MA 417 Tutorial

Tutorial-1:

Power series and analytic functions:

power series: $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ – power series in variable x

$x_0 \in \mathbb{R} \to$ centre

$a_n \in \mathbb{R}$ $(0^0=1)$

Radius of convergence:

$$R := \sup\{|x-x_0| \mid x\in D\},\ D=\{x\in\mathbb{R} \mid \sum_{n=0}^{\infty} a_n(x-x_0)^n \text{ converges}\}$$

(i) if $D=\{x_0\}$
$\Rightarrow R=0$

(ii) if $D=\mathbb{R}$
$\Rightarrow R=\infty$

(iii) if $D\neq\mathbb{R}$, $D\neq\{x_0\}$
then
$0<R<\infty$

lemma: $\sum a_n(x-x_0)^n$ power series s.t $\exists x_1\in\mathbb{R}\neq x_0$ $\sum a_n(x_1-x_0)^n$ – converges then for $I=\{x \mid |x-x_0|<|x_1-x_0|\}$, then $\sum a_n(x-x_0)^n$ conv absolutely

proof: if $b_n\to 0$
then $\exists N$ s.t $|b_n|<N$ $\forall n\in\mathbb{N}$

so,

$$\sum |a_n| \underbrace{\frac{|x-x_0|^n}{|x_1-x_0|^n}}_{<1} |x_1-x_0|^n \leq M\sum \rho^n < \infty$$

this seq conv

as $\rho<1$ it converges

converges given $\sum a_n(x_1-x_0)^n$
(we can take modulus
$\Rightarrow \sum|a_n|\,|x_1-x_0|^n$ conv)

Lemma: $(x_0-R, x_0+R)\subset D\subseteq [x_0-R, x_0+R]$

lemma: R be radius of conv. $\sum a_n(x-x_0)^n$

Radius test (a) if $L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ exist then $R=\frac{1}{L}$

Root test (b) if $L=\lim_{n\to\infty}\sqrt[n]{a_n}$ exist then $R=\frac{1}{L}$

Eg: $\sum_{n\geq 0} 4^n x^n$ $L=\lim_{n\to\infty}\left|\frac{4^{n+1}}{4^n}\right|=4 \Rightarrow R=\frac{1}{L}$ so $|x|<\frac{1}{4}$

if $\sum 4^n(-1)^n x^n$ $\sqrt[n]{a_n}=4(-1)^n$ so $\overline{\lim}\ 4(-1)^n=4$ so, $R=\frac{1}{4}$

Note: $R=\dfrac{1}{\overline{\lim}\sqrt[n]{a_n}}$

Theorem: let $R>0$ be radius of convergence of $\sum a_n(x-x_0)^n$, $f(x):=\sum_{n=0}^{\infty} a_n(x-x_0)^n$ $\forall x\in I=(x_0-R, x_0+R)$

then: (a) f is $C^\infty(I)$: $=\{f: I\to\mathbb{R} \mid f^{(k)}$ exist $\forall k\in\mathbb{N}\}$

(b) $f^{(k)}(x)=\sum_{n=k}^{\infty}\frac{\underline{n}}{\underline{n-k}} a_n(x-x_0)^{n-k}$ $\forall x\in I$

$f^{(k)}(x_0)=\underline{k}\,a_k \Rightarrow a_k=\frac{f^{(k)}(x_0)}{k!}$

(c) $f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(x_0)}{\underline{k}}(x-x_0)^k$

(d) R of $f(x)=R$ of $f^{(k)}(x)$ $\forall k\in\mathbb{N}$

Real analytic funtions:

f is real analytic when $f: \mathbb{R} \to \mathbb{R}$, let $x_0 \in \mathbb{R}$ f is real ana. at x_0 and if $\exists R>0$ s.t

$$f(x) = \sum_n a_n (x-x_0)^n \quad \forall x \in (x_0-R, x_0+R)$$

f is real analytic if it is real analytic at all $x \in$ Domain(f)

Exe: $f(x) = \sum a_n (x-x_0)^n$, show that f is real analytic $\forall x \in \mathbb{R}$ → done

Eg: $P(x)$, $\sin x$, $\cos x$, $f(x) = \frac{1}{x}$ when $x \neq 0$

Theorm: $f: I \to \mathbb{R}$ be real analytic then

(i) f is $C^\infty(I)$

(ii) for $x_0 \in I$, $f(x) = \sum_{n=0} a_n (x-x_0)^n \quad \forall x \in (x_0-R, x_0+R)$

$$a_n = \frac{f^{(n)}(x_0)}{\lfloor n}$$

(iii) If for some $x_0 \in I$, $f^{(n)}(x_0) = 0 \ \forall n \geq 0$ then $f(x) \equiv 0$ on I

proof: $A := \{x \in I \mid f^{(k)}(x) = 0 \ \forall k = 0, 1 \ldots \}$

as $x_0 \in A \Rightarrow A \neq \phi$

now, if $x \in A^c$ then $f^{(k)}(x) \neq 0$ for some k

and so as funtion is continous, $\exists$ a ball around x s.t $f^{(k)}(x) \neq 0$, so A^c is open $\Rightarrow$ A is closed

now, for $y_0 \in A$

then as f is anlytic $\exists I_{y_0}$ s.t $\forall y \in I_{y_0} = (y_0 - R_{y_0}, y_0 + R_{y_0})$ the funtion $f(y) = \sum_{n=0}^{\infty} a_n' (y-y_0)^n$

$\Rightarrow f(y) = 0$ as $a_n' = 0 = \frac{f^{(n)}(y_0)}{\lfloor n}$ as $f^{(n)}(y_0) = 0$

$\Rightarrow I_{y_0} \subset A \Rightarrow A$ is open

so A is open and closed

$\Rightarrow A = I$

Exe: $f(x) = \begin{cases} 0 & ; \text{if } x \leq 0 \\ e^{-1/x^2}; & \text{if } x > 0 \end{cases}$ called bumpy funtion

f is not real anlytic at $x=0$ but f is C^∞

Exe: $f(x) = \sum a_n (x-x_0)^n$, show that f is real analytic $\forall x \in \mathbb{R}$

Ans: now from condition of $f(x)$, $R = \infty$ for x_0

so f is analytic at x_0 with $R = \infty$ now for some x_1

as f is continious everywhere $f \in C^\infty$ (from above theorms)

let A = set of all points where f is analytic, then $x_0 \in A$, now

for $x \in A$, $\exists \delta > 0$ s.t $B_\delta(x) \subseteq A$ from continuity

and for $x \in A^c$, $\exists \delta > 0$ s.t $B_\delta(x) \subseteq A^c$ from fact that if at some x' not convergent then nbd also not conv

$\Rightarrow$ A is open and A is closed

$\Rightarrow A = \mathbb{R}$

$\therefore$ f is analytic everywhere

Tutorial - 2 :
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First order ODE :

we consider ODE $\begin{cases} \dfrac{dx}{dt} = f(x,t), \ f: \Omega \subset \mathbb{R}^2 \to \mathbb{R} \end{cases}$ —— ①
is continious Ω is domain (open + connected)

Defn : A fn $\varphi : I \subset \mathbb{R} \to \mathbb{R}$ is called a soln of ① if φ is diff in I, $(\varphi(t), t) \in \Omega$ and $\varphi'(t) = f(\varphi(t), t) \ \forall t \in I$

Eg :

$x'(t) = f(t) \quad f : I \subset \mathbb{R} \to \mathbb{R}$ cont
ϕ is a soln $\Leftrightarrow \varphi(t) = \int_{t_0}^{t} f(s)\,ds + c$
$I \to \mathbb{R}$

Eg : (First order ODE with const coeff)

$x'(t) = a x(t) \quad \phi$ is soln $\Leftrightarrow \varphi(t) = c e^{at}$

$\Rightarrow e^{-at} x'(t) = a x(t) e^{-at}$ (integrating factor method)

$\Rightarrow \dfrac{d}{dt}\left(e^{-at} x(t)\right) = 0$

$\Rightarrow e^{-at} x(t) = c \Rightarrow \varphi(t) = c e^{at}$

Eg : $x'(t) = \alpha(t)\, x(t) + \beta(t)$
$\alpha, \beta : I \to \mathbb{R}$ cont

$\varphi(t)$ is a soln let $\psi(t) = \int_{t_0}^{t} \alpha(s)\,ds$

now $\dfrac{d}{dt}\left(e^{-\psi(t)} x(t)\right) = e^{-\psi(t)} x'(t) - \psi'(t) e^{-\psi(t)} x(t)$
$= e^{-\psi(t)}[\alpha x + \beta] - \alpha[e^{-\psi(t)}][x]$
$= \beta e^{-\psi(t)}$

$\Leftrightarrow e^{-\psi(t)} x(t) = \int_{t_0}^{t} \beta e^{-\psi(t)} + c$

$\Leftrightarrow \varphi(t) = e^{\psi(t)}\left[\int_{t_0}^{t} \beta e^{-\psi(t)} + c\right]$

$\Updownarrow$
$\varphi(t)$ is a solution

Eg : (seperation of variables)

$x'(t) = f(x) g(t)$
f, g cont on I, $f(x) \neq 0 \ \forall x \in I$

$\varphi(t) = \psi^{-1}\left(\int_{t_0}^{t} g(s)\,ds + c\right), \quad \psi(u) = \int_{u_0}^{u} \dfrac{1}{f(y)}\,dy$

$\psi' = \dfrac{1}{f}$ non zero

ψ is monotonic
& so we can invert it

assume φ is a solution, then $\varphi'(t) = f(\varphi(t)) g(t)$

$\dfrac{\varphi'(t)}{f(\varphi(t))} = g(t) \Rightarrow \psi'(\varphi(t))\varphi'(t) = g(t)$

$\psi'(\varphi(t)) = \dfrac{1}{f(\varphi(t))}$

$$\Rightarrow \frac{d}{dt}\left(\psi(\varphi(t))\right) = g(t)$$

$$\Rightarrow \psi(\varphi(t)) = \int_{t_0}^{t} g(s)\,ds + c$$

so, $\varphi(t) = \psi^{-1}\left(\int_{t_0}^{t} g(s)\,ds + c\right)$

Note: X, x in above defn/examples are u(t) in our orignal defination

Eg: $\frac{dx}{dt} = [x(t)]^3$ $\quad \psi(u) = \int_{u_0}^{u} \frac{1}{y^3}\,dy = \frac{1}{2}\left[\frac{1}{u_0^2} - \frac{1}{u^2}\right]$ (By integration)

$$\psi(\varphi(t)) = \int_{t_0}^{t} \cancel{dy} \times \frac{dt}{\cancel{dy}} = (t - t_0)$$

$$\frac{1}{2}\left(\frac{1}{\varphi(t_0)^2} - \frac{1}{\varphi(t)^2}\right) = (t - t_0)$$

$$2t_0 + \frac{1}{\varphi(t_0)^2} - 2t = \frac{1}{\varphi(t)^2}$$

$$\Rightarrow |\varphi(t)| = \frac{|\varphi(t_0)|}{\sqrt{2(t_0-t)\varphi(t_0)^2+1}}$$

$$2(t_0-t)\varphi(t)^2 + 1 > 0$$

$\Rightarrow t < t_0 + \frac{1}{2\varphi(t_0)^2}$ for a solution

for any t_0 $t <$ some value, to solve this we make bound $\to \infty$ as $u(t_0) \to 0$ is a soln

Eg: $\frac{dx}{dt} = \sqrt{|x(t)|}$

$$\varphi_{(a,b)}(t) = \begin{cases} \frac{(t-a)^2}{4} & ; t \geq a \\ 0 & ; b \leq t \leq a \\ -\frac{(t-b)^2}{4} & ; t < b \end{cases}$$

$\int |u|^{-1/2} du = \int dt$

$\Rightarrow |u|^{1/2} 2 = t + c$

$\Rightarrow |u| = \frac{1}{4}(t+c)^2$

So from this we can get an idea to solve above

Assignment - 1

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Assignment-1

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1.1 $y:[a,b]\to\mathbb{R}$ is continious and
$$\frac{1}{2}y^2(t)\leq\frac{1}{2}y_0^2+\int_a^t\psi(s)y(s)ds \quad \forall t\in[a,b]$$
$a,b,y_0\in\mathbb{R}$, $\psi(t)\geq 0$ $\forall t\in[a,b]$ and is continious

To prove: $|y(t)|\leq|y_0|+\int_a^t\psi(s)ds$

proof: as $\frac{1}{2}y^2(t)\leq\frac{1}{2}y_0^2+\int_a^t\psi(s)y(s)ds$

$$\Rightarrow \frac{1}{2}|y(t)|^2\leq\frac{1}{2}|y_0|^2+\int_a^t\psi(s)\,y(s)ds\leq\frac{1}{2}|y_0|^2+\int_a^t\psi(s)|y(s)|ds$$

$$\Rightarrow |y(t)|^2\leq|y_0|^2+2\int_a^t\psi(s)|y(s)|ds$$

$$\Rightarrow |y(t)|\leq\sqrt{|y_0|^2+2\int_a^t\psi(s)|y(s)|ds}$$

if $K(t)=|y_0|^2+2\int_a^t\psi(s)|y(s)|ds$

$\Rightarrow K'(t)=2\psi(t)|y(t)|$ ($\because$ FTC)

so $|y(t)|\leq(K(t))^{1/2}$ and $K'(t)=2\psi(t)|y(t)|$

$\Rightarrow |y(t)|\psi(t)\leq\psi(t)\sqrt{K(t)}$

$\Rightarrow \frac{K'(t)}{2}\leq\psi(t)\sqrt{K(t)}$ $\left(\because K'(t)=2\psi(t)|y(t)|\right)$

now, if $\forall t\in[a,b]$, $K(t)\neq 0$
$\Rightarrow K(t)>0$ $\left(\because K(t)\geq 0\right)$

$$\Rightarrow \frac{K'(t)}{2(K(t))^{1/2}}\leq\psi(t)$$

$$\Rightarrow \int_a^t\frac{K'(s)}{2(K(s))^{1/2}}ds\leq\int_a^t\psi(s)ds$$

$$\Rightarrow (K(t))^{1/2}-(K(a))^{1/2}\leq\int_a^t\psi(s)ds$$

$$\Rightarrow |y(t)|-|y_0|\leq\int_a^t\psi(s)ds$$

$\left(\because |y(t)|\leq|K(t)|^{1/2},\ |y_0|=|K(a)|^{1/2}\text{ by const}\right)$

$$\Rightarrow |y(t)|\leq|y_0|+\int_a^t\psi(s)ds$$

now if $K(t)=0$ at some point then let $\varepsilon>0$
s.t let $K(t)=|y_0|^2+2\varepsilon+2\int_a^t\psi(s)y(s)ds$

$$\frac{K'(t)}{2(K(t))^{1/2}}\leq\psi(t) \qquad \frac{1}{2}y^2(t)\leq\frac{1}{2}y_0^2+\int_a^t\psi(s)y(s)ds\leq\frac{1}{2}y_0^2+\varepsilon+\int_{t_0}\psi(s)y(s)ds$$

and so similarly from above equation $(\because\varepsilon>0)$

$$|y(t)|\leq|y_0|+\int_a^t\psi(s)ds+\sqrt{|\varepsilon/2|}$$

this is true as $\varepsilon > 0$ so

$k(t) > 0 \quad \forall t \in (a,b)$

and so from above calculations we get the inequality

Putting $\varepsilon \to 0$ we get

$$|y(t)| \leq |y_0| + \int_a^t \psi(s)\,ds$$

1.2 $x(t), \Phi(t)$ cont. functions

$$x(t) \leq x(\tau) + \int_\tau^t \Phi(s)x(s)\,ds \quad \forall\, t, \tau \in (a,b)$$

To prove: for $t_0 \in (a,b)$ fixed $\quad x(t) = x(t_0)\, e^{\int_{t_0}^t \Phi(s)ds}$

proof: let $\quad k(t) = x(\tau) + \int_\tau^t \Phi(s)\, x(s)\,ds$

now, keeping τ fixed t variable:

$$k'(t) = \Phi(t)\, x(t) \quad (\because FTC)$$

$$\Rightarrow k'(t) = \Phi(t)x(t) \leq \Phi(t)k(t) \quad (\because x(t) \leq k(t))$$

$$\Rightarrow k'(t) \leq \Phi(t)k(t)$$

now, $\quad k'(t)\, e^{\int_\tau^t -\Phi(s)ds} \leq \Phi(t)\, k(t)\, e^{\int_\tau^t -\Phi(s)ds}$

$\quad (\because$ multiplying by $e^{\int_\tau^t -\Phi(s)ds}$ on both sides as $e^x \geq 0)$

$$\Rightarrow k'(t)e^{\int_\tau^t -\Phi(s)ds} - \Phi(t)\,k(t)\,e^{\int_\tau^t -\Phi(s)ds} \leq 0$$

$$\Rightarrow \left(k(t)e^{\int_\tau^t -\Phi(s)ds}\right)' \leq 0$$

$$\Rightarrow k(t)e^{\int_\tau^t -\Phi(s)ds} - k(\tau)e^{\int_\tau^\tau -\Phi(s)ds} \leq 0$$

$$\Rightarrow k(t)e^{\int_\tau^t -\Phi(s)ds} \leq k(\tau)e^0 \quad (\because FTC)$$

$$\Rightarrow k(t) \leq k(\tau)e^{\int_\tau^t \Phi(s)ds}$$

now $K(T) = x(T) + \int_T^T \Phi(s)ds$

$= x(T)$

and $x(t) \leq K(t)$ so,

$$x(t) \leq K(t) \leq x(T) e^{\int_T^t \Phi(s)ds}$$

$\Rightarrow x(t) \leq x(T) e^{\int_T^t \Phi(s)ds}$ (true for all $t \in (a,b)$, fixed $T \in (a,b)$)

now let $T = t_0$ as T was our choice, choosing it to be some fixed t_0

$\Rightarrow x(t) \leq x(t_0) e^{\int_{t_0}^t \Phi(s)ds}$

now as this is true for any $t_0, t \in (a,b)$

replacing t_0 and t we get

$$x(t_0) \leq x(t) e^{\int_t^{t_0} \Phi(s)ds}$$

$\Rightarrow e^{\int_t^{t_0} -\Phi(s)ds} x(t_0) \leq x(t)$

$\Rightarrow e^{\int_{t_0}^t \Phi(s)ds} x(t_0) \leq x(t)$

$\Rightarrow x(t_0) e^{\int_{t_0}^t \Phi(s)ds} \leq x(t) \leq x(t_0) e^{\int_{t_0}^t \Phi(s)ds}$

($\because x(t) \leq x(t_0) e^{\int_{t_0}^t \Phi(s)ds}$ proved above)

$\Rightarrow x(t_0) e^{\int_{t_0}^t \Phi(s)ds} = x(t)$ for any t, t_0

so for any fixed $t_0 \in (a,b)$

$$x(t_0) e^{\int_{t_0}^t \Phi(s)ds} = x(t)$$

1.4 By definition

$$W(u_1, u_2, \ldots u_m)(t) = \det \begin{bmatrix} u_1(t) \cdots u_m(t) \\ u_1'(t) \cdots u_m'(t) \\ \vdots \\ u_1^{m-1}(t) \cdots u_m^{m-1}(t) \end{bmatrix}$$

u_j are $(m-1)$ diff $1 \leq j \leq m$

here f, g s.t $W(f,g)(t) = 3e^{4t}$ $f(t) = e^{2t}$

$\Rightarrow f'(t) = \frac{d}{dt} e^{2t} = 2e^{2t}$

$$W(f,g)(t) = \det \begin{bmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{bmatrix} \quad (\because \text{By defn})$$

$= f(t) g'(t) - g(t) f'(t)$

$3e^{4t} = e^{2t} g'(t) - g(t)(2e^{2t})$

$\Rightarrow 3e^{4t} e^{-2t} = e^{2t-2t} g'(t) - g(t) \times 2 e^{2t-2t}$

$\Rightarrow 3e^{2t} = g'(t) - 2g(t)$

now, $g'(t) - 2g(t) = 0$ is a linear homogeneous ODE

let $u(t)$ be solution to it

then $u'(t) = 2u(t)$

$\Rightarrow u'(t)e^{-2t} = 2u(t)e^{-2t}$

$\Rightarrow u'(t)e^{-2t} - 2u(t)e^{-2t} = 0$

$\Rightarrow (u(t)e^{-2t})' = 0$

$\Rightarrow u(t)e^{-2t} = c$

$\Rightarrow u(t) = ce^{2t}$

so general solution is $u(t) = ce^{2t}$ now
if $U_p(t)$ is a particular solution to $g'(t) - 2g(t) = 3e^{2t}$ then
by theorm done in class

$g(t) = u(t) + u_p(t)$ is solution to hetrogenious ODE

now we also know if $u(t) = \sum_{i=1}^{m} \alpha_i c_i(t)$

as $m = 1$ (degree 1 ODE)

$\Rightarrow u(t) = ce^{2t}$ so $c_1(t) = e^{2t}$

now for $\alpha(t)$ function
$U_p(t) = \alpha(t)e^{2t}$ for some $\alpha(t)$ ($\because$ done in class)

so, as $u_p'(t) - 2u_p(t) = 3e^{2t}$

$\Rightarrow (\alpha(t)e^{2t})' - 2\alpha(t)e^{2t} = 3e^{2t}$

$\Rightarrow \alpha'(t)e^{2t} + \cancel{2\alpha(t)e^{2t}} - \cancel{2\alpha(t)e^{2t}} = 3e^{2t}$

$\Rightarrow \alpha'(t) = 3$

$\Rightarrow \alpha(t) = 3t + c$

so, $U_p(t) = \alpha(t)e^{2t} = (3t+c)(e^{2t})$

$= 3te^{2t} + ce^{2t}$

now as $g(t) = u_p(t) = 3te^{2t} + ce^{2t}$ $\forall c \in \mathbb{R}$
there will be more than one $g(t)$ as $\forall c \in \mathbb{R}$

lets check the $W(f,g)(t) = \det \begin{bmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{bmatrix}$

$= \det \begin{bmatrix} e^{2t} & 3te^{2t} + ce^{2t} \\ 2e^{2t} & 3e^{2t} + 6te^{2t} + 2ce^{2t} \end{bmatrix}$

$= 3e^{4t} + \cancel{6te^{4t}} + \cancel{2ce^{4t}} - \cancel{6te^{4t}} - \cancel{2ce^{4t}}$

$= 3e^{4t}$

so $g(t) = 3te^{2t} + ce^{2t}$ $\forall c \in \mathbb{R}$

1.7 $a, b \in \mathbb{R}$, we have to find necessary and sufficient s.t

$u''(t) + au'(t) + bu(t) = 0$ $t \in \mathbb{R}$ admits bounded, non trivial solution now $L(u) = \left[\left(\frac{d}{dt}\right)^2 + a\frac{d}{dt} + b\right]u = 0$ is given to us

now for $P(\lambda) = \lambda^2 + a\lambda + b$ let λ_1, λ_2 be roots ($\lambda_1, \lambda_2 \in \mathbb{C}$)

then $P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)$

similarly $L(u) = \left(\frac{d}{dt} - \lambda_1\right)\left(\frac{d}{dt} - \lambda_2\right) u = 0$

now it is proved in class that solution space of u will be

case I: $\{e^{\lambda_1 t}, e^{\lambda_2 t}\}$ if $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 \neq \lambda_2$

case II: $\{e^{\lambda_1 t}, te^{\lambda_1 t}\}$ if $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 = \lambda_2$

case III: if $\lambda_1, \lambda_2 \in \mathbb{C}$ then $\lambda_1 = \overline{\lambda_2} = x_1 + iy_1$

& solution space: $\{e^{x_1 t}\sin y_1 t, e^{x_1 t}\cos y_1 t\}$

in case I: $u(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$, here to make $u(t)$ bonded we need as $t \to \infty$ $u(t) \not\to \infty$

& $t \to -\infty$ $u(t) \not\to \infty$ only when $\lambda_1 = \lambda_2 = 0$, or $u(t) = c_1 + c_2$ (non trivial)

only when $\lambda_1 = \lambda_2 = 0$ or

$L(u) = \left(\frac{d}{dt}\right)^2 u = u''$ i.e $a = b = 0$

case II: for $u(t) = c_1 e^{\lambda_1(t)} + c_2 te^{\lambda t}$

$= e^{\lambda_1 t}[c_1 + c_2 t]$

this will never be bonded as for $t \to \infty$ for $c_2 \neq 0$ or $t \to -\infty$ for $c_2 \neq 0$ the function will not be bonded (depending on sign of λ_1)

so in case II, no such λ exist and so no such a, b exist

case III: $u(t) = e^{x_1 t}[c_1 \sin y_1 t + c_2 \cos y_1 t]$, this will be bonded when $x_1 = 0$ as for $x_1 \neq 0$ wlog $x_1 > 0$ then for $t \to \infty$ not bonded

so, $u(t) = c_1 \sin y_1 t + c_2 \cos y_2 t$, this is always bonded as

$-1 \leq \cos y_2 t \leq 1$

$-1 \leq \sin y_1 t \leq 1$

so $|u(t)| = |c_1 \sin y_1 t + c_2 \cos y_1 t| \leq |c_1| + |c_2|$ $\forall t \in \mathbb{R}$

now this means $\lambda_1 = iy_1$

$\lambda_2 = -iy_1$ or $y_1 \in \mathbb{R}$

$L(u) = \left(\frac{d}{dt} - iy_1\right)\left(\frac{d}{dt} + iy_1\right)u = \left(\left(\frac{d}{dt}\right)^2 + iy_1\frac{d}{dt} - iy_1\frac{d}{dt} - i^2 y_1^2\right)u$

$$= \left(\left(\frac{d}{dt}\right)^2 - 1 \times (-1)\, y_1^2 \right) u$$
$$= u'' + y_1^2 u$$
$$= u'' + au' + bu$$
$$= 0$$
$$\Rightarrow y_1^2 = a,\ b = 0$$

so $a = y_1^2,\ b = 0 \ \forall y_1 \in \mathbb{R}$

$\therefore$ combining case 1, 2, 3 $a \geq 0$, $b = 0$ gives us our required u to be bounded and nontrivial

1.8 $u_{i,j}(t) = t^j e^{\lambda_i t} \quad 1 \leq i \leq m,\ 0 \leq j \leq m_i$

$v_{k,l}(t) = t^l e^{x_k t} \sin(y_k t),\ 1 \leq k \leq n,\ 0 \leq l \leq n_k$

$w_{k,l}(t) = t^l e^{x_k t} \cos(y_k t),\ 1 \leq k \leq n,\ 0 \leq l \leq n_k$

To prove: for any i, j, k, l $\{u_{i,j}, v_{k,l}, w_{k,l}\}$ are linearly independent

proof: lets assume they are linearly dependent then $\exists c_1, c_2, c_3 \in \mathbb{R}$ not all zero s.t
$$c_1 u_{i,j} + c_2 v_{k,l} + c_3 w_{k,l} = 0 \ \forall t \in \mathbb{R}$$

now $c_1 t^j e^{\lambda_i t} + c_2 t^l e^{x_k t} \sin(y_k t) + c_3 t^l e^{x_k t} \cos(y_k t) = 0$

I am assuming x_k and y_k are s.t $x_k \pm i\, y_k$ is a root which is complex so $y_k \neq 0$ (as if $y_k = 0$ then for $l = j$, $\lambda_i = x_k$, $n_k = m_i$ we get lin dep solution)

case I : $j > l$, then
$$c_1 t^{j-l} e^{\lambda_i t} + c_2 e^{x_k t} \sin(y_k t) + c_3 e^{x_k t} \cos(y_k t) = 0$$
($\because$ taking t^l common)

putting $t = 0$ we get:
$$c_1(0) + c_2 e^0 \sin(0) + c_3 e^0 \cos(0) = 0$$
$$\Rightarrow c_3 = 0$$

also, putting $t = \pi / y_k$ ($\because y_k \neq 0$)
$$c_1 \left(\frac{\pi}{y_k}\right)^{j-l} e^{\lambda_i (\pi / y_k)} + 0 = 0 \quad (\because \sin \pi = 0)$$
$$\Rightarrow c_1 = 0$$

then $c_2 e^{x_k t} \sin y_k t = 0 \ \forall t \in \mathbb{R}$
$$\Rightarrow c_2 = 0 \quad \left(\because \text{for } t = \frac{\pi}{2 y_k},\ c_2 e^{x_k \pi / 2 y_k} = 0 \Rightarrow c_2 = 0\right)$$

case II : $j = l$: by taking t^l common we get:
$$c_1 e^{\lambda_i t} + c_2 e^{x_k t} \sin(y_k t) + c_3 e^{x_k t} \cos(y_k t) = 0$$

putting $t=0$: $c_1+c_2(0)+c_3=0$

$$c_1+c_3=0$$

as $y_k \neq 0$

$t=\frac{\pi}{y_k}$: $c_1 e^{\lambda_i(\pi/y_k)}+0+c_3 e^{x_k(\pi/y_k)}(-1)=0$

$$\Rightarrow c_1 - c_3 e^{(x_k-\lambda_i)(\pi/y_k)}=0$$

$$\Rightarrow c_1 = c_3 \underbrace{e^{(x_k-\lambda_i)(\pi/y_k)}}_{\alpha} = c_3\alpha$$

$$\& \ c_1+c_3=0$$

$$\Rightarrow c_1=-c_3$$

so, $-c_3=c_3\alpha$ where $\alpha>0$ as $\alpha=e^{(x_k-\lambda_i)(\pi/y_k)}$

$$\Rightarrow c_3\alpha+c_3=0$$

$$\Rightarrow c_3=0$$

then $c_1=0$ and so $c_2 e^{x_k t}\sin y_k t=0$

for $t=\frac{\pi}{2y_k}$

$$\Rightarrow c_2 e^{x_k\pi/2y_k}=0$$

$$\Rightarrow c_2=0$$

<u>case III</u>: $j<l$ by taking t^j common we get:

$$c_1 e^{\lambda_i t}+c_2 t^{l-j}e^{x_k t}\sin y_k t+c_3 t^{l-j}e^{x_k t}\cos y_k t=0$$

putting $t=0$: $c_1+0+0=0$

$$\Rightarrow c_1=0$$

then equation becomes: (by taking t^{l-j} common)

$$c_2 e^{x_k t}\sin y_k t+c_3 e^{x_k t}\cos y_k t=0$$

putting $t=\pi/y_k$ as $y_k\neq 0$:

$$c_2 e^{x_k\pi/y_k}(0)-c_3 e^{x_k\pi/y_k}=0$$

$$\Rightarrow -c_3=0$$

$$\Rightarrow c_3=0$$

then for $t=\pi/2y_k$:

$$c_2 e^{x_k\pi/2y_k}=0 \Rightarrow c_2=0$$

so in every case, $c_1=0=c_2=c_3$, so this contradicts fact that the three functions are linearly dependent as by defn $\exists c_i\neq 0$, here all three zero

so, $\{u_{i,j}, v_{k,l}, w_{k,l}\}$ are lin ind for any choice of i,j,k,l

now, $\sum c^1_{i,j}u_{i,j}+\sum c^2_{k,l}v_{k,l}+\sum c^3_{k,l}w_{k,l}=0$

for above $\{u_{i,j}, v_{k,l}, w_{k,l}\}$ if we show all

$u_{i,j}$ are lin ind, $v_{k,l}$ lin ind & $w_{k,l}$ lin ind then done as then all $\{u_{i,j}\}$ lin ind

$\{v_{k,l}\}$ lin ind

$\{w_{k,l}\}$ lin ind and group wise lin ind

now for $\{u_{i,j}\}$ lin ind is trivial as

$u_{i,j}(t) = t^j e^{\lambda_i t}$

wlog for 2:

$\alpha t^j e^{\lambda_i t} + \beta t^l e^{\lambda_k t} = 0$

for $t=0$, $t=1$, $t=2$ we get 3 unique equation

only 2 sols $\Rightarrow \alpha = \beta$

Similar to all other cases

we do same for $\{v_{k,l}\}, \{w_{k,l}\}$ so all individually for different values are lin ind

$\Rightarrow$ all funtions lin ind as they are pairwise lin ind

Tutorial-3:

Assignment 1 left questions

Tutorial-4:

family $dx/dt = f(at+bx+c)$:

$\frac{dx}{dt} = f(at+bx+c)$, $b \neq 0$, then $at+bx+c = y$

→ family of this ODEs

$a + b\frac{dx}{dt} = \frac{dy}{dt}$

$$\frac{\frac{dy}{dt} - a}{b} = f(y)$$

$$\Rightarrow \frac{1}{b}\frac{dy}{dt} - \frac{a}{b} = f(y)$$

$$\Rightarrow \frac{dy}{dt} = bf(y) + a$$

now if $f(y) = -\frac{a}{b}$ then $\frac{dy}{dt} = 0 \Rightarrow y = c$

if $\frac{dy}{dt} = y \Rightarrow \frac{1}{y}dy = dt \Rightarrow \log y = t + c' \Rightarrow y = ce^t$

Note: If $\frac{dx}{dt} = 2\sqrt{x}$

$$\Rightarrow \int \frac{1}{\sqrt{x}}dx = \int 2\,dt$$

$\sqrt{x} = t + c$

$\Rightarrow x = (t+c)^2$

as $\sqrt{x} > 0$

$\Rightarrow t > -c$

family $dx/dt = f(x/t)$:

let $x = ty \Rightarrow \frac{dx}{dt} = y + t\frac{dy}{dt}$

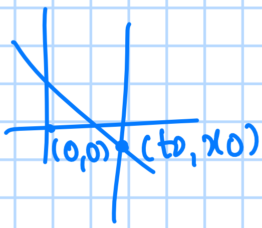
$$\Rightarrow \frac{dy}{dt} = \frac{f(y) - y}{t}$$

and then solve accordingly

family $\frac{dx}{dt} = f\left(\frac{at+bx+c}{\alpha t+\beta x+\varphi}\right)$:

the $\frac{at+bx+c}{\alpha t+\beta x+\varphi}$ has to become homogeneous first

if $a\beta \neq \alpha b$ then let $t = t_0$, $x = x_0$ solution of $at+bx+c=0$, $\alpha t+\beta x+\varphi = 0$



lets transform variables to get

$y = x - x_0$
$s = t - t_0$

then we will convert problem into $\frac{dy}{dt} = \tilde{f}\left(\frac{at+bY}{\alpha t+\beta Y}\right)$

$$= \tilde{f}\left(\frac{a+bY/t}{\alpha+\beta Y/t}\right)$$

$\frac{dy}{dt} = \tilde{\tilde{f}}(Y/t)$ → we already know this solution

Eg: $\frac{dx}{dt} = \frac{x+1}{t+2} e^{\frac{x+1}{t+2}}$

$t \in I \subseteq \mathbb{R} \setminus \{-2\}$

let $s = t+2$
$y = x+1$

$\frac{dx}{dt} = \frac{dy}{ds}$

$\frac{dy}{ds} = \frac{y}{s} e^{\frac{y}{s}}$ let $sz = y$

$\frac{dy}{ds} = z + s\frac{dz}{ds}$

$\Rightarrow z + s\frac{dz}{ds} = ze^{z}$

$\Rightarrow z' = \frac{z(e^{z}-1)}{s}$

i.e $\frac{dx}{dt} = \frac{x+1}{t+2} e^{\frac{x+1}{t+2}} = f\left(\frac{x+1}{t+2}\right)$

$f(x) = xe^{x}$

$y(t) = x(t)+1$
$t+2 = s$
$y(s-2) = x(s-2)+1$
$y(s) = x(s)+1$

$\frac{dy}{ds} = \frac{y}{s} e^{y/s}$ let $z = y/s$

$\Rightarrow z + s\frac{dz}{ds} = ze^{z}$

$\frac{dx}{dt} = \frac{t+x+4}{t-x+2}$ Now $x=-1$, $t=-3$

$y = x+1$
$s = t+3$

$\Rightarrow \frac{dy}{ds} = \frac{y+s}{s-y}$

$\Rightarrow \frac{dy}{ds} = \frac{y/s+1}{1-y/s}$

Assignment - 2

Dhairya Kantawala

Assignment - 2

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1.

(a) $u''(t) - tu'(t) + u(t) = 0$, then let $u(t) = \sum_{k=0}^{\infty} c_k t^k$, we can find as coeff are real analytic

then $u'(t) = \sum_{k=1}^{\infty} k c_k t^{k-1}$

$= \sum_{k=0}^{\infty} (k+1) c_{k+1} t^k$

$u''(t) = \sum_{k=1}^{\infty} (k+1)(k) c_{k+1} t^{k-1}$

$= \sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} t^k$

now, $u''(t) - tu'(t) + u(t) = \left(\sum_{t=0}^{\infty} (k+2)(k+1) c_{k+2} t^k\right) - t\left(\sum_{t=0}^{\infty} (k+1) c_{k+1} t^k\right) + \left(\sum^{\infty} c_k t^k\right) = 0$

$\Rightarrow \left(\sum_{t=0}^{\infty} (k+2)(k+1) c_{k+2} t^k\right) - \left(\sum_{t=0}^{\infty} (k+1) c_{k+1} t^{k+1}\right) + \left(\sum_{k=0}^{\infty} c_k t^k\right) = 0$

$\Rightarrow \left(\sum_{t=0}^{\infty} (k+2)(k+1) c_{k+2} t^k\right) - \left(\sum_{t=0}^{\infty} (k) c_k t^k\right) + \left(\sum_{k=0}^{\infty} c_k t^k\right) = 0$

$\Rightarrow (k+2)(k+1) c_{k+2} - k c_k + c_k = 0$ $\left(\because \text{every } t^k \text{ term } 0 \text{ as } \{t^k\}_{k=0}^{\infty} \text{ is lin ind}\right)$

$\Rightarrow c_{k+2} = \dfrac{(k-1) c_k}{(k+2)(k+1)}$

where c_0 and c_1 are some constants so

$c_2 = \dfrac{(-1) c_0}{(2)(1)}$

$c_{2+2} = \dfrac{(2-1)}{(4)(3)} c_2$

$c_{4+2} = \dfrac{(\cancel{3})}{(6)(5)} \cdot \dfrac{(2-1)}{(4)(\cancel{3})} c_2$

$c_{6+2} = \dfrac{(\cancel{5})}{(8)(7)} \times \dfrac{(\cancel{3})}{(6)(\cancel{5})} \times \dfrac{(2-1)}{(4)(\cancel{3})} c_2$

$c_{8+2} = \dfrac{(\cancel{7})}{(10)(9)} \times \dfrac{1}{(8)(\cancel{7})} \times \dfrac{1}{3.2} \times \dfrac{(2-1)}{2.2} c_2$

$\vdots$

$c_{2n+2} = \dfrac{1}{(2n+2)} \times \dfrac{1}{(2n+1)} \times \dfrac{1}{(2)^{n-1}} \times \dfrac{1}{n!} \times \left(\dfrac{-1}{2}\right) c_0$

so, $c_{2n+2} = \dfrac{1}{2^n (n+1)!} \times \dfrac{1}{(2n+1)} \left(\dfrac{-1}{2} c_0\right)$

similarly $c_{3+2} = \dfrac{(3-1)}{(5)(4)} \times c_3 = 0$ $c_3 = (1-1) c_1 = 0$

$c_{5+2} = \dfrac{(\cancel{5-1})}{(7)(6)} \times \dfrac{(3-1)}{(5)(\cancel{4})} c_3 = 0$

$c_{7+2} = \dfrac{(\cancel{7-1})}{(9)(8)} \times \dfrac{(3-1)}{(7)(\cancel{6})} \times \dfrac{1}{(5)} c_3 = 0$

$\vdots$

$$C_{2n+4} = \frac{(2)}{(2n+3)(2n+2)} \times \frac{3}{1 \times 2 \times \cdots \times (2n+1)} \times 2 \times 4 \times \cdots 2n \; c_3 = 0$$

$$= \frac{2}{(2n+3)(2n+2)} \times \frac{3 \times (2)^n \times n!}{(2n+1)!} c_3 = 0$$

$$C_{2n+3} = \frac{1}{(2n+3)(n+1)} \times \frac{3 \times (2)^n \times n!}{(2n+1)!} c_3 = 0$$

so, $u(t) = \sum_{k=0}^{\infty} c_k t^k$, two solution are:

$$u_1(t) = c_0 + \sum_{k=0}^{\infty} c_{2k+2} t^{2k+2} \quad \text{where} \quad c_{2n+2} = \frac{1}{2^n (n+1)!} \times \frac{1}{(2n+1)} \left(\frac{-1}{2} c_0\right)$$

$$u_2(t) = \sum_{n=0}^{\infty} \frac{1}{(2n+3)(n+1)} \times \frac{3 \times (2)^n \times n!}{(2n+1)!} c_1 t^{2n+1} = c_1 t$$

where $c_0, c_1 \in \mathbb{R}$

now to calculate radius of convergence for $u_1(t)$:

if $\sum_{i=0}^{\infty} a_i$ then it cvg if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ ($\because$ ratio test)

here $u_1(t) = \sum_{i=0}^{\infty} c_{2i} t^{2i}$

then $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|c_{2n+4}||t^2|}{|c_{2n+2}|}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{2^{\cancel{n+1}}(n+2)!} \frac{1}{(2n+3)} \cancel{c_0} \cancel{\left(-\frac{1}{2}\right)} \, |t|^2}{\frac{1}{\cancel{2^n}(\cancel{n+1})!} \times \frac{1}{(2n+1)} \cancel{c_0} \cancel{\left(-\frac{1}{2}\right)}}$$

$$= \lim_{n\to\infty} \frac{(2n+1)}{(2n+3)} \times \frac{1}{n+2} |t|^2$$

as $n \to \infty$ the above always $\to 0$

so $|t| < \infty$

or $R = \infty$ is the radius of convergence

similarly for $u_2(t)$: $u_2(t) = \sum_{i=0}^{\infty} c_{2i+1} t^{2i+1} = c_1 t$

trivial convergence

as its just a const polynomial

(b) $u''(t)+t^3u'(t)+t^2u(t)=0$, we can find such series soln as coeff are real analytic from (a)

$$u(t)=\sum_{i=0}^{\infty} c_i t^i$$

$$u'(t)=\sum_{i=0}^{\infty}(i+1)c_{i+1}t^i$$

$$u''(t)=\sum_{i=0}^{\infty}(i+2)(i+1)c_{i+2}t^i$$

now

$$u''(t)+t^3u'(t)+t^2u(t)=0$$

$$\Rightarrow \sum_{i=0}^{\infty}(i+2)(i+1)c_{i+2}t^i+\sum_{i=0}^{\infty}(i+1)c_{i+1}t^{i+3}+\sum_{i=0}^{\infty}c_i t^{i+2}=0$$

now as $\{t^n\}_{n=0}^{\infty}$ is lin ind, coef of t^n $\forall n=0$

so $(2)(1)c_2t^0+(3)(2)c_3t^1+(4)(3)c_4t^2+\sum_{i=0}^{\infty}(i+2+3)(i+1+3)c_{i+2+3}t^{i+3}$

$$+\sum_{i=0}^{\infty}(i+1)c_{i+1}t^{i+3}+c_0t^2+\sum_{i=0}^{\infty}c_{i+1}t^{i+3}=0$$

$\Rightarrow 2c_2=0$ —— ①

$6c_3=0$ —— ②

$c_4=-\dfrac{c_0}{12}$ —— ③

$(i+5)(i+4)c_{i+5}+(i+1)c_{i+1}+c_{i+1}=0$ —— ④

$\Rightarrow c_{i+5}=-\dfrac{(i+2)c_{i+1}}{(i+5)(i+4)}$ —— ⑤

as $c_2=0$ we get $i=1$ ④ s.t

$c_6=0,\ c_{10}=0,\ c_{14}=0\ldots\ldots c_{4n+2}=0\ \forall n\in\mathbb{Z}_{\geq 0}$

similarly as $c_3=0$ we get

$c_7=0,\ c_{11}=0,\ldots$ or $c_{4n+3}=0\ \forall n\in\mathbb{Z}_{\geq 0}$

and let $c_0,c_1\in\mathbb{R}$ then

$c_4=-\dfrac{c_0}{4\times 3}$

putting $i=3$: $\quad c_8=\dfrac{+(5)}{(8)(7)}\times\left(+\dfrac{c_0}{4\times 3}\right)$

Similarly we get c_{4n} in terms of c_0 $\forall n\in\mathbb{Z}_{\geq 0}$

now let $c_1 \in \mathbb{R}$ then putting $i=0$ in ⑤:

$$c_5 = \frac{-(2)}{(5)(4)} c_1$$

$i=4$

$$c_9 = \frac{+(11)}{(14)(13)} \times \frac{(2)}{(5)(4)} c_1$$

and similarly we get
c_{4n+1} in terms of c_1 $\forall$ $n \in \mathbb{Z}_{\geq 0}$

so by above calculations we get 2 solutions of u depending of $c_0, c_1 \in \mathbb{R}$

s.t $u_1(t) = \sum_{n=0}^{\infty} c_{4n} t^{4n}$ where c_{4n} depends of c_0

$u_2(t) = \sum_{n=0}^{\infty} c_{4n+1} t^{4n+1}$ where c_{4n+1} depends of c_1

now, to find radius of conv, lets we use ratio test,

for $\sum_{i=1}^{\infty} a_i$ it conv if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

here $u_1(t) = \sum_{i=0}^{\infty} c_{4i} t^{4i}$

$$\lim_{n\to\infty} \frac{|c_{4n+4}| |t|^4}{|c_{4n}|} = \lim_{n\to\infty} \left|\frac{(4n-1+2)}{(4n-1+5)(4n-1+4)}\right| |t|^4$$

$(\because$ from ⑤ $i=4n-1)$

$$= \lim_{n\to\infty} \left|\underbrace{\frac{4n+1}{(4n+4)(4n+3)}}_{\text{as } n\to\infty \text{ term} \to 0}\right| |t|^4 < 1$$

$\Rightarrow |t| < \infty$
$\Rightarrow R \to \infty$ is radius of convergence

now for $u_2(t)$, let's use ratio test again where

$u_2(t) = \sum_{n=0}^{\infty} c_{4n+1} t^{4n+1}$

then we want $\lim_{n\to\infty} \frac{|c_{4n+5}| |t|^4}{|c_{4n+1}|} < 1$

putting $i=4n$ in ⑤ we get

$$\lim_{n\to\infty} \left|\underbrace{\frac{(4n+2)}{(4n+5)(4n+4)}}_{\to 0 \text{ as } n\to\infty}\right| |t|^4 < 1$$

$\Rightarrow |t| < \infty$
$\Rightarrow R \to \infty$ is radius of conv

so in both (a),(b) $u_1(t), u_2(t)$ conv $\forall$ $t \in \mathbb{R}$ as $R \to \infty$ for both cases

2. $(1+t^2)u''(t) + u(t) = 0$, $u(0)=1$, $u'(0)=1$ given
as real analytic coeff, we can find solution of the form $\sum_{k=0}^{\infty} c_k t^k$

now as $u(t) = \sum_{k=0}^{\infty} c_k t^k$

$$u'(t) = \sum_{k=1}^{\infty} k c_k t^{k-1} \quad (\because \text{normal diff})$$

$$u'(t) = \sum_{k=0}^{\infty} (k+1) c_{k+1} t^k$$

$$\& \quad u''(t) = \sum_{k=1}^{\infty} (k+1) k c_{k+1} t^{k-1}$$

$$= \sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} t^k$$

now, $(1+t^2)u''(t) + u(t) = 0$

$$\Rightarrow (1+t^2)\left(\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2} t^k\right) + \left(\sum_{k=0}^{\infty} c_k t^k\right) = 0$$

now as $\{t^k\}_{k=0}^{\infty}$ is lin ind, every coeff of t^k $\forall k =$

now

$$\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2} t^k + \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2} t^{k+2} + \sum_{k=0}^{\infty} c_k t^k = 0$$

$$\Rightarrow (2)(1)c_2 t^0 + (3)(2)c_3 t^1 + \sum_{k=0}^{\infty}(k+2+2)(k+1+2)\, c_{k+4} t^{k+2}$$

$$+ \sum_{k=0}^{\infty}(k+2)(k+1)\, c_{k+2} t^{k+2} + c_0 t^0 + c_1 t^1$$

$$+ \sum_{k=0}^{\infty} c_{k+2} t^{k+2} = 0$$

$$\Rightarrow 2c_2 + c_0 = 0 \quad \text{— ①}$$

$$6c_3 + c_1 = 0 \quad \text{— ②}$$

$$(k+4)(k+3)c_{k+4} + (k+2)(k+1)c_{k+2} + c_{k+2} = 0$$

$$\Rightarrow c_{k+4} = -\frac{c_{k+2}(1+(k+2)(k+1))}{(k+4)(k+3)} \quad \text{— ③}$$

as $u(0) = c_0 = 1$ given

& $u'(0) = c_1 = 1$ given

from ① we get: $c_2 = -\frac{1}{2}$

from ② we get: $c_3 = -\frac{1}{6}$

putting $k=0$ in ③:

$$c_4 = -\left(\frac{-1}{2}\right)\left(\frac{\cancel{1+(2)(1)}}{(4)\cancel{(3)}}\right)$$

$$c_4 = \frac{1}{2\times 4}$$

$$i=2: \quad c_6 = \frac{1}{2\times 4} \times (-1) \times \frac{(1+4\times 3)}{(6)(5)}$$

Similarly c_{2n} can be calculated $\forall n \in \mathbb{Z}_{\geq 0}$

putting $i=1$ $c_5 = \left(\frac{-1}{6}\right)(-1)\left(1+3\times 2\right) \times \frac{1}{(5)(4)}$, similarly c_{2n+1} $\forall n \in \mathbb{Z}_{\geq 0}$ can be calc.

as we calcutad c_i $\forall i \in \mathbb{Z}_{\geq 0}$ uniquely, we get
$u(t) = \sum_{i=0}^{\infty} c_i t^i$ as the solution

3. $(1-t^2)u''(t) - tu'(t) + \alpha^2 u(t) = 0$

(a) as analytic coeff, $u(t) = \sum_{k=0}^{\infty} c_k t^k$ is the form of solution

then $u'(t) = \sum_{k=1}^{\infty} k c_k t^{k-1} = \sum_{k=0}^{\infty} (k+1) c_{k+1} t^k$ from basic diff

& $u''(t) = \sum_{k=1} (k+1)(k) c_{k+1} t^{k-1}$

$$= \sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} t^k$$

$$= \sum_{k=2}^{\infty} (k)(k-1) c_k t^{k-2}$$

then $(1-t^2)u''(t) - tu'(t) + \alpha^2 u(t) = 0$

$$\Rightarrow u''(t) - t^2 u''(t) - tu'(t) + \alpha^2 u(t) = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} t^k - t^2 \left(\sum_{k=2}^{\infty} (k)(k-1) c_k t^{k-2} \right) - t \left(\sum_{k=1}^{\infty} (k) c_k t^{k-1} \right) + \alpha^2 \left(\sum_{k=0}^{\infty} c_k t^k \right) = 0$$

$$\Rightarrow (2)(1) c_2 t^0 + (3)(2) c_3 t^1 + \sum_{k=2}^{\infty} t^k [(k+2)(k+1) c_{k+2} - (k)(k-1) c_k - (k) c_k + \alpha^2 c_k] - t(c_1 t^0) + \alpha^2 c_0 t^0 + \alpha^2 c_1 t^1 = 0$$

as $\{t^k\}_{k=0}^{\infty}$ is lin independent every t^k has coeff $= 0$

so, $2c_2 + \alpha^2 c_0 = 0$ —— ①

$6c_3 - c_1 + \alpha^2 c_1 = 0$ —— ②

$(k+2)(k+1) c_{k+2} - (k)(k-1) c_k - (k) c_k + (\alpha^2) c_k = 0$ —— ③

from ①: $c_2 = -\frac{\alpha^2}{2} c_0$

from ②: $c_3 = \frac{(1-\alpha^2)}{6} c_1$

now, $c_{k+2} = \frac{(k^2 - \cancel{k} + \cancel{k} - \alpha^2) c_k}{(k+2)(k+1)}$

$c_{k+2} = \frac{(k^2 - \alpha^2) c_k}{(k+2)(k+1)}$ —— ④

for $k = 2$ we get:

$c_4 = \frac{(4^2 - \alpha^2) c_2}{(4)(2)}$ terms of c_0

similarly we can get c_{2n} in terms of c_0 $\forall n \in \mathbb{Z}_{\geq 0}$

for $k = 3$

$c_5 = (5^2 - \alpha^2) c_3 \times \frac{1}{(7)(6)}$

Similarly we get c_{2n+1} $\forall n \in \mathbb{Z}_{\geq 0}$ in terms of c_1

so, $u_1(t) = \sum_{n=0}^{\infty} c_{2n} t^{2n}$ $\quad u_1(t) = \sum_{n=0}^{\infty} c_{2n+1} t^{2n+1}$ are two solutions s.t

$u_1(t)$ depends of c_0 & $u_1(t)$ on c_1

now, we will find radius of conv for both $u_1(t)$ and $u_2(t)$ using root test

for $u_1(t)$: we want $\lim_{n\to\infty} \frac{|c_{2n+2}|}{|c_{2n}|} |t|^2 < 1$

$$\Rightarrow \lim_{n\to\infty} \frac{|((2n)^2 - \alpha^2)|}{|(2n+2)(2n+1)|} |t|^2 \text{ from } 4 \text{ putting } k = 2n$$

$$\Rightarrow |t|^2 < 1$$

$$\Rightarrow |t| < 1 \text{ or } R = 1 \text{ is radius of conv}$$

Similarly, for $u_2(t)$: $\lim_{n\to\infty} \frac{|c_{2n+3}|}{|c_{2n+1}|} |t|^2 < 1$

$$\Rightarrow \lim_{n\to\infty} \frac{|(2n+1)^2 - \alpha^2|}{|(2n+1+2)(2n+1+1)|} |t|^2 < 1 \quad (\because \text{putting } k = 2n+1 \text{ in } 4)$$

$$\Rightarrow |t|^2 < 1$$

$$\Rightarrow |t| < 1 \Rightarrow R = 1 \text{ is radius of conv}$$

so, $u_1(t), u_2(t)$ are two lin ind power series sol in $(-1,1)$

(b) $\alpha = n$ then there are two cases:

Case I: n = even i.e $\exists k$ s.t $n = 2k$

then

$$c_{2k+2} = \frac{c_{2k} \times ((2k)^2 - \alpha^2)}{(2k+2)(2k+1)} \quad (\because \text{putting } 2k \text{ in } 4)$$

as $\alpha^2 = (2k)^2$

$$\Rightarrow c_{2k+2} = 0 \text{ as } c_{2k+2} = 0,\ c_{2k+4} = 0, \ldots$$

by 4

and so, $c_0, c_2, c_4, \ldots c_{2k}$ are only non zero, else all 0 of form c_{2n}

from (a)

$$\Rightarrow u_1(t) = \sum_{n=0}^{\infty} c_{2n} t^{2n}$$

$$= c_0 t^0 + c_2 t^2 + \ldots c_{2k} t^{2k}$$

as $2k = n$ degree of $u_1(t) = n$

so, $u_1(t)$ is a polynomial soln of degree n given $c_0 \neq 0$

as $c_0 \neq 0 \Rightarrow c_2 \neq 0 \ldots c_{2k} \neq 0$ & so n degree polynomial case II : n is odd so $\exists k$ s.t $n = 2k+1$

now from ④ putting $2k+1$:

$$C_{2k+3} = \frac{((2k+1)^2 - \alpha^2)}{(2k+1+2)(2k+1+1)} C_{2k+1}$$

as $\alpha = (2k+1)$

$\Rightarrow C_{2k+3} = 0$ then $C_{2k+5} = 0$, $C_{2k+7} = 0 \ldots\ldots$

if $C_1 \neq 0$ then from ①

$C_3 \neq 0, \ldots . C_{2k+1} \neq 0$

& so $u_2(t) = \sum_{n=0}^{\infty} C_{2n+1} t^{2n+1}$

$= C_1 t^1 + C_3 t^3 + \cdots C_{2k+1} t^{2k+1}$

or $u_2(t)$ is polynomial of degree $2k+1 = n$

4. $u''(t) - 2tu'(t) + 2\alpha u(t) = 0$

(a) as coeff are analytic, $u(t) = \sum_{k=0}^{\infty} C_k t^k$ is a form of solution

now $u'(t) = \sum_{k=1}^{\infty} k C_k t^{k-1}$

$u''(t) = \sum_{k=2}^{\infty} (k)(k-1) C_k t^{k-2}$

$= \sum_{k=0}^{\infty} (k+2)(k+1) C_{k+2} t^k$

By Basic diff

now, $u''(t) - 2tu'(t) + 2\alpha u(t) = 0$

$\Rightarrow \sum_{k=0}^{\infty} (k+2)(k+1) C_{k+2} t^k - 2 \sum_{k=0}^{\infty} (k) C_k t^k + 2\alpha \sum_{k=0}^{\infty} C_k t^k = 0$

now as $\{t^k\}_{k=0}^{\infty}$ is lin ind, coeff of t^k $\forall k = 0$

$\Rightarrow (k+2)(k+1) C_{k+2} - 2k C_k + 2\alpha C_k = 0$

$$\Rightarrow \quad C_{k+2} = \frac{(2k - 2\alpha) C_k}{(k+2)(k+1)} \quad \text{——} ①$$

now, if $C_0, C_1 \in \mathbb{R}$ then

$k = 0$ in ①

$C_2 = \frac{(-2\alpha)}{(2)(1)} C_0$

$C_4 = \frac{(4 - 2\alpha)}{(4)(3)} C_2 = \frac{(4-2\alpha)}{(4)(3)} \times \frac{(-2\alpha)}{(2)(1)} C_0$

$C_6 = \frac{(2\times 4 - 2\alpha)}{(6)(5)} \times \frac{(2\times 2 - 2\alpha)}{(4)(3)} \times \frac{(-2\alpha)}{(2)(1)} C_0$

$k = 4$

$\vdots$

$$C_{2n} = \frac{\prod_{i=0}^{n-1} (2\times(2i) - 2\alpha)\, C_0}{(2n)!} \quad \text{——} ②$$

for $n \geq 1$

& $C_0 = C_0$

similarly

$K=1$ in ① $C_3 = \frac{(2\times1-2\alpha)C_1}{(3)(2)}$

$K=3$ $\quad C_5 = \frac{(2\times3-2\alpha)}{(5)(4)} \times \frac{(2\times1-2\alpha)C_1}{(3)(2)}$

$\vdots$

$$C_{2n+1} = \prod_{i=0}^{n-1} \frac{(2\times(2i+1)-2\alpha)\, C_1}{(2n+1)!} \quad \text{③}$$

for $n \geq 1$

for $n=0$: $\quad C_1 = C_1 \in \mathbb{R}$

then from ② & ③ we get two solutions of $u(t)$ depending on $C_0, C_1 \in \mathbb{R}$

$$u_1(t) = \sum_{i=0}^{\infty} C_{2i} t^{2i} \quad \text{s.t } C_{2i} \text{ from ②}$$

$$u_2(t) = \sum_{i=0}^{\infty} C_{2i+1} t^{2i+1} \quad \text{s.t } C_{2i+1} \text{ from ③}$$

now for $\sum_{k=0}^{\infty} a_k$ it converges for $\lim_{n\to\infty} \frac{|a_{k+1}|}{|a_k|} < 1$

then for $u_1(t)$, we get $\lim_{n\to\infty} \frac{|C_{2n+2}|}{|C_{2n}|} |t|^2 < 1$

for values of t to converge

also putting $k=2n$ in ① we get

$$\left|\frac{C_{2n+2}}{C_{2n}}\right| = \left|\frac{2(2n-\alpha)}{(2n+2)(2n+1)}\right| \to 0 \text{ as } n\to\infty$$

$\Rightarrow \lim_{n\to\infty} \left|\frac{C_{2n+2}}{C_{2n}}\right| |t|^2 \to 0$ as $\left|\frac{C_{2n+2}}{C_{2n}}\right| \to 0$

$\Rightarrow |t|^2 < \infty$

$\Rightarrow |t| < \infty$ or $\forall t \in \mathbb{R}$ $u_1(t)$ converges

similarly for $u_2(t)$:

$$\lim_{n\to\infty} \frac{|C_{2n+3}|}{|C_{2n+1}|} |t|^2 < 1$$

then putting $k = 2n+1$ in ① we get:

$$\frac{C_{2n+3}}{C_{2n+1}} = \frac{2(2n+1-\alpha)}{(2n+1+2)(2n+1+1)} \to 0 \text{ as } n\to\infty$$

$\Rightarrow \lim_{n\to\infty} \left|\frac{C_{2n+3}}{C_{2n+1}}\right| |t|^2 \to 0$

$\Rightarrow |t|^2 < \infty$

$\Rightarrow |t| < \infty$ so $\forall t \in \mathbb{R}$, $u_2(t)$ converges

so, we got two lin ind power series sol $u_1(t), u_2(t)$ in $\mathbb{R}$

(b) given $n \in \mathbb{Z}_{\geq 0}$, there are two cases:

Case I: n is even, then $\exists k$ s.t $n = 2k$, now $k \in \mathbb{Z}_{\geq 0}$

putting $n=k+1$ in ② we get:

$$C_{2k+2} = \frac{\prod_{i=0}^{k} (2\times(2i)-2\alpha)\, c_0}{(2k+2)!}$$

as $\alpha = n$

$$= \frac{\prod_{i=0}^{k-1} (2\times(2i)-2\alpha)}{(2k+2)!} \times (2\times 2k - 2\alpha) C_0$$

$$= \prod_{i=0}^{k-1} \frac{(2\times(2i)-2\alpha)}{(2k+2)!} (2n-2n) C_0 \qquad (\because 2k=n=\alpha)$$

$$=0$$

so, $C_{2k+2}=0$

similarly from ① recursion

$C_{2k+4} = 0$

$\vdots$

so on, so $u_1(t) = \sum_{i=0}^{\infty} C_{2i} t^{2i} = C_0 + C_2 t^2 + \dots C_{2k} t^{2k}$

also trivial to see that $C_{2k} \neq 0$ as from ② no term 0 (given $C_0 \neq 0$)

so, $u_1(t) = C_0 + C_2 t^2 + \dots C_{2k} t^{2k}$ as $n=2k$, $u_1(t)$ is a n^{th} degree polynomial solution

Case II: n is odd, $\exists k \in \mathbb{Z}_{\geq 0}$ s.t $n=2k+1$

now, in ③ we put $n=k+1$ to get:

$$C_{2k+3} = \frac{\prod_{i=0}^{k} (2\times(2i+1)-2\alpha)\, C_0}{(2n+3)!}$$

$$= \prod_{i=0}^{k-1} \frac{(2\times(2i+1)-2\alpha)}{(2n+3)!} \underbrace{(2\times(2k+1)-2\alpha)}_{0 \text{ as } 2\alpha = 2n,\ 2\times(2k+1)=2n} C_0$$

$$=0$$

so $C_{2k+3}=0$

similarly $C_{2k+5}=0$ from ① recursion
and all next odd terms similarly 0

now, $C_{2k+1} \neq 0$ given $C_1 \neq 0$ as from ③ there is no zero making term

so

$$u_2(t) = \sum_{i=0}^{\infty} C_{2i+1} t^{2i+1} = C_1 t + C_3 t^2 + \dots C_{2k+1} t^{2k+1}$$

as $2k+1=n$, $u_2(t)$ is an n degree polynomial

(c) given $\alpha = n$,

$H_n = (-1)^n e^{t^2} \frac{d^n}{dt^n}(e^{-t^2})$ is a polynomial solution

now first let's show the above is a polynomial, then we put $u(t) = H_n(t)$ to show it's a solution.

for $n=0$ $H_0(t) = (-1)^0 e^{t^2}(e^{-t^2}) = 1$
so $H_0(t)$ is a polynomial of degree 0

now let's assume for some $n=n$, H_n is true n^{th} degree polynomial
then by induction let's show H_{n+1} $(n+1)^{th}$ degree

now, $H_{n+1}(t) = (-1)^{n+1} e^{t^2} \frac{d^{n+1}}{dt^{n+1}}(e^{-t^2})$

$$= (-1)^{n+1} e^{t^2} \frac{d}{dt}\left(\frac{d^n}{dt^n}(e^{-t^2})\right)$$

as $H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n}(e^{-t^2})$

$$\Rightarrow H_n(t)(-1)^{-n} e^{-t^2} = \frac{d^n}{dt^n}(e^{-t^2})$$

putting this above:

$$H_{n+1}(t) = (-1)^{n+1} e^{t^2} \frac{d}{dt}\left(H_n(t)(-1)^{-n} e^{-t^2}\right)$$

$$H_{n+1}(t) = -e^{t^2}\frac{d}{dt}\left(H_n(t)e^{-t^2}\right) \quad \text{—— } ①$$

now from induction Hypotesis $H_n(t) = a_0 + a_1 t + \cdots a_n t^n$
where $a_n \neq 0$

then

$$H_{n+1}(t) = -e^{t^2}\frac{d}{dt}\left(a_0 e^{-t^2} + a_1 t e^{-t^2} + a_2 t^2 e^{-t^2} + \cdots a_n t^n e^{-t^2}\right)$$

$$= -e^{t^2}\left(a_0 \frac{d}{dt}(e^{-t^2}) + a_1\left(e^{-t^2} + t\frac{d}{dt}(e^{-t^2})\right) + \cdots + a_n\left(nt^{n-1}e^{-t^2} + t^n\frac{d}{dt}(e^{-t^2})\right)\right)$$

($\because$ product rule)

now $\frac{d}{dt}(e^{-t^2}) = -2te^{-t^2}$

so

$$H_{n+1}(t) = -e^{t^2}\left(-2ta_0 e^{-t^2} + a_1 e^{-t^2} - 2t^2 e^{-t^2} a_1 + \cdots + na_n t^{n-1} e^{-t^2} - 2t^{n+1} a_n e^{-t^2}\right)$$

$$= 2ta_0 - a_1 + 2t^2 a_1 + \cdots + na_n t^{n-1} - 2t^{n+1} a_n$$

so $H_{n+1}(t)$ is a $n+1$ degree polynomial as $-2a_n \neq 0$
so from induction th, $H_n(t)$ is n degree polynomial

now, let's show $H_n(t)$ is solution to given ODE
where $H_n(t) = (-1)^n e^{t^2}\frac{d^n}{dt^n}(e^{-t^2})$

then $H_n'(t) = (-1)^n e^{t^2}\frac{d^{n+1}}{dt^{n+1}}(e^{-t^2})$

$$+ (-1)^n 2te^{t^2}\frac{d^n}{dt^n}(e^{-t^2}) \quad (\because \text{product rule})$$

$$H_n''(t) = (-1)^n e^{t^2}\frac{d^{n+2}}{dt^{n+2}}(e^{-t^2}) + (-1)^n 2te^{t^2}\frac{d^{n+1}}{dt^{n+1}}(e^{-t^2})$$

$$+ (-1)^n\left[2e^{t^2} + 2t \times 2te^{t^2}\right]\frac{d^n}{dt^n}(e^{-t^2})$$

$$+ (-1)^n 2te^{t^2}\left[\frac{d^{n+1}}{dt^{n+1}}(e^{-t^2})\right] \quad (\because \text{product rule})$$

now, $H_n''(t) - 2t H_n'(t) + 2n H_n(t)$

$$= (-1)^n \Big[e^{t^2} \frac{d^{n+2}}{dt^{n+2}}(e^{-t^2}) + 2te^{t^2} \frac{d^{n+1}}{dt^{n+1}} e^{-t^2} + 2e^{t^2} \frac{d^n}{dt^n} e^{-t^2} + \cancel{4t^2 e^{t^2} \frac{d^n}{dt^n} e^{-t^2}} + \cancel{2te^{t^2} \frac{d^{n+1}}{dt^{n+1}}} e^{-t^2}$$

$$- \cancel{2te^{t^2} \frac{d^{n+1}}{dt^{n+1}}} e^{-t^2} - \cancel{4t^2 e^{t^2} \frac{d^n}{dt^n} e^{-t^2}} + 2ne^{t^2} \frac{d^n}{dt^n}(e^{-t^2}) \Big]$$

now if $W(t) = e^{-t^2}$ then

$$= (-1)^n e^{t^2} \left[\omega^{n+2}(t) + 2t\omega^{n+1}(t) + (2+2n)\omega^n(t) \right]$$

now, $\omega'(t) = -2te^{-t^2}$ from chain rule

$$= -2t\omega(t)$$

$\Rightarrow \omega'(t) + 2t\omega(t) = 0$

now diff again:

$$\omega^2(t) + 2t\omega'(t) + 2\omega(t) = 0$$

and again:

$$\omega^3(t) + 2t\omega^2(t) + 2\omega'(t) + 2\omega'(t) = 0 \quad \text{——— } (2)$$

and again:

$$\omega^4(t) + 2t\omega^3(t) + 2\omega^2(t) + 2\omega^2(t) + 2\omega^2(t) = 0$$

so, lets show that $\omega^{n+2}(t) + 2t\omega^{n+1}(t) + (2+2n)\omega^n(t) = 0$

for $n=1$ from prev (2)

we got

$$\omega^3(t) + 2t\omega^2(t) + (2+2)\omega'(t) = 0$$

so true for $n=1$,

lets suppose true for $n=r$, then using induction

$$\omega^{r+2}(t) + 2t\omega^{r+1}(t) + (2+2r)\omega^r(t) = 0$$

diff:

$$\omega^{r+3}(t) + 2t\omega^{r+2}(t) + 2\omega^{r+1}(t) + (2+2r)\omega^{r+1}(t) = 0$$

$\Rightarrow$ true for $r+1$ case, so by induction true $\forall r$

$\Rightarrow H_n''(t) - 2t H_n'(t) + 2n H_n(t)$

$$= (-1)^n e^{t^2} \left[\omega^{n+2}(t) + 2t\omega^{n+1}(t) + (2+2n)\omega^n(\right.$$

$$= 0$$

so, $H_n(t)$ is solution to given ODE.

5. $t^2 u''(t) + 2tu'(t) - 6u(t) = 0 \quad t > 0$

this is of form $t^2 u''(t) + atu'(t) + bu(t) = 0$

where $a, b \in \mathbb{R}$, so we can use euler equation to solve, where

$q(r) = r(r-1) + ar + b$ is the indicial polynomial

$a = 2,\ b = -6$

$q(r) = r(r-1) + 2r - 6$
$= r^2 - r + 2r - 6$
$= r^2 + r - 6$

then $q(r) = 0 = r^2 + r - 6$
$r^2 + 3r - 2r - 6 = 0$
$\Rightarrow r(r+3) - 2(r+3) = 0$
$\Rightarrow (r+3)(r-2) = 0$

$Lu = t^2 u^2 + a t u' + u$

now, if solution is in form of $L(t^r) = (r(r-1) + ar + b)t^r$ (done in class)
$(\because t > 0)$
$\Rightarrow L(t^r) = (r+3)(r-2)t^r$

so, $L(t^2) = 0$
$\& \; L(t^{-3}) = 0$

for $c_1, c_2 \in \mathbb{R}$
$c_1 t^2 \,\&\, c_2 t^{-3}$ will solve the given ODE as:

$u_1(t) = c_1 t^2$

$u_1'(t) = 2c_1 t$

$u_1''(t) = 2c_1$

then $2c_1(t^2) + 2t(2c_1 t) - 6c_1 t^2 = 0$

& similarly
$$\left.\begin{aligned} u_2(t) &= t^{-3} c_2 \\ u_2'(t) &= -3t^{-4} c_2 \\ u_2''(t) &= 12 t^{-5} c_2 \end{aligned}\right\} t > 0$$

so, $u_2''(t) \times t^2 + 2t u_2'(t) - 6 u_2(t)$
$= (12t^{-3} - 6t^{-3} - 6t^{-3}) c_2$
$= 0$

lin ind as $\{t^2, t^{-3}\}$ lin ind

so, $u_1(t) = c_1 t^2$
$u_2(t) = c_2 t^{-3}$ are two solutions of the given ODE for some $c_1, c_2 \in \mathbb{R}$

6. $u(t) = |t|^r \sum_{k=0}^{\infty} c_k t^k$ solutions for ODE: $t^2 u''(t) + t u'(t) + \left(t^2 - \frac{1}{4}\right) u(t) = 0$ given $|t| > 0$

now if we let $t > 0$
then we want to find $u(t) = t^r \sum_{k=0}^{\infty} c_k t^k$

then $u'(t) = r t^{r-1} \sum_{k=0}^{\infty} c_k t^k$
$+ t^r \sum_{k=1}^{\infty} k c_k t^{k-1}$ ($\because$ product rule)

Similarly $u''(t) = (r)(r-1)t^{r-2} \sum_{k=0}^{\infty} c_k t^k + r t^{r-1} \sum_{k=1}^{\infty} k c_k t^{k-1}$
$+ r t^{r-1} \sum_{k=1}^{\infty} k c_k t^{k-1}$
$+ t^r \sum_{k=2}^{\infty} (k)(k-1) c_k t^{k-2}$

($\because$ product rule)

now putting above in $t^2 u''(t) + t u'(t) + \left(t^2 - \frac{1}{4}\right) u(t) = 0$

$$\Rightarrow (r)(r-1)t^r\sum_{k=0}^{\infty}c_k t^k + r t^k\sum_{k=0}^{\infty}k c_k t^k$$
$$+ r t^k \sum_{k=0}^{\infty} k c_k t^k$$
$$+ t^r \sum_{k=0}^{\infty}(k)(k-1)c_k t^k$$
$$+ r t^r \sum_{k=0}^{\infty} c_k t^k + t^r \sum_{k=0}^{\infty} k c_k t^k + \left(t^2-\frac{1}{4}\right)\left(t^r\sum_{k=0}^{\infty} c_k t^k\right) = 0$$

now as $\{t^r\}_{r=0}^{\infty}$ is linearly independent, we get every coef of $t^k = 0$ $\forall k$

for $t^0 \times t^r$: $(r)(r-1)c_0 + r c_0 - \frac{1}{4}c_0 = 0$

$$\Rightarrow c_0\left(r^2-\frac{1}{4}\right)=0$$

now $t^1 \times t^r$:

$$(r)(r-1)c_1 + r c_1 + r c_1 + r c_1 + c_1 - \frac{1}{4}c_1 = 0$$
$$\Rightarrow c_1\left[r^2 - \cancel{r} + \cancel{r} + r + r + \frac{3}{4}\right] = 0$$
$$\Rightarrow c_1\left[(r+1)^2 - \frac{1}{4}\right] = 0$$

for $k \geq 2$: $t^k \times t^r$:

$$(r)(r-1)c_k + r k c_k + r k c_k + (k)(k-1)c_k + r c_k + k c_k - \frac{1}{4}c_k + c_{k-2} = 0$$
$$\Rightarrow c_k\left[r^2 - \cancel{r} + r k + r k + k^2 - \cancel{k} + \cancel{r} + \cancel{k} - \frac{1}{4}\right] + c_{k-2} = 0$$
$$\Rightarrow c_k\left[(r+k)^2 - \frac{1}{4}\right] + c_{k-2} = 0$$

let $q(r) = (r)^2 - \frac{1}{4}$ then

we got $q(r)c_0 = 0$ —— ①
$q(r+1)c_1 = 0$ —— ②

& $k \geq 2$: $q(r+k)c_k = -c_{k-2}$ —— ③

now roots of the $q(r) = 0 \Rightarrow r^2 - \frac{1}{4} = 0 \Rightarrow r = \pm\frac{1}{2}$

so, putting $r = 1/2$ we get $q(\frac{1}{2})c_0 = 0$
$c_0 \in \mathbb{R}$
$q(\frac{1}{2}+2)c_2 = -c_0$

as $q(1/2+2) \neq 0 \Rightarrow c_2 = \dfrac{-c_0}{q(\frac{1}{2}+2)}$

similarly $q\left(\frac{1}{2}+1\right)c_1 = 0$

as $q\left(\frac{1}{2}+1\right) \neq 0 \Rightarrow c_1 = 0$

and so $c_{2n+1} = 0$ $\forall n \in \mathbb{Z}_{\geq 0}$

& $c_4 = \dfrac{-c_2}{q(1/2+4)} = \dfrac{c_0}{q(1/2+4)\,q(1/2+2)}$

so, $c_{2n} = \dfrac{(-1)^n c_0}{\prod_{i=1}^{n} q\left(\frac{1}{2}+2i\right)}$

and so $u_1(t) = |t|^{1/2} \sum_{i=0}^{\infty} c_{2i} t^{2i}$ depending on $c_0 \in \mathbb{R}$ $\left(\because u_1(t) = |t|^r \sum_{i=0}^{\infty} c_i t^i\right)$

$$= c_0 |t|^{1/2} \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{\prod_{i=1}^{k} q\left(\frac{1}{2}+2i\right)}$$

when $t > 0$

now similarly if we put $r = -1/2$ we get

$q(-1/2) c_0 = 0 \Rightarrow c_0 \in \mathbb{R}$

$q\left(-1/2+1\right) c_1 = 0 \Rightarrow \underset{\text{Any value}}{c_1 \in \mathbb{R}}$

and so now, we get $c_{2k} = \dfrac{(-1)^k c_0}{\prod_{i=1}^{k} q\left(\frac{-1}{2}+2i\right)}$ —— (4)

$k \geq 1$

and from (3) we get

$$c_3 = \frac{-c_1}{q\left(\frac{-1}{2}+3\right)} \qquad q\left(\tfrac{-1}{2}+3\right) \neq 0$$

$$c_5 = \frac{-c_3}{q\left(\frac{-1}{2}+5\right)} = \frac{+c_1}{q\left(\frac{-1}{2}+5\right) q\left(\frac{-1}{2}+3\right)} \qquad q\left(\tfrac{-1}{2}+5\right) \neq 0$$

$\vdots$

$k \geq 1$

$$c_{2k+1} = \frac{(-1)^k c_1}{\prod_{i=1}^{k} q\left(\frac{-1}{2}+2i+1\right)} \quad \text{—— (5)}$$

and so we get for some $c_1, c_2 \in \mathbb{R}$

$$u_2(t) = |t|^{-1/2} \sum_{i=0}^{\infty} c_i t^i$$

$$= |t|^{-1/2} \left(c_0 + c_1 t + \sum_{k=1}^{\infty} \underbrace{\left(\frac{(-1)^k c_0}{\prod_{i=1}^{k} q\left(\frac{-1}{2}+2i\right)} t^{2k} + \frac{(-1)^k c_1}{\prod_{i=1}^{k} q\left(\frac{-1}{2}+2i+1\right)} t^{2k+1} \right)}_{\text{from (4), (5)}} \right)$$

so for $t > 0$, we get $u_1(t), u_2(t)$, now i will show that for $t < 0$ we get some $u_1(t), u_2(t)$ by doing similar calculations

for $t < 0$, we need to find $u(t) = (-t)^r \sum_{k=0}^{\infty} c_k t^k$ solution lets find $u'(t)$, $u''(t)$ of the above form by normal diff and product rule:

$$u(t) = (-t)^r \sum_{k=0}^{\infty} c_k t^k$$

$$u'(t) = r(-t)^{r-1}(-1) \sum_{k=0}^{\infty} c_k t^k + (-t)^r \sum_{k=1}^{\infty} k c_k t^{k-1}$$

$$u''(t) = (r)(r-1)(-t)^{r-2} \sum_{k=0}^{\infty} c_k t^k + r(-t)^{r-1}(-1) \sum_{k=1}^{\infty} k c_k t^{k-1}$$

$$+ r(-t)^{r-1}(-1) \sum_{k=1}^{\infty} k c_k t^{k-1} + (-t)^r \sum_{k=2}^{\infty} (k)(k-1) c_k t^{k-2}$$

then $t^2 u''(t) + t\, u'(t) + \left(t^2 - \frac{1}{4}\right) u(t) = 0$

$$\Rightarrow (r)(r-1)(-t)^r \sum_{k=0}^{\infty} c_k t^k + r(-t)^r \sum_{k=0}^{\infty} k c_k t^k$$

$$+ r(-t)^r \sum_{k=0}^{\infty} k c_k t^k + (-t)^r \sum_{k=0}^{\infty} k(k-1) c_k t^k$$

$$+ r(-t)^r \sum_{k=0}^{\infty} c_k t^k + (-t)^r \sum_{k=0}^{\infty} k c_k t^k$$

$$+ \left(t^2 - \frac{1}{4}\right)(-t)^r \sum_{k=0}^{\infty} c_k t^k = 0$$

as $\{t^r\}_{r=0}^{\infty}$ is lin ind, we get coeff of $t^r = 0 \ \forall r$

for $(-t)^r \times t^0$: $c_0\left[(r)(r-1) + r - \frac{1}{4}\right] = 0$

$$\Rightarrow c_0\, q(r) = 0$$

from before

$(-t)^r \times t^1$: $c_1\left[(r)(r-1) + r + r + r + 1 - \frac{1}{4}\right] = 0$

$$\Rightarrow c_1\left[r^2 + 2r + 1 - \frac{1}{4}\right] = 0$$

$$\Rightarrow c_1\, q(r+1) = 0$$

for $k \geq 2$: $(-t)^r (t)^k$:

$$c_k\left[r^2 + 2kr + k^2 - \frac{1}{4}\right] = -c_{k-2}$$

$$\Rightarrow c_k(q(r+k)) = -c_{k-2}$$

so, we get same equations as $t>0$ case, so following similar calculations we will get 2 solutions for $c_0, c_1 \in \mathbb{R}$ s.t

$$u_1(t) = c_0(-t)^{1/2} \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{\prod_{i=1}^{k} q\left(\frac{1}{2} + 2i\right)} = c_0 |t|^{1/2} \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{\prod_{i=1}^{k} q\left(\frac{1}{2} + 2i\right)}$$

$$u_2(t) = (-t)^{-1/2}\left(c_0 + c_1 t + \sum_{k=1}^{\infty}\left(\frac{(-1)^k c_0}{\prod_{i=1}^{k} q\left(\frac{-1}{2}+2i\right)} t^{2k} + \frac{(-1)^k c_1}{\prod_{i=1}^{k} q\left(\frac{-1}{2}+2i+1\right)} t^{2k+1}\right)\right)$$

$$= |t|^{-1/2}\left(c_0 + c_1 t + \sum_{k=1}^{\infty}\left(\frac{(-1)^k c_0}{\prod_{i=1}^{k} q\left(\frac{-1}{2}+2i\right)} t^{2k} + \frac{(-1)^k c_1}{\prod_{i=1}^{k} q\left(\frac{-1}{2}+2i+1\right)} t^{2k+1}\right)\right)$$

so we got $u_1(t), u_2(t)$

s.t $u(t) = |t|^k \sum_{k=0}^{\infty} c_k t^k$

Tutorial-5 :

Exe: $f:[-2,2]\to\mathbb{R}$, $f\in C^2$, $f(1)=f(-1)=1$ & $\int_{-1}^{1} f(x)\,dx=0$, then show that

$$\int_{-1}^{1}(f''(x))^2\,dx \geq 15$$

Ans:

$$\left|\int_{-1}^{1} fg\right| \leq \left|\int_{-1}^{1}|f|^2\right|^{1/2}\left|\int_{-1}^{1}|g|^2\right|^{1/2}$$

$$\Rightarrow \underbrace{\int_{-1}^{1} f''g}_{} \leq \left(\int_{-1}^{1} f''^2\right)^{1/2}\left(\int_{-1}^{1}|g|^2\right)^{1/2}$$

$$= gf'\Big|_{-1}^{1} - \int g'f'$$

$$= gf'\Big|_{-1}^{1} - \left[\cancel{g'f\Big|_{-1}^{1}} - \int_{-1}^{1} g''f\right]$$

$g=\frac{(x-1)(x+1)}{2}=\frac{x^2-1}{2}$

$g'=\frac{2x}{2}=x$

$g''=1$

so, $=0-\left[(1)(1)+(-1)(1)\right]+0$

$=2$

$$2\leq\left(\int_{-1}^{1}(f'')^2\right)^{1/2}\left(\int_{-1}^{1}\left(\frac{x^2-1}{2}\right)^2 dx\right)^{1/2}$$

$$\int_{-1}^{1}\left(\frac{x^2-1}{2}\right)^2 dx=\int_{-1}^{1}\frac{x^4-2x^2+1}{4}\,dx=\frac{\frac{2}{5}-\frac{2\cdot2}{3}+2}{4}=\frac{\cancel{2}\left(\frac{-7}{15}+1\right)}{2}=\frac{8}{30}$$

so $\int_{-1}^{1}(f'')^2 \geq \left(2\times\left(\frac{30}{8}\right)^{1/2}\right)^2 = 4\times\frac{30}{8} = 15 \Rightarrow \int_{-1}^{1}(f'')^2\geq 15$

Exact equations :

An ODE of the form $P(x,y)\frac{dy}{dx}+Q(x,y)=0$ ① is called exact on Ω (open, basic conn domain)

if $\exists$ a fn $F(x,y)\in C^2(\mathbb{R})$ s.t. $\begin{cases}\partial_y F(x,y)=P(x,y)\\ \partial_x F(x,y)=Q(x,y)\end{cases}$ $\forall\, x,y\in\Omega$

Note: if $y=y(x)$ is a solution of the ODE ① then $\frac{d}{dx}F(x,y(x))=(\partial_x F)(x,y(x))+(\partial_y F)(x,y(x))\times\frac{dy}{dx}$

chain rule

$=Q(x,y)+P(x,y)\frac{dy}{dx}=0$

so, $h(x)=F(x,y(x))$

$h'(x)=0\Rightarrow \underbrace{F(x,y(x))=c}_{\text{this is the solution}}$

also as $\partial x\partial y F=\partial y\partial x F\Rightarrow \partial_x P=\partial_y Q$

Integrating factor :

$P(x,y)\frac{dy}{dx}+Q(x,y)=0$, A function $\mu(x,y)$ is called an IF of ① if

$\mu P\frac{dy}{dx}+\mu Q=0$ is an Exact equation

so, $\partial x(\mu P)=\partial y(\mu Q)$

$\Rightarrow P\mu_x+\mu P_x=\mu_y Q+Q_y\mu$

we try to get: $\mu_y Q=0$

$\mu_x=\frac{(Q_y\mu-\mu P_x)}{P}$

$\mu_x=\mu\frac{(Q_y-P_x)}{P}$ to find μ

Eg: $\frac{dy}{dx}\underbrace{(3y^2+x)}_{P(x,y)} + \underbrace{(2x^2+2xy^2+1)y}_{Q(x,y)} = 0$

$\partial_x P(x,y) = 1$

$\partial_y Q(x,y) = (2x^2+2xy^2+1)+(4xy^2)$

done match

let $\partial_x \mu(x,y) P(x,y) = \partial_y \mu(x,y) Q(x,y)$

$$\mu_x = \mu \frac{(2x^2+2xy^2+4xy^2)}{(3y^2+x)} = \mu(2x)$$

$$\mu = ce^{x^2}$$

so now $\partial_x F(x,y) = Q(x,y)\mu(x)$

$\partial_y F(x,y) = P(x,y)\mu(x)$

$\partial_x F(x,y) = (2x^2+2xy^2+1)ye^{x^2}$

$\partial_y F(x,y) = (3y^2+x)e^{x^2}$

$\Rightarrow F(x,y) = (y^2+xy)e^{x^2} + \phi(x)$

$\partial_x F = ye^{x^2} + (y^3+xy)2xe^{x^2} + \phi'(x)$

$\phi'(x) = 0$

$F(x,y) = (y^3+xy)e^{x^2}$

so $(y^3+xy)e^{x^2} = C$ is solution of ODE

Qualitative Behaviour:

Phase plane & phase portal: consider the system $\frac{dx}{dt} = f(x,y)$, $\frac{dy}{dt} = -g(x,y)$

Autonomus system

now, $y^{(n)}(t) = f(x, y, y', \dots y^{(n-1)})$

Note: If $(x(t), y(t))$ solves system, then $(x(t+c), y(t+c))$ also solves for any constant c

A solution $(x(t), y(t))$ can be interperated as a parametrisation of the corr. solution in the x-y plane. This (x-y) plane is said the phase-plane. The solution curves are called trajectories/ orbits of ODE

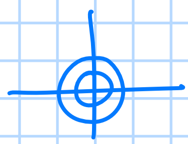
A sketch of several trajectories / orbits is called phase-point

this phase point gives an overview of the qualitative properties of solution

Eg: $\frac{dx}{dt} = y$ $\frac{dy}{dt} = -x$ along $y = y(x)$, ODE form

$y\frac{dy}{dt} + x\frac{dx}{dt} = 0$, $y\frac{dy}{dx} + x = 0$

$F(x, y(x)) = x^2 + y^2 = C$



Tutorial-6:

Exe: $u:[0,1] \to [0,\infty)$ cont

$M = \sup_{[0,1]} u(t)$

$$u(t) \leq 1 + \int_0^t \left[3s^2 u(s) + \left(\frac{u(s)}{M}\right)^4 \right] ds \quad \forall t \in [0,1]$$

show $u(t) < 4 \ \forall t$

Ans: $u(t) \leq 1 + \int_0^t u(t)\left[3s^2 + \left(\frac{u(s)^3}{M^4}\right)\right] ds \leq 1 + \int_0^t u(t)\left(3s^2 + \frac{1}{M}\right) ds$

$$\Rightarrow u(t) \leq e^{\int_0^t (3s^2 + \frac{1}{M}) ds}$$

$$= e^{t^3 + \frac{t}{M}}$$

$$\leq e^1 \times e^{\frac{1}{M}} = e^{1+\frac{1}{M}}$$

$\Rightarrow u(t) \leq e^{1+1/M}$

so $\sup u(t) \leq e^{1+1/M}$

$\Rightarrow M \leq 1 + 1/M$

$\Rightarrow M^2 \leq M+1$

$M < 4$ as inequality invalid for $M \geq 4$

so, $u(t) < 4$

we had H_n in assignment, we find H_n'', H_n' and use leibnitz rule by

$H_{n+1}' = 2(n+1) H_n$

and so we get: $H_n'' - 2tH_n' + 2nH_n = 2(n+1)H_n - H_{n+1}'$

$= 0$

Exe: $u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \in \mathbb{R}^2$ be $C^1[0,\infty)$ $\frac{du}{dt} = A(t)u + f(t)$

$u(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$A(t) = \begin{pmatrix} \sin t & 0 \\ 0 & \cos t \end{pmatrix}$ $f(t) = \begin{pmatrix} e^t \\ 0 \end{pmatrix}$

show that

$$\sqrt{u_1^2(t) + u_2^2(t)} \leq (1+t)e^t \quad \forall t \geq 0$$

Ans: Here $h(t) = \|u(t)\|$

then

$\frac{d}{dt} h^2(t) = 2h(t)h'(t)$ dot product

$= \frac{d}{dt}(u(t) \cdot u(t))$ $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

$\leq 2u(t) \cdot \frac{d}{dt} u(t)$ $A = \begin{pmatrix} \sin t & 0 \\ 0 & \cos t \end{pmatrix}$

$= 2\langle u, Au + f \rangle$ $Au = \begin{pmatrix} \sin t \ u_1 \\ \cos t \ u_2 \end{pmatrix}$

$$\langle u, Au \rangle = \sin t \, u_1^2 + \cos t \, u_2^2$$
$$\le \|u\|^2$$

$$\frac{d}{dt} h^2(t) \le 2\|u\|^2 + 2\|u\| e^t$$
$$\cancel{2} \frac{d}{dt} h(t) \le \cancel{2}\|u\|^{\cancel{2}} + \cancel{2}\|\cancel{u}\| e^t$$
$$h'(t) \le h(t) + e^t$$

now as $h'(t) \le h(t) + e^t$

$$e^{-t} h'(t) \le e^{-t} h(t) + 1$$

$$\frac{d}{dt}\left(e^{-t} h(t)\right) = e^{-t} h'(t) - e^{-t} h(t)$$

$$\frac{d}{dt}\left(e^{-t} h(t)\right) \le 1$$

$$e^{-t} h(t) \le t + c$$

$$e^{-t} \sqrt{u_1^2(t) + u_2^2(t)} \le t + c$$

$t = 0$

$$1 \le c \Rightarrow \sqrt{u_1^2(t) + u_2^2(t)} \le (1+t) e^t$$

Tutorial - 3

Dhairya

Assignment-3

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1.(a) $y' = y^{1/2}$ $y(0)=1$

let $y_1(t) = \frac{(t+2)^2}{4}$ for $t \in [-2,\infty)$, $y_1(0)=1$

then $y_1'(t) = \underbrace{\frac{(t+2)}{2}}_{\geq 0 \text{ as } t+2 \geq 0} = (y_1(t))^{1/2}$

$\Rightarrow y(t)$ is a solution to given IVP

so we have shown existance, now

for $I = (-2,\infty)$
$\Omega = (0,\infty)$

$f: I \times \Omega \to \mathbb{R}$ as $I \times \Omega$ is open
$f(t,x) = (x)^{1/2}$

$f(t,x) = (x)^{1/2}$ as $x \in (0,\infty)$
we have $\forall x \in (0,\infty)$

$$\lim_{\substack{h\to 0\\ r\to 0}} f(t+h, x+r) = \lim_{r\to 0}(x+r)^{1/2}$$
$$= x^{1/2}$$
as $x \in (0,\infty)$

so, $f(t,x)$ is continious from defn of continuity on $I \times \Omega$

now for $\varepsilon > 0$ s.t $[x-\varepsilon, x+\varepsilon] \subset (0,\infty)$
and all $t \in (-2,\infty)$

$\forall y_1, y_2 \in (x-\frac{\varepsilon}{2}, x+\frac{\varepsilon}{2}) \subset [x-\varepsilon, x+\varepsilon] \subset (0,\infty)$

$$|f(t,y_1) - f(t,y_2)| = |y_1^{1/2} - y_2^{1/2}|$$
$$= |y_1^{1/2} - y_2^{1/2}| \frac{|y_1^{1/2} + y_2^{1/2}|}{|y_1^{1/2} + y_2^{1/2}|}$$
$$= \frac{|y_1 - y_2|}{|y_1^{1/2} + y_2^{1/2}|}$$

now as $[x-\varepsilon, x+\varepsilon]$ is closed and bounded so compact

$$|y_1^{1/2} + y_2^{1/2}| \geq |y_1^{1/2}| \geq (x-\varepsilon)^{1/2}$$

as $y_1 > 0$, $y_2 > 0$

so, $|f(t,y_1) - f(t,y_2)| \leq |y_1 - y_2| \times \frac{1}{(x-\varepsilon)^{1/2}}$

so, $L_x = \frac{1}{(x-\varepsilon)^{1/2}}$

be lipschitz constant

this is for every interval of $(0,\infty)$ so, f is locally lipschitz

so on interval I, $y_1(t) = (\frac{t-2}{4})^{1/2}$ is unique $t \in (-2,\infty)$ from the theorm done in class (u_1 on I_1, u_2 on I_2 then $u_1 = u_2$ on $I_1 \cap I_2$)

now we extend y_1 to $(-\infty,-2]$ i.e

$$y_1(t) = \begin{cases} \frac{(t+2)^2}{4} & ; t > -2 \\ 0 & ; t \leq -2 \end{cases}$$

then y_1 is continious at $t=-2$ (trivial)

this is unique as any solution to $y^{1/2} = y'$ for $t \in (-\infty,-2]$ should be s.t

$y(-2)=0$ ($\because$ continuity)

and $y^{1/2}$ to exist $\Rightarrow$ $y \geq 0$

and so $y' \geq 0$ $\forall$ $t \in (-\infty,-2]$

i.e y to be non dec on $(-\infty,-2]$

and $y \geq 0$, where $y(-2)=0$

$\Rightarrow y=0$ as if $y(\alpha)>0$ for some $\alpha<-2$ then not non-dec

so, $y_1(t) = \begin{cases} \frac{(t+2)^2}{4} & ; t > -2 \\ 0 & ; t \leq -2 \end{cases}$ is unique solution to IVP

also, maximal interval of existance is $\mathbb{R}$ ($\because$ trivial from y_1)

(b) $y' = y^{1/3}$ $y(0)=1$

let $y_1(t) = \frac{(2t+3)^{3/2}}{3\sqrt{3}}$ for $t \in (-3/2,\infty)$

then $y_1'(t) = \frac{\cancel{3}}{\cancel{2}} \frac{(2t+3)^{1/2}}{\cancel{3}\sqrt{3}} \cancel{(2)}$

$= \frac{(2t+3)^{1/2}}{\sqrt{3}}$

$(y_1(t))^{1/3} = \frac{1}{\sqrt{3}}(2t+3)^{1/2} = y_1'(t)$, so $\exists$ a solution

let $I = (-3/2,\infty)$

$\Omega = (0,\infty)$ then

$f(t,x) = x^{1/3}$ is s.t for $h,\gamma \to 0$

$$\lim_{h,\gamma \to 0} f(t+h, x+\gamma) = \lim_{\gamma \to 0} (x+\gamma)^{1/3}$$
$$= x^{1/3}$$

so, f is continious on $I \times \Omega$, $I \times \Omega$ is open

now, as $f(t,x)$, now $\frac{\partial f}{\partial t} = 0$, $\frac{\partial f}{\partial x} = \frac{1}{3}(x)^{-2/3}$

as $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x}$ is continious $\Rightarrow$ f to be differentiable on $I \times \Omega$

so, $f \in C^1$ and so f is locally lipschitz ($\because$ exercise done in class)

then from theorm done in class for $I = (-\frac{3}{2},\infty)$ y_1 is unique

now, let's extend $y_1(t) = \begin{cases} \frac{(2t+3)^{3/2}}{3\sqrt{3}} & ; t > -\frac{3}{2} \\ 0 & ; t \leq -\frac{3}{2} \end{cases}$

then y_1 is cont at $t=\frac{-3}{2}$ ($\because$ trivial)

and for $t \leq -3/2$

$y_1' = 0 \quad y_1^{1/3} = 0$

$\Rightarrow y_1' = y_1^{1/3}$

so, y_1 is solution to IVP

now as y is s.t $y^{1/3} = y'$ for $t \in (-\infty, -\frac{3}{2}]$ we need $y^{1/3}$ and y' to have same sign

i.e $y^{1/3} > 0 \Leftrightarrow y' > 0$

or $y > 0 \Leftrightarrow y' > 0$, similarly for other

now, as y is cont ($\because f$ is C^1) if for some $\alpha \in (-\infty, -\frac{3}{2})$ $y(\alpha) > 0$, and $y'(\alpha) > 0$

then from continuing onwards possible if

$\forall x > \alpha$, $y(x) > 0, y'(x) > 0$

this is not possible as $y(-\frac{3}{2}) = 0$

similarly for $y(\alpha) < 0$, $y'(\alpha) < 0$ case

so, $y(t) = 0 \ \forall t \in (-\infty, -\frac{3}{2}]$, so we have shown y_1 is unique on $\mathbb{R}$, and maximal interval will be $\mathbb{R}$.

2. (a) σ be $C^1(\mathbb{R})$

$|\sigma'(x)| \leq K\sigma(x)$; $-\infty < x < \infty$ where $K > 0$ is const

To prove: $\sigma(a)e^{-K|x-a|} \leq \sigma(x) \leq \sigma(a)e^{K|x-a|}$ for any $a \in (-\infty, \infty)$

proof: Case I: if $\exists a \in (-\infty, \infty)$ s.t $\sigma(a) = 0$, then

$|\sigma'(a)| \leq K(0) \Rightarrow \sigma'(a) = 0$, now,

as $\sigma'(x) \leq K\sigma(x) \Rightarrow \sigma'(x)e^{-Kx} - K\sigma(x)e^{-Kx} \leq 0$

$\Rightarrow (\sigma(x)e^{-Kx})' \leq 0$

from differentiation and IF multiplication

so, $f(x) = \sigma(x)e^{-Kx}$ then $f'(x) \leq 0$

similarly

$g(x) = \sigma(x)e^{Kx}$ will be s.t

$g'(x) = \sigma'(x)e^{Kx} + K\sigma(x)e^{Kx}$

$= e^{Kx}\underbrace{(\sigma'(x) + K\sigma(x))}_{\geq 0 \text{ from given}}$

$\sigma'(x) \geq -K\sigma(x)$

so, $g'(x) \geq 0$

so, $f' \leq 0$ and $g' \geq 0$, now at $x = a$ $f(a) = 0$

$g(a) = 0$

so, $\forall x \geq a$, $f(x) \leq f(a) \Rightarrow f(x) \leq 0$
$\Rightarrow \sigma(x) e^{-kx} \leq 0$
$\Rightarrow \sigma(x) \leq 0$
but as $|\sigma'(x)| \leq k\sigma(x)$
$\Rightarrow \sigma(x) \geq 0$, so $\sigma(x) = 0$ for $\forall x \geq a$

similarly $\forall x \leq a$, $g(x) \leq g(a) = 0$
$\Rightarrow g(x) \leq 0$
$\Rightarrow \sigma(x) e^{kx} \leq 0$
$\Rightarrow \sigma(x) \leq 0$
but $\sigma(x) \geq 0$, so $\sigma(x) = 0$ $\forall x \leq \alpha$

so we get $\sigma(x) = 0$ and so statement trivially true

Case II: $\sigma(x) < 0$ $\forall x \in (-\infty, \infty)$, this is not possible as $\sigma(x) \geq 0$ from previous case

Case III: $\sigma(x) > 0$ $\forall x \in (-\infty, \infty)$, then

as $\sigma(x) > 0$ let $f(x) = \ln \sigma(x)$

then as $\dfrac{\sigma'(x)}{\sigma(x)} \leq k$
$\Rightarrow f'(x) \leq k$

similarly $\dfrac{\sigma'(x)}{\sigma(x)} \geq -k \Rightarrow f'(x) \geq -k$

so we have $-k \leq f'(x) \leq k$

now for $a \in (-\infty, \infty)$ $\forall x \neq a$ we get

$\dfrac{f(x) - f(a)}{x - a} = f'(c)$ ($\because$ rolles, as $\sigma \in C^1$ so $f \in C^1$)

then $-k \leq \dfrac{f(x) - f(a)}{x - a} \leq k$ ($\because -k \leq f'(x) \leq k$)

$\Rightarrow -k|x-a| \leq f(x) - f(a) \leq k|x-a|$

also for $x = a$, $f(x) - f(a) = 0$
so $0 \leq 0 \leq 0$ $\therefore$ true trivially

so, $\forall x \in (-\infty, \infty)$ we get

$-k|x-a| \leq f(x) - f(a) \leq k|x-a|$
$\Rightarrow -k|x-a| \leq \ln(\sigma(x)) - \ln(\sigma(a)) \leq k|x-a|$
$\Rightarrow -k|x-a| \leq \ln\left(\dfrac{\sigma(x)}{\sigma(a)}\right) \leq k|x-a|$
$\Rightarrow e^{-k|x-a|} \leq \dfrac{\sigma(x)}{\sigma(a)} \leq e^{k|x-a|}$

$\Rightarrow \sigma(a) e^{-k|x-a|} \leq \sigma(x) \leq \sigma(a) e^{k|x-a|}$
as $\sigma(a) > 0$

so, in all cases we get $\sigma(a) e^{-k|x-a|} \leq \sigma(x) \leq \sigma(a) e^{k|x-a|}$

(b) $y'(x) = f(x, y(x))$ $y(x_0) = y_0$ and

$|f(x, y_1) - f(x, y_2)| \leq K|y_1 - y_2|$ $\forall x, y_1, y_2 \in \mathbb{R}$

and f is cont on x, as from above we see f is lipschitz cont y we get f is cont on y ($\because$ lipschitz $\Rightarrow$ continuity)

so, f is continuous on $\mathbb{R} \times \mathbb{R}$, so $f: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is cont on every point from peano existence,

$y'(t) = f(t, y(t))$ has a solution

$\forall$ $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ and $x \in K$ $\forall t_0$, every K

(ε, K, x from peano's existence)

$\Rightarrow$ so, $\exists$ a solution

now to show its unique, lets assume not, then $\exists y_1, y_2$ solving IVP, both C^1 ($\because f$ is cont) then on a small nbd around t_0

$y_1(t_0) = x_0$

$y_2(t_0) = x_0$

as y_1, y_2 cont on a small nbd if not zero other then to lets assume

on $(t_0, t_0 + \varepsilon)$ for some small ε

wlog $y_1(t) > y_2(t)$ $\forall t \in (t_0, t_0 + \varepsilon)$

then $\sigma(t) = y_1(t) - y_2(t)$ is C^1

and $\sigma'(t) = y_1'(t) - y_2'(t)$

$= f(t, y_1) - f(t, y_2)$

so, $|\sigma'(t)| \leq K|y_1 - y_2| = K\sigma(t)$

$\Rightarrow |\sigma'(t)| \leq K\sigma(t)$ and $K > 0$ (otherwise trivially done)

so, from (a) we get for $t \in (t_0, t_0 + \varepsilon)$

$\sigma(t_0)e^{-K|t - t_0|} \leq \sigma(t) \leq \sigma(t_0)e^{K|t - t_0|}$

$\Rightarrow \sigma(t) = 0$, so $y_1(t) = y_2(t)$ $\forall t \in (t_0, t_0 + \varepsilon)$

we get same for $(t_0 - \varepsilon, t_0)$

so, $\forall t \in$ nbd of t_0 $y_1(t) = y_2(t)$, then we can extend this to all points in $\mathbb{R}$ by taking $t_0 + \varepsilon/2$, $t_0 + \varepsilon/2 + \varepsilon/2$ .... same on $t_0 - \varepsilon/2$...

so, $y_1 \equiv y_2$ $\therefore$ IVP has unique solution

(c) First let's show f is cont wrt y

$$\lim_{h \to 0} |f(x, y+h) - f(x, y)| \leq \lim_{h \to 0} K|h| |\log|h||$$

as $|h| \to 0$ faster then $\log|(h)| \to -\infty$ ($\because$ L'Hospital)

$$\Rightarrow \lim_{h\to 0} |f(x, y+h) - f(x, y)| \leq 0$$

$$\Rightarrow \lim_{h\to 0} f(x, y+h) = f(x, y)$$

so, f is cont wrt y also and from (b) we get $\exists$ solution to IVP.

Now let $w(t) = k|t||\log|t||$ $t>0$

then we see $\lim_{t\to 0^+} w(t) = 0$ as from prev $t \to 0$ faster

then $|f(x,y_1) - f(x,y_2)| \leq w(|y_1-y_2|)$

and $\int_{0^+}^{\infty} \frac{dt}{k|t||\log|t||}$

$= \int_{0^+}^{\infty} \frac{dt}{kt|\log|t||}$

$= \int_{0^+}^{1} \frac{-dt}{kt\log t} + \int_{1}^{\infty} \frac{dt}{kt\log t}$

let $\log t = r$

$= \int_{-\infty}^{0} \frac{-dr}{kr} + \int_{0}^{\infty} \frac{dr}{kr}$

$= 2\int_{0}^{\infty} \frac{dr}{kr} = \infty$ so, from Osgood condition

we get solution to be unique

3. $y' = 3y^{2/3}$ $y(0)=0$, now f is cont

as $\lim_{h\to 0} 3(y+h)^{2/3} = 3y^{2/3}$, also similarly $y^{2/3}$ is diff

but $|f(t,y_1) - f(t,y_2)| \leq L|y_1-y_2|$ is not satisfied in a nbd of $y=0$ as

$f'(y) = 2y^{-1/3} \to \pm\infty$ as $y \to 0$ ($\because f$ is diff)

so, Lipschitz not satisfied near $y=0$

now, if $y = t^3$

then $y' = 3t^2 = 3y^{2/3}$, so $y(t) = t^3$ is a solution

$y(t) = t^3$ is s.t. $y(0)=0$ so its a valid solution to IVP

but similarly, $y \equiv 0$ is also a valid solution as $y' = 3y^{2/3}$ and $y(0)=0$

so, solution exist but not unique

now, if $y(0)=1$, then $y(t) = (t+1)^3$ is a solution as

$y'(t) = 3(t+1)^2 = 3(y(t))^{2/3}$

now, $y(t) = \begin{cases} (t+1)^3; & t > -1 \\ 0; & t \leq -1 \end{cases}$ is also a solution to above IVP as

y is continious (trivial)

and $y'(t) = \begin{cases} 3(t+1)^2 ; & t > -1 \\ 0 & ; t \leq -1 \end{cases}$ $\rightarrow$ also continious (trivial)

so, $y'(t) = y(t)$ for $t \leq -1$

and same for $t > -1$, so here also solution is not unique

4. $y' = 1 + y^{2/3}$ with $y(0) = 0$

now first let's show above function is not lipschitz

$f(t,y) = 1 + y^{2/3}$ then

as $y^{2/3}$ is continious

we get $f(t,y)$ to be continious, moreover diff

so, $\frac{d}{dy} f(t,y) = \frac{2y^{-1/3}}{3}$ i.e as $y \to 0$ $\frac{\partial}{\partial y} f(t,y) \to \pm\infty$, this means f is not lipschitz on a nbd of 0 wrt y

now as $\frac{dy}{dt} = 1 + y^{2/3} \Rightarrow \frac{dy}{1+y^{2/3}} = dt$

$$\Rightarrow \int \frac{dy}{1+y^{2/3}} = \int dt$$

let $y = \tan^3\theta$

then

$dy = 3\tan^2\theta \sec^2\theta \, d\theta$

and $y^{2/3} = \tan^2\theta$

$1 + y^{2/3} = \sec^2\theta$

so, $\int \frac{dy}{1+y^{2/3}} = \int \frac{3\tan^2\theta \cancel{\sec^2\theta} d\theta}{\cancel{\sec^2\theta}}$

$$= \int 3\tan^2\theta \, d\theta$$

$$\Rightarrow \int 3\sec^2\theta \, d\theta - \int 3 \, d\theta = t + c$$

$$\Rightarrow 3\tan\theta - 3\theta = t + c$$

as $\theta = \tan^{-1} y^{1/3}$ we get:

$3\tan\theta - 3\theta = t + c$

as $3\cancel{\tan(\tan^{-1}} y^{1/3}) - 3\tan^{-1} y^{1/3} = t + c$

$\Rightarrow 3y^{1/3} - 3\tan^{-1} y^{1/3} = t + c$

and as $y(0) = 0$ $\Rightarrow 3(0) - 3(0) = (0) + c$

$\Rightarrow c = 0$

so, $3y^{1/3} - 3\tan^{-1}(y)^{1/3} = t$ as our solution

now if $F(u) = 3u - 3\tan^{-1}(u)$

then as $u, \tan^{-1}(u) \in C^1$

$$F'(u) = 3 - \frac{3}{1+u^2} \quad \text{as } 1+u^2 \geq 1$$

$$\Rightarrow \frac{1}{1+u^2} \leq 1$$

$$\Rightarrow -\frac{1}{1+u^2} \geq -1$$

$$\Rightarrow 3 - \frac{3}{1+u^2} \geq 0$$

so, $F'(u) \geq 0 \Rightarrow F$ is non dec function

and as $u \to \infty$ $F(u) \to \infty$

$u \to -\infty$ $F(u) \to -\infty$

F has rage $\mathbb{R}$, domain $\mathbb{R}$

so, F is an inc function, one-one onto

$$F: \mathbb{R} \to \mathbb{R}$$

so, $\forall\, t \in \mathbb{R}, \exists\, u$ s.t

$$3u - 3\tan^{-1} u = t$$

or for $u = y^{1/3}$ we have a unique solution $\forall t \in \mathbb{R}$

so, $3y^{1/3} - 3\tan^{-1} y^{1/3} = t$ has a unique solution

$\Rightarrow$ our IVP has unique solution

so, not lipschitz but unique solution

5. To prove: $y'(t) = \frac{e^{-y^2(t)} \sin^2 t}{1+t^2}$; $y(0)=1$ has unique solution and maximal interval is $\mathbb{R}$

proof: for $f(t,y) = \frac{e^{-y^2} \sin^2 t}{1+t^2}$

$$\lim_{\substack{h \to 0 \\ \gamma \to 0}} f(t+h, y+\gamma) = \lim_{\substack{h \to 0 \\ \gamma \to 0}} \frac{e^{-(y+\gamma)^2} \sin^2(t+h)}{1+(t+h)^2}$$

$$= \frac{e^{-y^2} \sin^2(t)}{1+t^2} \quad (\because 1+t^2, \sin^2 t, e^{-y^2} \text{ all cont})$$

so, f is continious on $\mathbb{R} \times \mathbb{R}$ and so,

$f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is s.t for any $[a,b] \subset \mathbb{R}$ and $K \subset \mathbb{R}$ s.t $1 \in K$

from peano's existance, $\exists$ a solution for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ for $t_0 = 0$

now $\left|\frac{\partial f}{\partial y}\right| = \left|\frac{\partial}{\partial y}\left(\frac{e^{-y^2} \sin^2 t}{1+t^2}\right)\right| = \left|\frac{-2y e^{-y^2} \sin^2 t}{1+t^2}\right|$

now as $\sin^2 t \leq 1$

$\frac{1}{1+t^2} \leq 1$

$$\Rightarrow \left|\frac{\partial f}{\partial y}\right| \leq 2|ye^{-y^2}|$$

now ye^{-y^2} is s.t

$$\frac{d}{dy}(ye^{-y^2}) = 0 \Rightarrow e^{-y^2} - 2y^2e^{-y^2} = 0$$

$$\Rightarrow y = \frac{\pm 1}{\sqrt{2}} \rightarrow \text{when } ye^{-y^2} \text{ is max or min}$$

i.e $|ye^{-y^2}| \leq \left(\frac{1}{\sqrt{2}}\right)e^{\frac{-1}{2}}$

$$\Rightarrow 2|ye^{-y^2}| \leq \sqrt{2}e^{\frac{-1}{2}}$$

$\Rightarrow \left|\frac{\partial f}{\partial y}\right| \leq \sqrt{2}e^{-1/2}$, so $\frac{\partial f}{\partial y}$ is bounded $\forall y \in \mathbb{R}$

and so f is lipschitz wrt y as $f \in C^1$

so, $f : \mathbb{R}\times\mathbb{R} \rightarrow \mathbb{R}$ is s.t it is lipschitz $\forall y \in \mathbb{R}$

then from theorm done in class as
1. f is continious
2. f is lipschitz wrt y

if $f : I \times \Omega \rightarrow \mathbb{R}^m$ where $I \subseteq \mathbb{R}$ is open
$\Omega \subseteq \mathbb{R}^m$ is open

then IVP has solution
u_1 on I_1
u_2 on I_2
s.t $u_1 = u_2$ for $I_1 \cap I_2$

now if $I = \mathbb{R}$, $\Omega = \mathbb{R}$, $\mathbb{R}^m = \mathbb{R}$, then any solution on $\mathbb{R}$ is unique

also, $|f(t,y)| \leq \frac{1}{1+t^2}$ $\quad (e^{-t^2} \leq 1, \sin^2 t \leq 1)$

$$\Rightarrow |y'(t)| \leq \frac{1}{1+t^2}$$

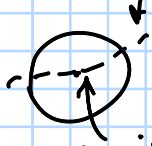
$$\Rightarrow |y(t) - 1| \leq \int_0^t \frac{ds}{1+s^2} = \tan^{-1} t \leq \frac{\pi}{2} \quad (\text{for } t > 0)$$

(same for $t \leq 0$)

so, solution never blows up and f can be defined for any $t \in \mathbb{R}$
so, maximal interval of existance is $\mathbb{R}$

so IVP ha unique solution on $\mathbb{R}$ and we see it exist for all of $\mathbb{R}$ as it does not shoot up (so we can just keep extending our existance bonds)

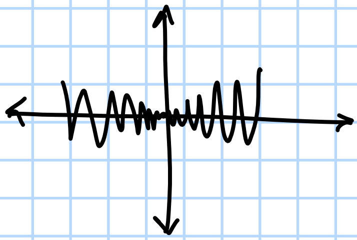
Extending to $\mathbb{R}$ as no blowups



By using previous every nbd

Existance and uniquen locally

6. $x' = \begin{cases} x\sin\left(\frac{1}{x}\right) ; & x \neq 0 \\ 0 & ; x = 0 \end{cases}$



$x(0) = 0$

now, let $f(t, x) = x'$

then

as for $x \neq 0$ $f(t, x)$ is continous as $\sin\frac{1}{x}$ is cont and x is cont

$$\lim_{x\to 0} f(t, x) = \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$$

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

$$\Rightarrow -x \leq x\sin\left(\frac{1}{x}\right) \leq x$$

$$\Rightarrow \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0 = f(t, 0)$$

so, f is continous at $\forall x \in \mathbb{R}$ and trivially $\forall t \in \mathbb{R}$

now $x_1 \equiv 0$ is a soultion as $x_1(0) = 0$,

now as $x' = \begin{cases} x\sin\frac{1}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$

x' is cont $\Rightarrow x$ is C'

so, $x' \leq x$

and $x' \geq -x$ as $-1 \leq \sin\frac{1}{x} \leq 1$

$\Rightarrow |x'| \leq x$

then from 2(a) $\sigma(t) = x'(t)$, as σ is C'

$K = 1$

we get

$$x(a)e^{-K|t-a|} \leq x(t) \leq x(a)e^{K|t-a|}$$

$K = 1$

$$\Rightarrow x(a)e^{-|t-a|} \leq x(t) \leq x(a)e^{|t-a|}$$

putting $a = 0 \Rightarrow x(a) = x(0) = 0$

$\Rightarrow x(t) = 0 \ \forall t \in \mathbb{R}$, this shows that any soution is $x(t) = 0$

or soultion is unique

7. $y' = p(t)\cos y + q(t)\sin y$, p, q are continious functions on $-\infty < t < \infty$ has unique soultion ϕ on $-\infty < t < \infty$ s.t $\phi(t_0) = y_0$

$$\left.\begin{aligned} f(t, y) &= p(t)\cos y + q(t)\sin y = y' \\ y(t_0) &= y_0 \end{aligned}\right\} \text{IVP}$$

now, as $p(t), q(t), \cos y, \sin y$ are continious

$$\lim_{h\to 0, \gamma\to 0} f(t+h, y+\gamma) = \lim_{h\to 0, \gamma\to 0} p(t+h)\cos(y+\gamma) + q(t+h)\sin(y+\gamma)$$

$= f(t, y)$ so, f is continious $\forall t, y \in \mathbb{R}$

so, $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continious
then for K compact $\subset \mathbb{R}$
s.t $y_0 \in K$
from picard's existance, IVP has a solution
for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$
from theorm

now, we have existance at a small nbd around (t_0, y_0) — ①

also $\frac{\partial f}{\partial y} = -P(t)\sin y + q(t)\cos y$

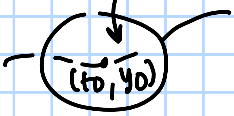
$\Rightarrow \left|\frac{\partial f}{\partial y}\right| \leq |P(t)| + |q(t)|$

then let $M_L = \sup_{|t-t_0| \leq L} \{|P(t)| + |q(t)|\}$

as P, q are cont

we get $\left|\frac{\partial f}{\partial y}\right| \leq M_L$ and so
f is locally lipscitz around t_0, wrt y
then from picard-lindeloff, $\exists$ unique local solution
in nbd (t_0, y_0) — ②

locally unique solution



then we can keep on shifting t_0 and y_0 to keep extending this for nbd of all points

as for any $(t_0 y_0)$ we get
both f is cont (global) and f is locally lipscitz
from M_L s.t
$M_L = \sup_{|t-t_0| \leq L} |P(t)| + |q(t)|$

we can keep shifting (t_0, y_0)
as

$|y'(t)| \leq M_L$
then $|y(t)| \leq M_L(L)$ interval length

so, $y(t)$ does not shoot up anywhere — ③

from ①, ②, ③ we get unique solution on $\mathbb{R}$

Tutorial-6:

'L' with BC $B_i u = 0$, $\langle u, Lv\rangle = \langle Lu, v\rangle$ $i=1,2$ $\begin{cases} Lu=0 \\ B_i u = 0\end{cases}$ $\begin{pmatrix} HBVP \\ HBC \end{pmatrix}$ was trivial solution

then $\exists$ unique G s.t $G(x,t) = G(t,x)$

now if $Lu = a_2 u'' + a_1 u' + a_0 u$

$\langle Lu, v\rangle = \int_a^b Lu\, v$ ← defn of $\langle Lu, v\rangle$

if we need complex: $\langle Lu, v\rangle = \int_a^b Lu\,\bar{v}$

$A_{n\times n}$: $A: \mathbb{R}^n \to \mathbb{R}^n$ is a matrix, $\langle Ax, y\rangle = (Ax)\cdot y$ ← dot product

$= x\cdot(A^t y)$

$= \langle x, A^t y\rangle$

so, when a linear operator satisfies above property we call it adjoint of operator A was A^t

lemma: (a) $Ax = b$; if it has solution iff $b\cdot v = 0$ $\forall v \in \ker(A^t)$

(b) $Ax = b$, has unique solution iff $Ax = 0$ has the only solution $x = 0$

$\int_a^b (a_2 u'' + a_1 u' + a_0 u)v = \langle Lu, v\rangle$ ← we want to write this in form of $\langle u, L^* v\rangle$

if we use integration by parts we get:

$$\int_a^b (a_2 u'' + a_1 u' + a_0 u)v = \int_a^b a_2 u'' v + \int_a^b a_1 u' v + \int_a^b a_0 u v$$

where $\int_a^b a_2 u'' v = a_2 v u'\Big|_a^b - \int_a^b (a_2 v)' u'$

$$= a_2 v u'\Big|_a^b - (a_2 v)' u\Big|_a^b + \int_a^b (a_2 v)'' u$$

So $\langle Lu, v\rangle$ becomes:

$$\langle Lu, v\rangle = \int_a^b [(a_2 v)'' - (a_1 v)' + a_0 v] u + [a_2(vu' - uv') + (a_1 - a_2')vu]_a^b$$

if $L^* v = (a_2 v)'' - (a_1 v)' + a_0 v$

then L^* is linear operator

we will call L^* formal adjoint of L if

$$[a_2(vu' - uv') + (a_1 - a_2')vu]_a^b = 0$$

So, $\begin{rcases} Lu = 0 \\ B_i u = 0 \end{rcases} \xrightarrow{L^*} \begin{cases} L^* v = 0 \\ B_i^* v = 0 \end{cases}$ ← $[a_2(vu' - uv') + (a_1 - a_2')vu]_a^b = 0$

$L^* \triangleq (a_2 v)'' - (a_1 v)' + (a_0 v)$

is formal adjoint of u with boundary condition as above

Note: $Lu = f$, $B_i u = c_i$ now becomes $\begin{matrix} Lu = f \\ B_i u = 0\end{matrix}$ and $\begin{matrix} Lu = 0 \\ B_i u = c_i \end{matrix}$ and we considered $\begin{matrix} Lu = 0 \\ B_i u = 0 \end{matrix}$ has unique solution

if non trivial solution exist for $\begin{matrix} Lu = 0 \\ B_i u = 0 \end{matrix}$ then we use L^* method

Theorm: $L: H \longrightarrow H$ be a bdd linear operator on Hilbert sp H, then $Lu = f$
$f \in H$, has a solution iff $\langle f, v \rangle = 0$ $\forall v \in \ker(L^*)$

Theorm: suppose the HBVP, HBC has a non-trivial solution, then the NHBVP, HBC has a solution iff $\int f v = 0$ i.e
$\langle f, v \rangle = 0$ $\forall v \in \ker(L^*)$