Fair Division with Subsidy

Daniel Halpern
University of Toronto

Nisarg Shah
University of Toronto

Example

		
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	\$50	\$110

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  - It satisfies several notions of fairness (EF1, the only Leximin and the only Max Nash Welfare allocation,...)

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- Alice values Bob's item $90 more than her own, so she needs to receive at least $90 to be happy.
- But if she receives anything over $60, Bob would envy Alice as he would be happier with the house and the money!

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  - We are assuming the valuations are in terms of money.
- Alice values Bob's item \$90 more than her own, so she needs to receive at least \$90 to be happy.
- But if she receives anything over \$60, Bob would envy Alice as he would be happier with the house and the money!
- A better way to do it would be to give the car to Alice, the house to Bob, and give Bob \$60

The big picture

- In Fair Division with Indivisible Goods, exact envy-freeness is impossible to guarantee
  - An envy-free allocation is one in which no-one prefers someone else's allocation to their own
- Much research has been on approximating fairness guarantees (EF1,...), but these can sometimes be unsatisfactory
  - There may be situations where we want full envy-freeness
- Perhaps the addition of some amount of a divisible good (money) can help!
  - When is this possible? How much do we need?

Fair Division Model

Standard Model:

- N: set of n agents
- M: set of m indivisible private goods
- $v_{i,g} \in \mathbb{R}_{\geq 0}$: value of agent i for good g
  - $v_i(S) = \sum_{g \in S} v_{i,g}$ for all $S \subseteq M$ (additive preferences)
- $\mathbf{A}$: allocation of goods in M to agents in N
  - A_i: subset of goods allocated to agent i
  - $\forall i, j A_i \cap A_j = \emptyset$; $\bigcup_i A_i = M$

New Feature:

- $\mathbf{p}$: the payment vector, an element of $\mathbb{R}^n$
  - For now we assume each $p_i \geq 0$

Definitions

- Envy-freeness (EF)
  - Each agent values her own allocation at least as much as anyone else's allocation
  - $(\mathbf{A}, \mathbf{p})$ is EF if $\forall i, j (v_i(A_i) + p_i \geq v_i(A_j) + p_j)$

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Example: From the last example, giving the house to Alice and the car to Bob was envy-freeable, the reverse allocation was not.

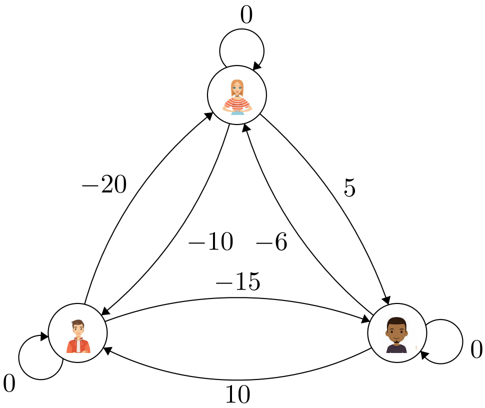
First setting

- Is a given allocation $\mathbf{A}$ envy-freeable?
- If so, what is the minimum amount of money required to make it envy-free?

A key tool: the envy graph

The envy graph of $\mathbf{A}$ is the complete weighted graph with nodes representing agents, and edge weights $w(i,j) = v_i(A_j) - v_i(A_i)$

				
	20	4	25	6
	10	10	15	20
	9	5	15	20



First setting

Theorem 1: The following are equivalent for an allocation $\mathbf{A}$

1. $\mathbf{A}$ is envy-freeable.
2. $\mathbf{A}$ maximizes the social welfare across all *reassignments* of bundles to agents.
    - A reassignment of an allocation A is an allocation $B = (A_{\sigma(1)}, \ldots, A_{\sigma(n)})$ where σ is some permutation of $\{1, \ldots, n\}$
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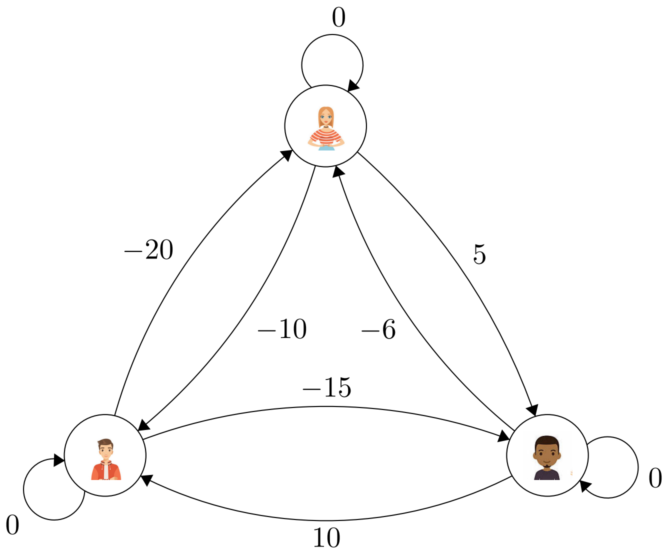
- This shows that any allocation that maximizes social welfare is envy-freeable!
  - There often are many other allocations that are envy-freeable.
  - We prefer allocations that require less total payment

First setting

Theorem 2:

- Paying each agent the value of the heaviest path beginning at their node in the envy-graph makes the allocation envy-free.
- These payments are optimal in the sense that paying any agent less than this path weight can never be envy-free.

Example with envy graph



Second setting: Allocation A can be chosen

- Given an allocation problem (agents, goods, and their values), compute an allocation that minimizes the subsidy needed, and bound the subsidy needed.
  - Note when determining bounds, we normalize all item values so that they fall in $[0, 1]$.

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- Given an allocation problem (agents, goods, and their values), compute an allocation that minimizes the subsidy needed, and bound the subsidy needed.
  - Note when determining bounds, we normalize all item values so that they fall in $[0, 1]$.
- Given an allocation problem, it's NP-hard to compute the minimum subsidy required.
  - It's NP-hard to determine whether an EF allocation exists [Liption et al., 2004]
  - An EF allocation exists iff the minimum subsidy required is 0

So what can we say about the minimum subsidy required?

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  - binary valuations (0 or 1), identical valuations, only two agents
- We've checked our conjecture experimentilly

Experiments

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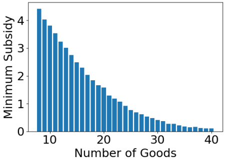
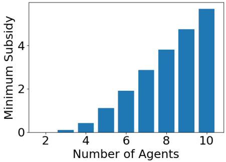
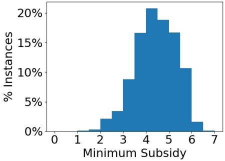


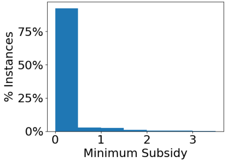
(e) Avg,n=8



(f) Avg,m=10



(g) Dist,n=8,m=8



(h) Dist,n=8,m=40

Conclusion

- We've shown how to determine if a specific allocation is envy-freeable, and how to compute the optimal subsidy.
- In the worst case, a subsidy of at least $\$(n-1)$ is needed
- We conjecture that such an allocation always exists, and in many special cases prove this.
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- Much of our work can be extended to *balance-budget transfers*
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- Current work: adding *truthfulness*.