Mathematics I (BSM 101)

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Outlines

- Introduction to Matrix
- Types of Matices
- Matrix Operations
- Transpose and Its Properties
- Determinant and Its Properties
- Inverse of a Matrix
- System of Equations: Solution by Cramer's Rule

Introduction to Matrix

Matrix: A matix is a rectangular array of numbers (real and Complex) that is represented as $A = (a_{ij})_{m \times n}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

here $a_{11}, a_{12} \cdots a_{33}$ are entries or elements of the matrix.
The number of row followed by the number of comumn is called size or order of the matrix.
The order of above matirx A is 3×3

Types of Matrices

- Row Matrix
- Column Matirx
- Square Matrix
- Rectangular Matrix
- Diagonal Matrix
- Scalar Matrix
- Identity Matrix
- Null Matrix
- Triangular Matrix
- Equal Matrix
- Symmetric Matrix
- Skew Symmetrix Matrix

Matirx Operations

Addition of Matrices: If matrix A and matrix B are of the same order and have elements a_{ij} and b_{ij}, respectively, then their sum $A + B$ is a matrix C whose elements are $c_{ij} = a_{ij} + b_{ij}$ for all i and j. That is,

$$A + B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & & & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix} = C$$

Matrix Multiplication

Multiplication by a Scalar: Multiplying a matrix by a real number (called a scalar) results in a matrix in which each entry of the original matrix is multiplied by the real number. Thus, if

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \text{ then } cA = \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix}$$

Product of Two Matrices: Given an $m \times n$ matrix A and an $n \times p$ matrix B, the matrix product AB is an $m \times p$ matrix C, with the ij entry of C given by the formula

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

Transpose of a Matrix

Transpose of A: Let A be a matrix, then the matrix obtained by interchanging rows and columns of A is called the transpose of A. It is denoted by A' or A^T.

If A is an $m \times n$ matrix, then its transpose will be an $n \times m$ matrix.

Example: If

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 3 & 4 & 2 \\ 7 & 8 & 3 \end{bmatrix}$$

then,

$$A^T = \begin{bmatrix} 1 & 3 & 7 \\ 0 & 4 & 8 \\ 3 & 2 & 3 \end{bmatrix}$$

Types of Matrix Define with Transpose

Orthogonal Matrix: A square matrix A is called an orthogonal matrix if $AA^T = I$, where I is an identity matrix.
Symmetric Matrix: A square matrix A is called a symmetric matrix if $A = A^T$.
Skew Symmetric: A square matrix A is called a skew-symmetric matrix if $A^T = A$.
Properties of Transpose:

- The transpose of the sum of two matrices is the sum of their transposes;
  i.e., $(A + B)^T = A^T + B^T$
- The transpose of the transpose of a matrix is the matrix itself;
  i.e., $\left(A^T\right)^T = A$.
- The transpose of the product of matrices is the product of theit transposes taken in reverse order;
  i.e., $(AB)^T = B^T A^T$

Freely choose matrices of different size to perform the operations.
For Example:

$$A = \begin{bmatrix} -1 & 3 & 7 \\ 0 & -4 & 8 \\ 3 & 2 & -3 \end{bmatrix} B = \begin{bmatrix} 1 & 0 & -3 \\ 0 & -1 & -4 \\ 3 & 2 & -2 \end{bmatrix}$$

Find $(i)A + B \quad (ii)A - B, \quad (iii)AB, \quad (iv)BA \quad (v)(AB)^T$

The inverse of matrix is another matrix, which on multiplication with the given matrix gives the multiplicative identity.
For a matrix A, its inverse is A^{-1}, and $A \cdot A^{-1} = A^{-1} \cdot A = 1$, where l is the identity matrix.
The matrix whose determinant is non-zero and for which the inverse matrix can be calculated is called an invertible matrix.
For a 2×2 matrix: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The inverse matrix of A is given by the formula, $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
For example: The inverse of $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$ is $\begin{bmatrix} 1 & 1/2 \\ 0 & 1/2 \end{bmatrix}$
Since, $A \cdot A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \cdot$
$A^{-1} \cdot A = \begin{bmatrix} 1 & 1/2 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$

Inverse of a Matrix

Procedure and Formula The inverse of matrix A can be computed using the inverse of matrix formula,

$$A^{-1} = \frac{1}{|A|} \cdot \operatorname{Adj} A$$

Since $|A|$ is the determinant of a matrix A that is in the denominator of the above formula, the inverse of a matrix exists only if the determinant of the matrix is a non-zero value. i.e., $|\mathrm{A}| \neq 0$.
The inverse of a matrix can be calculated by following the given steps:

- Calculate the minors of all elements of A.
- Then compute the cofactors of all elements and write the cofactor matrix by replacing the elements of A by their corresponding cofactors.
- Find the adjoint of A (written as adj A) by taking the tranpose of cofactor matrix of A.
- Multiply adj A by reciprocal of determinant.

Example:

Find the inverse of a matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 2 & 9 \end{pmatrix}$

Solution: Determinant of the given matrix is

$$\begin{aligned} |A| = \det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 2 & 9 \end{pmatrix} &= 1 \cdot 33 - 2(-6) + 3(-27) \\ &= -36 \end{aligned}$$

Now, the minors of the given matrix as given below:

$$M_{1,1} = \det \begin{pmatrix} 5 & 6 \\ 2 & 9 \end{pmatrix} = 33$$

$$M_{1,2} = \det \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} = -6$$

$$M_{1,3} = \det \begin{pmatrix} 4 & 5 \\ 7 & 2 \end{pmatrix} = -27$$

Continue...

$$M_{2,1} = \det \begin{pmatrix} 2 & 3 \\ 2 & 9 \end{pmatrix} = 12$$

$$M_{2,2} = \det \begin{pmatrix} 1 & 3 \\ 7 & 9 \end{pmatrix} = -12$$

$$M_{2,3} = \det \begin{pmatrix} 1 & 2 \\ 7 & 2 \end{pmatrix} = -12$$

$$M_{3,1} = \det \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} = -3$$

$$M_{3,2} = \det \begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix} = -6$$

$$M_{3,3} = \det \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} : -3$$

Continue. . .

Now, to form a cofactor matrix of A, change the values according to sign matrix then place each entries in correct positions

$$\text{Cofactors Matrix } = \begin{pmatrix} 33 & 6 & -27 \\ -12 & -12 & 12 \\ -3 & 6 & -3 \end{pmatrix}$$

Now, find the adjoint of a matrix by taking the transpose of cofactors of the given matrix.

$$\begin{pmatrix} 33 & 6 & -27 \\ -12 & -12 & 12 \\ -3 & 6 & -3 \end{pmatrix}^T = \begin{pmatrix} 33 & -12 & -3 \\ 6 & -12 & 6 \\ -27 & 12 & -3 \end{pmatrix}$$

Now, $A^{-1} = \frac{1}{|A|} \cdot \text{Adj}\, A$

Hence, the inverse of the given matrix is:= $\begin{pmatrix} -\frac{11}{12} & \frac{1}{3} & \frac{1}{12} \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{3}{4} & -\frac{1}{3} & \frac{1}{12} \end{pmatrix}$

System of Linear Equations

A system of linear equations is a collection of several linear equations, For example:

$$\begin{cases} x + 2y + 3z = & 6 \\ 2x - 3y + 2z = & 14 \\ 3x + y - z = & -2 \end{cases}$$

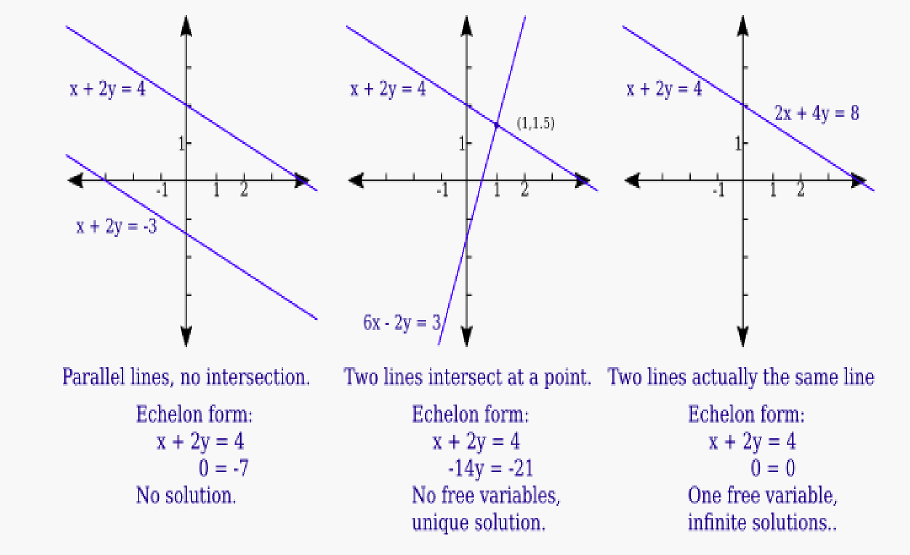
Solution sets:

- A solution of a system of equations is a list of values of $x, y, z, \ldots$ that make all of the equations true simultaneously.
- The solution set of a system of equations is the collection of all solutions.
- Solving the system means finding all solutions with formulas involving some number of parameters.

Note:

A system of equations is called **inconsistent** if it has no solutions. It is called **consistent** otherwise.

Types of Solutions to the System of Linear Equations:



Example with Unique Solution:

Solve the following system of linear equations using matrix method:

$3x + y + z = 1$

$2x + 2z = 0$

$5x + y + 2z = 2$

Solution: Writing the given equation in the matrix form AX = B as

$$\begin{bmatrix} 3 & 1 & 1 \\ 2 & 0 & 2 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

where, $A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 0 & 2 \\ 5 & 1 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$$\therefore |\mathrm{A}| = \begin{vmatrix} 3 & 1 & 1 \\ 2 & 0 & 2 \\ 5 & 1 & 2 \end{vmatrix} = 3(0-2) - 1(4-10) + 1(2-0) = 2 \neq 0$$

$\therefore$ A is non-singular $\therefore$ The system has the unique solution $\mathrm{X} = \mathrm{A}^{-1}\ \mathrm{B}$

$$\text{here, Adj (A)} = \begin{bmatrix} -2 & -1 & 2 \\ 6 & 1 & -4 \\ 2 & 2 & -2 \end{bmatrix}$$

$$\text{A}^{-1} = \frac{\text{Adj} \quad (\text{A})}{|\text{A}|} = \frac{1}{2} \begin{bmatrix} -2 & -1 & 2 \\ 6 & 1 & -4 \\ 2 & 2 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -1/2 & 1 \\ 3 & 1/2 & -2 \\ 1 & 1 & -1 \end{bmatrix}$$

Now,

$$\text{AX} = \text{B} \Rightarrow \text{X} = \text{A}^{-1} \text{ B}$$

$$\begin{bmatrix} \text{x} \\ \text{y} \\ \text{z} \end{bmatrix} = \begin{bmatrix} -1 & -1/2 & 1 \\ 3 & 1/2 & -2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$$x = 1, y = -1, z = -1$$

System of Equations

System with No Solution: $\begin{cases} x + 2y - z = 3 \\ 3x + y = 4 \\ 2x - y + z = 2 \end{cases}$

Solution: The augmented matrix is $\left[\begin{array}{rrr|r} 1 & 2 & -1 & 3 \\ 3 & 1 & 0 & 4 \\ 2 & -1 & 1 & 2 \end{array}\right]$

This can be reduced as follows:

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 3 \\ 3 & 1 & 0 & 4 \\ 2 & -1 & 1 & 2 \end{array}\right] \frac{R_2 \to (-3)R_1 + R_2}{R_3 \to (-2)R_1 + R_3} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 3 \\ 0 & -5 & 3 & -5 \\ 0 & -5 & 3 & -4 \end{array}\right]$$

$$R_2 \to -\frac{1}{5}R_2 \left[\begin{array}{rrr|r} 1 & 2 & -1 & 3 \\ 0 & 1 & -\frac{3}{5} & 1 \\ 0 & -5 & 3 & -4 \end{array}\right] \frac{R_1 \to (-2)R_2 + R_1}{R_3 \to 5R_2 + R_3} \left[\begin{array}{rrr|r} 1 & 0 & \frac{1}{5} & 1 \\ 0 & 1 & -\frac{3}{5} & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The system of equations corresponding to the reduced matrix is

$\begin{cases} x + \frac{1}{5}z = 1, \quad y - \frac{3}{5z} = 1, \quad 0 = 1, \end{cases}$ This is impossible, thus this system has no solution.

Thank You