Binary Fair Division

Daniel Halpern[1] , **Ariel Procaccia**[2] , **Alexandros Psomas**[3] and **Nisarg Shah**[1]
[1]University of Toronto
[2]Carnegie Mellon University
[3]Simons Institute for the Theory of Computing
daniel.halpern@mail.utoronto.ca, arielpro@cs.cmu.edu, alexpsomi@cs.berkeley.edu, nisarg@cs.toronto.edu

1 Introduction

2 Preliminaries

For $k \in \mathbb{N}$, let $[k] = \{1, \ldots, k\}$. Let $\mathcal{N} = [n]$ denote the set of *agents*, and let $\mathcal{M}$ denote the set of m indivisible *goods*. Each agent i is endowed with a *valuation* function $v_i : 2^{\mathcal{M}} \to \mathbb{R}_{\geq 0}$ such that $v_i(\emptyset) = 0$. It is assumed that valuations are additive: $\forall S \subseteq \mathcal{M}$, $v_i(S) = \sum_{g \in S} v_i(\{g\})$. To simplify notation, we write $v_i(g)$ instead of $v_i(\{g\})$. In addition, we assume the valuations are binary: $\forall g \in \mathcal{M}$, $v_i(g) \in \{0, 1\}$. We'll say an agent *likes* a good g if $v_i(g) = 1$. Sometimes it is easier to think of a valuation functions as a set $V_i \subseteq \mathcal{M}$, the set of goods that the agent likes. In this case, $v_i(A) = |V_i \cap A|$ For a set of agents $S \subseteq \mathcal{N}$, let $V_S := \bigcup_{i \in S} V_i$ be the set of goods that at least one agent in S likes. We denote the vector of valuations by $\mathbf{v} = (v_1, \ldots, v_n)$. We define an *allocation problem* to be the tuple $\mathcal{A} = (\mathcal{N}, \mathcal{M}, \mathbf{v})$.

For a set of goods $M \subseteq \mathcal{M}$ and $k \in \mathbb{N}$, let $\Pi_k(M)$ denote the partitions of M into k bundles. Given an allocation problem $\mathcal{A}$, an allocation $\mathbf{A} = (A_1, \ldots, A_n) \in \Pi_n(M)$ for some $M \subseteq \mathcal{M}$ is a partition of some subset of the goods into n bundles, where A_i is the bundle allocated to agent i. For a set of agents $S \subseteq \mathcal{N}$, let $A_S := \bigcup_{i \in S} A_i$ be the union of their allocations. Under this allocation, the *utility* to agent i is $v_i(A_i)$. Let $\mathbb{A} = \bigcup_{S \subseteq \mathcal{M}} \Pi_n(S)$ be the set of all possible allocations. We'll say a good g is *non-valued* if no agent likes it, $v_i(g) = 0$ for all agents i, the set of such goods we'll denote $N_{\mathbf{v}}$. We'll say an allocation $\mathbf{A}$ is *fully allocated* if the only goods it doesn't allocate are non-valued, $V_{\mathcal{N}} \subseteq \mathbf{A}_{\mathcal{N}}$.

The following fairness notion is central to our work.

Definition 1 (Envy-Freeness)**.** *An allocation* $\mathbf{A}$ *is called* envy-free *(EF) if* $v_i(A_i) \geq v_i(A_j)$ *for all agents* $i, j \in \mathcal{N}$.

Envy-freeness requires that no agent prefer another agent's allocation over her own allocation. This cannot be guaranteed to exist when goods are indivisible, even when they are binary and additive. Prior literature focuses on its relaxations, such as envy-freeness up to one good [Lipton *et al.*, 2004; Budish, 2011], which can be guaranteed.

Definition 2 (Envy-Freeness up to One Good)**.** *An allocation* $\mathbf{A}$ *is called* envy-free up to one good *(EF1) if, for all agents* $i, j \in \mathcal{N}$*, either* $v_i(A_i) \geq v_i(A_j)$ *or there exists* $g \in A_j$ *such that* $v_i(A_i) \geq v_i(A_j \setminus \{g\})$*. That is, it should be possible to remove envy between any two agents by removing a single good from the envied agent's bundle.*

In addition, we use the following notion for efficiency.

Definition 3 (Pareto Optimality)**.** *An allocation* $\mathbf{A}$ *is called* pareto optimal *(PO) if there does not exist an allocation* $\mathbf{A}'$ *such that for all* i*,* $v_i(A_i') \geq v_i(A_i)$ *and there exists some* i *such that this inequality is strict. If such an allocation* $\mathbf{A}'$ *exists, we'll say it Pareto Dominates* $\mathbf{A}$.

When valuations are binary, Pareto Optimality is equivalent to maximizing social welfare.

For many of the proofs in the paper we will make use of the *graph of an allocation* $\mathbf{A}$, $G(\mathbf{A})$ as it was used in [Barman *et al.*, 2018]. $G(\mathbf{A})$ is defined as a directed graph of n vertices, each representing an agent. There is a directed edge (i, j) if there is a good in j's bundle that i likes, i.e. $A_j \cap V_i \neq \emptyset$, or equivalently, $v_i(A_j) \geq 1$. If $G(\mathbf{A})$ contains a path $P = (u_1, \ldots, u_k)$ from agent i to j, so $i = u_1$ and $j = u_k$, we can consider the allocation $\mathbf{A}'$ obtained by *passing back from* i *to* j. $\mathbf{A}'$ computed by for each (u_l, u_{l+1}), picking some good $g \in A_{u_{l+1}}$ that u_l liked, and giving it to u_l. If $\mathbf{A}$ was PO, the resulting allocation has utilities such that $u_k = u_k'$ for all $k \neq i, j$, $u_i' = u_i + 1$ and $u_j' = u_j - 1$ (if it was not PO then it is possible that some of the middle agents on the path increased or that j did not decrease).

3 Properties of Binary Maximum Nash Welfare

We begin by highlighting a few important properties an allocation can have:

Definition 4 (Pigou-Dalton Principle)**.** *An allocation* $\mathbf{A}$ *satisfies the* Pigou-Dalton principle*(PDP) if for all agents* i *and* j*, if* $v_i(A_i) < v_j(A_j)$*, there does not exist another allocation* $\mathbf{A}'$ *such that* $v_i(A_i) + v_j(A_j) = v_i(A_i') + v_j(A_j')$*,* $v_k(A_k) = v_k(A_k')$ *for all* $k \neq i, j$*,* $v_i(A_i) < v_i(A_i') < v_j(A_j)$*, and* $v_i(A_i) < v_j(A_j') < v_j(A_j)$.

The pigou dalton principle is generally seen in welfare economics as a notion of equality. If an allocation does not satisfy the PDP, then it's possible to make two agents more equal without changing the utility of any other agent or changing total social welfare.

Definition 5 (Max Nash Welfare allocation)**.** *An allocation* $\mathbf{A}$ *is a* Maximum Nash Welfare Allocation *(MNW allocation) if* *among possible allocations $\mathbb{A}$, it maximizes the size of the set of agents receiving positive utility, S, and, given that, maximizes the product of the utilities of agents in S.*

Nash Welfare (the product of the utilities) is yet another notion of efficiency and fairness, one especially good at balancing the two. It becomes maximized when agent utilities are high, but no agent has utility too low. Even among general combinatorial valuations, an MNW allocation always satisfies the Pigou Dalton Principle and is Pareto Optimal. For arbitrary additive valuations, it has even been shown to be EF1 [Caragiannis *et al.*, 2016].

Definition 6 (Leximin allocation)**.** *An allocation $\mathbf{A}$ is a* Leximin Allocation *if among all possible allocations $\mathbb{A}$, its sorted utility vector $(u_1, \ldots, u_n)$ is maximized in lexicographic order. Similarly to PO, if $\mathbf{A}'$ has a strictly larger sorted utility vector, we'll say it leximin dominates $\mathbf{A}$.*

Another way to describe a Leximin allocation is as a more efficient extension of the *egalitarian solution*, an allocation that maximizes the minimum utility received by any agent. A leximin allocation takes this a step further. It first only considers allocations that are egalitarian, maximize the minimum utility any agent receives. Among allocations that satisfy that, it then only considers allocations that maximize the utility of the agent with the second smallest utility, and so on. Similar to MNW allocations, for general combinatorial valuations, Leximin allocations always satisfy PDP and are PO.

Both MNW and Leximin allocations have been used heavily in part due to the many properties they satisfy. For example, with additive valuations, it can be hard to find valuations that are both EF1 and PO at the same time, finding an allocation that is MNW achieves this. And perhaps in general, if we are allocating goods and our goal is fairness, we may want to restrict ourselves to allocations that at least satisfy the PDP in order to ensure that a similar but seemingly more equal solution should not exist. But beyond that, the property of begin MNW or Leximin is even stronger. They are fairness and efficiency notions in themselves. Unfortunately, for general, even additive, valuations, if we are in search for some "most fair" allocation, there are trade offs. There is no need for an allocation to exist that would satisfy all the properties we would want. Allocations can be one of MNW or Leximin, sometimes both, and there are allocations that are both Pareto Optimal and satisfy the Pigou Dalton Principle, but are neither MNW or Leximin. It is interesting then that in the binary world, all three in fact coincide.

Theorem 1. *For a fixed allocation $\mathbf{A}$, the following are equivalent:*

1. *$\mathbf{A}$ is a Maximum Nash Welfare allocation.*
2. *$\mathbf{A}$ is a Leximin allocation.*
3. *$\mathbf{A}$ is Pareto Optimal and satisfies the Pigou-Dalton principle.*

Proof. $2 \implies 3$ (and in fact $1 \implies 3$ as well) are known properties of such allocations. We will show $1 \implies 2$ and $3 \implies 1$.

$1 \implies 2$: Given some allocation $\mathbf{A}$ that is a Maximum Nash Welfare allocation, we'll show it is also a Leximin allocation.

In this proof we'll making use of the graph of $\mathbf{A}$, $G(\mathbf{A})$. First we'll need to show a few facts about the graph which we will use several times throughout the proofs in this paper.

Lemma 1. *If $\mathbf{A}$ is fully allocated, and $U \subseteq \mathcal{N}$ is a subset of agents with no outgoing edges in $G(\mathbf{A})$, an agent in U can only like goods that are given to agents in U. That is, $V_U \subseteq A_U$.*

Proof. Otherwise there would be an outgoing edge to the agent with that good. □

Lemma 2. *If $\mathbf{A}$ is an MNW allocation and there is a path from agent i to agent j, then $v_j(A_j) \leq v_i(A_i) + 1$*

Proof. If not, we could pass back an item from j to i and increase the Nash Welfare as shown in [Barman *et al.*, 2018]. □

Corollary 1. *If $\mathbf{A}$ is a Maximum Nash Welfare allocation, then given some set of agents S, if $R_S := \{j \in \mathcal{N} | \text{there is a path from some } i \in S \text{ to } j\}$ is the set of agents reachable from S in $G(\mathbf{A})$, then:*

1. *All items liked by agents in R_S are given to agents in R_S. More formally, if $i \in R_S$ and $v_i(g) = 1$ for some good g, then $g \in A_j$ for some $j \in R_S$.*
2. *The largest utility received by an agent in R_S is at most 1 more than the largest utility received by an agent in S. More formally, $\max\{v_i(A_i) | i \in R_S\} \leq \max\{v_i(A_i) | i \in S\} + 1$.*

Proof.
1. R_S has no outgoing edges, as if it did, then they agent reached by that edge would clearly also be reachable from an agent in S. Since $\mathbf{A}$ is an MNW allocation, it is PO, and therefore fully allocated, so this holds by Lemma 1
2. Otherwise there would be a path from some agent $i \in S$ to the agent j receiving at least $\max\{v_i(A_i) | i \in S\} + 2$. This would contradict Lemma 2

□

Define $U_t := \{i \in \mathcal{N} | v_i(A_i) < t\}$ to be the agents with utility at most t in $\mathbf{A}$. Let R_{U_t} as defined above be the set of agents reachable from U_t in $G(\mathbf{A})$. To save space, we'll let $R_t := R_{U_t}$. We have that for each R_t, $U_t \subseteq R_t$ so it contains all agents with utilities at most $t-1$. By Lemma 1, R_t may contain agents with utility t, but cannot contain agents with utility greater than t. In addition, by the second part of Lemma 1, agents in R_t can only like goods given to agents in R_t. Since $\mathbf{A}$ is MNW and therefore PO, it is also the case that if an agent likes a good, it must be given to an agent that also likes it (otherwise we could easily get a PO improvement). Therefore, the agents in R_t collectively like $\sum_{i \in R_t} v_i(A_i)$ items in total, i.e. $|V_{R_t}| = \sum_{i \in R_t} v_i(A_i)$.

For a set of agents $S \subseteq \mathcal{N}$, let the *vector of S in* $\mathbf{A}$ be the sorted utility vector of the utilities of agents in S. Let the *k-prefix* of a vector $\mathbf{u}$ be the vector of the first k values, $(u_1, \ldots, u_k)$. We'll show by induction on t that for all t the following statement holds: The vector of R_t in $\mathbf{A}$ is Leximin dominant. That is, if the vector of R_t in $\mathbf{A}$ is $(u_1, \ldots, u_{|R_t|})$, there is no allocation $\mathbf{A}'$ with vector of R_t $(u'_1, \dots, u'_{|R_t|})$ and a k such that $u_i = u'_i$ for $i < k$ and $u_k < u'_k$.

This is quite a strong statement as this is independent of the utilities of agents not in R_t, so $\mathbf{A}'$ is allowed to allocate *all* items to agents in R_t, but it should still be impossible to increase the sorted utility of such agents. Once we've proven this for all t, if we take some t larger than the maximum utility of any agent, then $\mathcal{N} = U_t = R_t$, so this proves that $\mathbf{A}$ is in fact a Leximin allocation.

For $t = 0$, $U_t = \emptyset$ so $R_t = \emptyset$, therefore the statement is vacuously true.

Suppose the statement is true for t and consider $t + 1$. Let $\mathbf{u} = (u_1, \dots, u_{|R_{t+1}|})$ be the vector of R_{t+1} in $\mathbf{A}$. We have that R_t contains all the agents with utility at most $t - 1$ and potentially some with utility t. In addition, the agents of $R_{t+1} \setminus R_t$ either have utility t or $t + 1$. Therefore, the largest utility in R_t is at most the smallest utility in $R_{t+1} \setminus R_t$, so the vector of R_t in $\mathbf{A}$ in fact matches the $|R_t|$-prefix, $(u_1, \dots, u_{|R_t|})$.

Suppose for a contradiction there was some allocation $\mathbf{A}'$ with vector of R_{t+1}, $\mathbf{u}' = (u'_1, \dots, u'_{|R_{t+1}|})$ that lexicgraphically dominated $(u_1, \dots, u_{|R_{t+1}|})$. Now, if the vector R_t of $\mathbf{A}$, which is simply the $|R_t|$-prefix of $\mathbf{u}$, dominated the sorted vector of those agents in $\mathbf{A}'$, then it surely would also dominate the $|R_t|$-prefix of $\mathbf{u}'$, which would imply $\mathbf{u}$ actually dominated $\mathbf{u}'$, a contradiction. Therefore, since the vector of R_t in $\mathbf{A}'$ cannot dominate the one on $\mathbf{A}$ by the induction hypothesis, the vector of R_t must match; in particular the prefix of $\mathbf{u}'$must match, as otherwise $\mathbf{u}$ would dominate $\mathbf{u}'$. So we have that $u_i = u'_i$ for $i \le |R_t|$. Therefore, the only place $\mathbf{u}'$ could be different is for u_k, $k > |R_t|$. This means some u_k was turned from a t to at least a $t + 1$ with all u_i for $i > k$ being at least $t + 1$, or u_k was turned from a $t + 1$ to at least a $t + 2$ with all u_i for $i > k$ being at least $t + 2$. In either case, $\sum_{i \in R_{t+1}} v_i(A'_i) > \sum_{i \in R_t} v_i(A_i) = |V_{R_t}|$, a contradiction as a set of agents cannot get larger utility than the number of goods they like.

$3 \implies 1$: We will prove the contrapositive. Suppose $\mathbf{A}$ is not a Maximum Nash Welfare allocation. If $\mathbf{A}$ was not PO, then we're done, so suppose in addition it is PO, we'll show it does not satisfy the PDP. Then, as shown in Algorithm 2 in [Barman *et al.*, 2018], there is some path P in $G(\mathbf{A})$ from i to j such that passing a back along P increases Nash welfare. Let $\mathbf{A}'$ be the resulting allocation. Note that the only difference in utilities between $\mathbf{A}$ and $\mathbf{A}'$ is that $u'_i = u_i + 1$ and $u'_j = u_j - 1$. The only way the resulting allocation has a strictly higher Nash Welfare is if $(u_i + 1)(u_j - 1) > u_i u_j$, but this is true precicely when $u_j \ge u_i + 2$, which implies $u_i < u'_i \le u'_j < u_j$, so $\mathbf{A}'$ is a pigou improvement over $\mathbf{A}$, meaning $\mathbf{A}$ did not satisfy the PDP. $\square$

From now on, we will refer to such allocations as MNW and Leximin interchangeably. The following Lemma shows a sense of equality even among different MNW allocations.

Lemma 3. *For two Maximum Nash Welfare allocations $\mathbf{A}^1$, $\mathbf{A}^2$, the values for an agent i in each differ by at most 1. That is, $|v_i(A^1_i) - v_i(A^2_i)| \le 1$*

The purpose of this section is to show why we should care about these allocations. There are many notions in which these are the "fairest" possible allocations in the binary setting, in the sense that no other allocation can satisfy any of them. Ideally, we may hope that in a binary world, we would only allocate items in these ways.

4 Deterministic Setting

A *deterministic mechanism F* is a function that given valuations $\mathbf{v}$, outputs an allocation $\mathbf{A}$. We'll say a deterministic mechanism is EF, EF1, or PO if at always outputs an allocation that is EF, EF1, or PO respectively. In addition to satisfying fairness and efficiency properties, we may care that a mechanism has some game-theoretic property, in a sense, agents are not incentivized to lie about their values.

Definition 7 (Strategyproofness)**.** *A deterministic mechanism F is* strategyproof *if there do not exist possible valuations $\mathbf{v}$, an agent i, and an alternative valuation for i v'_i such that $v_i(F(\mathbf{v}_{-i}, v'_i)_i) > v_i(F(\mathbf{v})_i)$.*

In words, it is impossible for an agent i to lie about their values and end up with an allocation that they strictly prefer. Similarly:

Definition 8 (Group Strategyproofness)**.** *A deterministic mechanism F is* group strategyproof *if there do not exist possible valuations $\mathbf{v}$, a group of agents $C \subset \mathcal{N}$, and alternative valuations for each $j \in C$, v'_j such that $v_j(F(\mathbf{v}_{-C}, \mathbf{v}'_C)_j) > v_j(F(\mathbf{v})_j)$ for all j.*

Similar to above, it is impossible for a group of agents to all lie about their values in a way that makes all of them strictly better off. Note that strategyproofness is a special case of group strategyproofness where the group is simply a single agent.

Consider the following mechanism we'll call MNW^{tie}. Each vector of valuations $\mathbf{v}$ is mapped to some arbitrary allocation $\mathbf{A}$ such that:

1. $\mathbf{A}$ is an MNW allocation
2. $\mathbf{A}$ does not allocate any non-valued goods $N_{\mathbf{v}}$
3. Given the first two constraints, $\mathbf{A}$ maximizes the lexicographic value of $(v_1(A_1), \dots, v_n(A_n))$. This is simply a tie-breaking rule that prefers lower valued agents.

Since non-valued goods do not affect utilities, any MNW allocation with non-valued goods is still one without them. Therefore, MNW^{tie} can always be well-defined. We'll provide a few examples now of how MNW^{tie} works. We'll represent values as a matrix, with rows representing agents, columns goods, and a specific row and column being that agent's value for that good. Consider the valuations: $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. The only possibility for MNW^{tie} is $A_1 = \{g_1\}$ and $A_2 = \emptyset$. This is because g_2 cannot be allocated as it is non-valued, and MNW^{tie} must prefer the allocation of giving g_1 to agent 1 than to agent 2. Note that possible allocations for MNW^{tie} need not be unique. Consider $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Any allocation that gives two items to agent

1 and one item to agent 2 is acceptable, and can be chosen arbitrarily (although consistently).

Theorem 2. *MNW^{tie} is group strategyproof.*

Proof. First, we'll need to prove a few more properties of MNW^{tie}.

Lemma 4. *If an allocation $\mathbf{A}$ of allocation problem $\mathcal{A}$ is PO and does not allocate non-valued goods, then if agent i received an item g, they must have liked it, $v_i(g) = 1$. In other words, $A_i \subseteq V_i$ or equivalently $v_i(A_i) = |A_i|$. This further applies any subset $U \subseteq \mathcal{N}$, $A_U \subseteq V_U$, a group of agents can only receive items at least one of them likes.*

Proof. If $g \in A_i$ and $v_i(g) = 0$. Since $\mathbf{A}$ did not allocate any non valued goods, $v_j(g) = 1$ for some j. But then the allocation obtained by taking $\mathbf{A}$ and giving g to j is a pareto improvement, contradicting that $\mathbf{A}$ was PO. For the group part, if $A_i \subseteq V_i$ for all i, then for each $i \in U$, $A_i \subseteq V_i \subseteq V_U$, so clearly $A_U = \bigcup_{i \in U} A_i \subseteq V_U$. □

Corollary 2. *Lemma 4 applies to any allocation output by MNW^{tie}.*

For a given allocation $\mathbf{A}$ and agent $i \in \mathcal{N}$, define $L^i_{\mathbf{A}} = \{j \in \mathcal{N} | v_j(A_j) < v_i(A_i)\}$ to be the set of agents who have strictly less utility than i. Define $S^i_{\mathbf{A}} = \left\{j \in \mathcal{N} | j \text{ is reachable from } L^i_{\mathbf{A}} \cup \{i\} \text{ in } G(\mathbf{A})\right\}$ to be the set of agents reachable from i and any agent with utility less than i.

Lemma 5. *If $\mathbf{A}$ is fully allocated, pareto optimal, and does not allocate non-valued goods (i.e. it allocates $\mathcal{M} \setminus N_{\mathbf{v}}$ in a non-wasteful way), then the goods liked by agents in $S^i_{\mathbf{A}}$ are the exact same as the goods received by agents in $S^i_{\mathbf{A}}$, $V_{S^i_{\mathbf{A}}} = A_{S^i_{\mathbf{A}}}$.*

Proof. Combine Lemma 1 and and Lemma 4, the agents of $S^i_{\mathbf{A}}$ only like allocated goods given to agents in $S^i_{\mathbf{A}}$, and they must like all such items. □

Corollary 3. *Lemma 5 applies to any allocation output by MNW^{tie}.*

Lemma 6. *For any allocation problem $\mathcal{A}$, if $MNW^{tie}(\mathbf{v}) = \mathbf{A}$, then for any agent $i \in \mathcal{N}$ and all agents $j \in S^i_{\mathbf{A}} \setminus \{i\}$, $v_j(A_j) \leq v_i(A_i) + 1$ and if $j > i$, $v_j(A_j) \leq v_i(A_i)$.*

Proof. Recall that for any $k \in L^i_{\mathbf{A}}$, $v_k(A_k) < v_i(A_i)$, so even if $k \in L^i_{\mathbf{A}} \cup \{i\}$, $v_k(A_k) \leq v_i(A_i)$. Now if $j \in S^i_{\mathbf{A}} \setminus \{i\}$, then j is reachable from some $k \in L^i_{\mathbf{A}} \cup \{i\}$. But by Lemma 1, we have that $v_j(A_j) \leq v_k(A_k) \leq v_i(A_i) + 1$ as needed. Consider now when $j > i$. There are two cases, either there is a path from i to j or not. If there was a path from i to j, then we claim that $v_j(A_j) \leq v_i(A_i)$ as otherwise, passing back from j to i would have at least the same Nash Welfare, but would be strictly better lexicographically. If there was no path from i to j, then j must instead be reachable from some $k \in L^i_{\mathbf{A}}$. But then by Lemma 1, $v_j(A_j) \leq v_k(A_k) + 1 \leq v_i(A_i)$. □

Suppose for a contradiction MNW^{tie} was not group strategyproof. That is, there is some allocation problem $\mathcal{A}$ such that there exists a set of agents $C \subset \mathcal{N}$ and alternative valuations v'_j for $j \in C$ such that $v_j(MNW^{tie}(\mathbf{v}_{-C}, \mathbf{v}'_C)_j) > v_j(MNW^{tie}(\mathbf{v})_j)$ for all $j \in C$. Define $\mathbf{A}^{Truth} := MNW^{tie}(\mathbf{v})$ to be the allocation when agents report their true values and $\mathbf{A}^{Lie} := MNW^{tie}(\mathbf{v}_{-C}, \mathbf{v}'_C)$ be the allocation when agents misreport. Let $i \in C$ be the agent of C with minimum utility in the truthful allocation and given that constraint has the lowest index. Formally, $i = \min(\arg\min_{j \in C} v_j(A^{Truth}_j))$. Let $S = S^i_{\mathbf{A}^{Truth}}$ to save space on notation. We have for every $j \in C$, $|V_j \cap A^{Truth}_j| < |V_j \cap A^{Lie}_j|$ and since $A^{Truth}_j \subseteq V_j$ by Corollary 2, this simplifies to $|A^{Truth}_j| < |V_j \cap A^{Lie}_j|$. If that agent also happens to be in S ($j \in S \cap C$, this is the case for i and potentially others), since $V_j \subseteq V_S$ by definition, we can simplify again to $|A^{Truth}_j| < |V_j \cap A^{Lie}_j| \leq |V_S \cap A^{Lie}_j|$. Define $R \subseteq S$ to be the set of agents in S that can reach *some* misreporting agent $j \in C$ in $G(\mathbf{A}^{Lie})$.

Claim 1. *There exists an agent $j \in R \setminus C$ such that $|A^{Lie}_j| < |A^{Truth}_j|$*

Proof. Suppose for a contradiction for all $j \in R \setminus C$, $|A^{Lie}_j| \geq |A^{Truth}_j|$. For all such j, since $j \notin C$, j is reporting their true values so by Corollary 2, $A^{Lie}_j \subseteq V_j$. Since $j \in R \subseteq S$, $V_j \subseteq V_S$ by definition. Putting that together, $A^{Lie}_j \subseteq V_j \subseteq V_S$, so $|A^{Lie}_j \cap V_S| = |A^{Lie}_j| \geq |A^{Truth}_j|$. Additionally, for each $j \in R \cap C \subseteq C$, we have that $|A^{Lie}_j \cap V_S| \geq |A^{Lie}_j \cap V_j| > |A^{Truth}_j|$. Since the bundles for a specific allocation are disjoint, we can add these inequalities together for all agents in $R \setminus C$ and $R \cap C$ (which will together form just R) to get $|A^{Lie}_R \cap V_S| > |A^{Truth}_R|$. The inequality is strict as there must be at least one agent in $R \cap C$, namely i. Since $A^{Truth}_S = V_S$ by Corollary 3, this becomes $|A^{Lie}_R \cap A^{Truth}_S| > |A^{Truth}_j|$. This implies there must be some good that is in both A^{Lie}_R and A^{Truth}_S but not A^{Truth}_R. Therefore, there exists an agent $j \in R$, and $k \in S \setminus R$ such that $g \in A^{Lie}_j$ and $g \in A^{Truth}_k$. Since $g \in A^{Truth}_k$ this means $v_k(g) = 1$ by Corollary 2. Since $k \notin R$, $k \notin C$ as every node can reach itself. Therefore, k must still like this item even in the misreported allocation problem, meaning there is an edge from k to j in $G(\mathbf{A}^{Lie})$, k can reach whatever coalition agent j could reach. This contradicts that $k \notin R$. □

Claim 2. *The allocation computed by passing back to j from an agent $k \in C$ that is reachable from j is strictly preferred to $\mathbf{A}^{Lie}$ by MNW^{tie}.*

Proof. We have

$$\begin{aligned}
v'_k(A^{Lie}_k) &= |A^{Lie}_k| && \text{(By Corollary 2)} \\
&\geq |A^{Lie}_k \cap V_k| \\
&> |A^{Truth}_k| && \text{(By assumption for contradiction)} \\
&\geq |A^{Truth}_i| && \text{(By choice of } i\text{)}
\end{aligned}$$

Additionally, $v_j(A^{Lie}_j) < v_j(A^{Truth}_j)$ from applications of Corollary 2 to Claim 1. After passing back, since $\mathbf{A}^{Lie}$ is

PO, all utilities are the same except agent k's is replaced with $v'_k(A_k^{Lie}) - 1$ and agent j's is replaced with $v_j(A_j^{Lie}) + 1$. Note that the same set of goods has been allocated, so it must be the case that this new allocation is fully-allocated and does not allocate goods that no agent likes (otherwise this would not be the case for $\mathbf{A}^{Lie}$).

If $v_j(A_j^{Lie}) + 1 \leq v'_k(A_k^{Lie}) - 1$, then this new allocation has a strictly higher Nash welfare. If not, note that $v_j(A_j^{Lie}) + 1 \leq v_j(A_j^{Truth})$ and $v_j(A_j^{Truth}) \leq v_i(A_i^{Truth}) + 1$ from Lemma 6. Additionally, $v'_k(A_k^{Lie}) \geq v_k(A_k^{Truth}) + 1 \geq v_i(A_i^{Truth}) + 1$ by choice of i. Therefore, the only way this inequality could not hold is if $v_j(A_j^{Lie}) + 1 = v_j(A_j^{Truth}) = v_i(A_i^{Truth}) + 1 = v_k(A_k^{Truth}) + 1 = v'_k(A_k^{Lie})$. By the second equality, $j < i$ by Lemma 6. By the third equality, $k \geq i$ since their values in the truthful allocation are equal. Therefore, $k > j$. In this case, the Nash welfare remains exactly the same, but the utility vector is larger lexigraphically. $\square$

This contradicts the fact that $MNW^{tie}(\mathcal{N}, \mathcal{M}, (\mathbf{v}_{-i}, v'_i)) = \mathbf{A}^{Lie}$. $\square$

Proposition 1. *MNW^{tie} can be computed in polynomial time.*

5 Randomized Setting

As in many subfields of computer science, it's common to ask the question, can randomness help? In our case, as in many others, it can. First we'll extend many of the definitions from before to make sense when we talk about randomization. A *randomized mechanism F* is a function that given valuations $\mathbf{v}$, outputs a distribution over allocations $\mathcal{D}$. Many of the properties of deterministic mechanisms can be extended to the randomized world. A randomized mechanism F is called ex-ante EF or ex-ante PO if all possible outputted distributions are EF or PO in expectation,. That is, they are EF and PO if we replace the values of the agent by their expected values. Similarly, strategyproofness and group-strategyproofness can be defined in terms of the expectation of the agent. A randomized mechanism is ex-post EF, ex-post EF1, or ex-post PO if all allocations in the support of any outputted distribution is EF, EF1, or PO.

Theorem 3. *There is a randomized mechanism that is group strategyproof, ex-ante envy-free, whose support contains only Maximum Nash Welfare allocations (which implies it is ex-post Pareto Efficient and ex-post EF1) that can be computed in polynomial time.*

6 Conclusion

References

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Appendix