S1419 tutorial

Tutorial-1:

12. $\{\{1,2\},\{3,4\},\ldots\{2n-1,2n\}\}$

18. ommit 'either' typo

3. $m=2^2\cdot3^3\cdot5^5\cdot7^7$

$\longrightarrow 3\times4\times6\times8$

2. $\boxed{\;|\;|\;|\;}\quad\boxed{\;|\;|\;|\;}$

$9\times10\times10\times2-8\times9\times9\times2$ No need to write actual answer

7. $9\times10^3-9\times9\times8\times7$

8. $8\times8\times7\times5$

10. $k\in[n]\cup\{0\}$, $T\subseteq[n]$ with $|T|=k$

$|\mathbb{P}(T)|=2^k$

$$\sum_{k=0}^{n}\binom{n}{k}2^k=(2+1)^n=3^n$$

11. $A\subseteq B\subseteq C$

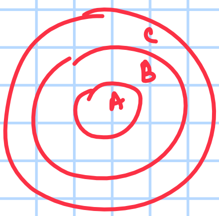
$E=\{(A,B,C)\mid A\subseteq B\subseteq C\subseteq[n]\}$

$F=\{(x_1,\ldots x_n)\mid x_i\in\{0,1,2,3\}\}$

$f:E\longrightarrow F$

$f(A,B,C)=(x_1,\cdots x_n)$

$$x_i=\begin{cases}3 ; & x_i\notin C\\ 2 ; & x_i\in C-B\\ 1 ; & x_i\in B-A\\ 0 ; & x_i\in A\end{cases}$$



$f^{-1}(x_1,\ldots x_n)=(A,B,C)$

$A=\{i\in[n]\mid x_i=0\}$

$B=\{i\in[n]\mid x_i\in\{0,1\}\}$

$C=\{i\in[n]\mid x_i\in\{0,1,2\}\}$

17. 101 out of $[200]$

$n_1=2^{k_1}a_1$ $\quad k\in\mathbb{N}\cup\{0\}$

$n_2=2^{k_2}a_2$ $\quad a$ is odd

$\vdots$

$n_{101}=2^{k_{101}}a_{101}$ $\}$ fundamental theorem of arithmetic

$a_1,\ldots a_{101}\in\underbrace{\{1,3,\ldots 199\}}_{100 \text{ numbers}}$

$\exists\, i,j\in[101]$ s.t

$i<j$ $\quad a_i=a_j\Rightarrow 2^{k_j}>2^{k_i}$ (wlog)

$n_j=2^{k_j-k_i}n_i$

$\Rightarrow n_j \bmod n_i=0$ or $n_i\mid n_j$

19. $\{7, 77, \ldots\}$

then $n \bmod 17 = \{0, \ldots 16\}$

then taking first 18, $\exists\, i,j$ s.t $\underbrace{77\cdots7}_{i \text{ times}} \bmod 17 = \underbrace{77\cdots7}_{j \text{ times}} \bmod 17$

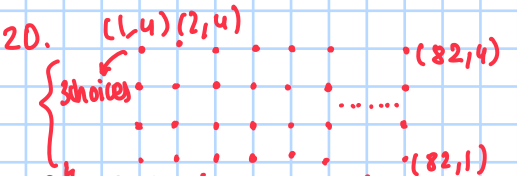
now, wlog $j>i$, $\underbrace{77\cdots7}_{j \text{ times}} - \underbrace{77\cdots7}_{i \text{ times}} = \underbrace{77\cdots7}_{(j-i) \text{ times}} \times 10^{i}$

$= 17 \times k$ for some p

now as $\gcd(17, 10^{p}) = 1$

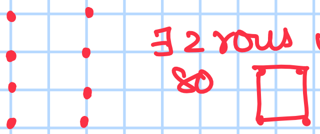
$\Rightarrow 17 \mid \underbrace{77\cdots7}_{j-i \text{ times}}$

20.



$3^4 = 81$ choices per column

$\exists$ 2 columns with same configuration



$\exists$ 2 rows with same configuration

so $\square$

22. $P_n(r,b)$ = prob of getting red on n^{th} trial given urn initially has r red and b black

claim: $P_n(r,b) = \frac{r}{r+b}$ $\forall\, n, r, b \in \mathbb{N}$

1. 3 legs, 8 colors, 4 diameters

$S = \{(l, c, d) \mid l \in \{1,2,3\},\ c \in \{1\ldots8\},\ d \in \{1\ldots4\}\}$

$\Rightarrow |S| = 3 \times 8 \times 4$ by multiplication rule

2. $\underset{9}{\_}\ \underset{10}{\_}\ \underset{10}{\_}\ \underset{\uparrow}{\_}$ 5 or 0

Total nums divisible by 5
− Total nums divisible by 5 not cont 3
= Ans from subtraction rule

$\Rightarrow 9 \times 10^2 \times 2 - 8 \times 9^2 \times 2$

3. $2^2 \cdot 3^3 \cdot 5^5 \cdot 7^7$

so let $n = 2^a 3^b 5^c 7^d$

s.t $a \in \{0,1,2\}$,
$b \in \{0, \ldots 3\}$,
$c \in \{0, \ldots 5\}$
$d \in \{0, \ldots 7\}$

total $= 3.4.6.8$ by multiplication rule

4. 6 o
9 a

no of oranges $= \{0,1,2 \ldots 6\}$
no of apples $= \{0,1, \ldots 9\}$
total $= 7 \times 10 - 1$ ← No fruit case

by multiplication rule

5. $\_\ \_\ \_\ \_\ \_\ \_$

repeated symbols = total − no repeated symbols ← from subtraction rule

total $= (10+26)^6$

no repeated $= (10+26) \times (10+26-1)^5$

↑ option not to be repeated from multiplication principle

so total repeated $= 36^6 - 36 \times 35^5$

6. 0 to 1000 have only 1 digit = 5

Case I: $\_\ \_\ \underline{5}$ (9, 9)

Case II: $\_\ \underline{5}\ \_$ (9, 9)

Case III: $\underline{5}\ \_\ \_$ (9, 9)

} total 3×9^2 from multiplication and addition rule

7. $\_\ \_\ \_\ \_$

Total 4 digit pos. ints $= 9 \times 10^3$

4 digit pos. ints with no digit same $= 9 \times 9 \times 8 \times 7$ } multiplication rule

Total 4 digit pos. ints with atleast one digit rep $= 9 \times 10^3 - 9 \times 9 \times 8 \times 7$ from subtraction rule

8. 1000, 9999 odd nums s.t distint digits

$\_\ \_\ \_\ \_$ odd $\{1,3,5,7,9\}$

any → 8, any → 8, any → 7

by multiplicative rule $= 8 \times 8 \times 7 \times 5$

9. As no restriction, no of ways to choose S = no of ways to choose T

$S = \{(S,T) \mid S, T \subseteq [n]\}$

$T = \{(x_1, x_2 \dots x_n) \mid x_i \in \{0,1,2,3\}\ \forall i \in [n]\}$

then $f: S \to T$

$f(S,T) = (x_1, x_2 \dots x_n)$ s.t $x_i = 0$ if $i \notin S \cup T$

$x_i = 1$ if $i \in S,\ i \notin T$

$x_i = 2$ if $i \notin S,\ i \in T$

$x_i = 3$ if $i \in S \cap T$, $\forall i \in [n]$

and $f^{-1}(x_1, \dots x_n) = (S,T)$

when $S = \{i \mid x_i = 1 \text{ or } 3\}$

$T = \{i \mid x_i = 2 \text{ or } 3\}$

$\Rightarrow |S| = |T|$

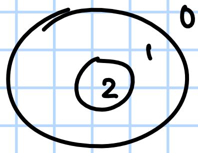
$\Rightarrow |S| = 4^n$ by multiplication rule as x_i has 4 choices and $i \in [n]$

10. $S = \{(S,T) \mid S \subseteq T \subseteq [n]\}$

$T = \{(x_1, x_2 \dots x_n) \mid x_i \in \{0,1,2\} \ \forall i \in [n]\}$

$f: S \longrightarrow T$

$f(S,T) = (x_1, \dots x_n)$ s.t $x_i = 0$ if $i \notin S, i \notin T$
$x_i = 1$ if $i \notin S, i \in T$
$x_i = 2$ if $i \in S, i \in T$



then $|T| = 3 \times 3 \times \dots 3$
$= 3^n$ from multiplication rule

and $f^{-1}: T \longrightarrow S$

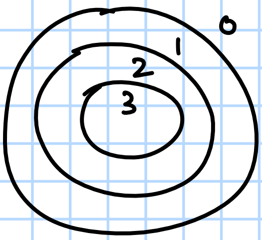
$f^{-1}(x_1 \dots x_n) = (S,T)$ s.t $S = \{i \mid x_i = 2\}$
$T = \{i \mid x_i = 1 \text{ or } 2\}$

$\Rightarrow |T| = |S|$
$\Rightarrow |S| = 3^n$

11. A, B, C

$S = \{(A, B, C) \mid A \subseteq B \subseteq C \subseteq [n]\}$

$T = \{(x_1, \dots x_n) \mid x_i \in \{0,1,2,3\} \ \forall i \in [n]\}$



now $f: S \longrightarrow T$

$f(A,B,C) = (x_1, \dots x_n)$
s.t $x_i = 0$ if $i \notin A \cup B \cup C$
$x_i = 1$ if $i \in C, i \notin A \cup B$
$x_i = 2$ if $i \in B \cap C, i \notin A$
$x_i = 3$ if $i \in A \cap B \cap C$

and $f^{-1}: T \longrightarrow S$

$f^{-1}(x_1 \dots x_n) = (A, B, C)$
s.t if $x_i = 3$ then $i \in A, i \in B, i \in C$
$x_i = 2$ then $i \notin A, i \in B, i \in C$
$x_i = 1$ then $i \notin A, i \notin B, i \in C$
$x_i = 0$ then $i \notin A, i \notin B, i \notin C$

so, $|T| = |S| \Rightarrow |S| = 4^n$ as x_i has 4 choices, then by multiplication rule

12. $(n+1)$ $[2n]$

$\left.\begin{matrix}(1,2)\\(3,4)\\ \vdots \\(2n-1,2n)\end{matrix}\right\}$ n boxes $n+1$ differt integers by PHP $\exists$ one box s.t $(i, i+1)$ selected
so, $\exists$ integers s.t gcd = 1

13. differ by atmost 2

let $S_i = (3i+1, 3i+2, 3i+3)$
for $i = 0, \dots n-1$

then n sets and by PHP $\exists i \in \{0, \dots n-1\}$ s.t
2 integers from set i
as elements differ by atmost 2 in S_i, this will be true.

14. To prove: $\exists k, l$ $0 \leq k < l \leq m$ s.t $(a_{k+1}, \dots a_l) \bmod m = 0$ where $a_i \in \mathbb{Z} \ \forall i \in [m]$

proof: let

$r_i = \sum_{j=0}^{i} a_j$ then as $i \in [m]$, m differt r_i exist

also $r_i \bmod m \in \{0,1,\ldots m-1\}$ from defn

if $\exists\, i \in [m]$ s.t

$r_i \bmod m = 0$ then $k=0$
$l = i$

if not then as m numbers but $\{1,\ldots m-1\}$ choices

$\exists\, i, j \in [m]$ s.t $i \neq j$

$r_i \bmod m = r_j \bmod m$ (PHP)

wlog $i > j$ then

$(r_i - r_j) \bmod m = 0$

$\Rightarrow (a_0 + a_1 + \cdots a_i) - (a_0 + \cdots a_j) \bmod m = 0$

$\Rightarrow \sum_{k=j+1}^{i} a_k \bmod m = 0$

so $\exists$ $k = j$, $l = i$

15. To prove: $a_i \in \mathbb{Z}$, $i \in [n+1]$, $\exists$ $i \neq j$, $i, j \in [n+1]$ s.t $|a_i - a_j| \bmod n = 0$

proof:

now let $a_i \bmod n = r_i$

then

$r_i \in \{0,1\ldots n-1\}$ but $i \in [n+1]$

so by PHP $\exists\, i \neq j$ s.t $r_i = r_j$

$\Rightarrow a_i \bmod n = a_j \bmod n$

$\Rightarrow |a_i - a_j| \bmod n = 0$

16. 42 days 70 Hours atleast one per day

x_i = no of hours studied on i^{th} day

$i \in [42]$

$x_i \geq 1 \;\forall\, i \in [42]$

$\sum_{i=1}^{42} x_i = 70$

now let $y_i = \sum_{j=1}^{i} x_j$ then

$y_1 = x_1$, $y_2 = x_1 + x_2$, $\ldots$ $y_{42} = \sum_{i=1}^{42} x_i = 70$

$\Rightarrow 1 \leq y_1 < y_2 < y_3 \cdots < y_{42} = 70$

now, $1 + 13 \leq y_1 + 13 < y_2 + 13 < y_3 + 13 \cdots < y_{42} + 13 = 83$

$\Rightarrow 14 \leq y_1 + 13 < y_2 + 13 < \cdots < y_{42} + 13 = 83$

let $S_1 = \{y_1, y_2 \ldots y_{42}\}$

$S_2 = \{y_1 + 13, y_2 + 13, \ldots y_{42} + 13\}$ s.t No element same in S_1 or S_2

as $S_1 \subseteq [83]$, $S_2 \subseteq [83]$

$|S_1| = 42$, $|S_2| = 42$ so, $\exists\, x \in S_1$, $y \in S_2$ s.t $x = y \Rightarrow y_i = 13 + y_j$

$\Rightarrow y_i - y_j = 13$

so consecutive b/w j and i days she studies 13 hours

17. 101 ints from [200], so let

n_i be chosen $\forall i \in [101]$
then let k_i be s.t
$n_i = 2^{k_i} a_i$ where $a_i \bmod 2 = 0$
(from fundamental theorem of arithmetic)

then $a_i \in \{1, 3, 5, \ldots 199\} = \{2k-1 \mid k \in [100]\}$

now as $i \in [101]$ but a_i has 100 options
$\exists i \neq j$ s.t
$a_i = a_j$ from PHP

$\Rightarrow n_i 2^{-k_i} = n_j 2^{-k_j}$
wlog $k_i > k_j$

$\Rightarrow n_i = 2^{k_i - k_j} n_j$

$\Rightarrow n_i \bmod n_j = 0$

so, $\exists$ a pair s.t one divides another

18. $x_i \in \mathbb{N}$, $i \in [n]$, $\sum_{i=1}^{n} x_i - n + 1$ no of balls into n boxes

let y_i = no of balls in i^{th} box

$\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} x_i - n + 1$

To prove: $\exists i \in [n]$ s.t $y_i \leq x_i$

proof: $n-1 = \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} y_i$

$\Rightarrow n-1 = \sum_{i=1}^{n} (x_i - y_i)$

now as $x_i \in \mathbb{N}$, $x_i - y_i \in \mathbb{Z}$
and also as

$n-1 \geq 0$ $\sum_{i=1}^{n} (x_i - y_i) \geq 0$

so by PHP $\exists i$ s.t $x_i - y_i \geq 0$ (as all neg, then sum<0)
$\Rightarrow x_i \geq y_i$

19. To prove: $\exists$ number $n \in \{7, 77, \ldots\}$ divisible by 17
proof:

lets take first 17 number in seq

$\{7, 77, 777, \ldots \underbrace{77\ldots7}_{17 \text{ times } 7}\}$

now let $a_i = \underbrace{77\ldots7}_{i \text{ times}}$

then $a_i \bmod 17 \in \{0, 1 \ldots 16\}$

now if $\exists i \in [17]$ s.t

$a_i \bmod 17 = 0$ then we are done

or else, $\exists i \neq j$ s.t

$a_i \bmod 17 = a_j \bmod 17$ (PHP)

wlog $i > j$ then

$(a_i - a_j) \bmod 17 = 0$

$\Rightarrow (\underbrace{77\ldots7}_{i \text{ times}} - \underbrace{77\ldots7}_{j \text{ times}}) \bmod 17 = 0$

$\Rightarrow (a_{i-j} \times 10^j) \bmod 17 = 0$

as $\gcd(10^j, 17) = 1$

$\Rightarrow a_{i-j} \bmod 17 = 0$

$\Rightarrow \exists\ \underbrace{77\ldots7}_{i-j} \bmod 17 = 0$

20. (1,1) . . . .(1,82)

(2,1) . . . . (2,82)

(3,1) . . . . (3,82)

(4,1) . . . . (4,82)

now no of rows = 4

no of colums = 82

then total diffent colums = 3.3.3.3 = 81 (multiplication rule)

then $\exists i \neq j$ from PHP s.t

$\text{column}_i = \text{column}_j$

i.e $(1,i) = (1,j)$

$(2,i) = (2,j)$

$(3,i) = (3,j)$

$(4,i) = (4,j)$

also as 3 colors $\exists\ a \neq b$ from PHP s.t

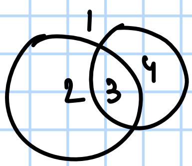
$(a,i) = (b,i)$ color

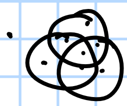
so, $(a,i), (a,j), (b,i), (b,j)$ same colors

21. n mutually overlapping circles

$C_1, C_2, \ldots C_n$

for $n=1$  2 regions

$n=2$  4 regions

$n=3$  8 regions let inside region be s.t point $p=(x,y)$

$$C_i(P) \leq 0$$

outside region: $C_i(P) > 0$

then every point P on plane has 2 options per C_i

$\Rightarrow$ total $(2)^n$ options

$\Rightarrow 2^n$

22. b black balls
r red balls

$n=1$ then $P_1(\text{red ball}) = \dfrac{r}{b+r}$

$n=2$ $P_2(r) = P(r \mid \text{prev } r) \times P(\text{prev } r) + P(r \mid \text{prev } b) \times P(\text{prev } b)$

$$= \frac{r}{b+r} \times \frac{r+c}{r+b+c} + \frac{r}{r+b+c} \times \frac{b}{r+b}$$

$$= \frac{r\,[\cancel{b+r+c}]}{b+r\,(\cancel{r+b+c})} = \frac{r}{b+r}$$

to prove: $P_n(r,b) = \dfrac{r}{r+b}$

proof: $n=1$ true, then if true for k then

$$P_{k+1}(r,b) = \frac{\text{no of red}}{\text{total no of balls}}$$

let no of red $= r'$ black $= b'$ at k^{th} pick

$$\frac{r'}{r'+b'} = \frac{r}{r+b} \quad \text{or} \quad \frac{r'}{r} = \frac{b'}{b}$$

now, $P_{k+1}(r,b) = P(\text{picking } r \mid r \text{ picked in } k^{th}) \times P(r \text{ picked in } k)$
$+ P(\text{piking } r \mid b \text{ picked in } k^{th}) \times P(b \text{ picked in } k)$

$$= \left(\frac{r'+c}{r'+b'+c}\right)\left(\frac{r}{r+b}\right) + \left(\frac{r'}{r'+b'+c}\right)\left(\frac{b}{r+b}\right)$$

$$b = \frac{r\,b'}{r'}$$

$$= \left(\frac{r'+c}{r'+b'+c}\right)\left(\frac{r}{r+b}\right) + \left(\frac{1}{r'+b'+c}\right)\frac{(r\,b')}{r+b}$$

$$= \frac{r}{r+b}\left[\frac{\cancel{r'+c+b'}}{\cancel{r'+b'+c}}\right]$$

$$P_{k+1}(r,b) = \frac{r}{r+b}$$

so, by induction true for $k+1$

$$\forall n \quad P_n(r,b) = \frac{r}{r+b}$$

Tutorial-2:

1. product rule

[7]

_ _ _ _ _ _ _

$\uparrow\ \uparrow$

3 6 . . . . . .

$6! \times 3$

2. Inclusion / Exclusion

3. tie up 1 and then permute

$\{1,1,2,2,3,4,5\}$

1 2 3 4 5 6

$\frac{6!}{2!}$

4. first & last element is diff = total − first & last same

$$\frac{7!}{2!\cdot 2!} \qquad \frac{6!}{2!}+\frac{6!}{2!} = 6!$$

5. _ 10 10 10 10 _

o/e \quad o/e

$\{\cancel{1},2,\cancel{3},4,\cancel{5},6,\cancel{7},8,\cancel{9}\}$

$*\ 5\,(5\times 5 + 4\times 5)\,10^4$

$X = X_1 \sqcup X_2$

o/...o \quad e/...e

6.

$\underline{a}\ \underline{b}\ 6\ \underline{d}\ \underline{e}$

$a+b+d+e \pmod 3 \equiv 0$

10 10 10

a \quad 3 6 5 7 \quad if $(b,d,e)=(3,5,7)$ $\rightarrow 6 \rightarrow a = (3$ choices$)$

divisible by 3

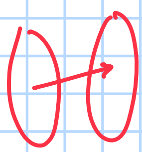
has to be divisible by 3 $\rightarrow$ 3 choices

$b+d+e \bmod 3 = (-a) \bmod 3$

3 choices

$10.10.10.3 = 3000$ ways

7. $f:[n]\rightarrow[n]$ $\exists P \in [n]$ $f(i)=i$



$f:[n-1]\rightarrow[n-1]$

$i \rightarrow n$ choices the fixed point then functions from $n-1 \rightarrow n-1$

$(n-1)^{n-1}$

total: $(n)\times(n-1)^{n-1}$

8. $[n] \longrightarrow [n]$
not one-one = total − (one-one)

9. ◯ - - - ◯
↓
when is odd → o e
oeoe . . . oe . .
eoeo . . . .

$2n \rightarrow n$ odd, n even: $\frac{n!\cdot n! \times 2}{2n}$ or $2n+1 \rightarrow n+1$ odds, $\rightarrow n$ even: $\frac{(n+1)!\,n!}{2n+1}$

10. (a) $2^n \rightarrow$ total grouping

$\binom{n}{k}$ choosing k from n

$$\sum \binom{n}{k} = 2^n$$

(b) $k\binom{n}{k} \rightarrow$ choose 1 from $\binom{n}{k}$

$n\binom{n-1}{k-1}$ ⟼ $\times \binom{n-1}{k-1}$
choose any one from n

(c) LHS = choose k from $[n]$
RHS = n present choose $k-1$ from $[n-1]$
+ n not present choose k from $[n-1]$

(d) $\sum k\binom{n}{k} = \sum n\binom{n-1}{k-1} = n\sum\binom{n-1}{k-1} = n2^{n-1}$

(e) $k^2\binom{n}{k}$ k = choose president
k = choose president

$n \times 2^{n-1} + n(n-1)2^{n-2}$
$= (n^2 - n + 2n)\,2^{n-2}$
$= (n)(n+1)\,2^{n-2}$

(g) $\binom{n}{k}$ = no of binary strings of len n and no of ones $= k$

$\sum_{j=0}^{k} \binom{n}{j}$ = no of binary strings of wt $\leq k$
= # of n binary strings with atleast $n-k$ '0's

(h) $\sum a_1 \binom{n}{a_1 a_2 a_3} = n3^{n-1} \longrightarrow$ one leader, 3 groups
choosing leader among 3 groups

11. $\sum_{k=0}^{n} 3^k 2^{n-k}\binom{n}{k} = 1 + \sum_{k=0}^{n} 3^k 2^{n-k}\binom{n}{k}$
$n-k$ even $\qquad n-k$ odd

$1 = (3-2)^n$ combinatorially

12. $\sum_{a_1+a_2+a_3=n} (-1)^{a_2} \binom{n}{a_1 a_2 a_3}$ where $(x_1+x_2+x_3)^n = \sum_{a_1+\dots+a_3=n} \binom{n}{a_1 \dots a_3} x_1^{a_1} x_2^{a_2} x_3^{a_3}$

$$(x - x + 1)^n = \sum (-1)^{a_2} \binom{n}{a_1\, a_2\, a_3}$$

$$\Rightarrow 1^n = \sum (-1)^{a_2} \binom{n}{a_1 \cdots a_3}$$ combinatorial through bijection

Tutorial-3:

13. Lex order: 75 base 2

$75 = 64+8+2+1$
$= 2^6+2^3+2^1+2^0$
$= 01001011$

$\exists$ bijection from binary to set

$75 = 01001011 \Leftrightarrow \{1,2,4,7\}$
87654321

$\{1,2,6\} \leftrightarrow 00100011 = 35$
so $\{1,2,6\} < \{1,2,4,7\}$ as $35 < 75$

$\{1,2,7\} > \{1,3\}$
largest   same

$\{1,2,7\} < \{1,3,7\}$
same   larger

$\{a_1 \dots a_k\} \subseteq [n] \quad a_i \in \mathbb{N}$

picking k from $\{a_k+1, \dots n\}$ so, $\binom{n-a_k}{k}$

or $k-1$ from $\{a_{k-1}+1 \dots, n\}$ and compulsory picking a_k $\binom{n-a_{k-1}}{k-1}$

14. $k,n \in \mathbb{N}$, $k<n$ $\quad m \in \{0, \dots \binom{n}{k}-1\}$

$f: \{0, \dots \binom{n}{k}-1\} \longrightarrow$ subsets of size k of $[n]$

$f(m) = \{i \mid a_i = 1, m \text{ base } 2 = a_n a_{n-1} \cdots a_1, \ a_i = 1, \text{or } 0\}$

15. 15 first part not 1

$x_1 + \dots x_k = 15$

$x_i \geq 1, x_1 \geq 2$

so for k: $\binom{n-1}{k-1} = \binom{14-1}{k-1} = \binom{13}{k-1}$

$k = 1, 2, \dots .14$

$\sum_{k=1}^{14} \binom{13}{k-1} = 2^{13}$

17. $x_1+x_3+x_3+x_4 = n$

$\Rightarrow y_1+y_2+y_3+y_4 = n-4$
$\Rightarrow \binom{n-4-1}{4-1} = \binom{n-5}{3}$

18. $x_1 + \dots x_5 = 10$ ← choose which 2 are zero
$\binom{5}{2}\binom{10+3-1}{3-1} = \binom{5}{2}\binom{12}{2}$

19. $[10]$ 1 not in any block

$\{2, \dots, 10\}$ any partition so $\sum_{k=1}^{9} k\, S(9,k) =$ total   (k partitions)

$B_{10} - B_9 =$ total no of partitions − singleton 1 block ∪ 9 blocks

20. $S(n,k)$ $[n]$ in k blocks

$[n]$ — A^c

k-subset of A → Double counting

$$\frac{2^n - 2}{2} = S(n,2)$$

$$S(n, n-1) = \binom{n}{2}$$

choose any 2 then all singletons

$S(n, n-2)$

at end 2 sets left

$$(n-2)(n-3)\binom{n}{2}$$

1. [7] even if first pos

$2, 4, 6$

$3 \times 6 \times 5 \times \cdots 1 = 3 \times 6!$

2. [8] $A_2 = 2$ in natural position

$|A_2| = 7!$ $|A_{2n}| = 7!$ $n = 1, 2 \dots 4$

$A_2 \cap A_4 = 6!$

now, $\cup A_{2i}$ = atleast one even integer in natural pos

$\Rightarrow (\cup A_{2i})^c$ = no even integer in natural pos

$$\Rightarrow |\text{total}| - |\cup A_{2i}| = 8! - \left[\binom{4}{1}7! - \binom{4}{2}6! + \binom{4}{3}5! - \binom{4}{4}4!\right]$$

3. $\{1, 1, 2, 2, 3, 4, 5\}$

$\{1^*, 2, 2, 3, 4, 5\}$

$$\frac{6!}{2!}$$

4. $\{1, 1, 2, 2, 3, 4, 5, 6\}$

\# first and last element same $= \cancel{2} \times \frac{6!}{\cancel{2!}} = 6!$

\# total $= \frac{7!}{2!\,2!}$

$|(\text{first \& last same})^c| = \frac{7!}{2!\,.\,2!} - 6! = |\text{first \& last diff}|$

5. _ _ _ _ _ _

o/e o/e

$\{1,3,5,7,9\}$ $\{2,4,6,8\}$ $\{0,2,4,6,8\}$

$\underbrace{5\times5\times10^4}_{\text{odd}} + \underbrace{4\times5\times10^4}_{\text{even}} = |X_1 \sqcup X_2|$

X_1 → first & last odd

X_2 → first & last even

6. a b 6 c d

$a+b+6+c+d \,\%\, 3 = 0$

now $\Rightarrow a+b+c+d \,\%\, 3 = 0$

$\Rightarrow a \,\%\, 3 = \underbrace{-b-c-d}_{\text{any option}} \,\%\, 3$

then
- 0 → 3, 6, 9
- 1 → 1, 4, 7
- 2 → 8, 5, 2

options for a

b – 10 options, c – 10 options, d - 10 options

a → 3 options ⇒ $10\times10\times10\times3$

7. $f:[n] \longrightarrow [n]$

exactly one i s.t. $f(i)=i$ for any i

now $\binom{n}{1}$ choose i

and then $\tilde{f}:[n]\setminus\{i\} \longrightarrow [n]\setminus\{i\}$

s.t.

$f(i)=i$, $k \neq i$

$f(k)=\tilde{f}(k)$

$\tilde{f}$ in $(n-1)^{n-1}$ ways

& so $\binom{n}{1}(n-1)^{n-1}$ = total ways

8. $f:[n] \longrightarrow [n]$

one-one maps $= n \times n-1 \times \cdots \times 1$

n options for 1 — n-1 options for 2

$= n!$

now, total $= n^n$

total − (one-one) $= n^n - n!$

9. $[n]$ odd sum every consecutive

o e . . . . . . _

e o . . . . . . _

now, if n is even: o e o e .... o e → $((n/2)(n/2-1)\cdots1)^2$

e o e o .... e o → same

so $2\times\left(\left(\frac{n}{2}\right)!\right)^2$ for case of n even

n is odd:

o e o e . . . . . o

e o e o . . . . . e o e $\rightarrow$ not possible as more odd then even

so o e o . . . o case

o: $\frac{n+1}{2}$ choices

e: $\frac{n-1}{2}$ choices

$\left(\frac{n+1}{2}\right)!\left(\frac{n-1}{2}\right)!$

10. (a) $\sum_{k=0}^{n}\binom{n}{k}=2^n$

2^n = total groups s.t we choose someone or not from $[n]$ per person $i \rightarrow 2$ options so $2\times2\ldots\times2=2^n$

$\binom{n}{k}$ = choosing any k from $[n]$

$\sum_{k=0}^{n}\binom{n}{k}$ = choosig $0/1/2\ldots/n$ same as no of groups $=2^n$

(b) $k\binom{n}{k}$ = choosing k people from n people class and then choosing a leader

$n\binom{n-1}{k-1}$ = choosing a leader from n peoile, and then addig $k-1$ people as team from $n-1$

$\Rightarrow k\binom{n}{k}=n\binom{n-1}{k-1}$

(c) $\binom{n}{k}$ = choose k from $[n]$

2 cases: if n present then choose $k-1$ from $n-1$

if n not present then choose k from $n-1$

$\Rightarrow \binom{n}{k}=\binom{n-1}{k-1}+\binom{n-1}{k}$

(d) $n\times2^{n-1}$ = one caption choose (compulsory in group) $\times$ any group formation from rest $n-1$

so $n\times2^{n-1}$

Similarly chose k from n ($k\geq1$) and choose a caption

$\sum_{k=1}^{n}k\binom{n}{k}=n\times2^{n-1}$

(e) choose k from n ($k\geq1$) and assign one caption & one president (can be same) so

$\sum_{k=1}^{n}k\times k\times\binom{n}{k}$ = total possible groups now, if caption = president then n ways $\times 2^{n-1}$ groups

caption $\neq$ president then $(n)(n-1) \times 2^{n-2}$

$$\text{total} = (n)2^{n-1} + (n)(n-1)2^{n-2}$$

$$= \cancel{2}n\, 2^{n-2} + (n^2 - \cancel{n})2^{n-2}$$

$$\sum_{k=1}^{n} k^2 \binom{n}{k} = (n)(n+1)2^{n-2}$$

(f) given $n \geq 2$

$2^{n-2}(n)(n-1)$ = choose a president & vice president then make any group of any no of people

$(k)(k-1)\binom{n}{k} \to$ group of k people $k \geq 2$

$$\Rightarrow \sum_{k=2}^{n} (k)(k-1)\binom{n}{k} = (n)(n-1)2^{n-2}$$

(g) $0 \leq k \leq n-1$

$\binom{n}{k}$ = no of binary $\underbrace{\text{— — . . —}}_{n \text{ length}}$ strings with $1^s = k$

$\sum_{k=0}^{k} \binom{n}{k}$ = no of binary strings len n, $1^s \leq k$

$1^s + 0^s = n \Rightarrow 0^s = n - 1^s \geq n-k$

= no of binary strings of len n, $0^s \geq n-k$

$0's = \underbrace{n-k,\ n-k+1,\ n-k+2 \ldots n}_{\text{total cases}}$

$1^s \leq k$

fix 0 $\quad k=3$

$$\begin{array}{cccc|l} \multicolumn{4}{c}{k-i} & \\ 1 & 1 & 1 & 0 & i=0 \\ 1 & 1 & 0 & * & i=1 \\ 1 & 0 & * & * & i=2 \\ 0 & * & * & * & i=3 \end{array}$$

$\vdash\!\!\text{———}\!\!\dashv$ $n-k-1$

$A_i = \{$ $n-k-1$ zeros first part of bin string, then $k-i$ ones & then 1 0 then i random $\}$

no of zeros $\geq n-k$

also $\cup A_i = \{0^s \geq n-k\} \wedge \{\underbrace{\bigcup_{i=0}^{k}(n-i-1)^{th} \text{ zero}}\}$ all sets as $0^s \geq 1$ by PHP

$$\Rightarrow \cup A_i = \{0^s \geq n-k\}$$

& $A_i \cap A_j = \phi$ as last $k+1$ string is unique for all cases

$$\text{so } \sum_{j=0}^{k} \binom{n}{j} = \sum_{j=0}^{k} \binom{n-1-j}{k-j} 2^j$$

(h) $\binom{n}{a_1}$ = choose a_1 from n

$\binom{n-a_1}{a_2}$ = choose a_2 from left

$\binom{n-a_1-a_2}{a_3}$ = choose a_3 from left

$a_1\binom{n}{a_1}\binom{n-a_1}{a_2}\binom{n-a_1-a_2}{a_3}$ = make 3 groups s.t from group 1 a leader is chosen

now $a_1+a_2+a_3=n$

$$\Rightarrow a_1\binom{n}{a_1}\binom{n-a_1}{a_2}\binom{n-a_1-a_2}{a_3} = a_1\binom{n}{a_1,a_2,a_3}$$

and $\displaystyle\sum_{a_1+a_2+a_3=n} a_1\binom{n}{a_1,a_2,a_3}$

if $a_1=0$ then the whole term becomes 0

so $\displaystyle\sum_{a_1+a_2+a_3=n} a_1\binom{n}{a_1,a_2,a_3} = \sum_{\substack{a_1+a_2+a_3=n \\ a_1\geq 1}} a_1\binom{n}{a_1,a_2,a_3} = \underbrace{n}_{\text{make a leader from } n \text{ of group 1}}\times \underbrace{3^{n-1}}_{\text{then } 3^{n-1} \text{ ways}}$

11. $1^n = (3-2)^n$

$= \underbrace{(3-2)\times(3-2)\cdots(3-2)}_{n \text{ times}}$

$= \displaystyle\sum_{k=0}^{n} \underbrace{3^k}_{\text{choose } k \text{ many } 3}\underbrace{(-2)^{n-k}}_{\text{choose rest } 2} \times \underbrace{\binom{n}{k}}_{\text{choose } k \text{ 3's from } n}$

$$1^n = 1 = \sum_{k=0}^{n}\binom{n}{k}3^k2^{n-k}(-1)^{n-k}$$

$$\Rightarrow 1+\sum_{\substack{k=0 \\ n-k=\text{odd}}}^{n} 3^k2^{n-k}\binom{n}{k} = \sum_{\substack{k=0 \\ n-k=\text{even}}}^{n} 3^k2^{n-k}\binom{n}{k}$$

12. $(1-1+1)^n = \underbrace{(1-1+1)\times(1-1+1)\cdots\times(1-1+1)}_{n \text{ times}}$

$= \displaystyle\sum_{a_1+a_2+a_3=n} \underbrace{\binom{n}{a_1\,a_2\,a_3}}_{\text{ways of making 3 groups from } n} \times (1)^{a_1}(-1)^{a_2}(1)^{a_3}$

choose a_1 many 1

a_2 many -1

a_3 many 1 (second one)

$$\Rightarrow 1 = \sum_{a_1+a_2+a_3=n}\binom{n}{a_1,a_2,a_3}(-1)^{a_1}$$

13. $\{x_1 \dots x_k\}$ of $[n]$

$x_1 < x_2 \dots < x_k$
$A = \{a_1 \dots a_k\}$
$B = \{b_1 \dots b_k\}$

no of k subsets succeeding k

$a_1 < a_2 < \dots < a_k$

$A_1 = \{$all B's s.t all elems in it $> a_1\}$

$A_2 = \{$all B's s.t one a_1 all otr $> a_2\}$

⋮

$A_k = \{$all B's s.t $a_1, a_2 \dots a_{k-1}$ in B all others $> a_k\}$

all $A_i \cap A_k = \phi$ (trivial)
& $\cup A_i$ is required
$= \binom{n-a_1}{k} + \binom{n-a_2}{k-1} + \dots + \binom{n-a_k}{1}$

14. $k, n \in \mathbb{N}$ $k < n$ $m \in \{0, \dots \binom{n}{k} - 1\} = M$

let $B = \{b_1, b_2 \dots b_k\}$ s.t $b_i \neq b_j$, $i \neq j$
and $b_i \in [n]$
i.e $\binom{n}{k}$ such ways to convert B

now let $n' = \mathbb{I}_1 \times 2^{1-1} + \mathbb{I}_2 \times 2^{2-1} + \mathbb{I}_3 \times 2^{3-1} + \dots \mathbb{I}_n \times 2^{n-1}$

s.t $\mathbb{I}_i = \begin{cases} 1; & i \in B \\ 0; & i \notin B \end{cases}$

the above is a function
$n'(B) =$ some number unique to every B
as if $n'(B) = n'(C)$
then

$\mathbb{I}_{1_B} \times 2^0 + \mathbb{I}_{2_B} \times 2^1 + \dots \mathbb{I}_{n_B} \times 2^{n-1}$
$= \mathbb{I}_{1_C} \times 2^0 + \mathbb{I}_{2_C} \times 2^1 + \dots \mathbb{I}_{n_C} \times 2^{n-1}$

taking mod 2
$\Rightarrow \mathbb{I}_{1B} = \mathbb{I}_{1C}$

then div by 2 taking mod again and repeating

$\Rightarrow \mathbb{I}_{i_B} = \mathbb{I}_{i_C} \; \forall i \Rightarrow B = C$

so, n' is a unique map from $\binom{n}{k}$ such groups to a set of numbers of cardinality $\binom{n}{k}$

now that we have a map from $\{(a_1, a_2 \dots a_k) \mid 1 \leq a_1 < a_2 < \dots < a_k \leq n\} = S$
to $\{b_1 \times 2^0 + b_2 \times 2^1 + \dots b_n \times 2^{n-1} \mid b_i = 0$ or 1 & only k many $b_i = 1\} = R$

$|S| = |R|$ as there is a bijection shown

now if we pick any $b \in R$, $b = b_1 \times 2^0 + b_2 \times 2^1 + \dots + b_n \times 2^{n-1}$

now as $|B| = \binom{n}{k}$

let total nubes more then it be m

then

$m \in \{0, 1, \ldots, \binom{n}{k} - 1\}$

now let $C = \{c \in R \mid b < c\}$ then $|C| = m$

now, as b corresponds to a particular group let it be

$\{b_1', b_2', \ldots b_k'\}$ then any group with lex compare more will have a wider corresp in C

so $|C| = \binom{n-b_1'}{k} + \ldots + \binom{n-b_k'}{1}$ from prev question

15. first part is not one

$\sum x_i = 15 \quad x_i \geq 1, x_1 \geq 2$

$\Rightarrow \sum x_i = 14$

$x_i \geq 1$

$\Rightarrow \binom{14-1}{k-1}$

$$\sum_{k=1}^{14} \binom{14-1}{k-1} = \sum_{k=0}^{13} \binom{13}{k} = 2^{13}$$

16. no of comp of $n = \sum_{k=1}^{n} \binom{n-1}{k-1} = 2^{n-1}$

first part $= 2$

then $\binom{n-2-1}{k-1}$ compositions

$$\sum_{k=1}^{n-2} \binom{n-3}{k-1} = 2^{n-3}$$

so total $= 2^{n-1} - 2^{n-3}$

17. $\sum_{i=1}^{4} x_i = n$ s.t $x_i \geq 2$

$\Rightarrow \sum y_i = n - 1 \times 4$

$\Rightarrow \binom{n-4-1}{4-1} = \binom{n-5}{3}$

5

18. $\sum_{i=0}^{10} x_i = 10$ s.t $x_i \geq 0 \; \forall i$

$\exists i, j \; i \neq j$ s.t $x_i = 0, x_j = 0$

so $\binom{5}{2}$ ways to choose i & j

then $x_1' + x_2' + x_3' + 0 + 0 = 10$

$\Rightarrow \binom{10+3-1}{3-1}$ for weak comps

total $= \binom{5}{2} \times \binom{12}{2}$

19. [10] $S(n,k)$ = total partions of n of size k

5 (1,1,2) (2,1,1) both are partition

now, $\sum_{k=1}^{n} S(n,k) = B(n) \rightarrow$ Bell number

now $B(10)$ = total partions of 10

$B(9)$ = total partions of 9 or {1} is seperated and then
other 9 made into partions

$\Rightarrow B(10) - B(9) = \text{total}$

20. $s(n,k)$ = no of partitions of $[n]$ into k parts, closed formula of
$S(n,2)$ and $s(n,n-2)$ for $n \geq 2$

now, $s(n,2) = \left(\underbrace{\binom{n}{0}}_{}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}-1-1\right)/2$

→ choose 1

removing $\binom{n}{0}+\binom{n}{n}$

the above are choosing k for
group 1 and rest for other group
but as groups not unique divide by 2

$$s(n,2) = \frac{2^n-2}{2} = 2^{n-1}-1$$

now, $s(n,n) = 1$
$s(n,n-1) = |\{\text{choose any 2 and keep them together, rest singletons}\}|$

$$= \binom{n}{2}$$

$s(n,n-2) = |\{\text{choose 3 keep them together, rest all singletons}\}$

$\sqcup$ {choose 2, keep them differently with other $n-2$ singletons }|

$$s(n,n-2) = \binom{n}{3} + \binom{n}{2} \times (n-2) \times (n-3)$$

Tutorial-4:

1. case I: $n=2$, $k=1 \Rightarrow f(2)=0$
$k=2 \Rightarrow f(2)=1$

case II: $n \geq 3$
$k=1 \quad f(n)=0$
$k=n \quad f(n)=1$
$k=n-1 \quad f(n) = \overbrace{\underbrace{\binom{n}{2}}_{S(n,n-1)}}^{\text{total cases}} - \underbrace{1}_{\text{case of } \{1,2\}}$

$k \in \{2,\dots,n-2\}$

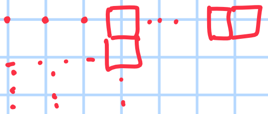
$$f(n) = \underbrace{S(n,k)}_{\text{total cases}} - \underbrace{S(n-2,k)}_{\substack{n-2 \text{ numbers} \\ \text{removing } \{1,2\}}} \times \underbrace{k}_{\substack{\text{select one out of } k \\ \text{to place } \{1,2\}}} - \underbrace{S(n-2,k-1)}_{\{1,2\} \text{ is seperate}}$$

remark: $P(n)-P(n-1)$ = no of partions of n with last part ≥ 2

3. last row $=2$ $\quad a_1 \geq a_2 \dots \geq \underbrace{a_{k-1}}_{\text{this is } 2} \geq 2$

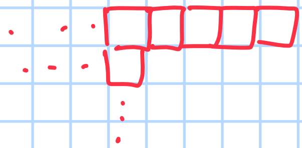
so $P(n-2) - \underbrace{P(n-3)}_{\text{atleast } 2\ a_{k-1} \text{ case}}$

5. a)



if we take conjugate last 2 rows-entry 1
so $\underbrace{a_1 \geq a_2 \dots \geq a_{k-2}}_{\text{any}} \geq 1 \geq 1$
$\Rightarrow P(n-2)$

b)



$P(n-4) - P(n-5)$

7. To prove: $P(n)^2 < P(n^2+n)$
proof: case I: $n=1$
$P(1)^2 = 1 \quad P(1+1)=2$
$\Rightarrow 1 < 2$

case II: $n \geq 2$
$\tilde{P}(n) = \{(a_1,\dots,a_k) \mid \sum_{i=1}^{k} a_i = n,\ a_1 \geq \dots \geq a_k \geq 1,\ k \in [n]\}$

consider $f: \tilde{P}(n) \times \tilde{P}(n) \to \tilde{P}(n^2+n)$
given by $f((a_1,a_2\dots a_k),(b_1,b_2\dots b_l)) = (na_1,\dots na_k, b_1,\dots b_l)$

f is one-one

8. 8 hours of classes
let x_i = no of hours of i^{th} day

$$\sum_{i=1}^{5} x_i = 8 \quad \text{s.t } x_i \geq 0,\ x_4 > x_5$$

if $x_5 = 0$ then $x_4 \geq 1$ & so $\tilde{x}_4 = x_4 - 1 \geq 0$

$\Rightarrow x_1 + x_2 + x_3 + \tilde{x}_4 = 7$

total $\binom{n+k-1}{k-1} = \binom{7+4-1}{4-1} = \binom{10}{3}$

if $x_5 = 1 \Rightarrow x_4 \geq 2$ $\quad \tilde{x}_4 = x_4 - 2$

so, $x_1 + x_2 + x_3 + \tilde{x}_4 + 1 = 8 - 2$

$$\Rightarrow x_1 + x_2 + x_3 + \tilde{x}_4 = 5$$

$$\Rightarrow \binom{5+4-1}{4-1} = \binom{8}{3}$$

similarly $x_5 = 2 \Rightarrow \binom{6}{3}$

$x_5 = 3 \Rightarrow \binom{4}{3}$

By addition priciple we will get $\displaystyle\sum_{k=2}^{5} \binom{2k}{3}$

9. To prove: $B(n) = F(n) + F(n+1)$

proof:

$S = \{B \mid B$ is a partition of $[n]$ with singletons$\}$

$B(n) - F(n) = |S|$

Suffices to show

$F(n+1) = |S|$

consider $f : S \longrightarrow F(n+1)$

$f(\{a_1\}, \{a_2\} \dots \{a_k\}, B_2, B_3, \dots B_l) = (B_1, \dots B_l)$

$B_1 = \{a_1, a_2 \dots a_k, n+1\}$

$f^{-1}(\{a_1 \dots a_k, n+1\}, B_2, \dots B_l) = (\{a_1\}, \dots \{a_k\}, B_2, \dots B_l)$

10. a_n = no of compositions of n into parts larger than 1

$$a_n = a_{n-2} + a_{n-1}$$

11. $S_n = \{\sigma : [n] \longrightarrow [n] \mid \sigma$ is bijective$\}$

$|S_n| = n!$ $\quad D_n = \{\sigma \in S_n \mid \sigma(i) \neq i \ \forall i \in [n]\}$

$\hookrightarrow$ Dearrangements of S_n

$A_i = \{\sigma \in S_n \mid \sigma(i) = i\}$

$$D_n = S_n - \bigcup_{i=1}^{n} A_i$$

$$\Rightarrow |D_n| = |S_n| - |\bigcup_{i=1} A_i|$$

$$= |S_n| - \left[\sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_k \leq n} |A_{i_1} \cap \dots \cap A_{i_k}|\right]$$

$$= n! - \left[\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (n-k)!\right]$$

$$= n! + \sum_{k=1}^{n} (-1)^k \frac{n!}{k!}$$

$\Rightarrow |D_n| = \displaystyle\sum_{k=0}^{n} (-1)^k \frac{n!}{k!}$ $\quad$ so, $|D_n|^2$ is what we would get

13. 12 $S = \{4.a, 3.b, 4.c, 5.d\}$

$M = \{T \mid T$ is size 12 multiset over $\{a, b, c, d\}\}$

$\forall T \in M$, set
$a(T) =$ no of a in T

$0 \leq a(T) \leq 4$
$0 \leq b(T) \leq 3$
$0 \leq c(T) \leq 4$
$0 \leq d(T) \leq 5$

$a(T) + b(T) + c(T) + d(T) = 12$

$A_1 = \{T \in M \mid a(T) \geq 5\}$
$A_2 = \{T \in M \mid b(T) \geq 4\}$
$A_3 = \{T \in M \mid c(T) \geq 5\}$
$A_4 = \{T \in M \mid d(T) \geq 6\}$

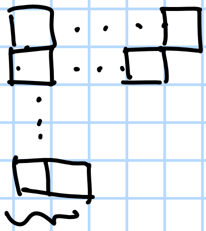
1. $f(n)$ = no of partitions of $[n]$ into k blocks where 1 & 2 seperate

$S(n,k)$ = no of partions of $[n]$ for k blocks (for $n \geq 2$)

total $= S(n,k)$ — for $n < 2$ trivial cases or $k \leq 1$
where $\{1,2\}$ part of some block $= S(n-2, k) \times k$
where $\{1,2\}$ is a block in itself $= S(n-2, k-1)$

Total − together $\{1,2\} = f(n,k) = S(n,k) - k\,S(n-2, k) - S(n-2, k-1)$

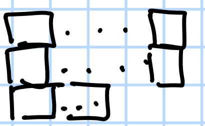
2. $T = \{$int partion of n s.t all parts $\geq 2\}$

T is of form 

atleast 2 columns
& as $x_1 \geq x_2 \ldots \geq x_k \geq 2$

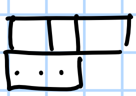
T^c = no of partions s.t last part < 2
= no of partios s.t last part $= 1$
$= P(n-1)$
so $|T^c| = P(n-1)$
$\Rightarrow |T| = P(n) - P(n-1)$

now conjugate diagrams of coluns $\geq 2 \Rightarrow$ first two rows same
or

 $\}$ the top two parts same

so $|T| = P(n) - P(n-1)$
= no of partitions of n in which two largest parts are equal

3. last colp $= 2$

 so last row size $= 2$

$x_1 \geq x_2 \cdots \geq x_{k-1} \geq 2 \geq 1$
s.t $x_{k-1} \geq 2$ & $x_1 + \cdots x_{k-1} + 2 = n$
$\Rightarrow x_1 + \cdots x_{k-1} = n-2$
where $x_{k-1} \geq 2$
so from previous problem instead of n, we have $n-2$
so $P(n-2) - P(n-3) = |T|$

4. $|S_k| = P(k)$
if $i \neq j$ then
$S_i \cap S_j = \phi$
so $\left|\bigcup_{k=1}^{n} S_k\right| = \sum_{k=1}^{n} |S_k|$

now let $f: \bigcup_{k=1}^{n} S_k \longrightarrow S_{2n}$ be s.t
$$f(a_1, a_2 \ldots a_k) = (2n - \sum_{i=1}^{k} a_i, a_1, a_2 \ldots a_k)$$
then as $a_1 \geq a_2 \geq \cdots \geq a_k$
$$n \geq a_1 \geq a_2 \ldots \geq a_k \geq 1$$
& $\sum_{i=1}^{k} a_i = r \in \{1 \ldots n\}$
we have $n - r \geq 0$
$\Rightarrow 2n - r \geq n \geq a_1 \geq \cdots \geq a_k \geq 1$
& $\sum_{i=1}^{k} a_i + 2n - r = 2n + \cancel{r} - \cancel{r} = 2n$
so $(2n-r, a_1, \ldots a_k) \in S_{2n}$
where $(a_1, \ldots, a_k) \in S_r$

now if $f(a) = f(b)$
when $a \in S_i$, $b \in S_j$
then
$f(a_1, a_2 \ldots, a_{k_a}) = f(b_1, \ldots b_{k_b})$
$\Rightarrow (2n - i, a_1, \ldots a_{k_a}) = (2n - j, b_1 \ldots b_{k_b})$
$\Rightarrow i = j$ & $a_1 = b_1, a_2 = b_2 \ldots a_{k_a} = b_{k_b}$ and so $k_a = k_b$
$\Rightarrow a = b \in S_i$

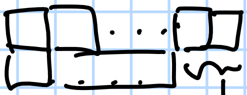
so f is one-one & so, $\left|\bigcup_{i=1}^{n} S_i\right| = \sum_{k=1}^{n} P(k) \leq P(2n)$
now this is strictly dec as $\exists \underbrace{(1,1,1 \ldots 1)}_{2n \text{ times}}$ s.t $\forall a \in \bigcup_{i=1}^{n} S_i$ $f(a) \neq (1,1 \ldots 1)$
as if $f(a) = (1,1 \ldots 1)$
then $a \in S_{2n-1}$
not in domain

so, $\sum_{k=1}^{n} P(k) < P(2n)$

5. $n \geq 5$ no of patitions of n

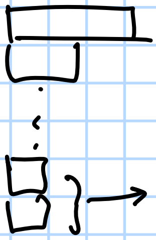
(a) $x_1 - x_2 \geq 2$

i.e



min 2 diff   i.e last two row $=1$ for conjugate

i.e



$\rightarrow x_1 \geq x_2 \geq \cdots \geq x_{k-2} \geq 1 \geq 1$

$\underbrace{x_1 \geq x_2 \geq \cdots \geq x_{k-2}}_{\sum x_i = n}$

$\Rightarrow x_1 + \cdots + x_{k-2} = n-2$

$\Rightarrow p(n-2)$

(b) $x_1 - x_2 = 4$ or conjugate rows s.t last $4 = 1$

& 5^{th} last ≥ 2

or $x_1 \geq x_2 \cdots \geq x_k \geq 2$ then $1, 1, 1, 1$

or $x_1 \geq x_2 \geq \cdots \geq x_k \geq 2$

& $x_1 + \cdots + x_k = n-4$

or $p(n-4) - \underbrace{p(n-5)}_{\text{cases where last} = 1}$

6. Let $\tilde{S}_{2n} = \{$int partition of $2n$ with all even parts$\}$ ← as no odd parts ⇒ all even

$S_n = \{$int partition of $n\}$

$f : \tilde{S}_{2n} \longrightarrow S_n$

s.t

$f(a_1, a_2 \ldots a_k) = \left(\frac{a_1}{2}, \frac{a_2}{2}, \ldots \frac{a_k}{2}\right)$

as if $(a_1 \ldots a_k) \in \tilde{S}_{2n}$

$\Rightarrow a_i = 2b_i$ & $\sum a_i = 2n$

$\Rightarrow \sum 2b_i = 2n$

$\Rightarrow \sum b_i = n$

$a_1 \geq a_2 \cdots \geq a_k$

$\Rightarrow b_1 \geq b_2 \cdots \geq b_k$

so $(b_1, b_2 \ldots b_k) = \left(\frac{a_1}{2}, \frac{a_2}{2} \ldots \frac{a_k}{2}\right) \in S_n$

also let $f^{-1} : S_n \longrightarrow \tilde{S}_{2n}$

be

$f^{-1}(b_1, \ldots b_k) = (2b_1, \ldots 2b_k)$

trivial to see $(2b_1 \ldots 2b_k) \in \tilde{S}_{2n}$

and so $|\tilde{S}_{2n}| = |S_n|$

$\Rightarrow p(n) = |\tilde{S}_{2n}|$

7. case I: $n = 1$

then $P(1) = 1$

$\Rightarrow P(1)^2 = 1$

$P(1+1) = P(2) = 2$

so $P(1)^2 < P(1+1)$

Case II: $n \geq 2$

here let $S_n = \{$int partions of $n\}$

then let $f: S_n \times S_n \to S_{n^2+n}$

s.t $f((a_1, a_2, \dots a_k), (b_1, b_2 \dots b_\ell)) = (na_1, na_2, \dots na_k, b_1, b_2, \dots b_\ell)$

as if $(a_1 \dots a_k) \in S_n$
$(b_1, \dots b_\ell) \in S_n$

then $a_1 \geq a_2 \geq \dots \geq a_k \geq 1$
$\Rightarrow na_1 \geq na_2 \geq \dots \geq na_k \geq n \geq b_1 \geq b_2 \dots \geq b_\ell$
$\Rightarrow na_1 \geq na_2 \geq \dots \geq na_k \geq b_1 \dots \geq b_\ell$

and also $\Sigma a_i = n$
$n \Sigma a_i = n^2$
$\Sigma b_i = n$
$\Rightarrow n\Sigma a_i + \Sigma b_i = n^2 + n$

so, $(na_1, \dots na_k, b_1, \dots b_\ell) \in S_{n^2+n}$

also f is one-one as for $a, b, c, d \in S_n$

s.t $f(a, b) = f(c, d)$
$\Rightarrow (na_1, na_2 \dots na_{k_a}, b_1 \dots b_{k_b}) = (nc_1, nc_2 \dots nc_{k_c}, d_1, d_2 \dots, d_{k_d})$

$\Rightarrow a = c$ & $b = d$

$\Rightarrow |S_n \times S_n| \leq |S_{n^2+n}|$

strictly dec as $(1,1,1 \dots 1) \in S_{n^2+n}$ but no domain maps it

so $P(n)^2 = |S_n \times S_n| < P(n^2+n) = |S_{n^2+n}|$

8. x_i = no of hours i^{th} day $i \in [5]$
s.t $x_4 > x_5$ $\Sigma x_i = 8$
$x_i \geq 0$ and so
if $x_5 = 0$ $x_4 \geq 1$
$\Rightarrow \Sigma x_i = 8$
$\Rightarrow \Sigma x_i' = 7$
or $\binom{7+4-1}{4-1} = \binom{10}{3}$

if $x_5 = 1$
then $x_4 \geq 2$ & so
$\binom{5+4-1}{4-1} = \binom{8}{3}$

similarly $\binom{6}{3}, \binom{4}{3}$ and so $\sum_{i=2}^{5} \binom{2i}{3}$

9. $F(n)$ = no of all partion with no singleton

$S = \{$paton of $[n]$ s.t atleast one singleton$\}$

$|S^c| = F(n) \Rightarrow |S| = B(n) - F(n)$

let $T = \{$partions of $[n+1]$ s.t no singleton$\}$

then $f: S \longrightarrow T$

$$f(\underbrace{\{a_1\}, \ldots \{a_\ell\}, B_2, B_3, \ldots B_K}_{b}) = (\{a_1, a_2, \ldots a_\ell, n+1\}, B_2, B_3, \ldots B_K) = \tilde{b}$$

as $\ell \geq 1$ $b \in S$

as $\ell \geq 1$ $\{a_1 \ldots a_\ell, n+1\}$ is not singleton so $\tilde{b} \in T$

$$f^{-1}(\{a_1 \ldots a_\ell, n+1\}, B_2, \ldots B_K) = (\{a_1\} \ldots \{a_\ell\}, B_2, \ldots B_K)$$

$\ell \geq 1$

$f^{-1}(\tilde{b}) = b$

so f^{-1} exist and

$|S| = |T|$

$\Rightarrow F(n+1) = B(n) - F(n)$

$\Rightarrow F(n+1) + F(n) = B(n)$

10. a_n = no of comp of n into parts ≥ 1

$x_1 + \cdots + x_K = n$

s.t $x_i \geq 2, \forall i \in [k]$

then $(x_1 - 1) + \cdots (x_K - 1) = n - k$

s.t $\tilde{x}_i \geq 1$

$\Rightarrow \binom{n-k-1}{k-1}$ = total

$$a_n = \sum \binom{n-k-1}{k-1}$$

now $\binom{n-k-1}{k-1} = \binom{n-k-2}{k-1} + \binom{n-k-2}{k-1-1}$ ($\because$ pascales formula)

$$= \binom{(n-1)-k-1}{k-1} + \binom{(n-2)-(k-1)-1}{k-1-1}$$

or $\sum \binom{n-k-1}{k-1} = \sum \binom{(n-1)-k-1}{k-1} + \sum \binom{(n-2)-(k-1)-1}{k-1-1}$

$$\Rightarrow a_n = a_{n-1} + a_{n-2}$$

11. let $S_n = \{\sigma \mid \sigma \text{ is a bijection } \sigma: [n] \to [n]\}$

$|S_n| = n!$

$A_i = \{\sigma \in S_n \mid \sigma(i) = i\}$

then $D_n = \{\sigma \in S_n \mid \sigma(i) \neq i \; \forall i \in [n]\}$

$$= \bigcap_{i=1}^{n} A_i^c$$

$$= S_n \setminus \bigcup_{i=1}^{n} A_i$$

$$|D_n| = |S_n| - \left|\bigcup_{i=1}^{n} A_i\right|$$

$$= n! - \left(\sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| \ldots + \cdots + (-1)^{n+1} |A_1 \cap A_2 \ldots \cap A_n|\right)$$

choosing i

$$= n! - \left[\binom{n}{1} \times (n-1)! - \binom{n}{2}(n-2)! + \cdots + (-1)^{n+1}\binom{n}{n}(n-n)!\right]$$

$$= n! - \left[\frac{n!}{1!} - \frac{n!}{2!} + \frac{n!}{3!} \cdots + (-1)^{n+1}\frac{n!}{n!}\right]$$

$$|D_n| = \sum_{i=0}^{n} \frac{n!(-1)^i}{i!}$$

$$\text{or ans} = |D_n|^2$$

12. $A_4 = \{n \in [10000] \mid n \bmod 4 = 0\}$

$$|A_4| = \left\lfloor \frac{\cancel{10{,}000}}{\cancel{4}} \right\rfloor = 2500$$

$$|A_5| = \left\lfloor \frac{10000}{5} \right\rfloor = 2000$$

$$|A_6| = \left\lfloor \frac{10000}{6} \right\rfloor = \lfloor 1666.66 \rfloor = 1666$$

$$|A_4 \cap A_5| = \left\lfloor \frac{10000}{20} \right\rfloor \quad (\because \text{lcm}(4,5)=20)$$
$$= 500$$

$$|A_4 \cap A_6| = \left\lfloor \frac{10000}{12} \right\rfloor = 833$$

$$|A_5 \cap A_6| = \left\lfloor \frac{10000}{30} \right\rfloor = 333$$

$$|A_4 \cap A_5 \cap A_6| = \left\lfloor \frac{10000}{60} \right\rfloor = 166$$

now total $= 10{,}000 - |A_4 \cup A_5 \cup A_6|$

$$= 10{,}000 - [2500 + 2000 + 1666 - 500 - 833 - 333 + 166]$$
$$= 5{,}334$$

13. size 12

$4 \geq x_a \geq 0$
$3 \geq x_b \geq 0$
$4 \geq x_c \geq 0$
$5 \geq x_d \geq 0$

s.t $x_a + x_b + x_c + x_d = 12$

now if $x_i \geq 0$ only

then $\binom{12+4-1}{4-1} = \binom{15}{3}$ general cases

If $x_a > 4 \Rightarrow x_a \geq 5$ then

$$\binom{12+4-1-5}{4-1} = \binom{10}{3}$$

similarly for others

$$\binom{11}{3}, \binom{10}{3}, \binom{9}{3}$$

for $x_a \geq 5, x_b \geq 4$ we get

$$\binom{12+4-1-5-4}{3} = \binom{6}{3}$$

for other cases: $\binom{5}{3}, \binom{4}{3}, \binom{6}{3}, \binom{5}{3}, \binom{4}{3}$

for $x_a \geq 5, x_b \geq 4, x_c \geq 5$

we get: $\binom{15-5-4-5}{3} = 0$

others: $\binom{15-4-5-6}{3} = 0$

$\binom{15-5-5-6}{3} = 0$

$\binom{15-5-4-6}{3} = 0$

so total $= \binom{15}{3} - \binom{10}{3} - \binom{11}{3} - \binom{10}{3} - \binom{9}{3} + \binom{6}{3} + \binom{5}{3} + \binom{4}{3} + \binom{6}{3} + \binom{5}{3} + \binom{4}{3}$

by inclusion-exclusion

14. $0, \dots 9999$

each 1, 3 & 5

1,3,5 compulsory

$\_\_\_\_$ $A = \{\_\_\_\_ \mid 1 \wedge 3 \wedge 5 \text{ in}\}$

$A^c = \{\_\_\_\_ \mid 1^c \vee 3^c \vee 5^c \text{ in}\}$

so $|A^c| = |\bigcup A_i^c|$ $\quad A_i^c = \{\_\_\_\_ \mid i^c \text{ in}\}$

$= |A_1| + |A_3| + |A_5| - |A_1 \cap A_3| - |A_1 \cap A_5| - |A_3 \cap A_5| + |A_1 \cap A_3 \cap A_5|$

$= 9^4 + 9^4 + 9^4 - 8^4 - 8^4 - 8^4 + 7^4$ by inclusion-exclusion

Tutorial-5:

1. $K_{1,1} \cong K_2$ consider $K_{p,q}$ with $m=\max\{p,q\}>1$
$\Rightarrow$ Not K_n for any $n \in \mathbb{N}$

2. $(4,4,3,2,2)$ Not exist as $\sum_{i \in V} d_i = 2|E(G)|$

3. $(4,4,4,2,2)$ order = no. of vertices

4. $G=(V,E)$ $\sum_{i=1}^{9} d_i = 2|E(G)| \geq 27$
$\Rightarrow 2|E(G)| \geq 28$
use pigeon hole

5. $(d_1 \dots d_n)$ n non-negative integers

every vertex - $\frac{d_i}{2}$ self loops

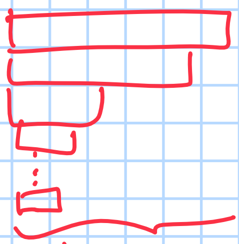
6. $n \geq 2$ same as quiz 1, q1, part 2 is no as • O

7. $|V| \cdot d = \sum_{i \in V} d = 2|E| = 44 = 11^1 \cdot 2^2$

also $d \leq |V|-1$
$\Rightarrow |V| = 11$ & $d=4$
or $|V|=22$ & $d=2$

8. WLOG assume $d_k \geq 1$
$\sum_{i=1}^{n} d_i = 2|E| = 2k$

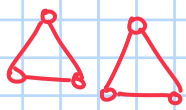
G is basic graph then π is not self-conjugate
$d_1 \leq n-1$



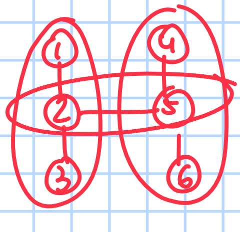
$d_1 \leq n-1$ max size

but if π was self conjugate
then $d_1 = n$ (from property)

9.

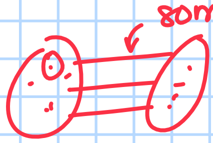


10.



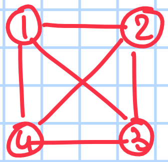
11. Suffices to show C_n is not bipartite $\forall$ n odd

some edges



odd starts on right side
even jumps on left side

14. K_4



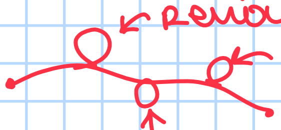
a) trivial
b) not possible

$1 \cdots\cdots 1 \cdots\cdots 1$

closed trail

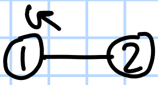
we have 4 edges that corresponds to 1
so not possible by pigeonhole principle

12. we use constructive process



algoritm is: $x-y-y-z-b-y-y-a$
$\Rightarrow x-y-a$

1. $K_{1,1}$:



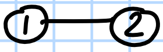
as $V = \{1\} \cup \{2\}$ $\quad E = \{\{1,2\}\}$

disjoint independent sets

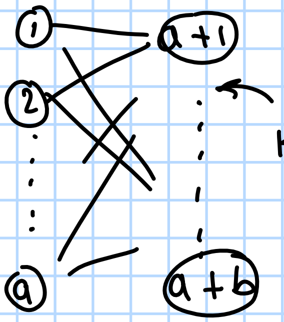
so, $G = (V, E)$ is $K_{1,1}$

and also trivial to see $K_{1,1} \cong K_2$

as K_2:



now if $K_{a,b}$ s.t $m = \max(a,b)$
for $m > 1$ ($\because$ if $m=1$ then $K_{1,1} \cong K_2$ shown)



$K_{a,b}$ for $m>1$
as $m>1$, so $\exists$ atleast 2 vertices s.t
they are not connected as on any one
side vertices $>1 \Rightarrow$
vertices ≥ 2
and as bipartate, not connected
as independent set from defn of bipartate

2. as for any $G = (V,E)$

$$\sum_{i \in V} d_i = 2|E|$$

we $4+4+3+2+2 = 15$
but as 15 is odd $\sum_{i \in V} d_i$ is even so not
possible $(4,4,3,2,2)$

3. as basic graph, if it exist then as $(4,4,4,2,2)$ given
for 3 vertices degree $=4$, so these 3 vertices connect to every
other vertice, so, every vertice degree ≥ 3 ($\because$ 3 vertices
connect to
every vertice)
but given 2,2 degree of rest which is not possible
this contradicts, so not possible

4. $G=(V,E)$ $|V|=9$ and G is a basic graph
also
$$\sum_{i\in V} d_i = \underbrace{2|E|}_{\text{this is even}} \geq 27$$
$$\Rightarrow 2|E| \geq 28$$

so, $\sum_{i\in V} d_i \geq 28$

then as $|V|=9$ let $V=[9]$
then
$$\sum_{i=1}^{9} d_i \geq 28 \quad \{d_1 \ldots d_9\}$$
and as basic graph $d_i \geq 0$

as $d_i \in \mathbb{Z}_{\geq 0}$ and $\sum_{i=1}^{9} d_i \geq 28 = 3\times 9+1$
$$\sum_{i=1}^{9} d_i > 27 = 9\times 3$$
$$\text{so } \sum_{i=1}^{9} d_i > 3\times 9$$
or $\exists i \in \{1,\ldots 9\}$ from pigeonhole principle s.t
$$d_i > 3 \Rightarrow d_i \geq 4$$

5. $(d_1, d_2 \ldots d_n)$ s.t $d_i \in \mathbb{Z}_{\geq 0}$ and $d_i \bmod 2 = 0$

To prove: $\exists G=(V,E)$ s.t vertices have degree $d_1, d_2 \ldots d_n$ and $|V|=n$

proof: lets construct a G s.t
$$V=\{1,2 \ldots n\}$$
$$E=\Big\{ \underbrace{\tfrac{d_i}{2}\cdot\{i,i\}}_{\frac{d_i}{2} \text{ times } \{i,i\} \text{ or loop}} \;\Big|\; i\in\{1,2\ldots n\}\Big\}$$
as d_i is even

so every vertex has $\frac{d_i}{2}$ loops $\Rightarrow \frac{d_i}{2}\times 2$ degree
$$= d_i \text{ degree } \forall i \in V$$

6. To prove: A basic graph of order $n\geq 2$ always has two vertices of same degree
proof:
let $G=(V,E)$ be a basic graph with $|V|=n\geq 2$
then
$$\sum_{i\in V} d_i = 2|E|$$
d_i ← degree of i^{th} vertex

as basic
$$0 \leq d_i \leq |V|-1 = n-1$$
$$0 \leq d_i \leq n-1$$

so $d_i \in \{0, 1 \ldots n-1\}$

now there are n vertices

If $\exists i$ s.t $d_i = 0$ then for all other degrees $n-1$ not possible as, one vertex no degree, then some other vertex cannot connect to all vertex

so $\forall i \in V$, $\underbrace{d_i}_{n \text{ options}} \in \underbrace{\{0, \ldots n-2\}}_{n-1 \text{ values}}$

then by PHP $\exists i \neq j$ s.t

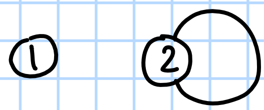
$d_i = d_j$ so $\exists$ 2 vertex with same degree

again if $\exists i$ s.t $d_i = n-1$, then for every other d_i, $d_i \neq 0$ as if one degree connects to all then no vertex has degree 0 so

$\forall i \in V$, $\underbrace{d_i}_{n \text{ options}} \in \underbrace{\{1, 2 \ldots n-1\}}_{n-1 \text{ values}}$

by PHP, $\exists i \neq j$ s.t $d_i = d_j$ so $\exists$ 2 vertex with same degree

For graph not true as



$d_1 = 0$
$d_2 = 2$ $n = 2$
$d_1 \neq d_2$

7. $G = (V, E)$ G is normal i.e basic and $d_i = d$ $\forall i \in V$

$$\sum_{i \in V} d = |V| d = 2|E|$$

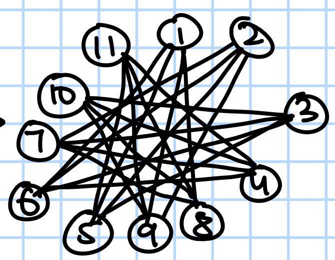
as $|E| = 22$
$\Rightarrow |V| d = 44$

and

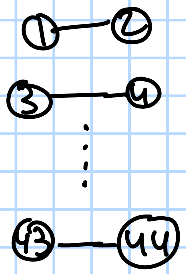
$d \leq |V| - 1$ ($\because$ basic graph)

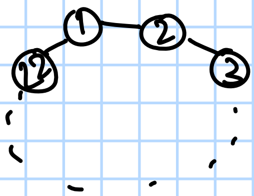
$|V| d = 11 \cdot 2^2$

11	2.2
11.2	2
11.2.2	1



$|V| = 11$, $d = 4$
$|V| = 22$, $d = 2$
$|V| = 44$, $d = 1$



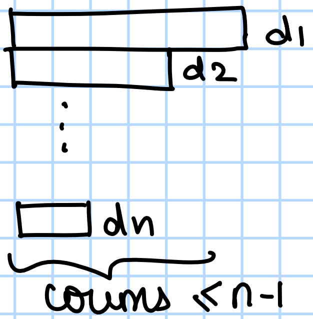


8. wlog $d_i \geq 1 \ \forall i$
and $d_1 \geq d_2 \geq \cdots \geq d_n$ n vertices

$$\sum_{i \in V} d_i = 2|E| = 2K$$

$\sum_{i \in V} d_i = 2K$, then from defn of integer partition
$\pi = (d_1 \ldots d_n)$ is int part of $2K$

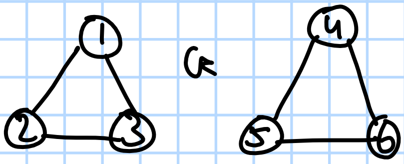
also from young's digram:



and as basic graph
$d_1 \leq n-1$

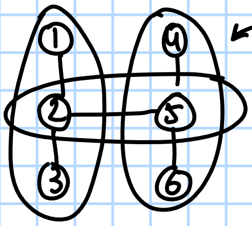
for $\pi = \pi^c$
$\pi^c = (d_1, d_2 \ldots d_n)$
or greater n rows in π^c
or greater n colns in π
this is a contradiction, so not possible

9.



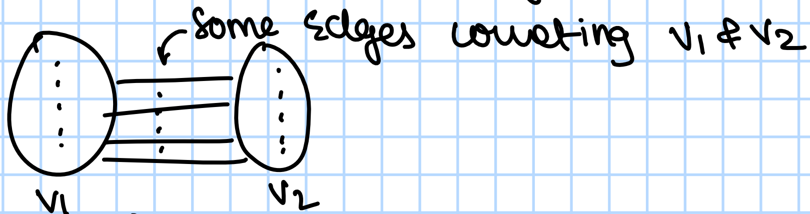
Basic graph, every vertex has degree 2

10.



← graph with 6 vertex, all odd degree, decomposed into 3 paths

11. <u>To prove</u>: A bipartite graph cant have cycle of odd length
<u>proof</u>:
let a bipartite graph be s.t $V = V_1 \sqcup V_2$ s.t vertices between V_1 & V_2 are connected by edges in between



now for any cycle in this graph let it start from
$i \in V$ wlog $i \in V_1$
then if odd length path then we will end upon V_2
as $V_1 \cap V_2 = \phi$ we get no odd path ends in V_i from V_i
$\Rightarrow$ no odd cycle

12. Now let $a - x_1 - \cdots x_n - b$ be a walk, algorithm is going from left

let x_i be s.t

$$a - x_1 \cdots \underset{\substack{\uparrow \\ \text{first} \\ x_i}}{x_i} \cdots \underset{\substack{\uparrow \\ \text{last} \\ x_i}}{x_i} - \cdots x_n - b$$

$$\Rightarrow a - x_1 \cdots x_i \cdots x_n - b$$

we do this for every $x_i \in (a, x_1 \ldots x_n, b)$

so we get $a - x_1 \cdots x_n - b$ s.t no vertex same

so its a path

13. for $x_0 - x_1$ i.e length $= 1$ true as trivial to see

if for length $< m$ then for $x_0 - x_1 \cdots - x_m$

either it is a path so we are done, or else

$\exists\, i$ (smallest) s.t $x_0 \cdots \underset{\substack{\uparrow \\ \text{first} \\ \text{seen}}}{x_i} \cdots \underset{\substack{\uparrow \\ \text{last} \\ \text{seen}}}{x_i} \cdots x_m$

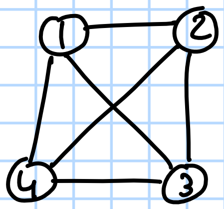
then we apply on $x_0 \cdots x_i \cdots x_m$ len $< m$

& cycle $x_i \cdots x_i$ len $< m$

and by induction done

either cycle or walk

14. K_4:



(a) $1-2-1$ is a walk, not a trail ($\because$ $1-2$, $2-1$ same edge)

(b) wlog closed trail be $1 - \cdots - 1$, if it is not a cycle then

$\exists\, 1$ in b/w trail

i.e trail looks like: $1 - \cdots - 1 - \cdots - 1$

by defn of trail 1 should have minimum 4 degree, but

degree of 1 is 3, so not cycle assumption is false

every closed trail is a cycle for K_4

Tutorial – 6 :

1. $|V| = 10$
   $|E| = 28$ basic graph G

$$\sum d_i = 2|E| = 56$$

now as $\sum d_i = 56$
every vertex degree ≤ 9
≥ 0

$0 \leq d_i \leq 9$

$\exists\, i, j \in [10]$ s.t $d_i + d_j \geq 12$ by PHP

8 other vertices, even if connected, $\exists$ 2 u, v s.t all connected

2. $(\underbrace{4,4,4}, 2, 1, 1)$ sum of degrees of last 3 vertices

adjacent sum of last 3 vertices
$\geq 2+2+2 = 6$
but $2+1+1 < 6$

3. $G = (V, E)$ $(x \sim y)$

(i) by defn
(ii) trivial
(iii)

x —— y —— z so from walk, we extract path

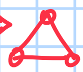
x — y — z

4. $x \in V$
$C(x) = \{z \in V \mid x \sim z\}$

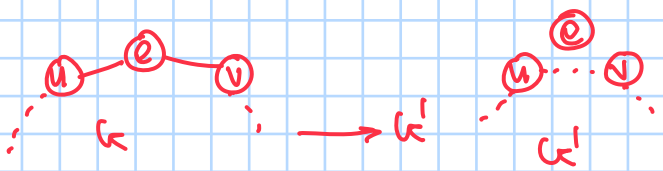
(a) if $C(x) \cap C(y) \neq \phi$, then $\exists\, v_1 \in C(x) \cap C(y) \Rightarrow x \sim v_1$ & $v_1 \sim y$
$\Rightarrow x \sim y$
$\Rightarrow C(x) = C(y)$

(c) $\forall i \in [k]$ $V_i = C(v_i)$ for some $v_i \in V$, as true $\forall k$ we have $\bigcup_{i=1}^{k} V_i \subseteq V$
$V_i \subseteq V$
if $x \in V$, then $x \in C(x) \Rightarrow V \subseteq \bigcup_{i=1}^{k} V_i$

5. G is a cycle every vertex of connected basic graph G
2 arguments, Induction :

$|V| = 3 \Rightarrow$ 

Induction step : suppose claim holds for $|V| = n$
now assume $|V| = n+1$ and let $e \in V$, with neighbours
u, v



Card of $|V_{G'}| = n$
and degree is 2
$\Rightarrow G'$ is a cycle and
So, G is a cycle for $|V_G| = n+1$

so, $\exists$ path in G'', $u - \cdots - v \Rightarrow \exists$ path $e - u - \cdots - v - e$ in G

7. n $\quad \underbrace{\frac{(n-1)(n-2)}{2}}_{\binom{n-1}{2}} + 1$

connected

K_{n-1} $\cdot \leftarrow$ connected

if graph not connected, then we get upper bound of edges it has 2 connected components

K_{r_1} K_{r_2} $r_1 + r_2 = n$

max

$\binom{r_1}{2} + \binom{r_2}{2} \longrightarrow$ max $\binom{n-1}{2}, 0$

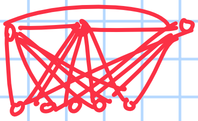
$\exists$ partition $V_1, \ldots V_k$ of V with $k \geq 2 \Rightarrow |E(G)| = \sum |E(G_i)|$
$\leq \binom{n-1}{2}$

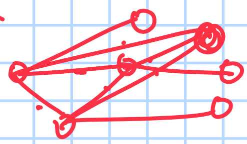
8. G is Basic $\quad$ 10 V, 38 E contains K_4 as induced

76 $\quad \binom{10}{2} = 45$

removed 7 edges

$3 \rightarrow$ connected to all





does not work

suppose $\{u, v\} \in K_{10}$, then $\{u, v\}$ is a part of atmost $\binom{8}{2} = 28$ in K_{10}

for each edge - we are removing atmost 28 K_4

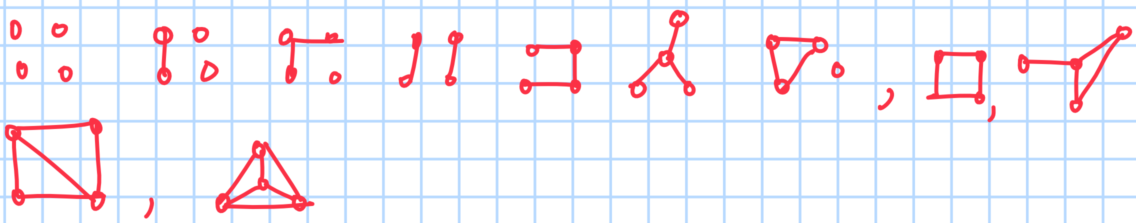
$196 = 28 \times 7$ atmost K_4 removed

total K_4 in $K_{10} = \binom{10}{4} = 210$

10. non-isomorphic basic graph order 3

$|V| = 3$

$|V| = 4$



11. $K, K \geq 1$ are there not basic, ∞ graphs

12. automorphism

(a) $n!$

(b) 1 vertex - fix to itself $(n-1)!$

(C) $2n$

13. G has open eulerian trail

14. K_n for all n odd as $\deg(v)$ even $\forall v \in K_n$

$K_2 \to$ open

1. G is basic graph $|V|=10$
$|E|=28$

To prove: G contains a cycle of length 4

proof: Now let $\{v_1, v_2 \dots v_{10}\}$ be 10 vertices

$$\sum_{i \in [10]} d_i = 2|E| = 2 \times 28 = 56$$

now $(d_1+d_2)+(d_3+d_4)+\dots+(d_9+d_{10}) = 56$

s.t. (d_{2i-1}, d_{2i}) $\forall i \in [5]$ is $d_j \geq 0$, $d_j \leq 9$

so, $0 \leq (d_{2i-1}+d_{2i}) \leq 18$

$56 = 55+1$
$= 5 \times 11 + 1$, so, $\sum_{i=1}^{5}(d_{2i-1}+d_{2i}) = 5 \times 11 + 1$

so, $\exists i \in [5]$ s.t
$d_{2i-1}+d_{2i} \geq 12$

let $i, j \in [10]$ s.t $i \neq j$ where
$d_i + d_j \geq 12$

now as basic graph, now $d_i \leq 9$, $d_j \leq 9$, $\Rightarrow d_i, d_j \geq 3$

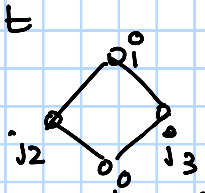
wlog $d_i \geq d_j$ then following can happen:

Case 1: $d_j = 3$ $d_i \geq 9$ then as $d_i \leq 9 \Rightarrow d_i = 9$ and so v_i connected to all, v_j connected to v_i and 2 others wlog v_x, v_y where $x \neq y \neq i \neq j$, so $\exists$ cycle v_i, v_j, v_x, v_y of length 4

Case 2: $d_j = 4$, $d_i \geq 8$ then let j connect to j_1, j_2, j_3, j_4 then from PHP as $d_i \geq 8$, 9 options, we select atleast 3 from j_1, j_2, j_3

so, even if $v_1 = i$, we get j_2, j_3 s.t

j, j_2, j_3, i form a cycle



Case 3: $d_i = 5$, $d_j \geq 7$ then let j_1, j_2, j_3, j_4, j_5 same by PHP as 9 options $\exists$ atleast 3

Case 4: $d_i = 6$, $d_j \geq 6$, same

2. $(4,4,4,2,1,1)$ Basic graph, now as $d_1=4, d_2=4, d_3=4$, minimum sum of last 3 degree will be when v_1, v_2, v_3 connected $\to$ uses 2 degree each then minimum sum of last 3 degree $\geq (4-2)+(4-2)+(4-2) = 6$

but $2+1+1 < 6$ so this is a contradiction

3. $G=(V,E)$ is a graph $x \sim y$ if $x = y$,

(a) $x \sim x$ is trivial as x connected to itself

(b) if $x \sim y$ then $\exists$ a path $x - v_1 - v_2 \cdots v_n - y$ then $y - v_n - \cdots - v_1 - x$ so, $y \sim x$

(c) If $x\sim y$ and $y\sim z$ then $x-v_1-v_2-\cdots-v_n-y$ and $y-\tilde{v}_1-\cdots-\tilde{v}_k-z$

so, $x-v_1-\cdots-v_n-y-\tilde{v}_1-\cdots-\tilde{v}_k-z$ is a walk from $x-z$

as $\exists$ walk, $\exists$ a path from $x-z$ so $x\sim z$

4. $\forall x\in V$ $\quad C(x)=\{z\in V \mid x\sim z\}$

(a) if $C(x)\cap C(y)\neq\phi$ then $\exists z\in C(x)$ and $z\in C(y)$

so, $z\sim x$ and $z\sim y$

$\Rightarrow x\sim y$

$\Rightarrow C(x)=C(y)$

so, either $C(x)\cap C(y)=\phi$ or $C(x)=C(y)$

(b) $C(x)\cap C(y)=\phi$, then for $\tilde{x}\in C(x)$, $\tilde{y}\in C(y)$ as $C(x)\cap C(y)=\phi$ we have $x\nsim y$

but as $\tilde{x}\sim x$ & $\tilde{y}\sim y \Rightarrow \tilde{x}\nsim\tilde{y}$

so, $\forall \tilde{x}\in C(x), \forall \tilde{y}\in C(y)$ $\tilde{x}\nsim\tilde{y}$, so no path between any two vertex of $C(x)$ and $C(y)$

as no path between 2 vertex of $C(x)$, $C(y)$, no path of length 1 also, ie no edge between $C(x)$ and $C(y)$ vertex

(c) $V_1,V_2,\dots V_k$ different sets $V_1,V_2,\dots,V_k$ to form partions of vertex set V of G

i.e $\forall x\in V, \exists p\in[k]$ s.t $x\in V_p$ and $\bigcup_{i=1}^{k} V_i=V$

now, if $x\in V$, then $x\in C(x)$ from construction of $C(x)$

$\Rightarrow \exists p\in[k]$ s.t $x\in V_p$

also, as $\forall x\in V, \exists p$ s.t $x\in V_i$ we get
let p be $p(x)$

$$V\subseteq \bigcup_{x\in V} V_{p(x)} \subseteq \bigcup_{i\in[k]} V_i$$

$$\Rightarrow V\subseteq \bigcup_{i\in[k]} V_i$$

$$\Rightarrow V=\bigcup_{i\in[k]} V_i \quad \text{as} \quad \bigcup_{i\in[k]} V_i\subseteq V \quad \text{(trivial)}$$

(d) $G_1=(V_1,E_1)$

$\vdots$

$G_k=(V_k,E_k)$ induced by $V_1,V_2\dots V_k$

now $\forall i\in[k]$, $G_i=(V_i,E_i)$ is s.t
$\forall x\in V_i$ we get
$C(x)=V_i$

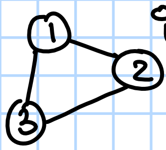
i.e $\forall x,y\in V_i \Rightarrow C(x)=C(y)$

$\Rightarrow x\sim y$

so, $\forall$ $x,y\in V_i$, $\exists$ a path between x and y

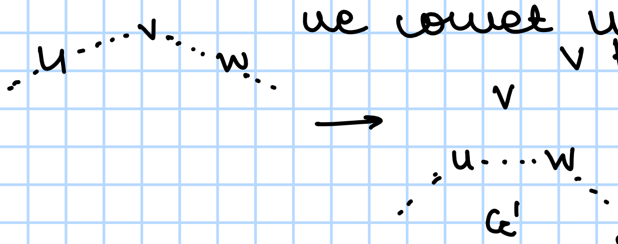
so, by defination G_i is connected

5. To prove: If every vertex of a basic connected graph G has degree 2, then G is a cycle

proof: for case of $|V|=3$, we see  is only such graph we can have, so true for $|V|=3$

let this be true for $|V|=n$, then for $|V|=n+1$

as $n \geq 3$ we get u, v, w s.t $u \neq v \neq w \in V$ s.t v connects u and w as degree of $v = 2$, now if G is like:



we connect u and w and remove v to get G': , $G' = (V', E')$ then $|V| = n$ for G' and every vertex has degree 2 so by induction hypothesis we get G' to be a cycle, now when we add back v, we get a cycle G

so, true for $|V| = n+1$

6. <u>To prove</u>: G is connected iff G' is connected

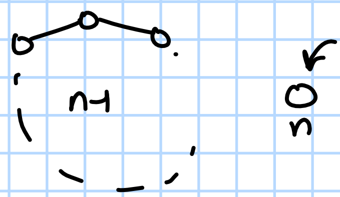
<u>proof</u>:

($\Rightarrow$) If G is connected, then $\forall x, y \in V$, $\exists$ path from x to y. As path does not have loops or repeated vertex, same path works for G', so $\forall x, y \in V'$, G' is connected

($\Leftarrow$) as G' is connected, G is connected, this is trivial

7. If a graph is not connected, then maximum edges it can have is s.t there is one comp with all vertices connected, another with same s.t max edges, when $|V_1| = n-1$, $|V_2| = 1$ i.e K_{n-1} and K_1 where total edges are $\binom{n-1}{2} = \frac{(n-1)(n-2)}{2}$ so if $|E| > \frac{(n-1)(n-2)}{2}$, we have G to be connected

disconnected basic graph of order n, with $n-1$ edges:



8. $|V| = 10$

$|E| = 38$

<u>To prove</u>: G contains K_4

<u>proof</u>: now for K_{10}, $\binom{10}{2} = \frac{10 \cdot 9}{2} = 45$ $|E| = 45$

K_{10} contains $\binom{10}{4} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210$ K_4

as $|E| = 38$ $45 - 38 = 7$

or we remove 7 edges from K_{10} to get G

every edge corresponds to $\binom{8}{2} = \frac{8 \cdot 7}{2 \cdot 1} = 28$ K_4 in K_{10}

so, atmost $28 \times 7 = 196$ K_4 removed

as $210 - 196 = 14$, we have atleast 14 K_4 in G

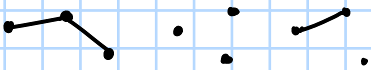
9. as every vertex of basic connected graph has vertex 2, this is a cycle from spart

(a) $|V| = 1$: $\bullet$

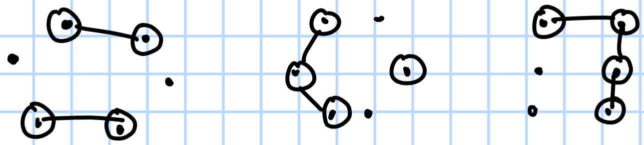
$|V|=2$:

 or 

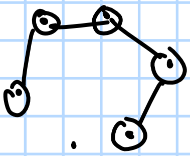
$|V|=3$



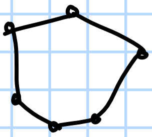
$|V|=4$:



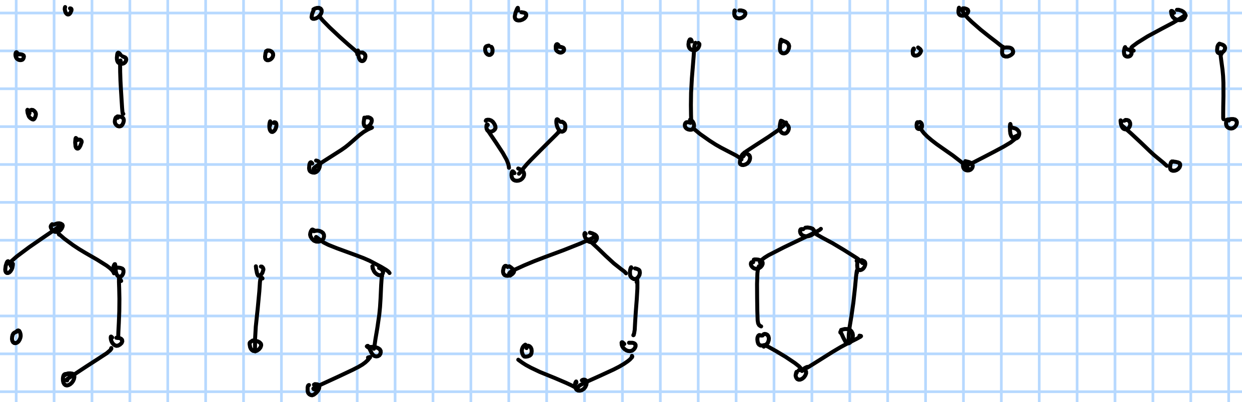
$|V|=5$:



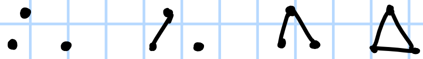
$|V|=6$:



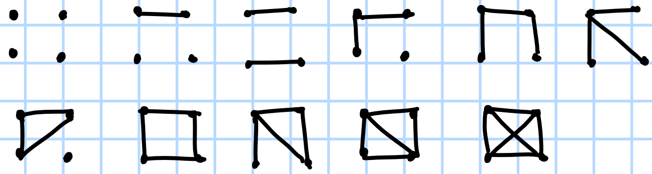
(b) and (c) will be same



10. $|V|=3$



$|V|=4$

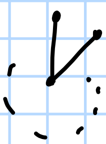


11. $K\geq 1$, non-isomorphic

 so ∞ many

12. (a) K_n on n vertices
then $1\to n$ options to map
$2\to n-1$ options
⋮
1
total $n!$.

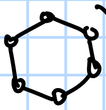
(b)



so, centre $\to$ centre
every else $\to$ any $\to (n-1)!$.

total $=(n-1)!$.

(c)



$\to n$ rotational
and flip $\to n$ rotational, total $=2n$

13. G is connected

<u>To prove</u>: G has open eulerian trail $\Leftrightarrow$ $\exists u, v$ with odd degree, else all even

and every open eulerian trail in G joins u, v

<u>proof</u>: As any closed eulerian trail iff all deg even, if we join u, v we get $\exists$ closed eulerian, removing $u - v$ we get required

14. $K_n \to K_1, K_3, K_5 \dots$ as all even degree

$K_2 \to$ open

K_{2k+2} $\forall k \in \mathbb{N}$ is s.t all deg odd so not possible from 13 part

Tutorial - 7 :

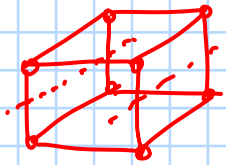
1. $V = [10]$

	connected	Bipartite	Eulerian trail	Hamilton Path
(a) $K_{5,5}$	yes	yes	NO (all odd deg)	yes
(b) all even connected	yes	NO (triangle)	NO (degree)	yes
(c) all odd and all even disconnected $K_5 \cup K_5$	NO	NO	NO	NO

2. $n \in \mathbb{N}$ ntuples of 0s and 1s - Hypercube

if $n=2$, $(0,0) - (1,0)$, $(0,0) - (1,1)$

$n=3$:



$Q_n = V_1 \cup V_2$

$V_1 = \{x \in Q_n \mid x \text{ has odd no of 1's}\}$

$V_2 = \{x \in Q_n \mid x \text{ has even no of 1's}\}$

3. $(\Rightarrow)$ $G = (V, E)$
$V = B_1 \cup B_2$ is bipartition of G

$V = V_1 \cup \cdots \cup V_k$ partition of G
into connected comp

then $\forall i \in [k]$, $V_i = (V_i \cap B_1) \cup (V_i \cap B_2)$ is bipartition of V_i

$(\Leftarrow)$ $V_1 = B_{k,1} \cup B_{k,2}$

$V_j = B_{j,1} \cup B_{j,2}$ $\rightarrow$ $B_1 = \bigcup_{j=1}^{k} B_{j,1}$ $\quad B_2 = \bigcup_{j=1}^{k} B_{j,2}$

$\vdots$

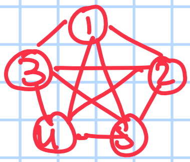
$V_k = B_{k,1} \cup B_{k,2}$

then $V = B_1 \cup B_2$ is bipartite

4. case:

case	Hamiton cycle	Hamiton path	$\min \in \mathbb{Z}$ with $n \geq m$
$n \geq m+2$	x	x	≤
$n = m+1$	x	✓	
$n = m$	✓ for $n \geq 2$ x for $n=1$	✓	

5. smallest



~~1 2 3~~

~~1 2 4~~

~~1 2 5~~

~~1 3 4~~

~~1 3 5~~

~~1 4 5~~

~~2 3 4~~

~~2 3 5~~

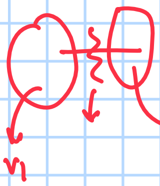
2 4 5

~~3 4 5~~

Remove 1,2
Remove 3,4
Remove 1,5
Remove 2,5
↓
No more triagles

optimal as removing any 3 edges will remove atmost $3 \times 3 \to 9$

6. $n \geq 3$ yes it does not have closed eulerian trail



→ V_2 we have to cross cut edge

V_1

7. G is connected
odd degree → $m > 0$
$m \geq 0 \to m$ is even
so $m \geq 2$ (Handshaking lemma)

$m=2 \to$ 2 odd degree → open eulerian trail

$m=4$ u v $u,v,x,y \to$ odd degree

G

x y

$G' \to u - v \to$ even

G

$x - y \to$ even

so we have closed eulerian traile

G'': $u \overset{*}{-} v$ → open eulerian trail

G'' G ε

x y

$E = E_1 \cup E_2$ → edge set of G

$E_1 = \{e \in E \mid e$ appears before $\{u,v\}^*$ in $\varepsilon\}$

$E_2 = \{e \in E \mid e$ appears after $\{u,v\}^*$ in $\varepsilon\}$

Assume claim holds for $m=2k$ $k \in \mathbb{Z}_{\geq 0}$

consider $m=2k+2$

u v $\quad$ $u \overset{*}{-} v$

u $\longrightarrow$ u $\quad$ then $2k$ odd and so k trails

trail contain $* \to$ split

now $\exists$ unique $i \in [k]$ s.t $\{u, \overset{*}{v}\} \in E_i$

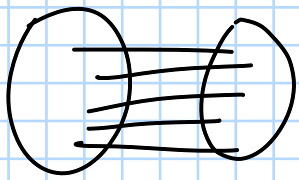
$E_i \to E_{i,1}, E_{i,2}$

1. $V = [10]$

(a) $E = \{(a,b) \mid a+b \text{ is odd}\}$

as odd + even = odd
odd + odd = even
even + even = even

we will have $V = \{1,3,\ldots 9\} \cup \{2,4,\ldots 10\}$ bipartite



(i) it is connected as for $x,y \in V$ if $x+y$ odd $\Rightarrow$ connected
otherwise if both even then 1 is common
if both odd then 2 is common

(ii) It is bipartite as $X = \{1,\ldots 9\}$ for any $x_i, x_j \in X$ $i \neq j$
we get x_i and x_j distinct so X is pairwise
not connected, X is independent set

similarly, $Y = \{2,4,\ldots 10\}$ is independent set
$V = X \cup Y$ $\quad$ $X \cap Y = \phi$, we have G as bipartite

(iii) as G is connected we know G has Eulerian trail iff
degree of every vertex even except 2
which has odd degree

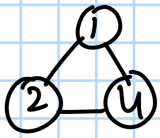
Here degree of $i \in X$ is 5 and so is of $j \in Y$
so, all degrees odd and so no Eulerian trail

(iv) Yes, $1-2-3-\cdots-10$ by construction is Hamilton path

(b) $E = \{(a,b) \mid a \times b \text{ is even}\}$

now $a \times b$ is even iff a even or b even
so, we will have an edge between 2 if one of
them is even

(i) Yes as for any $x, y \in V$ if x or y is even $\exists$ a path
otherwise if both odd we have 2 connecting both
of them so, $x-2-y$ is a path
so, G is connected

(ii) No, as  is in G, $\exists$ a closed cycle of odd length
$\Rightarrow G$ is not bipartite

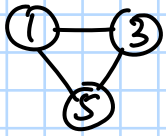
(iii) deg odd $= |\{2,4,6,\dots 10\}| = 5$
deg even $= |\{1,2\dots10\}| - 1 = 9$
as all deg odd, similar from (a)(iii) there is not an eulerian trail

(iv) Yes, $1-2-3-4-5\cdots-10$ by construction is hamilton path

(c) $E = \{(a,b) \mid a+b \text{ is even}\}$
a and b both even connected, a, b both odd connected

(i) There is no path between odds and evens, so G is not connected
connected components are like K_5 and K_5 as all odds and all even connected

(ii), (iii), (iv) not possible as G is not connected and $\exists$ a cycle of odd length



2. $n \in \mathbb{N}$ $\quad S = \{(x_1, x_2 \dots x_n) \mid x_i = 0 \text{ or } 1 \ \forall i \in [n]\}$
$|S| = 2 \times 2 \cdots 2 = 2^n$ ($\because$ multiplicative rule)
$Q_n = (S, E)$ $\quad E = \{(a,b) \mid a = (a_1, \dots a_n) \in S,\ b = (b_1, \dots b_n) \in S, \text{ and } \exists i \in [n] \text{ s.t } a_i \neq b_i, \forall j \neq i\ a_j = b_j\}$

let $S_1 = \{(x_1, x_2, \dots x_n) \in S \mid \sum_{i=1}^{n} x_i \text{ is even}\}$
$S_2 = \{(x_1, x_2, \dots x_n) \in S \mid \sum_{i=1}^{n} x_i \text{ is odd}\}$

now for any $x, y \in S_1$ if they had an edge between them then
$\exists i$ s.t $x_i \neq y_i$ wlog $x_i = 1$, $y_i = 0$
$\forall j \neq i$ $x_j = y_j$
then $\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} y_i$ is even
$\Rightarrow x_i - y_i$ is even
$\Rightarrow 1 - 0$ is even
this is a contradiction, so S_1 is pairwise disjoint
similarly S_2 is pairwise disjoint
now $S_1 \cup S_2 = S$ is trivial
$S_1 \cap S_2 = \emptyset$ as if $x \in S_1 \cap S_2$ then $\sum x_i$ is odd & even (contradiction)

so, $S' = S_1 \cup S_2$, Q_n is bipartite, with bipartition S_1 and S_2

3. To prove: A graph is bipartite iff each of its connected components are bipartite

proof:

($\Rightarrow$) $G = (V, E)$ is bipartite and $G_i = (V_i, E_i)$ are connected components for $i \in [n]$

if $V = B_1 \cup B_2 \rightarrow$ Bipartition and $\exists i \in [n]$ s.t G_i is not bipartite, then $\exists$ a cycle of odd length in G_i

if $\exists$ a cycle of odd length in $G_i \Rightarrow \exists$ a cycle of odd length in G
$\Rightarrow G$ is not bipartite
this is a contradiction

so, $\forall i \in [n]$, G_i is bipartate

($\Leftarrow$) if $G_i = (V_i, E_i)$ $\forall i \in [n]$ $V_i = B_{i,1} \cup B_{i,2}$ as G_i is bipatate

then
$$B_1 = \bigcup_{i=1}^{n} B_{i,1} \,,\; B_2 = \bigcup_{i=1}^{n} B_{i,2}$$

s.t $B_1 \cup B_2 = \bigcup (B_{i,1} \cup B_{i,2}) = \bigcup V_i = V$

$$\begin{aligned} B_1 \cap B_2 &= (\cup B_{i,1}) \cap (\cup B_{i,2}) \\ &= (\cup B_{i,1} \cap B_{1,2}) \cup (\cup B_{i,1} \cap B_{2,2}) \ldots \\ &= \phi \cup \phi \ldots \phi = \phi \end{aligned}$$

as all $B_{i,1}, B_{i,2}$ are disconnected

now, $x, y \in B_j$ then $\exists i_1, i_2$ s.t $x \in B_{i_1,j}$, $y \in B_{i_2,j}$

if $i_1 \neq i_2$ then x and y does not have an edge b/w them (disconnected comp)

if $i_1 = i_2$ then as $B_{i,j}$ is Bipartiton, by defn x and y does not have edge b/w

so, $B_{i,j}$ is bipartaute

$\Rightarrow G$ is bipatite

4. $K_{m,n}$ Hamilton cycles if $n = m$, then

let $V = V_1 \cup V_2$ $V_1 = \{x_1 \ldots x_n\}$
$V_2 = \{y_1 \ldots y_n\}$

if $n = 1$, then $x_1 - y_1$ so Hamiltonion path not cycle, if $n \geq 2$

$x_1 - y_1 - x_2 - y_2 \ldots - x_n - y_n \} x_1$ ← Hamitaon path + cycle if $n=m+1$ (wlog $n \geq m$)

then $V_1 \cup V_2 = V$

$V_1=\{x_1 \dots x_{m+1}\}$

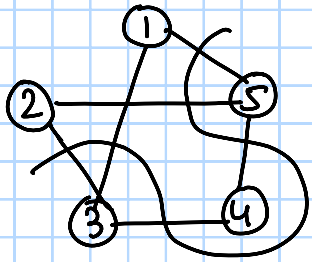
$V_2=\{y_1 \dots y_m\}$

$x_1-y_1-x_2-y_2 \cdots - x_m - y_m - x_{m+1}$ is hamilton path

no hamilton cycle

if $n \geq m+2$, then no Hamilton cycle/path

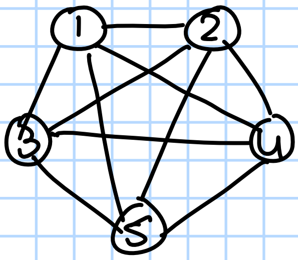
5.



if we remove ①–②, ③–⑤, ②–④, ①–④ we get this graph

we can trivially see graph is bipartite $\{1,2,4\} \cup \{3,5\}$

also, K_5:



has 1 x y
$x \neq 4$ $\quad {}^4C_2 = \frac{4 \cdot 3}{2} = 6$ triangles

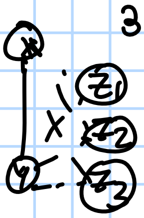
2 x y $\quad x \neq y \neq 1$
${}^3C_2 = \frac{3!}{2!} = 3$ triangles

3 x y $\quad x \neq y \neq 1, 2$

${}^2C_2 = 1$

total $6+3+1=10$ triangles in K_4

if we remove any 3 edges, atmost 3×3 triangles removed

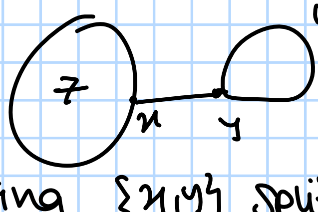


$\Rightarrow$ 3 edges removes atmost 9 triangles
so, $\exists$ one cycle of odd length left

so min no of edges ≥ 4, so 4 is minimum

6. To prove: A graph of order $n \geq 3$ with a cut edge does not have a closed eulerian trail

proof: let the cut edge be $\{x,y\} \in E$ then if $\exists$ a closed eulerian trail as $n \geq 3$, $\exists$ z s.t (wlog on x side)



cutting $\{x,y\}$ splits graph where z and x in same component

if $\exists$ some eulerian cost trail then from z we go to y so $z \cdots - x - y$ then to close it we need to go from y to z for which we have use same edge twice, so no eulerian trail closed exist

7. as $\sum d_i = 2|E|$

if no of vertices with odd degree m, we get

$$\underbrace{\sum 2n_i}_{\text{even}} + \underbrace{\sum(2n_i+1)}_{\text{odd}} = \underbrace{2|E|}_{\text{even}}$$

$\Rightarrow \underbrace{\sum(2n_i+1)}_{\text{even}}$ only when $m=$ even as sum of m odd numbers is even

now, as m is even, $m>0 \Rightarrow m \geq 2$

if $m=2$ then 2 vertices with odd degree say x, y rest all even

if we add an edge b/w x and y all degree even so, $\exists$ eulerian closed trail

then cutting the trail on this new edge we get $\frac{m}{2}=1$ open trail partion of G

let's assume true for general $m<2k$ i.e for $m=2, m=4 \ldots m=2k-2$

then for $m=2k$: if we connect x and y with odd degree, new graph has $2k-2$ many even degree vertices, so from induction $k-1$ many open trails to partion

and one such partion will contain $x-y$ new edge cutting it we get k many trails which partition

Also it is not possible to partion G into fewer than $\frac{m}{2}$ open trails as if so then less then m no of vertices has odd degree (contradiction)

Tutorial-8:

Trees:

$T = (V, E)$ is a tree if any of the following hold

T is a connected acyclic graph

T is minimally connected

$|E| = |V| - 1$ and T is connected / T is acyclic

$\forall x, y \in V$ with $x \neq y$ $\exists$ unique path from x to y in T

1. Draw 25 graphs

2. Forest is just union of trees

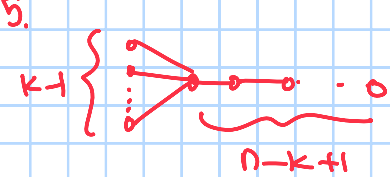
   ($\Rightarrow$) let F be a forest and $T_1, T_2, \dots T_k$ be connected components of F, then $\forall i \in [k]$, T_i is acyclic, thus F is acyclic

   ($\Leftarrow$) let G be an acyclic graph, let $G_1, G_2, \dots G_k$ be connected component of G, then $\forall i \in [k]$, G_i is a tree, thus G is a forest

3. Trees have atleast 2 leaves
   $\Rightarrow$ no closed ETs
   only open ETs are path graphs

4. cut edge removal (trivial)

5.



6. $k \geq 3$ Spc there at most $k-1$ leaves in T and $\exists v \in V$ with $\deg(v) = k$
   $|T| = n \geq 4$

   By handshaking lemma

   $$2n - 2 = 2|E| = \sum_{i=1}^{n} d_i \geq 2(n-k) + k + 1.(k-1) = 2n-1 \text{ (contradiction)}$$

   Remaining one vertex; atleast 1 degree

7. Join leaves, we get path of trees

8. ($\Rightarrow$) Handshaking

   ($\Leftarrow$) $n = 2$, $(d_1, d_2) \in (\mathbb{Z}^+)^2$ deg of seq for basic graph on $[2]$

   $\Rightarrow d_1 = d_2 = 1$

   suppose claim holds for $n = k$

   Suppose $(d_1 \dots d_{k+1}) \in (\mathbb{Z}^+)^{k+1}$ s.t

   $\sum_{i=1}^{k+1} d_i = 2k$ let G be corresponding graph

   by PHP, $\exists i \in [k+1]$ s.t $d_i = 1$, wlog assume $i = k+1$, G is connected so, $\exists$ unique $i \in [k]$ s.t i is adjacent to

$k+1$ in G, AND $\deg(i) > 1$

let $G' = G-(k+1)$, let $d_i' = d_i - 1$, then G' has deg seq: $(d_1, d_2 \ldots d_{i-1}, d_i', d_{i+1}, \ldots d_k) \in (\mathbb{Z}^+)^k$ s.t

$$\sum_{j=1}^{i-1} d_j + d_i' + \sum_{j=i+1}^{k} d_j = 2k-2 = 2(k-1)$$ by IH, G' is a tree

thus G is a tree

9. contradicate, we get unique path

10. Let's say $G_1, \ldots G_k$ be connected components of G

spanning trees: given graph G we say T is a spanning tree of G if T is a subgraph of G and $V(T) = V(G)$; T is a tree

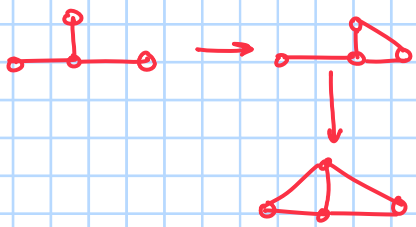
let $(T_1 \ldots T_k)$ be spanning tree of $G_1, \ldots G_k$; $|V(T_i)| = j_i \ \forall i \in [k]$

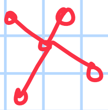
$$\sum_{i=1}^{k} j_i = n = |G|$$

$$\sum_{i=1}^{k} |E(T_i)| = n-k$$

$F = \bigsqcup_{i=1}^{k} T_i$, $\forall e \in E(G) - E(F)$

let C_e be the unique cycle in $F+e$, thus G has atleast $m-n+k$ cycles

so like if 

11.  & d  question 2, contradicate

12. Trivial

13. $n \geq 3$, let T be spanning tree of G and

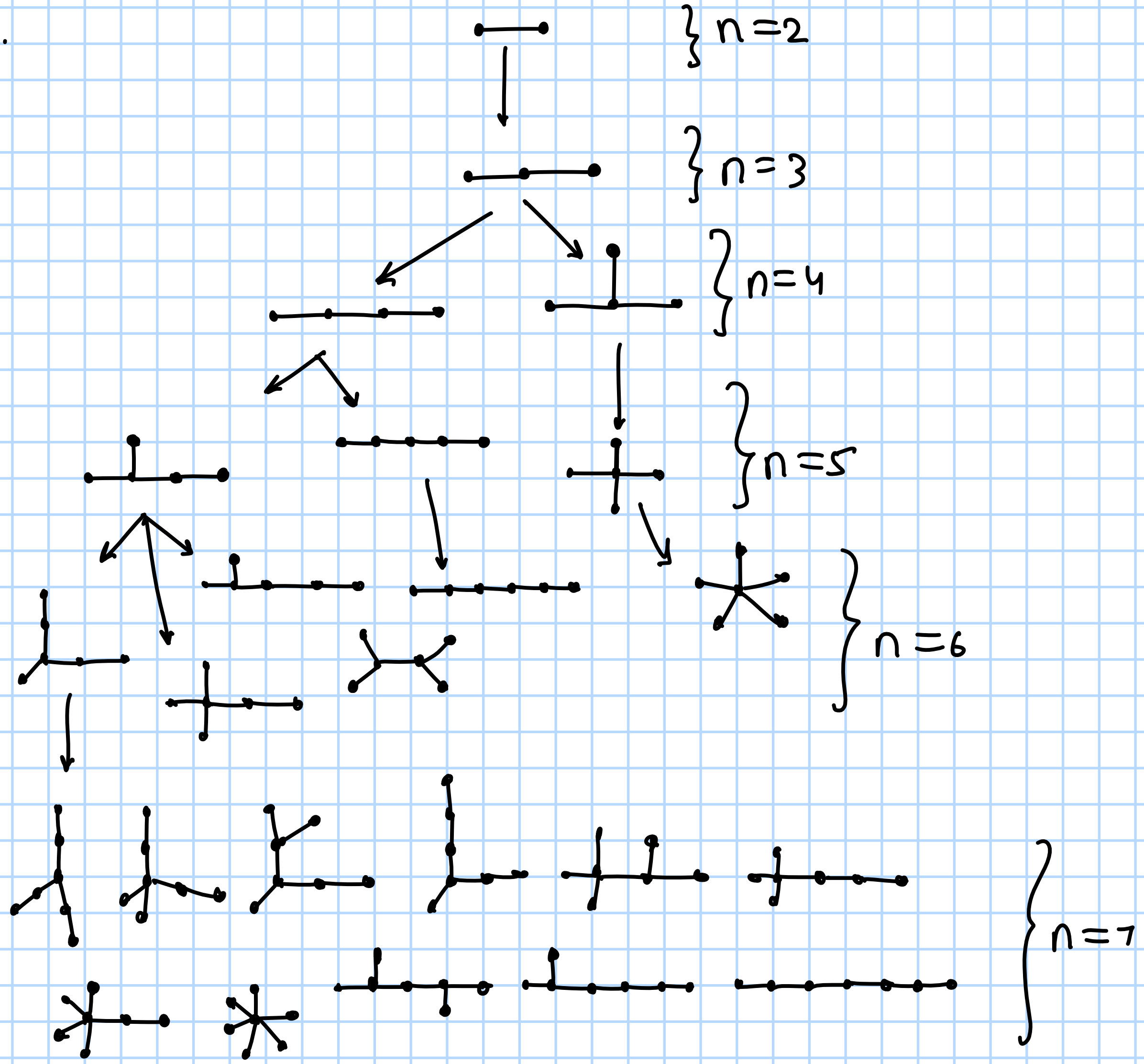
let v_1, v_2 be 2 leaves of T, then $(T-v_1)-v_2$ is a tree and $E((T-v_1)-v_2) \subseteq E((G-v_1)-v_2)$ thus

v_1, v_2 are not cut vertices

14. $|E| = m.k$, $|E| = m+\ell-1$ where ℓ = no of leaves of T

$$m+\ell-1 = mk$$
$$\Rightarrow \ell = m(k-1)+1$$

1.



2. Forest is a union of trees

($\Rightarrow$) F be a forest, and $T_1, T_2 \dots T_k$ be connected components of F, then, $\forall i \in [k]$, T_i is acyclic as T_i is a tree
$\Rightarrow F$ is acyclic

($\Leftarrow$) if G is acyclic graph, and $G_1 \dots G_k$ are connected components of G, then $\forall i \in [k]$, G_i is a tree, so G is a forest

3. As tree has atleast 2 leaves (degree 1) for eulerian trail we need all even degree and 2 odd degree

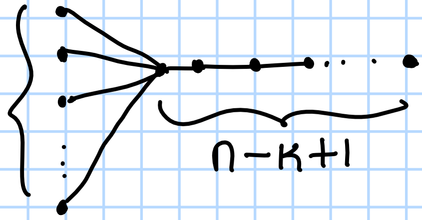
a tree with 2 leaves will have eulerian trail

2 leave trees are only of form $1-2 \dots -n$, so trees of this form have open eulerian trails

4. let 'e' be the edge we remove from tree T, then from definition of tree, e is a cut edge

Cutting e, will make T into two connected components, also as Tree does not have any cycles, connected comp will also won't have any cycles, so the two comp are trees

5. $\forall n \geq 3$, $2 \leq k \leq n-1$ order $= n$, leaves $= k$



6. To prove: If a tree has a vertex of degree k, then it has atleast k leaves

proof: if atmost k-1 many leaves in T & $\exists i \in V$ $d_i = k$

$|T| = n$

By handshaking lemma:

$$2|E| = 2(n-1) = \sum_{i=1}^{n} d_i \geq k + (k-1) + 2(n-k)$$

↑ from theorm i.e $|V| = n \Rightarrow |E| = n-1$; ↑ d_i; ↑ k-1 leaves; ↑ 2 deg of n-k

$\Rightarrow 2(n-1) \geq \cancel{2k} - 1 + 2n - \cancel{2k}$

$\Rightarrow \cancel{2n} - 2 \geq \cancel{2n} - 1$

$\Rightarrow 1 \geq 2$ this is a contradiction

so, min leaves $\geq k$

7. k-1 many, we number trees as $T_1, T_2 .. T_k$, as every tree has minimum 2 leaves, we join T_1 and T_2 by a leaf

T_2 to T_3 by another leaf of T_2 (not used) and so on.

This will make a tree as all leaves joined by $e_1, e_2 .. e_{k-1}$ are all cut edges and so by def we have a tree (rest e are already cut edges)

8. $(d_1, d_2, \ldots d_n)$ sequence of integers

To prove: There is a tree of order n with degree sequence iff $d_1, d_2, .. d_n \in \mathbb{N}$, $\sum d_i = 2(n-1)$

proof:

$(\Rightarrow)$ As tree of $(d_1, .. d_n)$ degrees, and $|V| = n \Rightarrow |E| = n-1$

$\Rightarrow \sum d_i = 2|E| = 2(n-1)$ ($\because$ Handshaking lemma)

$(\Leftarrow)$ $n=2$, $(d_1, d_2) \in (\mathbb{N})^2$ as $d_1 + d_2 = 2(1) = 2$

$d_1 \geq 1$, $d_2 \geq 1 \Rightarrow d_1 = d_2 = 1$

and so for $n=2$ we have a tree

if true for $n=k$ then for $n = k+1$:

$(d_1, \ldots d_{k+1}) \in (\mathbb{N})^{k+1}$, $\sum d_i = 2(k)$

as $d_i \geq 1$ $\forall i \in [k+1]$ By PHP, $\exists i \in [k+1]$

s.t $d_i = 1$, let $i = k+1$ (wlog)

then $\exists j$ wlog $j = k$ s.t j is connected to $i = k+1$

if we remove this edge between k and k+1

$$\sum d_i = 2(k)$$

$$\sum d_i - d_{k+1} - 1 = 2k-2$$

one extra edge of k

$\Rightarrow G' = G \setminus (k+1)$ is a tree from induction hypothesis

$\Rightarrow$ and as $G = G' \cup (k+1)$, we have G as a tree

so, by induction true for any n

9. A tree has a unique path between any two vertices say v_1, v_2 if we connect v_1 and v_2 there will be two paths between them. so one cycle is added.

also if it adds more than one cycle then cutting this e should give more than 2 paths, so exactly one cycle

10. To prove: $G=(V,E)$ basic, $|V|=n$, $|E|=m$, $m \geq n-1$, then G has atleast $m-n+1$ cycles

proof: let $G_1, G_2 \ldots G_k$ be connected components of G

let T_i be a spanning tree of G_i then

$$|E_{T_i}| = |V_{G_i}| - 1$$

total edges $\sum |E_{T_i}| = \sum |V_{G_i}| - k$
$= n-k$

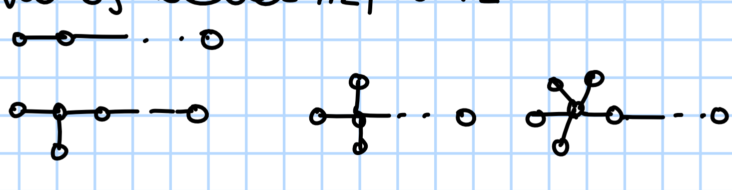
$k \geq 1$, now if we have $m-(n-k)$
$= m+k-n$ many edges remaining

from 9 we know adding one edge increases cycle by one

so $m+k-n$ many cycles
$\geq m+1-n$ minimum cycles

11. $d = \max\{d_1 \ldots d_n\}$, $|V| = n$, $|E| = n-1$ as tree T
l = no of leaves

now, $l \geq d$ from 6
and
$l = d$ then no of leaves = max degree
as no of leaves ≥ 2, $d \geq 2$
so,



... so on

12. $B(n)$ no of forest on vertex set $[n]$

$\{1,2..n\} \rightarrow$ if S is a particular set partion of $[n]$
let S be of k blocks, then we can just make k many lines

such that lowest number on block is leftmost and then inc for tree
this is a valid tree, we will have $B(n)$ such trees

13. let T be spanning tree of G, then T atleast 2 leaves v_1, v_2
removing them will not inc no of components
(assuming G is connected)
adding back all vertices no of components will not changed as $\exists$ a path always

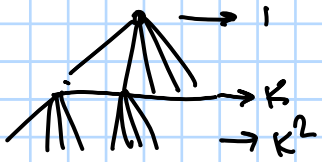
if G not connected, we do this on its connected component

so, atleast 2 vertices

14. every vertex – k or 0 children

m non-leaf vertex

$$\underset{\text{non-leaf verte}}{m} + \overset{\text{no of leaves}}{l} - 1 = |E| \quad \text{as } |V| = m + l$$



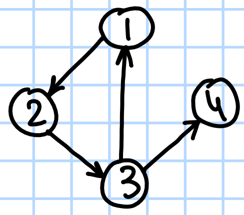
also $|E| = mk$ ← every non leaf vertex produces m many edges

↑ total no of non-leaf vertex

$\Rightarrow mk = m + l - 1$
$\Rightarrow l = m(k-1) + l$

Tutorial-9 :

1. $V=[4]$ $E=\{(1,2),(2,3),(3,1),(3,4)\}$



in degree (1) = 1
out degree (1) = 1

in degree (2) = 1
out degree (2) = 1

indegree (3) = 1
out degree (3) = 2

in degree (4) = 1
out degree (4) = 0

$1 \to 2 \to 3 \to 1$ is a cycle for G as no edges/vertex repeats

2. $G=(V,E)$ is a digraph $V=\{v_1, v_2, \ldots v_n\}$ as n vertices

now, $\forall e \in E, \exists i,j \in [n]$ s.t $e=\{v_i, v_j\}$ then

if x = total no of indegree
y = total no of outdegree

removing e, removes one indegree (of v_j) and one outdegree of v_i
so, $x \to x-1$, $y \to y-1$ (both will decrease by one)

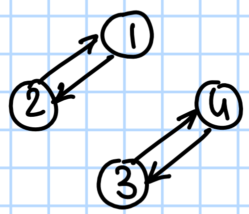
If we remove edges, wlog $|E|=m$, then

$x \to x-m$, $y \to y-m$ where $x-m=0$, $y-m=0$ as we removed all edges

$\Rightarrow x-m=0=y-m$
$\Rightarrow x=y$

so, no of indegree = no of outdegree

3. $G=(V,E)$ $|V|=4$ also no loop in G, and indegree(v) = outdegree$(v)=1$ $\forall v \in V$



This is one such case of G
trivial to see G is not strictly connected

4. To prove : A digraph is strongly connected $\Rightarrow$ it is weakly connected

Proof :

let $G=(V,E)$ with $V=\{v_1, v_2, \ldots v_n\}$ be a strongly connected graph, then

$\forall i,j \in [n], i \neq j$ $\exists$ a path from v_i to v_j

let $P: v_i \to \tilde{v}_1 \to \tilde{v}_2 \cdots \to \tilde{v}_k \to v_j$

then in its associated graph for v_i, v_j

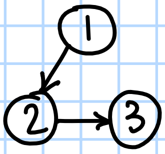
$\tilde{\mathcal{C}}: v_i - \tilde{v}_1 - \tilde{v}_2 - \dots \tilde{v}_k - v_j$ will be a path as direction does not matter
no $v_i, \tilde{v}_1 \dots \tilde{v}_k, v_j$ repeats
($\tilde{\mathcal{C}}$ is a trail with no vertices repeating)

So, $\forall\, i,j \in [n]$ $i \neq j$, $\exists$ a path b/w v_i and v_j for associated graph of G

$\Rightarrow$ associated graph is connected

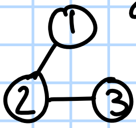
$\Rightarrow$ G is weakly connected

converse is not true as:



this graph is not strongly connected as no path from ① to ③

but



is connected (trivial), so G is weakly connected

5. To prove: If Adjacency matrix of a directed graph G is symmetric then G is balanced

proof:

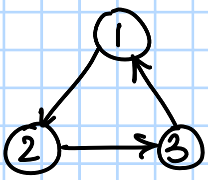
$A = A^T$ for $G = (V, E)$, $V = \{v_1, v_2, \dots v_n\}$

then $\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ji}$ $(\because a_{ij} = a_{ji}\ \forall i,j)$
$\forall i \in [n]$

$\Rightarrow$ indegree(v_i) = out degree(v_i) $\forall i \in [n]$

$\Rightarrow$ G is balanced

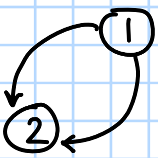
converse is not true as



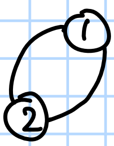
is balanced but

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$A^T \neq A$ or A is not symmetric

6. 

is a digraph with no closed eulerian trail (trivial)

but 

has a closed eulerian trail ①—②—①

7. In a tournanet, every vetex either connected by or connected to all other vertex

so, $\text{indegree}(v_i) + \text{outdegree}(v_i) = n-1$

for $V = \{v_1, v_2, \ldots v_n\}$ (n vertices)

for $n \in \mathbb{N}$

now, if $\text{indegree}(v_i) = \text{outdegree}(v_i)$

$\Rightarrow 2\,\text{indegree}(v_i) = n-1$

as $\text{indegree}(v_i) \in \mathbb{Z}_{\geq 0}$ $\forall i$

$\Rightarrow 2 \mid n-1$

$\Rightarrow n-1$ is even

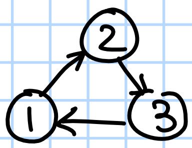
$\Rightarrow n$ is odd

so if tournanet is balaced then n is odd

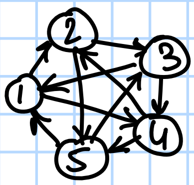
n is even $\Rightarrow$ no tournanet is balanced

<u>$n=1$</u>:

<u>$n=3$</u>:



<u>$n=5$</u>:



now to show $\forall n$ odd, $\exists$ a balaced tournament, lets use induction

for $n = 2k+1$, for $k=0$ it's trivially true

lets assume true for $n = 2k+1$ for some k

then for $n = 2(k+1)+1 = 2k+3$

let G be the balaced tournamet for $|V| = 2k+1$
$V = \{v_1, v_2, \ldots v_{2k+1}\}$

$\tilde{G}$ be our constructed graph s.t

$v_{2k+2} \rightarrow v_i$ and $v_{2k+3} \leftarrow v_i$ for $i = 2, 4, \ldots 2k$

$v_{2k+2} \leftarrow v_j$ and $v_{2k+3} \rightarrow v_j$ for $j = 1, 3, \ldots 2k+1$

from this we see that indegree and outdegree of $v_1, v_2, \ldots v_{2k+1}$ will be incresed by one (so all balanced)

$\text{indegree}(v_{2k+2}) = k+1 = \text{outdegree}(v_{2k+3})$

$\text{indegree}(v_{2k+3}) = k = \text{outdegree}(v_{2k+2})$

and now $v_{2k+2} \to v_{2k+3}$ makes v_{2k+2}, v_{2k+3} balanced

so $\tilde{G}$ by construction is a balanced tournament
$\Rightarrow$ for all n odd, $\exists$ a balanced tournament

8. $V=[n]$ now $\forall i,j \in [n]$ s.t $i \neq j$

$(i) \to (j)$ or $(j) \to (i)$

i.e $\{i,j\}$ or $\{j,i\}$ in E

we can choose 2 from $[n]$ by ${}^nC_2$ many ways

so by multiplication principle

$\underbrace{2 \times 2 \times 2 \times \cdots \times 2}_{{}^nC_2 \text{ times}}$ = total no of tournaments $V=[n]$

$\Rightarrow 2^{{}^nC_2}$ = total no of tournaments $V=[n]$

${}^nC_2 = \frac{(n)(n-1)}{2}$ so $2^{{}^nC_2} = 2^{(n)(n-1)/2}$

9. $G=(V,E)$, $V=[n]$

$\forall i,j \in [n]$, $i \neq j$ we will have 3 cases

Case I: $\{i,j\} \in E$, $\{j,i\} \notin E$ ($\because G$ is basic)

Case II: $\{j,i\} \in E$, $\{i,j\} \notin E$ ($\because G$ is basic)

Case III: $\{i,j\} \notin E$, $\{j,i\} \notin E$

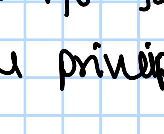
and we can choose i,j from $[n]$ in ${}^nC_2$ many ways

so by multiplication principle $= \underbrace{3 \times 3 \times \cdots \times 3}_{{}^nC_2 \text{ times}} = (3)^{(n)(n-1)/2}$

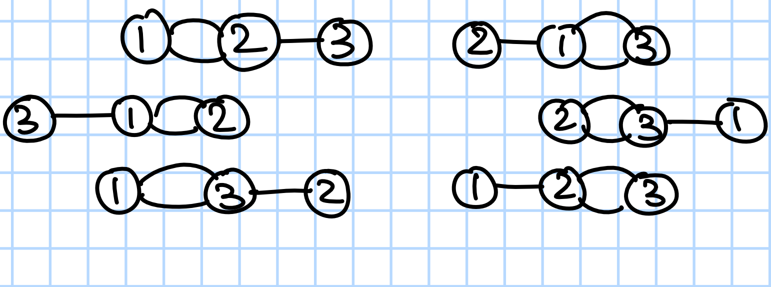
10. weakly connected digraph

$V=\{1,2,3\}$ $|E|=3$, as $|E|=2$ for a tree, we will have a triangle or one multiedge

first let's make underlying of this, we need it to be connected



$\to 2 \times 2 \times 2$ many digraphs (every edge 2 options)



every $\to 2 \times 2 \times 2$

so total $= 2 \times 2 \times 2 \times 7$
$= 8 \times 7$
$= 56$