Answers of the question paper

1. Let A and B be two events with P(A)>0, P(B|A)=0.3, and $P(A \cap B^c) = 0.2$. Then find P(A).

Answer: $P(B|A) = 0.3 \Rightarrow P(A \cap B) = 0.3P(A)$. And we know that $P(A) = P(A \cap B) + P(A \cap B^c) \Rightarrow P(A) = 0.3P(A) + 0.2$. So, P(A)=2/7=0.28

2. A system with m components functions if and only if at least one of m components functions. Suppose all the m components of the system functions independently, each with probability 3/4. If the probability of functioning of the system is 63/64, then find the value of m.

Answer: P(at least one component is working) = 63/64. Now,

1-P(no component is working) = 63/64. So, 1-$(1/4)^m = 63/64$. We get

$1 - 63/64 = (1/4)^m \Rightarrow 1/64 = (1/4)^m \Rightarrow (1/4)^3 = (1/4)^m$. So, m=3.

3. Consider a telephone operator who, on average, handles five calls every three minutes. What is the probability there will be no call in the next minute?

Answer: Use Poisson distribution to solve this problem. Five calls every three meets so rate is 5/3.

P(no call in the next minute) = P(X=0) = $\frac{e^{-5/3}(5/3)^0}{0!} = e^{-5/3} = 0.189$

4. Let X be the random variable with density function $f(x) = 7e^{-7x}, 0 < x < \infty$. Let Y = 4X+3, then find the density function of Y.

Answer: We know that $F(y) = P(Y \leq y) = P(4X + 3 \leq y) = P\left(X \leq \frac{y-3}{4}\right) = F_X\left(\frac{y-3}{4}\right)$. Now taking derivative on both sides we get,

$$f(y) = \frac{1}{4} f_X\left(\frac{y-3}{4}\right). \text{ So, } f(y) = \frac{7}{4} e^{-\frac{7}{4}(y-3)}; 3 < y < \infty.$$

5. Consider the discrete bivariate random vector with the joint probability mass function is given by f(10,1) = f(20,1) = f(20,2) = 1/10, f(10,2) = f(10,3) = 1/5, and f(20,3) = 3/10. Show that random variables X and Y are not independent.

Answer: $\boldsymbol{f(10,3) \neq f_X(10) f_Y(3)}$

6. Let the joint probability mass function of (X, Y) is given by f(0,10) = f(0,20) = 2/18, f(1,10) = f(1,30) = 3/18, f(1,20) = 4/18, and f(2,30) = 4/18 and zero elsewhere. Then find f(y=10|x=0).

Answer: $f(y = 10|x = 0) = \frac{f(x=0,y=10)}{f_X(x=0)}$. So, $f_X(x = 0) = f(x = 0, y = 10) + f(x = 0, y = 20) + f(x = 0, y = 30) = 4/18$

And f(x=0,y=10)=2/18. Hence, we get $f(y = 10|x = 0) = \frac{2/18}{4/18} = 1/2.$

7. Let be the partitions of the sample space, and let B be any set. Then, for each i=1,2,3,… write the Bayes' rule.

Answer: $$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{\sum_{j=1}^{\infty} P(B|A_j)P(A_j)}$$

8. Write the necessary and sufficient condition for cumulative distribution function.

Answer:

a) $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$

b) F(x) is a nondecreasing function of x.

a) F(x) is right continuous.

9. Suppose we are experiencing an extreme heatwave and there are two weather phenomena that can occur: Heatstroke (H) and Dehydration (D). Let's assume that the probability of experiencing a Heatstroke is 80% and the probability of experiencing Dehydration is 65%. We also know that the probability of experiencing both Heatstroke and Dehydration is 60%. If someone is experiencing Heatstroke, what is the probability that they are also experiencing Dehydration?

Answer: We know that P(H) = 0.8, P(D) = 0.65, and P(H and D) = 0.60

And P(D|H) = P(H and D) / P(H)

Substituting the given probabilities, we get: P(D|H) = 0.6 / 0.8 = 0.75

10. Suppose we have a team of 10 scientists who are conducting research in Antarctica during an extreme cold wave. The team needs to select two scientists from their group to be the lead researcher and the assistant lead researcher. How many ways can this be done?

Answer: Let n1 = the number of ways the lead researcher can be chosen = 10. Let n2 = the number of ways the assistant lead researcher can be chosen once the chair has been chosen = 9.

Then N = n1*n2 = (10)(9) = 90

11. What is the probability that a committee of 10 people chosen from a group consisting of 40 principals, 35 teachers, and 25 students, will include three principals, five teachers, and two students?

Answer: Let X be the event that a committee of 10 people chosen from a group consisting of 40 principals, 35 teachers, and 25 students, will include three principals, five teachers, and two students. Here, X follows hypergeometric distribution and it's probability can be given by:

$$P(X) = \frac{(40C3)(35C5)(25C2)}{(100\ C\ 10)} = 0.0556$$

This means there is a 0.0556 chance that precisely 3 principals, five teachers, and two students will be chosen for the committee.

12. A catalyst producer produces a device for testing defects in a certain electrocatalyst (EC). The catalyst producer claims that the test is 97% reliable if the EC is defective and 99% reliable when it is flawless. However, 4% of said EC may be expected to be defective upon delivery. What is the probability that EC found flawless given that it is tested defective?

Answer: Let A: EC is defective; $\bar{A}$: the EC is flawless;

B: the EC is tested to be defective; $\bar{B}$: the EC is tested to be flawless.

The probabilities would be

B/A: EC is (known to be) defective, and tested defective, P(B/A) = 0.97,

$\bar{B}$/A: EC is (known to be) defective, but tested flawless, P($\bar{B}$/A)=1-P(B/A)=0.03,

B/$\bar{A}$: EC is (known to be) defective, but tested defective, P(B/$\bar{A}$)= 1- P($\bar{B}$/$\bar{A}$)=0.01

P(EC found flawless given that it is tested defective)

$$= P(\bar{A}/B) = \frac{P(B/\bar{A})\ P(\bar{A})}{P(B/\bar{A})\ P(\bar{A}) + P(B/A)\ P(A)} = \frac{0.01 * 0.96}{0.01*0.96 + 0.97*0.04} = 0.1983$$

13. Suppose that 5 people, including you and a friend, line up at random. Let the random variable X denote the number of people standing between you and a friend. Determine the probability mass function of X in tabular form. Also, verify that the p.m.f. is a valid p.m.f.

Answer: Given, X denotes the number of people standing between you and a friend.

Then, X = 0,1,2,3

$P(X=0) = \frac{4*2!*3!}{5!} = 4/10 = 2/5$

(where 4 is no. of positions you and your friend can stand, 2! is no. of ways you and your friend can be arranged, 3! is no. of ways remaining people will be arranged.)

$P(X=1) = \frac{3*2!*3!}{5!} = 3/10$

$P(X=2) = \frac{2*2!*3!}{5!} = 2/10 = 1/5$

$P(X=3) = \frac{1*2!*3!}{5!} = 1/10$

Now,

$P(X=0)+P(X=1)+P(X=2)+P(X=3) = 2/5 + 3/10 + 1/5 + 1/10 = 1$

So, p.m.f is valid p.m.f.

14. Let Jim and his fiance stay together. Let

C = Event that Jim had Covid
T = Event that Jim tests positive
F = Event that his fiance had Covid
$P(C)= 0.1$, $P(T|\overline{C})= 0.005$, $P(\overline{T}|C)= 0$, $P(F|\overline{C})= 0$, $P(F|C)=0.95$
Find the probability that Jim had Covid given that Jim tests positive and his fiance had not covid.

Answer: P(Jim had Covid given that Jim tests positive and his fiance had not covid)

$$= P(C|T,\overline{F}) = \frac{P(C,T,\overline{F})}{P(C,T,\overline{F})+ P(\overline{C},T,\overline{F})} = \frac{0.1*1*0.05}{0.1*1*0.05+0.9*1*0.005} = \frac{0.005}{0.005+0.0045} = 0.526 = 52.6\%$$

There is 52.6% that Jim had Covid if he tests positive and his fiance did not have covid.

15. How many different ways can the letters of the word TRIANGLE be arranged

a) If the order of the vowels IAE cannot be changed, though their placement may (IAETRNGL and TRIANGEL are acceptable but EIATRNGL and TRIENGLA are not)?
b) If the order of the vowels IAE can be changed, though their placement may not?

Answer: a) Step one is to choose the places that the vowels go. Here we are picking three places out of eight, and the order that we do this is not important. This is a combination and there are a total of $8C3 = 56$ ways to perform this step. The remaining five letters may be arranged in $5! = 120$ ways. This gives a total of 56 x 120 = 6720 arrangements.

b) We arrange three letters in 3! = 6 ways and the other five letters in 5! = 120 ways. The total number of ways for this arrangement is 6 x 120 = 720.

16. The joint probability of random variable X, Y is given below

$$f_{x,y}(x,y) = \begin{cases} xy\, e^{\frac{-(x2+y2)}{2}} & x \geq 0, y \geq 0 \\ 0 & elsewhere \end{cases}$$

(a) Find $f_x(x)$.
(b) Find $f_{Y/X}(y/x)$.

Answer: Solution 1: (a) $f_x(x) = \int_{-\infty}^{\infty} fx,y(x,y)dy$

$$= \int_0^\infty xy\, e^{\frac{-(x2+y2)}{2}}\, dy$$

$$= x\, e^{-x2/2} \int_0^\infty y\, e^{\frac{-y2}{2}}\, dy$$

$$= x\, e^{-x2/2} \qquad (\int_0^\infty y\, e^{\frac{-y2}{2}}\, dy = 1)$$

(b) $f_{Y/X}(y/x) = \frac{fx,y(x,y)}{fx(x)}$

$$= \frac{xy\, e^{\frac{-(x2+y2)}{2}}}{x\, e{-}(x2/2)}$$

$$= y\, e^{-y2/2}$$

17. The two random variables X and Y have a joint CDF.

$$F_{x,y}(x,y) = \begin{cases} (1-\frac{1}{x2})(1-\frac{1}{y2}) & x \geq 1, y \geq 1 \\ 0 & elsewhere \end{cases}$$

(a) Find the marginal CDF of X and Y.
(b) Are X and Y independent?
(c) Find the probability for $\{X \leq 5, Y \leq 5\}$.
(d) Using the results obtained in above parts, find probability for $\{X>3, Y>3\}$

Answer: (a) $F_X(x) = F_{X,Y}(x,\infty)$

$$= (1-1/x^2)$$

(b) $F_Y(y) = F_{X,Y}(\infty, y)$

$= (1-1/y^2)$

(c) To prove random variable X and Y independent we need to show

$F_{X,Y}(x,y) = F_X(x)F_Y(y)$

$F_{X,Y}(x,y) = (1-1/x^2)(1-1/y^2)$

$F_X(x) = (1-1/x^2)$

$F_Y(y) = (1-1/y^2)$

Which is true in this case so X and Y are independent

(d) $P(x \leq 5, y \leq 5) = F_{X,y}(5,5) = (1-1/5^2)(1-1/5^2) = 0.9216$

(e) $P(x>3, y>3) = 1-F_{X,y}(3,3) = 1-(1-1/3^2)(1-1/3^2) = 0.209$

18. Let Y be a random variable having the density function as

$$f(y; y_0, \beta) = \frac{\beta y_0^{\beta}}{y^{(\beta+1)}} \; if \; y > y_0.$$

Where $\beta > 0, y_0 > 0$. If $X = \log\left(\frac{Y}{y_0}\right)$, then find range of X, density function of X i.e., f(x), P(X>3), and P(X=3).

Answer:

(a) We are given that $X = log\left(\frac{Y}{y_0}\right) \Rightarrow Y = y_0 e^X$. Since

$Y > 0 \Rightarrow \; y_0 e^X > 0$. So, X>0.

(b) $F_X(x) = P(X \leq x) = P\left(log\left(\frac{Y}{y_0}\right) \leq x\right) = P(Y \leq y_0 e^X) = F_Y(y_0 e^X)$. Now taking derivative with respect to x on both sides we get

$$f_X(x) = y_0 e^X f_Y(y_0 e^X) \Longrightarrow f_X(x) = \beta e^{-\beta x}.$$

(c) $P(X > 3) = 1 - P(X \leq 3) = 1 - \int_0^3 \beta e^{-\beta x} dx = e^{-3\beta}$

(d) P(X=3) = 0, probability at a single point is zero for continuous distribution.