Distortion in Fair Division

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Abstract

We study the fundamental problem of allocating indivisible goods to agents with additive preferences. We assume that each agent only submits a ranking of the goods by their value to her. The distortion measures the worst-case multiplicative loss in social welfare due to this lack of information. We analyze the distortion of deterministic and randomized allocation rules with or without additional fairness constraints such as envy-freeness up to one good and approximate maximin share guarantee. Along the way, we settle an open question due to Amanatidis et al. [2016] and show that the best possible approximation to the maximin share guarantee given the ordinal information is logarithmic in the number of agents.

1 Introduction

The theory of fair division studies how goods (or bads) should be fairly divided between individuals (a.k.a. agents) with different preferences over them. While the pioneering fair division research in economics, starting with the work of Steinhaus [30], focused on *divisible* goods which can be split between the agents, a significant body of recent research within computer science has focused on allocation of *indivisible* goods [12].

In the standard formulation of this problem, a set of indivisible goods M is to be partitioned among a set of agents N, and each agent $i \in N$ has a valuation function $v_i : 2^M \rightarrow \mathbb{R}_{\geqslant 0}$ assigning a non-negative value to every possible bundle of goods she can receive. Various classes of valuation functions have been studied, but the class of *additive* valuation functions has arguably had the most impact. Under additive valuations, an agent's value for a bundle of goods is simply the sum of her values for the individual goods in the bundle. Theoretically, this valuation class gives way to algorithms achieving strong fairness guarantees [17, 18, 21, 20]. Practically, additive valuations are much simpler to elicit than fully combinatorial valuations, which has led to their adoption by popular fair division tools such as Spliddit and Adjusted Winner.[1]

However, expressing additive valuations still requires placing an exact numerical value on each good, which can sometimes be difficult. This has led researchers to study *ordinal algorithms* which only ask agents to *rank* the goods in the order of their preference [10, 13, 4, 1]. These rankings can be viewed as partial information regarding the underlying additive valuations. A system designer deliberating on whether to elicit such partial information or to elicit the full additive valuations needs to know what the *price* of the missing information would be.

A natural quantitative goal in fair division is to maximize the social welfare, i.e., the sum of values that the agents derive from their allocations. Given only the ordinal preference information, we cannot always find an allocation that maximizes social welfare. The *distortion*, originally proposed by Procaccia and Rosenschein [29], Boutilier et al. [11] in the context of voting, captures the worst-case (multiplicative) loss in social welfare incurred by when only ordinal preferences are available. The focus of this work is to analyze distortion in the context of fair division.

In addition to the quantitative goal, one may be interested in achieving qualitative fairness guarantees — it is, after all, *fair* division. Two popular fairness guarantees for allocation of indivisible goods are envy-freeness up to one good (EF1) [25, 14] and approximate maximin share guarantee (MMS) [24], which we define in Section 2. It is known that EF1 can be achieved via an ordinal algorithm called round robin [25, 15], under which agents take turns in a cyclic fashion, picking one good each time. In case of MMS, Amanatidis et al. [1] show that using only the ordinal

[1] www.spliddit.org, www.nyu.edu/projects/adjustedwinner/

Fairness requirement	Deterministic	Randomized
None	n	n
EF1	$\Theta(n^2)$	n
Balancedness	$\Theta(n^2)$	n
α-MMS for $\alpha \leqslant 1/(2H_n)$	$\Theta(n^2)$	n

Table 1: The table summarizes the optimal distortion of deterministic and randomized ordinal allocation rules with and without fairness requirements. In case of a fairness requirement, the lower bound holds even when comparing to the social optimum subject to the fairness requirement. Additionally, the following fairness requirements cannot be achieved via ordinal allocation rules: α-MMS for $\alpha > 1/H_n$, EFX, and EQ1 (thus EQX as well).

information, it is not possible to achieve better than $1/H_n$ approximation of MMS, where $H_n = \Theta(\log n)$ is the n^{th} harmonic number and n is the number of agents; given full additive valuations, even $3/4$-MMS can be achieved [21, 20]. What is the best MMS approximation that can be achieved given only ordinal information? More importantly, what distortion do we incur if we require the ordinal algorithm to further guarantee EF1 or approximate MMS? We answer these questions in our work.

1.1 Our Contribution

A bit more formally, a deterministic (resp. randomized) ordinal allocation rule takes as input the preference rankings of the agents and returns an allocation (resp. a distribution over allocations) of the goods to the agents. The distortion of the rule is the ratio of the maximum social welfare of any allocation to the (expected) social welfare of the allocation returned, in the worst case over all possible problem instances. As is common in the literature on distortion, we assume normalized valuations: the total value each agent places on all goods is normalized to 1. We are interested in the best distortion that any deterministic or randomized allocation rules can achieve with or without imposing fairness constraints that the rule must satisfy. Note that in case of a randomized rule, we require that the fairness constraint be met by *all* allocations in the support of the distribution returned.

Our results are summarized in Table 1. Without any fairness constraint, the simple deterministic rule that allocates all the goods to a single agent achieves distortion n. We show that not even a randomized rule can achieve distortion better than n. Next, we consider three fairness requirements: envy-freeness up to one good (EF1), balancedness (i.e. all agents receive an approximately equal number of goods), and approximate maximin share (MMS). EF1 and balancedness are already known to be achievable via ordinal allocation rules. We design an ordinal allocation rule (more specifically, what is known as a picking sequence rule), which achieves $1/(2H_n)$-MMS, asymptotically matching the upper bound of $1/H_n$ established by Amanatidis et al. [1]. Thus, we establish, for the first time, that the best approximation to MMS that ordinal allocation rules can achieve scales logarithmically in the number of agents. We also show that when ordinal allocation rules are required to guarantee any of EF1, balancedness, and α-MMS for any $\alpha > 0$, the lowest distortion achievable by deterministic rules worsens to $\Theta(n^2)$, while for randomized rules, it stays n. Thus, imposing fairness does come at an additional price, unless randomization is allowed. We also show that various other fairness guarantees studied in the literature cannot be achieved via ordinal allocation rules.

1.2 Related Work

There has been a substantial amount of work on using ordinal preferences in fair allocation of indivisible goods. For example, Aziz et al. [4] consider the question of checking the existence of allocations that possibly or necessarily satisfy certain fairness guarantees such as envy-freeness given only ordinal preferences of the agents over the goods. Bouveret et al. [13] study similar questions, but given partial ordinal preferences of the agents over *bundles of goods*.

Some of the work does not assume any underlying cardinal preferences; instead, it aims to obtain guarantees defined directly in terms of the ordinal preferences. For example, Baumeister et al. [8], Nguyen et al. [28] use the so-called scoring vectors to convert agents' ordinal preferences into numerical proxies for their utility and then consider maximizing various notions of social welfare or guaranteeing various fairness properties in terms of such utilities.

More closely related to ours are the papers that use ordinal allocation rules (such as picking sequence rules) in settings with cardinal valuations. For example, Aziz et al. [5] focus on the complexity of checking what social welfare such rules can possibly or necessarily achieve. Amanatidis et al. [1] seek to use picking sequence rules to obtain approximation of the maximin fair share guarantee; indeed, as mentioned earlier, we settle a question left open in their work. However, their main focus is on ensuring truthfulness, i.e., preventing agents from manipulating their preferences. Manipulations under picking sequence rules have received significant attention [7, 6].

Finally, our work is heavily inspired by the growing literature on distortion in voting, where voters express ordinal preferences (instead of numerical utilities) over candidates, and the goal is to pick a candidate maximizing social welfare. Procaccia and Rosenschein [29] introduce the idea of distortion in voting, and Boutilier et al. [11] identify the lowest distortion achievable via any randomized voting rule. This basic idea has been extended in a number of different directions, from selecting a set of candidates [16] to imposing interesting structural properties on the voter valuations [3] to analyzing the exact tradeoff between the amount of information available and the distortion [26, 27, 23, 2]. We consider the possibility of extending our work along similar directions in Section 5.

2 Model

For $k \in \mathbb{N}$, let $[k] = \{1, \ldots, k\}$. Let $N = [n]$ be a set of n *agents* and $M = [m]$ be a set of m *goods*. Each agent i is endowed with a *valuation* function $v_i : 2^M \to \mathbb{R}_{\geqslant 0}$. We assume that valuations are *additive*: $v_i(S) = \sum_{g \in S} v_i(\{g\})$ for all $i \in N, S \subseteq M$; and *unit-sum*: $v_i(M) = 1$ for all $i \in N$. To simplify notation, we write $v_i(g)$ instead of $v_i(\{g\})$ for a good $g \in M$. We refer to $v = (v_1, \ldots, v_n)$ as the *valuation profile*, and the tuple (N, M, v) as an *instance*.

For a set of goods $S \subseteq M$ and $k \in \mathbb{N}$, let $\Pi_k(S)$ denote the set of ordered partitions of S into k bundles. An allocation $A = (A_1, \ldots, A_n) \in \Pi_n(M)$ is a partition of the goods into n bundles, where A_i is the bundle allocated to agent i. Under this allocation, the *utility* to agent i is $v_i(A_i)$. Given a valuation profile v, the *social welfare* of an allocation A is $\mathrm{sw}(A, v) = \sum_{i \in N} v_i(A_i)$; we simply write $\mathrm{sw}(A)$ when the valuation profile v is clear from the context.

Given an instance (N, M, v), a *preference ranking* σ_i of agent $i \in N$ is a ranking of the goods in a non-decreasing order of their value to agent i. We say that v_i is consistent with σ_i, denoted $v_i \triangleright \sigma_i$, if $v_i(g) \geqslant v_i(g')$ for all $g, g' \in M$ such that $g \succ_{\sigma_i} g'$. Note that goods with equal value can be ranked arbitrarily by the agent. We refer to $\sigma = (\sigma_1, \ldots, \sigma_n)$ as the *preference profile*, and say that v is consistent with σ, denoted $v \triangleright \sigma$, if $v_i \triangleright \sigma_i$ for each $i \in N$.

Let $\mathcal{L}(M)$ denote the set of linear orders (rankings) over M. A (randomized) *ordinal allocation rule* (hereinafter, simply a rule) f takes an input (N, M, σ) — for simplicity, we refer to σ as the sole input to f — and returns a distribution over the set of allocations $\Pi_n(M)$. We say that f is *deterministic* if it always returns a distribution with singleton support. We will sometimes refer to a distribution over allocations as a *randomized allocation*. The *distortion* of an ordinal allocation rule f, denoted $\mathrm{dist}(f)$, is the worst-case approximation ratio it provides to the social welfare:

$$\mathrm{dist}(f) = \sup_{\sigma} \sup_{v : v \triangleright \sigma} \frac{\max_{A \in \Pi_n(M)} \mathrm{sw}(A, v)}{\mathbb{E}[\mathrm{sw}(f(\sigma), v)]},$$

where the expectation is over possible randomization in the ordinal allocation rule. Following prior work, we treat the number of agents as fixed, but seek rules that can operate on instances with any number of goods. Hence, the worst case is over instances consisting of any number of goods and n agents. Consequently, the distortion is also a function of n. We are interested in the lowest distortion that deterministic and randomized ordinal allocation rules can achieve.

A *fairness property* P maps every instance $I = (N, M, v)$ to a (possibly empty) set of allocations $P(I)$; every allocation in $P(I)$ is said to satisfy P in instance I. We say that a rule f satisfies property P if for all σ and v such that $v \triangleright \sigma$, every allocation in the support of $f(\sigma)$ satisfies P in the instance (N, M, v). We are also interested in determining whether ordinal allocation rules can satisfy prominent fairness properties from the fair division literature, and when they can, determining the lowest possible distortion they can achieve subject to such properties.

Given an instance $I = (N, M, v)$, we are interested in the following fairness properties.

Definition 1 (EF1)**.** An allocation A is called *envy-free up to one good (EF1)* if for every pair of agents i, j, either $v_i(A_i) \geqslant v_i(A_j)$ or there exists a good $g \in A_j$ such that $v_i(A_i) \geqslant v_i(A_j \setminus \{g\})$.

Definition 2 (Balancedness)**.** An allocation A is called *balanced* if $|A_i|-|A_j|\leqslant 1$ for all $i, j \in N$, i.e., if the agents receive approximately an equal number of goods.

Definition 3 (MMS)**.** The *maximin share* of agent i is defined as

$$\mathrm{MMS}_i = \max_{A\in\Pi_n(M)} \min_{A_j\in A} v_i(A_j).$$

Given $\alpha \in [0, 1]$, an allocation A is called *α-maximin share fair* (α-MMS) if $v_i(A_i) \geqslant \alpha \cdot \mathrm{MMS}_i$ for all agents $i \in N$. When $\alpha = 1$, we simply say that A is an MMS allocation.

Additionally, we will frequently make use of the following class of ordinal allocation rules.

Definition 4 (Picking Sequence Rules)**.** A *picking sequence* of length m is a sequence of agents $\pi_m = p_{i_1}p_{i_2}\dots p_{i_m}$, where $p_{i_k} \in [n]$ for each $k \in [m]$. A *picking sequence rule* is a deterministic ordinal allocation rule which is parameterized by a picking sequence of every possible length, and given an instance with m goods, uses the picking sequence π_m of length m as follows.[2] It first gives to agent p_{i_1} their favorite good, then gives to agent p_{i_2} their favorite good among the ones remaining, and so on, until all goods are allocated. Note that the rule only needs the preference profile σ as input. The popular round robin rule chooses an arbitrary ordering of the agents and repeats it to form the picking sequence (the last repetition may be truncated).

Definition 5 (Uniform Picking Sequence Rules)**.** A *uniform picking sequence rule* is a randomized ordinal allocation rule which is also parameterized by a picking sequence of every possible length. Given an instance with m goods, it uses the picking sequence $\pi_m = p_{i_1}p_{i_2}\dots p_{i_m}$ of length m as follows. It first picks a permutation of the agents τ uniformly at random, and then runs the (deterministic) picking sequence rule with the picking sequence $\tau(p_{i_1})\tau(p_{i_2})\dots\tau(p_{i_m})$.

3 Distortion of Ordinal Allocation Rules

We begin by analyzing the lowest distortion that deterministic and randomized ordinal allocation rules can achieve in the absence of any fairness requirement. This precisely captures value of cardinal preference information, or the loss incurred in social welfare due to the availability of only ordinal preference information.

A trivial *deterministic* way to achieve distortion n is to ignore any preference information whatsoever and simply allocate all the goods to an arbitrary single agent: indeed, the social welfare of such an allocation is 1, while the maximum social welfare cannot be larger than n since valuations are unit-sum ($v_i(M) = 1$ for all $i \in M$). Our main result of this section, presented next, shows that not even *randomized* ordinal allocation rules can achieve lower distortion.

Theorem 1. *The lowest possible distortion of deterministic as well as randomized ordinal allocation rules is n.*

Proof. Given the observation above, we only need to show that the distortion of an arbitrary randomized ordinal allocation rule f is at least n.

Fix an integer x, and let $\varepsilon = 1/x$. Let us consider a preference profile σ under which every agent has the same preference ranking: lower indexed goods are preferred to higher indexed goods. We construct a family of consistent valuations $\mathcal{V}$, and show that regardless of the randomized allocation returned by f given σ, the worst-case loss in social welfare across just this family of valuations already approaches n as x grows large.

We note that our construction below is similar to the one used by Bei et al. [9] for lower bounding the social welfare loss incurred by a specific deterministic ordinal allocation rule, but our analysis significantly more intricate as it applies to all randomized ordinal allocation rules.

For $k \in [n]$, we say that an agent is of type t_k if they like goods 1 through x^k equally, at value $1/x^k$ each, and have value 0 for the remaining goods. Let $T = \{t_1, \dots, t_n\}$ be the set of all types. The family $\mathcal{V}$ consists of valuation profiles under which there is exactly one agent of each type. Such valuations can be represented by a bijection $\tau : [n] \mapsto T$ mapping agents to types.

[2] Recall that the rule must be able to operate on instances with any number of goods.

Consider the partition of goods $W = (W_1, \ldots, W_n)$ such that $W_1 = \{1, \ldots, x\}$, and $W_k = \left\{x^{k-1}+1, \ldots, x^k\right\}$ for $2 \leqslant k \leqslant n$. First, note that under every valuation in $\mathcal{V}$, there exists an allocation with social welfare at least $(1-\varepsilon)n$. This is achieved by assigning each W_k to the agent of type t_k, which gives them value at least $(1/x^k) \cdot |W_k| \geqslant (1/x^k) \cdot (x^k - x^{k-1}) \geqslant 1 - 1/x = 1 - \varepsilon$, resulting in social welfare at least $(1-\varepsilon)n$. In contrast, we show that under some valuation in $\mathcal{V}$, the randomized allocation returned by f generates poor social welfare.

First, note that under any valuation in $\mathcal{V}$, all agents are indifferent between the goods in W_k, for each $k \in [n]$. Hence, we can succinctly describe a randomized allocation by the expected fraction of goods from W_k assigned to the different agents, for each $k \in [n]$. In other words, we can describe a randomized allocation $\mathcal{A}$ by an $n \times n$ matrix X such that $X_{i,k}$ is the expected fraction of goods from W_k assigned to agent i. Note that for the randomized allocation to be feasible, it must be the case that $\sum_{i=1}^n X_{i,k} = 1$ for all $k \in [n]$.

Let X be the matrix corresponding to the randomized allocation $\mathcal{A}$ returned by f on input σ constructed above. Our goal is to find a valuation $v \in \mathcal{V}$, characterized by a bijection $\tau : [n] \to T$, under which the expected social welfare of $\mathcal{A}$ is not much more than 1. First, we claim that if agent i is of type t_k, then their expected utility under $\mathcal{A}$ is at most $X_{i,k} + \varepsilon$. To see this, note that agent i is in expectation receiving an $X_{i,k}$ fraction of the goods from W_k, and W_k consists of at least a $1-\varepsilon$ fraction of the goods they like, as argued above. Hence, the expected utility to agent i under such an allocation is at most $X_{i,k}$ from the goods in W_k and at most ε from the goods outside of W_k, which is at most $X_{i,k} + \varepsilon$ in total. Thus, the expected social welfare of A is at most $\sum_{i,k:\tau(i)=t_k} X_{i,k} + n\varepsilon$. Our goal is to show that there exists a bijection τ for which this quantity is at most $1 + n\varepsilon$.

Note that if we choose τ uniformly at random, then the expected value of $\sum_{i,k:\tau(i)=t_k} X_{i,k}$ is equal to $1/n \cdot \sum_{i\in[n],k\in[n]} X_{i,k} = 1$. Hence, there must exist a bijection τ for which $\sum_{i,k:\tau(i)=t_k} X_{i,k} \leqslant 1$. Under the corresponding valuation profile, the expected social welfare of $\mathcal{A}$ is at most $1 + n\varepsilon$. This shows that the distortion of f is at least $\frac{(1-\varepsilon)n}{1+n\varepsilon}$. As x approaches ∞, ε approaches 0, which establishes that the distortion of f is at least n, as desired. □

Earlier, we observed that distortion n can be achieved trivially by allocating all the goods to an arbitrary single agent. The following result shows that a more interesting class of rules — uniform picking sequence rules — also achieves distortion n. This will be useful in Section 4, where we will show that some rules in this family achieve interesting fairness notions in addition to achieving distortion n. This result is a well-known folklore in the literature, but we present a proof for completeness. A proof for the special case of round robin method was given by Freeman et al. [19].

Proposition 1. *Every uniform picking sequence rule has distortion n.*

Proof. Fix an arbitrary $I = (N, M, v)$ with m goods. Let the picking sequence be $p_{i_1} p_{i_2} \ldots p_{i_m}$. We show that the expected social welfare under the rule is at least 1. Since the optimal social welfare is upper bounded by n, this gives the desired upper bound of n on distortion. The tightness follows from Theorem 1.

To show the expected social welfare is at least 1, we show that the expected utility to each agent i is at least $1/n$. Suppose agent i has preference ranking $g_1 \succ \ldots \succ g_m$. Because utilities are additive, we can decompose the expected utility to agent i as the sum of the expected utilities received by agent i from all m picks.

Note that agent i only receives a nonzero utility from the k^{th} pick if $\tau(p_{i_k}) = i$, where τ is the random permutation chosen by the rule. This happens with probability $1/n$, and with this probability, agent i picks a good at least as valuable to her as g_k (since one of $g_1, \ldots, g_k$ must be available during k^{th} pick), thus receiving utility at least $v_i(g_k)$. Therefore, the expected utility of agent i from the k^{th} pick is at least $1/n \cdot v_i(g_k)$ for each k. Using linearity of expectation, we get that the expected utility to agent i is at least $\sum_{k=1}^m 1/n \cdot v_i(g_k) = 1/n \cdot v_i(M) = 1/n$, as needed. □

4 Distortion of Fair Ordinal Allocation Rules

In this section, we analyze the lowest distortion that ordinal allocation rules can achieve when they are required to satisfy some fairness constraints. This captures the *combined price* of the lack of cardinal preference information and the imposition of fairness constraints. As such, we will contrast it with the sole price of the former analyzed in Section 3 and the sole price of the latter from known results in the literature. Perhaps not surprisingly, it turns out that the two together lead to a much greater loss in social welfare than each individually.

Another interesting consequence of our results is that while randomized ordinal allocation rules are no more powerful than deterministic ones in the absence of any fairness requirements (Theorem 1), imposing fairness requirements makes their powers diverge.

Keeping aside the question of distortion, we are also interested in determining which fairness properties ordinal allocation rules can or cannot satisfy, as a negative answer can also be interpreted as a qualitative price of the lack of cardinal information.

We begin by establishing a lower bound on the distortion of deterministic ordinal allocation rules that holds when any fairness property from a broad class is imposed; later, we argue that the fairness properties of our interest belong to this class. Recall that we require the allocation returned by the rule to satisfy the fairness property, regardless of the unobserved cardinal valuations as long as they are consistent with the observed ordinal preferences.

Theorem 2. *Let P be a fairness property such that when the number of goods equals the number of agents, for every preference profile σ, an allocation satisfies P for all valuations consistent with σ if and only if each agent receives a single good. Then, the distortion of every deterministic ordinal allocation rule satisfying P is $\Omega(n^2)$.*

Proof. Fix such a fairness property P, a number of agents n, and a deterministic ordinal allocation rule f satisfying P. First, let us suppose that n is even. We construct an instance with n goods, that is, with $m = n$. We split the goods into three different categories and construct a preference profile σ as follows. The first category consists of a single good g^* that is ranked highest by all agents. The next category consists of $n/2$ goods labeled $g_{\{1,2\}}, g_{\{3,4\}}, \ldots, g_{\{n-1,n\}}$. For each $k \in [n/2]$, good $g_{\{2k-1,2k\}}$ is ranked second by both agents $2k-1$ and $2k$. The final category consists of the remaining $n/2 - 1$ goods. The construction above identifies the two most preferred goods for all agents; their preference rankings from the third rank onward can be arbitrary.

Let A be the allocation returned by f given σ. By the assumption of the theorem statement, each agent must receive exactly one good in A. Without loss of generality, let us assume that agent 1 receives g^*. In addition, for each $k \in \{2, \ldots, n/2\}$, at least one of agents $2k-1$ and $2k$ does not receive good $g_{\{2k-1,2k\}}$; without loss of generality, assume that agent $2k-1$ does not receive it. Let us construct a consistent valuation profile as follows:

- Agent 1 has value $1/n$ for each good.
- Agent 2 has value 1 for g^* and 0 for all other goods.
- For $k \in \{2, \ldots, n/2\}$, agent $2k-1$ has value $1/2$ for g^*, $1/2$ for $g_{\{2k-1,2k\}}$, and 0 for all other goods; and agent $2k$ has value 1 for g^* and 0 for all other goods.

Under A, the only agent receiving positive utility is agent 1, who receives utility $1/n$. Therefore, the social welfare is $1/n$. In contrast, consider the allocation that gives g^* to agent 2, $g_{\{2k-1,2k\}}$ to agent $2k-1$ for each $k \in \{2, \ldots, n/2\}$, and the remaining goods arbitrarily such that each agent receives a single good. It is easy to check that its social welfare is at least $1 + (n/2 - 1) \cdot 1/2 = n/4 + 1/2$. Therefore, the distortion of f is at least $(n/4 + 1/2)/(1/n) \in \Omega(n^2)$.

If n is odd, we can construct the described instance with $n-1$ agents and $n-1$ goods, add a good ranked last by all agents, and add an agent whose preference ranking matches that of one of the other agents. Using similar arguments as above, regardless of the allocation A chosen by f, we can construct a consistent valuation profile in which the social welfare of A is $1/n$, while the optimal social welfare is at least $(n-1)/4 + 1/2$, resulting in $\Omega(n^2)$ distortion. □

Notice that in the proof of Theorem 2, we contrast the social welfare achieved by the rule satisfying P against that of an allocation that assigns each agent a single good, thus also satisfying P. That is, the $\Omega(n^2)$ lower bound continues to hold even when comparing to the optimal social welfare subject to P. On the other hand, our matching upper bounds presented later in the section hold even when comparing to the optimal social welfare without any fairness constraints.

4.1 EF1 and Balancedness

We consider envy-freeness up to one good (EF1) and balancedness together for a simple reason. While these two are incomparable fairness notions in general, they become related in the ordinal world: every EF1 allocation computed using only ordinal preference information must be balanced.[3] Our results for these properties follow easily from known results in the literature along with the results derived so far in the paper.

Proposition 2. *The lowest possible distortion of any deterministic ordinal allocation rule satisfying* $P \in \{\textit{EF1}, \textit{balancedness}, \textit{EF1+balancedness}\}$ *is* $\Theta(n^2)$.

Proof. It is easy to check that EF1 and balancedness each satisfy the requirements of Theorem 2, giving us the $\Omega(n^2)$ lower bound. Bei et al. [9] show that the round robin rule uses only ordinal information, achieves both EF1 and balancedness, and has distortion $\Theta(n^2)$, giving us the desired upper bound. □

In contrast to Proposition 2, randomized rules can still achieve the same (lower) distortion n that they were able to achieve without any fairness requirements.

Proposition 3. *The lowest possible distortion of any deterministic ordinal allocation rule satisfying* $P \in \{\textit{EF1}, \textit{balancedness}, \textit{EF1+balancedness}\}$ *is* $\Theta(n)$.

Proof. Theorem 1 shows that the lower bound of $\Omega(n)$ holds even without any fairness requirements. For the upper bound, note that the uniform version of the round robin rule, commonly called randomized round robin, uses only ordinal information, and satisfies both EF1 and balanced. Further, Proposition 1 shows that it has distortion n, giving us the desired upper bound. □

4.2 MMS

We now turn to our most technical results, which are regarding approximate maximin share (MMS) guarantee. Before we consider distortion subject to approximate MMS, we need to know what approximation to MMS is possible to achieve. Given full cardinal information, it is known that exact MMS cannot be achieved [24], but $4/5$-MMS can [21, 20]. Given only ordinal information, Amanatidis et al. [1] show that it is not possible to achieve α-MMS for $\alpha > 1/H_n$, where $H_n = \Theta(\log n)$ is the n^{th} Harmonic number. On the opposite end, they only establish a weaker $\Omega(1/\sqrt{n})$ lower bound, leaving open the question of what the best possible MMS approximation is given ordinal information. Our constructive result presented next settles this question by showing that their upper bound is asymptotically tight.

Theorem 3. *There exists an ordinal allocation rule satisfying* $\frac{1}{2H_n}$*-MMS.*

Proof. In this proof, we borrow and significantly build upon ideas from the proof of the $1/H_n$ upper bound due to Amanatidis et al. [1].

We construct a picking sequence rule achieving the desired MMS approximation. Fix some value of the number of agents n and the number of goods m.

Fix an agent. Suppose the first time this agent appears in the picking sequence is at the k-th position (we call this the agent's 0-th appearance), and then, the agent's j-th appearance occurs at or before position $(k + \lfloor j \cdot 2H_n(n-k+1) \rfloor)$ in the picking sequence for every j (as long as this quantity does not exceed m). Then, we claim that the agent must be guaranteed at least $1/(2H_n)$ fraction of her MMS value. To see this, note that the agent picks a good at least as valuable as her k-th most favorite good in her 0-th appearance, and then an additional good at least as valuable as her $(k + \lfloor j \cdot 2H_n(n-k+1) \rfloor)$-th favorite good in her j-th appearance for each j. Let S denote the total value the agent places on her $k-1$ most valuable goods. Then, this picking sequence guarantees the agent utility at least $\frac{1-S}{2H_n(n-k+1)}$. On the other hand, note that the MMS value of the agent is at most $\frac{1-S}{n-k+1}$; this is because regardless of how the agent partitions the goods into n bundles, ignoring the (at most) $k-1$ bundles containing her $k-1$ most valuable goods, even the average value across the remaining (at least $n-k+1$) bundles is at most $\frac{1-S}{n-k+1}$. Hence, it follows that the agent is guaranteed at least a $1/2H_n$ fraction of her MMS share.

Our picking sequence gives a guarantee of this style to each agent, albeit for different values of k. In particular, for each agent $i \in [n]$, the picking sequence provides this guarantee with $k = i$. The construction is very simple.

[3] To see this, note that the valuation profile in which each agent derives equal value from all goods is consistent with any preference profile, and EF1 under this valuation profile would be violated unless the allocation is balanced.

1. For $1 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant \left\lfloor \frac{m-i}{2H_n(n-i+1)} \right\rfloor$, we create the pair $(i, i + \lfloor j \cdot 2H_n(n-i+1) \rfloor)$, indicating that agent i's j-th appearance must occur at or before the position indicated in the second component — we refer to this as the deadline.

2. We sort the pairs with respect to their second coordinate.

3. The first coordinates with respect to the above sorting are a prefix of the picking sequence.

4. If the length of the above sequence is m, we are done; otherwise we arbitrarily assign the remaining picks.

The idea of Steps 2–4 is to produce a picking sequence that meets all the deadlines by using earliest-deadline-first scheduling. It is known that if all the deadlines can be met, then this greedy scheduling procedure is guaranteed to return a sequence meeting them. To show that all deadlines are met, we want to show that there are at most d pairs introduced in Step 1 with the second coordinate (deadline) at most d, for all $d \leqslant m$. Note that in particular, this implies that there are at most m pairs in total, so Step 3 would not produce a sequence of length more than m.

To prove this, let us first consider $d \leqslant n$. Observe that the 1-st appearance deadline of any agent is at or after position $n+1$: this is because $i + \lfloor 2H_n(n-i+1) \rfloor \geqslant 1 + \lfloor n-i+1 \rfloor = n+1$ for all $i \in [n]$. This implies that the only pairs with deadline at most n are the n pairs of the form (i,i) for $i \in [n]$ corresponding to the 0-th appearances of all the agents, which immediately implies the desired goal holds for all $d \leqslant n$.

Next, consider $d \geqslant n+1$.The number of pairs for agent i with the second coordinate at most d is at most $1 + \left\lfloor \frac{d-i}{2H_n(n-i+1)} \right\rfloor$. Therefore, the number of total pairs with second coordinate at most d is at most $\sum_{i=1}^{n} 1 + \left\lfloor \frac{d-i}{2H_n(n-i+1)} \right\rfloor = n + \sum_{i=1}^{n} \left\lfloor \frac{d-i}{2H_n(n-i+1)} \right\rfloor$. Our goal is to show that this value is at most d, which is equivalent to the following lemma.

Lemma 1. *For all $n \in \mathbb{N}$ and for all $d \geqslant n+1$, $\sum_{i=1}^{n} \left\lfloor \frac{d-i}{2H_n(n-i+1)} \right\rfloor \leqslant d-n$.*

Proof. There are two cases, either $d \geqslant 2n$ or $d < 2n$. First suppose that $d \geqslant 2n$. Then,

$$\begin{aligned}
\sum_{i=1}^{n} \left\lfloor \frac{d-i}{2H_n(n-i+1)} \right\rfloor &\leqslant \sum_{i=1}^{n} \frac{d-i}{2H_n(n-i+1)} \\
&\leqslant \frac{d}{2H_n} \sum_{i=1}^{n} \frac{1}{n-i+1} \\
&= \frac{d}{2} \leqslant d-n.
\end{aligned}$$

Next, assume that $n+1 \leqslant d < 2n$. Let $\ell = d-n$. Note that $1 \leqslant \ell < n$. The inequality in the lemma is therefore equivalent to (note the cancelling $+1$ and -1 in the numerator) $\sum_{i=1}^{n} \left\lfloor \frac{\ell+(n-i+1)-1}{2H_n(n-i+1)} \right\rfloor \leqslant \ell$. Note that summing over i ranging from 1 through n is equivalent to summing over $n-i+1$ ranging from 1 through n; substituting i to denote $n-i+1$, we need to prove $\sum_{i=1}^{n} \left\lfloor \frac{\ell+i-1}{2H_n \cdot i} \right\rfloor \leqslant \ell$.

Call the summand value $f(i) = \left\lfloor \frac{\ell+i-1}{2H_n i} \right\rfloor$. First, we show that $f(i) \leqslant \ell$ for all $i \in [n]$. For this, it is sufficient to show $\ell + i - 1 \leqslant 2H_n \cdot i \cdot \ell$. We have that $\ell + i - 1 \leqslant 2\max(i,\ell) \leqslant 2H_n \cdot i \cdot \ell$ because H_n, i, and ℓ are all at least 1.

Let $g(j)$ be the number of values of i such that $f(i) \geqslant j$; more formally, $g(j) = \sum_{i=1}^{n} \mathbb{1}[f(i) \geqslant j]$. Now since $f(i) \leqslant \ell \leqslant n$, we have that $\sum_{i=1}^{n} f(i) = \sum_{j=1}^{n} g(j)$ as each agent i is counted precisely $f(i)$ times in the RHS sum. Next, we bound $g(j)$ for each $j \in [n]$. To do this, we show that if i is such that $f(i) \geqslant j$, then i must be bounded by some value, B. This would imply that at most B values of i have $f(i) \geqslant j$, implying $g(j) \leqslant B$.

$$\begin{aligned}
& f(i) = \left\lfloor \frac{\ell+i-1}{2H_n \cdot i} \right\rfloor \geqslant j \\
\implies & \ell + i - 1 \geqslant 2H_n \cdot i \cdot j && (2H_n \cdot i > 0) \\
\implies & \ell - 1 \geqslant (2H_n j - 1) \cdot i
\end{aligned}$$

$$\implies \frac{\ell - 1}{2H_n j - 1} \geqslant i. \qquad (2H_n j - 1 > 0 \text{ as } H_n \cdot j \geqslant 1)$$

Hence, $g(j) \leqslant \frac{\ell-1}{2H_n j-1}$ for all j. In addition, since $H_n \cdot j \geqslant 1$, $H_n \cdot j \leqslant 2H_n \cdot j - 1$, so $\frac{\ell-1}{2H_n \cdot j-1} \leqslant \frac{\ell-1}{H_n \cdot j} \leqslant \frac{\ell}{H_n \cdot j}$.

Plugging this into our earlier equation:

$$\sum_{i=1}^{n} f(i) = \sum_{j=1}^{n} g(j) \leqslant \sum_{j=1}^{n} \frac{\ell}{H_n \cdot j} = \frac{\ell}{H_n} \sum_{j=1}^{n} \frac{1}{j} = \ell,$$

as desired. □

This completes the proof of the theorem. □

Next, we study the lowest distortion achievable via deterministic and randomized ordinal allocation rules satisfying α-MMS for some $\alpha > 0$.

Proposition 4. *For any $\alpha > 0$, every deterministic ordinal allocation rule satisfying α-MMS has distortion $\Omega(n^2)$.*

Proof. It is easy to check that for any $\alpha > 0$, the α-MMS requirement satisfies the conditions of Theorem 2. Hence, the lower bound follows. □

Strikingly, while $\Omega(n^2)$ distortion is unbeatable subject to α-MMS for *any* $\alpha > 0$, we can achieve a matching $O(n^2)$ distortion even with $1/(2H_n)$-MMS, which is the best approximation that is currently known to be achievable due to Theorem 3.

Theorem 4. *There exists a deterministic ordinal allocation rule satisfying $\frac{1}{2H_n}$-MMS with distortion $O(n^2)$.*

Proof. The construction is very similar to Theorem 3 except that we add one more constraint: we want agent 1 to also have additional appearances, once at or before position $2nj+1$ for each j. Because agent 1 gets the first overall pick as well, this ensures that the utility to agent 1 is at least $1/2n$ fraction of her value for all the goods, i.e., at least $1/2n$. Since n is a trivial upper bound on the social welfare of any allocation, this implies the distortion upper bound of $O(n^2)$.

Now, as in the proof of Theorem 3, we must show that for all d, there are at most d pairs with second coordinate at most d, also counting the additional pairs generated due to the extra constraint above. Since we have not added any pairs with second coordinate at most n, the part of the proof for $d \leqslant n$ still holds. For $d \geqslant n+1$, the number of new pairs added due to the above constraint is $\left\lfloor \frac{d-1}{2n} \right\rfloor$. Hence, we need the following strengthening of Lemma 1.

Lemma 2. *For all $n \in \mathbb{N}$ and for all $d \geqslant n+1$, $\left\lfloor \frac{d-1}{2n} \right\rfloor + \sum_{i=1}^{n} \left\lfloor \frac{d-i}{2H_n(n-i+1)} \right\rfloor \leqslant d - n$.* □

Proof. First note that when $n = 1$, this is equivalent to showing $\left\lfloor \frac{d-1}{2} \right\rfloor + \left\lfloor \frac{d-1}{2} \right\rfloor \leqslant d - 1$ which trivially holds for all $d \in \mathbb{N}$. Therefore, from now on, we restrict to the case of $n \geqslant 2$.

Similarly to Lemma 1, we consider two cases: $d \geqslant 4n$ and $d < 4n$. First, suppose $d \geqslant 4n$. Then,

$$\begin{aligned}
\left\lfloor \frac{d-1}{2n} \right\rfloor + \sum_{i=1}^{n} \left\lfloor \frac{d-i}{2H_n(n-i+1)} \right\rfloor &\leqslant \frac{d-1}{2n} + \sum_{i=1}^{n} \frac{d-i}{2H_n(n-i+1)} \\
&\leqslant \frac{d}{4} + \sum_{i=1}^{n} \frac{d}{2H_n(n-i+1)} \\
&= \frac{d}{4} + \frac{d}{2H_n} \cdot \sum_{i=1}^{n} \frac{1}{n-i+1} \\
&= \frac{d}{4} + \frac{d}{2H_n} \cdot H_n \\
&= \frac{3d}{4} \leqslant d - n,
\end{aligned}$$

where the second transition holds because $n \geqslant 2$, and the final transition holds because $d \geqslant 4n$.

Next, suppose $d < 4n$. First, we tackle the special case of $n = 2$. Note that when $d \leqslant 2n$, we have $\left\lfloor \frac{d-1}{2n} \right\rfloor = 0$, in which case the desired inequality is already implied by Lemma 1. For $2n < d < 4n$, i.e., $d \in \{5, 6, 7\}$, we have that $\left\lfloor \frac{d-1}{2n} \right\rfloor = 1$. Hence, the desired inequality holds as follows:

$$d = 5: \quad 1 + \sum_{i=1}^{2} \left\lfloor \frac{5-i}{2H_2(2-i+1)} \right\rfloor = 2 \leqslant 5 - 1$$
$$d = 6: \quad 1 + \sum_{i=1}^{2} \left\lfloor \frac{6-i}{2H_2(2-i+1)} \right\rfloor = 2 \leqslant 6 - 1$$
$$d = 7: \quad 1 + \sum_{i=1}^{2} \left\lfloor \frac{7-i}{2H_2(2-i+1)} \right\rfloor = 3 \leqslant 7 - 1$$

From now on, we restrict to the case of $n \geqslant 3$. Because we are in the case of $d < 4n$, this implies $\left\lfloor \frac{d-1}{2n} \right\rfloor \leqslant 1$. Hence, it is sufficient to show that

$$\sum_{i=1}^{n} \left\lfloor \frac{d-i}{2H_n(n-i+1)} \right\rfloor \leqslant d - n - 1.$$

As in Lemma 1, letting $\ell = d - n$, the above statement is equivalent to

$$\sum_{i=1}^{n} \left\lfloor \frac{\ell + i - 1}{2H_n \cdot i} \right\rfloor \leqslant \ell - 1.$$

As in Lemma 1, define $f(i) = \left\lfloor \frac{\ell+i-1}{2H_n \cdot i} \right\rfloor$ and $g(j) = \sum_{i=1}^{n} \mathbb{1}[f(i) \geqslant j]$. In Lemma 1, we had $d < 2n$, giving us $f(i) \leqslant \ell \leqslant n$ for all $i \in [n]$. In this case, we still have that $f(i) \leqslant \ell$ for all $i \in [n]$, but since we only have $d < 4n$, we have $\ell \leqslant 3n$. We note that $g(j) \leqslant \frac{\ell-1}{2H_n j - 1}$ can still be derived as in the proof of Lemma 1. Thus, we have that

$$\sum_{i=1}^{n} f(i) = \sum_{j=1}^{3n} g(j) \leqslant \sum_{j=1}^{3n} \frac{\ell - 1}{2H_n j - 1} = (\ell - 1) \cdot \sum_{j=1}^{3n} \frac{1}{2H_n j - 1}.$$

To complete the proof, we simply need to show that $\sum_{j=1}^{3n} \frac{1}{2H_n j - 1} \leqslant 1$ for all $n \geqslant 3$. When $n = 3$, we have

$$\sum_{j=1}^{3n} \frac{1}{2H_n j - 1} = \sum_{j=1}^{9} \frac{1}{2H_3 j - 1} = \frac{19{,}657{,}653{,}727}{21{,}402{,}806{,}880} \leqslant 1.$$

Suppose $n \geqslant 4$. First, note that

$$\sum_{j=1}^{3n} \frac{1}{2H_n j - 1} \leqslant \sum_{j=1}^{3n} \frac{1}{(2H_n - 1)j} = \frac{1}{2H_n - 1} \sum_{j=1}^{3n} \frac{1}{j} = \frac{H_{3n}}{2H_n - 1},$$

where the first transition holds because $j \geqslant 1$. To show that this final quantity is at most 1, we need to show $H_{3n} \leqslant 2H_n - 1$ for all $n \geqslant 4$. We prove this by induction on n.

For the base case of $n = 4$, we have that $H_{12} = \frac{86{,}021}{27{,}720} \leqslant \frac{19}{6} = 2H_4 - 1$. Suppose $H_{3n} \leqslant 2H_n - 1$ holds for some $n \geqslant 4$. We need to show that $H_{3(n+1)} \leqslant 2H_{n+1} - 1$. We have that

$$\begin{aligned} H_{3(n+1)} &= H_{3n} + \frac{1}{3n+1} + \frac{1}{3n+2} + \frac{1}{3n+3} \\ &\leqslant H_{3n} + \frac{3}{3n} \end{aligned}$$

$$
\begin{aligned}
&= 2H_n - 1 + \frac{1}{n} \\
&\leqslant 2H_n - 1 + \frac{2}{n+1} \\
&\leqslant 2H_{n+1} - 1
\end{aligned}
$$
□

Given Lemma 2, the theorem follows.

Finally, we show that randomized rules can still achieve distortion n subject to $\frac{1}{2H_n}$-MMS.

Proposition 5. *The lowest possible distortion of any randomized ordinal allocation rule satisfying $\frac{1}{2H_n}$-MMS is n.*

Proof. Note that the rule from Theorem 3 was a picking sequence rule. Therefore, by Proposition 1, its uniform version has distortion n, and still satisfies $\frac{1}{2H_n}$-MMS. The lower bound follows from Theorem 1. □

4.3 Impossible Fairness Properties

Finally, we show that some fairness properties studied in the literature are impossible to guarantee given only the ordinal information. Consider an instance $I = (N, M, v)$.

Definition 6. An allocation A is said to be *envy-free up to any good (EFX)* if for every pair of agents i, j and every good $g \in A_j$, we have $v_i(A_i) \geqslant v_i(A_j \setminus \{g\})$.

Definition 7. An allocation A is said to be *equitable up to one good (EQ1)* if for every pair of agents i, j such that $A_j \neq \emptyset$, there exists some good $g \in A_j$ such that $v_i(A_i) \geqslant v_j(A_j \setminus \{g\})$. A is said to be *equitable up to any good (EQX)* if for every pair of agents i, j such that $A_j \neq \emptyset$ and for every good $g \in A_j$ such that $v_j(g) > 0$, we have that $v_i(A_i) \geqslant v_j(A_j \setminus \{j\})$.

Proposition 6. *There does not exist an ordinal allocation rule satisfying EFX even for two agents.*

Proof. Suppose there are two agents and four goods (denoted $g_1, \ldots, g_4$ for notational convenience). Consider the preference profile wherein both agents have the same preference ranking over the goods, given by $g_1 \succ g_2 \succ g_3 \succ g_4$.

We show that regardless of the allocation chosen by a deterministic ordinal allocation rule, EFX cannot be satisfied with respect to every valuation profile consistent with this preference profile.

Suppose the rule picks allocation $A = (A_1, A_2)$. There are two cases: $|A_1| \neq |A_2|$ and $|A_1| = |A_2|$.

First, suppose $|A_1| \neq |A_2|$, and without loss of generality, suppose $|A_1| > |A_2|$. Since there are four goods, this implies that $|A_1| \geqslant |A_2| + 2$. In this case, if agent 2 has equal value of $1/4$ for all four goods, then agent 2 would envy agent 1 even after removing any single good from A_1, violating EFX (and in fact, EF1 too).

Next, suppose that $|A_1| = |A_2|$, which implies that each is equal to 2. Without loss of generality, suppose $g_1 \in A_1$. Then, consider the valuation function of agent 2 which places value 1 on good g_1 and $\epsilon < 1/3$ on every other good. Then, agent 2 would envies agent 1 even after removing the single good in $A_1 \setminus \{g_1\}$, violating EFX. □

Proposition 7. *There does not exist an ordinal allocation rule satisfying EQ1 or EQX even for two agents.*

Proof. Consider the construction used in the proof of Proposition 6, in which two agents have the same preference ranking over four goods given by $g_1 \succ g_2 \succ g_3 \succ g_4$.

We show that regardless of the allocation chosen by a deterministic ordinal allocation rule, EQ1 cannot be satisfied with respect to all valuation profiles consistent with this preference profile. This implies the desired result for EQX as well.

Suppose the rule picks allocation $A = (A_1, A_2)$. There are two cases: $|A_1| \neq |A_2|$ and $|A_1| = |A_2|$.

First, suppose $|A_1| \neq |A_2|$, and without loss of generality suppose $|A_1| > |A_2|$. Since there are four goods, this implies that $|A_1| \geqslant |A_2| + 2$. Now, consider the valuation profile in which both agents have an equal value of $1/4$ for all the goods. Then, even after removing any single good from A_1, the utility to agent 1 will be more than the utility to agent 2, violating EQ1.

Next, suppose $|A_1| = |A_2|$, which implies that each is equal to 2.Without loss of generality, suppose $g_1 \in A_1$. Consider the valuation profile in which agent 2 has value 1 for g_1 and 0 for every other good, while agent 1 has value $1/4$ for all the goods. Thus, the utility to agent 2 is 0, while the utility to agent 1 remains higher than that, even after removing any single good from A_1, violating EQ1. □

5 Discussion

Our work explores the value of cardinal preference information in fair division by analyzing the distortion (loss incurred in social welfare) and identifying fairness properties which cannot be satisfied when only ordinal preference information is available.

This is inspired by a growing literature on distortion in voting [11]. One line of recent work in this direction has focused on imposing additional structure on the underlying cardinal preferences, for example, by assuming that the preferences are induced by a metric [3, 22]. It would be interesting to study distortion in fair division under natural restrictions on cardinal preferences, such as a limit on the number of goods that an agent can derive positive utility from or on the difference between the values two agents derive from a good. It would also be interesting to analyze the distortion in the average case when the cardinal preferences are drawn from a distribution.

Another thread of research on distortion in voting has focused on the tradeoff between distortion and the amount of information available regarding agent preferences by allowing the designer to either elicit arbitrary — not necessarily ordinal — information about the preferences but limiting the number of bits elicited [26, 27, 23], or make queries to elicit information on top of the ordinal information already available [2]. An interesting direction for the future is to study such a tradeoff between communication and distortion (or the best possible approximation to fairness properties like the maximin share guarantee) in fair division.

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