Mathematics I (BSM 101)

Manju Subedi

Gandaki University
Bachelor in Information Technology(BIT)
BSM 101

manjusubedi.061@gmail.com

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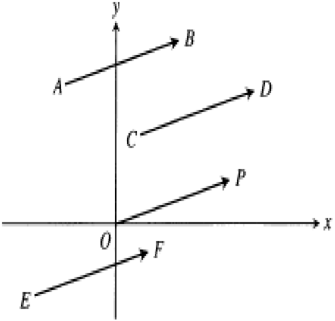

Figure: Equality of Vectors

Vector Operations

Definitions: If $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$, the vectors $a\mathbf{i}$ and $b\mathbf{j}$ are the vector components of $\mathbf{v}$ in the directions of $\mathbf{i}$ and $\mathbf{j}$.
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Equality of Vectors: (Algebraic Definition)

$$a\mathbf{i} + b\mathbf{j} = a'\mathbf{i} + b'\mathbf{j} \quad \Leftrightarrow \quad a = a' \text{ and } b = b'$$

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Algebraic Addition: Vectors may be added algebraically by adding their corresponding scalar components. If $\mathbf{v}_1 = a_1\mathbf{i} + b_1\mathbf{j}$ and $\mathbf{v}_2 = a_2\mathbf{i} + b_2\mathbf{j}$, then

$$\mathbf{v}_1 + \mathbf{v}_2 = (a_1 + a_2)\,\mathbf{i} + (b_1 + b_2)\,\mathbf{j}.$$

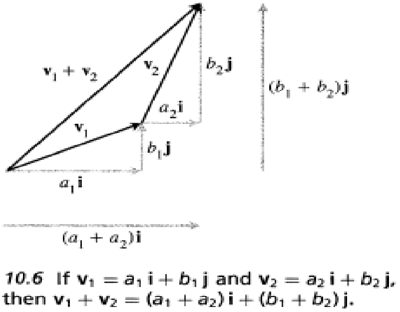

Figure: Vector Addition

Example: Given $\mathbf{v_1} = 2\mathbf{i} - 4\mathbf{j}$, $\mathbf{v_2} = 5\mathbf{i} + 3\mathbf{j}$
then

$$\mathbf{v_1} + \mathbf{v_2} = (2\mathbf{i} - 4\mathbf{j}) + (5\mathbf{i} + 3\mathbf{j}) = (2+5)\mathbf{i} + (-4+3)\mathbf{j} = 7\mathbf{i} - \mathbf{j}$$

Vector Operations

Subtraction: The negative of a vector $\mathbf{v}$ is the vector $-\mathbf{v} = (-1)\mathbf{v}$. It has the same length as $\mathbf{v}$ but points in the opposite direction. To subtract a vector $\mathbf{v}_2$ from a vector $\mathbf{v}_1$, we add $-\mathbf{v}_2$ to $\mathbf{v}_1$.
If $\mathbf{v}_1 = a_1\mathbf{i} + b_1\mathbf{j}$ and $\mathbf{v}_2 = a_2\mathbf{i} + b_2\mathbf{j}$, then
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Position Vector: We find the components of the vector from a point $P_1\,(x_1, y_1)$ to a point $P_2\,(x_2, y_2)$ by subtracting the components of position vector $\overrightarrow{OP}_1 = x_1\mathbf{i} + y_1\mathbf{j}$ from the components of position vector $\overrightarrow{OP}_2 = x_2\mathbf{i} + y_2\mathbf{j}$. The vector from $P_1\,(x_1, y_1)$ to $P_2\,(x_2, y_2)$ is

$$\overrightarrow{P_1P_2} = (x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j}.$$

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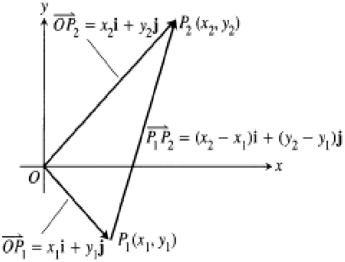

10.8 $\overrightarrow{P_1P_2} = (x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j}$

Magnitude or Length

Example: The vector from $P_1(3,4)$ to $P_2(5,1)$ is

$$\overrightarrow{P_1P_2} = (5-3)\mathbf{i} + (1-4)\mathbf{j} = 2\mathbf{i} - 3\mathbf{j}.$$

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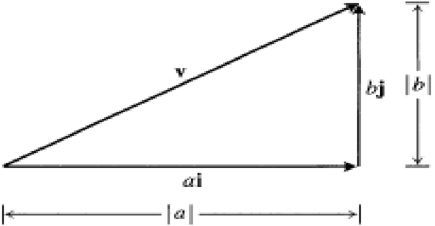
Magnitude: The magnitude or length of $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ is $|\mathbf{v}| = \sqrt{a^2+b^2}$. We arrive at this number by applying the Pythagorean theorem to the right triangle determined by $\mathbf{v}$ and its two vector components (Fig. 10.9). The bars in $|\mathbf{v}|$ (read"the magnitude")



10.9 The length of $\mathbf{v}$ is $\sqrt{|a|^2+|b|^2} = \sqrt{a^2+b^2}$.

Properties and Types of Vectors

If c is a scalar and $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ is a vector, then

$$c\mathbf{v} = c(a\mathbf{i} + b\mathbf{j}) = (ca)\mathbf{i} + (cb)\mathbf{j}.$$

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The Zero Vector: In terms of components, the zero vector is the vector

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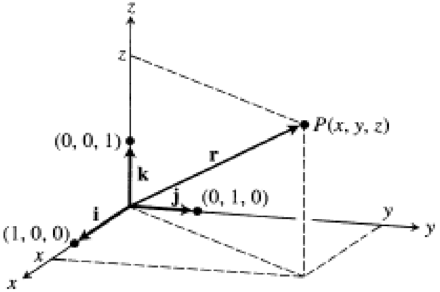
Unit Vectors: Any vector whose length is 1 is a unit vector. The vectors $\mathbf{i}$ and $\mathbf{j}$ are unit vectors.

$$|\mathbf{i}| = |1\mathbf{i} + 0\mathbf{j}| = \sqrt{1^2 + 0^2} = 1, \quad |\mathbf{j}| = |0\mathbf{i} + 1\mathbf{j}| = \sqrt{0^2 + 1^2} = 1$$

The vectors represented by the directed line segments from the origin to the points $(1, 0, 0), (0, 1, 0)$, and $(0, 0, 1)$ are the basic vectors (Fig. 10.18, on the following page). We denote them by $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$. The position vector $\mathbf{r}$ from the origin O to the typical point $P(x, y, z)$ is

$$\mathbf{r} = \overrightarrow{OP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$



Vectors in Space

Addition and Subtraction for Vectors in Space: For any vectors $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$,

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The Vector Between Two Points: We can express the vector $\overrightarrow{P_1P_2}$ from the point $P_1\,(x_1, y_1, z_1)$ to the point $P_2\,(x_2, y_2, z_2)$ in terms of the coordinates of P_1 and P_2.

$$\begin{aligned}\overrightarrow{P_1P_2} &= \overrightarrow{OP_2} - \overrightarrow{OP_1} \\ &= (x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}) - (x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}) \\ &= (x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j} + (z_2 - z_1)\,\mathbf{k}.\end{aligned}$$

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The **magnitude (length)** of $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is

$$|\mathbf{A}| = |a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

Scalar Multiplication: If c is a scalar and $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is a vector, then

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Unit Vectors: A unit vector in space is a vector of length 1 . The basic vectors are unit vectors because

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Magnitude and Direction: If $\mathbf{A} \neq \mathbf{0}$, then $\mathbf{A}/|\mathbf{A}|$ is a unit vector in the direction of $\mathbf{A}$

Distance in Space:

The **distance** between two points P_1 and P_2 in space is the length of $\overrightarrow{P_1P_2}$. The Distance Between $P_1\left(x_1, y_1, z_1\right)$ and $P_2\left(x_2, y_2, z_2\right)$

$$\left|\overrightarrow{P_1P_2}\right| = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 + \left(z_2 - z_1\right)^2}$$

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Example: Find a unit vector **u** in the direction of the vector from $P_1(1, 0, 1)$ to $P_2(3, 2, 0)$

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Solution: We divide $\overrightarrow{P_1P_2}$ by its length:

$$\overrightarrow{P_1P_2} = (3-1)\mathbf{i} + (2-0)\mathbf{j} + (0-1)\mathbf{k} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$\left|\overrightarrow{P_1P_2}\right| = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{4+4+1} = \sqrt{9} = 3$$

$$\mathbf{u} = \frac{\overrightarrow{P_1P_2}}{\left|\overrightarrow{P_1P_2}\right|} = \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}.$$

Examples

Example: Find a vector 6 units long in the direction of $\mathbf{A} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

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Solution:

The vector we want is

$$6\frac{\mathbf{A}}{|\mathbf{A}|} = 6\frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{\sqrt{2^2 + 2^2 + (-1)^2}} = 6\frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} = 4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}.$$

Example: The distance between $P_1(2, 1, 5)$ and $P_2(-2, 3, 0)$ is

$$\begin{aligned} \left|\overrightarrow{P_1P_2}\right| &= \sqrt{(-2-2)^2 + (3-1)^2 + (0-5)^2} \\ &= \sqrt{16 + 4 + 25} \\ &= \sqrt{45} = 3\sqrt{5}. \end{aligned}$$

Scalar/ Dot Product of Two Vectors

Definition: The scalar product (dot product) $\mathbf{A} \cdot \mathbf{B}$ ("A dot $\mathbf{B}$ ") of vectors $\mathbf{A}$ and $\mathbf{B}$ is the number

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos \theta,$$

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To calculate $\mathbf{A} \cdot \mathbf{B}$ from the components of $\mathbf{A}$ and $\mathbf{B}$ in a Cartesian coordinate system with unit vectors $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$, we let

$$\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$
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NOTE:
$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$
$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$

Angle Between two Vectors

The angle between two nonzero vectors **A** and **B** is

$$\theta = \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|}\right).$$

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Solution:

$$\begin{aligned}
\mathbf{A}\cdot\mathbf{B} &= (1)(6) + (-2)(3) + (-2)(2) = 6 - 6 - 4 = -4 \\
|\mathbf{A}| &= \sqrt{(1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3 \\
|\mathbf{B}| &= \sqrt{(6)^2 + (3)^2 + (2)^2} = \sqrt{49} = 7 \\
\theta &= \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|}\right) \\
&= \cos^{-1}\left(\frac{-4}{(3)(7)}\right) = \cos^{-1}\left(-\frac{4}{21}\right)
\end{aligned}$$

Properties of Scalar/Dot Product

Here we have, $\mathbf{A} \cdot \mathbf{B} = a_1 b_1 + a_2 b_2 + a_3 b_3$,

1. $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$. In other words, the dot product is commutative.
2. $(c\mathbf{A}) \cdot \mathbf{B} = \mathbf{A} \cdot (c\mathbf{B}) = c(\mathbf{A} \cdot \mathbf{B})$.
3. $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$. and $(\mathbf{A} + \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{C} + \mathbf{B} \cdot \mathbf{C}$.

Properties of Scalar/Dot Product

Here we have, $\mathbf{A} \cdot \mathbf{B} = a_1 b_1 + a_2 b_2 + a_3 b_3$,

1. $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$. In other words, the dot product is commutative.
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Perpendicular (Orthogonal) Vectors: Two nonzero vectors $\mathbf{A}$ and $\mathbf{B}$ are perpendicular or orthogonal if the angle between them is $\pi/2$. For such vectors, we automatically have

$$\mathbf{A} \cdot \mathbf{B} = 0$$

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because $\cos(\pi/2) = 0$. The converse is also true. If $\mathbf{A}$ and $\mathbf{B}$ are nonzero vectors $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\theta = 0$, then $\cos\theta = 0$ and $\theta = \cos^{-1} 0 = \pi/2$.

Nonzero vectors $\mathbf{A}$ and $\mathbf{B}$ are perpendicular (orthogonal) if and only if $\mathbf{A} \cdot \mathbf{B} = 0$

Example: $\mathbf{A} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{B} = 2\mathbf{j} + 4\mathbf{k}$ are orthogonal because

$$\mathbf{A} \cdot \mathbf{B} = (3)(0) + (-2)(2) + (1)(4) = 0$$

Exercise

1. Find the angles between the vectors in following exercise.

   1. $\mathbf{A} = 2\mathbf{i} + \mathbf{j}, \quad \mathbf{B} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$
   2. $\mathbf{A} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}, \quad \mathbf{B} = 3\mathbf{i} + 4\mathbf{k}$
   3. $\mathbf{A} = \sqrt{3}\mathbf{i} - 7\mathbf{j}, \quad \mathbf{B} = \sqrt{3}\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
   4. $\mathbf{A} = \mathbf{i} + \sqrt{2}\mathbf{j} - \sqrt{2}\mathbf{k}, \quad \mathbf{B} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$
   5. $\mathbf{A} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{B} = 2\mathbf{i} + 10\mathbf{j} - 11\mathbf{k}$

Cross/Vector Product of two Vectors in Space

Definition: Take two nonzero vectos $\mathbf{A}$ and $\mathbf{B}$ then vector product of $\mathbf{A}$ and $\mathbf{B}$ is derfined as

$$\mathbf{A} \times \mathbf{B} = (|\mathbf{A}||\mathbf{B}| \sin\theta)\mathbf{n}$$

Where $|A|$ and $|B|$ are magnitudes/length and $\mathbf{n}$ is a unit vector perpendicular to the plane and then to both $\mathbf{A}$ and $\mathbf{B}$.
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Unlike the dot product, the cross product is not commutative.

Vector Products

When we apply the definition to calculate the pairwise cross products of $\mathbf{i}, \mathbf{j}$, and k we find below and $|\mathbf{A} \times \mathbf{B}|$ is the Area of a Parallelogram
Because $\mathbf{n}$ is a unit vector, the magnitude of $\mathbf{A} \times \mathbf{B}$ is

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When we apply the definition to calculate the pairwise cross products of $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ we have

$\mathbf{i} \times \mathbf{j} = -(\mathbf{j} \times \mathbf{i}) = \mathbf{k}$

$\mathbf{j} \times \mathbf{k} = -(\mathbf{k} \times \mathbf{j}) = \mathbf{i}$

$\mathbf{k} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{k}) = \mathbf{j}$

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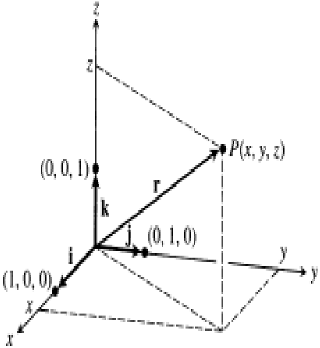

If $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then $\mathbf{A} \times \mathbf{B}$ from components of $\mathbf{A}$ and $\mathbf{B}$ is defined as the dereminant of components.

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example: Find $\mathbf{A} \times \mathbf{B}$ and $\mathbf{B} \times \mathbf{A}$ if

$$\mathbf{A} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}, \quad \mathbf{B} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}.$$

Solution

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} \mathbf{k} \\ &= -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k} \\ \mathbf{B} \times \mathbf{A} &= -(\mathbf{A} \times \mathbf{B}) = 2\mathbf{i} + 6\mathbf{j} - 10\mathbf{k} \end{aligned}$$

Example

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Firts find the position vectors

$\overrightarrow{OP} = 1\mathbf{i} - 1\mathbf{j}$

$\overrightarrow{OQ} = 2\mathbf{i} + 1\mathbf{j} - 1\mathbf{k}$

$\overrightarrow{OR} = -1\mathbf{i} + 1\mathbf{j} + 2\mathbf{k}$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (2-1)\mathbf{i} + (1+1)\mathbf{j} + (-1-0)\mathbf{k} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = (-1-1)\mathbf{i} + (1+1)\mathbf{j} + (2-0)\mathbf{k} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix}\mathbf{k}$$

$$= 6\mathbf{i} + 6\mathbf{k}.$$

Example: Find a unit vector perpendicular to the plane of $P(1,-1,0)$, $Q(2,1,-1)$, and $R(-1,1,2)$

Solution: Since $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane, its direction $\mathbf{n}$ is a unit vector perpendicular to the plane, from above example $\overrightarrow{PQ} \times \overrightarrow{PR} = 6\mathbf{i} + 6\mathbf{k}$ and $|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{6^2 + 6^2} = 6\sqrt{2}$ then,

$$\mathbf{n} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{|\overrightarrow{PQ} \times \overrightarrow{PR}|} = \frac{6\mathbf{i} + 6\mathbf{k}}{6\sqrt{2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{k}$$

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Example: Find the area of the triangle with vertices $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$.

Solution: The area of the parallelogram determined by P, Q, and R is

$$\begin{aligned} |\overrightarrow{PQ} \times \overrightarrow{PR}| &= |61 + 6\mathbf{k}| \quad \text{Values from Example 3} \\ &= \sqrt{(6)^2 + (6)^2} = \sqrt{2 \cdot 36} = 6\sqrt{2}. \end{aligned}$$

The triangle's area is half of this, i.e. $3\sqrt{2}$.

The Triple Scalar or Box Product

The product $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$ is called the triple scalar product of $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ (in that order). As you can see from the formula

$$|(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}| = |\mathbf{A} \times \mathbf{B}||\mathbf{C}|| \cos \theta|$$

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The absolute value of the product is the volume of the parallelepiped (parallelogramsided box) determined by $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ The number $|\mathbf{A} \times \mathbf{B}|$ is the area of the base parallelogram.

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By treating the planes of $\mathbf{B}$ and $\mathbf{C}$ and of $\mathbf{C}$ and $\mathbf{A}$ as the base planes of the parallelepiped determined by $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$, we see that

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = (\mathbf{B} \times \mathbf{C}) \cdot \mathbf{A} = (\mathbf{C} \times \mathbf{A}) \cdot \mathbf{B} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

.

Volume: The number $|(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}|$ is the volume of a parallelepiped. The dot product is commutative, then above equality gives

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$$

Example

Example: Find the volume of the box (parallelepiped) determined by $\mathbf{A} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}, \mathbf{B} = -2\mathbf{i} + 3\mathbf{k}$, and $\mathbf{C} = 7\mathbf{j} - 4\mathbf{k}$.
Solution:

$$\begin{aligned}\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix} = \begin{vmatrix} 0 & 3 \\ 7 & -4 \end{vmatrix} - 2 \begin{vmatrix} -2 & 3 \\ 0 & -4 \end{vmatrix} - \begin{vmatrix} -2 & 0 \\ 0 & 7 \end{vmatrix} \\ &= -21 - 16 + 14 = -23\end{aligned}$$

The volume is $|\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})| = 23$.

Triple Vector Products

A triple vector products of three vectors **A** ,**B** and **C** is
$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = (\mathbf{A}\cdot\mathbf{C})\cdot\mathbf{B} - (\mathbf{B}\cdot\mathbf{C})\cdot\mathbf{A}$
$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\cdot\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\cdot\mathbf{C}$
A triple vector products $(\mathbf{A}\times\mathbf{B})\times\mathbf{C})$ and $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ are usually not equal.
Note: In print, boldcase letters are used to denote vectors but in writing we use vector notation $\rightarrow$ above the letter.
Example: Take $\vec{A} = \vec{i} - \vec{j} + \vec{k}, \;\; \vec{B} = 2\vec{i} + \vec{j} - 2\vec{k}, \; \vec{C} = -\vec{i} + 2\vec{j} - \vec{k}$
Verify the above definition. (done in class)

Exercise

1. Form the given given coordinates below:
a) Find the area of the triangle determined by the points P, Q, and R. (area of the triangle is half of the area of the parallelopiped determined by P,Q,R
b) Find a unit vector perpendicular to plane PQR.
(i) $P(1,-1,2)$, $Q(2,0,-1)$, $R(0,2,1)$
(ii)$P(1,1,1)$, $Q(2,1,3)$, $R(3,-1,1)$
(iii) $P(2,-2,1)$, $Q(3,-1,2)$, $R(3,-1,1)$
(iv) $P(-2,2,0)$, $Q(0,1,-1)$, $R(-1,2,-2)$
2. For and vectors $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ by using scalar triple products verify that: $[\vec{A}\ \vec{B}\ \vec{C}] = [\vec{B}\ \vec{C}\ \vec{A}] = [\vec{C}\ \vec{A}\ \vec{B}]$
Hint: $[\vec{A}\ \vec{B}\ \vec{C}] = (\vec{A} \times \vec{B}) \cdot \vec{C})$ first find $\vec{A} \times \vec{B}$ then dot product with $\vec{C}$ or keep the components in dererminant form in given order in the box.
3. For any vector $\vec{A}, \vec{B}$ and $\vec{C}$ show that
$\vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) + \vec{C} \times (\vec{A} \times \vec{B}) = 0$

Condition for Collinear: Two vectors $\vec{a}$ and $\vec{b}$ are considered to be collinear vectors if there exists a scalar ' n ' such that $\vec{a} = n \cdot \vec{b}$

OR

Two vectors $\vec{a}$ and $\vec{b}$ are considered to be collinear vectors if their cross product is equal to the zero vector. This condition can be applied only to three-dimensional or for vector in Space.

If $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then $\mathbf{A}$ and $s\mathbf{B}$ are collinear if

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0$$

Example: Show that the vectors $2\vec{i} - 3\vec{j} + 4\vec{k}$ and $-4\vec{i} + 6\vec{j} - 8\vec{k}$ are collinear.

Coplanar Vectors

Definition: Three or more vectors lying on the same plane or parallel to the plane are said to be coplanar.

OR

Three vectors are said to be coplanar if one of them is expressible as a linear combination of others two.

i.e. vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar if $\vec{c} = \vec{a}x + \vec{b}y$

Thank You