Binomial Distribution

In probability theory and statistics, the **binomial distribution** is the discrete probability distribution that gives only two possible results in an experiment, either **Success or Failure**. For example, if we toss a coin, there could be only two possible outcomes: heads or tails, and if any test is taken, then there could be only two results: pass or fail. This distribution is also called a binomial probability distribution.

There are two parameters n and p used here in a binomial distribution. The variable 'n' states the number of times the experiment runs, and the variable 'p' tells the probability of any one outcome.

Definition: A discrete random variable X is said to follow binomial distribution if it takes only the positive integer's values i.e., 0, 1, 2, …, n and probability mass function is as follows:

$$P(X=x) = p(x) = {}^{n}C_{x}\ p^{x}q^{n-x}$$

$$= \frac{n!}{x!(n-x)!}\ p^{x}\ q^{n-x}, \text{ for } x = 0, 1, 2, \ldots, n$$

$$= 0, \text{ otherwise}$$

Where, p is probability of success and q=1-p, x the number of success(say) in a series of 'n' independent trials ($x \geq 0$)

Conditions for Binomial Distribution

We get the Binomial Distribution under the following experimental conditions:

- The number of trials `*n*' is finite
- The trials are independent of each other
- The probability of success `*p*' is same for each trial
- Each trial must result in a success or a failure.

Characteristics of Binomial Distribution

- Binomial distribution is a discrete distribution i.e., *X* can take values 0, 1, 2, ... *n* where `*n*' is finite.

- Constants of the distributions are:

Mean = np; Variance = npq; Standard deviation = √npq

$$\text{Skewness} = \frac{q-p}{\sqrt{npq}} \; ; \quad \text{Kurtosis} = \frac{1-6pq}{npq}$$

- It may have one or two modes.
- If $X \sim B(n_1, p)$ and $Y \sim B(n_2, p)$ and that X and Y are independent then $X+Y \sim B(n_1+n_2, p)$
- If `*n*' independent trials are repeated *N* times the expected frequency of `*x*' successes are $N \times {}^nC_x \, p^x \, q^{n-x}$
- If $p = 0.5$, the distribution is symmetric.

Some examples of Binomial Distribution

1)Comment on the following 'The mean of binomial distribution is 5 and its variance is 9'.

Solution:

Given mean np = 5 and variance npq = 9

$$\therefore \frac{\text{Value of variance}}{\text{Value of mean}} = \frac{npq}{np} = \frac{9}{5} \therefore q = \frac{9}{5} > 1 \quad \text{is not possible}$$

as $0 \leq q \leq 1$ and hence the given statement is wrong.

2)Eight coins are tossed simultaneously. Find the probability of getting at least six heads.

Solution:

Here n=8 $\quad p = P\text{ (head)} = \frac{1}{2} \qquad q = 1 - \frac{1}{2} = \frac{1}{2}$

Trials satisfy conditions of Binomial distribution

$$\text{Hence} \quad P(X = x) = nC_x \, p^x \, q^{n-x} \qquad x = 0, 1, 2, \ldots n$$

$$= 8C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{8-x} \qquad x = 0,1,2,\ldots.8$$

$$= 8c_x \left(\frac{1}{2}\right)^{x+8-x}$$

$$= 8C_x \left(\frac{1}{2}\right)^8$$

$$\therefore P(X = x) = \frac{8C_x}{256}$$

P (getting atleast six heads)

$$= P\,(x \geq 6)$$

$$= P\,(x = 6) + P\,(x = 7) + P\,(x = 8)$$

$$= \frac{8C_6}{256} + \frac{8C_7}{256} + \frac{8C_8}{256}$$

$$= \frac{28}{256} + \frac{8}{256} + \frac{1}{256}$$

$$= \frac{37}{256}$$

3)Ten coins are tossed simultaneously. Find the probability of getting (i) at least seven heads (ii) exactly seven heads (iii) at most seven heads.

Solution:

X denote the number of heads appear

$$P(X = x) = nC_x\, p^x\, q^{n-x}, \; x = 0,1,2,\ldots n$$

Given: $p = P(\text{head}) = \frac{1}{2}$ $\quad q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$ and $n = 10$

$$\therefore X \sim B\left(10, \frac{1}{2}\right)$$

$$= 10C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{10-x}$$

$$P(X = x) = \frac{10C_x}{1024}$$

(i) P (atleast seven heads)

$$P(X \geq 7) = P(x = 7) + P(x = 8) + P(x = 9) + P(x = 10)$$

$$= \frac{10C_7}{1024} + \frac{10C_8}{1024} + \frac{10C_9}{1024} \frac{10C_{10}}{1024}$$

$$= \frac{120}{1024} + \frac{45}{1024} + \frac{10}{1024} + \frac{1}{1024}$$

$$= \frac{176}{1024}$$

(ii) P (exactly 7 heads)

$$P(x{=}7) = \frac{10C_7}{1024} = \frac{120}{1024}$$

(iii) P (atmost 7 heads)

$$= P(x \leq 7) = 1 - P(x > 7)$$

$$= 1 - \{P(x = 8) + P(x = 9) + P(x = 10)\}$$

$$= 1 - 1 - \left\{\frac{10C_8}{1024} + \frac{10C_9}{1024} + \frac{10C_{10}}{1024}\right\}$$

$$= 1 - \frac{56}{1024}$$

$$= \frac{968}{1024}$$

4)With usual notation find p for Binomial random variable X if $n = 6$ and $9\ P\ (x = 4) = P\ (x = 2)$

Solution:

$$P\ (X = x) \ = \ nC_x\, p^x\, q^{n\text{-}x}\ ,\ x = 0,1,2,\ldots n$$

$$X \sim B\ (6, p) \ \Rightarrow\ P(X = x) = 6C_x p^x\ q(6\text{-}x)$$

Also $\quad 9 \times P\ (X = 4) \ = P\ (X = 2)$

$\Rightarrow \quad 9 \times 6C_4 p^4 q^2 \ = 6C_2\ p^2\ q^4$

$\Rightarrow \quad 9\ p^2 \ = q^2$

$\Rightarrow \quad 3p \ = q \qquad$ as $p, q > 0$

$3p \ = 1\text{-}\ p$

$4p \ = 1$

$\Rightarrow \quad p \ = \dfrac{1}{4} \ = 0.25$

5)A Binomial distribution has parameters n=5 and p=1/4. Find the Skewness and Kurtosis.

Solution:

Here we are given $n=5$ and $p=\frac{1}{4}$

$$\text{Skewness} = \frac{q-p}{\sqrt{npq}}$$

$$= \frac{\frac{3}{4}-\frac{1}{4}}{\sqrt{5\times\frac{1}{4}\times\frac{3}{4}}}$$

$$= \frac{\frac{2}{4}}{\sqrt{\frac{15}{16}}}$$

$$= \frac{2}{\sqrt{15}}$$

Finding: The distribution is positively skewed.

Kurtosis

$$\text{Kurtosis} = \frac{1-6pq}{npq}$$

$$= \frac{1-6\times\frac{1}{4}\times\frac{3}{4}}{\frac{15}{16}}$$

$$= \frac{\frac{-2}{16}}{\frac{15}{16}}$$

$$= \frac{-2}{15}$$

$$= -0.1333$$

Finding: The distribution is Platykurtic.

6)In a Binomial distribution with 7 trials, $P(X=3)$ $=P(X=4)$ Check whether it is a symmetrical distribution?

Solution:

A Binomial distribution is said to be symmetrical if $p = q = \frac{1}{2}$

Given: $P(X=3) = P(X=4)$

$X \sim B(n, p)$

$P(X = x) = nC_x\, p^x\, q^{n-x}$, $x = 0,1,2,\ldots n$

$nC_3\, p^3 q^{n-3} = nC_4\, p^4\, q^{n-4}$

$7C_3\, p^3 q^4 = 7C_4\, p^4 q^3$ note that $7C_3 = 7C_4$

On simplifying, we have

$$q = p$$

$$1-p = p$$

$$1 = 2p$$

$$p = \frac{1}{2}$$

$$q = \frac{1}{2}$$

Hence the given Binomial distribution is symmetrical.

7)From a pack of 52 cards 4 cards are drawn one after another with replacement. Find the mean and variance of the distribution of the number of kings.

Solution:

Success *X*=event of getting king in a draw

p=probability of getting king in a single trial

$$p = \frac{4}{52} = \frac{1}{13}$$

This is constant for each trial.

Hence, it is a binomial distribution with $n=4$ and $p = \frac{1}{13}$

$$\text{Mean} = np = 4\times\frac{1}{13} = \frac{4}{13}$$

$$\text{Variance} = npq = \frac{4}{13}\times\frac{12}{13} = \frac{48}{169}$$

8)In a street of 200 families, 40 families purchase the Hindu newspaper. Among the families a sample of 10 families is selected, find the probability that

i. Only one family purchase the news paper

ii. No family purchasing

iii. Not more than one family purchase it

Solution:

X ~ B (n, p)

$P(X = x) = nC_x p^x q^{n-x}$, x= 0,1,2,...n

Let *X* denote the number of families purchasing Hindu Paper

p = Probability of their family purchasing the Hindu

$$p = \frac{40}{200} = \frac{1}{5}$$

$$q = \frac{4}{5}$$

$$n = 10$$

(i) Only one family purchase the Hindu

$$P(X=1) = nC_1\, p^1\, q^{n-1}$$

$$=10C_1 \times \frac{1}{5} \times \left(\frac{4}{5}\right)^9$$

$$= 2 \times \left(\frac{4}{5}\right)^9$$

(ii) No family purchasing the Hindu

$$P(X=0) = 10C_0 (\frac{1}{5})^0 (\frac{4}{5})^{10}$$

$$= (\frac{4}{10})^{10}$$

(iii) Not more than one family purchasing The Hindu means that $X \leq 1$

$$P(X \leq 1) = P[x=0] + P[x=1]$$

$$= 10C_0 (\frac{1}{5})^0 (\frac{4}{5})^{10} + 10C_1 (\frac{1}{5})^1 (\frac{4}{9})^9$$

$$= (\frac{4}{5})^{10} + 10 \times (\frac{1}{5})^1 (\frac{4}{5})^9$$

$$= (\frac{4}{5})^9 \left[(\frac{4}{5}) + 2\right]$$

$$= (\frac{4}{5})^9 (\frac{14}{5})$$

9)In a tourist spot, 80% of tourists are repeated visitors. Find the distribution of the numbers of repeated visitors among 4 selected peoples visiting the place. Also find its mode or the maximum visits by a visitor.

Solution:

Let the random variable *X* denote the number of repeated visitors.

X ~ B (n, p)

$P(X = x) = nC_x p^x q^{n-x}$, x= 0,1,2,...n

It is a Binomial Distribution with n=4

$$p = \frac{80}{100} = \frac{4}{5} \qquad q = 1\text{-}p = 1 - \frac{4}{5} = \frac{1}{5}$$

$$P(x = 0) = 4C_0(\tfrac{4}{5})\ (\tfrac{1}{5})^4 = (\tfrac{1}{625})$$

$$P(x=1) = 4C_1\ (\tfrac{4}{5})(\tfrac{1}{5})^3 = \frac{16}{625}$$

$$P(x=2) = 4C_2\ (\tfrac{4}{5})^2(\tfrac{1}{5})^1 = (\tfrac{96}{625})$$

$$P(x = 3) = 4C_3(\tfrac{4}{5})^3(\tfrac{1}{5})^1 = (\tfrac{256}{625})$$

$$P(x = 4) = 4C_4\left(\frac{4}{5}\right)^4\left(\frac{1}{5}\right)^0 = \left(\frac{256}{625}\right)$$

The probability distribution is given below.

X=x	0	1	2	3	4
P(X=x)	$\frac{1}{625}$	$\frac{16}{625}$	$\frac{96}{625}$	$\frac{256}{625}$	$\frac{256}{625}$

10)In a college, 60% of the students are boys. A sample of 4 students of the college, is taken, find the minimum number of boys should it have so that probability up to that number is ≥ 1/2.

Solution:

It is given that 60% of the students at the college are boys and the selection probability for a boy is 60% or 0.6 As we are taking four samples, the number of trials n = 4. The selection process is independent.

$$X \sim B\ (n,\ p)$$

$$P\ (X = x) = \quad nC_x\, p^x\, q^{n-x}\ ,\ x = 0,1,2,\ldots n$$

Let X be the number of boys so that $P(X \le x) \ge \frac{1}{2}$

If $x = 0$ $\quad P(X \le 0) = 4C_0 (\frac{3}{5})^0 \ (\frac{2}{5})^4 = \frac{16}{625} < \frac{1}{2}$

$x=1$ $\quad P(X \le 1) = P\ (x = 0) + P\ (x=1)$

$$= 4C_0 \left(\frac{3}{5}\right)^0 \left(\frac{2}{5}\right)^4 + 4C_1 \left(\frac{3}{5}\right)^1 \left(\frac{2}{5}\right)^3$$

$$= \frac{16}{625} + \frac{96}{625} = \frac{112}{625} < \frac{1}{2}$$

$x=2$ $\quad P(X \le 2) = P(x=0) + P(x=1) + P(x=2)$

$$= \frac{112}{625} + P(x=2) = \frac{112}{625} + 4C_2 \left(\frac{3}{5}\right)^2 \left(\frac{2}{5}\right)^2$$

$$= \frac{112}{625} + \frac{216}{625} = \frac{328}{625} > \frac{1}{2}$$

Therefore, the sample should contain a minimum of 2 boys.