Discrete Math

Logic and Induction

Predicate Logic

- *Predicate logic* is an extension of propositional logic that permits concisely reasoning about whole *classes* of entities.

  E.g., “$x>1$”, “$x+y=10$”

- Such statements are neither true or false when the values of the variables are not specified.

Predicate

- Predicate is a part of declarative sentences describing the properties of an object or relation among objects. For example: “is a student” is a predicate as ‘A is a student’ and ‘B is a student’.
- A predicate logic is a formal system that uses objects/ variables and quantifiers ($\forall$, $\exists$) to formulate propositions.

Quantification

- A quantifier is a symbol that permits one to declare the range or scope of variables in a logical expression.
- The process of binding propositional variable over a given domain is called quantification.
- Two common quantifier are the existential quantifier (“there exists or for some or at least one”) and universal quantifier (“for all or for each or for any or for every and or for arbitrary”).

Universal Quantifier (∀: For All)

- It is denoted by ∀ and used for universal quantification.
- The universal quantification of p(x) denoted by $\forall x\ p(x)$ is proposition that is true for all values in universal set
- The universal quantifier is read as:
  - For all x, p(x) holds
  - For each x, p(x) holds
  - For every x, p(x) holds

• Definition

Let $Q(x)$ be a predicate and D the domain of x. A **universal statement** is a statement of the form "$\forall x \in D, Q(x)$." It is defined to be true if, and only if, $Q(x)$ is true for every x in D. It is defined to be false if, and only if, $Q(x)$ is false for at least one x in D. A value for x for which $Q(x)$ is false is called a **counterexample** to the universal statement.

Example

a. Let $D = \{1, 2, 3, 4, 5\}$, and consider the statement

$$\forall x \in D, x^2 \geq x.$$

Show that this statement is true.

b. Consider the statement

$$\forall x \in \mathbf{R}, x^2 \geq x.$$

Find a counterexample to show that this statement is false.

a. Check that "$x^2 \geq x$" is true for each individual x in D.

$$1^2 \geq 1, \qquad 2^2 \geq 2, \qquad 3^2 \geq 3, \qquad 4^2 \geq 4, \qquad 5^2 \geq 5.$$

Hence "$\forall x \in D, x^2 \geq x$" is true.

b. *Counterexample:* Take $x = \frac{1}{2}$. Then x is in **R** (since $\frac{1}{2}$ is a real number) and

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4} \ngeq \frac{1}{2}.$$

Hence "$\forall x \in \mathbf{R}, x^2 \geq x$" is false.

Existential Quantifier(∃ : For Some)

- It is denoted by ∃ and used for existential quantification.
- The existential quantification of p(x) denoted by $\exists x\ p(x)$ is proposition that is true for some values in universal set.
- The existential quantifier is read as:
  - There is an x, such that p(x)
  - There is at least one x such that p(x)
  - For some x, p(x)

• **Definition**

Let $Q(x)$ be a predicate and D the domain of x. An **existential statement** is a statement of the form "$\exists x \in D$ such that $Q(x)$." It is defined to be true if, and only if, $Q(x)$ is true for at least one x in D. It is false if, and only if, $Q(x)$ is false for all x in D.

a. Consider the statement

$$\exists m \in \mathbf{Z}^+ \text{ such that } m^2 = m.$$

Show that this statement is true.

b. Let $E = \{5, 6, 7, 8\}$ and consider the statement

$$\exists m \in E \text{ such that } m^2 = m.$$

Show that this statement is false.

a. Observe that $1^2 = 1$. Thus "$m^2 = m$" is true for at least one integer m. Hence "$\exists m \in \mathbf{Z}$ such that $m^2 = m$" is true.

b. Note that $m^2 = m$ is not true for any integers m from 5 through 8:

$$5^2 = 25 \neq 5, \qquad 6^2 = 36 \neq 6, \qquad 7^2 = 49 \neq 7, \qquad 8^2 = 64 \neq 8.$$

Thus "$\exists m \in E$ such that $m^2 = m$" is false.

Questions:

- The domain for all variables in the expressions below is the set of real numbers. Determine whether each statement is true or false. Justify your answer.

(a) $\forall x \, \exists y \, (x + y = 0)$

(b) $\exists x \, \forall y \, (x + y = 0)$

(c) $\exists x \, \forall y \, (xy = y)$

Q. Convert into FOPL

a. All men are people.

b. Marcus was Pompeian.

c. All Pompeian were Roman.

d. Ram tries to assassinate Hari.

e. All Romans were either loyal to caser or hated him.

f. Socrates is a man. All men are mortal; therefore Socrates is mortal.

g. Some student in this class has studied mathematics.

a. **All men are people.**

$\Rightarrow \forall x \; MAN(x) \rightarrow PEOPLE(x)$

b. **Marcus was Pompeian.**

$\Rightarrow POMPEIAN(Marcus)$

c. **All Pompeian were Roman.**

$\Rightarrow \forall x \; POMPEIAN(x) \rightarrow ROMAN(x)$

d. **Ram tries to assassinate Hari.**

$\Rightarrow ASSASSINATE(Ram, Hari)$

e. All Romans were either loyal to caser or hated him.

$\Rightarrow \forall x$ ROMAN (x) $\rightarrow$ LOYAL (x, caser) $\vee$HATES (x, caser)

f. Socrates is a man. All men are mortal; therefore Socrates is mortal.

$\Rightarrow$ MAN (Socrates), $\forall$ x MAN (x) $\rightarrow$ MORTAL (x), MORTAL (Socrates)

g. Some student in this class has studied mathematics.

$\Rightarrow$ Let

- S(x) = "x is a student in this class"
- M(x) = "x has studied mathematics"

Hence, required expression is: $\exists x\ [S(x) \wedge M(x)]$

Introduction to Induction

- Mathematical induction is an extremely important proof technique that can be used to proof mathematical theorems or statements.
- It is a technique for proving results or establishing statements for natural numbers.

Mathematical Induction

- Mathematical Induction is a mathematical technique which is used to prove a statement, a formula or a theorem is true for every natural number.
- The technique involves two steps to prove a statement, as stated below –
- To prove a result P(n) using the principle of mathematical induction

    Step 1 (**Base step**) − It proves that a statement is true for the initial value. i.e. prove that P(1) holds.

    Step 2 (**Inductive step**) − Then we assume that P(k) holds for some natural number k, and using this hypothesis, we prove that P (k+ 1) is true. If P(k+1) holds true, then the statement P(n) is true.

Example

Q. Use mathematical induction to prove that n^3-n is divisible by 3 where n is a positive integer.

Solution:

Let p(n) denote the proposition: n^3-n is divisible by 3.

Base step: Let p(n) is true for n=1

i.e. p(1): $1^3-1=0$, is divisible by 3.

Induction step:

Let p(n) is true for n=k.

i.e. p(k): k^3-k is divisible by 3.(Assumption)

Now, using p(k) we try to show that p(n) is true for n=k+1.

$P(k+1)= (k+1)^3-(k+1)$

$= k^3+ 3k^2+3k+1-k-1$

$= (k^3-k)+3(k^2+k)$

Since, both terms in this sum are divisible by 3.

Therefore, using principle mathematical induction, n^3-n is divisible by 3 whenever n is a positive integer.

Question

- 1. Prove that 3 divides n^3+2n whenever n is a nonnegative integer.

Problem 2

$1+3+5+...+(2n-1)=n^2$ for $n=1,2,...$

Solution

Step 1 – For $n=1, 1=1^2$, Hence, step 1 is satisfied.

Step 2 – Let us assume the statement is true for $n=k$.

Hence, $1+3+5+\cdots+(2k-1)=k^2$ is true (It is an assumption)

We have to prove that $1+3+5+...+(2(k+1)-1)=(k+1)^2$ also holds

$$1+3+5+\cdots+(2(k+1)-1)$$

$$=1+3+5+\cdots+(2k+2-1)$$

$$=1+3+5+\cdots+(2k+1)$$

$$=1+3+5+\cdots+(2k-1)+(2k+1)$$

$$=k^2+(2k+1)$$

$$=(k+1)^2$$

So, $1+3+5+\cdots+(2(k+1)-1)=(k+1)^2$ hold which satisfies the step 2.

Hence, $1+3+5+\cdots+(2n-1)=n^2$ is proved.

Problem 3

Prove that $(ab)^n = a^n b^n$ is true for every natural number n

Solution

Step 1 – For $n = 1, (ab)^1 = a^1 b^1 = ab$, Hence, step 1 is satisfied.

Step 2 – Let us assume the statement is true for $n = k$, Hence, $(ab)^k = a^k b^k$ is true (It is an assumption).

We have to prove that $(ab)^{k+1} = a^{k+1} b^{k+1}$ also hold

Given, $(ab)^k = a^k b^k$

Or, $(ab)^k (ab) = (a^k b^k)(ab)$ [Multiplying both side by 'ab']

Or, $(ab)^{k+1} = (aa^k)(bb^k)$

Or, $(ab)^{k+1} = (a^{k+1} b^{k+1})$

Hence, step 2 is proved.

So, $(ab)^n = a^n b^n$ is true for every natural number n.

Problem 4

Prove that $1 + 2 + 3 + 4 + 5 + ... + n = n(n+1)/2$.

Solution: Suppose $P(n)$: $1 + 2 + 3 + 4 + 5 + ... + n = n(n+1)/2$

Base Step: To prove $P(1)$ is true.

For $n = 1$, LHS = 1

RHS = $1(1+1)/2 = 2/2 = 1$

Hence LHS = RHS $\Rightarrow$ $P(1)$ is true.

Assumption Step: Assume that $P(n)$ holds for $n = k$, i.e., $P(k)$ is true

$\Rightarrow 1 + 2 + 3 + 4 + 5 + ... + k = k(k+1)/2$ --- (1)

Induction Step: Now we will prove that $P(k+1)$ is true.

To prove: $1 + 2 + 3 + 4 + ... + (k+1) = (k+1)(k+2)/2$

Consider LHS $= 1 + 2 + 3 + 4 + ... + (k+1)$

$= 1 + 2 + 3 + 4 + ... k + (k+1)$

$= (1 + 2 + 3 + 4 + ... + k) + k+1$

$= k(k+1)/2 + k+1$ [Using (1)]

$= [k(k+1) + 2(k+1)]/2$

$= (k+1)(k+2)/2$

= RHS

$\Rightarrow$ $P(n)$ is true for $n = k+1$

Hence, by the principle of mathematical induction, $P(n)$ is true for all natural numbers n.