Fair Division with Binary Valuations: One Rule to Rule Them All

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Abstract

We study fair allocation of indivisible goods among agents. Prior research focuses on additive agent preferences, which leads to an impossibility when seeking truthfulness, fairness, and efficiency. We show that when agents have binary additive preferences, a compelling rule — maximum Nash welfare (MNW) — provides all three guarantees.

Specifically, we show that deterministic MNW with lexicographic tie-breaking is group strategyproof in addition to being envy-free up to one good and Pareto optimal. Along the way, we establish several structural properties of deterministic MNW allocations, which may be of independent interest. We also prove that fractional MNW — known to be group strategyproof, envy-free, and Pareto optimal — can be implemented as a distribution over deterministic MNW allocations, which are envy-free up to one good. Our work establishes maximum Nash welfare as the ultimate allocation rule in the realm of binary additive preferences.

1 Introduction

Fair division [Moulin, 2003; Brams and Taylor, 1996] is a sprawling field that cuts across scientific disciplines. Among its many challenges, the division of indivisible goods — an ostensible oxymoron — is arguably the most popular in recent years. The goods are "indivisible" in the sense that each must be allocated in its entirety to a single agent (think of pieces of jewelry or tickets to different football games in a season). Each agent has her own *valuation function*, which represents the benefit the agent derives from bundles of goods.

A fully expressive model of valuation functions would have to account for combinatorial preferences. Classic examples include a right shoe that is worthless without its matching left shoe (complementarities), and two identical refrigerators (substitutes). However, rich preferences can be difficult to elicit. It is often assumed, therefore, that the valuation functions are *additive*, that is, that each agent's value for a bundle of goods is the sum of her values for individual goods in the bundle. Additive valuations strike a balance between expressiveness and ease of elicitation; in particular, each agent need only report her value for each good separately.

Another advantage of additive valuations is that they admit a practical rule that is both (economically) efficient and fair. Specifically, the Maximum Nash Welfare (MNW) solution — which maximizes the product of valuations and, therefore, is obviously Pareto optimal (PO) — is envy-free up to one good (EF1): for any two agents i and j, it is always the case that i prefers her own bundle to that of j, possibly after removing a single good from the latter bundle [Caragiannis *et al.*, 2019].

The MNW solution, however, is not *strategyproof*, that is, agents can benefit by misreporting their preferences. In fact, under additive valuations, the only rule that is Pareto optimal and strategyproof is *serial dictatorship*, which is patently unfair [Klaus and Miyagawa, 2001]. This profound clash between efficiency and truthfulness holds true even when agents can only have three possible values for goods!

The only hope for reconciling efficiency, fairness and truthfulness, therefore, is to assume that agents' values for goods are *binary*. This assumption is not just a theoretical curiosity: while it obviously comes at a significant cost to expressiveness, it leads to extremely simple elicitation. In this sense, it arguably represents another natural point on the conceptual expressiveness-elicitation Pareto frontier. The same bold tradeoff has long been considered sensible in the literature on voting, where binary values are implicitly represented as *approval* votes [Brams and Fishburn, 2007]; in fact, the assumption underlying some of the recent work on approval-based multi-winner elections [Cheng *et al.*, 2019; Lackner and Skowron, 2019] is nothing but that of binary additive valuations. It is not surprising, therefore, that several papers in fair division pay special attention to the case of binary additive valuations [Bouveret and Lemaître, 2016; Darmann and Schauer, 2015; Barman *et al.*, 2018; Aleksandrov *et al.*, 2015; Freeman *et al.*, 2019].

With this rather detailed justification for binary additive valuations in mind, our primary research question is this: do such valuations admit rules that are efficient, fair, and truthful? We provide a positive answer — and then some. Specifically, Theorem 3 asserts that, under binary additive valuations, a particular form of the MNW solution is Pareto optimal, EF1, *group* strategyproof (even a coalition of agents cannot misreport its members' preferences in a way that benefits them all) and polynomial-time computable. Furthermore, we show (Theorem 6) that by randomizing over MNW allocations we can achieve ex ante envy-freeness (each agent's expected value for their random allocation is at least as high as for any other agent's), ex ante Pareto optimality, ex ante group strategyproofness, and ex post EF1 simultaneously in polynomial time. In other words, randomization allows us to circumvent the mild unfairness that is inherent in deterministic allocations of indivisible goods without losing the other guarantees. Along the way, we characterize the MNW solution under binary additive valuations (Theorem 1), and, in particular, establish that it is equivalent to the leximin solution, which lexicographically maximizes the sorted vector of agent values.

In our view, these results are essentially the final word on how to divide indivisible goods under binary additive valuations.

2 Preliminaries

For $k \in \mathbb{N}$, let $[k] = \{1, \ldots, k\}$. Let $\mathcal{N} = [n]$ denote a set of *agents*, and $\mathcal{M}$ denote a set of m indivisible *goods*. Each agent i is endowed with a *valuation* function $v_i : 2^{\mathcal{M}} \to \mathbb{R}_{\geq 0}$ such that $v_i(\emptyset) = 0$. It is assumed that valuations are additive: $\forall T \subseteq \mathcal{M}$, $v_i(T) = \sum_{g \in T} v_i(\{g\})$. To simplify notation, we write $v_i(g)$ instead of $v_i(\{g\})$.

We focus on a subclass of additive valuations known as binary additive valuations, under which $v_i(g) \in \{0, 1\}$ for all $i \in \mathcal{N}$ and $g \in \mathcal{M}$. We say that agent i *likes* good g if $v_i(g) = 1$. Sometimes it is easier to think of the valuation function of agent i as the set of goods that agent i likes, denoted $V_i = \{g \in \mathcal{M} : v_i(g) = 1\}$. Note that $v_i(T) = |V_i \cap T|$ for $T \subseteq \mathcal{M}$. For a set of agents $S \subseteq \mathcal{N}$, let $V_S = \bigcup_{i \in S} V_i$ be the set of goods that at least one agent in S likes. The vector of agent valuations $\mathbf{v} = (v_1, \ldots, v_n)$ is called the *valuation profile*. A problem instance is given by the tuple $(\mathcal{N}, \mathcal{M}, \mathbf{v})$.

For a set of goods $T \subseteq \mathcal{M}$ and $k \in \mathbb{N}$, let $\Pi_k(T)$ denote the set of partitions of T into k bundles. We say that $\mathbf{A} = (A_1, \ldots, A_n)$ is an allocation if $\mathbf{A} \in \Pi_n(T)$ for some $T \subseteq \mathcal{M}$. Here, A_i is the bundle of goods allocated to agent i, and $v_i(A_i)$ is the *utility* to agent i. Let $A_S = \bigcup_{i \in S} A_i$ for $S \subseteq \mathcal{N}$. Let $\mathbb{A} = \bigcup_{T \subseteq \mathcal{M}} \Pi_n(T)$ denote the set of all allocations.

We say that good g is *non-valued* if $v_i(g) = 0$ for all agents i; all remaining goods are *valued*. Let $\mathcal{Z}$ denote the set of non-valued goods. We say that allocation $\mathbf{A}$ is *complete* if it allocates every valued good, i.e., if $\mathcal{M} \setminus \mathcal{Z} \subseteq A_{\mathcal{N}}$. We say that an allocation $\mathbf{A}$ is *minimally complete* if it is complete and does not allocate any non-valued goods, i.e., if $\mathcal{M} \setminus \mathcal{Z} = A_{\mathcal{N}}$.

We are interested in *fair* allocations. One of the most prominent notions of fairness is envy-freeness [Foley, 1967].

Definition 1 (Envy-freeness)**.** An allocation $\mathbf{A}$ is called *envy-free* (EF) if, for all agents $i, j \in \mathcal{N}$, $v_i(A_i) \geq v_i(A_j)$.

Envy-freeness requires that no agent prefer another agent's bundle over her own. This cannot be guaranteed (imagine two agents liking a single good). Prior literature focuses on its relaxations, such as envy-freeness up to one good [Lipton *et al.*, 2004; Budish, 2011], which can be guaranteed.

Definition 2 (Envy-freeness up to one good)**.** An allocation $\mathbf{A}$ is called *envy-free up to one good* (EF1) if, for all agents $i, j \in \mathcal{N}$ such that $A_j \neq \emptyset$, there exists $g \in A_j$ such that $v_i(A_i) \geq v_i(A_j \setminus \{g\})$.

EF1 requires that it should be possible to remove envy between any two agents by removing at most one good from the envied agent's bundle. We remark that there is a stronger fairness notion called envy-freeness up to the least positively valued good (EFX) [Caragiannis *et al.*, 2019], which coincides with EF1 under binary additive valuations.

Another classic desideratum in resource allocation is Pareto optimality, which is a notion of economic efficiency.

Definition 3 (Pareto optimality)**.** An allocation $\mathbf{A}$ is called *Pareto optimal* (PO) if there does not exist an allocation $\mathbf{A}'$ such that for all agents $i \in \mathcal{N}$, $v_i(A'_i) \geq v_i(A_i)$, and at least one inequality is strict.

It is easy to see that with binary additive valuations, Pareto optimality is equivalent to ensuring that each valued good is allocated to one of the agents who likes it, i.e., that the utilitarian social welfare (sum of utilities) is maximized and is equal to the number of valued goods.

3 Maximum Nash Welfare Allocations

In this section, we define maximum Nash welfare allocations as introduced by Caragiannis *et al.* [2019], and show that they admit an elegant characterization in the case of binary additive valuations. In Sections 4 and 5, we establish compelling properties of such allocations.

Definition 4 (Maximum Nash welfare allocation)**.** We say that an allocation $\mathbf{A}$ is a *maximum Nash welfare* (MNW) allocation if, among the set of allocations $\mathbb{A}$, it maximizes the number of agents receiving positive utility and, subject to that, maximizes the product of positive utilities. Formally, let $P(\mathbf{A}) = \{i \in \mathcal{N} : v_i(A_i) > 0\}$ and $\mathcal{F} = \operatorname{argmax}_{\mathbf{A} \in \mathbb{A}} |P(\mathbf{A})|$. Then, $\mathbf{A}$ is an MNW allocation if $\mathbf{A} \in \operatorname{argmax}_{\mathbf{A}' \in \mathcal{F}} \prod_{i \in P(\mathbf{A}')} v_i(A'_i)$.

Even for general additive valuations, maximum Nash welfare allocations satisfy EF1 and PO [Caragiannis *et al.*, 2019]. The concept of leximin allocations is equally classic.

Definition 5 (Leximin allocation)**.** For an allocation $\mathbf{A}$, let its *utility vector* be $(v_1(A_1), \ldots, v_n(A_n))$, and its *utility profile* be the utility vector sorted in a non-descending order. We say that $\mathbf{A}$ is a *leximin allocation* if, among all allocations, it lexicographically maximizes the utility profile, i.e., maximizes the minimum utility, subject to that maximizes the second minimum, and so on. If $\mathbf{A}'$ has a lexicographically greater utility profile than $\mathbf{A}$, we say that $\mathbf{A}'$ leximin-dominates $\mathbf{A}$.

Leximin is a refinement of the traditional Rawlsian fairness, which requires maximization of the minimum utility. Plaut and Roughgarden [2018] and Freeman *et al.* [2019] study leximin allocations (and variants of this definition), and show that they have related fairness properties as well.

Even for fully combinatorial valuations, it is easy to check that both maximum Nash welfare allocations and leximin allocations are not only Pareto optimal, but also satisfy the Pigou-Dalton principle.

Definition 6 (Pigou-Dalton principle)**.** We say that an allocation $\mathbf{A}$ satisfies the *Pigou-Dalton principle* (PDP) if, for all agents $i, j \in \mathcal{N}$ with $v_i(A_i) < v_j(A_j)$, there does not exist another allocation $\mathbf{A}'$ such that $v_k(A_k) = v_k(A'_k)$ for all $k \in$

$\mathcal{N} \setminus \{i, j\}$, $v_i(A_i) + v_j(A_j) = v_i(A'_i) + v_j(A'_j)$, and $v_i(A_i) < \min\{v_i(A'_i), v_j(A'_j)\} \leq \max\{v_i(A'_i), v_j(A'_j)\} < v_j(A_j)$.

Intuitively, this says that it should not be possible to "transfer" some utility from a "richer" agent to a "poorer" agent without changing the utility to any other agent. This is seen as a weak notion of equitability in welfare economics.

For general additive valuations, it is difficult to reconcile these notions. An allocation may be MNW but not leximin, or leximin but not MNW. Or it may satisfy the Pigou-Dalton principle and Pareto optimality, but not be MNW or leximin. Under binary additive valuations, we show that these three coincide. We first observe three useful properties.

Lemma 1. *Any two leximin allocations have the same utility profile. Further, any allocation with this utility profile is a leximin allocation.*

Proof. This is because lexicographic comparison is a total order among utility profiles, and leximin allocations, by definition, are those whose utility profile is its greatest element. □

Lemma 2 (Lemma 21 of Freeman *et al.* [2019])**.** *Under binary additive valuations, any two maximum Nash welfare allocations have the same utility profile. Further, any allocation with this utility profile is a maximum Nash welfare allocation.*

Finally, we define a concept that we will repeatedly use in the paper. The *graph of an allocation* $\mathbf{A}$ is a directed graph $G(\mathbf{A}) = (V, E)$, where V contains a vertex for each agent, and there is a directed edge $(i, j) \in E$ if and only if there is a good in j's bundle that i likes (i.e., $A_j \cap V_i \neq \emptyset$). Given a path $P = (u_1, \ldots, u_k)$ in $G(\mathbf{A})$, let $P(\mathbf{A})$ denote an allocation obtained by transferring a good $g \in A_{u_{\ell+1}} \cap V_{u_\ell}$ from $u_{\ell+1}$ to u_ℓ for each $\ell \in [k-1]$; we refer to this operation as *passing back along P*. We characterize MNW allocations in terms of non-existence of a special path in their graph.

Lemma 3. *Let $\mathbf{A}$ be a Pareto optimal allocation, P be a path from agent i to agent j in $G(\mathbf{A})$, and $\mathbf{A}' = P(\mathbf{A})$ be obtained by passing back along P. Then $v_j(A'_j) = v_j(A_j) - 1$, $v_i(A'_i) = v_i(A_i) + 1$, and $v_k(A'_k) = v_k(A_k)$ for all $k \in \mathcal{N} \setminus \{i, j\}$.*

Proof. Note that if good g is being passed from agent $u_{\ell+1}$ to agent u_ℓ on path P, then by definition u_ℓ must like g. Hence, g is a valued good. Thus, by PO, $u_{\ell+1}$ must like g as well. Thus, each agent on P except i and j loses a good she likes and receives a good she likes, agent j only loses a good she likes, and agent i only receives a good she likes. □

Lemma 4. *A Pareto optimal allocation $\mathbf{A}$ is an MNW allocation if and only if there is no directed path from an agent i to an agent j in $G(\mathbf{A})$ such that $v_j(A_j) > v_i(A_i) + 1$.*

Proof. Lemma 3 of Barman *et al.* [2018] establishes that $\mathbf{A}$ is an MNW allocation if and only if there is no directed path P such that passing back along P strictly increases Nash welfare.[1] Given that $\mathbf{A}$ is PO, Lemma 3 implies that this is equivalent to $(v_j(A_j) - 1) \cdot (v_i(A_i) + 1) > v_j(A_j) \cdot v_i(A_i)$, which is equivalent to $v_j(A_j) > v_i(A_i) + 1$. □

[1] Technically, either more agents receive positive utility, or the product of positive utilities increases.

We are now ready to characterize MNW allocations.

Theorem 1. *Under binary additive valuations, the following are equivalent for an allocation $\mathbf{A}$.*

1. *$\mathbf{A}$ is a maximum Nash welfare allocation.*
2. *$\mathbf{A}$ is a leximin allocation.*
3. *$\mathbf{A}$ is Pareto optimal and satisfies the Pigou-Dalton principle.*

Proof. If $\mathbf{A}$ is not PO, then all three sentences are trivially false, and thus equivalent. Hence, let us assume that $\mathbf{A}$ is PO. We want to show the following cyclic implication: $\mathbf{A}$ is not an MNW allocation $\Rightarrow$ $\mathbf{A}$ violates the PDP $\Rightarrow$ $\mathbf{A}$ is not a leximin allocation $\Rightarrow$ $\mathbf{A}$ is not an MNW allocation.

For the first implication, if $\mathbf{A}$ is not an MNW allocation, then by Lemmas 3 and 4, there exists a path P from an agent i to an agent j such that $v_j(A_j) > v_i(A_i) + 1$, and passing back along P decreases the utility to j by 1, increases the utility to i by 1, and preserves the utility to every other agent. This shows a violation of the PDP under $\mathbf{A}$.

The second implication is trivial because, as we mentioned earlier, every leximin allocation trivially satisfies the PDP.

We have thus established that $\mathbf{A}$ not being an MNW allocation implies $\mathbf{A}$ not being a leximin allocation, i.e., the set of leximin allocations is a subset of the set of MNW allocations. Let U^* denote the unique utility profile of leximin allocations (Lemma 1), so at least one MNW allocation has utility profile U^*. Thus, by Lemma 2, the set of MNW allocations is the set of allocations with utility profile U^*, i.e., the set of leximin allocations, establishing the remaining implication. □

Henceforth, we will use the terms "MNW allocation" and "leximin allocation" interchangeably.

4 Deterministic Setting

In this section, our main goal is to establish the existence of a deterministic rule that is fair, efficient, and truthful under binary additive valuations. Formally, fix the set of agents $\mathcal{N}$ and the set of goods $\mathcal{M}$. A *deterministic rule* f takes a valuation profile $\mathbf{v}$ as input and returns an allocation $\mathbf{A}$. Note that f is not allowed to return ties. We say that f is EF1 (resp. PO) if it always outputs an allocation that is EF1 (resp. PO). The game-theoretic literature offers the following strong desideratum to prevent strategic manipulations by agents.

Definition 7 (Group strategyproofness)**.** A deterministic rule f is called *group strategyproof* (GSP) if there do not exist valuation profiles $\mathbf{v}$ and $\mathbf{v}'$, and a group of agents $C \subseteq \mathcal{N}$ such that $v'_k = v_k$ for all $k \in \mathcal{N} \setminus C$, and $v_j(A'_j) > v_j(A_j)$ for all $j \in C$, where $\mathbf{A} = f(\mathbf{v})$ and $\mathbf{A}' = f(\mathbf{v}')$.

A weaker requirement, which only imposes the above property for groups of size 1 (i.e. prevents manipulations by a single agent) is commonly known as strategyproofness (SP).

We are now ready to define our rule, which chooses a special MNW allocation.

Definition 8 (MNW^{tie})**.** The deterministic rule MNW^{tie} returns an allocation $\mathbf{A}$ such that:

1. $\mathbf{A}$ is a maximum Nash welfare allocation, which lexicographically maximizes the utility vector $(v_1(A_1), \dots, v_n(A_n))$ among all maximum Nash welfare allocations (i.e., among all MNW allocations, it maximizes $v_1(A_1)$, subject to that maximizes $v_2(A_2)$, and so on); and
2. $\mathbf{A}$ is minimally complete (i.e. $\mathcal{M} \setminus \mathcal{Z} = A_{\mathcal{N}}$).

If there are several allocations satisfying both conditions, MNW^{tie} arbitrarily picks one.

First, observe that MNW^{tie} is well-defined, i.e., that the set of allocations satisfying both conditions is non-empty. Indeed, the set of allocations satisfying the first condition is trivially non-empty. And for any allocation in this set, there is a corresponding minimally complete allocation — obtained by throwing away all non-valued goods — which has the same utility vector, and therefore still satisfies the first condition.

Next, we argue that MNW^{tie} can be computed efficiently. The key idea is as follows. Darmann and Schauer [2015] show that an MNW allocation can be computed using min-cost max-flow. By slightly modifying the costs of the network they construct, we can ensure that ties are broken in favor of lower-indexed agents receiving higher utility. Alternatively, Barman *et al.* [2018] provide a different algorithm for computing an MNW allocation, which greedily keeps finding a path as in Lemma 4 and passing back along it to improve Nash welfare. Once their algorithm terminates, we start moving to lexicographically better MNW allocations by finding a path from an agent i to an agent $j > i$ with $v_j(A_j) = v_i(A_i) + 1$ and passing back along it. We defer the proof to the full version due to space constraints.

Theorem 2. *Under binary additive valuations, MNW^{tie} can be computed in polynomial time.*

We are now ready to prove the main result of this section, which establishes the desired properties of MNW^{tie}.

Theorem 3. *Under binary additive valuations, MNW^{tie} is envy-free up to one good, Pareto optimal, and group strategyproof.*

Proof. Caragiannis *et al.* [2019] already establish that all MNW allocations are EF1 and PO, even for general additive valuations. Hence, MNW^{tie} is also trivially EF1 and PO. We now establish that it is GSP. Note that this holds regardless of how ties are broken among allocations satisfying the two conditions in the definition of MNW^{tie}.

First, notice that if $\mathbf{A} = MNW^{\text{tie}}(\mathbf{v})$, then $\mathbf{A}$ is minimally complete and PO. Hence, if agent i receives good g, she must like it. In other words, $A_i \subseteq V_i$, and thus, $v_i(A_i) = |A_i|$ for each agent $i \in \mathcal{N}$. Consequently, $A_U \subseteq V_U$ for every subset of agents $U \subseteq \mathcal{N}$. We will use this observation repeatedly.

Next, for an allocation $\mathbf{A}$ and agent $i \in \mathcal{N}$, define $L^i_{\mathbf{A}} = \{j \in \mathcal{N} \mid v_j(A_j) < v_i(A_i)\}$ to be the set of agents who have strictly less utility than agent i, and define $S^i_{\mathbf{A}}$ to be the set of agents reachable from $L^i_{\mathbf{A}} \cup \{i\}$ in $G(\mathbf{A})$. The following lemma shows that agents in $S^i_{\mathbf{A}}$ must collectively receive all the goods that they like.

Lemma 5. *If $\mathbf{A} = MNW^{tie}(\mathbf{v})$, then for each agent $i \in \mathcal{N}$, we have $A_{S^i_{\mathbf{A}}} = V_{S^i_{\mathbf{A}}}$.*

Proof. We have already established that $A_{S^i_{\mathbf{A}}} \subseteq V_{S^i_{\mathbf{A}}}$. Suppose for contradiction that there exists a good $g \in V_{S^i_{\mathbf{A}}} \setminus A_{S^i_{\mathbf{A}}}$. Then, by the construction of $G(\mathbf{A})$, there would have been an edge from an agent in $S^i_{\mathbf{A}}$ who likes g to an agent outside of $S^i_{\mathbf{A}}$ who is allocated g under $\mathbf{A}$ (note that g is valued, so it must be allocated under $\mathbf{A}$). However, the definition of $S^i_{\mathbf{A}}$ implies that it cannot have any outgoing edges, otherwise the set of agents reachable from $L^i_{\mathbf{A}} \cup \{i\}$ could be expanded. Hence, we have $A_{S^i_{\mathbf{A}}} = V_{S^i_{\mathbf{A}}}$. ☐

Next, we show that even though $S^i_{\mathbf{A}}$ contains all agents reachable from $L^i_{\mathbf{A}} \cup \{i\}$, an agent in $S^i_{\mathbf{A}}$ cannot have much higher utility than agent i does. The proof is straightforward, but deferred to the full version due to space constraints.

Lemma 6. *If $\mathbf{A} = MNW^{tie}(\mathbf{v})$, then for each agent $i \in \mathcal{N}$ and each agent $j \in S^i_{\mathbf{A}}$, we have that $v_j(A_j) \leq v_i(A_i) + 1$, and if $j \geq i$, then $v_j(A_j) \leq v_i(A_i)$.*

We are now ready to show that MNW^{tie} is GSP. Suppose for contradiction that there exist valuation profiles $\mathbf{v}$ and $\mathbf{v}'$, and a set of agents $C \subseteq \mathcal{N}$ such that $v_t = v'_t$ for all $t \notin C$ and $v_j(A^{\text{lie}}_j) > v_j(A^{\text{truth}}_j)$ for each $j \in C$, where $\mathbf{A}^{\text{truth}} = MNW^{\text{tie}}(\mathbf{v})$ and $\mathbf{A}^{\text{lie}} = MNW^{\text{tie}}(\mathbf{v}')$.

Let $i = \min\left[\operatorname{argmin}_{t \in C} v_t(A^{\text{truth}}_t)\right]$ be the agent in C who has the lowest index among all agents in C having the minimum utility under honest reporting. For simplicity, let us denote $S = S^i_{\mathbf{A}^{\text{truth}}}$. We have that for every $j \in C$, $|V_j \cap A^{\text{truth}}_j| < |V_j \cap A^{\text{lie}}_j|$. Further, since $A^{\text{truth}}_j \subseteq V_j$, this simplifies to $|A^{\text{truth}}_j| < |V_j \cap A^{\text{lie}}_j|$. When $j \in S \cap C$, we get $|A^{\text{truth}}_j| < |V_j \cap A^{\text{lie}}_j| \leq |V_S \cap A^{\text{lie}}_j|$ because $V_j \subseteq V_S$.

Let $R \subseteq S$ be the set of agents in S from which some agent in C is reachable in $G(\mathbf{A}^{\text{lie}})$. We now establish that some non-manipulating agent in R must receive strictly fewer goods under $\mathbf{A}^{\text{lie}}$ than under $\mathbf{A}^{\text{truth}}$.

Lemma 7. *There exists $j^* \in R \setminus C$ with $|A^{lie}_j| < |A^{truth}_j|$.*

Proof. Suppose for a contradiction that for all $j \in R \setminus C$, $|A^{\text{lie}}_j| \geq |A^{\text{truth}}_j|$. Take a $j \in R \setminus C$. Since $j \notin C$, she reports $v'_j = v_j$. Hence, we have $A^{\text{lie}}_j \subseteq V'_j = V_j$. Further, since $j \in R \subseteq S$, we have $V_j \subseteq V_S$ by definition. We conclude that for each $j \in R \setminus C$, $A^{\text{lie}}_j \subseteq V_S$, so $|A^{\text{lie}}_j \cap V_S| = |A^{\text{lie}}_j| \geq |A^{\text{truth}}_j|$.

Additionally, for each $j \in R \cap C \subseteq C$, we have that $|A^{\text{lie}}_j \cap V_S| \geq |A^{\text{lie}}_j \cap V_j| > |A^{\text{truth}}_j|$. Since bundles of an allocation are disjoint, we can add these inequalities over all $j \in (R \setminus C) \cup (R \cap C) = R$ to get $|A^{\text{lie}}_R \cap V_S| > |A^{\text{truth}}_R|$. The inequality is strict because $R \cap C \neq \emptyset$ as $i \in R \cap C$ by definition. Now, recall that by Lemma 5, $A^{\text{truth}}_S = V_S$. Hence, this becomes $|A^{\text{lie}}_R \cap A^{\text{truth}}_S| > |A^{\text{truth}}_R|$.

This implies that there must exist a good g that is in both A^{lie}_R and A^{truth}_S but not in A^{truth}_R. Therefore, there exist agents $t \in R$ and $k \in S \setminus R$ such that $g \in A^{\text{lie}}_t$ and $g \in A^{\text{truth}}_k$. The latter implies $v_k(g) = 1$ due to Pareto optimality of $\mathbf{A}^{\text{truth}}$.

Since $k \notin R$, by definition k does not have a path to an agent in C under $G(\mathbf{A}^{\text{lie}})$. This trivially implies $k \notin C$ since every vertex is reachable from itself. Since only members of

C changed their reported valuations, $v'_k(g) = v_k(g) = 1$. It follows that there must be an edge from agent k to agent t in $G(\mathbf{A}^{\text{lie}})$. Thus, all vertices reachable from t are also reachable from k. But then, $t \in R$ implies $k \in R$, which is a contradiction. ☐

Consider an agent $j^* \in R \setminus C$ as per Lemma 7. Since $j^* \in R$, there must exist a path P from j^* to some agent $k \in C$. Let $\mathbf{A}'$ denote the allocation obtained by passing back along path P. We show that $\mathbf{A}'$ must be preferred to $\mathbf{A}^{\text{lie}}$ by MNW^{tie} given valuation profile $\mathbf{v}'$, contradicting the fact that $MNW^{\text{tie}}(\mathbf{v}') = \mathbf{A}^{\text{lie}}$.

Note that since $\mathbf{A}^{\text{lie}}$ is PO under valuation profile $\mathbf{v}'$, when constructing $\mathbf{A}'$ from $\mathbf{A}^{\text{lie}}$, we get $v'_t(A'_t) = v'_t(A^{\text{lie}}_t)$ for all $t \neq j^*, k$, $v'_k(A'_k) = v'_k(A^{\text{lie}}_k) - 1$, and $v_{j^*}(A'_{j^*}) = v_{j^*}(A^{\text{lie}}_{j^*}) + 1$ due to Lemma 3; recall that $j^* \notin C$, so $v_{j^*} = v'_{j^*}$. Further, the set of goods allocated does not change. Hence, $\mathbf{A}'$ remains minimally complete.

If $v_{j^*}(A^{\text{lie}}_{j^*}) + 2 \leq v'_k(A^{\text{lie}}_k)$, then it can be checked that $\mathbf{A}^{\text{lie}}$ violates the Pigou-Dalton principle due to the existence of $\mathbf{A}'$, which, given Theorem 1, contradicts the fact that $\mathbf{A}^{\text{lie}}$ is an MNW allocation. Hence, we must have

$$\begin{aligned} v_{j^*}(A^{\text{lie}}_{j^*}) + 1 &\geq v'_k(A^{\text{lie}}_k) = |A^{\text{lie}}_k| \geq v_k(A^{\text{lie}}_k) \\ &\geq v_k(A^{\text{truth}}_k) + 1 \geq v_i(A^{\text{truth}}_i) + 1, \end{aligned} \tag{1}$$

where the second transition is because $\mathbf{A}^{\text{lie}}$ is minimally complete and PO under valuation profile $\mathbf{v}'$, the fourth transition is because $k \in C$, and the last transition is due to the choice of i. On the other hand, we also have

$$v_{j^*}(A^{\text{lie}}_{j^*}) + 1 \leq v_{j^*}(A^{\text{truth}}_{j^*}) \leq v_i(A^{\text{truth}}_i) + 1, \tag{2}$$

where the first transition holds because, due to Lemma 7, $v_{j^*}(A^{\text{lie}}_{j^*}) \leq |A^{\text{lie}}_{j^*}| < |A^{\text{truth}}_{j^*}| = v_{j^*}(A^{\text{truth}}_{j^*})$, and the second transition holds due to Lemma 6 and the fact that $j^* \in R \subseteq S$.

Putting Equations (1) and (2) together, we have

$$\begin{aligned} v_{j^*}(A^{\text{lie}}_{j^*}) + 1 &= v_{j^*}(A^{\text{truth}}_{j^*}) = v_i(A^{\text{truth}}_i) + 1 \\ &= v_k(A^{\text{truth}}_k) + 1 = v'_k(A^{\text{lie}}_k). \end{aligned}$$

By the second equality and Lemma 6, we must have $j^* < i$. By the third equality, the fact that k and i have the same utility under $\mathbf{A}^{\text{truth}}$, and the definition of i, we have that $k \geq i$. Therefore, $k > j^*$. Then, as argued in the proof of Lemma 6, under the valuation profile $\mathbf{v}'$, $\mathbf{A}'$ has the same utility profile as $\mathbf{A}^{\text{lie}}$, and thus, by Lemma 2, it is an MNW allocation. Further, it is lexicographically better than $\mathbf{A}^{\text{lie}}$ under $\mathbf{v}'$, which contradicts the fact that $\mathbf{A}^{\text{lie}} = MNW^{\text{tie}}(\mathbf{v}')$. ☐

5 Randomized Setting

In the previous section, we established the existence of a deterministic rule which is EF1, PO, and GSP. For deterministic rules, it is necessary to relax EF to EF1. For example, in case of a single good that is liked by two agents, giving it to either agent would be EF1 but not EF. However, if one is willing to randomize, the natural solution of assigning the good to an agent chosen at random would be "ex ante EF", in addition to being "ex post EF1" because each deterministic allocation in the support is EF1, but in expectation, no agent envies the other. This leads to some natural questions. *Can randomness help us achieve ex ante EF, without losing ex post EF1? Can we achieve them in addition to PO and GSP?* In this section, we answer these questions affirmatively.

Let us first formally extend our framework to include randomness. A *fractional allocation* $\mathbf{A} = (A_1, \ldots, A_n)$ is such that $A_i(g) \in [0, 1]$ denotes the fraction of good g allocated to agent i and $\sum_{i \in \mathcal{N}} A_i(g) \leq 1$ for each good g. A *randomized allocation* $\overline{\mathbf{A}}$ is a probability distribution over deterministic allocations. There is a natural fractional allocation $\mathbf{A}$ associated with each randomized allocation $\overline{\mathbf{A}}$, where $A_i(g)$ is the probability of good g being allocated to agent i under $\overline{\mathbf{A}}$. In this case, we say that the randomized allocation *implements* the fractional allocation. There may be several randomized allocations implementing a given fractional allocation. A randomized rule takes a valuation profile as input and returns a randomized allocation.

We refer to the expected utility of agent i under a randomized allocation $\overline{\mathbf{A}}$ as simply the utility of agent i under $\overline{\mathbf{A}}$. For consistency, define the utility of agent i from the corresponding fractional allocation $\mathbf{A}$ as $v_i(A_i) = \sum_{g \in \mathcal{M}} A_i(g) \cdot v_i(g)$. With this utility definition, notions of ex ante envy-freeness, ex ante Pareto optimality, and ex ante group strategyproofness carry over directly to randomized allocations/rules. However, since an agent's utility under a randomized allocation is fully determined by the fractional allocation it implements, when talking about these ex ante guarantees, we will sometimes think of a randomized rule as simply returning a fractional allocation without referring to a specific randomized allocation that implements it. One notion that would require specifying the exact randomized allocation used is ex post EF1.

Definition 9 (Ex post EF1)**.** We say that a randomized allocation $\overline{\mathbf{A}}$ is *ex post envy-free up to one good* if each deterministic allocation in its support is EF1. A randomized rule is ex post EF1 if it always returns an ex post EF1 allocation.

Fractional leximin allocations, like their deterministic counterpart, lexicographically maximize the utility profile among all fractional allocations. The same can be said about fractional MNW allocations; however, we can skip the first step of maximizing the number of agents who receive positive utility because in the fractional case we can simultaneously give positive utility to every agent who likes at least one good (and thus can possibly get positive utility).

Definition 10 (Fractional MNW allocations)**.** We say that a fractional allocation is a fractional maximum Nash welfare allocation if it maximizes the product of utilities of agents who do not have zero value for every good.

Bogomolnaia and Moulin [2004], Bogomolnaia *et al.* [2005], and Kurokawa *et al.* [2018] study fractional leximin allocations under an assignment setting, and establish several desirable properties. In addition, fractional MNW allocations, also known as competitive equilibria with equal incomes (CEEI), are widely studied in fair division with additive valuations [Varian, 1974; Orlin, 2010; Cole *et al.*, 2013; Cole and Gkatzelis, 2018]. Our first result shows that under binary additive valuations, these two fundamental concepts coincide. The proof is deferred to the full version.

Theorem 4. *Under binary additive valuations, the set of fractional leximin allocations coincides with the set of fractional maximum Nash welfare allocations. Further, all such allocations have identical utility vectors.*

Note that the identical utility vector guarantee in Theorem 4 is much stronger than the identical utility profile guarantee in the deterministic case (Lemmas 1 and 2).

Even under general additive valuations, it is known that every fractional MNW allocation is ex ante EF and ex ante PO [Varian, 1974], and one such allocation can be computed in strongly polynomial time [Orlin, 2010; Végh, 2013]. Hence, these properties carry over to our binary additive valuations domain, and due to Theorem 4, also apply to fractional leximin allocations.

For ex ante GSP, we build on the literature on fractional leximin allocations. Kurokawa *et al.* [2018] show that returning a fractional leximin allocation satisfies ex ante EF, ex ante PO, and ex ante GSP whenever four key requirements are satisfied. We describe them in the full version, and show that they are easily satisfied under binary additive valuations, if we return a minimally complete leximin allocation. Hence, we define our fractional leximin/MNW rule to always return a minimally complete fractional leximin/MNW allocation (like our deterministic rule MNW^{tie}). The proof of the next result is straightforward, but deferred to the full version.

Definition 11 (Fractional maximum Nash welfare rule)**.** The fractional maximum Nash welfare rule returns a minimally complete fractional maximum Nash welfare allocation.

Theorem 5. *Under binary additive valuations, every fractional maximum Nash welfare (equivalently, leximin) allocation is ex ante envy-free and ex ante Pareto optimal. Further, the fractional maximum Nash welfare rule is ex ante group strategyproof.*

The only missing property at this point is ex post EF1. Therefore, the main question we seek to answer in this section is the following: Can every fractional MNW allocation be implemented as a distribution over deterministic EF1 allocations? We go one step further and show that it can in fact be implemented using deterministic MNW allocations, which are in turn EF1. Our main tool is the bihierarchy framework introduced by Budish *et al.* [2013]. At a high level, the framework allows implementing any fractional allocation $\mathbf{A}$ using deterministic allocations which satisfy a set of constraints, as long as the set of constraints forms a bihierarchy structure and the fractional allocation itself satisfies those constraints.

In our case, we start with a minimally complete fractional MNW allocation $\mathbf{A}^*$. Let u_i^* denote the utility to agent i under this allocation. We impose the following constraints on a deterministic allocation $\mathbf{A}$ used in the support, where $\mathbf{A}$ is represented as a matrix in which $A_i(g) \in \{0,1\}$ indicates whether good g is allocated to agent i.

$$\begin{aligned} &\mathcal{H}_1 : \sum_{i \in \mathcal{N}} A_i(g) = \sum_{i \in \mathcal{N}} A_i^*(g), \forall g \in \mathcal{M}, \\ &\mathcal{H}_2 : \lfloor u_i^* \rfloor \leq \sum_{g \in \mathcal{M}} A_i(g) \cdot v_i(g) \leq \lceil u_i^* \rceil, \forall i \in \mathcal{N}. \end{aligned} \tag{3}$$

Thus, under each deterministic allocation $\mathbf{A}$, the set of goods allocated matches that under $\mathbf{A}^*$ by the first set of constraints, and, crucially, each agent has utility that is either the floor or the ceiling of her utility under $\mathbf{A}^*$. That is, $\mathbf{A}$ is not allowed to stray far from $\mathbf{A}^*$.

It can be checked that these constraints form a bihierarchy (each of $\mathcal{H}_1$ and $\mathcal{H}_2$ is a hierarchy). Hence, the polynomial-time algorithm of Budish *et al.* [2013] computes a random allocation such that (a) it implements the fractional allocation $\mathbf{A}^*$, and (b) each deterministic allocation $\mathbf{A}$ in its support satisfies the constraints in Equation (3). We show that in this case, every deterministic allocation in the support must be a deterministic MNW allocation, yielding the desired result. The proof is deferred to the full version of the paper.

Theorem 6. *Under binary additive valuations, given any fractional maximum Nash welfare allocation, one can compute, in polynomial time, a randomized allocation which implements it and has only deterministic maximum Nash welfare allocations in its support.*

Let us amend the definition of the fractional MNW rule so that it uses Theorem 6 to implement a minimally complete fractional MNW allocation. Then, we have the following.

Corollary 1. *Under binary additive valuations, the fractional maximum Nash welfare rule is ex ante envy-free, ex ante Pareto optimal, ex ante group strategyproof, ex post envy-free up to one good, and polynomial-time computable.*

6 Discussion

To recap, we showed that under binary additive valuations a deterministic variant of the maximum Nash welfare rule is envy-free up to one good (EF1), Pareto optimal (PO), and group strategyproof (GSP). We also demonstrated that its randomized variant is ex ante EF, ex ante PO, ex ante GSP, and ex post EF1. All our rules are polynomial-time computable.

Amanatidis *et al.* [2017] show that under general additive valuations, there is no deterministic rule that is envy-free up to one good (EF1) and strategyproof, even with two agents and $m \geq 5$ goods. At first glance, Theorem 3, which establishes MNW^{tie} as both GSP and EF1, seems to show that this impossibility result does not hold for the special case of binary additive valuations. However, the impossibility result of Amanatidis *et al.* [2017] only applies to rules that allocate all the goods; by contrast, MNW^{tie} does not allocate non-valued goods. This begs the following question: Under binary additive valuations, is there a deterministic rule that allocates all the goods and achieves EF1, PO, and GSP? In the full version of the paper, we show that this cannot be achieved by any variant of MNW.

Another open question is whether the ex ante GSP guarantee of Corollary 1 can be strengthened to ex post GSP, which would require the randomized rule to be implementable as a probability distribution over deterministic GSP rules.

Modulo these minor caveats, though, our results are the strongest one could possibly hope for in the domain of binary additive valuations.

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