(p) = g''(p) = 0$, given $f(p)=0$

proof: $g'(x) = 1 - \frac{d}{dx}\left(\frac{f(x)}{f'(x)}\right) - \frac{1}{2}\frac{d}{dx}\left(\frac{f''(x)(f(x))^2}{(f'(x))^3}\right)$

$$= 1 - \left[\frac{f'(x)f'(x) - f(x)f''(x)}{(f'(x))^2}\right]$$

$$-\frac{1}{2}\left[f'''(x)(f(x))^2 + 2f''(x)f(x)f'(x)\right]\frac{1}{(f'(x))^3}$$

$$-\frac{1}{2}\left[f''(x)(f(x))^2\right] \times \frac{(-3)}{(f'(x))^4} \times (f''(x))$$

$$g'(x) = \cancel{1} - \cancel{1} + \frac{f(x)\cancel{f''(x)}}{(\cancel{f'}(x))^2} - \frac{\cancel{f''(x)}f(x)}{\cancel{(f'(x))^2}} - \frac{1}{2}\frac{f'''(x)(f(x))^2}{(f'(x))^3} + \frac{3}{2}\frac{(f''(x))^2(f(x))^2}{(f'(x))^4}$$

$$g'(x) = [f(x)]^2 \underbrace{\left[\frac{3}{2}\frac{(f''(x))^2}{(f'(x))^4} - \frac{1}{2}\frac{f'''(x)}{(f'(x))^3}\right]}_{r(x)}$$

$$g'(x) = (f(x))^2 r(x)$$

so $g'(p) = (f(p))^2 r(p) = 0$ as $f(p)=0$

now, $g''(x) = 2f(x)r(x)f'(x) + (f(x))^2 r'(x)$

$$g''(p) = f(p)\left[2r(p)f'(p) + f(p)r'(p)\right] = 0$$

as $f(p)=0$

To prove: order of conv is 3

proof:

$$p_{n+1} - p = p_n - p - \frac{f(p_n)}{f'(p_n)} - \frac{f''(p_n)}{2f'(p_n)}\left(\frac{f(p_n)}{f'(p_n)}\right)^2 \text{ — ① (Subtracting } p \text{ from both sides)}$$

from given make $n \to n+1$ and subtract p

$$f(p) = f(p_n) + (p-p_n)f'(p_n) + (p-p_n)^2\frac{f''(p_n)}{2} + \frac{(p-p_n)^3}{6}f'''(\gamma_n) \text{ — ②}$$

taylor expansion for some γ_n b/w p & p_n

$$0 = f(p_n) + (p-p_n)f'(p_n) + (p-p_n)^2\frac{f''(p_n)}{2} + \frac{(p-p_n)^3}{6}f'''(\gamma_n)$$

as $f(p)=0$

$$\Rightarrow 0 = \frac{f(p_n)}{f'(p_n)} + (p-p_n)\frac{f'(p_n)}{f'(p_n)} + (p-p_n)^2\frac{f''(p_n)}{2f'(p_n)} + \frac{(p-p_n)^3}{6}\frac{f'''(\gamma_n)}{f'(p_n)}$$

$\frac{f(p_n)}{f'(p_n)}$ in ①:

$$p_{n+1} - p = \cancel{p_n - p} + \cancel{(p-p_n)\frac{f'(p_n)}{f'(p_n)}} + (p-p_n)^2\frac{f''(p_n)}{2f'(p_n)} + \frac{(p-p_n)^3}{6}\frac{f'''(\gamma_n)}{f'(p_n)} - \frac{f''(p_n)}{2f'(p_n)}\left(\frac{f(p_n)}{f'(p_n)}\right)^2$$

$$p_{n+1} - p = \frac{f''(p_n)}{2f'(p_n)}\left[(p-p_n)^2 - \left(\frac{f(p_n)}{f'(p_n)}\right)^2\right] + \frac{(p-p_n)^3}{6}\frac{f'''(\gamma_n)}{f'(p_n)}$$

now putting $\frac{f(p_n)}{f'(p_n)}$ here from ②:

$$\Rightarrow p_{n+1} - p = \frac{f''(p_n)}{2f'(p_n)}\left[(p-p_n)^2 - \left((p-p_n) + (p-p_n)^2\frac{f''(p_n)}{2f'(p_n)} + \frac{(p-p_n)^3}{6}\frac{f'''(\gamma_n)}{f'(p_n)}\right)^2\right] + \frac{(p-p_n)^3}{6}\frac{f'''(\gamma_n)}{f'(p_n)}$$

$$= \frac{f''(p_n)}{2f'(p_n)}\left[(p-p_n)^2 - (p-p_n)^2\Big[\overbrace{1 + (p-p_n)\frac{f''(p_n)}{2f'(p_n)} + \frac{(p-p_n)^3}{6}\frac{f'''(\gamma_n)}{f'(p_n)}}^{1+(p-p_n)\gamma(p_n)}\Big]^2\right] + \frac{(p-p_n)^3}{6}\frac{f'''(\gamma_n)}{f'(p_n)}$$

$$= \frac{f''(p_n)}{2f'(p_n)}\left[(p-p_n)^2 - (p-p_n)^2\left[\cancel{1} + (p-p_n)^2\gamma^2(p_n) + 2(p-p_n)\gamma(p_n)\right]\right] + \frac{(p-p_n)^3}{6}\frac{f'''(\gamma_n)}{f'(p_n)}$$

$$= (p-p_n)^3\underbrace{\left[\frac{f''(p_n)}{2f'(p_n)}[h(p_n)] + \frac{f'''(\gamma_n)}{f'(p_n)}\right]}_{o(p_n)}$$

some fuction after facting out $(p-p_n)^3$

$$= (p-p_n)^3 o(p_n)$$

$$\Rightarrow \frac{p_{n+1}-p}{(p-p_n)^3} = o(p_n)$$

$$\Rightarrow \lim_{n\to\infty}\frac{|p_{n+1}-p|}{|p-p_n|^3} = |o(p)|$$ ← some finite value

so order $= 3$

9. $p(x) = a_0 + a_1 x + \cdots + x^n a_n$

$a_n \neq 0$

$R = \dfrac{|a_0| + \cdots |a_n|}{|a_n|} \geq \dfrac{|a_n|}{|a_n|} = 1$ — ①

now, if $p(x)=0$ is s.t $|x| \leq 1$

then $\Rightarrow |x| \leq 1 \leq R$ from ①

$\Rightarrow |x| \leq \max\{R, \sqrt[n]{R}\}$ ($\because |x| \leq R$ so $|x| \leq \max\{R, \sqrt[n]{R}\}$)

if $|x|>1$ then:

$p(x)=0 \Rightarrow a_0 + a_1 x + \cdots x^n a_n = 0$ ($\because$ root of $p(x)$)

$\Rightarrow x^n a_n = a_0 + a_1 x + \cdots a_{n-1} x^{n-1}$

$\Rightarrow x^n = \dfrac{a_0 + a_1 x + \cdots a_{n-1} x^{n-1}}{a_n}$

$\Rightarrow |x|^n = \dfrac{|a_0 + a_1 x + \cdots a_{n-1} x^{n-1}|}{|a_n|}$

$\leq \dfrac{|a_0| + |a_1 x| + \cdots |a_{n-1}||x|^{n-1}}{|a_n|}$

$\leq \dfrac{|a_0| + |a_1| + \cdots |a_{n-1}|}{|a_n|}$ ($\because |x|>1$ case)

$\Rightarrow |x|^n \leq \dfrac{|a_0| + |a_1| + \cdots |a_{n-1}| + |a_n|}{|a_n|}$

$\Rightarrow |x|^n \leq R$ ($\because$ given R)

$\Rightarrow |x| \leq \sqrt[n]{R}$

$\Rightarrow |x| \leq \max\{R, \sqrt[n]{R}\}$ (trivial as $|x| \leq \sqrt[n]{R}$)

so from both cases

$|x| \leq \max\{R, \sqrt[n]{R}\}$

10. $x = 1 + \tan^{-1}(x)$

now, let $f(x) = 1 + \tan^{-1}(x) - x$

$f'(x) = \dfrac{1}{1+x^2} - 1$

as $x^2 \geq 0 \Rightarrow 1 + x^2 \geq 1$

$\Rightarrow \dfrac{1}{1+x^2} \leq 1$

$\Rightarrow \dfrac{1}{1+x^2} - 1 \leq 1 - 1$

$\Rightarrow \dfrac{1}{1+x^2} - 1 \leq 0$

$\Rightarrow f'(x) \leq 0$ so non-increasing function also $\lim_{x\to-\infty} f(x) \to \infty$

$\lim_{x\to\infty} f(x) \to -\infty$

so by IVP, $\exists$ one root of the function, α

now, $f(1) = \cancel{1} + \tan^{-1}(1) - \cancel{1}$

$= \frac{\pi}{4} > 0$

$f(3) = 1 - 3 + \tan^{-1}(3)$

$= -2 + \tan^{-1}(3)$

$< -2 + \frac{\pi}{2} \leftarrow$ upper bond of $\tan^{-1}(x)$

< 0 as $\pi < 4$

$\Rightarrow \pi/2 < 2$

so $\alpha \in (1,3)$

now, let $[a,b] = [1,3]$ then

$g(x) = x = 1 + \tan^{-1}(x)$

now, $1 \leq x \leq 3$

$\tan^{-1}(1) \leq \tan^{-1}(x) \leq \tan^{-1}(3)$

as $\tan^{-1}(x)$ is monotonically inc

$\Rightarrow \frac{\pi}{4} \leq \tan^{-1}(x) \leq \frac{\pi}{2}$

$\Rightarrow 1 + \pi/4 \leq 1 + \tan^{-1}(x) \leq 1 + \pi/2 \leq 1 + 4/2 = 3$

as $\pi \leq 4$

$\Rightarrow 1 \leq 1 + \pi/4 \leq g(x) \leq 1 + \pi/2 \leq 3$

$\Rightarrow 1 \leq g(x) \leq 3$

$\forall x \in [1,3]$, $g(x) \in [1,3]$

also $g \in C^1[1,3]$ is trivial as $1 + \tan^{-1}(x) \in C^1(0,\infty)$

now $g'(x) = \frac{1}{1+x^2}$

as $1 \leq x \leq 3$

$\Rightarrow 1 \leq x^2 \leq 9$

$\Rightarrow 2 \leq 1 + x^2 \leq 10$

$\Rightarrow \frac{1}{10} \leq \frac{1}{1+x^2} \leq \frac{1}{2}$

$\Rightarrow |g'(x)| \leq \frac{1}{2} \forall x \in [1,3]$

< 1

so from theorm done in class (g has unique fixed point on $[a,b]$ if ① $g \in C^1[a,b]$
② $|g'(x)| \leq k < 1 \; \forall x \in [a,b]$
③ $g(x) \in [a,b] \; \forall x \in [a,b]$)

$g(x) = x$ has unique fixed point on $[1,3]$

here its α as we showed only

one root to $f(x)$

now, $x_{n+1} = 1 + \tan^{-1}(x_n)$
will converge to α on $[1,3]$ as α is unique fixed point for $x \in [1,3]$, from theorm done in class

($g \in C^1[a,b]$, $a \leq g(x) \leq b$, $|g'(x)| \leq k < 1$ $\forall$ $x \in [a,b]$
then $p_0 \in [a,b]$ seq $p_n = g(p_{n-1})$ conv to unique fixed point)

Rate of convergence :

$|p_{n+1} - \alpha| = |g(p_n) - g(\alpha)|$ (from condition)
$= |g'(r_n)(p_n - \alpha)|$ (from MVT)

r_n b/w p_n and α

$\Rightarrow \dfrac{|p_{n+1} - \alpha|}{|p_n - \alpha|} = |g'(r_n)|$

as $n \to \infty$ $r_n \to \alpha$

and as $g'(x) = \dfrac{1}{1+x^2}$ $g'(\alpha) = \dfrac{1}{1+\alpha^2}$

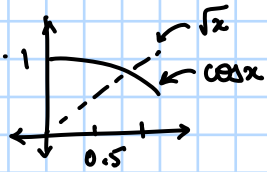
so $\lim\limits_{n \to \infty} \dfrac{|p_{n+1} - \alpha|}{|p_n - \alpha|} = |g'(\alpha)| = \dfrac{1}{1+\alpha^2}$

order of conv $= 1$ (from defn)

so, this has a linear rate of convergence

Tutorial-2:

3.



Tutorial-3:

1. $x_0=0 \quad y_0=0$
$x_1=0.5 \quad y_1=4$
$x_2=1 \quad y_2=3$
$x_3=2 \quad y_3=2$

$$P_3(x)=\sum_{i=0}^{3} y_i\, l_i(x) \qquad l_i(x)=\prod_{\substack{i\neq j\\ j=0}}^{3}\frac{(x-x_j)}{(x_i-x_j)}$$

$$l_0(x)=\frac{(x-0.5)(x-1)(x-2)}{(0-0.5)(0-1)(0-2)}$$

$$l_1(x)=\frac{(x-0)(x-1)(x-2)}{(0.5-0)(0.5-1)(0.5-2)}$$

2. $x_0=0$
$x_1=0.6$
$x_2=0.9$
lagrange interpolation
$f(0.45)$
$x_0, x_1 \rightarrow$ degree 1 (linear) as $x_0<0.45<x_1$
$x_0, x_1, x_2 \rightarrow$ degree 2

abs error $=|P_i(x)-f(0.45)|$ for $i=1,2$

3. $H_{2n+1}(x)$ is unique polynomial with degree agreeing with f & f'

$$H(x_i)=f(x_i)\ \forall\, i\in\{0,\dots n\}$$
$$H'(x_i)=f'(x_i)\ \forall\, i\in\{0,\dots n\}$$

if not then $\exists G$ same & so

$F=H-G\rightarrow 2n+1$ degree
s.t. $F(x_i)=0\ \forall\, i\in\{0,\dots n\}$
& $F'(x_i)=0\ \forall\, i\in\{0,\dots n\}$
So double root at $x_i\ \forall\, i$
$\Rightarrow 2n+2$ roots
$\Rightarrow$ this is not possible for $2n+1$ degree polynomial — so $F\equiv 0$
$\Rightarrow H$ is unique by contradiction

4. $$S(x)=\begin{cases} s_0(x)=3(x-1)+2(x-1)^2-(x-1)^3 & ;\ 1\leq x<2\\ s_1(x)=a+b(x-2)+c(x-2)^2+d(x-2)^3 & ;\ 2\leq x\leq 3\end{cases}$$

for S to be cubic spline, $S\in C^2[0,2]$

$$S''(0)=0$$
$$S''(2)=0$$

5. spline s for function f on $[1,3]$ is:
$s\in C^2[1,3]$
$f'(1)=f'(3)$
clamped

7. put values $\int_{-1}^{1} 1\,dx,\ \int_{-1}^{1} x\,dx,\ \int_{-1}^{1} x^2\,dx$

8. same as previous

9. (a) general form & then compute / satisfy all conditions & derive $p(x)$

(b) $x_0=-1 \quad x_1=0 \quad x_2=1$

let $\varphi(t)=(f(t)-p(t))-(f(x)-p(x))\dfrac{(t-1)(t)(t+1)}{(x-1)(x)(x+1)}$

φ has $-1,0,1,x$ as roots

φ' has 3 roots

φ'' has 2 roots

φ''' has 1 root say $\varphi'''(\xi_x)=0$ depends upon x

$(\xi_x)\in[-1,1]$

10. $0\leq t_0<t_1<\cdots<t_{2n}<2\pi$

$$l_j(t)=\prod_{\substack{k=0\\ j\neq k}}^{2n}\frac{\sin\left(\frac{1}{2}(t-t_k)\right)}{\sin\left(\frac{1}{2}(t_j-t_k)\right)} \qquad j=0,1\ldots 2n$$

$l_j(t_i)=\delta_{ij} \qquad 0\leq i,j\leq 2n$

$$l_j(t)=\frac{\prod_{\substack{k=0\\ j\neq k}}^{2n}\sin\left(\frac{1}{2}(t-t_k)\right)}{\alpha}$$

$$\sin(z)=\frac{e^{iz}-e^{-iz}}{2i}$$

$$\alpha\, l_j(t)=\prod_{\substack{k=0\\ j\neq k}}^{2n}\frac{e^{\frac{i(t-t_k)}{2}}-e^{\frac{-i(t-t_k)}{2}}}{2i}$$

11. (a) $\displaystyle\sum_{j=0}^{m-1} e^{(2\pi ijk)/m} \qquad k\equiv 0\ (\mathrm{mod}\ m)$

Assignment - 2

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Assignment - 2

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1. $P_3(x)$ is interpolating polynomial for $(0,0), (0.5, y), (1,3), (2,2)$

$$x_0 = 0 \quad y_0 = 0$$
$$x_1 = 0.5 \quad y_1 = y$$
$$x_2 = 1 \quad y_2 = 3$$
$$x_3 = 2 \quad y_3 = 2$$

$$P_3(x) = \sum_{i=0}^{3} y_i\, l_i(x) \qquad l_i(x) = \prod_{\substack{i \neq j \\ j=0}}^{3} \frac{(x - x_j)}{(x_i - x_j)}$$

then $l_0(x) = \dfrac{(x-0.5)(x-1)(x-2)}{(0-0.5)(0-1)(0-2)}$

$$l_1(x) = \frac{(x-0)(x-1)(x-2)}{(0.5-0)(0.5-1)(0.5-2)}$$

$$l_2(x) = \frac{(x-0)(x-0.5)(x-2)}{(1-0)(1-0.5)(1-2)}$$

$$l_3(x) = \frac{(x-0)(x-0.5)(x-1)}{(2-0)(2-0.5)(2-1)}$$

and so, $P_3(x) = 0 \times l_0(x) + y \times l_1(x) + 3 l_2(x) + 2 l_3(x)$

now coeff of x^3 is 6

so,

$$P_3(x) = y\left(\frac{(x-0)(x-1)(x-2)}{(0.5-0)(0.5-1)(0.5-2)}\right) + 3\left(\frac{(x-0)(x-0.5)(x-2)}{(1-0)(1-0.5)(1-2)}\right)$$
$$+ 2\left(\frac{(x-0)(x-0.5)(x-1)}{(2-0)(2-0.5)(2-1)}\right)$$

so coeff of $x^3 = \dfrac{y}{(0.5-0)(0.5-1)(0.5-2)} + \dfrac{3}{(1-0)(1-0.5)(1-2)} + \dfrac{2}{(2-0)(2-0.5)(2-1)}$

$= 6$ ($\because$ given)

$$\Rightarrow \frac{y}{0.5(-0.5)(-1.5)} + \frac{3}{(1)(0.5)(-1)} + \frac{2}{(2)(1.5)(1)} = 6$$

$$\Rightarrow \frac{y}{0.5 \times 0.5 \times 1.5} - \frac{3}{0.5} + \frac{2}{3} = 6$$

$$\Rightarrow 2 \times 2 \times \frac{2}{3} \times y - 3 \times 2 + \frac{2}{3} = 6$$

$$\Rightarrow \frac{8}{3}y - 6 + \frac{2}{3} = 6$$

$$\Rightarrow \frac{8}{3}y + \frac{2}{3} = 12$$

$$\Rightarrow \frac{4}{3}y + \frac{1}{3} = 6$$

$$\Rightarrow 4y + 1 = 18$$

$$\Rightarrow 4y = 17$$

$$\Rightarrow y = \frac{17}{4}$$

so $y = \dfrac{17}{4}$

2. $f(x)$, $x_0 = 0$, $x_1 = 0.6$, $x_2 = 0.9$

(a) for degree (1) linear $x_0 = 0$, $x_1 = 0.6$ as $0.45 \in [x_0, x_1]$

then $P_1(x) = \sum_{i=0}^{1} y_i \ell_i(x)$

when $y_0 = f(0)$
$y_1 = f(0.6)$

$$\ell_0(x) = \frac{(x-0.6)}{(0-0.6)} = \frac{0.6-x}{0.6} \qquad \ell_1(x) = \frac{(x-0)}{(0.6-0)} = \frac{x}{0.6}$$

$f(x) = \cos(x)$ ($\because$ given)

$y_0 = f(0) = \cos(0) = 1$
$y_1 = f(0.6) = \cos(0.6)$

then $y_1(x) = 1\,\ell_0(x) + \cos(0.6) \times \ell_1(x)$

$$y_1(x) = \frac{0.6-x}{0.6} + \frac{x}{0.6}\cos(0.6)$$

$$= 1 - \frac{x}{0.6} + \frac{\cos(0.6)}{0.6}x$$

$$y_1(x) = x\left(\frac{\cos(0.6)}{0.6} - \frac{1}{0.6}\right) + 1$$

and $y_1(0.45) = 0.45\left(\frac{\cos(0.6)}{0.6} - \frac{1}{0.6}\right) + 1 \approx 0.8690 \leftarrow$ approximation of $f(0.45)$

$f(0.45) = \cos(0.45) \approx 0.9004$

now $|y_1(0.45) - f(0.45)| \approx 0.0314$

now, for degree 2: $y_2(x) = \sum_{i=0}^{2} y_i \ell_i(x)$

when $y_0 = f(0)$ $y_1 = f(0.6)$ $y_2 = f(0.9)$

$$\ell_0(x) = \frac{(x-0.6)(x-0.9)}{(0-0.6)(0-0.9)} \qquad \ell_1(x) = \frac{(x-0)(x-0.9)}{(0.6-0)(0.6-0.9)} \qquad \ell_2(x) = \frac{(x-0)(x-0.6)}{(0.9-0)(0.9-0.6)}$$

and now $P_2(x) = \cos(0)\left[\frac{(x-0.6)(x-0.9)}{(0.6)(0.9)}\right] - \cos(0.6)\left[\frac{(x-0)(x-0.9)}{0.3 \times 0.6}\right]$
$+ \cos(0.9)\left[\frac{(x-0)(x-0.6)}{(0.9)\ (0.3)}\right]$

now approximating $f(0.45)$:

$P_2(0.45) \approx 0.8981$ (By plotting 0.45 in $P_2(x)$)

$f(0.45) \approx 0.9004$ (By plotting 0.45 in $\cos(x)$)

now abs error: $|P_2(0.45) - f(0.45)| \approx 0.9004 - 0.8981$
≈ 0.0023

(b) $f(x) = \sqrt{1+x}$, now we will do same as (a)

$$P_1(x) = y_0 l_0(x) + y_1 l_1(x) \quad \text{as } 0.45 \in [x_0, x_1]$$

$$= f(0)\left[\frac{(x-0.6)}{(0-0.6)}\right] + f(0.6)\left[\frac{(x-0)}{(0.6-0)}\right]$$

$f(0) = \sqrt{1+0} = 1$

$f(0.6) = \sqrt{1.6}$

then now plotting 0.45 we get:

$$P_1(0.45) = \sqrt{1}\left(\frac{0.45-0.6}{0-0.6}\right) + \sqrt{1.6}\left(\frac{0.45-0}{0.6-0}\right)$$

$$= 0.25 + \sqrt{1.6}\left(\frac{0.45}{0.6}\right)$$

$$\approx 1.1987$$

and $f(0.45) = \sqrt{1+0.45} \approx 1.2042$

now abs error $= |f(0.45) - P_1(0.45)| \approx 1.2042 - 1.1987$

≈ 0.0055

now similarly $P_2(x) = y_0 l_0(x) + y_1 l_1(x) + y_2 l_2(x)$

$$= f(0)\left[\frac{(x-0.6)(x-0.9)}{(0-0.6)(0-0.9)}\right] + f(0.6)\left[\frac{(x-0)(x-0.9)}{(0.6-0)(0.6-0.9)}\right] + f(0.9)\left[\frac{(x-0)(x-0.6)}{(0.9-0)(0.9-0.6)}\right]$$

now, plotting 0.45 in $P_2(0.45)$:

$$P_2(0.45) = \sqrt{1}\left[\frac{(0.45-0.6)(0.45-0.9)}{(0-0.6)(0-0.9)}\right] + \sqrt{1.6}\left[\frac{(0.45)(0.45-0.9)}{(0.6)(0.6-0.9)}\right]$$

$$+ \sqrt{1.9}\left[\frac{(0.45)(0.45-0.6)}{(0.9-0)(0.9-0.6)}\right]$$

$$= 0.125 + \sqrt{1.6} \times 1.125 + \sqrt{1.9} \times (-0.25)$$

$$\approx 1.2034$$

and $f(0.45) \approx 1.2042$

so, abs error $= |P_2(0.45) - f(0.45)| \approx 1.2042 - 1.2034$

≈ 0.0008

(c) now $f(x) = \ln(1+x)$

now, $P_1(x) = y_0 l_0(x) + y_1 l_1(x)$

$y_0 = \ln(1+0)$

$y_1 = \ln(1+0.6)$ as $0.45 \in [x_0, x_1]$

$$P_1(x) = \ln(1.6)\left[\frac{(x-0)}{(0.6)-0}\right]$$

now putting 0.45 we get

$$P_1(0.45) = \ln(1.6)\left[\frac{0.45}{0.6}\right] \approx 0.3525$$

now putting 0.45 in $\ln(1+x)$, we get $f(0.45) \approx 0.3716$

now, abs error $= |f(0.45) - P_1(0.45)| \approx 0.3716 - 0.3525$

≈ 0.0191

now for $P_2(x)$: $P_2(x) = y_0 l_0(x) + y_1 l_1(x) + y_2 l_2(x)$
$$= 0 + \ln(1.6)\left[\frac{(x-0)(x-0.9)}{(0.6-0)(0.6-0.9)}\right] + \ln(1.9)\left[\frac{(x-0)(x-0.6)}{(0.9-0)(0.9-0.6)}\right]$$

putting 0.45 we get:

$$P_2(0.45) = \ln(1.6)\times 1.125 + \ln(1.9)\times 0.25(-1)$$
$$\approx 0.3683$$

now abs error $= |f(0.45) - P_2(0.45)| \approx 0.3716 - 0.3683$
$$\approx 0.0033$$

3. To prove: $H_{2n+1}(x)$ is unique of least degree s.t f & f' agree at $x_0, \ldots x_n$

proof: first lets show $H_{2n+1}(x)$ is unique (it has degree $2n+1$ from it's defn)
if not then $\exists\, G_{2n+1}(x)$ (degree $2n+1$) s.t $G_{2n+1}(x) \neq H_{2n+1}(x)$

$$G_{2n+1}(x_i) = f(x_i) \quad i \in \{0, \ldots n\}$$
$$G'_{2n+1}(x_i) = f'(x_i)$$

let $F(x) = H_{2n+1}(x) - G_{2n+1}(x)$ then $F(x_i) = 0 \;\forall i \in \{0, \ldots n\}$

F is of degree $2n+1$ ($\because H, G$ of degree $2n+1$) and

$$F'(x) = H'_{2n+1}(x) - G'_{2n+1}(x)$$
$$\Rightarrow F'(x_i) = 0 \text{ for } i \in \{0, \ldots n\}$$

so as $F(x_i) = 0$, $F'(x_i) = 0$ for $i \in \{0, \ldots n\}$, F has double root at each x_i

$\Rightarrow F$ is of form $F(x) = g(x)\prod_{i=0}^{n}(x - x_i)^2$, some $g(x)$ is a polynomial

so degree of $F(x) \geq 2n+2$ ($\because$ atleast $2n+2$ roots counting multiplicity)
but $F(x)$ is a $2n+1$ degree polynomial, this is a contradiction

$$\Rightarrow G_{2n+1}(x) = H_{2n+1}(x)$$
so, $H_{2n+1}(x)$ is unique

now, to show $H_{2n+1}(x)$ is of least such degree as if not, then

$\exists\, G(x)$ s.t degree of $G(x) < 2n+1$
$$\Rightarrow \deg(G(x)) \leq 2n$$
and $G(x_i) = f(x_i) \;\forall i \in \{0, \ldots n\}$
$G'(x_i) = f'(x_i) \;\forall i \in \{0, \ldots n\}$

if $G(x) = \sum_{i=0}^{2n} \alpha_i x^i \quad \{\alpha_0, \alpha_1 \ldots \alpha_{2n}\}$

$$G'(x) = \sum_{i=1}^{2n} i\alpha_i x^{i-1}$$

we have $2n+1$ variables in form of $\{\alpha_0, \ldots \alpha_{2n}\}$ but
$2n+2$ equations, as $G(x_i) = f(x_i)$
$G'(x_i) = f'(x_i)$

so, this is a contradiction and minimum degree is $2n+1$

4. $$S(x) = \begin{cases} 1 + 2x - x^3 & ; 0 \leq x \leq 1 \\ 2 + b(x-1) + c(x-1)^2 + d(x-1)^3 & ; 1 \leq x \leq 2 \end{cases} \quad \text{on } [0,2]$$

as natural cubic spline $S \in C^2[0,2]$ & $S''(0) = 0$
$S''(2) = 0$

also $\left.\begin{array}{l} S_0(1)=S_1(1) \\ S'_0(1)=S'_1(1) \\ S''_0(1)=S''_1(1) \end{array}\right\}$ By defn of spline

now $S_0(1) = \cancel{1}+2-\cancel{1} = 2$

$S_1(1)=2$

$S'_0(x) = 2-3x^2 \Rightarrow S'_0(1) = 2-3 = -1$

$S''_0(x) = -6x \Rightarrow S''_0(1) = -6$ also $S''_0(0)=0$ (valid natral spline)

Similarly $S'_1(x) = b+2c(x-1)+3d(x-1)^2$

$S''_1(x) = 2c+6d(x-1)$

& $S''_1(2) = 2c+6d = 0$ —— ① (by defn of natral spline)

$S'_0(1) = S'_1(1) \Rightarrow -1 = b$ —— ②

$S''_0(1) = S''_1(1) \Rightarrow -6 = 2c$ —— ③

now from ①, ② ③ we get:

$c=-3$, $b=-1$ and

$2(-3)+6d=0$
$\Rightarrow d=1$

so, $b=-1$, $c=-3$, $d=1$

5. $S(x) = \begin{cases} S_0(x) = 3(x-1)+2(x-1)^2-(x-1)^3 ; & 1 \leq x \leq 2 \\ S_1(x) = a+b(x-2)+c(x-2)^2+d(x-2)^3 ; & 2 \leq x \leq 3 \end{cases}$

given its a clamped spline ie $f'(1)=f'(3)$

now by defn of spline:

$S'_0(1) = f'(1)$ $f'(3) = S'_1(3)$
$\Rightarrow S'_0(1) = S'_1(3)$ —— ①

now, $S_0(2) = S_1(2)$ —— ②
$S'_0(2) = S'_1(2)$ —— ③
$S''_0(2) = S''_1(2)$ —— ④

now, $S_0(x) = 3(x-1)+2(x-1)^2-(x-1)^3$
$S'_0(x) = 3+6(x-1)-3(x-1)^2$
$S''_0(x) = 6-6(x-1)$

$S_1(x) = a+b(x-2)+c(x-2)^2+d(x-2)^3$
$S'_1(x) = b+2c(x-2)+3d(x-2)^2$
$S''_1(x) = 2c+6d(x-2)$

now, $S_0(2) = 3+2-1 = 4$ $S_1(2)=a$
$S'_0(2) = 6$ $S'_1(2) = b$
$S''_0(2) = 0$ $S''_1(2) = 2c$
$\Rightarrow a=4$, $b=6$, $c=0$ (from ②, ③, ④)

now, $S_0'(1) = S_1'(3)$ (from ①)

$\Rightarrow 3 = b + 2c + 3d$
$\Rightarrow 3 = (6) + 2(0) + 3d$
$\Rightarrow -3 = 3d$
$\Rightarrow d = -1$

so we get $a = 4$
$b = 6$
$c = 0$
$d = -1$

6. $\int_{-1}^{1} f(x)\,dx = f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)$

if $f(x) = 1$ then

$\int_{-1}^{1} 1\,dx = 2$ and $2 = 1+1$ so $f(x) = 1$ true

if $f(x) = x$:

$\int_{-1}^{1} x\,dx = 0$ and $f\left(\frac{\sqrt{3}}{3}\right) + f\left(-\frac{\sqrt{3}}{3}\right) = 0$, so true

if $f(x) = x^2$:

$\int_{-1}^{1} x^2\,dx = \left|\frac{x^3}{3}\right|_{-1}^{1} = \frac{2}{3}$ $\quad f\left(\frac{\sqrt{3}}{3}\right) + f\left(-\frac{\sqrt{3}}{3}\right) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$ so true

if $f(x) = x^3$:

$\int_{-1}^{1} x^3\,dx = 0$ $\quad f\left(\frac{\sqrt{3}}{3}\right) + f\left(-\frac{\sqrt{3}}{3}\right) = 0$ so true

if $f(x) = x^4$:

$\int_{-1}^{1} x^4\,dx = \left|\frac{x^5}{5}\right|_{-1}^{1} = \frac{2}{5}$ & $f\left(\frac{\sqrt{3}}{3}\right) + f\left(-\frac{\sqrt{3}}{3}\right) = \frac{1}{6} + \frac{1}{6}$

as $\frac{2}{5} \neq \frac{2}{6}$ not true for x^4, so

for any polynomial of form $a + bx + cx^2 + dx^3 = f(x)$ we get

$$\int_{-1}^{1} f(x)\,dx = f\left(\frac{\sqrt{3}}{3}\right) + f\left(-\frac{\sqrt{3}}{3}\right)$$

so degree of precision is 3.

7. $\int_{-1}^{1} f(x)dx = c_0 f(-1) + c_1 f(0) + c_2 f(1)$ for degree ≤ 2

then as $\{1, x, x^2\}$ are lin independent

putting $1, x, x^2$ we will get c_0, c_1, c_2 value: ($\because$ for any polynomial of degree ≤ 2, exact)

$f(x) = 1$

$\int_{-1}^{1} 1\,dx = c_0 + c_1 + c_2 = 2$ —— ①

$f(x) = x$

$\int_{-1}^{1} x\,dx = -c_0 + c_2 = 0$ —— ②

$$\int_{-1}^{1} x^2 dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3} = c_0 + c_2 \quad \text{③}$$

then from ② & ③ $c_2 = c_0$
& $c_0 + c_2 = 2/3$
$\Rightarrow c_0 = c_2 = 1/3$

& from ①: $1/3 + c_1 + 1/3 = 2$
$\Rightarrow c_1 = 2 - 2/3 = 4/3$
$c_1 = 4/3$

8. as $\{1, (x-1), (x-1)^2\}$ is linearly independent, putting them we will get c_0, c_1, c_2 as exact ≤ 2 degrees:

$f(x) = 1 \qquad \int_0^2 f(x)\,dx = 2 = c_0 + c_1 + c_2 \quad \text{①}$

$f(x) = (x-1) \Rightarrow \int_0^2 (x-1)\,dx = \left|\frac{(x-1)^2}{2}\right|_0^2 = 0$

$0 = -c_0 + 0 + c_2 \quad \text{②}$

$f(x) = (x-1)^2 \Rightarrow \int_0^2 (x-1)^2 dx = \left|\frac{(x-1)^3}{3}\right|_0^2 = \frac{1}{3} + \frac{1}{3} = \frac{2}{3} = c_0 + c_2 \quad \text{③}$

from ②, ③: $c_2 = c_0$ & $c_2 + c_0 = 2/3$
$\Rightarrow c_0 = c_2 = 1/3$

from ①: $c_0 + c_1 + c_2 = 2 \Rightarrow 1/3 + 1/3 + c_1 = 2$
$\Rightarrow c_1 = 4/3$

9. let $P(x) = a + b(x-x_1) + c(x-x_1)^2 + d(x-x_1)^3$

$P'(x) = b + 2c(x-x_1) + 3d(x-x_1)^2$
$\Rightarrow P'(x_1) = b = f'(x_1) \quad \text{①}$

$P''(x) = 2c + 6d(x-x_1)$
$P''(x_1) = 2c = f''(x_1) \quad \text{②}$

so from ①, ②: $P(x) = a + f'(x_1)(x-x_1) + \frac{f''(x_1)}{2}(x-x_1)^2 + d(x-x_1)^3$

as $x_0 \neq x_2$

$P(x_0) = f(x_0)$ & $P(x_2) = f(x_2)$ we get:

$P(x_0) = a + f'(x_1)(x_0 - x_1) + \frac{f''(x_1)}{2}(x_0-x_1)^2 + d(x_0-x_1)^3 = f(x_0)$

$$p(x_2) = a + f'(x_1)(x_2-x_1) + \frac{f''(x_1)}{2}(x_2-x_1)^2 + d(x_2-x_1)^3 = f(x_2)$$

so we get
$$\underbrace{\begin{bmatrix} 1 & (x_0-x_1)^3 \\ 1 & (x_2-x_1)^3 \end{bmatrix}}_{\det = (x_2-x_1)^3 - (x_0-x_1)^3 \neq 0} \begin{bmatrix} a \\ d \end{bmatrix} = \begin{bmatrix} f(x_0) - f'(x_1)(x_0-x_1) - \frac{f''(x_1)}{2}(x_0-x_1)^2 \\ f(x_2) - f'(x_1)(x_2-x_1) - \frac{f''(x_1)}{2}(x_2-x_1)^2 \end{bmatrix}$$

as $x_2 \neq x_0$ given

so, $\exists$ a unique solution to $p(x)$ as (a,d) uniquely determined

let $g(x) = (x-x_1)^3$

$h(x) = f(x) - f'(x_1)(x-x_1) - \frac{f''(x_1)}{2}(x-x_1)^2$

then:

$$\begin{bmatrix} 1 & g(x_0) \\ 1 & g(x_2) \end{bmatrix} \begin{bmatrix} a \\ d \end{bmatrix} = \begin{bmatrix} h(x_0) \\ h(x_2) \end{bmatrix}$$

$$\begin{bmatrix} a \\ d \end{bmatrix} = \begin{bmatrix} 1 & g(x_0) \\ 1 & g(x_2) \end{bmatrix}^{-1} \begin{bmatrix} h(x_0) \\ h(x_2) \end{bmatrix}$$

$$= \frac{1}{g(x_2)-g(x_0)} \begin{bmatrix} g(x_2) & -g(x_0) \\ -1 & 1 \end{bmatrix} \begin{bmatrix} h(x_0) \\ h(x_2) \end{bmatrix}$$

$$= \frac{1}{g(x_2)-g(x_0)} \begin{bmatrix} g(x_2)h(x_0) - g(x_0)h(x_2) \\ h(x_2) - h(x_0) \end{bmatrix}$$

so $a = \frac{1}{g(x_2)-g(x_0)} \times (g(x_2)h(x_0) - g(x_0)h(x_2))$ — 3

$d = \frac{1}{g(x_2)-g(x_0)} (h(x_2) - h(x_0))$ — 4

so we have $p(x) = a + b(x-x_1) + c(x-x_1)^2 + d(x-x_1)^3$
with a, b, c, d value from 1, 2, 3, 4

(b) let $\varphi(t) = (f(t) - p(t)) - (f(x) - p(x)) \frac{(t^4-1)}{(x^4-1)}$

then from construction:

$f(-1) = p(-1)$
$f(1) = p(1)$

$f'(0) = p'(0)$
$f''(0) = p''(0)$

$\varphi(-1) = (0) - (f(x) - p(x))\, 0 = 0$

$\varphi(1) = (0) - (0) = 0$

also for $x \neq 0$ & $x \in (-1,1)$ we get

$\varphi(x) = \underbrace{f(x) \cancel{- p(x)} - (f(x) \cancel{- p(x)})}_{=0}$

so there are minimum 3 roots to $\varphi(t)$, -1, x & 1

now, if $x=-1$ or 1 it's trivial to see

$$f(1)-P(1)=0=\frac{f^{4}(\xi_1)(0)}{4!}$$ does not matter

& $$f(-1)-P(-1)=0=\frac{f^{4}(\xi_{-1})(0)}{4!}$$ does not matter

now for $x\in(-1,1)$ we have shown $\varphi(t)$ has 3 solutions $-1, x, 1$ then from rolles theorm $\exists$ 2 roots to $\varphi'(t)$ between $(-1,x)$ & $(x,1)$

case I: $x \geq 0$:
then $\exists$ root b/w $(x,1)$ & as

$$\varphi'(t)=\left(f'(t)-P'(t)\right)-(f(x)-P(x))\frac{(4t^3)}{(x^4-1)}$$

$$\Rightarrow \varphi'(0)=(f'(0)-P'(0))-0$$
$$=0 \quad (\because P'(0)=f'(0))$$

so 0 is also a root to $\varphi'(t)$

then $\exists$ 2 roots to $\varphi'(t)$, 0 & $x_1\in(x,1)$, then from rolles theorm $\exists$ a root $x_2\in(0,x_1)$ s.t $\varphi''(t)=0$

& $$\varphi''(t)=\left(f''(t)-P''(t)\right)-(f(x)-P(x))\frac{(4.3t^2)}{(x^4-1)}$$

$$\Rightarrow \varphi''(0)=f''(0)-P''(0)$$
$$=0 \quad (\because f''(0)=P''(0))$$

and so, $\exists$ 2 roots, 0 and $x_2\in(0,x_1)$ then from rolles theorm

$\exists\ x_3\in(0,x_2)$ s.t

$$\varphi''(x_3)=0 \text{ ——— } (1)$$

case II: $x<0$: then similar to case I, $\exists\ \tilde{x}_3\in(\tilde{x}_2,0)$ s.t $\varphi''(\tilde{x}_3)=0$, replacing x_i to $\tilde{x}_i$ and redoing same procedure ——— (2)

so, $\exists\ x_3$ & $\tilde{x}_3$ s.t $\varphi''(\tilde{x}_3)=0$, $\varphi''(x_3)=0$
& $x_3\neq\tilde{x}_3$ so by rolles theorm, as $\varphi(t)$ is 4 times diff ($\because f$ is 4 times diff & rest is polynomial)

$\Rightarrow \exists\ \xi_x\in(\tilde{x}_3, x_3)$ s.t

$$\varphi^{(4)}(\xi_x)=0$$

here ξ_x depends on x as $\tilde{x}_3$ & x_3 also depends on x
but as $\tilde{x}_3\in[-1,1]$
& $x_3\in[-1,1]$ we get:

$$\xi_x\in[-1,1] \text{ s.t } \varphi^{(4)}(\xi_x)=0$$

$$\varphi^{(4)}(t)=\left[f^{(4)}(t)-P^{(4)}(t)\right]-\left[f(x)-P(x)\right]\frac{4!}{(x^4-1)}$$

and as $P^{(4)}(t)=0$ ($\because$ 3 degree polynomial)

$\Rightarrow \quad \varphi^{(4)}(\xi_x) = f^{(4)}(\xi_x) - [f(x)-P(x)]\frac{4!}{(x^4-1)} = 0$

$\Rightarrow \quad \frac{f^{(4)}(\xi_x)}{4!}(x^4-1) = f(x)-P(x)$

so for $-1\le x\le 1$, $\exists\, \xi_x \in [-1,1]$ s.t

$$f(x)-P(x) = \frac{f^{(4)}(\xi_x)}{4!}(x^4-1)$$

10. $0\le t_0<t_1<\dots<t_{2n}<2\pi$

$$l_j(t) = \prod_{\substack{k=0\\ j\ne k}}^{2n} \frac{\sin\left(\frac{1}{2}(t-t_k)\right)}{\sin\left(\frac{1}{2}(t_j-t_k)\right)} \quad 0\le j\le 2n$$

<u>To prove</u> : $l_j(t_i) = \delta_{ij} = \begin{cases} 1 \; ; \; i=j \\ 0 \; ; \; i\ne j \end{cases} \quad 0\le i,j\le 2n$

<u>proof</u> : for $i=j$: $l_j(t_j) = \prod_{\substack{k=0\\ j\ne k}}^{2n} \frac{\cancel{\sin\left(\frac{1}{2}(t_j-t_k)\right)}}{\cancel{\sin\left(\frac{1}{2}(t_j-t_k)\right)}} = 1$

for $i\ne j$: $l_j(t_i) = \prod_{\substack{k=0\\ j\ne k\\ i\ne k}}^{2n} \frac{\sin\left(\frac{1}{2}(t_i-t_k)\right)}{\sin\left(\frac{1}{2}(t_j-t_k)\right)} \times \underbrace{\sin\left(\frac{1}{2}(t_i-t_i)\right)}_{0} = 0$

so, $l_j(t_i) = \delta_{ij}$ for $0\le i,j\le 2n$

<u>To prove</u>: $l_j(t)$ is a trignometric polynomial of degree $\le n$

<u>proof</u>: now as
$$\frac{\sin\left(\frac{1}{2}(t-t_k)\right)}{\sin\left(\frac{1}{2}(t_j-t_k)\right)} = \frac{\frac{e^{i(t-t_k)}-1}{e^{i(t/2-t_k/2)}}}{\frac{e^{i(t_j-t_k)}-1}{e^{i(t_j/2-t_k/2)}}}$$

$$= \frac{\frac{e^{it}-e^{it_k}}{e^{it/2}}}{\frac{e^{it_j}-e^{it_k}}{e^{it_j/2}}}$$

$$= \left(\frac{e^{it}-e^{it_k}}{e^{it_j}-e^{it_k}}\right)\times\frac{e^{it_j/2}}{e^{it/2}}$$

then $\prod_{\substack{k=0\\ k\ne j}}^{2n} \frac{(e^{it}-e^{it_k})}{(e^{it_j}-e^{it_k})} \times \frac{e^{it_j/2}}{e^{it/2}} = l_j(t)$

let $\alpha = \prod_{\substack{k=0 \\ k\neq j}}^{2n} \frac{e^{itj/2}}{(e^{itj}-e^{itk})}$

then $l_j(t) = \alpha \prod_{\substack{k=0 \\ k\neq j}}^{2n} \frac{(e^{it}-e^{itk})}{e^{nit}}$

now let $e^{it} = x(t)$ then:

$$l_j(t) = \frac{\alpha}{(x(t))^n} \prod_{\substack{k=0 \\ k\neq j}}^{2n} \underbrace{(x(t)-x(t_k))}_{2n \text{ times}}$$

$$= \frac{\alpha_0 + \alpha_1 x(t) + \alpha_2 (x(t))^2 + \dots \alpha_{2n}(x(t))^{2n}}{(x(t))^n}$$ from expanding numerator

where $\alpha_i \in \mathbb{C}$

then let $\alpha_0 = \beta_{-n}$

$\alpha_1 = \beta_{-n+1}$

$\vdots$

$\alpha_n = \beta_0$

$\alpha_{n+1} = \beta_1$

$\vdots$

$\alpha_{2n} = \beta_n$

then $l_j(t) = \sum_{k=-n}^{n} \beta_k (x(t))^k$

$$= \beta_0 + \sum_{k=1}^{n} (\beta_k e^{ikt} + \beta_{-k} e^{-ikt})$$

$$= \beta_0 + \sum_{k=1}^{n} (\beta_k \cos kt + i\beta_k \sin kt + \beta_{-k} \cos kt - i\beta_{-k} \sin kt)$$

let $\beta_k + \beta_{-k} = c_k$

$i\beta_k - i\beta_{n-k} = d_k$ for $k \in \{1, 2 \dots n\}$

then $l_j(t) = \beta_0 + \sum_{k=1}^{n} c_k \cos kt + \sum_{k=1}^{n} d_k \sin kt$

as $l_j(t) \in \mathbb{R}$, all coeffs $\in \mathbb{R}$ and not $\mathbb{C}$ and so we can write

$l_j(t) = \sum_{k=0}^{n} [a_k \cos(kt) + b_k \sin(kt)]$ for some constants a_k, b_k ← not as above

so $l_j(t)$ is a trignometric polynomial of degree $\leq n$

11.(a) To prove: $\sum_{j=1}^{m-1} \sin\left(\frac{2\pi jk}{m}\right) = 0$, $m \geq 2$ all integers k

and $$\sum_{j=0}^{m-1} \cos\left(\frac{2\pi jk}{m}\right) = \begin{cases} m; & k \text{ is a multiple of } m \\ 0; & k \text{ is not a multiple of } m \end{cases}$$

proof: $e^{\frac{i2\pi jk}{m}} = \cos\left(\frac{2\pi jk}{m}\right) + i\sin\left(\frac{2\pi jk}{m}\right)$

$$\sum_{j=0}^{m-1} e^{\frac{i2\pi jk}{m}} = \underbrace{\sum_{j=0}^{m-1}\left(e^{\frac{i2\pi k}{m}}\right)^j}_{\text{this is an gp}}$$

Case I: $k \bmod m \neq 0$

$$a + ar + \cdots ar^{n-1} = \frac{a(r^n - 1)}{(r-1)}$$

then $\sum_{j=0}^{m-1} e^{\frac{i2\pi jk}{m}}$ is a gp sum

and as $r = e^{i(2\pi k/m)} \neq 1$ we can use gp sum formula

so $a = 1$, $r = e^{i\left(\frac{2\pi k}{m}\right)}$

$$a + ar + \cdots\cdots ar^{m-1} = \frac{a(r^m - 1)}{r-1} = \frac{\left(e^{\frac{i2\pi k}{m}\cdot m} - 1\right)}{\left(e^{i2\pi k/m} - 1\right)}$$

so, $\sum_{j=0}^{m-1} e^{i\left(\frac{2\pi jk}{m}\right)} = \frac{\left(e^{2\pi ki} - 1\right)}{\left(e^{\frac{2\pi k}{m}i} - 1\right)} = 0$ as $e^{2\pi ki} = \cos(2\pi k) + i\sin(2\pi k) = 1$

so $\sum_{j=0}^{m-1}\left(\cos\left(\frac{2\pi jk}{m}\right) + i\sin\left(\frac{2\pi jk}{m}\right)\right) = 0$

when $k \bmod m \neq 0$

so, $\sum_{j=0}^{m-1} \cos\left(\frac{2\pi jk}{m}\right) = 0$ & $\sin\left(\frac{2\pi(0)k}{m}\right) + \sum_{j=1}^{m-1} \sin\left(\frac{2\pi jk}{m}\right) = 0$

$$\Rightarrow \sum_{j=1}^{m-1} \sin\left(\frac{2\pi jk}{m}\right) = 0$$

so in this case we get our desired output

Case II: $k \bmod m = 0$

then $r = 1$ and we cannot use gp sum method as $r = 1$

but

$$a + ar + \cdots ar^{m-1} = a + a + \cdots a = ma$$

as $a = 1$

$$\Rightarrow a + ar + \cdots ar^{m-1} = m$$

$$\Rightarrow \sum_{j=0}^{m-1} e^{\frac{i2\pi jk}{m}} = m$$

$$\Rightarrow \sum_{j=0}^{m-1} \cos\left(\frac{2\pi jk}{m}\right) + i\sum_{j=1}^{m-1} \sin\left(\frac{2\pi jk}{m}\right) = m + i0$$

as $j=0$ $\sin\left(\frac{2\pi(0)k}{m}\right) = 0$

$\Rightarrow \sum_{j=0}^{m-1} \cos\left(\frac{2\pi jk}{m}\right) = m$ for $k \bmod m = 0$

& $\sum_{j=1}^{m-1} \sin\left(\frac{2\pi jk}{m}\right) = 0$ for this case too

(b) $\sum_{j=0}^{m-1} e^{i\left(\frac{2\pi jk}{m}\right)} = 0$ if $k \bmod m \neq 0$

$\sum_{j=0}^{m-1} e^{i\left(\frac{2\pi jk}{m}\right)} = m$ if $k \bmod m = 0$ } shown in (a)

now, let $\sum_{j=0}^{m-1} e^{i\left(\frac{2\pi jk}{m}\right)} = m\,\mathbb{I}_{k \bmod m = 0}$

i.e $\mathbb{I}_{k \bmod m=0} = \begin{cases} 1 \,; & k \bmod m = 0 \\ 0 \,; & k \bmod m \neq 0 \end{cases}$

now, $\left(\sum_{j=0}^{m-1} e^{i\left(\frac{2\pi jk}{m}\right)}\right)\left(\sum_{j=0}^{m-1} e^{i\left(\frac{2\pi jl}{m}\right)}\right)$

$$= m^2\,\mathbb{I}_{k \bmod m=0}\,\mathbb{I}_{l \bmod m=0}$$

$$= \sum_{j=1}^{m-1}\left(\cos\left(\frac{2\pi jk}{m}\right) + i\sin\left(\frac{2\pi jk}{m}\right)\right)\left(\cos\left(\frac{2\pi jl}{m}\right) + i\sin\left(\frac{2\pi jl}{m}\right)\right)$$

$$= \sum_{j=0}^{m-1}\left(\cos\left(\frac{2\pi jk}{m}\right)\cos\left(\frac{2\pi jl}{m}\right) - \sin\left(\frac{2\pi jk}{m}\right)\sin\left(\frac{2\pi jl}{m}\right)\right)$$

$$+ \underbrace{2i\sum_{j=0}^{m-1}\sin\left(\frac{2\pi jk}{m}\right)\cos\left(\frac{2\pi jl}{m}\right)}_{0 \text{ as no complex term}}$$

so, $\underbrace{\sum_{j=0}^{m-1}\cos\left(\frac{2\pi jk}{m}\right)\cos\left(\frac{2\pi jl}{m}\right) - \sum_{j=0}^{m-1}\sin\left(\frac{2\pi jk}{m}\right)\sin\left(\frac{2\pi jl}{m}\right) = m^2\,\mathbb{I}_{k \bmod m=0} \times \mathbb{I}_{l \bmod m=0}}_{①}$

& $\underbrace{\sum_{j=0}^{m-1}\sin\left(\frac{2\pi jk}{m}\right)\cos\left(\frac{2\pi jl}{m}\right)}_{\text{replacing } k \,\&\, l \text{ we get}} = 0$

$\sum_{j=0}^{m-1}\cos\left(\frac{2\pi jk}{m}\right)\sin\left(\frac{2\pi jl}{m}\right) = 0$ ——— ②

also in ① if we replace l with $-l$ we get:

$\underbrace{\sum_{j=0}^{m-1}\cos\left(\frac{2\pi jk}{m}\right)\cos\left(\frac{2\pi jl}{m}\right)}_{\text{here sign same} = x} + \underbrace{\sum_{j=0}^{m-1}\sin\left(\frac{2\pi jk}{m}\right)\sin\left(\frac{2\pi jl}{m}\right)}_{\text{here the sign changes} = y} = m^2\,\mathbb{I}_{k \bmod m=0}\,\mathbb{I}_{-l \bmod m=0}$ ——— ③

so, $x - y = \begin{cases} m^2 \,; & \text{if } k \bmod m = 0 \,\&\, l \bmod m = 0 \\ 0 \,; & \text{otherwise} \end{cases}$ ——— ④

$x + y = \begin{cases} m^2 \,; & \text{if } k \bmod m = 0 \,\&\, -l \bmod m = 0 \\ 0 \,; & \text{otherwise} \end{cases}$ ——— ⑤

} from ①, ③

adding ④, ⑤ we get:

also $-l \bmod m = 0 \Leftrightarrow l \bmod m = 0$ ($\because$ multiply by -1 on both sides)

$$x = \begin{cases} m^2 ; & k \bmod m = 0 \text{ and } l \bmod m = 0 \\ 0 ; & \text{otherwise} \end{cases}$$

on ⑤ − ④ we get :

$y = 0$ (for all cases) so, finally we get :

$$x = \sum_{j=0}^{m-1} \cos\left(\frac{2\pi jk}{m}\right) \cos\left(\frac{2\pi jl}{m}\right) = \begin{cases} m^2 ; & k \bmod m = 0 \text{ and } l \bmod m = 0 \\ 0 ; & \text{otherwise} \end{cases}$$

$$y = \sum_{j=0}^{m-1} \sin\left(\frac{2\pi jk}{m}\right) \sin\left(\frac{2\pi jl}{m}\right) = \sum_{j=1}^{m-1} \sin\left(\frac{2\pi jk}{m}\right) \sin\left(\frac{2\pi jl}{m}\right) = 0 \quad \left(\begin{array}{l} \because \text{ for } j=0 \\ \sin(0) = 0 \end{array}\right)$$

$$\sum_{j=0}^{m-1} \cos\left(\frac{2\pi jk}{m}\right) \sin\left(\frac{2\pi jl}{m}\right) = 0 \quad (\because \text{ from } ②)$$

12. $f \in C^2[a,b]$ $h = \frac{b-a}{n}$ $x_j = a + jh$ $0 \le j \le n$, $j \in \mathbb{Z}_{\ge 0}$

<u>To prove</u>: $\exists\, \mu \in (a,b)$ s.t

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] - \frac{b-a}{12} h^2 f''(\mu)$$

<u>proof</u>: for some c, d s.t $a \le c \le d \le b$

let $h = d - c$

as $[c,d] \subseteq [a,b]$ so $f \in C^2[c,d]$, now

let $f(x) = P_1(x) + R(x)$

$R(x)$ ← Reminder term

$P_1(x)$ ← lagrange Polynomial of degree 1

then let $P_1(x) = f(c)\mathcal{L}_c(x) + f(d)\mathcal{L}_d(x)$

$$\mathcal{L}_c(x) = \frac{(x-d)}{(c-d)} \qquad \mathcal{L}_d(x) = \frac{(x-c)}{(x-d)}$$

& then $R(x) = \frac{f^2(\eta_{cd})}{2!}(x-c)(x-d)$ where $\eta_{cd} \in (c,d)$

(∵ $P_1(x)$, $R(x)$ done in class, from derivation)

$$\text{so, } \int_c^d f(x)\,dx = \int_c^d P_1(x)\,dx + \int_c^d R(x)\,dx$$

$$= \int_c^d f(c)\left(\frac{x-d}{c-d}\right) + f(d)\left(\frac{x-c}{d-c}\right) dx$$

$$+ \int_c^d \frac{f''(\eta_{cd})}{2!}(x-c)(x-d)\,dx$$

$$= \frac{f(c)}{c-d} \left.\frac{(x-d)^2}{2}\right|_c^d + \frac{f(d)}{d-c} \left.\frac{(x-c)^2}{2}\right|_c^d + \int_c^d \frac{f''(\eta_{cd})}{2!}(x-c)(x-d)\,dx$$

$$= \frac{f(c)}{\cancel{c-d}}\left[-\frac{(c-d)^2}{2}\right] + \frac{f(d)}{\cancel{d-c}}\left[\frac{(d-c)^2}{2}\right] + \int_c^d \frac{f''(\eta_{cd})}{2!}(x-c)(x-d)\,dx$$

$$= \frac{d-c}{2}\left[f(c)+f(d)\right] + \frac{f''(\eta_{cd})}{2}\int_c^d x^2 - cx - dx + cd \, dx$$

$$= \frac{h}{2}\left[f(c)+f(d)\right] + \frac{f''(\eta_{cd})}{2}\left[\frac{x^3}{3} - c\frac{x^2}{2} - d\frac{x^2}{2} + cdx\right]_c^d$$

$$= \frac{h}{2}\left[f(c)+f(d)\right] + \frac{f''(\eta_{cd})}{2}\underbrace{\left[\frac{d^3}{3} - \frac{c^3}{3} - \frac{cd^2}{2} + \frac{c^3}{2} - \frac{d^3}{2} + \frac{dc^2}{2} + cd^2 - c^2 d\right]}_{\frac{1}{6}\left(c^3 - d^3 + 3cd^2 - 3c^2 d\right)}$$

$$\int_c^d f(x)\,dx = \frac{h}{2}\left[f(c)+f(d)\right] + \frac{f''(\eta_{cd})(h)^3}{12} \qquad \left(\because (c-d)^3 = c^3 - d^3 + 3cd^2 - 3c^2 d\right)$$

now, for $x_j = a + \left(\frac{b-a}{n}\right)j$ $\quad h = \frac{b-a}{n}$

replacing $c = x_j$
$d = x_{j+1}$ for $j \in \{0, \dots n-1\}$

we get $\displaystyle\int_{x_j}^{x_{j+1}} f(x)\,dx = \frac{h}{2}\left[f(x_j) + f(x_{j+1})\right] + \frac{f''(\eta_{x_j x_{j+1}})}{12}\underbrace{(x_{j+1} - x_j)^3}_{h}$

now, $\displaystyle\sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}} f(x)\,dx = \int_{x_0}^{x_n} f(x)\,dx = \int_a^b f(x)\,dx$

$$= \sum_{j=0}^{n-1}\frac{h}{2}\left[f(x_j) + f(x_{j+1})\right] + \sum_{j=0}^{n-1}\frac{h^3}{12}f''(\eta_{x_j x_{j+1}})$$

$$\int_a^b f(x)\,dx = \underbrace{\frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right]}_{①} + \underbrace{\sum_{j=0}^{n-1}\frac{h^3}{12}f''(\eta_{x_j x_{j+1}})}_{②}$$

now, lets show if $g \in C^0[a,b]$ then for

$a \leq x_1 < x_2 < \dots < x_n \leq b$, $\exists x \in (a,b)$ s.t
$g(x_1) + g(x_2) + \dots + g(x_n) = n\,g(x)$

for this lets use IVT:

$$h(x) = g(x) - \frac{1}{n}\left(g(x_1) + g(x_2) + \dots + g(x_n)\right)$$

if $\forall i \in \{1, \dots n\}$ $h(x_i) > 0$ or $h(x_i) < 0$

$$\sum_{i=1}^{n} h(x_i) = \sum_{i=1}^{n} g(x_i) - \sum_{i=1}^{n} g(x_i) = 0$$

so $h(x_i) > 0$ $\forall i$ is not possible
similarly $h(x_i) < 0$ $\forall i$ is not possible

so, $\exists i \neq j$ s.t $h(x_i) \leq 0$, $h(x_j) \geq 0$, so by IVT as $h \in C^0[a,b]$
$\exists x \in (a,b)$ s.t $h(x) = 0$

$$\Rightarrow g(x) = \sum_{i=1}^{n}\frac{1}{n}g(x_i)$$

now ② becomes: $\sum_{i=1}^{n} \frac{h^3}{12} f''(\eta_i \eta_{i+1}) = \frac{nh^3}{12} f''(\mu)$ for some $\mu \in (a,b)$ from above proof

as $f \in C^2[a,b]$
$f'' \in C^0[a,b]$

so, $\exists \mu \in (a,b)$ s.t

$$\int_a^b f(x)\,dx = \frac{h}{2}\left(f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\right) + \frac{(b-a)}{12} h^2 f''(\mu)$$

13. $f \in C[0,1]$, $\int_0^1 t^k f(t)\,dt = 0 \ \forall k \in \mathbb{N}$

To prove: $f = 0$

proof: firstly let me state the weierstrass approximation theorem done in class, it states if f is cont fuction in $[a,b]$ then $\forall \varepsilon > 0$, $\exists$ polynomial $P(x)$ s.t

$$|f(x) - P(x)| < \varepsilon \ \forall x \in [a,b]$$

now here for the $f(x)$ given, using wiestras theorm $\forall \varepsilon > 0$, $\exists P(x)$ as f is cont on $[0,1]$ s.t

$$|f(x) - P(x)| < \varepsilon \ \forall x \in [0,1]$$

and as $p(x) = \sum_{i=0}^{n} \alpha_i x^p$ some n we get

$$\left|\int_0^1 f(x) - p(x)\,dx\right| \leq \int_0^1 |f(x) - P(x)|\,dx < \int_0^1 \varepsilon\,dx = \varepsilon$$

$$\Rightarrow \left|\int_0^1 f(x) - P(x)\,dx\right| < \varepsilon \quad \text{——} ①$$

now, $\int_0^1 (f(t))^2\,dt = \int_0^1 (f(t))^2 + f(t)P(t) - f(t)P(t)\,dt$

$$= \int_0^1 f(t)p(t)\,dt + \int_0^1 f(t)(f(t) - p(t))\,dt$$

now let $g(t) = f(t).t$ i.e

$$\int_0^1 t^k f(t)\,dt = \int_0^1 t^{k-1} g(t)\,dt = 0 \ \forall k-1 \geq 0$$

so, $\forall k \in \mathbb{Z}_{\geq 0}$, $\int_0^1 t^k g(t) = 0$

and as $f(t).t = g(t)$, $g(t) \in C[0,1]$ and we can apply weierstras theorm on g so $\forall \varepsilon > 0$, $\exists P(t)$ polynomial s.t

$$|g(t) - P(t)| < \varepsilon \quad (\because ① \text{ calculation})$$

now, $\int_0^1 g(t)P(t)\,dt = 0$ as $\int_0^1 g(t)t^k\,dt = 0$ $\forall_p k \geq 0$
and $p(t) = \sum \alpha_i t^p$

so, $\int_0^1 (g(t))^2\,dt = \int_0^1 (g(t))^2\,dt - \int_0^1 g(t)P(t)\,dt \quad (\because \int_0^1 g(t)P(t)\,dt = 0)$

$$= \int_0^1 g(t)(g(t) - P(t))\,dt$$

so, $\left|\int_0^1 (g(t))^2 dt\right| = \left|\int_0^1 g(t)(g(t) - P(t))\,dt\right|$

$\leq \int_0^1 |g(t)(g(t)-P(t))|\,dt$ ($\because$ property of integrals)

$\leq \int_0^1 |g(t)||g(t)-P(t)|\,dt$

$< \varepsilon \int_0^1 |g(t)|\,dt$ ($\because |g(t)-P(t)| < \varepsilon$)

so, $\left|\int_0^1 (g(t))^2 dt\right| < \varepsilon \underbrace{\int_0^1 |g(t)|\,dt}$

this is some constant as $g \in C[0,1]$

so as $\varepsilon \longrightarrow 0$, we get $\left|\int_0^1 (g(t))^2 dt\right| \leq 0$

$\Rightarrow \int_0^1 (g(t))^2 dt = 0$

then $g(t) = 0 \ \forall\ t \in [0,1]$, as if not then as g is continious and if $\exists \alpha$ s.t $g(\alpha) \neq 0$ $\forall\ t \in (-\delta+\alpha, \alpha+\delta)$ ($\because g$ is cont so δ s.t $g(t) \neq 0$) s.t $(g(t))^2 > 0$ so,

$\int_0^1 (g(t))^2 dt > 0$ but this is not the case

so, $g(t) = 0 \ \forall\ t \in [0,1]$

now, $g(t) = t f(t)$, $\forall\ t \in [0,1]$

for $t \neq 0$ we get $0 = t f(t)$
$\Rightarrow f(t) = 0$

so, $\forall\ t \in (0,1] \Rightarrow f(t) = 0$ and as f is continious in $[0,1]$

$\lim_{\delta \to 0^+} f(\delta) = f(0)$

$0 = f(0)$ as $f(\delta) = 0$

so, $\forall\ t \in [0,1]$

$f(t) = 0$

$\Rightarrow f = 0$

Tutorial-4:

1. $y'(t) = y\cos t \quad 0 \le t \le 1$
$y(0) = 1$

let $y'(t) = f(t, y(t))$
where $f(t, y(t)) = y(t)\cos t$

$f(t,y) = y\cos t$
$\frac{\partial f}{\partial y} = \underbrace{\cos t}_{\text{Bouded}} \le 1$ or $|f(t, y_1) - f(t, y_2)| = |(y_1 - y_2)\cos t|$
$\le |y_1 - y_2|$

so f is lipstitz

and f is cont so, unique soution

second problem same as

$y'(t) = \frac{2y}{t} + t^2 e^t$, linear futions always lipshitz

$y'(t) - \frac{2y}{t} = t^2 e^t \longrightarrow$ Bernoli

$\Rightarrow \underbrace{\frac{y'}{t^2} - \frac{2y}{t^3}} = e^t$

$\frac{d}{dt}\frac{y}{t^2} = \frac{y'}{t^2} - \frac{2y}{t^3}$

2. $1 \le t \le 2$
$h = 0.1$
$y(1) = 1$
$t_0 = 1$
$t_1 = 1.1$
$\vdots$
$t_9 = 1.9$
$t_{10} = 2$

$y(t_1) = y(1) + 0.1 f(1, y(1))$

$\Rightarrow y(1.1) = 1 + 0.1\left(1 - 1\right) = 1$

$y(1.1) = 1$

$y(1.1) = 1 + 0.1\left(\frac{1}{1.1} - \left(\frac{1}{1.1}\right)^2\right)$

3. $y' = f(t, y) \quad a \le t \le b \quad y(a) = \alpha$

$y_0(t) = \alpha \quad \forall t \in [a, b]$
$\{y_k(t)\} \quad y_k(t) = \alpha + \int_a^t f(\tau, y_{k-1}(\tau))d\tau \quad k = 1, 2, \dots$

$y' = -y + t + 1 \quad 0 \le t \le 1 \quad y(0) = 1$

$y_0(t) = 1 \quad y_1(t) = 1 + \int_0^t (-\cancel{1} + t + \cancel{1})\,dt = 1 + \frac{t^2}{2}$

$$y_2(t) = 1 + \int_0^t -\frac{t^2}{2} \underset{\cancel{1}}{+} t + \cancel{1} \, dt$$

$$= 1 - \frac{t^3}{6} + \frac{t^2}{2}$$

$$y_3(t) = 1 + \int_0^t \left(\frac{t^3}{6} - \frac{t^2}{2} - \cancel{1} + t + \cancel{1} \right) dt$$

$$= 1 + \frac{t^4}{4!} - \frac{t^3}{3!} + \frac{t^2}{2!}$$

Assignment - 3

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Assignment - 3 :

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1. A, B are non-singular $n \times n$ matrices, then
from defn $\det(A) \neq 0 \Rightarrow A$ is invertible
$\det(B) \neq 0 \Rightarrow B$ is invertible ($\because$ result covered in class)

so, $\exists A^{-1}, B^{-1}$ s.t $AA^{-1} = A^{-1}A = I$ ($\because$ from defn)
$BB^{-1} = B^{-1}B = I$ —— ①

now, two matrics are similar if $\exists P$ (invertible)
s.t $PM_1P^{-1} = M_2$
where M_1, M_2 are two matrices

here, as $BA = I.BA$
$= (A^{-1}AB\,A)$ ($\because$ ①)
$= (A^{-1})\,AB\,(A)$
$= P\,AB\,P^{-1}$
where $P = A^{-1}$ (invertible)

so, $\exists P$ s.t $P(AB)P^{-1} = BA$
$\Rightarrow AB$ and BA are similar

2. given $A \sim B$ (A is similar to B) so, $\exists P$ (invertible) s.t
$PAP^{-1} = B$ — ①
also as $B \sim C$ (B is similar to C) so, $\exists Q$ (invertible) s.t
$QBQ^{-1} = C$ —— ②

then $QPAP^{-1}Q^{-1} = Q(PAP^{-1})Q^{-1}$
$= Q(B)Q^{-1}$ ($\because$ ①)
$= C$ ($\because$ ②)

$\Rightarrow QPAP^{-1}Q^{-1} = C$

now, as P is invertible $\Leftrightarrow \det(P) \neq 0$ —— ③
Q is invertible $\Leftrightarrow \det(Q) \neq 0$ —— ④
from theorem done in class
$\det(QP) = \det(P)\det(Q) \neq 0$ ($\because$ ③, ④)
$\Rightarrow QP$ is invertible

moreover $(QP)(P^{-1}Q^{-1}) = Q(PP^{-1})Q^{-1} = QQ^{-1} = I$
$(P^{-1}Q^{-1})(QP) = P^{-1}(Q^{-1}Q)P = P^{-1}P = I$

$\Rightarrow (QP)^{-1} = P^{-1}Q^{-1}$

let $(QP) = R$, then $R^{-1} = P^{-1}Q^{-1}$ and
$RAR^{-1} = QPAP^{-1}Q^{-1}$
$= QBQ^{-1}$
$= C$
$\Rightarrow \exists R$ invertible s.t $RAR^{-1} = C$
so, A is similar to C

3. given A is similar to B, so $\exists P$ invertible s.t
$PAP^{-1} = B$ —— ①
(a) To prove : $\det(A) = \det(B)$
proof : as $PAP^{-1} = B$ from ①

$$
\begin{aligned}
\det(B) &= \det(PAP^{-1}) \\
&= \det(P)\det(AP^{-1}) \\
&= \det(P)\det(A)\det(P^{-1}) \\
&= \det(P)\det(P^{-1})\det A \\
&= \det(PP^{-1})\det(A) \\
&= \det(I)\det(A) \\
&= \det(A) \quad (\because \det(I)=1)
\end{aligned}
$$

(b) <u>To prove</u>: $\chi_A(t) = \chi_B(t)$ where $\chi_M(t)$ is characteristic polynomial for matrix M

<u>proof</u>: let's say both A, B are $n \times n$ matrix (necessary square from similarity argument)

now, $\chi_A(t) = \det(A - tI)$

$$
\begin{aligned}
\chi_B(t) &= \det(B - tI) \\
&= \det(PAP^{-1} - tI) \quad (\because ①)
\end{aligned}
$$

as $PP^{-1} = I$

$\Rightarrow PIP^{-1} = I$

$$
\begin{aligned}
\text{so, } \chi_B(t) &= \det(PAP^{-1} - tPIP^{-1}) \\
&= \det(P)(A-tI)(P^{-1}) \\
&= \det(P)\det(A-tI)\det(P^{-1}) \\
&= \det(P)\det(P^{-1})\det(A-tI) \\
&= \det(PP^{-1})\det(A-tI) \\
&= \det(A-tI) \\
&= \chi_A(t)
\end{aligned}
$$

(c) <u>To prove</u>: A is non singular iff B is non singular

<u>proof</u>: we know that

A is non-singular iff $\det(A) \neq 0$ ($\because$ theorm done in class)

as from (a) we have $\det(A) = \det(B)$ we get

$\det(A) \neq 0 \Leftrightarrow \det(B) \neq 0$

$\Rightarrow$ A is non-singular $\Leftrightarrow \det(A) \neq 0$

$\Leftrightarrow \det(B) \neq 0$

$\Leftrightarrow$ B is non-singular

(d) <u>To prove</u>: If A is non-singular then A^{-1} is similar to B^{-1}

<u>proof</u>:

From (c) we know A is non-singular $\Rightarrow$ B is non-singular

now, this means both A^{-1}, B^{-1} exist

now claim if M_1, M_2 are invertable then

$(M_1 M_2)^{-1} = M_2^{-1} M_1^{-1}$

this is true as

$(M_2^{-1} M_1^{-1}) M_1 M_2 = M_2^{-1} I M_2 = I$

and $M_1 M_2 (M_2^{-1} M_1^{-1}) = M_1 I M_1^{-1} = I$

$\Rightarrow M_2^{-1} M_1^{-1} = (M_1 M_2)^{-1}$ ——— ②

also as P, A are invertable, $\det(PA) = \det(P)\det(A) \neq 0$

$\Rightarrow PA$ also invertable

now in ② $M_1 = PA$
$M_2 = P^{-1}$ ($\because$ Both invertible)

$\Rightarrow (M_1 M_2)^{-1} = (PAP^{-1})^{-1}$
$= M_2^{-1} M_1^{-1}$
$= P(PA)^{-1}$
$= PA^{-1}P^{-1}$ ($\because$ applying claim on $(PA)^{-1}$)

and as $(PAP^{-1})^{-1} = PA^{-1}P^{-1}$
$\Rightarrow (B)^{-1} = PA^{-1}P^{-1}$ ($\because$ ①)

so, $\exists P$ (invertible) s.t
$PA^{-1}P^{-1} = B^{-1}$
$\Rightarrow A^{-1}$ is similar to B^{-1}

(e) <u>To prove</u>: A^T is similar to B^T

<u>proof</u>: now as P is invertible $\Rightarrow \det(P) \neq 0$
also $\det(P) = \det(P^T)$
$\Rightarrow \det(P^T) \neq 0$
$\Rightarrow P^T$ is invertible ——③

now, let's claim $(MN)^T = N^T M^T$ for any two matrices

as $(MN)_{ij} = (MN)^T_{ji}$ ($\because$ defn)

$\Rightarrow (MN)^T_{ij} = (MN)_{ji} = \sum_{k=1}^{n} M_{jk} N_{ki}$ ($\because$ matrix multiplication)
$= \sum_{k=1}^{n} N_{ki} M_{jk}$
$= \sum_{k=1}^{n} N^T_{ik} M^T_{kj}$
$= (N^T M^T)_{ij}$

$\Rightarrow (MN)^T_{ij} = (N^T M^T)_{ij}$
$\Rightarrow (MN)^T = N^T M^T$ ——④

now from ③ as P^T is invertible

$(P^T)(P^{-1})^T = (P^{-1}P)^T = I^T = I$ ($\because$ ④)
and
$(P^{-1})^T(P^T) = (PP^{-1})^T = I^T = I$ ($\because$ ④)
so, $(P^{-1})^T = (P^T)^{-1}$ by defn ——⑤

now, as $PAP^{-1} = B$ ($\because$ ①)

$\Rightarrow (PAP^{-1})^T = (P^{-1})^T (PA)^T$ ($\because$ ④)
$= (P^T)^{-1}(A^T P^T)$ ($\because$ ④ and ⑤)
$= (P^T)^{-1}(A^T)(P^T)$

and as $PAP^{-1} = B$
$\Rightarrow B^T = (P^T)^{-1}(A^T)(P^T)$

so, $\exists$ invertible $(P^T)^{-1}$ s.t $B^T = (P^T)^{-1} A^T (P^T)$
(P^T is invertible, so is $(P^T)^{-1}$)
$\Rightarrow A^T$ is similar to B^T

4. $A = \begin{bmatrix} -4 & 14 & 0 \\ -5 & 13 & 0 \\ -1 & 0 & 2 \end{bmatrix}$ let $X_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

then $\|X_0\|_\infty = 1$

$P_0 = 1$ as $(X_0)_1 = 1$

now $y_1 = AX_0 = \begin{bmatrix} -4 & 14 & 0 \\ -5 & 13 & 0 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 1 \end{bmatrix}$

now $\mu' = (y_1)_{P_0} = 10$

and $\|y_1\|_\infty = 10 \Rightarrow P_1 = 1$ and $X_1 = \dfrac{y_1}{|(y_1)_{P_1}|} = \begin{bmatrix} 1 \\ \frac{8}{10} \\ \frac{1}{10} \end{bmatrix}$

$\Rightarrow Y_2 = AX_1 = \begin{bmatrix} -4 & 10 & 0 \\ -5 & 13 & 0 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.8 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 7.2 \\ 5.4 \\ -0.8 \end{bmatrix}$

so, $\mu_2 = (Y_2)_{P_1} = 7.2$

and $P_2 = 1$ so, $X_2 = \dfrac{1}{7.2} \begin{bmatrix} 7.2 \\ 5.4 \\ -0.8 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.75 \\ -0.1111 \end{bmatrix}$

$\Rightarrow Y_3 = \begin{bmatrix} -4 & 14 & 0 \\ -5 & 13 & 0 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.75 \\ -0.1111 \end{bmatrix} = \begin{bmatrix} 6.5 \\ 4.75 \\ -1.222 \end{bmatrix}$

$\mu_3 = (Y_3)_{P_2} = 6.5$, $P_3 = 1$ and $X_3 = \dfrac{1}{6.5} \begin{bmatrix} 6.5 \\ 4.75 \\ -1.222 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.73077 \\ -0.18803 \end{bmatrix}$

$\Rightarrow Y_4 = AX_3 = \begin{bmatrix} -4 & 14 & 0 \\ -5 & 13 & 0 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.73077 \\ -0.18803 \end{bmatrix} = \begin{bmatrix} 6.23077 \\ 4.5 \\ -1.37607 \end{bmatrix}$

$\mu_4 = (Y_4)_{P_3} = 6.23077$, $P_4 = 1$ and $X_4 = \dfrac{1}{6.23077} \begin{bmatrix} 6.23077 \\ 4.5 \\ -1.37607 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.72222 \\ -0.22085 \end{bmatrix}$

$\Rightarrow Y_5 = AX_4 = \begin{bmatrix} -4 & 14 & 0 \\ -5 & 13 & 0 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.72222 \\ -0.22085 \end{bmatrix} = \begin{bmatrix} 6.11111 \\ 4.38889 \\ -1.4417 \end{bmatrix}$

$\mu_5 = (Y_5)_{P_4} = 6.1111$, $P_5 = 1$ and $X_5 = \dfrac{1}{6.1111} \begin{bmatrix} 6.11111 \\ 4.38889 \\ -1.4417 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.71818 \\ -0.23591 \end{bmatrix}$

$\Rightarrow Y_6 = AX_5 = \begin{bmatrix} -4 & 14 & 0 \\ -5 & 13 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.71818 \\ -0.23591 \end{bmatrix} = \begin{bmatrix} 6.05455 \\ 4.33636 \\ -1.47183 \end{bmatrix}$

$\mu_6 = (Y_6)_{P_5} = 6.05455$, $P_6 = 1$ and $X_6 = \dfrac{1}{6.05455} \begin{bmatrix} 6.05455 \\ 4.33636 \\ -1.47183 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.71622 \\ -0.24309 \end{bmatrix}$

$\Rightarrow Y_7 = AX_6 = \begin{bmatrix} -4 & 14 & 0 \\ -5 & 13 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.71622 \\ -0.24309 \end{bmatrix} = \begin{bmatrix} 6.02703 \\ 4.31081 \\ -1.48619 \end{bmatrix}$

$\mu_7 = (Y_7)_{P_6} = 6.02703$, $P_7 = 1$ and $X_7 = \frac{1}{6.02703}\begin{bmatrix} 6.02703 \\ 4.31081 \\ -1.48619 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.71525 \\ -0.24659 \end{bmatrix}$

$\Rightarrow Y_8 = AX_7 = \begin{bmatrix} -4 & 14 & 0 \\ -5 & 13 & 0 \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.71525 \\ -0.24659 \end{bmatrix} = \begin{bmatrix} 6.01345 \\ 4.29821 \\ -1.49318 \end{bmatrix}$

$\mu_8 = (Y_8)_{P_7} = 6.01345$, $P_8 = 1$ and $X_8 = \frac{1}{6.01345}\begin{bmatrix} 6.01345 \\ 4.29821 \\ -1.49318 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.71477 \\ -0.24831 \end{bmatrix}$

$\Rightarrow Y_9 = AX_8 = \begin{bmatrix} -4 & 14 & 0 \\ -5 & 13 & 0 \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.71477 \\ -0.24831 \end{bmatrix} = \begin{bmatrix} 6.00671 \\ 4.29195 \\ -1.49661 \end{bmatrix}$

$\mu_9 = (Y_9)_{P_8} = 6.00671$, $P_9 = 1$ and $X_9 = \frac{1}{6.00671}\begin{bmatrix} 6.00671 \\ 4.29195 \\ -1.49661 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.71453 \\ -0.24916 \end{bmatrix}$

$\Rightarrow Y_{10} = AX_9 = \begin{bmatrix} -4 & 14 & 0 \\ -5 & 13 & 0 \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.71453 \\ -0.24916 \end{bmatrix} = \begin{bmatrix} 6.00335 \\ 4.28883 \\ -1.49831 \end{bmatrix}$

$\mu_{10} = (Y_{10})_{P_9} = 6.00335$, $P_{10} = 1$ and $X_{10} = \frac{1}{6.00335}\begin{bmatrix} 6.00335 \\ 4.28883 \\ -1.49831 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.71441 \\ -0.24958 \end{bmatrix}$

$\Rightarrow Y_{10} = AX_{10} = \begin{bmatrix} -4 & 14 & 0 \\ -5 & 13 & 0 \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.71441 \\ -0.24958 \end{bmatrix} = \begin{bmatrix} 6.00168 \\ 4.28727 \\ -1.49916 \end{bmatrix}$

$\mu_{10} = (Y_{10})_{P_9} = 6.00168 \approx 6$

then from μ_m as m goes to 10, it seems $\mu_m \to 6$, so approximation of dominant eigenvalue of A is $\lambda = 6$

5. $\lambda_1, \lambda_2, \ldots \lambda_n$ are eigenvalues of A with $v^{(1)}, v^{(2)}, \ldots v^{(n)}$ eigenvectors and λ_1 has multiplicity $= 1$

x is a vector s.t $x^T v^{(1)} = 1$

To prove : matrix $B = A - \lambda_1 v^{(1)} x^T$ has eigenvalues $0, \lambda_2, \ldots \lambda_n$ eigenvectors $v^{(1)}, w^{(2)}, \ldots w^{(n)}$ where

$$v^{(i)} = (\lambda_i - \lambda_1) w^{(i)} + \lambda_1 (x^T w^{(i)}) v^{(1)}, \; i = 2, 3 \ldots n$$

proof : A is s.t it's characteristic polynomial

$$\chi_A(\lambda) = \det(A - \lambda I) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$

where

λ_1 is only one as its multiplicity is 1

now, $\chi_B(t) = \det(B - tI)$
$= \det(A - \lambda_1 v^{(1)} x^T - tI)$
$= \det(A - (\lambda_1 v^{(1)} x^T + t) I)$

and $Bv^{(1)} = (A - \lambda_1 v^{(1)} x^T) v^{(1)}$
$= Av^{(1)} - \lambda_1 v^{(1)} x^T v^{(1)}$
$= \lambda_1 v^{(1)} - \lambda_1 v^{(1)}$ $(\because \langle x, v^{(1)} \rangle = 1,\ Av^{(1)} = \lambda_1 v^{(1)})$

so, $Bv^{(1)} = 0 . v^{(1)}$

i.e 0 is an eigenvalue of B and corresponding $v^{(1)}$ is eigenvector

so, $\chi_B(t) = t(t-\tilde{\lambda}_2)(t-\tilde{\lambda}_3)\cdots(t-\tilde{\lambda}_n)$ (deg n as B is $n\times n$)
($\because \chi_B(0)=0$ as 0 is an eigenvalue of B)

and let $w^{(2)}, w^{(3)} \ldots w^{(n)}$ be corresponding eigenvectors to $\tilde{\lambda}_2, \tilde{\lambda}_3, \ldots \tilde{\lambda}_n$

then $Bw^{(i)} = \tilde{\lambda}_i w^{(i)}$ for $i=2,\ldots n$

now, if $\exists i$ s.t $\chi_B(\lambda_i) \neq 0$ then

$$\Rightarrow \lambda_i(\lambda_i-\tilde{\lambda}_2)(\lambda_i-\tilde{\lambda}_3)\cdots(\lambda_i-\tilde{\lambda}_n) \neq 0$$

or $\det(A-\lambda_1 v^{(1)}x^T - \lambda_i I) \neq 0$

i.e $A-\lambda_1 v^{(1)}x^T - \lambda_i I$ is invertible

let $C = A-\lambda_1 v^{(1)}x^T - \lambda_i I$

then $Cv^{(i)} = \cancel{Av^{(1)}} - \lambda_1 v^{(1)}x^T v^{(i)} - \cancel{\lambda_i v^{(1)}}$
$= -\lambda_1 v^{(1)}x^T v^{(i)}$

$(C+\lambda_1 v^{(1)}x^T)\,v^{(i)} = 0$

as from defn of eigenvalues as $v^{(i)} \neq 0$

$\Rightarrow \det(C+\lambda_1 v^{(1)}x^T) = 0$ (i.e its non invertible as otherwise if so then $v^{(i)}=0$)

$\Rightarrow \det(A-\cancel{\lambda_1 v^{(1)}x^T} - \lambda_i I + \cancel{\lambda_1 v^{(1)}x^T}) = 0$
$\Rightarrow \det(A-\lambda_i I) = 0$

this is a contradiction, so $\forall i=2,\ldots n$
$\chi_B(\lambda_i)=0$ i.e
wlog $\tilde{\lambda}_i = \lambda_i$ for $i=2,\ldots n$

and so eigenvalues of B are $0, \lambda_2, \lambda_3, \ldots \lambda_n$

so, $Bw^{(i)} = \lambda_i w^{(i)}$ for $i=2,3\ldots n$ —— ①

now, as $Bv^{(1)} = 0v^{(1)} = 0$
$\Rightarrow Bv^{(1)} = 0$ —— ②

also $A((\lambda_i-\lambda_1)w^{(i)} + \lambda_1(x^T w^{(i)})v^{(1)})$

$= (B+\lambda_1 v^{(1)}x^T)((\lambda_i-\lambda_1)w^{(i)} + \lambda_1(x^T w^{(i)})v^{(1)})$

$= (\lambda_i-\lambda_1)Bw^{(i)} + \cancel{\lambda_1\langle x, w^{(i)}\rangle Bv^{(1)}}$
$+ \lambda_1 v^{(1)}(\lambda_i-\cancel{\lambda_1})\langle x, w^{(i)}\rangle + \cancel{\lambda_1^2\langle x, w^{(i)}\rangle v^{(1)}}$

$= (\lambda_i-\lambda_1)Bw^{(i)} + \lambda_1 v^{(1)}\lambda_i\langle x, w^{(i)}\rangle$
($\because$ ①, ②)

$= (\lambda_i-\lambda_1)\lambda_i w^{(i)} + \lambda_1\lambda_i\langle x, w^{(i)}\rangle v^{(1)}$

$= \lambda_i((\lambda_i-\lambda_1)w^{(i)} + \lambda_1\langle x, w^{(i)}\rangle v^{(1)})$

so we get $(\lambda_i-\lambda_1)w^{(i)} + \lambda_1\langle x, w^{(i)}\rangle v^{(1)}$ to be a corresponding eigenvector of λ_i for A for $i=2,\dots n$

i.e $(\lambda_i-\lambda_1)w^{(i)}+\lambda_1(x^T w^{(i)})v^{(1)} \in \text{span}\{v^{(i)}\}$

if $(\lambda_i-\lambda_1)w^{(i)}+\lambda_1(x^T w^{(i)})v^{(1)} = \alpha v^{(i)}$ for $\alpha \neq 0$

$$\Rightarrow (\lambda_i-\lambda_1)\frac{w^{(i)}}{\alpha}+\lambda_1\left(x^T \frac{w^{(i)}}{\alpha}\right)v^{(1)} = v^{(i)}$$

and we make $w^{(i)}$ as $\frac{w^{(i)}}{\alpha}$ (just scaling $w^{(i)}$ it will still be eigenvector)

so we just choose our $w^{(i)}$'s by scaling and then we will get

$v^{(i)} = (\lambda_i-\lambda_1)w^{(i)}+\lambda_1(x^T w^{(i)})v^{(1)}$ for $i=2\dots n$ as $\frac{w^{(i)}}{\alpha}$ becomes new $w^{(i)}$

6. $B=\begin{bmatrix}1&3&4\\0&2&1\\0&0&-2\end{bmatrix}$ $\quad X_0=\begin{bmatrix}1\\1\\1\end{bmatrix}$

then $\|X_0\|_\infty=1$ and $P_0=1$

1st iteration: $y_1=AX_0=\begin{bmatrix}1&3&4\\0&2&1\\0&0&-2\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}8\\3\\-2\end{bmatrix}$

so, $\mu_1=(y_1)_{p_0}=8$ and $p_1=1$, $X_1=\frac{1}{8}\begin{bmatrix}8\\3\\-2\end{bmatrix}=\begin{bmatrix}1\\0.375\\-0.25\end{bmatrix}$

2nd iteration: $y_2=AX_1=\begin{bmatrix}1&3&4\\0&2&1\\0&0&-2\end{bmatrix}\begin{bmatrix}1\\0.375\\-0.25\end{bmatrix}=\begin{bmatrix}1.125\\0.5\\0.5\end{bmatrix}$

so, $\mu_2=(y_2)_{p_1}=1.125$ and $p_2=1$, $X_2=\frac{1}{1.125}\begin{bmatrix}1.125\\0.5\\0.5\end{bmatrix}=\begin{bmatrix}1\\0.44444\\0.44444\end{bmatrix}$

3rd iteration: $y_3=AX_2=\begin{bmatrix}1&3&4\\0&2&1\\0&0&-2\end{bmatrix}\begin{bmatrix}1\\0.44444\\0.44444\end{bmatrix}=\begin{bmatrix}4.11111\\1.33333\\-0.88889\end{bmatrix}$

so, $\mu_3=(y_3)_{p_2}=4.11111$ and $p_3=1$, $X_3=\frac{1}{4.11111}\begin{bmatrix}4.11111\\1.33333\\-0.88889\end{bmatrix}=\begin{bmatrix}1\\0.32432\\-0.21622\end{bmatrix}$

4th iteration: $y_4=AX_3=\begin{bmatrix}1&3&4\\0&2&1\\0&0&-2\end{bmatrix}\begin{bmatrix}1\\0.32432\\-0.21622\end{bmatrix}=\begin{bmatrix}1.10811\\0.43243\\0.43243\end{bmatrix}$

so, $\mu_4=(y_4)_{p_3}=1.10811$ and $p_4=1$, $X_4=\frac{1}{1.10811}\begin{bmatrix}1.10811\\0.43243\\0.43243\end{bmatrix}=\begin{bmatrix}1\\0.39024\\0.39024\end{bmatrix}$

5th iteration: $y_5=AX_4=\begin{bmatrix}1&3&4\\0&2&1\\0&0&-2\end{bmatrix}\begin{bmatrix}1\\0.39024\\0.39024\end{bmatrix}=\begin{bmatrix}3.73171\\1.17073\\-0.7804\end{bmatrix}$

so, $\mu_5=(y_5)_{p_4}=3.73171$ and $p_5=1$, $X_5=\frac{1}{3.73171}\begin{bmatrix}3.73171\\1.17073\\-0.7804\end{bmatrix}=\begin{bmatrix}1\\0.31373\\-0.20915\end{bmatrix}$

6th iteration: $y_6=AX_5=\begin{bmatrix}1&3&4\\0&2&1\\0&0&-2\end{bmatrix}\begin{bmatrix}1\\0.31373\\-0.20915\end{bmatrix}=\begin{bmatrix}1.10458\\0.4183\\0.4183\end{bmatrix}$

so, $\mu_6=(y_6)_{p_5}=1.10458$ and $p_6=1$

$$X_6 = \frac{1}{1.10458}\begin{bmatrix}1.10458\\0.4183\\0.4183\end{bmatrix} = \begin{bmatrix}1\\0.3787\\0.3787\end{bmatrix}$$

$\underline{7^{th}\ \text{itteration}}$: $y_7 = AX_6 = \begin{bmatrix}1&3&4\\0&2&1\\0&0&-2\end{bmatrix}\begin{bmatrix}1\\0.3787\\0.3787\end{bmatrix} = \begin{bmatrix}3.65089\\1.13609\\-0.7574\end{bmatrix}$

so, $\mu_7 = (y_7)_{p_6} = 3.65089$ and $p_7 = 1$

now, μ_m does not seem to be converging, maybe because condition required for power method is not being satisfied by matrix B

7. $w \in \mathbb{C}^n$ $\quad \|w\|_2^2 = w^* w = 1$

$A = I - 2ww^*$

(a) $w = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$ $\quad w^* = (\bar{w})^T = \frac{1}{\sqrt{3}}[1\ 1\ 1]$

so, $w^* w = \frac{1}{3} \times (1+1+1) = 1$

and $ww^* = \frac{1}{\sqrt{3}} \times \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}[1\ 1\ 1] = \frac{1}{3}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$

$$A = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \frac{2}{3}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix} = \begin{bmatrix}1/3&-2/3&-2/3\\-2/3&1/3&1/3\\-2/3&-2/3&1/3\end{bmatrix}$$

A is symmetric as $A^T = \begin{bmatrix}1/3&-2/3&-2/3\\-2/3&1/3&-2/3\\-2/3&-2/3&1/3\end{bmatrix} = A$

also A is ortogonal as $AA^T = A(A)$
$= A^T A$

so, $AA^T = A^T A = A^2$

now $A^2 = (I - 2ww^*)^2 = (I - 2ww^*)(I - 2ww^*)$
$= I - 2ww^* - 2ww^* + 4ww^* ww^*$
$= I - 4ww^* + 4w\underbrace{(w^* w)}_{1}w^*$

$= I - \cancel{4ww^*} + \cancel{4ww^*}$
$= I$

so, $AA^T = A^T A = I$
$\Rightarrow$ A is ortogonal, also from $\begin{bmatrix}1/3&-2/3&-2/3\\-2/3&1/3&-2/3\\-2/3&0&1/3\end{bmatrix}^2 = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

(b) $A = I - 2ww^*$
now let $w = \begin{bmatrix}w_1\\w_2\\ \vdots\\w_n\end{bmatrix}$ then $\sum_{i=1}^{n} |w_i|^2 = 1$
$w_i \in \mathbb{C}$

and $2ww^* = 2\begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}[\overline{w}_1 \quad \overline{w}_2 \cdots \overline{w}_n]$

$$= 2\begin{bmatrix} w_1\overline{w}_1 & w_1\overline{w}_2 & \cdots & w_1\overline{w}_n \\ w_2\overline{w}_1 & \cdot & \cdots & w_2\overline{w}_n \\ \vdots & & & \\ w_n\overline{w}_1 & \cdots & \cdots & w_n\overline{w}_n \end{bmatrix}_{n\times n}$$

and $I - 2ww^* = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{bmatrix}_{n\times n} - 2\begin{bmatrix} w_1\overline{w}_1 & \cdots & w_1\overline{w}_n \\ \vdots & & \\ w_n\overline{w}_1 & \cdots & w_n\overline{w}_n \end{bmatrix}_{n\times n}$

$$A = \begin{bmatrix} 1-2w_1\overline{w}_1 & -2w_1\overline{w}_2 & \cdots & -2w_1\overline{w}_n \\ \vdots & \ddots & & -2w_2\overline{w}_n \\ \vdots & & \ddots & \vdots \\ -2w_n\overline{w}_1 & \cdots & \cdots & 1-2w_n\overline{w}_n \end{bmatrix}_{n\times n}$$

$\leftarrow n\times n$ matrix

and so $A = [\delta_{ij} - 2w_j\overline{w}_i]$

for i^{th} column and j^{th} row

$$\delta_{ij} = \begin{cases} 1 ; & i=j \\ 0 ; & i\neq j \end{cases}$$

then $\overline{A} = \begin{cases} [\overline{-2w_j\overline{w}_i}] & ; i\neq j \quad (\because \overline{w_i\overline{w}_j} = \overline{w}_i w_j) \\ [\overline{1-2w_i\overline{w}_i}] & ; i=j \end{cases}$

but as $w_i\overline{w}_i = |w_i|^2 \in \mathbb{R}$

$\Rightarrow 1-2w_i\overline{w}_i \in \mathbb{R}$

$\Rightarrow \overline{1-2w_i\overline{w}_i} = 1-2w_i\overline{w}_i$

and so, $\overline{A} = \begin{cases} [-2\overline{w}_j w_i] ; & i\neq j \\ [1-2w_i\overline{w}_i] ; & i=j \end{cases}$

now, $(\overline{A})^T = \begin{cases} [-2w_j\overline{w}_i] & ; i\neq j \\ [1-2w_i\overline{w}_i] & ; i=j \end{cases}$

so, we see that $(\overline{A})^T = A^* = [\delta_{ij} - 2w_j\overline{w}_i] = A$

$\Rightarrow A^* = A$

so, A is Hermitian

now, $A^*A = AA = AA^* = A^2 \; (\because A = A^*)$

and $A^2 = (I - 2ww^*)(I - 2ww^*)$
$= I - 2ww^* - 2ww^* + 4\, w\underbrace{w^* w}_{1} w^*$
$= I - \cancel{2ww^*} - \cancel{2ww^*} + \cancel{4ww^*}$
$= I$

so, $AA^* = A^* A = I$
$\Rightarrow A$ is unitary

8.
To prove: $\text{Rank}(A) = \text{Rank}(A^T)$
proof: let $\text{rank}(A)$
$= \text{column rank}(A) = \dim\{Ax \mid x \in \mathbb{R}^n\}$

let A^t = traspose of A

claim: $A^t A x = 0 \Leftrightarrow Ax = 0$

($\Leftarrow$) This direction is trivial

($\Rightarrow$) $A^t A x = 0$
$\Rightarrow x^t A^t A x = 0$
$\Rightarrow (Ax)^t (Ax) = 0$
$\Rightarrow \|Ax\| = 0$
$\Rightarrow Ax = 0$

then if A is $m \times n$, then $A^t A$ is $n \times n$, so same number of columns as A

then $N(A^t A) = N(A)$ $N(\cdot)$ = Null space
as $\forall x \in N(A^t A)$ from our claim
$\Rightarrow x \in N(A)$
and $\forall x \in N(A) \Rightarrow x \in N(A^t A)$

$\Rightarrow N(A^t A) = N(A)$

so Nullity of $A^t A$ = Nullity of A

and we know from rank-nullity theorem

$\text{rank}(A) + \text{nullity}(A) = n$
& $\text{rank}(A^t A) + \text{Nullity}(A^t A) = n$

$\Rightarrow \text{rank}(A) + \cancel{\text{Nullity}(A)} = \text{rank}(A^t A) + \cancel{\text{Nullity}(A^t A)}$

$\Rightarrow \text{rank}(A) = \text{rank}(A^t A)$

and as $\text{rank}(A^t A) = \dim\{A^t A x \mid \forall x\}$
and $\text{rank}(A^t) = \dim\{A^t y \mid \forall y\}$
now for $\text{rank}(A^t A)$ $y = Ax$
so we are restricting span of colums of A^t
so, $\{A^t A x \mid \forall x\} \subseteq \{A^t y \mid \forall y\}$

$\Rightarrow \dim(A^tA) \leq \dim(A^t)$
$\Rightarrow \dim(A) \leq \dim(A^t)$

so we get $\dim(A) \leq \dim(A^t)$ ——— ①
now we repeat our argument
with $AA^t x = 0 \Leftrightarrow A^t x = 0$
$\Rightarrow \dim(A^t) \leq \dim(A)$ ($\because$ similar calculation but with A^t)
——— ②

so from ①, ② we get

$$\dim(A) = \dim(A^t)$$

To prove: $\text{Nullity}(A) = \text{Nullity}(A^T)$ iff A is a square matrix

proof: $(\Rightarrow)$ from previous result we have
$\text{Rank}(A) = \text{Rank}(A^T)$
if A is $m \times n$ matrix
then from rank-nullity theorem

$\text{Rank}(A) + \text{Nullity}(A) = n$
similarly as
A^t is $n \times m$
$\text{Rank}(A^t) + \text{Nullity}(A^t) = m$

$\Rightarrow \text{Rank}(A) + \text{Nullity}(A) = m$

($\because \text{Rank}(A) = \text{Rank}(A^t)$, $\text{Nullity}(A) = \text{Nullity}(A^t)$)

$\Rightarrow n = m$ i.e A is square matrix

$(\Leftarrow)$ Now if A is a square matrix i.e $n = m$
then
from previous result
$\text{Rank}(A) = \text{Rank}(A^t)$
and rank-nullity theorem

$\text{Rank}(A) + \text{Nullity}(A) = n = \text{Rank}(A^t) + \text{Nullity}(A^t)$
$\Rightarrow \text{Nullity}(A) = \text{Nullity}(A^t)$

($\because \text{Rank}(A) = \text{Rank}(A^t)$)

To prove: A has SVD $A = USV^T$ show that $\text{Rank}(A) = \text{Rank}(S)$

proof:
now $\text{Rank}(USV^T) = \text{colrank of } USV^T$

$= \dim\{USV^T x \mid \forall x\}$

but $\{USV^T x \mid \forall x\} \subseteq \{USy \mid \forall y\}$

$\Rightarrow \dim\{USV^T x \mid \forall x\} \leq \dim\{USy \mid \forall y\}$

$\Rightarrow \text{Rank}(USV^T) \leq \text{Rank}(US)$

and as $U^TU = UU^T = I_{m \times m}$ ($\because$ from SVD) we have U to be invertible, so

Claim: $USx = 0$ iff $Sx = 0$

($\Rightarrow$) $USx = 0 \Rightarrow Sx = U^{-1}0 = U^T 0 = 0$

($\Leftarrow$) $Sx = 0 \Rightarrow USx = U.0 = 0$

so, Null space of US = Null space of S

as $x \in N(US)$ iff $x \in N(S)$

$\Rightarrow$ Nullity(US) = Nullity(S)

and from Rank-nullity theorm, we get

Rank(US) + Nullity(US) $= n =$ Rank(S) + Nullity(S)

$\Rightarrow$ Rank(US) = Rank(S)

so, Rank$(A) \leq$ Rank(US) = Rank (S)

$\Rightarrow$ Rank$(A) \leq$ Rank(S) —— ①

now as $V^TV = VV^T = I_{n \times n}$ ($\because$ SVD)

and $A = USV^T$

$\Rightarrow U^TA = SV^T$

$\Rightarrow U^TAV = S$

doing similar argument for $S = U^TAV$, we will get

dim$(S) \leq$ dim(A) —— ②

from ①, ② we get

Rank(S) = Rank(A)

Expressing Nullity(A) as Rank(S):

as previously shown Rank(A) = Rank(S)

and from Rank-nullity theorm we know

Rank(A) + Nullity$(A) = n$

$\Rightarrow$ Rank(S) + Nullity$(A) = n$

$\Rightarrow$ Nullity$(A) = n -$ Rank(S)

To prove: $n \times n$ matrix A SVD: $A = USV^T$, show that A^{-1} exist iff S^{-1} exist, Find SVD of A^{-1}

proof:

($\Rightarrow$) let A^{-1} exist i.e $AA^{-1} = A^{-1}A = I$

then $V^TA^{-1}U$ is a matrix

s.t $(V^TA^{-1}U)(S) = V^TA^{-1}(US)$

now as from SVD and U, V being orthogonal:

$A = USV^T$

$\Rightarrow AV = US$, so

$V^TA^{-1}(US) = V^TA^{-1}AV = V^TV = I$

similarly $(S)(V^T A^{-1} U)$ is s.t

as $A = USV^T$

$\Rightarrow U^T A = SV^T$

so, $(SV^T)A^{-1}U$

$= (U^T A)A^{-1}U$

$= U^T U$

$= I$

so, $\exists$ matrix $V^T A^{-1} U = S^{-1}$ s.t

$$S^{-1}S = SS^{-1} = I$$

$(\Leftarrow)$ let S^{-1} exist then as

$$A = USV^T$$

let matrix $VS^{-1}U^T$

then $(VS^{-1}U^T)(A) = VS^{-1}U^T(USV^T)$

$= VS^{-1}ISV^T$

$= VV^T$

$= I$

and $(A)(VS^{-1}U^T) = (USV^T)(VS^{-1}U^T)$

$= USIS^{-1}U^T$

$= UU^T$

$= I$

so, $\exists$ matrix $VS^{-1}U^T = A^{-1}$ s.t

$$AA^{-1} = A^{-1}A = I$$

so A^{-1} exist, also from construction

$$A^{-1} = VS^{-1}U^T$$

so, SVD of $A^{-1} = VS^{-1}U^T$

as $VV^T = I = V^T V$

$U^T U = I = UU^T$

and S^{-1} is just a matrix of singular values as $n \times n$ so all values on diagonals

$$S = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \qquad S^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{\lambda_n} \end{bmatrix}$$

as S^{-1} exist $\lambda_i \neq 0$ $\forall i$ as $\det(S) \neq 0$

To prove: If A is $m \times n$ matrix, P is $n \times n$ orthogonal matrix then AP has same singular values as A

proof: let $A = USV^T$, the SVD of A which is $m \times n$ where

$UU^T = U^T U = I_{m \times m}$

$VV^T = V^T V = I_{n \times n}$

and P is s.t $PP^T = P^T P = I_{n \times n}$

then $AP = (USV^T)P = US(V^TP)$

where V^TP is s.t

$(V^TP)(P^TV) = V^T(PP^T)V = V^TV = I_{m\times m}$

& $(P^TV)(V^TP) = P^T(I)P = P^TP = I_{m\times m}$

$\Rightarrow (V^TP)^{-1} = P^TV$

also as $(V^TP)^T = P^TV$

we get V^TP is ortogonal

so, we have $U_{n\times n}$, $V^TP_{m\times m}$ as ortogonal matrices s.t

$AP = (U)S(V^TP) = US(P^TV)^T$

so, AP has SVD as $US(P^TV)^T$ as U is $m\times m$ ortogonal

P^TV is $n\times n$ ortogonal

and so both A and AP have same singular value matrix S

$\Rightarrow$ Both have same singular values

Tutorial- 5 :

1. $A \in M_{n \times n}$

To prove: $|\text{trace}(A)| \leq n\, r_\sigma(A)$

proof: as A is a square matrix, $\exists P$ (non-singular) $\in M_{n \times n}$ s.t

$$A = PJP^{-1}$$

J ↑ Jordan canonical matrix

$\Rightarrow \text{trace}(A) = \text{trace}(PJP^{-1})$
$\Rightarrow \text{trace}(A) = \text{trace}(P)\,\text{trace}(J)\,\text{trace}(P^{-1})$
$\Rightarrow \text{trace}(A) = \text{trace}(PP^{-1})\,\text{trace}(J)$
$\Rightarrow \text{trace}(A) = \text{trace}(J)$

$\Rightarrow \text{trace}(A) = \sum_{i=1}^{n} \lambda_i$

λ_i are eigenvalues of A (including 0 as Jordan form)

then $|\text{trace}(A)| = \left|\sum_{i=1}^{n} \lambda_i\right| \leq \sum_{i=1}^{n} |\lambda_i|$ ($\because$ triangle inequality)

$\leq \max_{1 \leq i \leq n} \{|\lambda_i|\} \times n$

$= n\, r_\sigma(A)$

so, $|\text{trace}(A)| \leq n\, r_\sigma(A)$

now, if A is symmetric and positive definite then $\exists P$ (non-singular) $\in M_{n \times n}$ s.t

$PAP^{-1} = \Lambda$ ($\because$ schur canonical form with $A = A^T$)

now as A is positive definate
$\forall x \neq 0$
$x^T A x > 0$

if v is an eigenvalue of A then

$$v^T A v = v^T \lambda v = \lambda\left(\sum_{i=1}^{n} v_i^2\right) > 0$$

$\Rightarrow \lambda\left(\sum_{i=1}^{n} v_i^2\right) > 0$

$\Rightarrow \lambda > 0$

so, all eigenvalues are positive

$\text{trace}(A) = \sum \lambda_i \geq \lambda_i \ \forall i$

$\Rightarrow \text{trace}(A) \geq \max_{1 \leq i \leq n}(\lambda_i)$

$\Rightarrow \text{trace}(A) \geq \max_{1 \leq i \leq n}(|\lambda_i|)$

$\Rightarrow \text{trace}(A) \geq r_\sigma(A)$

2. $e^A = I + \frac{A}{1!} + \frac{A^2}{2!} + \cdots \frac{A^n}{n!} + \cdots, A \in M_{n\times n}$

let $\|\cdot\| : M_{n\times n} \to \mathbb{R}_{\geq 0}$ be any norm on $M_{n\times n}$

then $\|A\| \in \mathbb{R}$ and so

$e^{\|A\|}$ is defined (e^x defination)

$$e^{\|A\|} = 1 + \frac{\|A\|}{1!} + \frac{\|A\|^2}{2!} + \cdots$$

$$\geq \left\| I + \frac{A}{1!} + \frac{A^2}{2!} + \cdots \right\|$$

$\Rightarrow \left\| I + \frac{A}{1!} + \frac{A^2}{2!} + \cdots \right\| = e^{\|A\|} < \infty$ ($\because$ triangle inequality)

$\Rightarrow$ series converges absolutely

$\Rightarrow e^A$ converges

(a) $A = P^{-1} B P$

<u>To prove</u> : $e^A = P^{-1} e^B P$

<u>proof</u>: $A = P^{-1} B P$

$$e^A = I + \frac{A}{1!} + \frac{A^2}{2!} + \cdots$$

now $A^n = \underbrace{(P^{-1}BP)(P^{-1}BP)\cdots(P^{-1}BP)}_{n \text{ times}}$

$= P^{-1}\underbrace{(B)(B)(B)\cdots(B)}_{n \text{ times}}P^{-1}$

$= P^{-1}B^n P$

so, $e^A = I + \frac{A}{1!} + \frac{A^2}{2!} + \cdots$

$= I + \frac{P^{-1}BP}{1!} + \frac{P^{-1}B^2P}{2!} + \cdots$

$= P^{-1}IP + \frac{P^{-1}BP}{1!} + \frac{P^{-1}B^2P}{2!} + \cdots = P^{-1}\left(I + B + \frac{B^2}{2!} + \cdots\right)P$

so, $P^{-1}e^B P = P^{-1}\left(I + \frac{B}{1!} + \frac{B^2}{2!} + \cdots\right)P^{-1} = e^A$

(b) $\lambda_1, \lambda_2, \ldots \lambda_n$ eigenvalues of A

<u>To prove</u>: eigenvalues of e^A are $e^{\lambda_1}, e^{\lambda_2}, \ldots e^{\lambda_n}$

<u>proof</u>: now as $A \in M_{n\times n}$, $\exists P$ (non-singular) s.t

$A = P^{-1} J P$

Jordan canonical form of A

then from (a)

$$e^A = P^{-1} e^J P$$

as $e^J = I + \frac{J}{1!} + \frac{J^2}{2!} + \frac{J^3}{3!} + \cdots \frac{J^n}{n!} + \cdots$

as $J_k(\lambda) = \lambda I_k + \underset{\uparrow}{N}$

nilpotent matrix

$$(J_k(\lambda))^r = (\lambda I_k + N)^r$$

trivial to see this will be

$\lambda^2 I_k + (\quad)$

from binomial expansion

upper triangular part diag = 0

so, e^J will be upper triangular with diagonal entries

$1 + \lambda + \frac{\lambda^2}{2!} + \cdots = e^\lambda$

so, $e^J = \begin{bmatrix} e^{\lambda_1} & & * \\ & e^{\lambda_2} & \\ 0 & & \ddots \; e^{\lambda_n} \end{bmatrix}$

now $\det(e^A - \lambda I) = \det(\underset{\uparrow}{e^J} - \lambda I)$

this is also upper triangular

$$\det(e^J - \lambda I) = (e^{\lambda_1} - \lambda)(e^{\lambda_2} - \lambda)\cdots(e^{\lambda_n} - \lambda)$$

($\because$ det of upper triangular matrix is product of diag)

so, $\det(e^A - \lambda I) = (e^{\lambda_1} - \lambda)(e^{\lambda_2} - \lambda)\cdots(e^{\lambda_n} - \lambda)$
$= 0$

$\Rightarrow \lambda = e^{\lambda_1}, e^{\lambda_2}, \ldots e^{\lambda_n}$

3. As $\int_0^\infty x^n e^{-x} dx = n!$ ($\because$ gamma)

$$\int_0^\infty \phi_n(x)\phi_m(x) w(x) dx$$

$$= \int_0^\infty \frac{(-1)^{n+m}}{n!m!} e^{-x}[nx^{n-1} - x^n][mx^{m-1} - x^m] dx$$

$$= \frac{(-1)^{n+m}}{n!m!}[nm(n+m-2)! - \cancel{(n+m)!} + \cancel{(n+m)!}]$$

$\neq 0$ for $n \neq m$ so, $\{\phi_n\}_{n=1}^\infty$ are not orthogonal

4.

(a) $\int_{-1}^{1} (x-ax^2)^2 dx$

$= \int_{-1}^{1} (x^2 + a^2x^4 - 2ax^3) dx$

$= \left| \frac{x^3}{3} + a^2\frac{x^5}{5} - 2a\frac{x^4}{4} \right|_{-1}^{1}$

$= \frac{2}{3} + \frac{2a^2}{5}$

$\min_a \left(\frac{2}{3} + \frac{2a^2}{5} \right) = \frac{2}{3} \qquad (\because a^2 \geq 0)$

so for $a=0$ (unique value of a)

$\min_a \int_{-1}^{1} (x - ax^2)^2 dx = \int_{-1}^{1} x^2 dx = \frac{2}{3}$

(b) $\int_{-1}^{1} |x - ax^2| dx$

for $-1 \leq a \leq 1$

for $-1 \leq x \leq 0 \qquad\qquad 0 \leq x \leq 1$

$\Rightarrow -1 \leq x - ax^2 \leq 0 \qquad\qquad 0 \leq x - ax^2 \leq 1$

$ax^2 \leq x^2 \leq 1$

$-ax^2 \geq -1$

so $\int_{-1}^{1} |x - ax^2| dx = \int_{-1}^{0} |x - ax^2| dx + \int_{0}^{1} |x - ax^2| dx$

$= \int_{-1}^{0} -(x - ax^2) dx - \int_{0}^{1} (ax^2 - x) dx$

$= -\left| \frac{x^2}{2} - a\frac{x^3}{3} \right|_{-1}^{0} - \left| \frac{ax^3}{3} - \frac{x^2}{2} \right|_{0}^{1}$

$= -0 + \left(\frac{1}{2} + \frac{a}{3} \right) - \left(\frac{a}{3} - \frac{1}{2} \right)$

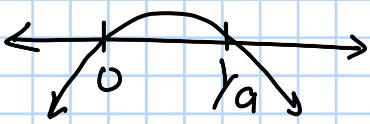
$= 1$

so, $\int_{-1}^{1} (x - ax^2) dx = 1 \ \forall \ a$ s.t $-1 \leq a \leq 1$

now if $a > 1$ then:

$x - ax^2 = 0$

$\Rightarrow x = 0, \ x = \frac{1}{a}$



so, $\int_{-1}^{1} |x - ax^2| dx = \int_{-1}^{0} (ax^2 - x) dx + \int_{0}^{1/a} (x - ax^2) dx + \int_{1/a}^{1} (ax^2 - x) dx$

$$= \left| \frac{ax^3}{3} - \frac{x^2}{2} \right|_{-1}^{0} + \left| \frac{x^2}{2} - \frac{ax^3}{3} \right|_{0}^{1/a} + \left| \frac{ax^3}{3} - \frac{x^2}{2} \right|_{1/a}^{1}$$

$$= +\left(+\frac{a}{3} + \cancel{\frac{1}{2}}\right) + \left(\frac{1}{6a^2}\right) + \left(\frac{a}{3} - \cancel{\frac{1}{2}}\right) - \left(\frac{1}{3a^2} - \frac{1}{2a^2}\right)$$

$$= \frac{2a}{3} + \frac{1}{3a^2}$$

as $a > 1$, let $f(a) = \frac{2a}{3} + \frac{1}{3a^2}$

$f'(a) = \frac{2}{3} - \frac{2}{3a^3} = 0$

$\Rightarrow a^3 = 1$ but $a > 1$

not posible

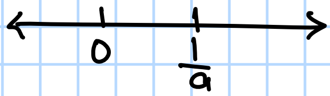
similarly for $a < -1$ case no values gives $f'(a) = 0$ so

minimum value is 1, same for all

$-1 \leq a \leq 1$

(c) as $x - ax^2$ has zeros

$0, \frac{1}{a}$



for $-1 \leq a \leq 1$

$0 \leq a \leq 1$ case:

$$\max_{[-1,1]} |x - ax^2| = \max\left\{ \max_{[0,1]} (x - ax^2), \max_{[-1,0]} (ax^2 - x) \right\}$$

$$= \max\left\{ \frac{1}{4a}, \max_{[-1,0]} (ax^2 - x) \right\}$$

by diff $x = \frac{1}{2a}$ for maxim

$2ax - 1 < 0$

diff

$$= \max\left\{ \frac{1}{4a}, a+1 \right\}$$

$= a + 1 \geq 1$ for $a = 0$ minima

similarly for $-1 \leq a < 0$ we will get $a = 0$ minima (symmetry)

now for $a > 1$ or $a < -1$

$a > 1$:

$$\max_{[-1,1]} |x - ax^2| \geq \max_{[-1,0]} |x - ax^2|$$

$\geq a + 1$ ($\because$ border)

> 2 as $a > 1$ so not minimum, same for $a < -1$

so, $\min_{a} \max_{[-1,1]} |x - ax^2| = 1$, for $a = 0$ (unique)

5. (a) $\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} (T_n(x))^2 dx = \int_{0}^{\pi} \cancel{\frac{1}{\sqrt{1-\cos^2\theta}}} (\cos(n\theta))^2 \cancel{\sin\theta}\, d\theta$

$x = \cos\theta$

$$= \int_0^\pi (\cos(n\theta))^2 d\theta$$

$$= \int_0^\pi \frac{1}{2}(1 + \cos 2n\theta)\, d\theta$$

$$= \frac{\pi}{2} + \left. \frac{\sin 2n\theta}{2n} \times \frac{1}{2} \right|_0^\pi = \frac{\pi}{2}$$

(b) $T_n(x)$ has n distinct zeros in $(-1,1)$

$T_n(x) = \cos(n \cos^{-1} x)$

let $x = \cos\theta$

as $x \in [-1,1]$

$\theta \in [0, \pi]$

now $\cos n\theta = T_n(\underbrace{\cos\theta}_{[0,\pi]})$

$\cos n\theta = 0$

$\theta = \frac{(2k+1)\pi}{2n} \Rightarrow \theta = \underbrace{\frac{\pi}{2n}, \frac{(3)\pi}{2n}, \frac{(5)\pi}{2n}, \dots \frac{(2n-1)\pi}{2n}}_{n \text{ values of } \theta}$

corresponding

$\cos(\theta) = \underset{\substack{\uparrow \\ \text{zeroes}}}{x} \in (-1,1)$

so, n zeros, $x_i = \cos\left(\frac{(2i-1)\pi}{2n}\right)$ $i = 1, 2, \dots n$

(c) from (b) $T_n(\cos\theta) = \cos(n\theta)$

$\frac{d}{dx} T_n(x) = \frac{d}{dx} \cos(n \cos^{-1} x)$

$T_n'(x) = -\sin(n \cos^{-1} x)\left(\frac{-n}{\sqrt{1-x^2}}\right)$

$= 0$

for $x \in (-1,1)$ $\sqrt{1-x^2} > 0$

$\Rightarrow \sin(n \cos^{-1} x) = 0$

$\Rightarrow \sin(n\theta) = 0$ $x = \cos\theta$, $\theta \in (0, \pi)$

$\Rightarrow \theta = \frac{\pi}{n}, \frac{2\pi}{n}, \frac{3\pi}{n}, \dots \frac{(n-1)\pi}{n}$

so, $x_i = \cos\left(\frac{i\pi}{n}\right)$ $i=1,2,\ldots n-1$

zeros of $T_n'(x)=0$

6. $$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx \qquad k=0,1,\ldots n$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx \qquad k=1,2,\ldots n-1$$

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n-1}(a_k\cos kx + b_k\sin kx) + a_n\cos nx$$

Here $f(x)=x^2$

$$\Rightarrow a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(0)\,dx = \frac{1}{\pi}\left|\frac{x^3}{3}\right|_{-\pi}^{\pi} = \frac{1}{\pi}\left(\frac{\pi^3}{3} - \left(\frac{-\pi^3}{3}\right)\right)$$
$$= \frac{2\pi^2}{3}$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos x\,dx$$
$$= \frac{1}{\pi}\Big[x^2\sin x\Big]_{-\pi}^{\pi} - \frac{2}{\pi}\int_{-\pi}^{\pi} 2x\sin x\,dx$$
$$= -\frac{4}{\pi}\int_{-\pi}^{\pi} x\sin x\,dx$$
$$= -\frac{4}{\pi}\left[x\frac{\cos x}{-1}\right]_{-\pi}^{\pi} \cancel{+ \frac{4}{\pi}\int_{-\pi}^{\pi}\cos x\,dx}$$
$$= \frac{4}{\pi}\Big[\pi(-1) - \pi(1)\Big]$$
$$= -4$$

$$a_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos 2x\,dx = \frac{1}{\pi}\left[-\int_{-\pi}^{\pi} \cancel{2}x\frac{\sin 2x}{\cancel{2}}\,dx\right]$$
$$= +\frac{1}{\pi}\left[x\frac{\cos 2x}{2}\right]_{-\pi}^{\pi}$$
$$= \frac{1}{\pi}\left[\frac{\pi+\pi}{2}\right] = 1$$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin x\,dx$$
$$= \frac{1}{\pi}\Big[x^2(-\cos x)\Big]_{-\pi}^{\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi} 2x\cos x\,dx$$

$$= \frac{1}{\pi}\left[\pi^2(1) - \pi^2(1)\right] + \frac{2}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x\sin x\right]_{-\pi}^{\pi} - \frac{2}{\pi}\int_{-\pi}^{\pi}\sin nx\,dx$$

$$= 0$$

So, $S_2(x) = \frac{\pi^2}{3} - 4\cos x + \cos 2x$