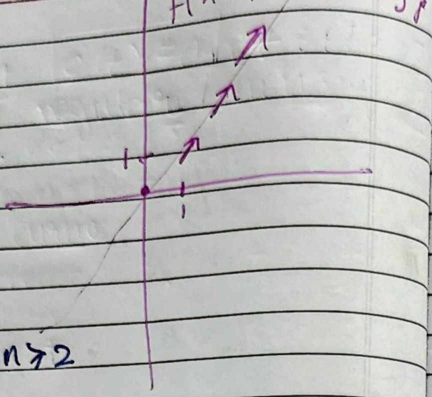
* **vector:**

$\|x\| = \left(\sum^{n} |x_i|^2\right)^{1/2}$ → norm $=1$



Scalar field on D

$f : D \to \mathbb{R}$

vector field on D

$\vec{F} : D \to \mathbb{R}^n \quad n \geq 2$

* **Del:**

$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

* **gradient vector field:**

$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$\vec{F}(x,y) = (2x, 2y) = \nabla(x^2+y^2)$

$\vec{F} = -\nabla v$ → $v = \varepsilon Q e/r$ is potential

→ force acting.

conservative vector field

work done → depends on initial and final.

$F = \nabla f$ → conservative vector field.

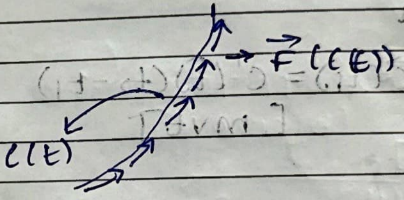
* **Flow lines:**

$\vec{F}$ is a vector field from $D \subset \mathbb{R}^n$ to $\mathbb{R}^n$, a flow line or integral curve is a path i.e a map $c : [a,b] \to D$ s.t

$\vec{c}'(t) = \vec{F}(c(t)) \quad \forall\, t \in [a,b]$

vector field.

tangent of c,

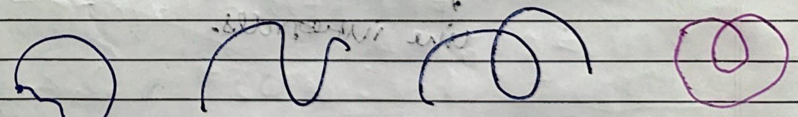


* **curve and path:**

path in $\mathbb{R}^n$ is a cont. map $\vec{c} : [a,b] \to \mathbb{R}^n$

A curve in $\mathbb{R}^n$ is an image of path $\vec{c}$ in $\mathbb{R}^n$.

both denoted by $\vec{c}$.



closed simple not-simple

$\vec{c}(t)$ → path

$\vec{c}'(t)$ → tangent vector to $\vec{c}(t)$.

* If a c' curve is s.t $\boxed{c'(t) \neq 0}$, $\forall t \in [a, b]$ the curve is called regular / non singular parametrized curve.

* work done along curve:

$\vec{F}.\vec{ds}$ → displacement

↓

dot product

$\Delta s = \vec{c}(t_2) - \vec{c}(t_1) = c'(t)(t_2 - t_1)$

[mvt]

total work:

$W = \sum_{i=1}^{t_2} \vec{F}(c(t_i))(c(t_{i+1}) - c(t_i))$

$$\boxed{\begin{aligned} W &= \int_{t_1}^{t_2} \vec{F}(c(t)).\,c'(t).dt \\ &= \int_c \vec{F}.\vec{ds} \end{aligned}}$$

↓

line integrals.

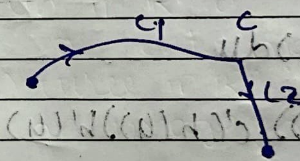
note:

$$\int_c \vec{F}.\vec{ds} = \int \vec{F}(c(t)).\,\vec{c}'(t).dt$$

$$= \int F_1(c(t)).\frac{dx(t)}{dt} + \int F_2(c(t))\frac{dy(t)}{dt} + \int F_3(c(t))\frac{dz(t)}{dt}$$

* two/three paths:

$$\int_c \vec{F}.\vec{ds} = \int_{c_1} \vec{F}.\vec{ds} + \int_{c_2} \vec{F}.\vec{ds}$$



* also note:

$c(a+b-t) = \tilde{c}$ → denoted by $-c$

$= -c$

$$\int_c \vec{F}.\vec{ds} + \int_{-c} \vec{F}.\vec{ds} = 0$$

$$\Rightarrow \int_c \vec{F}.\vec{ds} = -\int_{-c} \vec{F}.\vec{ds}$$

* paths vs curves :

$\vec{c_1}(t) = (\cos t, \sin t) \quad 0 \le t \le 2\pi$
$\vec{c_2}(t) = (\cos 2t, \sin 2t) \quad 0 \le t \le \pi$

path different → image same (curve is same)

* Reparametrisation :

$t = h(u)$
↳ C^1
diffeomorphism
(bijective, C^1)

$[\alpha, \beta] \to [a, b]$
$\varphi(u) = \vec{c}(h(u))$

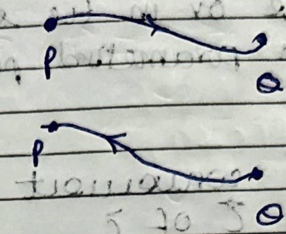
$$\int_{\varphi} \vec{F} \cdot d\vec{s} = \int_{\alpha}^{\beta} \vec{F}(\varphi(u)) \cdot \varphi'(u)\, du$$

$$= \int_{\alpha}^{\beta} \vec{F}(\vec{c}(h(u))) \cdot c'(h(u)) h'(u)\, du$$

$h'(u)\, du = dt$

$$= \int_{a}^{b} \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\, dt$$

* orientation of curves :



$P \to Q$

$Q \to P$

called orientation.

if : reparametrisation $\varphi(\cdot) = c(h(\cdot))$
preserves orientation :

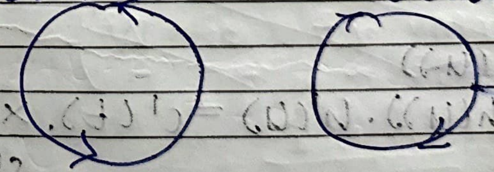
$$\int_{\varphi} \vec{F} \cdot d\vec{s} = \int_{c} \vec{F} \cdot d\vec{s}$$

if reverses orientation,

$$\int_{\varphi} \vec{F} \cdot d\vec{s} = -\int_{c} \vec{F} \cdot d\vec{s}$$

$(\cos\theta, \sin\theta)$ $\theta \in [0, 2\pi]$

$(\cos\theta, -\sin\theta)$ $\theta \in [0, 2\pi]$



simple closed planar curve anti-clock

simple closed planar curve clock

* generic curve C :
- set of points in plane or in the space that can be traversed by a parametrised path in given direction:

$\int_C \vec{F} \cdot d\vec{s}$ we choose convenient parametrisation $\vec{c}$ of $\vec{c}$

$\oint_C \vec{F} \cdot d\vec{s}$

$\hookrightarrow$ means closed curve C.

* Arc length parametrisation

$$\ell(\vec{c}) = \int_a^b \|\vec{c}'(t)\| dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt.$$

$$s(t) = \int_a^t \|c'(u)\| du$$

$$\boxed{ds = \|c'(t)\| dt}$$

$\tilde{c}(t) = c(h(u))$

$$\frac{d\tilde{c}(u)}{du} = c'(h(u)) \cdot h'(u) = c'(t) \times \frac{1}{s'(h(u))}$$

$h(u) = t$ $\qquad$ $= c'(t) \times \frac{1}{s'(t)}$

$s(t) = u$

$s(h(u)) = u$ $\qquad$ $= \boxed{\frac{c'(t)}{\|c(t)\|}}$

$s'(h(u)) \cdot h'(u) = 1$

$$\int_{\tilde{c}} F \cdot ds = \int_0^{\ell(c)} F(\tilde{c}(u))\, \tilde{c}'(u)\, du$$

$$= \int_0^{\ell(c)} F(c(h(u))) \cdot \frac{c'(h(u))}{\|c'(h(u))\|} du$$

$$= \int_0^{\ell(c)} F(c(h(u))) \cdot T(h(u))\, du$$

$$T(t) = \frac{c'(t)}{\|c'(t)\|}$$

unit tangent vector along the curve.

* Integrals of scalar functions along path:

$$\int_C f\, ds = \int_a^b f(c(t)) \|c'(t)\| dt$$

* $f : D \to \mathbb{R}$
$\hookrightarrow$ differentiable function
∇f be cont. on $\vec{c}$.

$$\int_{\vec{c}} \nabla f \cdot ds = f(\vec{c}(b)) - f(\vec{c}(a))$$

proof : $\cancel{\int_{\vec{c}} \frac{\partial f(c(t))}{\partial t} \frac{dc(t)}{dt}\, dt}$

$\cancel{\frac{\partial f(c(t))}{\partial t}}$ $\qquad$ $\int_a \frac{\partial}{\partial t} f(c(t))\, dt$

$= f(c(b)) - f(c(c

* ## cont. conservative field:

$\vec{F} = \nabla f$

$\int_C \vec{F} \cdot d\vec{s} = f(\vec{C}(b)) - f(\vec{C}(a))$

$\because f_x = y^2$

$f_y = x$

$F(x,y) = xy^2 + g(y)$

$F(x,y) =$

$\cancel{\vec{F} = y^2 \hat{i} + x \hat{j}}$

$\nabla f = \cancel{y^2 \, dx}$

eg:

$\vec{F}(x,y,z) = \dfrac{-mMG}{|r(x,y,z)|^3} \, \vec{r}(x,y,z)$

$= \dfrac{-mMG}{|r(x,y,z)|^3} \left[x\hat{i} + y\hat{j} + z\hat{k} \right]$

$\dfrac{\partial f}{\partial x} \hat{i} = \dfrac{-mMG}{|r|^3} \, x\hat{i}$

$\cancel{f = \dfrac{-mMG}{|r|^3} \left(\dfrac{x^2}{2} + \dfrac{y^2}{2} + \dfrac{z^3}{2} \right)}$

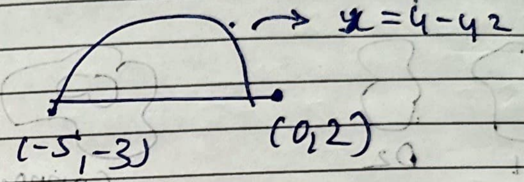
$f(x,y,z) = \dfrac{mMG}{\sqrt{x^2 + y^2 + z^2}}$

as $\vec{F} = -\dfrac{GmM}{r^2}$

$\nabla f = -\dfrac{GmM}{r^2}$

$f = \dfrac{GmM}{r}$

eg:



$\int_C y^2 \, dx + x \, dy$

for first one:

$\vec{C}(t) = (1-t)(-5,-3) + (t)(0,2)$

$\cancel{= (1-t)t}$

$= (-5+5t, -3+3t) + (0, 2t)$

$= (-5+5t, -3+5t)$

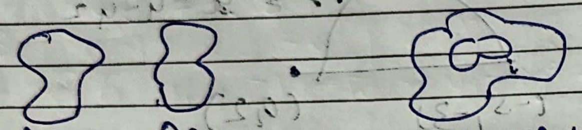
$0 \leq t \leq 1$

for second one:

$\vec{C}(t) = (4 - t^2, t)$

$-3 \leq t \leq 2$

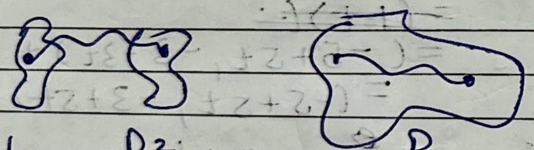
* **Connected:**
subset D of $\mathbb{R}^n$ is called connected if it cannot be written as a disjoint union of two-non-empty subsets $D_1 \cup D_2$, with $D_1 = D \cap U_1$ and $D_2 = D \cap U_2$ where U_1 and U_2 are open sets.



D_1 D_2
not connected

connected

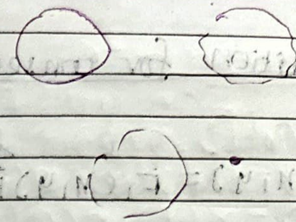
* **Path connected:**
subset D of $\mathbb{R}^n$ - if any two points in subset can be joined by a path inside D.



D_1 D_2
not path conned

D
Path connected.

Path conned $\Rightarrow$ conneted.



$x^2 + y^2 < a^2$
path conned ✓
$\Rightarrow$ conned ✓

$\{x^2+y^2<1\} \cup \{(2,2)\}$
Not conned.

$D = \left\{ (x, \sin(\frac{1}{x})) \in \mathbb{R}^2 \,\middle|\, x \in (0,1] \right\} \cup \{(0,0)\}$

Conneted ✓, not path conneted.

conneed open region

* **Theorm:**
If line integral of $\vec{F}$ is indepent of path in D, then F is a conservative vector field in D

proof:
$\vec{F}(x,y,z) = \nabla V(x,y,z)$
goal
find ∇V $\forall (x,y,z) \in D$

$V(x,y,z) = \int_{C_P} \vec{F} \cdot ds$ → any path from P_0 to P.

$\frac{\partial V}{\partial x} = F_1$
$\frac{\partial V}{\partial y} = F_2$
$\frac{\partial V}{\partial z} = F_3$

* Condition for conservative field:

$n=2$.

$\vec{F}(x,y) = F_1(x,y)\hat{i} + F_2(x,y)\hat{j}$.

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \text{ on } D.$$

$n=3$:

$\vec{F}(x,y,z) = F_1(x,y,z)\hat{i} + F_2(x,y,z)\hat{j} + F_3(x,y,z)\hat{k}$

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

$$\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$$

$$\frac{\partial F_1}{\partial z} = \frac{\partial F_2}{\partial x}$$

* finding ~~of~~ f from $\nabla f = F$

Ans: $\boxed{\text{integration}}$

* Green's theorem:

① D be bounded region $\mathbb{R}^2$, positively oriented simple boundary ∂D consisting of finite number of non-intersecting simple closed ~~curve~~ piecewise cont. diff curves.

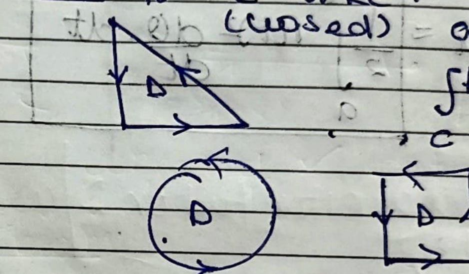
② Ω be open set in $\mathbb{R}^2$ s.t $(D \cup \partial D) \subset \Omega$

let $F_1 : \Omega \to \mathbb{R}$

$F_2 : \Omega \to \mathbb{R}$ - C^1 f.

$$\boxed{\int_{\partial D} F_1\,dx + F_2\,dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA}$$

* Note: for C like: (closed) and to find $\int_C F.ds$

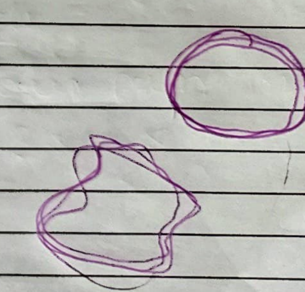


try using $\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$

$$\boxed{A(D) = \frac{1}{2}\int_C x\,dy - y\,dx}$$

$$\boxed{= \int_C x\,dy}$$

$$\boxed{= -\int_C y\,dx}$$



$$A = \iint_D dA = \frac{1}{2}\int x\,dy - y\,dx$$

* polar coordinate :

$C : (r(t), \theta(t))$

$x(t) = r(t)\cos(\theta(t))$

$y(t) = r(t)\sin(\theta(t))$

in area formula :

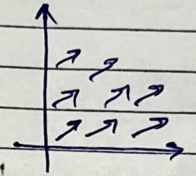
$$\frac{1}{2}\int x\,dy - y\,dx = \boxed{\frac{1}{2}\int_a^b r(t)^2 \frac{d\theta}{dt}\cdot dt}$$

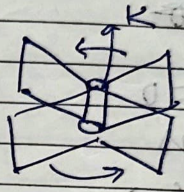
* Area of polar c :

$$A = \frac{1}{2}\int_a^b r^2\, d\theta$$

* curl $\vec{F}$:

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$





$\vec{v} = \vec{\omega} \times \vec{r}$

$\nabla \times \vec{v} = 2\omega\hat{k} = 2\vec{\omega}$

$$\boxed{\text{curl } \vec{v} = 2\vec{\omega}}$$

So if $\vec{F}$ rep. flow, then

curl $\vec{F}$ = 2 times rotation vector

if $\nabla \times F = 0$ → free from vortices / irrotational / curl free

* Polar coordinate :

$C : (r(t), \theta(t))$

$x(t) = r(t)\cos(\theta(t))$

$y(t) = r(t)\sin(\theta(t))$

in area formula :

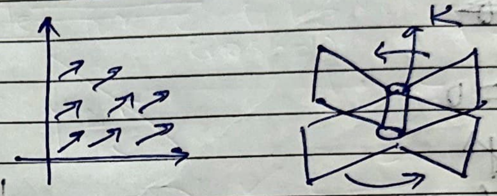
$$\frac{1}{2}\int x\,dy - y\,dx = \boxed{\frac{1}{2}\int_a^b r(t)^2 \frac{d\theta}{dt}\cdot dt}$$

* Area of polar c :

$$A = \frac{1}{2}\int_a^b r^2 \, d\theta$$

* curl $\vec{F}$:

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$



$\vec{V} = \vec{\omega} \times \vec{r}$

$\nabla \times \vec{V} = 2\omega \hat{k} = 2\vec{\omega}$

$$\boxed{\text{curl } \vec{V} = 2\,\vec{\omega}}$$

So if $\vec{F}$ rep. flow, then

curl $\vec{F}$ = 2 times rotation vector

if $\nabla \times F = 0$ → free from rotation / irrotational / curl free

* $\nabla \times \vec{F} = 0$

i.e curl $\vec{F} = 0$

then $F = \nabla f$

so gradient field.

also

~~$\nabla \times \nabla f = 0$~~

$\nabla \times (\nabla f) = 0$

always

* $\ker(\text{curl}) = \{\vec{F} \text{ is a vector fnd on } D \mid \text{curl}\,\vec{F} = 0\}$

* scalar curl :

$\vec{F} = (F_1, F_2)$

$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \underbrace{\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)}_{\text{scalar curl of } F}\hat{k}$

$$\int_{\partial D} \vec{F} \cdot d\vec{s} = \iint_D (\text{curl}\,\vec{F} \cdot \hat{k})\, dx\, dy$$

* other forms :

unit tangt to $\vec{C}$:

$\vec{T}(t) = \frac{c'(t)}{\|c'(t)\|}$

$\vec{n}(t) = \vec{T}(t) \times \hat{k} \quad \forall t \in [a, b]$

$$\int_{\partial D} \vec{F} \cdot \vec{T}\, ds = \iint_D (\text{curl}\,\vec{F}) \cdot \hat{k}\, dx\, dy$$

note : $ds = \|c'(t)\|\, dt$

so $\int_{\partial D} \vec{F} \cdot \vec{T}\, ds = \int_{\partial D} \vec{F}(c(t)) \cdot c'(t)\, dt$

so $\boxed{\text{curl}\,\vec{F} = 0}$

$\hookrightarrow$ F is consv.

* Divergence

$\text{div}\,\vec{F} := \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

amount of fluid flowing in

vs

amount of fluid flowing out.

* <u>change in area in a flow:</u>

$\vec{V} = (u, v)$

$t = 0 \quad P(x, y)$

under $\vec{v}$

at t $\quad X = X(x, y, t)$

$\quad Y = Y(x, y, t)$

$h(x, y) = (X, Y)$

$$J(x, y, t) = \begin{vmatrix} \frac{\partial X}{\partial x}(x, y, t) & \frac{\partial X}{\partial y}(x, y, t) \\ \frac{\partial Y}{\partial x}(x, y, t) & \frac{\partial Y}{\partial y}(x, y, t) \end{vmatrix}$$

$$\boxed{\frac{\partial J}{\partial t} = (\nabla \cdot \vec{v}) J}$$

when

$$\frac{dX}{dt}(x, y, t) = u(X(x, y, t), Y(x, y, t))$$

now, as $J = 1$ $(\nabla \cdot \vec{V} = 0)$ $\rightarrow$ const

$$\frac{\partial J}{\partial t} = 0$$

$$\Rightarrow \boxed{J = 1}$$

$$\text{Area}(D) = \iint_D dX\,dY = \iint_{D'} |J(x, y)|\,dx\,dy$$

$$= \iint_{D'} dx\,dy = \text{Area}(D')$$

change of area in flow $= 0$.

Called divergence free / incompressible field.

* $\text{div}(\text{curl}\ \vec{a}) = \nabla \cdot (\nabla \times \vec{a}) = 0$

$$\int_{\partial D} \vec{F} \cdot \vec{n}\,ds = \int_{\partial D} F_1 dy - F_2 dx$$

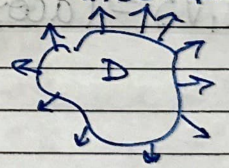
$$= \iint_D \frac{dF_1}{dx} + \frac{dF_2}{dy}\,dx\,dy$$

$$= \iint_D \text{div}\,F\,dx\,dy$$

fluid is incompressible $\rightarrow$ net flow across ∂D is zero.

$$\iint_D \text{div}\,F\,dx\,dy = 0$$

incompressible /



* **Parametrised surface:**

D be a path connected subset in $\mathbb{R}^2$.

A parametrised surface is a cont function

$\phi : D \to \mathbb{R}^2$

image $S = \phi(D) \to$ geometric surface

$(u,v) \in D$, $\phi(u,v)$ is a vector in $\mathbb{R}^3$.

$\phi(u,v) = (x(u,v), y(u,v), z(u,v))$

smooth parametrised surface $\to$ cont p derivate.

* note: ① $\phi(u,v) = (\underbrace{u,v}_{x,y}, \underbrace{f(u,v)}_{z})$

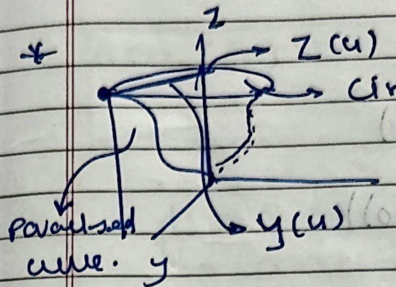
② cylinder:

$\phi(u,v) = (a\cos u, a\sin u, v)$

③ sphere:

$\phi(u,v) = (a\cos u\sin v, a\sin u\sin v, a\cos v)$

*



circle radius $= y(u)$

$x = y(u)\cos v$

$y = y(u)\sin v$

$z = z(u)$

$0 \le v \le 2\pi$

$a \le u \le b$

* **Tangent vector for a parametrised surface** -

$\phi(u,v)$ if $v = v_0$

$\vec{c}(u, v_0)$

$c(u) = x(u,v_0)\hat{i} + y(u,v_0)\hat{j} + z(u,v_0)\hat{k}$

$c'(u_0) = \frac{\partial x}{\partial u}(u_0,v_0)\hat{i} + \frac{\partial y}{\partial u}(u_0,v_0)\hat{j} + \frac{\partial z}{\partial u}(u_0,v_0)\vec{k}$

$\phi_u(u_0,v_0) = \frac{\partial \phi}{\partial u}(u_0,v_0) = c'(u_0)$

$\phi_v(u_0,v_0) = \frac{\partial \phi}{\partial v}(u_0,v_0) = c'(v_0)$

tangent plane:

$n(u_0,v_0) = \phi_u(u_0,v_0) \times \phi_v(u_0,v_0)$

$\hookrightarrow$ normal, then find tanget.

Date
Page

* $\phi(x,y) = (x, y, f(x,y))$

$\phi_x(x_0,y_0) = (1, 0, \frac{d}{dx} f(x_0,y_0))$

$\phi_y(x_0,y_0) = (0, 1, \frac{d}{dy} f(x_0,y_0))$

$$n(x_0,y_0) = \left(-\frac{\partial f}{\partial x}(x_0,y_0), -\frac{\partial f}{\partial y}(x_0,y_0), 1\right)$$

* non-singular surfaces:

ϕ is C^1 and $\phi_u \times \phi_v \neq 0$.

unit normal:

$$\hat{n}(u_0,v_0) = \frac{\phi_u(u_0,v_0) \times \phi_v(u_0,v_0)}{\| \phi_u(u_0,v_0) \times \phi_v(u_0,v_0) \|}$$

* surface area:

$P : \phi(u,v)$

$P_1 = \phi(u+h, v) \approx \phi(u,v) + h\,\phi_u(u,v)$

$P_2 = \phi(u, v+k) \approx \phi(u,v) + k\,\phi_v(u,v)$

$\|(P_1 - P) \times (P_2 - P)\| \approx \|\phi_u(u,v) \times \phi_v(u,v)\|\, |h|\,|k|$

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$$\boxed{\text{Area}(\phi) := \iint_E \|\phi_u \times \phi_v\|\, du\, dv}$$

$ds = \|\phi_u \times \phi_v\|\, du\, dv$

small area

$\text{Area}(\phi) = \iint_E ds$

* $(u, v, f(u,v))$

$\phi_u = (1, 0, \frac{\partial f}{\partial u}(u,v))$

$\phi_v = (0, 1, \frac{\partial f}{\partial v}(u,v))$

$\|\phi_u \times \phi_v\| = \sqrt{1 + (f_u)^2 + (f_v)^2}$

$\text{Area} = \iint_E \|(-f_u, -f_v, 1)\|\, du\, dv$

$= \iint_E \sqrt{1 + (f_u)^2 + (f_v)^2}\, du\, dv$

* Recolex:

$\phi(t,\theta) = (x(t),\ y(t)\cos\theta,\ y(t)\sin\theta)$

$\text{Area}(\phi) = 2\pi \int_{\alpha}^{\beta} y(t)\sqrt{x'(t)^2 + y'(t)^2}\,dt$

* area vector:

$d\vec{S} = (\phi_u \times \phi_v)\,du\,dv$

$dS = \|\phi_u \times \phi_v\|\,du\,dv$

so $\boxed{d\vec{S} = \hat{n}\,dS}$

* surface area:

$\boxed{\iint_S dS}$

* Surface integral:

$\iint_S f\,dS = \iint_E f(x,y,z)\,\hat{n}\,dS$

S → surface

$\iint_S f\,dS = \iint_{S_1} f\,dS + \iint_{S_2} f\,dS + \dots$

* So, $d\vec{S} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} du\,dv$

* Surface area:

Surface integral of scalar field:

$\iint_S f\,dS := \iint_E f(\phi(u,v))\,\|\phi_u \times \phi_v\|\,du\,dv$

vector:

$\iint_S \vec{F}\cdot d\vec{S} = \iint_E F(\phi(u,v))\cdot(\phi_u \times \phi_v)\,du\,dv$

$F\cdot\hat{n}\,dS$

* ~~Reparametrisation of surface~~ -

$h: \tilde{E} \to \mathbb{R}^2$ cont. diff one-one function

$h(\tilde{E}) = E$, $J(h)$ jacobian does not vanish on $\tilde{E}$.

$\tilde{\phi} = \phi \circ h$

$\cancel{(\tilde{\phi}_{\tilde{u}} \times \tilde{\phi}_{\tilde{v}}) = (\phi_u \times \phi_v)}$

$(\tilde{\phi}_{\tilde{u}} \times \tilde{\phi}_{\tilde{v}})(\tilde{u},\tilde{v}) = (\phi_u \times \phi_v)(h(\tilde{u},\tilde{v}))\,J(h)(h(\tilde{u},\tilde{v}))$

* Homeomorphism -

ψ function from $U_1 \subset \mathbb{R}^n$ to $U_2 \subset \mathbb{R}^m$

Homomorphism is cont, bijective map, ψ^{-1} is cont.

$\psi : U_1 \to U_2$

eg: $\psi : (-a, a) \to (-a^3, a^3)$

$\psi(x) = x^3$ is homeomorphism.

* Boundary of a surface :

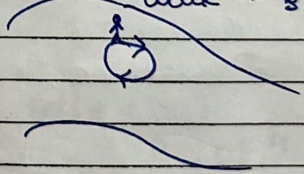
Boundary points:

P is point s.t i.e an open set U in $\mathbb{R}^3$ so $P \in U \cap S$ is homeomorphic to half disc in $\mathbb{R}^2$

Set of boundary points → boundary of S. ∂S denoted

orientation:

walk → surface on left



* Stokes theorem -

$S \to$ piece wise smooth oriented surface
non-empty boundary ∂S.

$\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ C^1

$$\int_{\partial S} \vec{F}\cdot d\vec{s} = \iint_S (\nabla \times \vec{F})\, ds$$

$$\boxed{\int_{\partial S} \vec{F}\cdot d\vec{s} = \iint_S (\text{curl } \vec{F})\, ds}$$

note: $\int_{\partial S} \vec{F} d\vec{s} = \iint_{S_1} (\text{curl } F)\, ds$

$= \iint_{S_2} (\text{curl } F)\, ds$

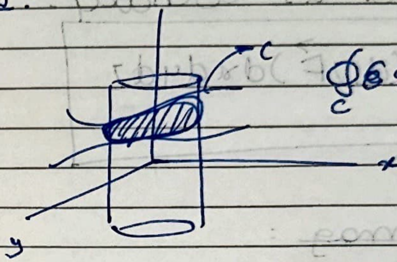
if $\partial S_1 = \partial S_2 = \partial S$

* note:

$S \to$ closed

$\iint_S \text{curl } F = 0$ $\quad \partial S = \phi$

eg:



$$\oint_C y\,dx + z\,dy + x\,dz$$

$$= \iint_S \text{curl } \vec{F}\cdot d\vec{s}$$

$$= \iint_S \text{curl } \vec{F}\cdot ds$$

$$\text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = (-1)\hat{i} - (1)\hat{j} - (1)\hat{k}$$

$$= -(\hat{i} + \hat{j} + \hat{k})$$

now $\quad z = \frac{xy}{6} \quad x \quad y$

$\left(x, y, \frac{xy}{6}\right)$

$(U_x \times U_y)$

$= \left(1, 0, \frac{y}{6}\right) \times \left(0, 1, \frac{x}{6}\right)$

$$\begin{vmatrix} i & j & k \\ 1 & 0 & y/6 \\ 0 & 1 & x/6 \end{vmatrix} = \left(-\frac{y}{6}, -\frac{x}{6}, 0\right)$$

$$\frac{1}{6}\iint_{x^2+y^2\leq a^2} (y+z-6)\,dx\,dy$$

* closed surfaces -

$S \rightarrow$ closed surface

3 d region $\omega \rightarrow S$ is its boundary.

$$\boxed{\iint_{\partial\omega} \vec{F}\cdot d\vec{S} = \iiint_{\omega} (\operatorname{div}\vec{F})\,dx\,dy\,dz}$$

* gauss law of electromag :

$$\vec{E}(x,y,z) = \frac{Q\,\vec{r}}{4\pi\varepsilon|\vec{r}|^3}$$

$$\nabla\cdot\vec{E} = 0$$

$$\iint_{S_0} \vec{E}\cdot d\vec{S} = \iint_{-S_i} \vec{F}\cdot d\vec{S} = \frac{Q}{\varepsilon}$$

flux across $\vec{\partial}$ ∂S is equal to triple integral of the divergence of vector field $\vec{F}$ over W.