Mathematics II (BSM 102)

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Outlines

- Continuity of Complex Variable Function(CVF)
- Differentiability of CVF
- Analyticity
- Cauchy-Riemann Equations

A function $f(z)$ is said to be **continuous** at $z = z_0$ if $f\left(z_0\right)$ is defined and

$$\lim_{z \rightarrow z_0} f(z) = f\left(z_0\right).$$

Note that by definition of a limit this implies that $f(z)$ is defined in some neighborhood of z_0.
$f(z)$ is said to be continuous in a domain if it is continuous at each point of this domain.
The **derivative** of a complex function f at a point z_0 is written $f'\left(z_0\right)$ and is defined by

$$f'\left(z_0\right) = \lim_{\Delta z \rightarrow 0} \frac{f\left(z_0 + \Delta z\right) - f\left(z_0\right)}{\Delta z}$$

provided this limit exists.Then f is said to be **differentiable** at z_0.
If we write $\Delta z = z - z_0$, we have $z = z_0 + \Delta z$ and takes the form

$$f'\left(z_0\right) = \lim_{z \rightarrow z_0} \frac{f(z) - f\left(z_0\right)}{z - z_0}$$

Analyticity

A function $f(z)$ is said to be **analytic** in a domain D if $f(z)$ is defined and differentiable at all points of D. The function $f(z)$ is said to be analytic at a point $z = z_0$ in D if $f(z)$ is analytic in a neighborhood of z_0.
Also, by an analytic function we mean a function that is analytic in some domain.
A more modern term for analytic in D is holomorphic in D.
The **Cauchy-Riemann equations** are the most important equations that provide a criterion (a test) for the analyticity of a complex function

$$f(z) = u(x, y) + iv(x, y).$$

with u and v real valued function which have continuous partial 1^{st} order derivatives then

$u_x = v_y, \quad u_y = -v_x \quad$ or $\quad \dfrac{\partial u}{\partial, x} = \dfrac{\partial v}{\partial y}, \quad \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

This is known as Cauchy-Riemann Equations.

Theorem on Cauchy-Riemann Equations

Theorem: If two real-valued continuous functions $u(x, y)$ and $v(x, y)$ of two real variables x and y have continuous first order partial derivatives that satisfy the Cauchy-Riemann equations in some domain D, then the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in D.
Example: Is $f(z) = u(x, y) + iv(x, y) = e^x(\cos y + i \sin y)$ analytic?
Solution: We have $u = e^x \cos y, v = e^x \sin y$ and by differentiation

$$\begin{aligned} u_x &= e^x \cos y, & v_y &= e^x \cos y \\ u_y &= -e^x \sin y, & v_x &= e^x \sin y. \end{aligned}$$

We see that the Cauchy-Riemann equations are satisfied and conclude that $f(z)$ is analylic for all z.
Exercise: Test for analyticity of given fucntion:
(i) $f(z) = z^2$ (ii) $f(z) = z^3$ (iii) $f(z) = \bar{z}$

Harmonic Function:

Laplace's Equation: If $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then both u and v satisfy Laplace's equation

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

$\left(\nabla^2\right.$ read "nabla squared") and

$$\nabla^2 v = v_{xx} + v_{yy} = 0,$$

in D and have contintious second order partial derivatives in D.
If any analytic function satisfy the above both condition (Laplace's equation) then the real and imaginary parts of the analytic function are called harmonic function.

If two harmonic functions u and v satisfy the Cauchy-Riemann equations in a domain D, then u and v are harmonic conjugate of each other in D.
Example: Is $f(z) = e^x(\cos y + i \sin y)$ is analytic? Is harmonic?

How to Find a Harmonic Conjugate Function by the Cauchy-Riemann Equations

Verify that $u = x^2 - y^2 - y$ is harmonic in the whole complex plane and find a harmonic conjugate function v of u. Also find the corresponding analytic function.

Solution: $\nabla^2 u = 0$ by direct calculation. Now $u_x = 2x$ and $u_y = -2y - 1$. Hence because of the Cauchy-Riemann equations a conjugate v of u musl satisfy

$$v_y = u_x = 2x. \quad v_x = -u_y = 2y + 1.$$

Integrating the first equation with respect to y and differentiating the result with respect to x. we obtain

$$v = 2xy + h(x), \quad v_x = 2y + \frac{dh}{dx}.$$

A comparison with the second equation shows that $\frac{dh}{dx} = 1$. This gives $h(x) = x + c$.

Hence $v = 2xy + x + c$ (c any real constant) is the most general harmonic conjugate of the given u.
The corresponding analytic function is

$$f(z) = u + iv = x^2 - y^2 - y + i(2xy + x + c) = z^2 + iz + ic.$$

Note: Here the solution steps are skiped, look at your class note for details.

Exercise

1. Are the following functions analytic?
(i) $f(z) = e^{-x}(\cos y - i \sin y)$
(ii) $f(z) = \operatorname{Re} z + \operatorname{Im} z$
(iii) $f(z) = e^{2x}(\cos y + i \sin y)$
2. Are the following functions harmonic? If your answer is yes, find a harmonic conjugate and corresponding analytic function
$f(z) = u(x, y) + iv(x, y)$.
(i) $u = x^3 - 3xy^2$
(ii) $v = xy$
(iii) $u = xy$
(iv) $u = x^2 - y^2$

Euler's Formula:

- The most common exponential function in complex variable which is analytic is e^z or $\exp(z) = e^x(\cos y + i \sin y)$
- A function $f(z)$ that is analytic for all z is called an entire function, Thus $f(z) = e^z$ is entire.
- **Properties:** $e^{z_1+z_2} = e^{z_1}e^{z_2}$, for any $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$
- If $z = iy$ then $e^z = e^{iy} = \cos y + i \sin y$
  Hence, the polar form of a complex number
  $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$
  Then, $e^{i\theta} = \cos\theta + i\sin\theta$ is known as Euler's formula.
  $|e^{i\theta}| = |\cos\theta + i\sin\theta| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$ Hence, for pure imaginary exponents the exponential function has absolute value 1.
  Example: Verify it $|e^z| = e^x$

Prove that: $\cos z = \cos x \cosh y - i \sin x \sinh y$

Proof:

$$
\begin{aligned}
\cos z &= \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \\
&= \frac{1}{2}\left(e^{i(x+iy)} + e^{-i(x+iy)}\right) \\
&= \frac{1}{2}e^{-y}(\cos x + i \sin x) + \frac{1}{2}e^{y}(\cos x - i \sin x) \\
&= \frac{1}{2}\left(e^{y} + e^{-y}\right)\cos x - \frac{1}{2}i\left(e^{y} - e^{-y}\right)\sin x.
\end{aligned}
$$

Since, $\cosh y = \frac{1}{2}\left(e^{y} + e^{-y}\right)$,
hence, $\cos z = \cos x \cosh y - i \sin x \sinh y$

Thank You