Graph Theory

Isomorphic graphs

Handshaking Theorem

- Handshaking Theorem is also known as **Handshaking Lemma** or **Sum of Degree Theorem**.
- Handshaking Theorem states in any given graph, Sum of degree of all the vertices is twice the number of edges contained in it.

$$\sum_{i=1}^{n} d(v_i) = 2 \times |E|$$

The following conclusions may be drawn from the Handshaking Theorem.
In any graph,

- The sum of degree of all the vertices is always even.
- The sum of degree of all the vertices with odd degree is always even.
- The number of vertices with odd degree are always even.

Question:

1. A simple graph G has 24 edges and degree of each vertex is 4. Find the number of vertices.

Solution:

Given,

Number of edges = 24

Degree of each vertex = 4

Let number of vertices in the graph = n

Using Handshaking Theorem, we have-

Sum of degree of all vertices = 2 x Number of edges

Substituting the values, we get : n x 4 = 2 x 24

n = 2 x 6

∴ n = 12

Thus, Number of vertices in the graph = 12.

Q. 2. A graph contains 21 edges, 3 vertices of degree 4 and all other vertices of degree 2. Find total number of vertices.

Solution:

Given-

Number of edges = 21

Number of vertices with degree 4 = 3

All other vertices are of degree 2

Let number of vertices in the graph = n.

Using Handshaking Theorem, we have-

Sum of degree of all vertices = 2 x Number of edges

$3 \times 4 + (n-3) \times 2 = 2 \times 21$

$12 + 2n - 6 = 42$

$2n = 42 - 6$

$2n = 36$

$\therefore n = 18$

Thus, Total number of vertices in the graph = 18

- Q. 3 A simple graph contains 35 edges, four vertices of degree 5, five vertices of degree 4 and four vertices of degree 3. Find the number of vertices with degree 2.

Given-

Number of edges = 35

Number vertices with degree 5 = 4

Number of vertices with degree 4 = 5

Number of vertices with degree 3 = 4

Let number of vertices with degree 2 in the graph = n.

Using Handshaking Theorem, we have-

Sum of degree of all vertices = 2 x Number of edges

$4 \times 5 + 5 \times 4 + 4 \times 3 + n \times 2 = 2 \times 35$

$20 + 20 + 12 + 2n = 70$

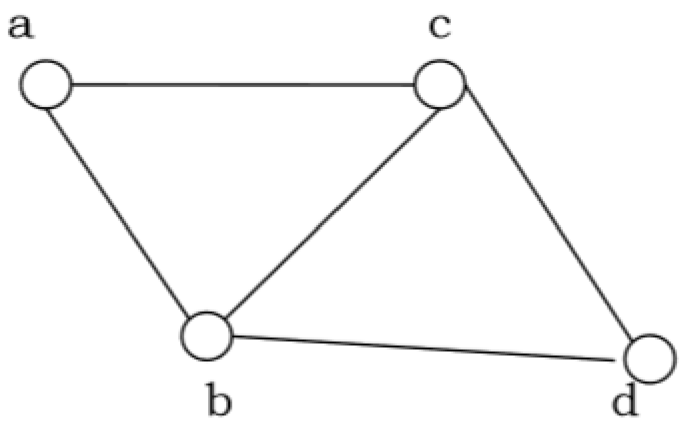
$2n = 70 - 52$

$2n = 18$

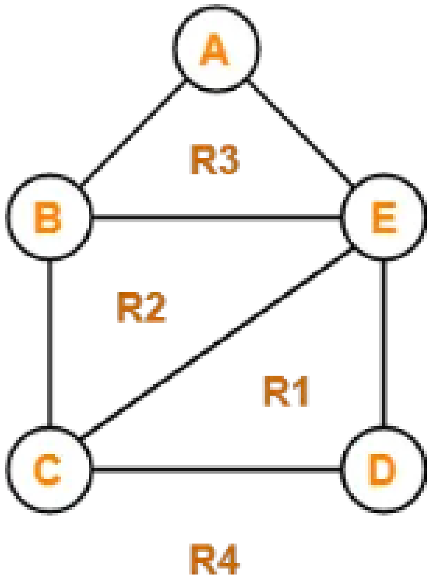
$\therefore n = 9$

Planar Graph-

In graph theory, Planar graph is a graph that can be drawn in a plane such that none of its edges cross each other.



- The following graph is an example of a planar graph-



Regions of Plane

Here, this planar graph splits the plane into 4 regions- R1, R2, R3 and R4

Euler’s Formula

- If G is a connected planar simple graph with ‘e’ edges, ‘v’ vertices and ‘r’ number of regions in the planar representation of G, then-

$$\mathbf{r = e - v + 2}$$

This is known as **Euler’s Formula**.

Question 1

Let G be a connected planar simple graph with 25 vertices and 60 edges. Find the number of regions in G.

Solution-

Given-

Number of vertices (v) = 25

Number of edges (e) = 60

By Euler's formula, we know $r = e - v + 2$.

Number of regions (r)

$= 60 - 25 + 2$

$= 37$

Thus, Total number of regions in G = 37.

Question 2

Let G be a connected planar simple graph with 20 vertices and degree of each vertex is 3. Find the number of regions in G.

Solution-

Given-

Number of vertices (v) = 20

Degree of each vertex = 3

Calculating Total Number Of Edges (e): By sum of degrees of theorem, we have-

Sum of degrees of all the vertices = 2 x Total number of edges

Number of vertices x Degree of each vertex = 2 x Total number of edges

20 x 3 = 2 x e ➔ e = 30

Calculating Total Number Of Regions (r)

By Euler’s formula, we know r = e – v + 2.

Number of regions (r)

= 30 – 20 + 2

= 12

Thus, Total number of regions in G = 12.

Isomorphic graphs

- A graph can exist in different forms having the same number of vertices, edges, and also the same edge connectivity. Such graphs are called isomorphic graphs.
- Conditions for being Isomorphic:

if two graph G_1 and G_2 are isomorphic then they must have:

i. same number of vertices i.e. $|V_1| = |V_2|$

ii. same number of edges i.e. $|E_1| = |E_2|$

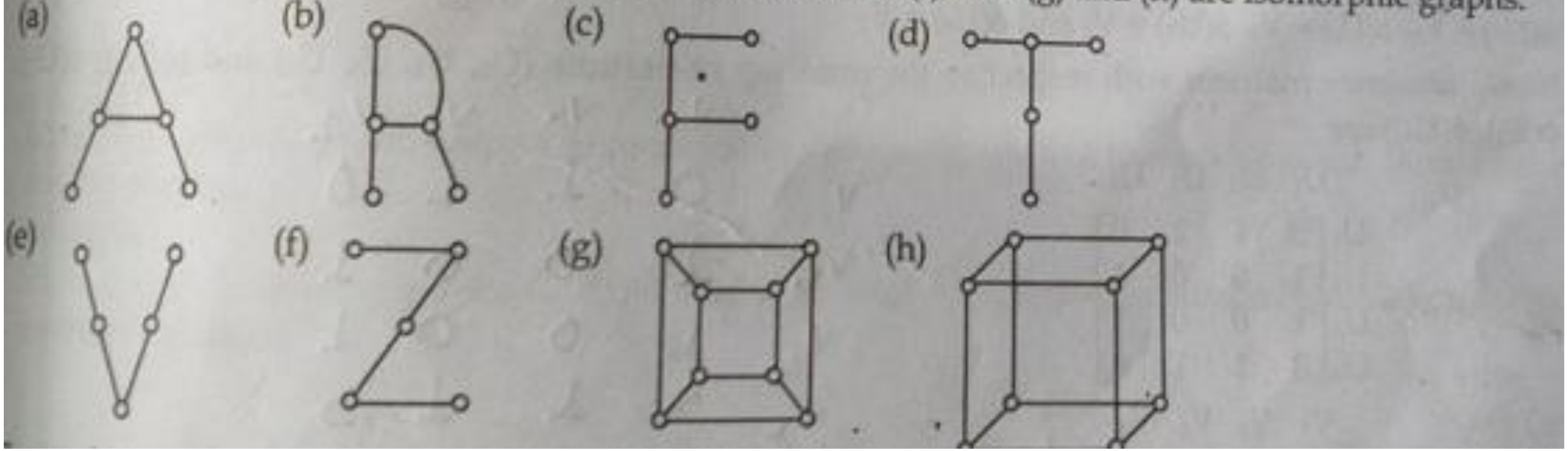
iii. if $\{u,v\}$ in $E_1 \rightarrow \{\phi(u), \phi(v)\}$ in E_2

iv. if $deg(u)=k$ in G_1 then $deg(\phi(u)) = k$ in G_2

v. $A(G_1)=A(G_2)$ with respect to orders of vertices $v_1, v_2, v_3, \ldots v_n$ and $\phi(v_1), \phi(v_2), \phi(v_3), \ldots\ldots \phi(v_n)$.

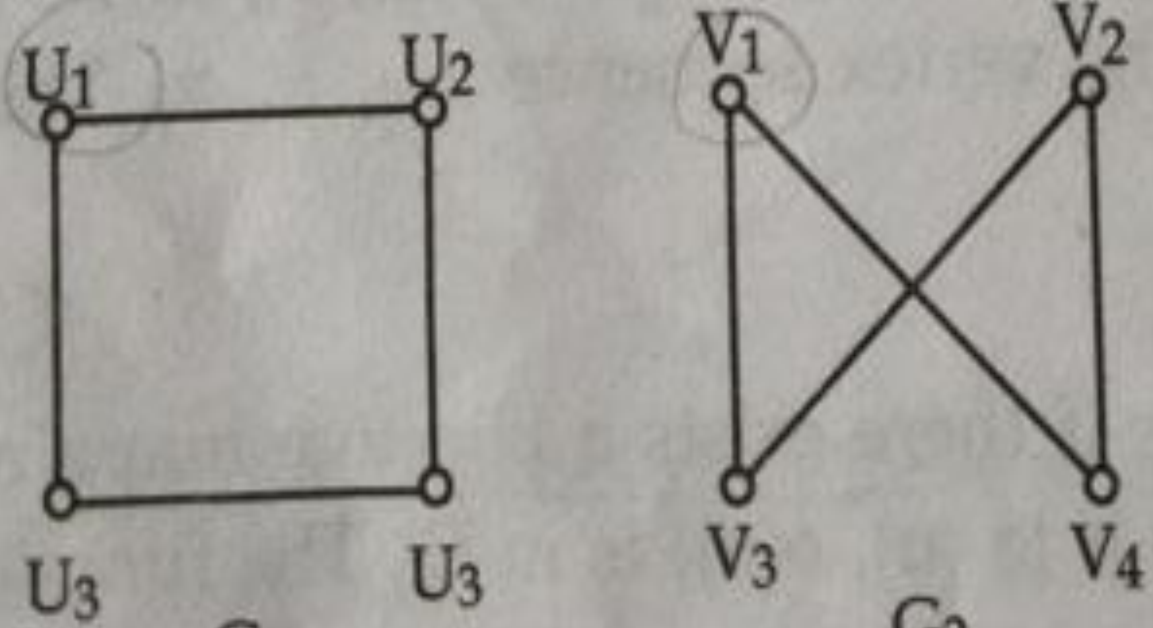
Example

In figure below, graphs in (a) and (b), (c) and (d), (e) and (f), and (g) and (h) are isomorphic graphs.



Example

Show that the graphs G_1 and G_2 are isomorphic.



G_1 G_2

Solution

The isomorphic invariants for two graphs are:

No. of vertices in G_1 $|V(G_1)| = 4$

No. of edges in G_1 $|E(G_1)| = 4$

No. of vertices in G_2 $|V(G_2)| = 4$

No. of edges in G_2 $|E(G_2)| = 4$

In graph G_1, there are four vertices each of degree 2 i.e. (2, 2, 2, 2) similar is true in graph G_2 also.

Since both graphs agree so many isomorphic invariants so, it is reasonable to find an isomorphison ϕ.

Let $\phi: V(G_1) \to V(G_2)$ defined by

$\phi(U_1) = V_1$, $\phi(U_2) = V_4$, $\phi(U_3) = V_3$ and $\phi(U_4) = V_2$

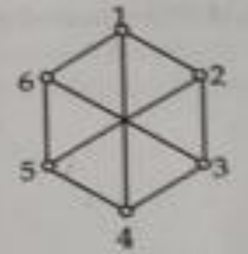
Now, adjacency matrices with respect to the ordering of vertices (U_1, U_2, U_3, U_4) and ($\phi(U_1)$, $\phi(U_2)$, $\phi(U_3)$, $\phi(U_4)$) are

$$A(G_1) = \begin{array}{c} \\ U_1 \\ U_2 \\ U_3 \\ U_4 \end{array}\begin{array}{c} \begin{array}{cccc} U_1 & U_2 & U_3 & U_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

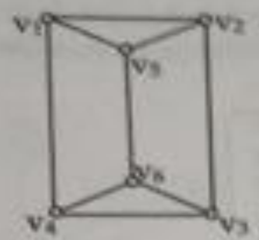
$$\begin{array}{c|cccc} & V_1 & V_4 & V_3 & V_2 \\ \hline V_1 & 0 & 1 & 1 & 0 \\ V_4 & 1 & 0 & 0 & 1 \\ V_3 & 1 & 0 & 0 & 1 \\ V_2 & 0 & 1 & 1 & 0 \end{array}$$

$$A(G_2) = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{array}\begin{array}{c} \begin{array}{cccc} V_1 & V_2 & V_3 & V_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Show that graphs G_1 and G_2 given below are not isomorphic.



G_1



G_2

Solution

The isomorphic invariants of two graphs are

$|V(G_1)| = |V(G_2)| = 6$

$|E(G_1)| = |E(G_2)| = 9$

Degree sequence of G_1: (3, 3, 3, 3, 3, 3)

Degree sequence of G_2: (3, 3, 3, 3, 3, 3)

Since both graphs agree so many invariants so, it is reasonable to find an isomorphism ϕ.

Let $\phi : V(G_1) \rightarrow V(G_2)$ defined by

$\phi(1) = V_1,\ \phi(2) = V_2,\ \phi(3) = V_3,\ \phi(4) = V_4,\ \phi(5) = V_6,\ \phi(6) = V_5$

Now,

Adjacency matrices with respect to ordering of vertices in ϕ are

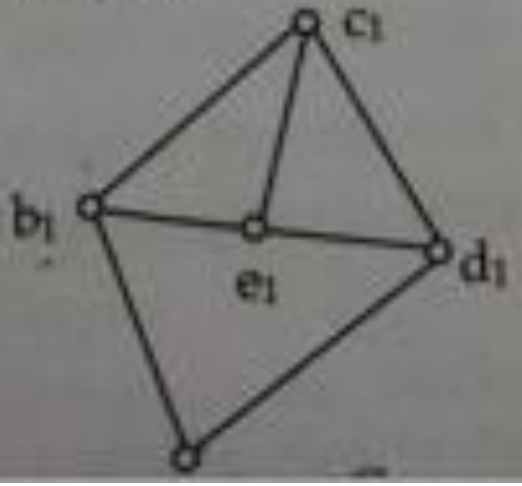
$$A(G_1) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

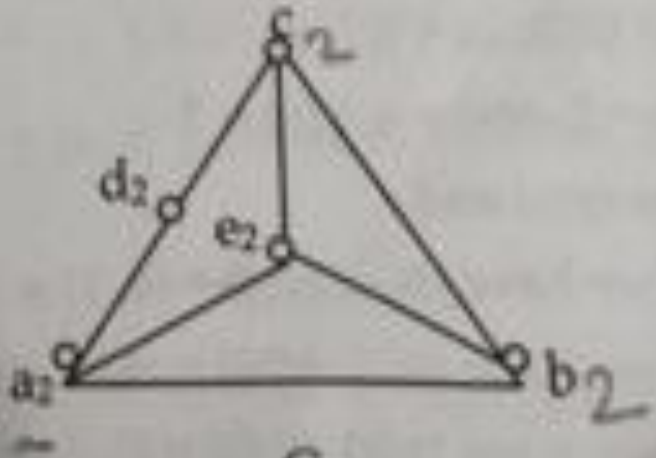
$$A(G_2) = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_6 \\ V_5 \end{array} \begin{array}{c} \begin{array}{cccccc} V_1 & V_2 & V_3 & V_4 & V_6 & V_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Since: $A(G_1) \neq A(G_2)$ with respect to ordering of vertices so, ϕ is not isomorphism and G_1 and G_2 a
not isomorphic.

Example

Show that the following graphs G_1 and G_2 are isomorphic by showing their corresponding adjacen matrices are equal.





G_2

Solution

Consider the map $\phi: G_1 \to G_2$ defined as $\phi(a_1) = d_2$, $\phi(b_1) = a_2$, $\phi(c_1) = b_2$, $\phi(d_1) = c_2$ and $\phi(e_1) = e_2$. The adjacency matrix of G_1 for the ordering a_1, b_1, c_1, d_1 and e_1 is

$$A(G_1) = \begin{array}{c} \\ a_1 \\ b_1 \\ c_1 \\ d_1 \\ e_1 \end{array} \begin{array}{c} \begin{array}{ccccc} a_1 & b_1 & c_1 & d_1 & e_1 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

The adjacency matrix of G_2 for the ordering d_2, a_2, b_2, c_2 and e_2 is;

$$A(G_2) = \begin{array}{c} \\ d_2 \\ a_2 \\ b_2 \\ c_2 \\ e_2 \end{array} \begin{array}{c} \begin{array}{ccccc} d_2 & a_2 & b_2 & c_2 & e_2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$\therefore$ G_1 and G_2 are isomorphic.

Example

Example

Show that the two graphs shown below are isomorphic