Distortion in Voting with Top-t Preferences

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Abstract

A fundamental question in social choice and multi-agent systems is aggregating ordinal preferences expressed by agents into a measurably prudent collective choice. A promising line of recent work views ordinal preferences as a proxy for underlying cardinal preferences. It aims to optimize distortion, the worst-case approximation ratio of the (utilitarian) social welfare. When agents rank the set of alternatives, prior work identifies near-optimal voting rules for selecting one or more alternatives. However, ranking all the alternatives is prohibitive when there are many alternatives.

In this work, we consider the setting where each agent ranks only her t favorite alternatives and identify almost tight bounds on the best possible distortion when selecting a single alternative or a committee of alternatives of a given size k. Our results also extend to approximating higher moments of social welfare. Along the way, we close a gap left open in prior work by identifying asymptotically tight distortion bounds for committee selection given full rankings.

1 Introduction

A common task in multi-agent systems is to make collective decisions that serve multiple agents well in a measurable sense, and voting is a frequently-used tool for this purpose [Shoham and Leyton-Brown, 2008; Pitt *et al.*, 2006], in applications such as human computation [Procaccia *et al.*, 2012], distributed sensor networks [Lesser *et al.*, 2003], meeting scheduling [Haynes *et al.*, 1997], planning [Ephrati and Rosenschein, 1997], and rank aggregation for the web [Dwork *et al.*, 2001].

Voting has been studied for centuries in social choice theory, dating back to the early work by Condorcet [1785], in which voters rank candidates, and the goal is to select one or more candidates. But the prominent approach for evaluating the efficacy of voting rules has been the axiomatic approach, which is more qualitative in nature and has resulted in celebrated impossibility results [Arrow, 1951]. Arguably, this has led to a lack of consensus, even among social choice theorists, on which voting rules are the "best". Further, Procaccia [2015] argues that axiomatic analysis is less-suited to multi-agent systems applications, where quantitative goals are often in priority.

A recent wave of interest in voting from computer science has provided a fundamentally new perspective for quantitatively evaluating voting rules. Procaccia and Rosenschein [2006] propose to view the ranked preferences submitted by voters over candidates as proxies for their underlying numerical utility functions. This assumption allows one to focus on a canonical quantitative goal: maximizing the (utilitarian) social welfare [Bentham, 1780]. They propose to judge voting rules by their distortion, the worst-case approximation ratio between the maximum possible social welfare given complete utility functions and the (expected) social welfare achieved by the voting rule given only the partial preference information. Hence, distortion is the "price" of missing information and acts as a yardstick for answering the age-old question: *Which voting rules are the best?*

Boutilier *et al.* [2015] identify a near-optimal randomized voting rule for selecting a single candidate given ranked preferences of voters. Caragiannis *et al.* [2017] extend their analysis to deterministic and randomized rules for selecting a committee of candidates of a given size k. Since then, the distortion literature has proliferated and the idea has been applied to settings even beyond voting; we refer the reader to the recent survey by Anshelevich *et al.* [2021] for a detailed overview of the results.

Of particular interest to us is the observation that once we surmise the existence of underlying utility functions, we do not need to stick with asking voters to rank candidates. As Benade *et al.* [2021] observe, distortion can be used to evaluate and compare different elicitation formats (i.e., ballot designs). Mandal *et al.*; Mandal *et al.* [2019; 2020] stretch this to the extreme, allowing arbitrary elicitation formats and studying the tradeoff between the number of bits they extract from each voter and the distortion they enable. However, this can lead to unintuitive elicitation formats, which may be difficult for humans to answer.

Another line of work focuses on intuitive elicitation formats that are either more expressive than ranked preferences [Amanatidis *et al.*, 2021] or less expressive [Gross *et al.*, 2017; Kempe, 2020b; Halpern and Shah, 2021]. A com-

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mon less expressive format is top-t preferences, where each voter ranks only her t most favorite candidates, instead of ranking all candidates. This is particularly well-suited in applications where we have far too many candidates to choose from [Procaccia *et al.*, 2012]. Kempe [2020b] studies the distortion under this type of preferences in the *metric framework*, where voters have costs rather than utilities. In this paper, our main goal is to study distortion with top-t preferences under the original utilitarian framework.

1.1 Our Results

We consider selecting a committee of a given size k given top-t preferences of the voters over the candidates, under the model in which a voter's utility for a committee is her maximum utility for any candidate in the committee. We consider approximating not only the social welfare but, more generally, the p-th power of the social welfare for $p \geqslant 1$, as advocated by Fain *et al.* [2020]. As described in Section 2, the results of Caragiannis *et al.* [2017] can be used to immediately settle this question for deterministic voting rules; hence, we focus exclusively on randomized rules in this work.

For single-winner selection ($k = 1$), we identify nearly-tight distortion bounds for all $p \geqslant 1$. We show that the best distortion is $O_p\left(\min\left(m, \max\left((\log t \cdot m)^{p/p+1}, ({m}/{t})^p\right)\right)\right)$, and this is tight up to the $\log t$ factor in it. For $p = 1$, we are able to eliminate the $\log t$ factor and prove a tight bound of $\Theta(\max(\sqrt{m}, {m}/{t}))$ using a technique introduced recently [Ebadian *et al.*, 2022].

For committee selection ($k \geqslant 1$) with the first moment ($p = 1$), we are able to extend the aforementioned single-winner selection bound to an upper bound of $O\left(\min\left({m}/{k}, \max\left(\sqrt{m}, {m}/{t}\right)\right)\right)$. For the case of full rankings ($t = m$), this matches the lower bound due to Caragiannis *et al.* [2017] and closes a gap of $m^{1/6}$ in their loose upper bound, giving a tight bound of $\Theta(\min(m/k, \sqrt{m}))$ and resolving the question of optimal distortion bounds for committee selection with ranked preferences. The lower bound of Caragiannis *et al.* [2017] continues to match our upper bound when $k \geqslant \sqrt{m}$ (with arbitrary t) or $k \leqslant \sqrt{m} \leqslant t$. However, when $k, t \leqslant \sqrt{m}$, our lower bound of $\Omega(\max(\sqrt{m}, {m}/{kt}))$ is weaker than our upper bound of $O({m}/{\max(k,t)})$. A visualization of these bounds is presented in the appendix.

Finally, we extend the committee selection ($k \geqslant 1$ bounds to higher moments ($p > 1$), but in this case, we leave open substantial gaps. In the appendix, we also present encouraging preliminary results for single-winner selection with higher moments in the metric framework.

1.2 Related Work

To the best of our knowledge, the only work that comes close to studying distortion under top-t preferences in the utilitarian framework is that of Mandal *et al.* [2019]. They propose a voting rule, PREFTHRESHOLD, which asks voters to report the *set* of their t most preferred candidates along with their approximate utilities for these candidates, by partitioning the utility space into discrete buckets and asking the voters to identify the appropriate buckets. Their distortion bound is only comparable to ours when their rule uses a single bucket, for which their bound is infinite.

In the metric framework, distortion under top-t preferences is better understood. For single-winner selection ($k = 1$) with the first moment ($p = 1$), Kempe; Kempe [2020b; 2020a] proves a lower bound of $(2m - t)/t$ and an asymptotically matching upper bound of $12m/t$. Recently, Anagnostides *et al.* [2021] improve the upper bound to $6m/t$ and show that this can be further improved to $(4m - t)/t$ if a generalization of a combinatorial lemma due to Gkatzelis *et al.* [2020] holds.

While top-t preferences are relatively less explored in the distortion setting, they are very well studied more broadly in voting [Oren *et al.*, 2013; Lee *et al.*, 2014; Lu and Boutilier, 2011; Filmus and Oren, 2014] and in settings beyond voting [Drummond and Boutilier, 2013; Hosseini *et al.*, 2021].

Finally, following Caragiannis *et al.* [2017], we use the model where the utility of a voter for a committee is her maximum utility for any candidate in the committee. This is in the style of common voting rules such as the Chamberlin-Courant rule and the Monroe rule (see [Lang and Xia, 2016] for definitions), which aim to select committees in which every voter has a candidate representing her. An alternative model would be to model the utility of a voter for a committee as the *sum* of her utilities for the candidates in the committee, which has also been considered in the literature [Benade *et al.*, 2021].

2 Preliminaries

For $t \in \mathbb{N}$, define $[t] = \{1, \ldots, t\}$. Let $V = [n]$ be a set of n voters and C be a set of m candidates. We use indices i, j to denote voters and letters a, b, c to denote candidates. A committee is a subset of candidates. In this work, we consider selecting a committee of a given size $k \in [m]$. Let $\mathcal{P}_k(C)$ denote the set of all committees of size k. When $k = 1$, we refer to it as single winner selection.

Voter utilities: Each voter i has a utility $u_i(c) \in \mathbb{R}_{\geqslant 0}$ for every candidate c; we assume the standard normalization that $\sum_{c \in C} u_i(c) = 1$ for every i [Aziz, 2020]. We refer to $u = (u_1, \ldots, u_n)$ as the utility profile. For a committee $X \in \mathcal{P}_k(C)$, we define, with slight abuse of notation, the utility of voter i for X as $u_i(X) = \max_{c \in X} u_i(c)$. This is a standard extension studied in the literature [Caragiannis *et al.*, 2017], whereby a voter cares about having some representative in the committee that they like. The (utilitarian) social welfare of X is then given by $\mathsf{sw}(X, u) = \sum_{i=1}^{n} u_i(X)$; for $X = \{c\}$, we simply write $\mathsf{sw}(c, u)$. When the utilities are clear from context, we may drop them from the notation and simply write $\mathsf{sw}(X)$ or $\mathsf{sw}(c)$.

Preference profile: We do not directly observe voters' underlying utility functions. Instead, we ask each voter i to submit a ranking of her t most preferred candidates, denoted by a one-to-one function $\sigma_i : [t] \to C$ satisfying $u_i(\sigma_i(1)) \geqslant \ldots \geqslant u_i(\sigma_i(t)) \geqslant u_i(c)$ for all $c \in C \setminus \sigma_i([t])$. We allow the voter to break any ties arbitrarily. We refer to $\sigma = (\sigma_1, \ldots, \sigma_n)$ as the preference profile. We use $u \triangleright \sigma$ to denote that preference profile σ is induced from the underlying utility profile u.

Voting rule: A (randomized) voting rule f takes as input a preference profile σ and outputs a distribution $f(\sigma)$ over committees of size k. We say that the voting rule is deterministic if it always returns a distribution with singleton support, in which case we use $f(\sigma)$ to denote the unique committee of size k in the support.

Distortion: An instance I in this model is given by the tuple (V, C, u). When evaluating distortion, we fix the number of candidates m. Let $\mathcal{I}$ denote the set of all instances with m candidates. Fix $p \in \mathbb{N}_{>0}$. Following Fain *et al.* [2017] and Fain *et al.* [2020], the p-th moment distortion of voting rule f on an instance $I = (V, C, u)$ is given by

$$\operatorname{dist}^p(f, I) = \sup_{\sigma : u \triangleright \sigma} \frac{\max_{Y \in \mathcal{P}_k(C)} [\mathsf{sw}(Y, u)]^p}{\mathbb{E}_{X \sim f(\sigma)}[(\mathsf{sw}(X, u))^p]}.$$

The p-th moment distortion of f is obtained by taking the worst case over all instances: $\operatorname{dist}^p(f) = \sup_{I \in \mathcal{I}} \operatorname{dist}^p(f, I)$.

Note that for deterministic rules, since there is no expectation in the denominator, the choice of p does not affect the distortion as it cancels out; hence, analyzing $p = 1$ is sufficient. For $p = 1$, Caragiannis *et al.* [2017] prove that a deterministic rule achieves distortion $1 + m(m-k)/k$ even with $t = 1$, and no deterministic rule can be asymptotically better even when $t = m$. Hence, this provides asymptotically optimal distortion bounds for all values of t, k, and p. Consequently, in this work, we focus exclusively on randomized voting rules.

3 Single Winner Selection

Let us begin by analyzing the distortion for single winner selection given top-t preferences. Given only plurality votes ($t = 1$), it is known that the best possible distortion is m, which can be achieved by selecting a uniformly random candidate (see, e.g. [Mandal *et al.*, 2019, Proposition 1]). On the other hand, given ranked preferences ($t = m$), Boutilier *et al.* [2015] pinpoint the optimal distortion to be between $O(\sqrt{m} \cdot \log^* m)$ and $\Omega(\sqrt{m})$, and Ebadian *et al.* [2022] close this gap to establish a $\Theta(\sqrt{m})$ bound.

In this section, we fill the gap between these two extremes. We show that the optimal distortion for top-t preferences is $\Theta(\max(m/t, \sqrt{m}))$. Hence, it first decreases from m to $\Theta(\sqrt{m})$ as ℓ increases from 1 to $\Theta(\sqrt{m})$, but then remains $\Theta(\sqrt{m})$ as ℓ increases further. In a sense, this shows that after eliciting the top-$\Theta(\sqrt{m})$ preferences of the voters, eliciting the rest of their preference ranking does not significantly help. Our analysis extends to the p-th moment with a logarithmic gap when $p > 1$.

3.1 Upper Bound

For ranked preferences, Boutilier *et al.* [2015] show that a simple rule achieves $O(\sqrt{m \log m})$ distortion, which is only logarithmically worse than the optimal distortion. They define the harmonic score of candidate a as $\mathsf{hsc}(a) = \sum_i {}^1\!/\sigma_i^{-1}(a)$; that is, candidate a gets ${}^1\!/r$ points whenever it appears in the r-th position in a voter's preference ranking. Then, their rule chooses each candidate a with probability $\frac{1}{2} \frac{\mathsf{hsc}(a)}{\sum_b \mathsf{hsc}(b)} + \frac{1}{2} \frac{1}{m}$.

We show that a natural extension of this rule to top-ℓ preferences achieves near-optimal distortion simultaneously for all p. We define the *truncated harmonic score* of candidate a, whereby the candidate still gets ${}^1\!/r$ points whenever it appears in the r-th position for $r \leqslant t$, but gets zero points if it does not appear in the top t positions; that is, $\mathsf{hsc_t}(a) = \sum_{i : a \in \sigma_i([t])} {}^1\!/\sigma_i^{-1}(a)$. Then, our rule, f^h, chooses every candidate a with probability $\frac{1}{2} \frac{\mathsf{hsc_t}(a)}{\sum_b \mathsf{hsc_t}(b)} + \frac{1}{2} \frac{1}{m}$.

Theorem 1. *For all $p \geqslant 1$ and $t \in [m]$, we have that*

$$\operatorname{dist}^p(f^h) \leqslant 2 \min\left(m, \frac{p}{W(p)} \max\left((H_t \cdot m)^{\frac{p}{p+1}}, \left(\frac{m}{t}\right)^p\right)\right).$$

Here, $H_t = \sum_{r=1}^t \frac{1}{r} = \Theta(\log t)$ is the t-th harmonic number and $W(p) = \Theta(\log p)$ is the solution of $W(p)e^{W(p)} = p$.[1]

Proof. Fix an arbitrary instance $I = (V, C, u)$ with top-t preference profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ induced by u. Fix an optimal candidate $a \in \arg\max_{c \in C} \mathsf{sw}(c)$. Let q_c be the probability by which f^h chooses candidate c on this profile.

We will show two separate upper bounds on the welfare approximation ratio $\frac{\mathsf{sw}(a)^p}{\sum_{c \in C} q_c \cdot \mathsf{sw}(c)^p}$; then, taking the minimum of the two ratios yields the bound stated in the theorem.

First, an upper bound of $2m$ follows directly from the fact that $q_a \geqslant {}^1\!/(2m)$. Hence,

$$\frac{\mathsf{sw}(a)^p}{\sum_{c \in C} q_c \cdot \mathsf{sw}(c)^p} \leqslant \frac{\mathsf{sw}(a)^p}{q_a \cdot \mathsf{sw}(a)^p} = \frac{1}{q_a} \leqslant 2m.$$

For the second upper bound, we consider two cases depending on the truncated harmonic score $\mathsf{hsc_t}(a)$ of the optimal candidate a. Fix $\tau = \frac{W(p)}{p} \cdot \left(\frac{H_t}{m^p}\right)^{1/(p+1)}$. We consider $\mathsf{hsc_t}(a) \geqslant n\tau$ and $\mathsf{hsc_t}(a) < n\tau$, and show that the desired upper bound holds in both cases.

Case 1: First, suppose $\mathsf{hsc_t}(a) \geqslant n\tau$. We have that

$$q_a \geqslant \frac{1}{2} \cdot \frac{\mathsf{hsc_t}(a)}{\sum_{c \in C} \mathsf{hsc_t}(c)} = \frac{1}{2} \cdot \frac{\mathsf{hsc_t}(a)}{nH_t} = \frac{\tau}{2H_t}.$$

Hence, by the same argument as above, the welfare approximation ratio is at most

$$\begin{aligned} \frac{1}{q_a} &\leqslant \frac{2H_t}{\tau} = \frac{2p}{W(p)} \cdot (H_t \cdot m)^{\frac{p}{p+1}} \\ &\leqslant \frac{2p}{W(p)} \cdot \max\left((H_t \cdot m)^{\frac{p}{p+1}}, \left(\frac{m}{t}\right)^p\right). \end{aligned}$$

Case 2: Next, suppose $\mathsf{hsc_t}(a) < n\tau$. Note that the utility of voter i for a is at most $1/r$ if the voter ranks a in the r-th position, for some $r \leqslant t$, and at most $1/t$ otherwise. Hence,

$$\mathsf{sw}(a) \leqslant \mathsf{hsc_t}(a) + n/t \leqslant n \cdot (\tau + 1/t).$$

The expected social welfare when picking a uniformly random candidate is n/m, which implies that, by Jensen's inequality, the expected p-th moment of social welfare is at least $(n/m)^p$. Since our rule f^h implements this with probability

[1] This is known as the Lambert W function.

$1/2$, we have $\sum_{c \in C} q_c \cdot \mathsf{sw}(c)^p \geqslant (1/2) \cdot (n/m)^p$. Together, these imply that

$$\begin{aligned}\frac{\mathsf{sw}(a)^p}{\sum_{c\in C} q_c \cdot \mathsf{sw}(c)^p} &\leqslant \frac{n^p \cdot (\tau + 1/t)^p}{(1/2) \cdot (n/m)^p} \\ &= 2 \cdot (m\tau + m/t)^p \\ &= 2 \cdot \left((W(p)/p) \cdot (H_t \cdot m)^{\frac{1}{p+1}} + \frac{m}{t} \right)^p \\ &\leqslant 2 \cdot (1 + W(p)/p)^p \cdot \max\left((H_t \cdot m)^{\frac{1}{p+1}}, \frac{m}{t} \right)^p \\ &\leqslant 2 \cdot e^{W(p)} \cdot \max((H_t \cdot m)^{\frac{p}{p+1}}, (m/t)^p) \\ &= \frac{2p}{W(p)} \cdot \max((H_t \cdot m)^{\frac{p}{p+1}}, (m/t)^p).\end{aligned}$$

Combining this with Case 1 yields the desired bound. $\square$

For $p = 1$, this bound is $O(\max(\sqrt{m \log t}, m/t))$. However, in this special case, we can eliminate the $\sqrt{\log t}$ factor by extending a recent technique due to Ebadian *et al.* [2022]. We defer the proof of this to Section 4 as it will follow from a more general result.

Proposition 1. *For $t \in [m]$ and $p = 1$, there exists a randomized rule whose distortion is $O(\max(\sqrt{m}, m/t))$.*

3.2 Lower Bound

Next, we show that the bound achieved in the previous subsection is tight up to the $(\log t)^{p/(p+1)}$ factor. For $p = 1$, the following lower bound is $\Omega(\max(\sqrt{m}, m/t))$, precisely matching the upper bound from Proposition 1.

Theorem 2. *Fix constant $p \geqslant 1$. Every randomized rule f for selecting a single winner given top-t preferences has*

$$\mathrm{dist}^p(f) = \Omega\left(\min\left(m, \max\left(m^{\frac{p}{p+1}}, \left(\frac{m}{t}\right)^p\right)\right)\right).$$

Proof. Note that we can rewrite the lower bound as $\Omega(\max(m^{p/(p+1)}, \min(m, (m/t)^p)))$. Fix an arbitrary rule f. We will prove two separate bounds: $\mathrm{dist}^p(f) = \Omega(m^{p/(p+1)})$ and $\mathrm{dist}^p(f) = \Omega(\min(m, (m/t)^p))$.

First bound: Since the first bound is independent of t, we only need to show that it holds even if $t = m$. Let $C' \subseteq C$ be a subset of the candidates with $|C'| = m^{p/(p+1)}$.[2] We construct a preference profile σ with $n = m^{p/(p+1)}$ voters, where each voter ranks a unique candidate $c \in C'$ first (we refer to this voter as i_c). The rest of the profile is arbitrarily chosen. Suppose the rule chooses every candidate c with probability q_c. There must exist $a \in C'$ with $q_a \leqslant m^{-p/(p+1)}$.

Next, set the utility profile u such that for voter i_a, we have $u_{i_a}(a) = 1$ and $u_{i_a}(c) = 0$ for all $c \neq a$, whereas for every other voter i, we have $u_i(c) = 1/m$ for all $c \in C$. That is, the voter who ranks a first intensely likes a, while the other voters are indifferent between the candidates. Note that $u \triangleright \sigma$, $\mathsf{sw}(a, u) \geqslant 1$, and $\mathsf{sw}(c, u) \leqslant n/m = m^{-1/(p+1)}$ for all $c \neq a$. Hence, we have

$$\begin{aligned}&\frac{\mathsf{sw}(a,u)^p}{\sum_{c\in C} q_c \cdot \mathsf{sw}(c,u)^p} \\ &\geqslant \frac{\mathsf{sw}(a,u)^p}{m^{-p/(p+1)} \cdot \mathsf{sw}(a,u)^p + \sum_{c \in C\setminus\{a\}} q_c \cdot m^{-p/(p+1)}} \\ &\geqslant \frac{\mathsf{sw}(a,u)^p}{\mathsf{sw}(a,u)^p + 1} \cdot m^{p/(p+1)} \geqslant (1/2) \cdot m^{p/(p+1)},\end{aligned}$$

where the first transition uses the upper bounds on q_a and $\mathsf{sw}(c, u)$ derived earlier, while the second transition uses $\mathsf{sw}(a, u) \geqslant 1$.

Second bound: Construct a preference profile σ with $n = \frac{m!}{(m-t)!}$ voters, where each voter submits a unique permutation of t out of m candidates. Suppose f chooses every candidate c with probability q_c. There must exist $a \in C$ such that $q_a \leqslant 1/m$. Next, we construct a consistent utility profile u (i.e., with $u \triangleright \sigma$) as follows. If voter i ranks a at position $j \leqslant t$, then we set $u_i(c) = 1/j$ for $c \in \sigma_i([j])$ and $u_i(c) = 0$ for all other c. If voter i does not rank a in the top t positions, then we set $u_i(c) = 1/(t+1)$ for $c \in \sigma_i([t]) \cup \{a\}$ and $u_i(c) = 0$ for all other c.

Note that due to the symmetry of the construction, a is ranked first in exactly n/m of the votes. In these votes, voters have utility 1 for a and in all others, voters have utility at least $1/(t+1)$. Hence,

$$\mathsf{sw}(a,u) \geqslant \frac{n}{m} \cdot 1 + \left(n - \frac{n}{m}\right) \cdot \frac{1}{t+1} = \frac{n \cdot (m+t)}{m \cdot (t+1)}. \quad (1)$$

By symmetry, all other candidates have the same welfare, so Equation (1) tells us that for all candidates $c \neq a$,

$$\mathsf{sw}(c,u) = \frac{n - \mathsf{sw}(a)}{m-1} \leqslant \frac{n - \frac{n\cdot(m+t)}{m\cdot(t+1)}}{m-1} = \frac{n \cdot t}{m \cdot (t+1)} \quad (2)$$

Next, using $q_a \leqslant 1/m$, the expected p-th moment of the welfare under f is

$$\begin{aligned}\sum_{c\in C} q_c \cdot \mathsf{sw}(c,u)^p &\leqslant 2 \cdot \max(q_a \cdot \mathsf{sw}(a,u)^p, \sum_{c \neq a} q_c \cdot \mathsf{sw}(c,u)^p) \\ &\leqslant 2 \cdot \max\left(\frac{\mathsf{sw}(a,u)^p}{m}, \left(\frac{n\cdot t}{m\cdot(t+1)}\right)^p\right).\end{aligned}$$

Thus, the approximation ratio is

$$\begin{aligned}\frac{\mathsf{sw}(a,u)^p}{\sum_{c\in C} q_c \cdot \mathsf{sw}(c,u)^p} &\geqslant \frac{\mathsf{sw}(a)^p}{2 \cdot \max\left(\frac{\mathsf{sw}(a)^p}{m}, \left(\frac{nt}{m(t+1)}\right)^p\right)} \\ &\geqslant (1/2) \cdot \min\left(m, (m/t)^p\right),\end{aligned}$$

where the last inequality holds due to Equation (1). $\square$

4 Committee Selection for the First Moment

We now turn our attention to selecting a committee of size k for $k \geqslant 1$ given top-t preferences. In this section, we focus on the first moment ($p = 1$), for which we are able to derive tight distortion bounds. The next section focuses on committee selection with higher moments ($p > 1$), for which our bounds are not tight.

[2] For simplicity, we avoid using floors and ceilings since this does not change the lower bound asymptotically.

4.1 Upper Bound

In order to derive the upper bound, we extend a recent approach introduced by Ebadian *et al.* [2022]. They use this approach to derive an optimal $\Theta(\sqrt{m})$ bound for winner selection ($k = 1$) given full rankings ($t = m$). We extend this approach to all $k, t \in [m]$.

The approach relies on another recent result due to Cheng *et al.* [2020]. They consider randomized committee selection that satisfies a compelling stability/fairness property. For a pair of committees $S, S' \subseteq C$, we say that $S' \succ_i S$ if voter i ranks her most preferred candidate in S' above her most preferred candidate in S. Let $V(S, S') = \{i \in V : S' \succ_i S\}$.

Definition 1 (Stable Lotteries)**.** Fix $\ell \in [m]$. A distribution $\mathcal{S}$ over committees of size ℓ is said to be *stable* if, for every committee S' with $|S'| \leqslant \ell$, we have $E_{S\sim\mathcal{S}}[|V(S, S')|] \leqslant n \cdot |S'|/\ell$.

Note that when a committee S is sampled from a stable lottery $\mathcal{S}$, the fraction of voters preferring any other fixed committee S' over S is bounded, in expectation, by the ratio of the sizes of S' and S. In other words, a small committee cannot be preferred by many voters. It is worth noting that if the property is satisfied for all S' with $|S'| = 1$, then it is satisfied for all S' with $|S'| \leqslant \ell$.

Theorem 3 (Cheng *et al.* [2020])**.** *Given ranked preferences and $\ell \in [m]$, a stable lottery over committees of size ℓ always exists.*

Given ranked preferences, Ebadian *et al.* [2022] show that if $\mathcal{S}$ is a stable lottery over committees of size $\ell = \sqrt{m}$, then picking a candidate uniformly at random from a committee $S \sim \mathcal{S}$ with probability $1/2$ and picking a uniformly random candidate from C with probability $1/2$ yields distortion $O(\sqrt{m})$ for single-winner selection with $p = 1$.

We want to extend this to select a committee of size k given only top-t preferences. Our rule, f^{mix}, is a combination of two rules.

- f^{unif} picks a uniformly random committee U of size k.
- f^{stable} arbitrarily completes the partial preference profile into a ranked preference profile, finds a stable lottery $\mathcal{S}$ committees of size $k\sqrt{m}$, samples $S \sim \mathcal{S}$, and then picks a uniformly random subset $S' \subseteq S$ of size k.

If $k > \sqrt{m}$, f^{mix} applies f^{unif}. Otherwise, it applies f^{stable} with probability $1/2$ and f^{unif} with probability $1/2$.

Note that while Ebadian *et al.* [2022] use a stable lottery over committees of size $\sqrt{m}$ to pick a single candidate, f^{stable} uses a stable lottery over committees of size $k\sqrt{m}$ to pick a committee of size k. While this approach does not work when $k > \sqrt{m}$ (since then $k\sqrt{m} > m$), that case turns out to be rather easy to address. Finally, note that we are able to deal with top-t preferences by arbitrarily extending them to ranked preferences!

Theorem 4. *For all $k, t \in [m]$, we have that*

$$\operatorname{dist}(f^{mix}) \leqslant \min\left(\frac{2m}{k}, 4\max\left(\frac{m}{t}, \sqrt{m}\right)\right).$$

Proof. We prove two separate upper bounds of $2m/k$ and $4\max(m/t, \sqrt{m})$ on $\operatorname{dist}(f^{\text{mix}})$. Fix an arbitrary instance (V, C, u) with top-t preference profile σ induced by u. Let $D = f^{\text{mix}}(\sigma)$ be the distribution return by our rule, and $q_a = \Pr_{S\sim D}[a \in S]$ be the marginal probability of candidate a being included in the chosen committee. Fix an optimal committee $S^* \in \arg\max_{S\in\mathcal{P}_k(C)} \mathsf{sw}(S)$.

First bound: Since f^{mix} executes f^{unif} with probability at least $1/2$, we have that $q_a \geqslant k/(2m)$ for all $a \in C$. Hence, we have $\mathbb{E}_{S\sim D}[\mathsf{sw}(S, u)] \geqslant (k/(2m)) \cdot \mathsf{sw}(S^*)$. Rearranging yields the desired distortion bound.

Second bound: Our desired bound is $4\max(m/t, \sqrt{m})$. We assume $k \leqslant \sqrt{m}$, otherwise $2m/k$ is already a stronger bound. Let $\hat{\sigma}$ denote the arbitrarily completed ranked preference profile, and let $\mathcal{S}$ be the stable lottery computed in f^{stable} for $\hat{\sigma}$. Fix a committee S in the support of $\mathcal{S}$. Let us partition the set of voters V into three:

- $V(S, S^*)$ includes every voter i for whom $S^* \succ_i S$ under $\hat{\sigma}$. From Definition 1, $E_{S\sim\mathcal{S}}[|V(S, S^*)|] \leqslant n/\sqrt{m}$.
- $G(S^*, S)$ includes every voter i for whom $S \succ_i S^*$ and she ranks her favorite candidate from S in the first t positions. This guarantees $u_i(S) \geqslant u_i(S^*)$.
- $N(S^*, S)$ includes every voter i for whom $S \succ_i S^*$ but she ranks her favorite candidate from S after the first t positions. In this case, $u_i(S^*) \leqslant 1/t$.

Now, we have

$$\begin{aligned}
&\mathsf{sw}(S^*, u)\\
&= \sum_{i\in V(S,S^*)} u_i(S^*) + \sum_{i\in N(S,S^*)} u_i(S^*) + \sum_{i\in G(S,S^*)} u_i(S^*)\\
&\leqslant |V(S, S^*)| \cdot 1 + n \cdot (1/t) + \sum_{i\in G(S,S^*)} u_i(S)\\
&\leqslant |V(S, S^*)| + n/t + \mathsf{sw}(S, u).
\end{aligned}$$

Next, we take the expectation over $S \sim \mathcal{S}$.

$$\begin{aligned}
\mathsf{sw}(S^*, u) &\leqslant \frac{n}{\sqrt{m}} + \frac{n}{t} + \mathbb{E}_{S\sim\mathcal{S}}[\mathsf{sw}(S, u)]\\
&\leqslant \frac{2n}{\min(\sqrt{m}, t)} + \mathbb{E}_{S\sim\mathcal{S}}[\mathsf{sw}(S, u)]. \qquad (3)
\end{aligned}$$

Let W_1 be the expected social welfare under f^{unif} and W_2 be the expected social welfare under f^{stable}. The expected social welfare under f^{mix} is $(W_1 + W_2)/2$. We express the RHS in Equation (3) in terms of W_1 and W_2.

First, Caragiannis *et al.* [2017] argue that W_1 is at least n/m. Next, consider $S' \subseteq S$ of size $|S'| = k$ chosen uniformly at random. For each voter i, her most favorite candidate in S is included in S' with probability $|S'|/|S| = 1/\sqrt{m}$. Hence, $\mathbb{E}_{S'}[u_i(S')] \geqslant u_i(S)/\sqrt{m}$. Summing over all voters and taking the expectation over $S \sim \mathcal{S}$, $W_2 = \mathbb{E}_{S,S'}[\mathsf{sw}(S', u)] \geqslant \mathbb{E}_S[\mathsf{sw}(S, u)]/\sqrt{m}$. Hence, $\mathbb{E}_S[\mathsf{sw}(S, u)] \leqslant \sqrt{m} \cdot W_2$.

Plugging these into Equation (3), we get

$$\begin{aligned}
\mathsf{sw}(S^*, u) &\leqslant \frac{2m \cdot W_1}{\min(\sqrt{m}, t)} + \sqrt{m} \cdot W_2\\
&\leqslant 2\max(\sqrt{m}, m/t) \cdot (W_1 + W_2),
\end{aligned}$$

which yields the desired distortion bound of $4\max(\sqrt{m}, m/t)$ upon rearranging. $\square$

4.2 Lower Bound

We now turn our attention to lower bounds. Caragiannis *et al.* [2017] already prove a lower bound of $\Omega(\min(m/k, \sqrt{m}))$ that holds even with fully ranked preferences ($t = m$), which obviously holds for all $t \leqslant m$. This matches the upper bound from Theorem 4 when $k \geqslant \sqrt{m}$ or when $k \leqslant \sqrt{m} \leqslant t$. In the remaining region of $k, t \leqslant \sqrt{m}$, the upper bound from Theorem 4 is $O(\min(m/k, m/t))$; for this case, we are able to establish a weaker lower bound of $\Omega(m/(kt))$. These bounds are illustrated in Figure 1 in the appendix. We do not provide a separate proof of the $\Omega(m/(kt))$ lower bound because it is implied by Theorem 6 in the next section.

Proposition 2. *Every randomized rule f for selecting a committee of size k given top-t preferences has*

$$\mathrm{dist}(f) = \Omega\left(\min\left(\frac{m}{k}, \max\left(\frac{m}{kt}, \sqrt{m}\right)\right)\right).$$

Crucially, note that there is no gap between our upper and lower bounds when $k = O(1)$, $t = O(1)$, $k \geqslant \sqrt{m}$, or $k \leqslant \sqrt{m} \leqslant t$. Particularly, for ranked preferences ($t = m$), we derive a tight distortion bound of $\Theta(\min(m/k, \sqrt{m}))$, which was posed as an open question by Caragiannis *et al.* [2017]. Their upper bound was loose by a factor of $O(m^{1/6})$. While our upper bound in Theorem 4 eliminates this completely using a technique very different from theirs, our upper bound in the next section would show that even their technique can be modified to eliminate this factor up to a logarithmic term.

5 Committee Selection for Higher Moments

Finally, we consider the p-th moment distortion, with $p > 1$, for selecting a committee of size k given top-t preferences. Unfortunately, the ingenious approach of Ebadian *et al.* [2022] to utilize stable lotteries to bound distortion seems to break down for higher moments. The problem is that having a committee sampled from such a lottery well approximate the optimal committee with respect to the p-th moment of the social welfare forces us to use a lottery over committees larger than kt, but this reduces the performance of subsampling of a committee of size k from such large committees, resulting in unappealing distortion bounds.

In contrast, we prove that the approach of Caragiannis *et al.* [2017], which extends the harmonic score based approach of Boutilier *et al.* [2015], continues to work reasonably well for higher moments. In doing so, we identify and improve upon a suboptimal step in their approach. For $p = 1$ and $t = m$, this is what reduces their $m^{1/6}$ gap to a logarithmic gap, as mentioned above.

5.1 Upper Bound

≪**Nisarg 5.1:** ——-POLISHED TILL HERE———≫

Let us define our harmonic score based rule f^{hc} for committee selection. The rule is independent of p. Given a top-t preference profile σ, the rule works as follows. With probability $1/2$, it picks a uniformly random committee of size k. With the remaining probability $1/2$, it does the following. First, it computes the truncated harmonic score $\mathsf{hsc}_t(c)$ of every candidate c, as defined in Section 3. At this point, Caragiannis *et al.* [2017] define a marginal probability $q_a = \alpha \cdot (k/m) + (1-\alpha) \cdot k \cdot \frac{(a)}{\sum_{c \in C}(c)}$, find α such that $q_a \leqslant 1$ for all a, and compute a distribution over committees matching these marginal probabilities (which can be done efficiently using an extension of the Birkhoff-von Neumann theorem due to Budish *et al.* [2013]). Instead, we compute a distribution over committees such that the marginal probability of each candidate a being included is *at least* $q_a = \min(k \cdot \frac{\mathsf{hsc}_t(a)}{\sum_{c \in C} \mathsf{hsc}_t(c)}, 1)$. It can be shown that this is always feasible (indeed, $\sum_{a \in C} q_a \leqslant k$ and $q_a \in [0,1]$ for all a) and efficiently computable. This change in the marginal probabilities allows us to improve upon their bounds.

Theorem 5. *For all $p \geqslant 1$ and $k, t \in [m]$, we have that*

$$\mathrm{dist}^p(f^{hc}) \leqslant 2 \cdot \min\left(\binom{m}{k},\ m \cdot k^{p-2},\right.$$

$$\left. 4^p \cdot \max\left((H_t \cdot m \cdot k^{p-1})^{\frac{p}{p+1}}, \left(\frac{m}{t}\right)^p\right)\right).$$

5.2 Lower Bounds

Next, we establish two lower bounds via different proof methodologies. The first bound is achieved using a more straightforward analysis.

Theorem 6. *Fix constant $p \geqslant 1$. Every randomized rule f for selecting a committee of size k given top-t preferences has*

$$\mathrm{dist}^p(f) = \Omega\left(\min\left(\frac{m}{k}, \left(\frac{m}{kt}\right)^p\right)\right).$$

The next bound requires a more intricate analysis; a proof sketch is presented below.

Theorem 7. *Let $c^* = \frac{p}{\ln 2} \cdot W\left(\frac{(2^{1/p}) \cdot \ln 2 \cdot (k+1)}{p}\right) - 1$ where W is the Lambert W function.* ≪**Daniel 5.2:** Technically W_0, the principle branch, the only real branch of the multi-valued W function that is defined for nonnegative real numbers≫ *For all rules f,* DH 5.2

1. *If $c^* \geqslant 2 \cdot \frac{k^2}{m-k}$,*

$$\mathrm{dist}^p(f) \geqslant 1/4 \cdot \left(e^{p \cdot W\left((2^{1/p}) \cdot \ln 2 \cdot \frac{k+1}{p}\right)}\right).$$

2. *Otherwise,*

$$\mathrm{dist}^p(f) \geqslant \left(\frac{(m-k)(k+1)}{k^2+m-k}\right)^p.$$

Recall that $W(x) \in \Theta(\log(x))$. Hence, as long as k/p is at least some universal constant, then $c^* \in \Theta(p \log(k/p))$. Additionally, as long as $m - k \in \Omega(m)$ (this is super mild, i.e., the proportion of candidates in the committee stays below a constant), the boundary is $\Theta(k^2/m)$. Under mild conditions on k/p, the first bound simplifies to $(k/p)^{\Theta(p)}$. The second bound simplifies to $\Theta\left(\min(m/k, k)\right)^p$. ≪**Daniel 5.3:** Weird things seem to happen if p is much larger than k. Although note that as p approaches infinity, the lower bound becomes $\binom{m}{k}$ which is tight.≫ DH 5.3

We now turn to the proof.

Proof sketch: The proof at a high level works as follows. The profile is very simple, there are $n = m$ voters, each ranking a different candidate first, and the rest of the rankings are arbitrary. Further, the utilities will always be such that there is a single optimal committee S^*. All voters that rank a candidate $a \in S^*$ first have utility 1 for a and 0 for all other candidates; all other voters have utility $1/m$ for all candidates. Hence under these utilities, there is always a committee (S^*) that has p-th moment social welfare $(k + \frac{m-k}{m})^p$. We then use an averaging argument to claim that, no matter what distribution a voting rule chooses on this profile, we can find some S^* so that the voting rule only achieves expected p-th moment social welfare at most $\frac{1}{\binom{m}{k}} \cdot \sum_{h=0}^{k} \binom{k}{h} \cdot \binom{m-k}{k-h} \cdot \left(h + \frac{m-k}{m}\right)^p$. The remainder of the proof is dedicated to bounding the ratio of these two values. Namely, we can use Chernoff bounds on binomial random variables to help upper bound the binomial sum. This finally allows us to make the following claim: For all $c \geqslant 2 \cdot \frac{k^2}{m-k}$, the distortion is at least

$$\frac{1}{2} \cdot \min\left(\left(\frac{k+1}{c+1}\right)^p, 2^c\right).$$

The final part of the proof is optimizing this value of c to get the tightest bound possible for each instance.

6 Discussion

Our work identifies exciting technical open questions. While we identify tight distortion bounds for single-winner selection ($k = 1$), for committee selection ($k > 1$) with the first moment ($p = 1$), there is a gap between our upper bound of $O(m/kt)$ and our lower bound of $O(m/\max(k,t))$ in the case where $k, t \leqslant \sqrt{m}$. We remark that in practice, it is common for k and t to be very small, for which the gap between our upper and lower bounds is also small (see Figure 1 in the appendix). We leave more room for improvement in committee selection with higher moments ($p > 1$). It would be interesting to close these gaps.

In the appendix, we provide encouraging preliminary results for the metric distortion framework. We identify tight distortion bounds for single-winner selection ($k = 1$) with an arbitrary moment p, but our bounds for committee selection are off by a polynomial factor. Closing this gap would also be of immediate interest.

More broadly, an interesting direction for future work is to study distortion under other realistic settings and thrifty elicitation methods. For example, what is the best possible distortion if we have access to the ranked (or top-ℓ) preferences of only a subset of randomly sampled voters? What if these voters are not sampled randomly? What if the preferences of the voters are not worst case, but instead stochastic (and possibly correlated)?

Casting an even broader net, the distortion framework is highly versatile and can shed a new light on quantitatively evaluating the effectiveness of more complex collective decision-making paradigms such as distributed elections [Filos-Ratsikas *et al.*, 2019], participatory budgeting [Benade *et al.*, 2021], and primaries [Borodin *et al.*, 2019]. An exciting direction for the future is to use this framework to analyze real-world decision-making paradigms such as sortition [Flanigan *et al.*, 2021] and liquid democracy [Brill, 2019].

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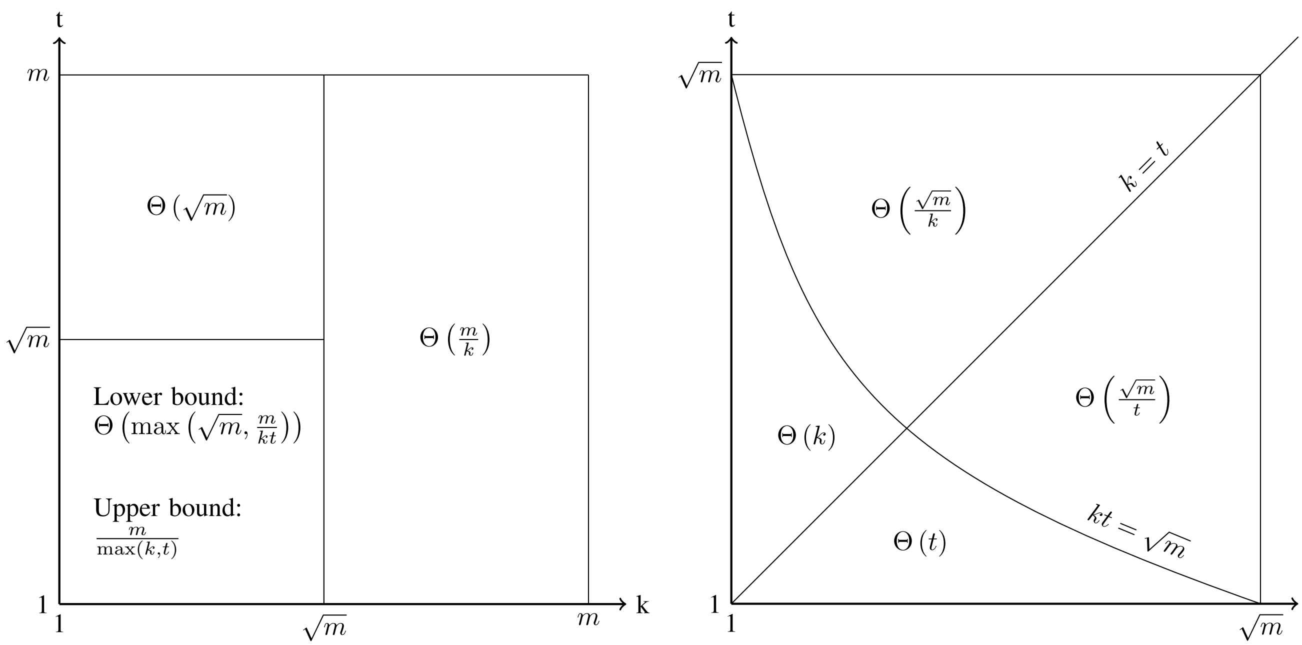
Appendix

A Summary & Visualization of Our Results

Single Winner Distortion	$\frac{1}{2} \cdot \max\left(\sqrt{m}, \frac{m}{t}\right)$	$38 \max\left(\sqrt{m}, \frac{m}{t}\right)$
Single Winner Higher Moment Distortion	$\frac{1}{2} \cdot \max(m^{p/(p+1)}, \min\left(m, \left(\frac{m}{t}\right)^p\right))$	$2 \min\left(m, \frac{p}{W(p)} \cdot \max\left(H_t \cdot m^{p/(p+1)}, \left(\frac{m}{t}\right)^p\right)\right)$
Committee Selection Distortion	$\min\left(\frac{m}{k}, \frac{m(m+t)}{kt(m-k)}\right) \geqslant \frac{m}{kt}$	$\min\left(\frac{2m}{k},\ 8 \max\left(\sqrt{mH_m}, \frac{m}{t}\right)\right)$
Committee Selection Higher Moment Distortion	$\min\left(\frac{m}{k}, \frac{m(m+t)^p}{kt^p(m-k)^p}\right)$	



(a) Upper and lower bounds for different values of k and t.

(b) Gap factor for $t, k \leqslant \sqrt{m}$.

Figure 1: Visualization of the results of committee selection for the first moment.

B Missing Proofs

B.1 Proof of Theorem 5

Proof. Fix an instance $I = (V, C, u)$ with top-t rankings $\sigma = (\sigma_1, \ldots, \sigma_n)$ induced by u. Fix an optimal committee $S^* \in \arg\max_{S \in \mathcal{P}_k(C)} \mathsf{sw}(S)$. Let $D = f^{hc}(\sigma)$ be the chosen distribution and let $p_a = \Pr_{S \sim D}[a \in S]$ be the marginal probability that candidate a is chosen.

The upper bound of $2\binom{m}{k}$ follows because we choose the optimal committee with probability at least $1/2\binom{m}{k}$.

We now show an upper bound of $2 \cdot m \cdot k^{p-2}$. For each $a \in S^*$, let $(N_a)_{a \in s^*}$ be a partition of the voters such that each voter in N_a is receiving maximal utility in S^* from candidate a, that is, for all $i \in N_a$, $u_i(a) = u_i(S^*)$. Let $T_a = \sum_{i \in N_a} u_i(a)$ be the total utility of each of these voters. Note that for a committee $S \subseteq S^*$, $\mathsf{sw}(S) \geqslant \sum_{a \in S \cap S^*} T_a$. Using this, we have that

$$\begin{aligned}
\mathbb{E}_{S\sim D}[\mathsf{sw}(S)^p] &= \sum_{S \in \mathcal{P}_k(C)} \Pr_{S\sim D}[S] \cdot \mathsf{sw}(S)^p \\
&\geqslant \sum_{S \in \mathcal{P}_k(C)} \Pr_{S\sim D}[S] \left(\sum_{a \in S \cap S^*} T_a \right)^p \\
&\geqslant \sum_{S \in \mathcal{P}_k(C)} \Pr_{S\sim D}[S] \cdot \sum_{a \in S \cap S^*} (T_a)^p \\
&= \sum_{a \in S^*} p_a \cdot (T_a)^p \\
&\geqslant \sum_{a \in S^*} 1/2 \cdot k/m \cdot (T_a)^p \\
&= 1/2 \cdot k/m \cdot \sum_{a \in S^*} (T_a)^p \\
&\geqslant 1/2 \cdot k/m \cdot 1/k^{p-1} \cdot \left(\sum_{a \in S^*} T_a \right)^p \\
&= \frac{1}{2 \cdot m \cdot k^{p-2}} \cdot \mathsf{sw}(S^*)^p.
\end{aligned}$$

This implies the $2mk^{p-2}$ distortion.

Next, define

$$\tau = 4 \cdot \max\left(H_t^{\frac{1}{p+1}} k^{\frac{p-1}{p+1}} m^{-\frac{p}{p+1}}, 1/t \right).$$

Case 1: $\mathsf{sw}(S^*) \leqslant n \cdot \tau$. Let D^k be the uniform distribution over committees of size k. Note that since in the Harmonic rule we pick uniformly at random with probability $1/2$,

$$\begin{aligned}
\mathbb{E}_{S\sim D}[\mathsf{sw}(S)^p] &\geqslant 1/2 \cdot \mathbb{E}_{S\sim D^k}[\mathsf{sw}(S)^p] \\
&\geqslant 1/2 \cdot \mathbb{E}_{S\sim D^1}[\mathsf{sw}(S)^p] \\
&\geqslant 1/2 \cdot \mathbb{E}_{S\sim D^1}[\mathsf{sw}(S)]^p \\
&\geqslant 1/2 \cdot (n/m)^p
\end{aligned}$$

where the second to last inequality holds by Jenson's inequality. Hence, the distortion is at most

$$\begin{aligned}
2(m\tau)^p &\leqslant 2 \cdot \left(4 \cdot \max\left(H_t^{\frac{1}{p+1}} k^{\frac{p-1}{p+1}} m^{\frac{1}{p+1}}, m/t \right) \right)^p \\
&= 2 \cdot 4^p \cdot \max\left((mH_t k^{p-1})^{\frac{p}{p+1}}, (m/t)^p \right),
\end{aligned}$$

as needed.

Case 2: Suppose $\mathsf{sw}(S^*) > n \cdot \tau$. We begin by distinguishing between candidates in S^* that have very high score and those that do not. In particular, we partition $S^* = S^L \cup S^H$ where $S^H = \{a \in S^* \mid k \cdot \frac{\mathsf{hsc_t}(a)}{\sum_{c \in C} \mathsf{hsc_t}(a)} \geqslant 1\}$ and $S^L = \{a \in S^* \mid k \cdot \frac{\mathsf{hsc_t}(a)}{\sum_{c \in C} \mathsf{hsc_t}(a)} < 1\}$. This is exactly the threshold for which a candidate will be chosen with probability 1 when we weight by scores. So, for $a \in S^H$, $p_a \geqslant 1/2$, and for $a \in S^L$, $p_a \geqslant 1/2 \cdot \frac{\mathsf{hsc_t}(a)}{\sum_{c \in C} \mathsf{hsc_t}(a)} \geqslant \frac{\mathsf{hsc_t}(a)}{2 \cdot n \cdot H_t}$.

Next, let $H = \{i \in V \mid u_i(S^H) = u_i(S^*)\}$ and let $R = V \setminus H$. In other words, H is the set of voters who have a favorite S^* candidate in S^H (there may be ties for their favorite S^* candidate, but at least one is in S^H). For a set $T \subseteq V$, let $\mathsf{sw}_T(S) = \sum_{i \in T} u_i(S)$ be the social welfare of voters in T. Note that $\mathsf{sw}_H(S^*) + \mathsf{sw}_R(S^*) = \mathsf{sw}(S^*)$. We now split into two subcases depending on whether $\mathsf{sw}_H(S^*) \geqslant \mathsf{sw}(S^*)/2$ or $\mathsf{sw}_R(S^*) \geqslant \mathsf{sw}(S^*)/2$. The first is relatively straightforward.

Subcase 2.1: Suppose $\mathsf{sw}_H(S^*) \geqslant 1/2 \cdot \mathsf{sw}(S^*)$. Note that

$$\mathsf{sw}(S^H) \geqslant \mathsf{sw}_H(S^H) = \mathsf{sw}_H(S^*) \geqslant 1/2 \cdot \mathsf{sw}(S^*).$$

Further, with probability $1/2$, D chooses a committee that contains S^H. Hence, the p'th moment expected social welfare is at least $1/2 \cdot \mathsf{sw}(S^*/2)^p$. Hence, the distortion is at most

$$2^{p+1} \leqslant 2 \cdot 4^p \leqslant 2 \cdot 4^p \cdot \max\left((mH_t k^{p-1})^{\frac{p}{p+1}}, (m/t)^p \right),$$

as needed.

Subcase 2.2: Suppose $\mathsf{sw}_R(S^*) \geqslant \mathsf{sw}(S^*)/2$ (and it is still the case that $\mathsf{sw}(S^*) > n \cdot \tau$). This implies that $\mathsf{sw}_R(S^*) \geqslant \mathsf{sw}(S^*)/2 \geqslant n \cdot \tau/2$. Next, note that all voters $i \in R$ have $u_i(S^H) < u_i(S^*)$. This implies that $u_i(S^L) = u_i(S^*)$, so we have that

$$\mathsf{sw}_R(S^L) = \mathsf{sw}_R(S^*) \geqslant n \cdot \tau/2 \geqslant 2 \cdot n/t.$$

For each $a \in S^L$, let N_a denote the subset of voters in R who rank a in their top-t above all other candidates in S^L, i.e.,

$$N_a = \{i \in R \mid \forall b \in S^L \setminus \{a\}, a \succ_i b\}.$$

Let T_a denote the total utility that voters in N_a have for alternative a, i.e., $T_a = \sum_{i \in N_a} u_i(a)$. For all $a \in A$, we have that $\mathsf{hsc}_\mathsf{t}(a) \geqslant T_a$ because $u_i(a) \leqslant 1/\sigma_i(a)$ and a is in their top t rankings.

Note that although the N_as are disjoint, unlike in the complete ranking case, they do not cover all voters, so do not form a partition. This is because it is possible for a voter in R to rank none of the candidates in S^L in their top t. Let $U = R \setminus \left(\bigcup_{a \in S^L} N_a\right)$ be the uncovered voters in R. We have that $\{U\} \cup \{N_a\}_{a \in S^L}$ do in fact form a partition of R. Further, for each $i \in U$, $u_i(S^L) \leqslant 1/t$ because they do not rank any of the candidates in S^L in their top t. This implies that

$$\mathsf{sw}_U(S^L) \leqslant n/t = 1/2 \cdot (2n/t) \leqslant \mathsf{sw}_R(S^L)/2.$$

Hence, voters in $R \setminus U$ account for more than half of the social welfare of S^L in R, so

$$\sum_{a \in S^L} T_a \geqslant \mathsf{sw}_R(S^L)/2 = \mathsf{sw}_R(S^*)/2 \geqslant \mathsf{sw}(S^*)/4.$$

We now have that

$$\begin{aligned}
\mathbb{E}_{S\sim D}[\mathsf{sw}(S)^p] &\geqslant \mathbb{E}_{S\sim D}\left[\left(\sum_{a \in S^L} \mathsf{sw}_{N_a}(S)\right)^p\right] \\
&\geqslant \mathbb{E}_{S\sim D}\left[\sum_{a \in S^L} \mathsf{sw}_{N_a}(S)^p\right] \\
&\geqslant \sum_{a \in S^L} \mathbb{E}_{S\sim D}[\mathsf{sw}_{N_a}(S)^p] \\
&\geqslant \sum_{a \in S^L} p_a \cdot (T_a)^p \\
&\geqslant \frac{k}{2 \cdot n \cdot H_t} \sum_{a \in S^L} (T_a)^{p+1} \\
&\geqslant \frac{1}{2 \cdot n \cdot H_t \cdot k^{p-1}} \left(\sum_{a \in S^L} T_a\right)^{p+1} \\
&\geqslant \frac{1}{2 \cdot n \cdot H_t \cdot k^{p-1}} (\mathsf{sw}(S^*)/4)^{p+1} \\
&= \frac{\mathsf{sw}(S^*)^{p+1}}{32 \cdot n \cdot H_t \cdot (4k)^{p-1}}.
\end{aligned}$$

Finally, let us consider the ratio

$$\begin{aligned}
\frac{\mathsf{sw}(S^*)^p}{\mathbb{E}_{S\sim D}[\mathsf{sw}(S)^p]} &\leqslant \frac{32 \cdot n \cdot H_t \cdot (4k)^{p-1} \cdot \mathsf{sw}(S^*)^p}{\mathsf{sw}(S^*)^{p+1}} \\
&= \frac{32 \cdot n \cdot H_t \cdot (4k)^{p-1}}{\mathsf{sw}(S^*)} \\
&\leqslant \frac{32 \cdot H_t \cdot (4k)^{p-1}}{\tau}
\end{aligned}$$

as needed. $\square$

B.2 Proof of Theorem 6

Proof. Assume we have $n = \frac{m!}{(m-t)!}$ voters, one for each permutation of t out of m candidates. Fix the voting rule f and let a be the candidate with the minimum probability of being selected in the committee. Let this probability be $p_f(a) \leqslant k/m$. Suppose that a appears in the top $t+1$ preferences of all the voters. Furthermore, voters who rank a as their top choice have utility 1 for a and zero for other candidates, and all other voters have utility $1/(t+1)$ for their top $t+1$ candidates.

In this scenario, the social welfare of a committee S that includes a is:

$$\mathsf{sw}(S) = \frac{1}{m} \cdot 1 + \frac{m-1}{m} \cdot \frac{1}{t+1} = \frac{m+t}{m(t+1)},$$

and the social welfare of a committee S' that does not includes a is at most:

$$\mathsf{sw}(S') \leqslant \frac{kt}{m} \cdot \frac{1}{t+1} = \frac{kt}{m(t+1)}.$$

For the p-th moment distortion of f we have:

$$\begin{aligned}
\operatorname{dist}^p(f) &\geqslant \frac{\mathsf{sw}(S)^p}{\frac{k}{m}\mathsf{sw}(S)^p + \frac{m-k}{m}\mathsf{sw}(S')^p} \\
&\geqslant \frac{\left(\frac{m+t}{m(t+1)}\right)^p}{\frac{k}{m}\left(\frac{m+t}{m(t+1)}\right)^p + \frac{m-k}{m}\left(\frac{kt}{m(t+1)}\right)^p} \\
&\geqslant \frac{m\,(m+t)^p}{k\,(m+t)^p + (m-k)k^p t^p} \\
&\geqslant \min\left(\frac{m}{2k}, \frac{m(m+t)^p}{2k^p t^p}\right).
\end{aligned}$$

□

B.3 Proof of Theorem 7

Proof. Fix a voting rule f. Assume we have $n = m$ voters, one for each candidate. We index these voters by the candidates $\{i_a\}_{a \in C}$. We construct a set of rankings σ as follows. Each voter i_a ranks candidate a first and the rest of the candidates arbitrarily. Let $D = f(\sigma)$ be the distribution over candidates returned by f.

The utilities will be chosen such that there is an optimal committee S^* (that we will later choose depending on D). For all voters i_a with $a \in S^*$, $u_a(a) = 1$ and $u_a(c) = 0$ for all $c \neq a$. For all remaining voters i_a with $a \notin S^*$, $u_a(c) = 1/m$ for all candidates $c \in C$. This allows us to pin down the social welfare for any specific committee S, which depends only on $|S \cap S^*|$. Indeed, we know that the $|S \cap S^*|$ voters $\{i_a \in V \mid a \in S \cap S^*\}$ receive utility 1, the $k - |S \cap S^*|$ voters $\{i_a \in V \mid a \in S^* \setminus S\}$ receive utility 0, and the remaining $m-k$ voters $\{i_a \in V \mid a \notin S^*\}$ receive utility $1/m$. Hence, the p'th moment social welfare of S is $(|S \cap S^*| + (m-k)/m)^p$. In particular, the p'th moment social welfare of the optimal committee S^* is $(k + \frac{m-k}{m})^p$.

We now show how to choose S^*. For each $S \in \mathcal{P}_k(C)$, let

$$g(S) = \sum_{h=0}^{k} \Pr_{S' \sim D}[|S \cap S'| = h] \cdot \left(h + \frac{m-k}{m}\right)^p.$$

Note that $g(S)$ exactly captures the social welfare of D if the optimal committee is S. We now show there is some S for which

g is not too large. This will follow from an averaging argument. We have that

$$
\begin{aligned}
\frac{1}{\binom{m}{k}} \cdot \sum_{S \in \mathcal{P}_k(C)} g(S) &= \frac{1}{\binom{m}{k}} \cdot \sum_{S \in \mathcal{P}_k(C)} \sum_{h=0}^{k} \Pr_{S' \sim D}[|S \cap S'| = h] \cdot \left(h + \frac{m-k}{m}\right)^p \\
&= \frac{1}{\binom{m}{k}} \cdot \sum_{S \in \mathcal{P}_k(C)} \sum_{h=0}^{k} \sum_{S' \in \mathcal{P}_k(C)} \Pr_{S' \sim D}[S'] \cdot \mathbb{I}[|S \cap S'| = h] \cdot \left(h + \frac{m-k}{m}\right)^p \\
&= \frac{1}{\binom{m}{k}} \cdot \sum_{S' \in \mathcal{P}_k(C)} \sum_{h=0}^{k} \sum_{S \in \mathcal{P}_k(C)} \Pr_{S' \sim D}[S'] \cdot \mathbb{I}[|S \cap S'| = h] \cdot \left(h + \frac{m-k}{m}\right)^p \\
&= \frac{1}{\binom{m}{k}} \cdot \sum_{S' \in \mathcal{P}_k(C)} \Pr_{S' \sim D}[S'] \cdot \sum_{h=0}^{k} \left(h + \frac{m-k}{m}\right)^p \cdot \sum_{S \in \mathcal{P}_k(C)} \mathbb{I}[|S \cap S'| = h] \\
&= \frac{1}{\binom{m}{k}} \cdot \sum_{S' \in \mathcal{P}_k(C)} \Pr_{S' \sim D}[S'] \cdot \sum_{h=0}^{k} \left(h + \frac{m-k}{m}\right)^p \cdot |\{S \in \mathcal{P}_k(C) \mid S' \cap S = h\}|.
\end{aligned}
$$

Note that for any fixed $S' \in \mathcal{P}_k(C)$, $|\{S \in \mathcal{P}_k(C) \mid S' \cap S = h\}| = \binom{k}{h} \cdot \binom{m-k}{k-h}$. For our purposes, we will never need this value exactly. We simply need to use the fact that it depends only on h (and m and k which we take as fixed for the instance), and not S'. Let $T_h = \binom{k}{h} \cdot \binom{m-k}{k-h}$. The value T_h can be described in the following way: If you fix a subset set $S^* \subseteq m$ of size k, T_h is the number of sets of subsets of size k that intersect S^* on exactly h elements. The above has shown that

$$
\begin{aligned}
\frac{1}{\binom{m}{k}} \cdot \sum_{S \in \mathcal{P}_k(C)} g(S) &= \frac{1}{\binom{m}{k}} \cdot \sum_{S' \in \mathcal{P}_k(C)} \Pr_{S \sim D}[S'] \cdot \sum_{h=0}^{k} T_h \cdot \left(h + \frac{m-k}{m}\right)^p \\
&= \frac{1}{\binom{m}{k}} \cdot \sum_{h=0}^{k} T_h \cdot \left(h + \frac{m-k}{m}\right)^p \cdot \sum_{S' \in \mathcal{P}_k(C)} \Pr_{S' \sim D}[S'] \\
&= \frac{1}{\binom{m}{k}} \cdot \sum_{h=0}^{k} T_h \cdot \left(h + \frac{m-k}{m}\right)^p
\end{aligned}
$$

Hence, an averaging argument tells us there is a specific S^*, such that

$$
g(S^*) \leqslant \frac{1}{\binom{m}{k}} \cdot \sum_{h=0}^{k} T_h \cdot \left(h + \frac{m-k}{m}\right)^p .
$$

We take this to be our S^*. Note that this implies that

$$
\operatorname{dist}^p(f) \geqslant \frac{(k + \frac{m-k}{m})^p}{g(S^*)} \geqslant \frac{(k + \frac{m-k}{m})^p}{\frac{1}{\binom{m}{k}} \cdot \sum_{h=0}^{k} T_h \cdot \left(h + \frac{m-k}{m}\right)^p}.
$$

The remainder of this proof will lower bound this right hand side.

First, we have that

$$\begin{aligned}
\frac{(k+\frac{m-k}{m})^p}{\frac{1}{\binom{m}{k}}\cdot\sum_{h=0}^{k} T_h\cdot\left(h+\frac{m-k}{m}\right)^p} &= \frac{(k+\frac{m-k}{m})^p}{\sum_{h=0}^{k}\frac{T_h}{\binom{m}{k}}\cdot\left(h+\frac{m-k}{m}\right)^p}\\
&= \frac{1}{\sum_{h=0}^{k}\frac{T_h}{\binom{m}{k}}\cdot\left(\frac{h+\frac{m-k}{m}}{k+\frac{m-k}{m}}\right)^p}\\
&\geqslant \frac{1}{\sum_{h=0}^{k}\frac{T_h}{\binom{m}{k}}\cdot\left(\frac{h+1}{k+1}\right)^p}\\
&= \frac{(k+1)^p}{\sum_{h=0}^{k}\frac{T_h}{\binom{m}{k}}\cdot(h+1)^p}.
\end{aligned}$$

Next, recall that by the definition of T_h, $\sum_{h=0}^{k} T_h = \binom{m}{k}$. Hence, the value $\frac{T_h}{\binom{m}{k}}$ form a pmf of a probability distribution, D'. This distribution D' is the distribution over values $\{0,\cdots,k\}$ where if we fix a subset $S^*\subseteq C$ of size k, $\Pr_{D'}[h]$ is the probability of a subset of size k chosen uniformly at random intersects S^* on exactly h elements. Suppose we bound the tail of D' such that for a value c with $0<c<k$, $\Pr[D'\geqslant c]\leqslant\alpha$. We will then have the following:

$$\begin{aligned}
\frac{(k+1)^p}{\sum_{h=0}^{k}\frac{T_h}{\binom{m}{k}}\cdot(h+1)^p} &= \frac{(k+1)^p}{\sum_{h=0}^{\lceil c\rceil-1}\frac{T_h}{\binom{m}{k}}\cdot(h+1)^p+\sum_{h=\lceil c\rceil}^{k}\frac{T_h}{\binom{m}{k}}\cdot(h+1)^p}\\
&\geqslant \frac{(k+1)^p}{\sum_{h=0}^{\lceil c\rceil-1}\frac{T_h}{\binom{m}{k}}\cdot\lceil c\rceil^p+\sum_{h=\lceil c\rceil}^{k}\frac{T_h}{\binom{m}{k}}\cdot(k+1)^p}\\
&= \frac{(k+1)^p}{(c+1)^p\cdot\sum_{h=0}^{\lceil c\rceil-1}\frac{T_h}{\binom{m}{k}}+(k+1)^p\sum_{h=\lceil c\rceil}^{k}\frac{T_h}{\binom{m}{k}}}\\
&= \frac{(k+1)^p}{(c+1)^p\cdot\Pr[D'<c]+(k+1)^p\Pr[D'\geqslant c]}\\
&\geqslant \frac{(k+1)^p}{(c+1)^p+(k+1)^p\cdot\alpha}\\
&\geqslant \frac{(k+1)^p}{2\max\left((c+1)^p,(k+1)^p\cdot\alpha\right)}\\
&= \frac{1}{2}\cdot\min\left(\left(\frac{k+1}{c+1}\right)^p,\frac{1}{\alpha}\right).
\end{aligned}$$

We now find a relationship between c and α as needed above. The first observation we will use is that D' is stochastically dominated by a Binomial $\left(k,\frac{k}{m-k}\right)$ random variable. This follows from a straightforward coupling argument. The mean of this binomial distribution is $\frac{k^2}{m-k}$. Hence, a Chernoff bound implies that as long as $c\geqslant 2\cdot\frac{k^2}{m-k}$, then

$$\Pr[D'\geqslant c]\leqslant\Pr\left[\text{Binomial}\left(k,\frac{k}{m-k}\right)\geqslant c\right]\leqslant 2^{-c}.$$

This gives us the following condition: for all $c\geqslant 2\cdot\frac{k^2}{m-k}$,

$$\operatorname{dist}^p(f)\geqslant\frac{1}{2}\cdot\min\left(\left(\frac{k+1}{c+1}\right)^p,2^c\right).$$

We now solve for the optimal value of c to maximize this quantity (subject to the constraint). Note that in the min, the left term is decreasing in c and the right term is increasing in c. Further at $c=0$, the left term is larger than the right, and at $c=k$, the right is larger than the left. Hence, if there were no constraint on c, the optimal would occur when these two terms are equal. On the other hand, if this optimal value of c were to be below the constraint $2\cdot\frac{k^2}{m-k}$, then the optimal bound would occur at he boundary $c=2\cdot\frac{k^2}{m-k}$, and the left hand side would be smaller.

We now solve for the optimal c. We set

$$\left(\frac{k+1}{c+1}\right)^p = 2^c.$$

We first rewrite this is

$$\left(\frac{k+1}{c+1}\right)^p = e^{\ln 2 \cdot c}.$$

Next, we take both sides of the equation to the $1/p$ which maintains equality:

$$\frac{k+1}{c+1} = e^{(\ln 2/p)\cdot c}.$$

Next, we do a substitution, allowing $x = -(\ln 2/p)\cdot(c+1)$. In particular, $c+1 = (-p/\ln 2)\cdot x$ and $(\ln 2/p)\cdot c = -x - \ln 2/p$. Plugging this in above yields

$$\frac{k+1}{(-p/\ln 2)\cdot x} = e^{-x-\ln 2/p}$$

Rearranging yields

$$\frac{(2^{1/p})\cdot \ln 2 \cdot (k+1)}{p} = -x \cdot e^{-x}.$$

This implies that

$$-x = W\left(\frac{(2^{1/p})\cdot \ln 2 \cdot (k+1)}{p}\right)$$

where W Lambert W function. Hence, the optimal c occurs at

$$c^* = \frac{p}{\ln 2}\cdot W\left(\frac{(2^{1/p})\cdot \ln 2 \cdot (k+1)}{p}\right) - 1.$$

Hence, if $c^* \geqslant 2 \cdot \frac{k^2}{m}$, our bound becomes

$$1/2 \cdot 2^{c^*} = 1/4 \cdot \left(e^{p\cdot W\left((2^{1/p})\cdot \ln 2 \cdot \frac{k+1}{p}\right)}\right).$$

When $c^* < 2 \cdot \frac{k^2}{m}$, our bound becomes

$$\left(\frac{k+1}{\frac{k^2}{m-k}+1}\right)^p = \left(\frac{(m-k)(k+1)}{k^2+m-k}\right)^p. \qquad \square$$

C Metric Space

Let V^c be the set of voters that have c as their top choice.

Theorem 8. *For any voting rule f we have:*

$$Dist^d(f) \in \Omega\left(\left(\frac{m}{k}\right)^{d-1}\right).$$

Proof. Divide candidates into m/k clusters of size k. For each cluster, nk/m of the voters rank the members of this cluster in their top k choice. For every voting rule, there exists a cluster that with probability at least $1/m$ one of its members is the winner. Let this cluster be $S^*(f)$ and the voters that have these candidates as their top choice be $V^*(f)$.

Consider an instance where member of $S^*(f)$ are located at point x_1, members of $V^*(f)$ are located at point x_2, and all other voters and candidates are located at x_3 where $d(x_1,x_2) = 1-\varepsilon$, $d(x_2,x_3) = 1+\varepsilon$, and $d(x_1,x_3) = 2$.

The social cost of members of $S^*(f)$ is $nk(1-\varepsilon)/m + 2n(m-k)/m$, and the social cost of other candidates is $nk(1+\varepsilon)/m$. When $\varepsilon \to 0$, for the d-th moment distortion of this rule we have:

$$Dist^d(f) = \frac{\frac{k}{m}(\frac{n(2m-k)}{m})^d + \frac{m-k}{m}(\frac{nk}{m})^d}{(\frac{nk}{m})^d} = \frac{k}{m}(\frac{2m-k}{k})^d + \frac{m-k}{m} \in \Omega\left(\left(\frac{m}{k}\right)^{d-1}\right). \quad \square$$

Remark 1. Using the deterministic rule with distortion m/k we can achieve d-th moment distortion of $(m/k)^d$.

Lemma 1. *For each candidate $c \in C$ we have:*

$$SC(c) \geqslant \frac{1}{2}\sum_{c' \in C} |V^{c'}| \cdot d(c, c').$$

Lemma 2. *Considering a voting rule f, for the d-th moment distortion of f we have:*

$$Dist^d(f) \leqslant 2^d + 4^d \max_{c \in C}[p_f(c) \frac{(n - |V^c|)^d}{|V^c|^d}].$$

In other words, if we define $s_f(a)$ to be the maximum probability of f choosing a voter that appears as the top choice of at most a portion of the voters as the winner, we have:

$$Dist^d(f) \leqslant 2^d + 4^d \max_{a}[s_f(a) \frac{(1 - a)^d}{a^d}].$$

Proof. Let c^* be the optimal candidate. We have

$$\begin{aligned}
Dist^d(f) &= \frac{\sum_{c \in C} p_f(c) \left(\sum_{v_i \in V} d(c, v_i)\right)^d}{\left(\sum_{v_i \in V} d(c^*, v_i)\right)^d} \\
&\leqslant \frac{\sum_{c \in C} p_f(c) \left(\sum_{v_i \in V^{c \succ c^*}} d(c^*, v_i) + \sum_{v_i \in V - V^{c \succ c^*}} d(c^*, v_i) + d(c^*, c)\right)^d}{\left(\sum_{v_i \in V} d(c^*, v_i)\right)^d} \\
&\leqslant \frac{\sum_{c \in C} p_f(c) \left(SC(c^*) + (n - |V^c|) d(c^*, c)\right)^d}{\left(\sum_{v_i \in V} d(c^*, v_i)\right)^d} \\
&\leqslant 2^d + 2^d \frac{\sum_{c \in C} p_f(c) (n - |V^c|)^d d(c^*, c)^d}{\left(\frac{1}{2} \sum_{c \in C} |V^c| \cdot d(c^*, c)\right)^d} \\
&\leqslant 2^d + 2^d \frac{\sum_{c \in C} p_f(c) (n - |V^c|)^d d(c^*, c)^d}{\frac{1}{2^d} \sum_{c \in C} |V^c|^d \cdot d(c^*, c)^d} \\
&\leqslant 2^d + 4^d \max_{c \in C} p_f(c) \frac{(n - |V^c|)^d}{|V^c|^d}
\end{aligned}$$

□

Theorem 9. *Proportional to d-th power has $\Theta\left(m^{d-1}\right)$ d-th moment distortion.*

Proof. Consider a candidate that is the top choice of a portion of the voters. The maximum probability for this candidate to win the election is $\frac{a^d}{m(\frac{1}{m})^d}$. By Lemma 2, we have:

$$Dist^d(f) \leqslant 2^d + 4^d \max_{a}[a^d m^{d-1} \frac{(1 - a)^d}{a^d}] \leqslant 2^d + 4^d \cdot m^{d-1}. \quad \square$$