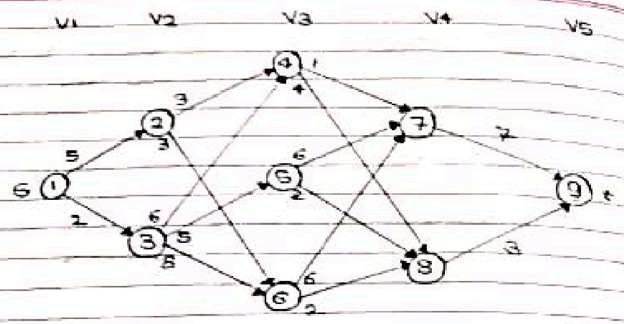


→ saving the multistage graph problem using forward approach:-

∴ $cost(i,j) = \min\{c(j,l) + cost(i+1,l)\}$

① $cost(5,9) = 0$

② $cost(4,7) = 7$ $\qquad d(4,7) = 9$

③ $cost(4,8) = 3$ $\qquad d(4,8) = 9$

④ $cost(3,4) = \min\{c(4,7) + cost(4,7) = 1+7 = 8$
$\qquad c(4,8) + cost(4,8) = 4+3 = \underline{7}\}$

∴ $cost(3,4) = 7$
∴ ~~cost(3~~ $d(3,4) = 8$

⑤ $cost(3,5) = \min\{c(5,7) + cost(4,7) = 6+7 = 13$
$\qquad c(5,8) + \underline{cost(4,8)} = 2+3 = \underline{5}\}$

∴ $cost(3,5) = 5$
∴ $d(3,5) = 8$

⑥ $cost(3,6) = \min\{c(6,7) + cost(4,7) = 6+7 = 13$
$\qquad c(6,8) + \underline{cost(4,8)} = 2+3 = \underline{5}\}$

∴ $cost(3,6) = 5$
$d(3,6) = 8$

⑦ $cost(2,2) = \min\{c(2,4) + cost(3,4) = 3+7 = 10$
$\qquad c(2,6) + \underline{cost(3,6)} = 3+5 = \underline{8}\}$

∴ $cost(2,2) = 8$
∴ $d(2,2) = 6$

⑧ $cost(2,3) = \min\{c(3,4) + cost(3,4) = 6+7 = 13$
$\qquad c(3,5) + \underline{cost(3,5)} = 5+5 = \underline{10}$
$\qquad c(3,6) + cost(3,6) = 8+5 = 13\}$

∴ $cost(2,3) = 10$
$d(2,3) = 5$

⑨ $cost(1,1) = \min\{c(1,2) + cost(2,2) = 5+8 = 13$
$\qquad c(1,3) + \underline{cost(2,3)} = 2+10 = \underline{12}\}$

∴ $cost(1,1) = 12$
∴ $d(1,1) = 3$

The path is-
$d(1,1) = 3$
$d(2,3) = 5$
∴ $d(3,5) = 8$
∴ $d(4,8) = 9$

∴ we get shortest path from s to t with minimum distance 12.

$1 \rightarrow 3 \rightarrow 5 \rightarrow 8 \rightarrow 9$

The problem is to allocate the resource to the r projects in such a way as to maximize total net profit. The problem can be formulated as an r+1 stage graph problem. Stage i, $1 \le i \le r$ represents project i. There are $n+1$ vertices $V(i,j)$, as is an associated with stage i, $2 \le i \le r$.

Q3. Write the expressions used for computation of minimum cost path in multistage graph with both approaches.

→ Forward Approach: calculate distance from target as reference.

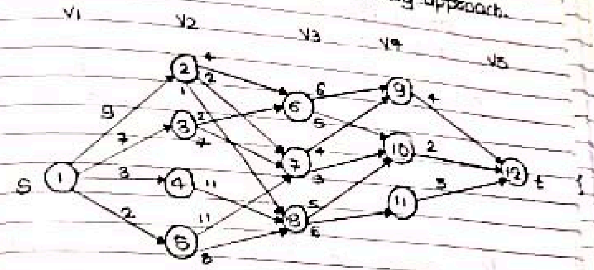
$cost(i,j) = \min\{c(j,l) + cost(i+1, l)\}$

where,
i = current level
j = current vertex number in that lab level
j = current vertex
l = vertex of next level

Backward approach: calculate distance from source as reference.

$bcost(i,j) = \min\{c(l,j) + bcost(i-1, l)\}$

Q4. Solve the multistage graph problem using dynamic programming approach. Find shortest path from s to t using dynamic programming approach.



→ Solving the multistage graph problem using forward approach-

$cost(i,j) = \min\{c(j,l) + cost(i+1, l)\}$

① $cost(5,12) = 0$

② $cost(4,9) = 4$

③ $cost(4,10) = 2$

④ $cost(4,11) = 5$

⑤ $cost(3,6) = \min\{c(6,9) + cost(4,9) = 6+4 = 10,\ c(6,10) + \underline{cost(4,10)} = 5+2 = \underline{7}\}$

$cost(3,6) = 7$

$d(3,6) = 10$

$A^3(4,1) = \min\{A^2(4,1), A^2(4,3) + A^2(3,1)\}$
$= \min\{\infty, 4+3 = 7\}$

$\therefore A^3(4,1) = 7$

$A^3(4,2) = \min\{A^2(4,2), A^2(4,3) + A^2(3,2)\}$
$= \min\{\infty, 4+7 = 11\}$

$\therefore A^3(4,2) = 11$

(4) To construct A^4 matrix → 4^{th} row and 4^{th} column keep as per the A^3 matrix.

$$A^4 = \begin{array}{c|cccc|} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & \underline{\underline{4}} & 10 & 6 \\ 2 & \underline{\underline{6}} & 0 & \underline{\underline{6}} & 2 \\ 3 & \underline{\underline{3}} & \underline{\underline{7}} & 0 & 9 \\ 4 & 7 & 11 & 4 & 0 \end{array}$$

$A^4(1,2) = \min\{A^3(1,2), A^3(1,4) + A^3(4,2)\}$
$= \min\{4, 6+11 = 17\}$

$\therefore A^4(1,2) = 4$

$A^4(1,3) = \min\{A^3(1,3), A^3(1,4) + A^3(4,3)\}$
$= \min\{11, 6+4 = 10\}$

$\therefore A^4(1,3) = 10$

$A^4(2,1) = \min\{A^3(2,1), A^3(2,4) + A^3(4,1)\}$
$= \min\{6, 2+7 = 9\}$

$\therefore A^4(2,1) = 6$

$A^4(2,3) = \min\{A^3(2,3), A^3(2,4) + A^3(4,3)\}$
$= \min\{17, 2+4 = 6\}$

$\therefore A^4(2,3) = 6$

$A^4(3,1) = \min\{A^3(3,1), A^3(3,4) + A^3(4,1)\}$
$= \min\{3, 9+7 = 16\}$

$\therefore A^4(3,1) = 3$

$A^4(3,2) = \min\{A^3(3,2), A^3(3,4) + A^3(4,2)\}$
$= \min\{7, 9+\infty = \infty\}$

$\therefore A^4(3,2) = 7$

$\therefore$ The final all pair shortest path matrix is:

$$A^4 = \begin{array}{c|cccc|} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 4 & 10 & 6 \\ 2 & 6 & 0 & 6 & 2 \\ 3 & 3 & 7 & 0 & 9 \\ 4 & 7 & 11 & 4 & 0 \end{array}$$

Q10 Explain solution to optimal binary search tree's problem using dynamic programming. OR write note on "optimal binary search tree."

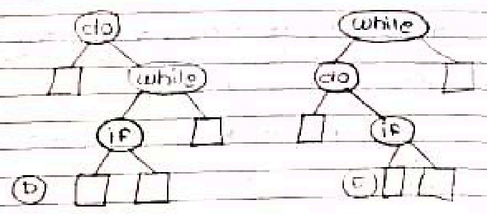
→ 1) Given set of identifiers, different binary trees could be formed.

2) Average number of comparisons needed for finding the identifiers will be different for different trees.

3) In the simplest form we assume

ⓐ probability of search of each element is equal.

ⓑ no unsuccessful searches made.



(d) (c)

$\therefore$ $cost = \sum_{1 \le i \le n} p(i) * level(a_i) + \sum_{0 \le i \le n} q(i) * (level(E_i) - 1)$

cost(A) = $p(i) = \frac{1}{7}$ $q(i) = \frac{1}{7}$

$cost(A) = \left[\frac{1}{7}*1 + \frac{1}{7}*2 + \frac{1}{7}*3\right] + \left[\frac{1}{7}*3 + \frac{1}{7}*3 + \frac{1}{7}*2 + \frac{1}{7}*1\right]$

$= \frac{1}{7} + \frac{2}{7} + \frac{3}{7} + \frac{3}{7} + \frac{3}{7} + \frac{2}{7} + \frac{1}{7}$

$\therefore cost(A) = \frac{15}{7}$

$cost(B) = \left[\frac{1}{7}*1 + \frac{1}{7}*2 + \frac{1}{7}*3\right] + \left[\frac{1}{7}*1 + \frac{1}{7}*2 + \frac{1}{7}*3 + \frac{1}{7}*3\right]$

$= \frac{1}{7} + \frac{2}{7} + \frac{3}{7} + \frac{1}{7} + \frac{2}{7} + \frac{3}{7} + \frac{3}{7}$

$\therefore cost(B) = \frac{15}{7}$

$cost(C) = \left[\frac{1}{7}*2 + \frac{1}{7}*1 + \frac{1}{7}*2\right] + \left[\frac{1}{7}*2 + \frac{1}{7}*2 + \frac{1}{7}*2 + \frac{1}{7}*2\right]$

$= \frac{2}{7} + \frac{1}{7} + \frac{2}{7} + \frac{2}{7} + \frac{2}{7} + \frac{2}{7} + \frac{2}{7}$

$\therefore cost(C) = \frac{13}{7}$

$cost(D) = \left[\frac{1}{7}*1 + \frac{1}{7}*2 + \frac{1}{7}*3\right] + \left[\frac{1}{7}*1 + \frac{1}{7}*3 + \frac{1}{7}*3 + \frac{1}{7}*2\right]$

$= \frac{1}{7} + \frac{2}{7} + \frac{3}{7} + \frac{1}{7} + \frac{3}{7} + \frac{3}{7} + \frac{2}{7}$

$\therefore cost(D) = \frac{15}{7}$

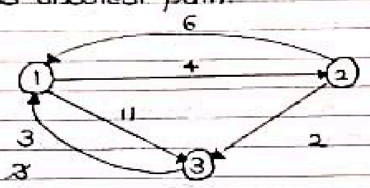
$cost(E) = \left[\frac{1}{7}*1 + \frac{1}{7}*2 + \frac{1}{7}*3\right] + \left[\frac{1}{7}*2 + \frac{1}{7}*1 + \frac{1}{7}*3 + \frac{1}{7}*3\right]$

$= \frac{1}{7} + \frac{2}{7} + \frac{3}{7} + \frac{2}{7} + \frac{1}{7} + \frac{3}{7} + \frac{3}{7}$

$\therefore cost(E) = \frac{15}{7}$

$\therefore$ the optimal binary search tree is (c)

Q7. Solve the following all pairs shortest path problem to obtain matrix representing all pairs shortest path.



→ The formula is:-

$A^k(i,j) = \min\{A^{k-1}(i,j), A^{k-1}(i,k) + A^{k-1}(k,j)\}$, $k \geq 1$

$$A^0 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 4 & 11 \\ 6 & 0 & 2 \\ 3 & \infty & 0 \end{bmatrix} \end{array}$$

→ in matrix A^0 we give direct distances

① To construct A^1 → 1st row and 1st column keep as per previous matrix i.e. A^0

$$A^1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 4 & 11 \\ 6 & 0 & 2 \\ 3 & 7 & 0 \end{bmatrix} \end{array}$$

$A^1(2,3) = \min\{A^0(2,3), A^0(2,1) + A^0(1,3)\}$

$= \min\{2, 6+11 = 17\}$

$\therefore A^1(2,3) = 2$

$A^1(3,2) = \min\{A^0(3,2), A^0(3,1) + A^0(1,2)\}$

$= \min\{\infty, 3+4 = 7\}$

$\therefore A^1(3,2) = 7$

② To construct A^2 → 2nd row and 2nd column keep as per previous matrix i.e. A^1

$$A^2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 4 & 6 \\ 6 & 0 & 2 \\ 3 & 7 & 0 \end{bmatrix} \end{array}$$

$A^2(1,3) = \min\{A^1(1,3), A^1(1,2) + A^1(2,3)\}$

$= \min\{11, 4+2 = 6\}$

$\therefore A^2(1,3) = 6$

$A^2(3,1) = \min\{A^1(3,1), A^1(3,2) + A^1(2,1)\}$

$\min\{3, 7+6 = 13\}$

$\therefore A^2(3,1) = 3$

③ To construct A^3 → 3rd row and 3rd column keep as per previous matrix i.e. A^2

$$A^3 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 4 & 6 \\ 5 & 0 & 2 \\ 3 & 7 & 0 \end{bmatrix} \end{array}$$

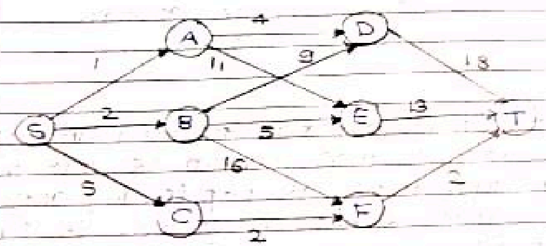
$A^3(1,2) = \min\{A^2(1,2), A^2(1,3) + A^2(3,2)\}$

$= \min\{4, 6+7 = 13\}$

$\therefore A^3(1,2) = 4$

2) Set V_1 and V_k are such that $|V_1| = |V_k| = 1$
3) Let s and t be the vertices in V_1 and V_k respectivly.
4) s is source and t is sink (destination)
5) Let $c(i,j)$ be the cost of edge (i,j)
6) The cost of all ~~the edges~~ on the path from s to t is the sum of the costs of all the edges on the path.
7) The multistage graph problem is to find a minimum cost path from s to t.
8) Each stage V_i defines a stage in the graph.
9) constraint is, every path from s to t starts in stage 1, goes to stage 2 and so on & terminate in n stage.

eg. The shortest path in multistage graph-



The greedy method can not be applied to this case:
$(S, A, D, T) \Rightarrow 1+4+18 = 23$
The real shortest path is:
(S, C, F, T) $5+2+2 = 9$

solution to this problem is dynamic programing approach-
Such problems can be solved using 2 approaches-
① Forward Approach (Backward reasoning)
② Backward Approach (Forward reasoning)

① Forward Approach-
It is also called as backward reasoning. Every s to t path is the result of a sequence of k-2 decisions. The ith decision involves determining which vertex in V_{i+1} is to be on the path.
calculate distance from target as reference-
$cost(i,j) = \min \{ c(j,l) + cost(i+1, l) \}$

② Backward Approach-
It is also called as forward reasoning. It calculate distance from source as reference-
$bcost(i,j) = \min \{ bcost(i-1, l) + c(l,j) \}$

Application of multistage graph- Resource allocation problem.

Example- consider a resource allocation problem n units of resource are to be considered to be allocated to 3 projects. If j, $0 \le j \le n$, units of the resources are allocated to project i, then the resulting net profit is $N(i,j)$.

$$A^3 = \begin{array}{c|ccc|} & 1 & 2 & 3 \\ \hline 1 & 0 & 3 & 8 \\ 2 & 4 & 0 & 5 \\ 3 & 10 & 6 & 0 \end{array}$$

$A^3(1,2) = \min\{A^2(1,2), A^2(1,3) + A^2(3,2)\}$
$= \min\{3, 8+6 = 14\}$

$\therefore A^3(1,2) = 3$

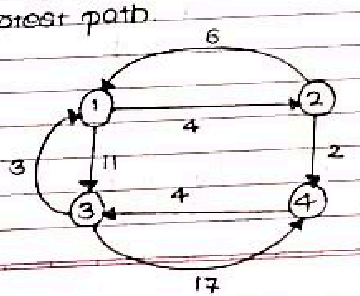
$A^3(2,1) = \min\{A^2(2,1), A^2(2,3) + A^2(3,1)\}$
$= \min\{4, 5+10 = 15\}$

$\therefore A^3(2,1) = 4$

$\therefore$ The final all pair shortest path matrix is,

$$\begin{array}{c|ccc|} & 1 & 2 & 3 \\ \hline 1 & 0 & 3 & 8 \\ 2 & 4 & 0 & 5 \\ 3 & 10 & 6 & 0 \end{array}$$

Q9. Solve the following all pairs shortest path problem to obtain matrix representing all pairs shortest path.



→ In matrix A^0 we give direct distances -

$$A^0 = \begin{array}{c|cccc|} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 4 & 11 & \infty \\ 2 & 6 & 0 & \infty & 2 \\ 3 & 3 & \infty & 0 & 17 \\ 4 & \infty & \infty & 4 & 0 \end{array}$$

① To construct A^1 matrix 1st row and 1st column keep as per the A^0 matrix.

$$A^1 = \begin{array}{c|cccc|} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 4 & 11 & \infty \\ 2 & 6 & 0 & \underline{17} & \underline{2} \\ 3 & 3 & \underline{7} & 0 & \underline{17} \\ 4 & \infty & \underline{\infty} & \underline{4} & 0 \end{array}$$

$A^1(3,2) = \min\{A^0(3,2), A^0(3,1) + A^0(1,2)\}$
$= \min\{\infty, 3+4 = 7\}$

$A^1(3,2) = 7$

$A^1(2,3) = \min\{A^0(2,3), A^0(2,1) + A^0(1,3)\}$
$= \min\{\infty, 6+11 = 17\}$

$A^1(2,3) = 17$

$A^1(2,4) = \min\{A^0(2,4), A^0(2,1) + A^0(1,4)\}$
$= \min\{2, 6+\infty = \infty\}$

$\therefore A^1(2,4) = 2$

$A^1(3,4) = \min\{A^0(3,4), A^0(3,1) + A^0(1,4)\}$
$= \min\{17, 3+\infty = \infty\}$

$\therefore A^1(3,4) = 17$

$A^1(4,2) = \min\{A^0(4,2),\ A^0(4,1) + A^0(1,2)\}$
$= \min\{\infty,\ \infty + 4 = \infty\}$
$\therefore A^1(4,2) = \infty$

$A^1(4,3) = \min\{A^0(4,3),\ A^0(4,1) + A^0(1,3)\}$
$= \min\{4,\ \infty + 11 = \infty\}$
$\therefore A^1(4,3) = 4$

② To construct A^2 matrix ⇒ 2nd row and 2nd column keep as per the A^1 matrix.

$$A^2 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 4 & \underline{11} & \underline{6} \\ 2 & 6 & 0 & 17 & 2 \\ 3 & \underline{3} & 7 & 0 & \underline{9} \\ 4 & \underline{\infty} & \infty & \underline{4} & 0 \end{array}$$

$A^2(1,3) = \min\{A^1(1,3),\ A^1(1,2) + A^1(2,3)\}$
$= \min\{11,\ 4 + 17 = 21\}$
$\therefore A^2(1,3) = 11$

$A^2(1,4) = \min\{A^1(1,4),\ A^1(1,2) + A^1(2,4)\}$
$= \min\{\infty,\ 4 + 2 = 6\}$
$\therefore A^2(1,4) = 6$

$A^2(3,1) = \min\{A^1(3,1),\ A^1(3,2) + A^1(2,1)\}$
$= \min\{3,\ 7 + 6 = 13\}$
$\therefore A^2(3,1) = 3$

$A^2(3,4) = \min\{A^2(3,4),\ A^1(3,2) + A^1(2,4)\}$
$= \min\{17,\ 7 + 2 = 9\}$
$\therefore A^2(3,4) = 9$

$A^2(4,1) = \min\{A^1(4,1),\ A^1(4,2) + A^1(2,1)\}$
$= \min\{\infty,\ \infty + 6 = \infty\}$
$A^2(4,1) = \infty$

$A^2(4,3) = \min\{A^1(4,3),\ A^1(4,2) + A^1(2,3)\}$
$= \min\{4,\ \infty + 17 = \infty\}$
$\therefore A^2(4,3) = 4$

③ To construct A^3 matrix ⇒ 3rd row and 3rd column keep as per the A^2 matrix

$$A^3 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & \underline{4} & 11 & \underline{6} \\ 2 & \underline{6} & 0 & 17 & \underline{2} \\ 3 & 3 & 7 & 0 & 9 \\ 4 & \underline{7} & \underline{11} & 4 & 0 \end{array}$$

$A^3(1,2) = \min\{A^2(1,2),\ A^2(1,3) + A^2(3,2)\}$
$= \min\{4,\ 11 + 7 = 18\}$
$\therefore A^3(1,2) = 4$

$A^3(1,4) = \min\{A^2(1,4),\ A^2(1,3) + A^2(3,4)\}$
$= \min\{6,\ 11 + 9 = 20\}$
$\therefore A^3(1,4) = 6$

$A^3(2,1) = \min\{A^2(2,1),\ A^2(2,3) + A^2(3,1)\}$
$= \min\{6,\ 17 + 3 = 20\}$
$\therefore A^3(2,1) = 6$

$A^3(2,4) = \min\{A^2(2,4),\ A^2(2,3) + A^2(3,4)\}$
$= \min\{2,\ 17 + 9 = 26\}$
$A^3(2,4) = 2$

Tutorial No:- 4

Q1. Write note on "Dynamic programming approach".

→ 1] Dynamic programming is an algorithmic technique for solving an optimization problem by breaking it down into simpler subproblems and utilizing the fact that the optimal solution to the overall problem depends upon the optimal

2] Dynamic programming drastically reduces the amount of time and space required.

3] It avoids the enumeration of some decision sequences, that cannot possibly be optimal.

4] In dynamic programming, many decision sequences can be generated.but sequences containing suboptimal subsequences can not be optimal.

5] Examples are

a] All pairs shortest job algorithm

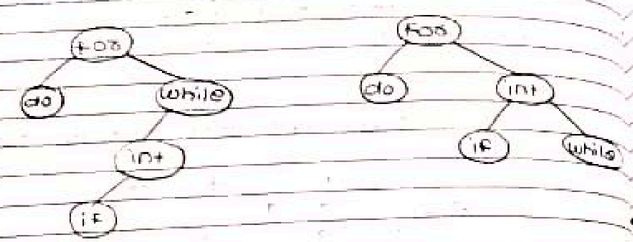
b] multistage graph problem.

Q2. Explain multistage graph problem and its use in resource allocation problem. OR what is multistage graph? OR Explain any one application of multistage graph. OR Explain dynamic programming solutions to multistage graph problem.

→ 1]A multistage graph $G=(V,E)$ is a direction graph, in which the vertices are pastitioned into $k \geq 2$ disjoint sets V_i, $1 \leq i \leq k$

Eg:- Let set of identifiers be {for, do, while, int, if} with do < for < if < int < while

∴ The possible trees are -



Comparison = 1+2+2+3+4/5 = $\frac{12}{5}$

Comparison = 1+2+2+3+3/5 = 11/5

Q11. Derive an expression for cost of optimal binary search tree.

→ 1) In general situation different identifiers can be searched with different probabilities and unsuccessful searches can also be made

2) Let set of identifiers be {a1, a2, ..., an} with a1 < a2 < a3 < ... < an

3) Let p(i) be the probability with which we search ai.

4) Let q(i) be the probability with which we search x such that $a_i < x < a_{i+1}$, $0 \le i \le n$

5) Assume $a_0 = -\infty$ and $a_{n+1} = \infty$ then

$\sum_{0 \le i \le n} q(i)$ is the probability of unsuccessful search

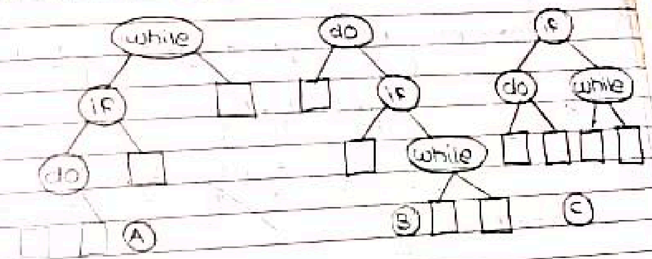
clearly

$$\sum_{1 \le i \le n} p(i) + \sum_{0 \le i \le n} q(i) = 1$$

$$\therefore \text{cost} = \sum_{1 \le i \le n} p(i) * \text{level}(a_i) + \sum_{0 \le i \le n} q(i) * (\text{level}(E_i) - 1)$$

Q12. Draw an optimal binary search tree where nodes are labeled as (do, if, while) and the events of successful and unsuccessful search are equi-probable

→ if do < if < while, int



GOODLUCK Page No. Date

6 cost(3,7) = min { c(7,9) + cost(4,9) = 4 + 4 = 8
c(7,10) + cost(4,10) = 3 + 2 = 5 }

∴ cost(3,7) = 5
∴ d(3,7) = 10

7 cost(3,8) = min { c(8,10) + cost(4,10) = 5 + 2 = 7
c(8,11) + cost(4,11) = 6 + 5 = 11 }

∴ cost(3,8) = 7
∴ d(3,8) = 10

8 cost(2,2) = min { c(2,6) + cost(3,6) = 4 + 7 = 11
c(2,7) + cost(3,7) = 2 + 5 = 7
c(2,8) + cost(3,8) = 1 + 7 = 8 }

∴ cost(2,2) = 7
∴ d(2,2) = 7

9 cost(2,3) = min { c(3,6) + cost(3,6) = 2 + 7 = 9
c(3,7) + cost(3,7) = 7 + 5 = 12 }

∴ cost(2,3) = 9
∴ d(2,3) = 6

10 cost(2,4) = min { c(4,8) + cost(3,8) = 11 + 7
= 18 }

∴ cost(2,4) = 18
∴ d(2,4) = 8

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11 cost(2,5) = min { c(5,7) + cost(3,7) = 11 + 5 = 16
c(5,8) + cost(3,8) = 8 + 7 = 15 }

∴ cost(2,5) = 15
∴ d(2,5) = 8

12 cost(1,1) = min { c(1,2) + cost(2,2) = 9 + 7 = 16
c(1,3) + cost(2,3) = 7 + 9 = 16
c(1,4) + cost(2,4) = 3 + 18 = 21
c(1,5) + cost(2,5) = 2 + 15 = 17 }

∴ cost(1,1) = 16
∴ d(1,1) = 2 and 3

The path is:-

d(1,1) = 2	d(1,1) = 3
d(2,2) = 7	d(2,3) = 6
d(3,7) = 10	d(3,6) = 10
d(4,10) = 12	d(4,10) = 12

∴ we get two shortest path from s to t with minimum distance 16.
the path are:-
1 → 2 → 7 → 10 → 12
1 → 3 → 6 → 10 → 12

Q5. Solve the multistage graph problem using dynamic programming approach to find shortest path from S to t.

Q6. Explain solution to all pairs shortest path problem using dynamic programming. Explain all pair shortest path algorithm with example.

→ 1] Let $G=(V,E)$ be a directed graph with n vertices.

2] Let cost be a adjancey matrix for G, such that $cost(i,i) = 0, 1 \le i \le n$

3] $cost(i,j)$ is the length (or cost) of edge (i,j) i.e. $(i,j) \in E(G)$

4] All pair shortest path problem is to determine a matrix A

5] such that $A(i,j)$ is the length of shortest path from i to j.

6] A can be obtained by solving n single source shortest path problems.

7] complexity $n*n^2$ i.e. $O(n^3)$.

8] Shortest path from i to j, $i \ne j$

a] Path may go through some intermediate vertex.

b] If k is the index of this intermediate vertex then path (i,k) and (k,j) must be shortest path.

c] If k is the intermediate vertex with highest index then

- i to k path is the shortest i to k path going through no vertex with index greater than k-1

- so is the k to j path.

9] We need to decide highest index intermediate vertex k

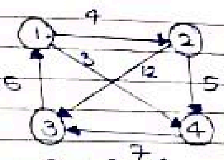
10] Let $A^k(i,j)$ be length of shortest path from i to j going through no vertex of index greather than k.

11] $A(i,j) = \min\{ \min_{1 \le k \le n} \{A^{k-1}(i,k) + A^{k-1}(k,j)\}, cost(i,j)\}$

12] A shortest path from i to j may or may not go through higest index vertex.

$$A^k(i,j) = \min\{A^{k-1}(i,j), A^{k-1}(i,k) + A^{k-1}(k,j)\} \quad k \ge 1$$

Eg:-



→ $A^0 =$

	1	2	3	4
1	0	4	∞	8
2	∞	0	12	5
3	5	∞	0	∞
4	∞	∞	7	0

$A^1 =$

	1	2	3	4
1	0	4	∞	8
2	∞	0	12	5
3	5	9	0	13
4	~~∞~~ 12	~~12~~ ∞	7	0

$A^2 =$

	1	2	3	4
1	0	4	16	8
2	~~∞~~ 17	0	12	5
3	5	9	0	13
4	12	∞	7	0

$A^3 =$

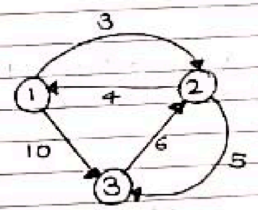
	1	2	3	4
1	0	4	16	8
2	17	0	12	5
3	5	9	0	13
4	12	16	7	0

$A^3(2,1) = \min\{A^2(2,1), A^2(2,3) + A^2(3,1)\}$
$= \min\{6, 2+3 = 5\}$
$\therefore A^3(2,1) = 5$

$\therefore$ Final matrix is,

All pairs shortest path is $\begin{bmatrix} 0 & 4 & 6 \\ 5 & 0 & 2 \\ 3 & 7 & 0 \end{bmatrix}$

Q2. Solve the following instance of all pairs shortest path problem to obtain matrix representing all pairs shortest path.



→ $A^0 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 3 & 10 \\ 2 & 4 & 0 & 5 \\ 3 & \infty & 6 & 0 \end{array}$ → in matrix A^0 we give direct distances

① To construct A^1 → 1^{st} row and 1^{st} column keep as per previous matrix i.e A^0

$A^1 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 3 & 10 \\ 2 & 4 & 0 & 5 \\ 3 & \infty & 6 & 0 \end{array}$

$A^1(2,3) = \min\{A^0(2,3), A^0(2,1) + A^0(1,3)\}$
$= \min\{5, 4+10 = 14\}$
$\therefore A^1(2,3) = 5$

$A^1(3,2) = \min\{A^0(3,2), A^0(3,1) + A^0(1,2)\}$
$= \min\{6, \infty + 3 = \infty\}$
$\therefore A^1(3,2) = 6$

② To construct A^2 → 2^{nd} row and 2^{nd} column keep as per previous matrix i.e A^1

$A^2 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 3 & 8 \\ 2 & 4 & 0 & 5 \\ 3 & 10 & 6 & 0 \end{array}$

$A^2(1,3) = \min\{A^1(1,3), A^1(1,2) + A^1(2,3)\}$
$= \min\{10, 3+5 = 8\}$
$\therefore A^2(1,3) = \cancel{10}\,8$

$A^2(3,1) = \min\{A^1(3,1), A^1(3,2) + A^1(2,1)\}$
$= \min\{\infty, 6+4 = 10\}$
$\therefore A^2(3,1) = 10$

③ To construct A^3 → 3^{rd} row and 3^{rd} column keep as per previous matrix i.e A^2