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PUSH DOWN AUTOMATA (PDA)

- Definition
- Notation
- Acceptance by final state
- Acceptance by empty stack
- Equivalence of PDA and CFG
  - → Grammar to PDA
  - → PDA to Grammar
- Deterministic PDA
- Non Deterministic PDA
- Applications of PDA

Introduction to PDA :

- A pushdown automata can be viewed as a finite automata with stack.
- An added stack provides memory and increases capability of the machine.

- Model of PDA

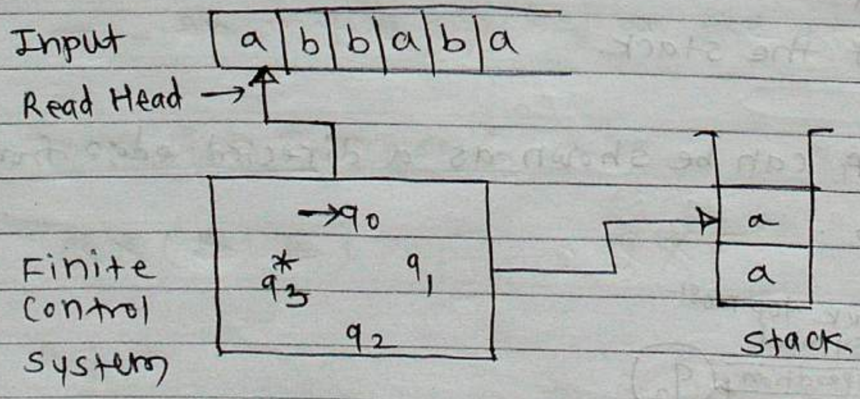


Figure 1: Model of PDA

- PDA can read input from input string as like FA.
- PDA performs operations on stack

- push
- pop
- check empty
- Nop

- PDA takes transitions from one state to another state.

- PDA is more powerful than FA.

Use of PDA :

1. To recognize context Free Languages (CFL)

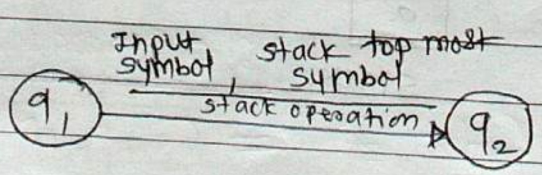
- The language of PDA is a context Free Language.

- For every CFL there exists PDA.

- A string of the form $a^n b^n$ can not be handled by FA but same can be handled by PDA.

- A transition in PDA depends on
  i. current state
  ii. current Input
  iii. Top symbol of the stack.

- A transition in PDA can be shown as a directed edge from the state q_1 to q_2.



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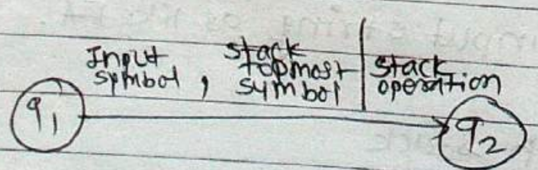

- Stack operations

i. Pop : It removes the top most symbol from the stack.

ii. Push : It inserts a symbol onto the top of the stack.

iii. NOP : It does nothing to stack.

Definition of PDA :

A pushdown automata, M is defined as, 7-tuple :

$$M = (Q, \Sigma, \Gamma, \delta, q_0, z_0, F)$$

where,

Q = Non empty finite set of states.

Σ = Non empty finite set of input alphabets.

Γ = stack symbols.

δ = Transition function.

It maps, $\delta : Q \times \{\Sigma \cup \epsilon\} \times \Gamma \rightarrow Q \times \Gamma^*$

q_0 = Initial / start state

$q_0 \in Q$

F = set of final states

$F \subseteq Q$

z_0 = An initial symbol on the stack.

Transition Function :

$$Q \times (\Sigma \cup \epsilon) \times \Gamma \rightarrow Q \times \Gamma^*$$

e.g.

$$\delta(Q, (\Sigma \cup \epsilon), \Gamma) \longrightarrow (Q, \Gamma^*)$$

current state

Next input including ϵ

stack topmost symbol

Next state

Replacement symbol (Push, POP, NOP)

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Stack operations in details:

1. Read input with NOP on stack:

$$\delta(q_1, a, b) = (q_2, b)$$

current state — current input symbol — current topmost symbol on the stack — Next state — stack replacement symbols

(stack symbol 'b' is replaced with b i.e no stack operation (NOP).

Thus, transition $\delta(q_1, a, b) = (q_2, b)$ does not modify a stack.

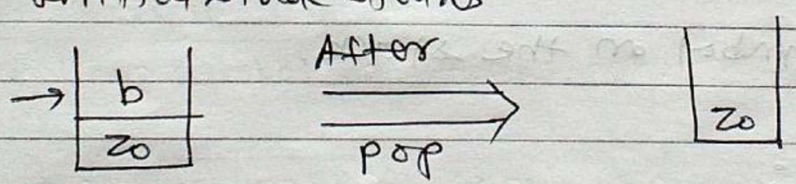
2. Pop:

$$\delta(q_1, a, b) = (q_2, \varepsilon)$$

It means from current state q_1 for current input symbol 'a' and top of stack contains 'b' then machine i.e. PDA should take transition to next state q_2 by replacing top most symbol 'b' on stack with ε.

i.e.

Initial stack status



$$\delta(q_1, a, b) \Rightarrow (q_2, \varepsilon)$$

3. Push operation:

$$\delta(q_1, a, b) = (q_2, ab)$$ Replace 'b' on top of stack by ab.

Add 'ab' on the top of stack by replacing 'b' and machine transit from 'q' state on current i/p alphabet a for topmost symbol 'b', to next state q_2 by pushing 'ab' on the stack by removing 'b'.

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Language of PDA:

The language 'L' can be accepted by a PDA in two ways:
1. Through Final state
2. Through Empty stack.

1. Acceptance by Final state:

Let the PDA, $M = (Q, \Sigma, \Gamma, \delta, q_0, F, z_0)$ can accept the language 'L' through a final state is given by,

$$L(M) = \{ \omega \mid (q_0, \omega, z_0) \xrightarrow[M]{*} (q_f, \epsilon, \alpha)$$

where,

$q_f \in F$

α = contents of stack

ϵ = Empty string

2. Acceptance by Empty stack:

Let the PDA, $M = (Q, \Sigma, \Gamma, \delta, q_0, z_0, F)$ then the language accepted through an empty stack is given by

$$L(M) = \{ \omega \mid (q_0, \omega, z_0) \xrightarrow[M]{*} (q_1, \underset{\text{I/p empty}}{\epsilon}, \underset{\text{stack empty}}{\epsilon}) \}$$

where,

q_1 = Any state belonging to Q

ϵ = Input is empty

ϵ = Stack is empty

It means when ~~input~~ PDA starts processing the input 'ω' from start state q_0 & current top most symbol on stack is z_0 after taking so many transitions ~~machine~~ input gets completely scanned i.e 'ω' contains 'ε', stack becomes empty i.e. ε and machine stops at ~~q_f~~ any state q_1 then that string is said to be accepted by PDA through stack empty way.

Representation of PDA:

1. Pictorial Representation / Flowchart
2. Transition Diagram

In this chapter we focus on PDA representation as per transition diagram.

Example 1: Construct PDA for Language,

$$L = \{a^n b^n \mid n \geq 1\}$$

a. Through stack empty condition
b. Through Final state condition

Sol: ⇒

Step 1:

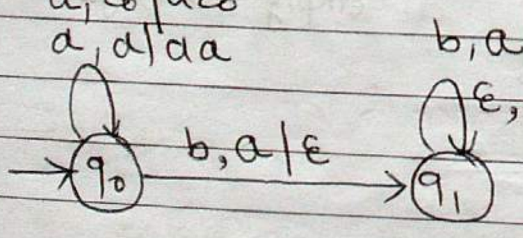
Logic:

$L = \{a^n b^n \mid n \geq 1\}$

$L = \{ab, aabb, aaabbb, \ldots\}$

$L = \{$ Equal number of a's followed by equal number of b's where $n \geq 1$ or number of a's & b's should be greater than 1 $\}$

Step 2: construct Transition diagram, as per stack empty condition.



Transition function / Rules:

$\delta(q_0, a, z_0) \Rightarrow (q_0, az_0)$

$\delta(q_0, a, a) \Rightarrow (q_0, aa)$

$\delta(q_0, b, a) \Rightarrow (q_1, \varepsilon)$

$\delta(q_1, b, a) \Rightarrow (q_1, \varepsilon)$

$\delta(q_1, \varepsilon, z_0) \Rightarrow (q_1, \varepsilon)$

Step 3: Transition diagram as per final state condition,

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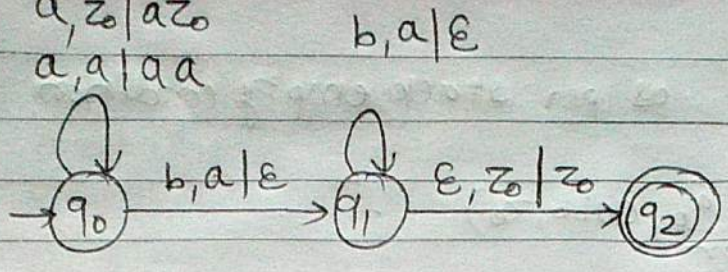

Transition function :

1. $\delta(q_0, a, z_0) \Rightarrow (q_0, az_0)$
2. $\delta(q_0, a, a) \Rightarrow (q_0, aa)$
3. $\delta(q_0, b, a) \rightarrow (q_1, \varepsilon)$
4. $\delta(q_1, b, a) \rightarrow (q_1, \varepsilon)$
5. $\delta(q_1, \varepsilon, z_0) \Rightarrow (q_2, z_0)$

Step 4: Example : $w = aaabbb$

$\delta(q_0, w, z_0) \Rightarrow \delta(q_0, aaabbb, z_0)$ [As per rule 1]

$\Rightarrow \delta(\delta(q_0, a, z_0), aabbb, z_0)$

$\Rightarrow \delta(q_0, aabbb, az_0)$ ∴ (As per rule 2)

$\Rightarrow \delta(\delta(q_0, a, a), abbb, az_0)$

$\Rightarrow \delta(q_0, abbb, aaz_0)$

$\Rightarrow \delta(\delta(q_0, a, a), bbb, aaz_0)$

$\rightarrow \delta(q_0, bbb, aaaz_0)$

As per rule 3

$\Rightarrow \delta(\delta(q_0, b, a), bb, aaaz_0)$

$\Rightarrow \delta(q_1, bb, aaz_0)$ [As per rule 4]

$\Rightarrow \delta(q_1, b, az_0)$ (As per rule 4)

$\Rightarrow \delta(q_1, \varepsilon, z_0)$ (As per rule 5)

$\Rightarrow (q_2, \varepsilon, \varepsilon)$ $\Rightarrow$ Accepted ∴ q_2 is final state & input is completely scanned

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as per stack empty condition.

Example : $w = aabb$

$\delta(q_0, w, z_0) \Rightarrow \delta(q_0, aabb, z_0)$

$\Rightarrow$ As per rule 1

$\Rightarrow \delta(q_0, abb, az_0)$

$\Rightarrow$ As per rule 2

$\Rightarrow \delta(q_0, bb, aaz_0)$

As per rule 3

$\Rightarrow \delta(q_1, b, az_0)$

As per rule 4

$\Rightarrow \delta(q_1, \varepsilon, z_0)$

As per rule 5

$\Rightarrow (q_1, \varepsilon, \varepsilon)$

Input completely scanned — stack is empty

$\Rightarrow$ <u>Accepted</u>

Example 2 : Construct PDA for Language,

$L = \{ 0^n 1^n \mid n \geq 1 \}$

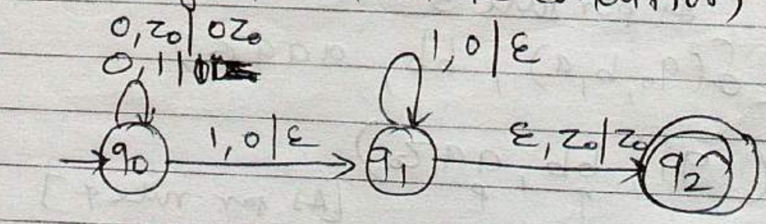
<u>Sol :</u> $\Rightarrow$

<u>Step 1 :</u>

Logic : $L = \{01, 0011, 000111, \ldots\}$

Step 2 : <u>Transition diagram</u>

a. Through final state condition



<u>Transition function :</u>

$\delta(q_0, 0, z_0) = (q_0, 0z_0)$

$\delta(q_0, 0, 1) = (q_0, 01)$

$\delta(q_0, 1, 0) = (q_1, \varepsilon)$

$\delta(q_1, 1, 0) = (q_1, \varepsilon)$

$\delta(q_1, \varepsilon, z_0) = (q_2, z_0)$

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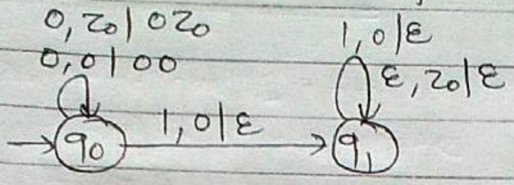
Step 3: Transition diagram as per stack empty condition.



Transition function:

1. $\delta(q_0, 0, z_0) = (q_0, 0z_0)$
2. $\delta(q_0, 0, 0) = (q_0, 00)$
3. $\delta(q_0, 1, 0) = (q_1, \varepsilon)$
4. $\delta(q_1, 1, 0) = (q_1, \varepsilon)$
5. $\delta(q_1, \varepsilon, z_0) = (q_1, \varepsilon)$

Step 4: Define Machine

1. As per stack empty condition,

$$M = (Q, \Sigma, \Gamma, \delta, q_0, F, z_0)$$

$$M = (\{q_0, q_1\}, \{0,1\}, \{0, z_0\}, \delta, q_0, \phi, z_0\}$$

2. As per final state condition,

$$M = (Q, \Sigma, \Gamma, \delta, q_0, F, z_0)$$

$$M = (\{q_0, q_1\}, \{0,1\}, \{0, z_0\}, \delta, q_0, \{q_2\}, z_0)$$

Step 5: Example: suppose $w = 0011$ As per final state condition

$$\delta(q_0, w, z_0) = \delta(q_0, \underset{\uparrow}{0}011, z_0)$$

$$\Rightarrow \delta(\underline{\delta(q_0, 0, z_0)}, 011, z_0)$$

As per rule 1

$$\Rightarrow \delta(q_0, 011, 0z_0)$$

$$\Rightarrow \delta(\delta(q_0, 0, 0), 11, 0z_0)$$

(As per rule 2.)

$$\Rightarrow \delta(q_0, 11, 00z_0)$$

$$\Rightarrow \delta(\delta(q_0, 1, 0), 1, 00z_0)$$

As per rule 3

$= \delta(q_1, 1, 0z_0)$

$\rightarrow \delta(\delta(q_1, 1, 0), \varepsilon, 0z_0)$

As per rule 4

$\Rightarrow \delta(q_1, \varepsilon, z_0)$

$\Rightarrow (q_2, \varepsilon, z_0)$ $\therefore q_2$ = final state, ε = Input is empty,

$\Rightarrow$ Accepted

Example 3: Give PDA to accept the language,

$L = \{0^n 1^m \mid n \le m\}$ 1. stack empty condition 2. final state condition

sol: $\Rightarrow$

step1:

Logic: $L = \{\varepsilon, 1, 01, 011, 0011, 111, 0111, 00111, 000111, \ldots\}$

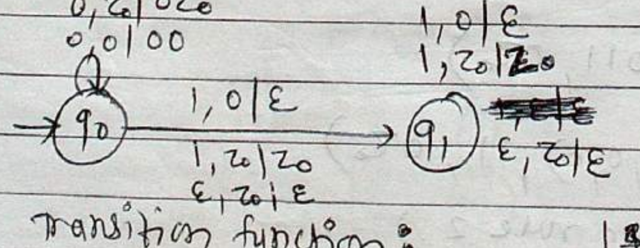
Here, $n \le m$ if

values of m	0^n ($\because n \le m$)	1^m	$0^n 1^m$
0	ε	ε	ε
1	ε, 0	1	1, 01
2	ε, 0, 00	11	11, 011, 0011
3	ε, 0, 00, 000	111	111, 0111, 00111, 000111
⋮	⋮	⋮	⋮

$L = \{\varepsilon, 1, 01, 11, 011, 0011, 111, 0111, 00111, 000111, \ldots\}$

step2: construction of Transition diagram

1. As per stack empty condition



Transition function:

1. $\delta(q_0, 0, z_0) \Rightarrow (q_0, 0z_0)$
2. $\delta(q_0, 0, 0) \Rightarrow (q_0, 00)$
3. $\delta(q_0, 1, 0) \Rightarrow (q_1, \varepsilon)$
4. $\delta(q_1, 1, 0) = (q_1, \varepsilon)$
5. $\delta(q_1, 1, z_0) = (q_1, z_0)$
6. $\delta(q_1, \varepsilon, z_0) = (q_1, \varepsilon)$
7. $\delta(q_0, 1, z_0) = (q_1, z_0)$
8. $\delta(q_0, \varepsilon, z_0) = (q_1, \varepsilon)$

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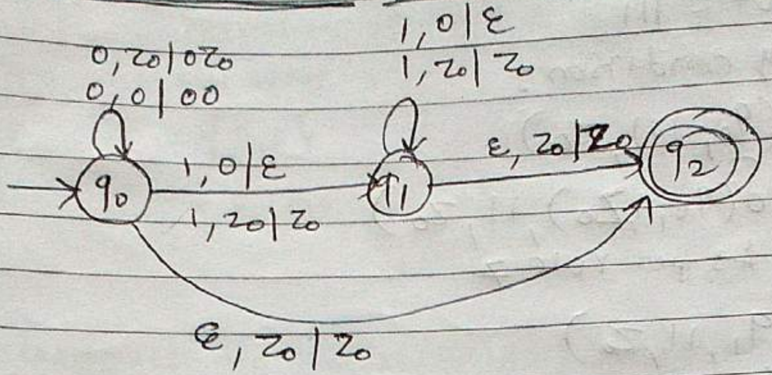
2. As per final state condition:



Transition function:

1. $\delta(q_0, 0, z_0) = (q_0, 0z_0)$

2. $\delta(q_0, 0, 0) = (q_0, 00)$

3. $\delta(q_0, 1, 0) = (q_1, \varepsilon)$

4. $\delta(q_0, 1, z_0) = (q_1, z_0)$

5. $\delta(q_0, \varepsilon, z_0) = (q_2, z_0)$

6. $\delta(q_1, 1, 0) = (q_1, \varepsilon)$

7. $\delta(q_1, 1, z_0) = (q_1, z_0)$

8. $\delta(q_1, \varepsilon, z_0) = (q_2, z_0)$

~~Exam~~ Step 3:

$M = (Q, \Sigma, \delta, \Gamma, q_0, F, z_0)$

1. stack empty condition

$M = (\{q_0, q_1\}, \{0, 1\}, \delta, \{0, z_0\}, q_0, \phi, z_0)$

2. Final state condition

$M = (\{q_0, q_1, q_2\}, \{0, 1\}, \delta, \{0, z_0\}, q_0, \{q_2\}, z_0)$

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Step 4 :

Example 1: suppose $w = 111$

As per stack empty condition :

$\delta(q_0, w, z_0) \Rightarrow \delta(q_0, 111, z_0)$

$\Rightarrow \delta(\delta(q_0, 1, z_0), 11, z_0)$

As per rule 7

$\Rightarrow \delta(q_1, 11, z_0)$

$\Rightarrow \delta(\delta(q_1, 1, z_0), 1, z_0)$

As per rule 7

$\Rightarrow \delta(q_1, 1, z_0)$ As per rule 7

$\Rightarrow \delta(q, \varepsilon, z_0)$

$\Rightarrow (q_1, \varepsilon, \varepsilon)$ $\therefore$ Input completely scanned & stack empty

$\Rightarrow$ Accepted

Example 2: suppose string $w = 011$ As per final state condition.

$\delta(q_0, w, z_0) = \delta(q_0, 011, z_0)$

$\Rightarrow \delta(\delta(q_0, 0, z_0), 11, z_0)$

As per rule 1

$\Rightarrow \delta(q_0, 11, 0z_0)$

$\Rightarrow \delta(\delta(q_0, 1, 0), 1, 0z_0)$

As per rule 3

$\Rightarrow \delta(q_1, 1, z_0)$

As per rule 4

$\Rightarrow \delta(q_1, \varepsilon, z_0)$ As per rule 8

Example 4 : Construct PDA to accept language containing equal number of a's & b's over $\Sigma = \{a, b\}$

Sol^n :→ 1. Final state 2. Empty stack [May-2009, May-2010]

Step 1: $\Sigma = \{0, 1\}$

Logic: $L = \{\varepsilon, ab, ba, aabb, baba, abab, baab, abba, \dots\}$

Step 2 : Construct transition table/diagram

a. Stack empty condition

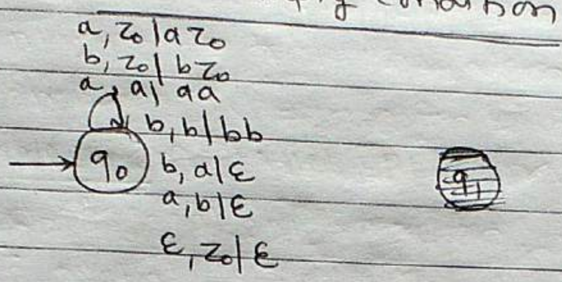
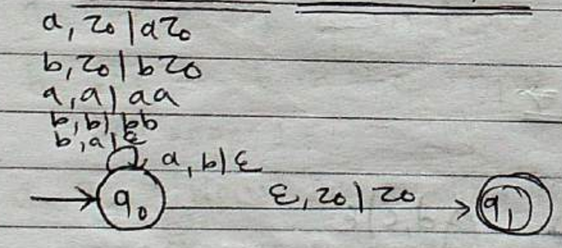


Figure 1: Transition Diagram.

Transition Function:

1. $\delta(q_0, a, z_0) = (q_0, az_0)$
2. $\delta(q_0, b, z_0) = (q_0, bz_0)$
3. $\delta(q_0, a, a) = (q_0, aa)$
4. $\delta(q_0, b, b) = (q_0, bb)$
5. $\delta(q_0, b, a) = (q_0, \varepsilon)$
6. $\delta(q_0, a, b) = (q_0, \varepsilon)$
7. $\delta(q_0, \varepsilon, z_0) = (q_0, \varepsilon)$

b. Final state condition :



Transition Function:

1. $\delta(q_0, a, z_0) = (q_0, az_0)$
2. $\delta(q_0, b, z_0) = (q_0, bz_0)$
3. $\delta(q_0, a, a) = (q_0, aa)$
4. $\delta(q_0, b, b) = (q_0, bb)$
5. $\delta(q_0, b, a) = (q_0, \varepsilon)$
6. $\delta(q_0, a, b) = (q_0, \varepsilon)$
7. $\delta(q_0, \varepsilon, z_0) = (q_1, z_0)$

Example 5 : Let $L = \{a^n b^n c^m d^m \mid n, m \geq 1\}$ find PDA that accepts L. [Dec-2009, May-2012]

Sol^n :→ Step 1 : Logic

1. The sequence of a's should be pushed on the stack.
2. For every 'b' as input, an 'a' should be poped from the stack.
3. Sequence of c's should be pushed on the stack.
4. For every 'd' as input, 'c' should be poped from the stack.

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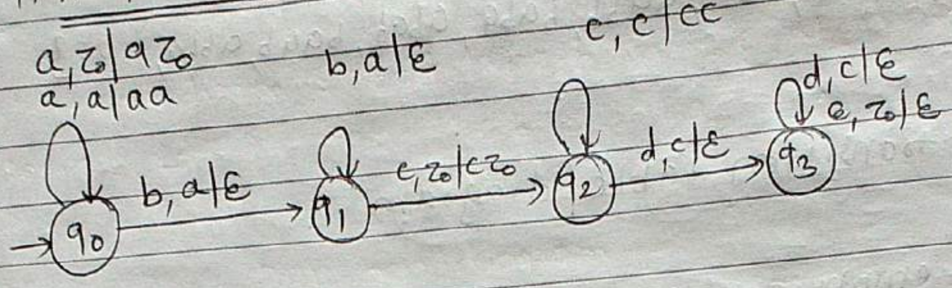
$L = \{abcd, abccdd, aabbcd, \ldots\}$

Step 2: construct transition table

1. <u>As per empty stack :</u>



<u>Transition function :</u>

$\delta(q_0, a, Z_0) = (q_0, aZ_0)$

$\delta(q_0, a, a) = (q_0, aa)$

$\delta(q_0, b, a) = (q_1, \varepsilon)$

$\delta(q_1, b, a) = (q_1, \varepsilon)$

$\delta(q_1, c, Z_0) = (q_2, cZ_0)$

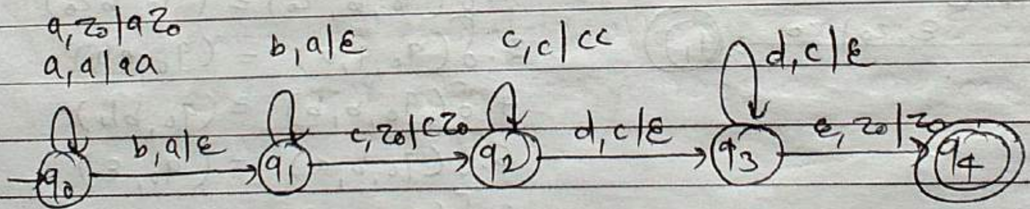
$\delta(q_2, c, c) = (q_2, cc)$

$\delta(q_2, d, c) = (q_3, \varepsilon)$

$\delta(q_3, d, c) = (q_3, \varepsilon)$

$\delta(q_3, \varepsilon, Z_0) = (q_3, \varepsilon)$

2. <u>As per final state condition :→</u>



Example 6 : construct PDA for $L = \{a^n b^m a^n \mid m, n \geq 1\}$ by null store.

[May-2011, Dec-2011, May-2012, Dec-2013]

Sol :⇒

Step 1:

<u>Logic :</u> $L = \{aba, abba, aabaa, aabbaa, \ldots\}$

a. push all 'a' on the stack

b. For all b's do not perform any operation on stack.

c. For each 'a' perform pop operation

d. once input is ε and stack contains Z_0 then string is accepted otherwise rejected.

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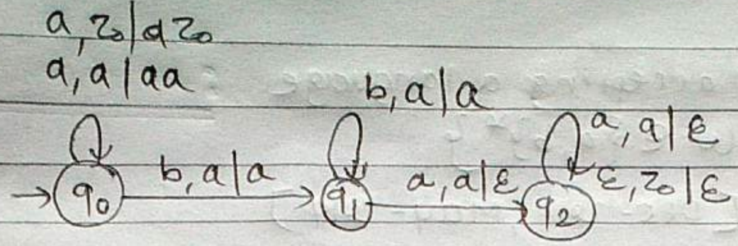


Fig 1: $L=\{a^n b^m a^n \mid m,n\geq 1\}$ PDA as per stack empty

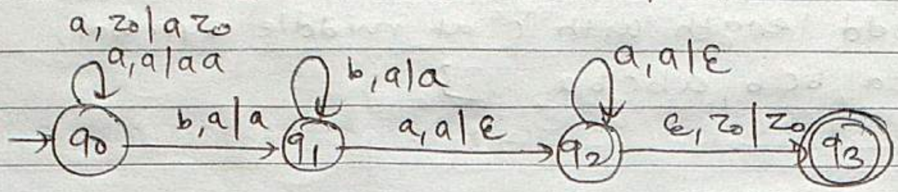


Fig 2: PDA for $L=\{a^n b^m a^n \mid m,n\geq 1\}$ for final state condition.

Example 7: construct PDA for $L=\{a^n.b^{2n} \mid n\geq 1\}$

sol^n :→

step1 :

Logic: $L=\{a^n.b^{2n} \mid n\geq 1\}$

$L=\{abb, aabbbb, aaabbbbbb, \ldots\}$

a. For every 'a' in input, push aa on the stack

b. For ~~a~~ 'b' in the input and 'a' on the top of stack should get popped.

c. If input is ε, symbol on stack z_0 then popped the stack then it accept the string through empty stack condition.

step2: Transition ~~tab~~ diagram

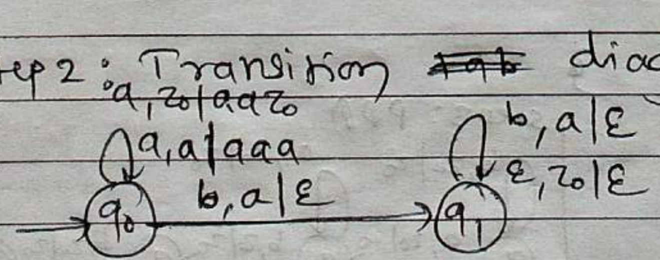
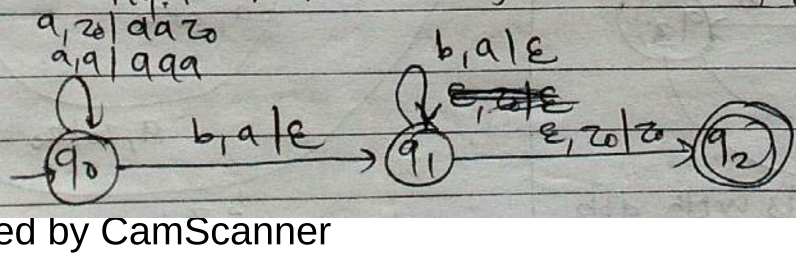


fig.1: PDA through stack empty.



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Example 8: Design PDA for accepting a language

$$L = \{ wcw^T \mid w \in \{a, b\}^* \}$$

[Dec-2011, May-2014]

Soln :⇒

step 1: A palindrome of odd length with 'c' at middle symbol.

Logic: $L = \{c, aca, bcb, abcba, \dots\}$

step 2: construct transition diagram

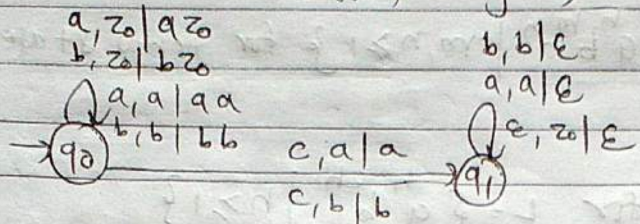
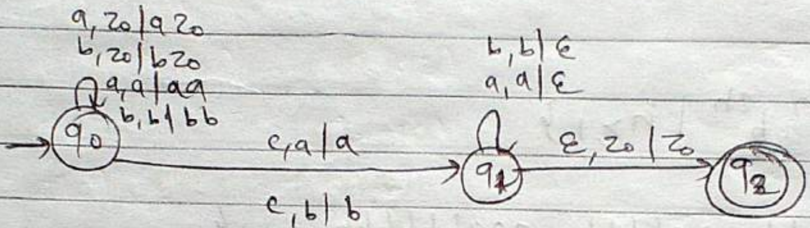


fig. 1: stack empty

$w = bbcbb$



Example 9: Design PDA to check whether a given string over $\{a, b\}$ ends in abb.

Soln :⇒

Step 1:

Logic: Language contains all strings over $\Sigma \{a, b\}$ which ends with abb. It is regular language and possible to construct ~~PDA~~ DFA.

$L = \{abb, aabb, babb, \dots\}$

step 2: DFA for the same,

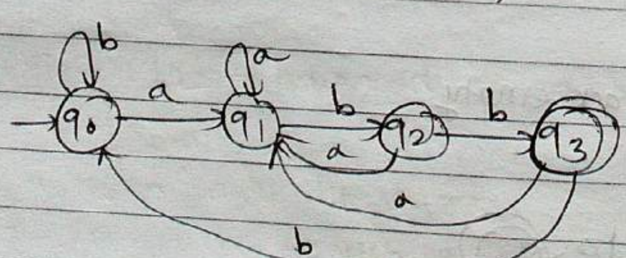
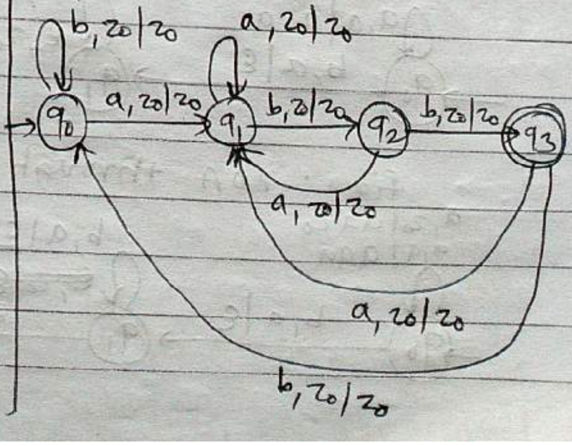


Fig. 1: DFA for string ends with abb.

step 3: PDA



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Example 10: Design a PDA for detection of palindrome over $\{a, b\}$.

[May-2007, Dec-2007, May-2008, Dec-2012]

Sol^n :→

Step1 :

Logic: A palindrome will be of the form:

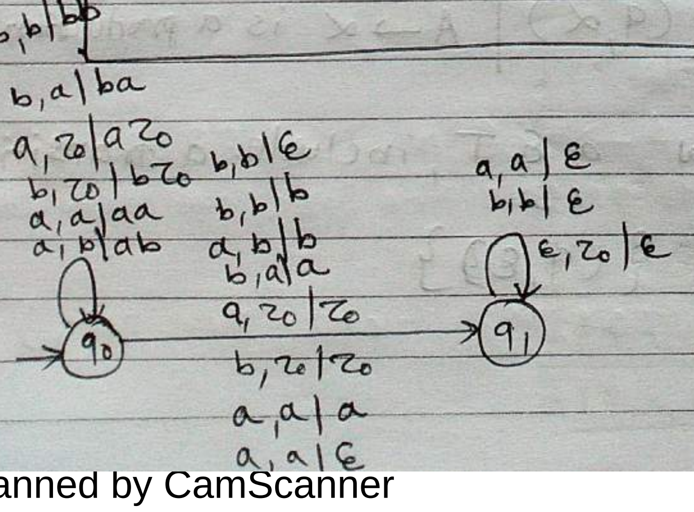
- ww^R (even palindrome)
- waw^R] (palindrome of odd length)
- wbw^R]

$$L = \{\epsilon, a, b, aa, bb, aba, bab, aaa, bbb, \ldots\}$$

Step2: Transition function

$$\delta(q_0, a, z_0) \Rightarrow \{(q_1, z_0), (q_0, az_0)\}$$
$$\delta(q_0, b, z_0) \Rightarrow \{(q_1, z_0), (q_0, bz_0)\}$$
$$\delta(q_0, a, a) \Rightarrow \{(q_0, aa), (q_1, a), (q_1, \epsilon)\}$$
$$\delta(q_0, a, b) \Rightarrow \{(q_0, ab), (q_1, b)\}$$
$$\delta(q_0, b, a) \Rightarrow \{(q_0, ba), (q_1, a)\}$$
$$\delta(q_0, b, b) \Rightarrow \{(q_0, bb), (q_1, b), (q_1, \epsilon)\}$$
$$\delta(q_1, a, a) \Rightarrow \{(q_1, \epsilon)\}$$
$$\delta(q_1, b, b) \Rightarrow \{(q_1, \epsilon)\}$$
$$\delta(q_1, \epsilon, z_0) \Rightarrow \{(q_1, \epsilon)\}$$ [Accept through an empty stack]

The transition $\delta(q_0, a, a) \Rightarrow \{\underbrace{(q_0, aa)}_{\text{Input 'a' is part of } w}, \underbrace{(q_1, a)}_{\text{Input 'a' is middle point of odd palindrome}}, \underbrace{(q_1, \epsilon)}_{\text{Input 'a' is first character of } w^R \text{ of even palindrome}}\}$



Pushdown Automata and context Free Languages :

- The class of languages accepted by PDA is exactly the cl
  of context free languages.

- the following three classes of languages are same
  1. Context Free Languages defined by CFG
  2. Languages accepted by PDA through final state.
  3. Languages accepted by PDA through empty stack.

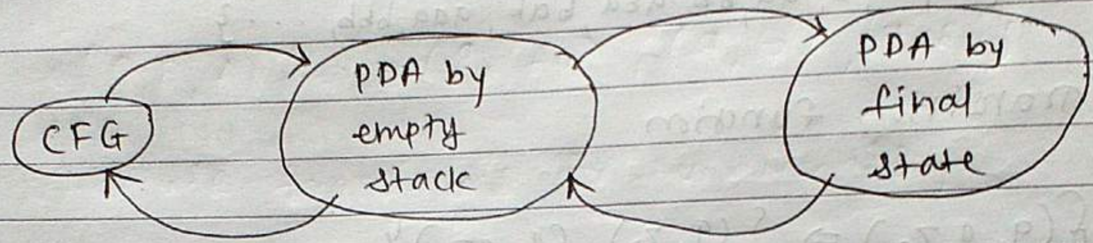


Fig 1: Equivalence of PDA and CFG

Context Free Grammar to PDA :

CFG : $G = (V, T, P, S)$
from given Grammar 'G' a PDA 'M' can be constructed which simulates leftmost derivation of G.

The PDA accepting a $L(G)$ by empty stack is given by

$$\begin{array}{ccccccc} & Q & \Sigma & \Gamma & \delta & q_0 \; z_0 & F \\ M = (& \{q\}, & T, & V \cup T, & \delta, & q, \; S, & \phi \;) \end{array}$$

where, δ is defined by :

1. For each variable (non terminal) $A \in V$, include a transition
$$\delta(q, \varepsilon, A) \Rightarrow \{ (q, \alpha) \mid A \to \alpha \text{ is a production in } G \}$$

2. For each terminal $a \in T$, include a transition.
$$\delta(q, a, a) \Rightarrow \{ (q, \varepsilon) \}$$

Example : Find PDA for given grammar,

$$S \rightarrow 0S1 \mid 00 \mid 11$$

Soln :→

Step 1 : Given CFG is,

$$S \rightarrow 0S1 \mid 00 \mid 11$$

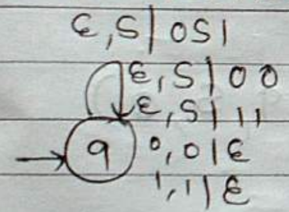
Step 2 : The equivalent PDA, M is given by :

$$M = (\{q\}, \{0,1\}, \{0,1,S\}, \delta, q, S, \phi)$$

where, δ is given by,

$$\delta(q, \varepsilon, S) = \{(q, 0S1), (q, 00), (q, 11)\}$$
$$\delta(q, 0, 0) = \{(q, \varepsilon)\}$$
$$\delta(q, 1, 1) = \{(q, \varepsilon)\}$$

Step 3 : Transition diagram



Step 4 : Example : suppose string w = 0011

$$\delta(q, 0011, S) \Rightarrow \xrightarrow{\delta(q,\varepsilon,S) = (q,0S1)} (q, 0011, 0S1)$$

$$\xrightarrow{\delta(q,0,0)} (q, 011, S1)$$

$$\xrightarrow{\delta(q,\varepsilon,S) = (q,0S1)} (q, 011, 0S11)$$

$$\xrightarrow{\delta(q,0,0)} (q, 11, S11)$$

$$\delta(q, \varepsilon, S) = (q, 11)$$

$$\xrightarrow{\delta(q,\varepsilon,S)} (q, 11, 1111)$$

$$\xrightarrow{\delta(q,1,1)} (q, 1, 111)$$

$$\xrightarrow{\delta(q,1,1)} (q, \varepsilon, 11)$$

$$\longrightarrow \text{Rejected}$$

Example 2: convert the grammar,

$S \rightarrow 0S1 \mid A$

$A \rightarrow 1A0 \mid S \mid \varepsilon$

to PDA that accept the same language by empty stack. [Dec-2005, May-2012, May-2014]

Soln:⇒

step 1: Include transitions for each variable/non terminal.

$$\delta(q, \varepsilon, S) \Rightarrow \{(q, 0S1), (q, A)\}$$

$$\delta(q, \varepsilon, A) \Rightarrow \{(q, 1A0), (q, S), (q, \varepsilon)\}$$

step 2: Include transitions for terminals,

$$\delta(q, 1, 1) = \{(q, \varepsilon)\}$$

$$\delta(q, 0, 0) = \{(q, \varepsilon)\}$$

step 3: PDA can be written as,

$M = (Q, \Sigma, \Gamma, \delta, q_0, z_0, \phi)$

$M = (\{q\}, \{0,1\}, \{0,1,S,A\}, \delta, q, S, \phi)$

δ:

$$\delta(q, \varepsilon, S) \Rightarrow \{(q, 0S1), (q, A)\}$$
$$\delta(q, \varepsilon, A) \Rightarrow \{(q, 1A0), (q, S), (q, \varepsilon)\}$$
$$\delta(q, 1, 1) \Rightarrow \{(q, \varepsilon)\}$$
$$\delta(q, 0, 0) \Rightarrow \{(q, \varepsilon)\}$$

Example 3: convert CFG to PDA.

$S \rightarrow aABB \mid aAA$, $A \rightarrow aBB \mid a$, $B \rightarrow bBB \mid A$

construct NPDA that accepts the language generated by this grammar.

[Dec-2006, May-2007, May-2008, Dec-2013]

Soln:⇒

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$M = (\{q\}, \{a,b\}, \{a,b,S,A,B\}, \delta, q, S, \phi)$

where, δ :

$\delta(q,\varepsilon,S) = \{(q,aABB), (q,aAA)\}$ ~~(q, aBB), (q,~~

$\delta(q,\varepsilon,A) = \{(q,aBB), (q,a)\}$

~~$\delta(q,\varepsilon,B) = \{(q,bBB), (q,A)\}$~~

$\delta(q,a,a) = (q,\varepsilon)$

$\delta(q,b,b) = (q,\varepsilon)$

Example 4: Construct PDA equivalent to the following CFG

$S \to 0BB$

$B \to 0S \mid 1S \mid 0$

Test if 010^4 is in language.

[DEC-2010, May-2011, DEC-2011, May-2012, May-2013]

Soln: ⇒

Step 1:

PDA

$M = (\{q\}, \{0,1\}, \{0,1,S,B\}, \delta, q, S, \phi)$

δ:

$\delta(q,\varepsilon,S) \Rightarrow \{(q,0BB)\}$

$\delta(q,\varepsilon,B) \Rightarrow \{(q,0S), (q,1S), (q,0)\}$

$\delta(q,0,0) \Rightarrow (q,\varepsilon)$

$\delta(q,1,1) \Rightarrow (q,\varepsilon)$

Step 2: Example, $w = 010^4$

$\therefore w = 010000$

$\delta(q,010000,S) \xrightarrow{\delta(q,\varepsilon,S)=(q,0BB)} (q,010000,0BB)$

$\xrightarrow{\delta(q,0,0)=(q,\varepsilon)} (q,10000,BB)$

$\xrightarrow{\delta(q,\varepsilon,B)=(q,1S)} (q,10000,1SB)$

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$\delta(q,1,1)=(q,\epsilon)$
$\longrightarrow (q, 0000, SB)$

$\delta(q,\epsilon,S)=(q,0BB)$
$\longrightarrow (q, 0000, 0BBB)$

$\delta(q,0,0)=(q,\epsilon)$
$\longrightarrow (q,000,BBB)$

$\delta(q,\epsilon,B)=(q,0)$
$\longrightarrow (q,000,0BB)$

$\delta(q,0,0)=(q,\epsilon)$
$\longrightarrow (q,00,BB)$

$\delta(q,\epsilon,B)=(q,0)$
$\longrightarrow (q,00,0B)$

$\delta(q,0,0)=(q,\epsilon)$
$\longrightarrow (q,0,B)$

$\delta(q,\epsilon,B)=(q,0)$
$\longrightarrow (q,0,0)$

$\delta(q,0,0)=(q,\epsilon)$
$\longrightarrow (q,\epsilon,\epsilon) \Rightarrow$ Accepted

Now input is completely scanned and stack is empty so PDA accepts the given string.

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PDA to context Free Grammar :

PDA, $M = (Q, \Sigma, \delta, q_0, F, \Gamma, z_0)$ [as per final state]

PDA $M = (Q, \Sigma, \delta, q_0, \phi, \Gamma, z_0)$ [as per stack empty condition]

Grammar G,

$$G = (V, T, P, S)$$

Then equivalent CFG is given by,

$$G = (V, T, P, S)$$
$$V = \{S, \{[q, z, q'] \mid q, q' \in Q \text{ and } z \in \Gamma\}\}$$

Steps to convert PDA to grammar :

~~Step 1:~~

If $Q = \{q_0, q_1\}$ and $\Gamma = \{a, b, z\}$

Step 1: Add the following productions for the start symbol S for all q' in Q. (i.e. for all states in Q).

$$S \rightarrow [q_0, z_0, q'] \quad \text{where } q' \text{ is every state in } Q$$

i.e. start symbol of grammar $\longrightarrow$ [start state of PDA, start/Initial symbol on stack, (Every) state in Q]

Step 2: For each transition of the form in PDA

$$\delta(q_i, a, B) \Rightarrow (q_j, C) \quad \text{Then,}$$

~~for~~ each $q' \in Q$, add the productions as follows:

$$[q_i, B, q'] \longrightarrow a\,[q_j, C, q']$$

here, $q_i, q_j \in Q$, $a \in (\Sigma \cup \varepsilon)$ & $B, C \in \Gamma$

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Step 3 : For each transition in PDA of the form

Note: In some cases $[\therefore \Delta = \epsilon]$

$$\delta(q, a, z) = (q', \epsilon)$$

then add production in Grammar as,

$$[q, z, q'] \to a$$

Step 4 : For each transition in PDA of the form,

$\delta(q_i, a, z) = (q_j, z_1 z_2)$ then add following productions in the grammar,

$$\boxed{[q_i, z, q'] \to a\,[q_j, z_1, q_1]\,[q_1, z_2, q']}$$

Example:

$$\delta(q_i, a, z) = (q_j, z_1 z_2)$$

$$[q_i, z, q'] \to a\,[q_j, z_1, q_1]\,[q_1, z_2, q']$$

(all states in Q)

Arrow indicates occurrences of particular state in same order. And q' is all states in Q

Example 1: If $Q = \{q_0, q_1\}$ and $\delta(q_0, a, z) \to (q_0, z_1 z_2)$

Sol^n :⇒ Productions needs ~~to~~ include in the grammar are,

for all states in Q — for all states in Q

$$[q_0, z, q_0] \to a\,[q_0, z_1, q_0]\,[q_0, z_2, q_0]$$

$$[q_0, z, q_0] \to a\,[q_0, z_1, q_1]\,[q_1, z_2, q_0]$$

$$[q_0, z, q_1] \to a\,[q_0, z_1, q_0]\,[q_0, z_2, q_1]$$

$$[q_0, z, q_1] \to a\,[q_0, z_1, q_1]\,[q_1, z_2, q_1]$$

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Example 1: Convert given PDA to CFG.

PDA is given by,

$M = \{ \{p, q\}, \{0, 1\}, \{x, z\}, \delta, q, z \}$

Transition function is,

$\delta(q, 1, z) = \{(q, xz)$

$\delta(q, 1, x) = (q, xx)$

$\delta(q, \Delta, x) = (q, \Delta)$

$\delta(q, 0, x) = (p, x)$

$\delta(p, 1, x) = (p, \Delta)$

$\delta(p, 0, z) = (q, z)$

~~Sol~~ $\Rightarrow$

Example 2: If $Q = \{q_0, q_1\}$ and $\Gamma = \{a, b, z\}$ then the possible set of variables in the corresponding CFG is given by:

Sol $\Rightarrow$

Possible set of variables in expected grammar are:

1. For start symbol: S
   S
2. for 'a' symbol on top of stack:
   $[q_0, a, q_0]$, $[q_0, a, q_1]$, $[q_1, a, q_0]$, $[q_1, a, q_1]$
3. For 'b' symbol on the top of stack
   $[q_0, b, q_0]$, $[q_0, b, q_1]$, $[q_1, b, q_0]$, $[q_1, b, q_1]$
4. For 'z' symbol on top of stack
   $[q_0, z, q_0]$, $[q_0, z, q_1]$, $[q_1, z, q_0]$, $[q_1, z, q_1]$

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Example 1 : Convert PDA to CFG -

$M = \{ \{p, q\}, \{0, 1\}, \{x, z\}, \delta, q, z \}$

$\uparrow Q$ $\uparrow \Sigma$ $\uparrow \Gamma$ $\uparrow \delta$ $\uparrow$ start state $\uparrow$ Initial symbol on stack

Transition function,

$\delta(q, 1, z) = (q, xz)$

$\delta(q, 1, x) = (q, xx)$

$\delta(q, \Delta, x) = (q, \Delta)$

$\delta(q, 0, x) = (p, x)$

$\delta(p, 1, x) = (p, \Delta)$

$\delta(p, 0, z) = (q, z)$

Soln :⇒

Step 1 : productions for start symbol 'S' in the grammar

start state, all states in Q.

$S \rightarrow [q, z, q]$

$S \rightarrow [q, z, p]$

Step 2 : For production $\delta(q, 1, z) = (q, xz)$ in PDA :

$[q, z, q] \rightarrow 1 [q, x, q] [q, z, q]$

$[q, z, q] \rightarrow 1 [q, x, p] [p, z, q]$

$[q, z, p] \rightarrow 1 [q, x, q] [q, z, p]$

$[q, z, p] \rightarrow 1 [q, x, p] [p, z, p]$

Step 3 : for $\delta(q, 1, x) = (q, xx)$

$[q, x, q] \rightarrow 1 [q, x, q] [q, x, q]$

$[q, x, q] \rightarrow 1 [q, x, p] [p, x, q]$

$[q, x, p] \rightarrow 1 [q, x, q] [q, x, p]$

$[q, x, p] \rightarrow 1 [q, x, p] [p, x, p]$

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step 4 :
for $\delta(q, \Delta, x) = (q, \Delta)$:
Grammar production,

$[q, x, q] \rightarrow \epsilon$
~~$[q, x, p] \rightarrow \epsilon$~~

Step 5: For $\delta(q, 0, x) = (p, x)$:
Grammar production,

$[q, x, q] \rightarrow 0 [p, x, q]$

$[q, x, p] \rightarrow 0 [p, x, p]$

step 6: for $\delta(p, 1, x) = (p, \Delta)$
Grammar productions,

$[p, x, p] \rightarrow 1$

step 7: for $\delta(p, 0, z) = (q, z)$:
Grammar productions,

$[p, z, q] \rightarrow 0 [q, z, q]$
$[p, z, p] \rightarrow 0 [q, z, p]$

step 8: Rename above all Nonterminals/variables by A, B, ...

$[q, z, q] \Rightarrow A$
$[q, z, p] \Rightarrow B$
$[p, z, q] = C$
$[p, z, p] = D$
$[q, x, q] = E$
$[q, x, p] = F$
$[p, x, p] = G$
$[p, x, q] = H$

step 9: Grammar for given PDA
$S \rightarrow A | B$
$A \rightarrow 1EA | 1FC$
$B \rightarrow 1EB | 1FH$
$E \rightarrow 1EE | 1FH$
$F \rightarrow 1EF | 1FG | 0G$
$E \rightarrow 0H | \epsilon$
$G \rightarrow 1$
$C \rightarrow 0A$
$D \rightarrow 0B$

step 10: simplify the grammar if required.
(Remove ϵ productions, unit productions, useless symbols)

Example 2: Find CFG for given PDA.

$M = (\{q\}, \{i, e\}, \{x, z\}, \delta, q, z)$

with,

(i) $\delta(q, i, z) = (q, xz)$

(ii) $\delta(q, e, x) = (q, \Delta)$

(iii) $\delta(q, \Delta, z) = (q, \Delta)$

Soln :=>

Step1 : For start symbol

$S \rightarrow [q, z, q]$

Step2 : for $\delta(q, i, z) = (q, xz)$

$[q, z, q] \rightarrow i\,[q, x, q]\,[q, z, q]$

Step3 : for $\delta(q, e, x) = (q, \Delta)$

$[q, x, q] \rightarrow e$

Step4 : for $\delta(q, \Delta, z) = (q, \Delta)$

$[q, z, q] \rightarrow \epsilon$

Step5 : Relabel the variables.

$[q, z, q] \Rightarrow A$

$[q, x, q] \Rightarrow B$

Step6:

$S \rightarrow A$

$A \rightarrow iBA \mid \epsilon$

$B \rightarrow e$

Step7: After Removing ϵ production & Unit production

$S \rightarrow iBA \mid iB$

$B \rightarrow e$

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