Mathematical Reasoning

Unit 2

- Rules of inference and proof (covered in chapter 1)

Direct and indirect proof

- **Direct Proof:**

The implication p->q can be proved by showing that if p is true then q must also be true.

To carry out such a proof, we assume that hypothesis p is true and using information already available if conclusion q becomes true then argument becomes valid.

Example-1

Prove that the sum of two odd integers is even.

Solution:

Let m and n be two odd integers. Then by definition of odd numbers

$m = 2k + 1$ for some $k \in Z$

$n = 2l + 1$ for some $l \in Z$

Now $m + n = (2k + 1) + (2l + 1)$

$= 2k + 2l + 2$

$= 2(k + l + 1)$

$= 2r$ where $r = (k + l + 1) \in Z$

Hence $m + n$ is even.

Hence Proved.

Example-2

Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd."

Solution:

Let p be the statement that n is an odd integer and q be the statement that n^2 is an odd integer. Assume that n is an odd integer, then by definition $n = 2k + 1$ for some integer k. We will now use this to show that n^2 is also an odd integer.

$n^2 = (2k + 1)^2$ since $n = 2k + 1$

$= (2k + 1)(2k + 1)$

$= 4k + 2k + 2k + 1$

$= 4k^2 + 4k + 1$

$= 2(2k^2 + 2k) + 1$

Hence proved n^2 is also odd integer.

Example -3

Use a direct proof to show that the product of two rational numbers is rational.

Solution:

First number $x = a/b$, $b \neq 0$

Second number $y = c/d$, $d \neq 0$

$X*y = a*c/b*d$ sense $b \neq 0$ and $d \neq 0$ then $b*d \neq 0$.

Let $a*c = l$ and $b*d = m$

so, $x = l/m$ s

Hence x is rational number.

Hence Proved.

Example-4

Prove that the product of an even integer and an odd integer is even.

Solution:

Suppose m is an even integer and n is an odd integer. Then

$m = 2k$ for some integer k and

$n = 2l + 1$ for some integer l

Now

$$\begin{aligned} m \cdot n &= 2k \cdot (2l + 1) \\ &= 2 \cdot k\,(2l + 1) \\ &= 2 \cdot r \quad \text{where } r = k(2l + 1) \text{ is an integer} \end{aligned}$$

Hence $m \cdot n$ is even. (Hence Proved)

Example-5

Prove that the square of an even integer is even.

Solution:

Suppose n is an even integer. Then $n = 2k$

Now

$$\begin{aligned} \text{square of } n = n^2 &= (2\cdot k)^2 \\ &= 4k^2 \\ &= 2\cdot(2k^2) \\ &= 2\cdot p \quad \text{where } p = 2k^2 \in Z \end{aligned}$$

Hence, n^2 is even. (Hence proved)

Example-6

- Using direct proof, prove that for every positive integer $n^3 + n$ is even.

Example-7

Prove that the sum of any three consecutive integers is divisible by 3.

Solution:

Let n, $n + 1$ and $n + 2$ be three consecutive integers.

Now

$$n + (n + 1) + (n + 2) = 3n + 3$$
$$= 3(n + 1)$$
$$= 3 \cdot k \quad \text{where } k=(n+1) \in Z$$

Hence, the sum of three consecutive integers is divisible by 3.

Hence Proved.

Indirect Proof

- We have p->q = ~q->~p
- Contrapositive and its implication is equivalent.
- The implication p-> q can be proved by showing that its contrapositive ~q-> ~p is true.
- Instead of proving $p \Rightarrow q$ directly, it is sometimes easier to prove it indirectly.
- The ***proof by contrapositive (form of indirect proof)*** is based on the fact that an implication is equivalent to its contrapositive. Therefore, instead of proving $p \Rightarrow q$, we may prove its contrapositive $\sim q \Rightarrow \sim p$.

Indirect Proof

- Since it is an implication, we could use a direct proof:
- Assume ~q is true (hence, assume q is false).
- Show that ~p is true (that is, show that p is false).
- The proof may proceed as follow:
- *Proof*: We want to prove the contrapositive of the stated result.
- Assume q is false, . . .
- .
- .
- .
- . . . Therefore p is false.

Example

Prove that if n is an integer and 3n+2 is odd, then n is odd.

Solution:

Assume that n is even (negation). So, n can be expressed as 2k for some integer k (definition of even).

$n = 2k$. Therefore,

$$
\begin{aligned}
3n + 2 &= 3(2k) + 2 \\
&= 6k + 2 \\
&= 2(3k + 1)
\end{aligned}
$$

So $3n + 2$ is even.

Hence proved.

- Example

Show that if n^2 is even, then n is also even where n is integer.

Solution:

Indirect Proof (Proof by contrapositive)

We want to prove that if n is odd, then n2 is odd.

If n is odd, then n=2k+1 for some integer k.

Hence,

n2=(2k+1)2

=4k2 + 2k+1

=2(2k2 +k) +1

=odd

This completes the proof.

Example

Let x be a real number. Prove that if $x^3-7x^2+x-7=0$ then $x=7$.

Solution:

Assume $x \neq 7$, then

$x^3-7x^2+x-7=0$

$=x^2(x-7) +(x-7)$

$=(x^2+1) (x-7) \neq 0$

Thus, if $x^3-7x^2+x-7=0$, then $x=7$.

Example

Prove that If $x=2$, then $3x-5 \neq 10$.

In an indirect proof the first thing you do is assume the conclusion of the statement is false. In this case, we will assume the opposite of "If $x=2$, then $3x-5 \neq 10$":

i.e. Assume that $3x-5=10$ is true and solve for x.

$3x-5=10$

$3x=15$

Therefore, $x=5$

But $x=5$ contradicts the given statement that $x=2$. Hence, our assumption is incorrect and $3x-5 \neq 10$ is true.