WORKING PAPER
Binary Fair Division

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1 Introduction

2 Preliminaries

For $k \in \mathbb{N}$, let $[k] = \{1, \ldots, k\}$. Let $\mathcal{N} = [n]$ denote the set of *agents*, and let $\mathcal{M}$ denote the set of m indivisible *goods*. Each agent i is endowed with a *valuation* function $v_i : 2^{\mathcal{M}} \rightarrow \mathbb{R}_{\geq 0}$ such that $v_i(\emptyset) = 0$. It is assumed that valuations are additive: $\forall S \subseteq \mathcal{M}$, $v_i(S) = \sum_{g \in S} v_i(\{g\})$. To simplify notation, we write $v_i(g)$ instead of $v_i(\{g\})$. In addition, we assume the valuations are binary: $\forall g \in \mathcal{M}$, $v_i(g) \in \{0, 1\}$. We'll say an agent *likes* a good g if $v_i(g) = 1$. Sometimes it is easier to think of a valuation functions as a set $V_i \subseteq \mathcal{M}$, the set of goods that the agent likes. There is a clear bijection between such sets and valuation functions, where a set V_i is equivalent to $v_i(g) = \begin{cases} 1 & g \in V_i \\ 0 & g \notin V_i \end{cases}$. For a set of agents $S \subseteq \mathcal{N}$, let $V_S := \bigcup_{i \in S} V_i$ be the set of goods that at least one agent in S likes. We denote the vector of valuations by $\mathbf{v} = (v_1, \ldots, v_n)$. We define an *allocation problem* to be the tuple $\mathcal{A} = (\mathcal{N}, \mathcal{M}, \mathbf{v})$.

For a set of goods $M \subseteq \mathcal{M}$ and $k \in \mathbb{N}$, let $\Pi_k(M)$ denote the partitions of M into k bundles. Given an allocation problem $\mathcal{A}$, an allocation $\mathbf{A} = (A_1, \ldots, A_n) \in \Pi_n(M)$ for some $M \subseteq \mathcal{M}$ is a partition of some subset of the goods into n bundles, where A_i is the bundle allocated to agent i. For a set of agents $S \subseteq \mathcal{N}$, let $A_S := \bigcup_{i \in S} A_i$ be the union of their allocations. Under this allocation, the *utility* to agent i is $v_i(A_i)$. Let $\mathbb{A} = \bigcup_{S \subseteq \mathcal{M}} \Pi_n(S)$ be the set of all possible allocations. We'll say a good g is *non-valued* if no agent likes it, $v_i(g) = 0$ for all agents i, the set of such goods we'll denote $N_{\mathbf{v}}$. We'll say an allocation $\mathbf{A}$ is *fully allocated* if the only goods it doesn't allocate are non-valued, $V_{\mathcal{N}} \subseteq \mathbf{A}_{\mathcal{N}}$.

The following fairness notion is central to our work.

Definition 1 (Envy-Freeness)**.** *An allocation* $\mathbf{A}$ *is called* envy-free *(EF) if* $v_i(A_i) \geq v_i(A_j)$ *for all agents* $i, j \in \mathcal{N}$.

Envy-freeness requires that no agent prefer another agent's allocation over her own allocation. This cannot be guaranteed to exist when goods are indivisible, even when they are binary and additive. Prior literature focuses on its relaxations, such as envy-freeness up to one good [Lipton *et al.*, 2004; Budish, 2011], which can be guaranteed.

Definition 2 (Envy-Freeness up to One Good)**.** *An allocation* $\mathbf{A}$ *is called* envy-free up to one good *(EF1) if, for all agents* $i, j \in \mathcal{N}$, *either* $v_i(A_i) \geq v_i(A_j)$ *or there exists* $g \in A_j$ *such that* $v_i(A_i) \geq v_i(A_j \setminus \{g\})$. *That is, it should be possible to remove envy between any two agents by removing a single good from the envied agent's bundle.*

In addition, we use the following notion for efficiency.

Definition 3 (Pareto Optimality)**.** *An allocation* $\mathbf{A}$ *is called* pareto optimal *(PO) if there does not exist an allocation* $\mathbf{A}'$ *such that for all* i, $v_i(A'_i) \geq v_i(A_i)$ *and there exists some* i *such that this inequality is strict. If such an allocation* $\mathbf{A}'$ *exists, we'll say it Pareto Dominates* $\mathbf{A}$.

For many of the proofs in the paper we will make use of the *graph of an allocation* $\mathbf{A}$, $G(\mathbf{A})$ as it was used in [Barman *et al.*, 2018]. $G(\mathbf{A})$ is defined as a directed graph of n verticies, each representing an agent. There is a directed edge (i, j) if there is a good in j's bundle that i likes, i.e. $A_j \cap V_i \neq \emptyset$, or equivalently, $v_i(A_j) \geq 1$.

3 Properties of Binary Maximum Nash Welfare

We begin by highlighting a few important properties an allocation can have:

Definition 4 (Pigou-Dalton priniciple)**.** *An allocation* $\mathbf{A}$ *satisfies the* Pigou-Dalton principle*(PDP) if for all agents* i *and* j, *if* $v_i(A_i) < v_j(A_j)$, *there does not exist another allocation* $\mathbf{A}'$ *such that* $v_i(A_i) + v_j(A_j) = v_i(A'_i) + v_j(A'_j)$, $v_k(A_k) = v_k(A'_k)$ *for all* $k \neq i, j$, $v_i(A_i) < v_i(A'_i) < v_j(A_j)$, *and* $v_i(A_i) < v_j(A'_j) < v_j(A_j)$.

The pigou dalton principle is generally seen in welfare economics as a notion of equality. If it's possible to make two agents more equal without decreasing social welfare or hurting another agent, we should prefer that.

Definition 5 (Max Nash Welfare allocation)**.** *An allocation* $\mathbf{A}$ *is a* Maximum Nash Welfare Allocation*(MNW allocation) if among possible allocations* $\mathbb{A}$, *it maximizes the size of the set of agents receving positive utility,* S, *and, given that, maximies the product of the utilities of agents in* S.

Definition 6 (Leximin allocation)**.** *An allocation* $\mathbf{A}$ *is a* Leximin Allocation *if among all possible allocations* $\mathbb{A}$, *its sorted utility vector* $(u_1, \ldots, u_n)$ *is maximized in lexicographic order. Similary to PO, if* $\mathbf{A}'$ *has a strictly larger sorted utility vector, we'll say it leximin dominates* $\mathbf{A}$.

Another way to describe leximin is that first it attempts to maximize the utility of the agent with the minimum utility. Among allocations that satisfies that, it then attempts to maximize the utility of the agent with the second smallest utility, and so on. Both MNW and Leximin allocations have been used heavliy in part due to the many properties they satisfy. In paticular, for any combinatorial valuations, they are both Pareto Optimal and satisfy the Pigou Dalton Principle. For general additive valuations, MNW allocations are even EF1. However, in addition to that, they are in themselves definitions fairness/efficiency notions. The product of utilities is seen as a nice balance of ensuring many agents get high utility while no agents get utility too low. The Leximin is an extension of the egalitarian solution where we just try to maximize the minimum utility, in order to get as much utility as possible while continuing to satisfy this for even smaller sets of agents. However, for general, even additive, valuations, there is no need for them to be equal. Allocations can be one of MNW, Leximin, or sometimes both, and there are allocations that are both Pareto Optimal and satisfy the Pigou Dalton Principle, but are neither MNW or Leximin. It is interesting then that in the binary world, all three in fact coincide.

Theorem 1. *For a fixed allocation $\mathbf{A}$, the following are equivalent:*

1. *$\mathbf{A}$ is a Maximum Nash Welfare allocation.*
2. *$\mathbf{A}$ is a Leximin allocation.*
3. *$\mathbf{A}$ is Pareto Optimal and satisfies the Pigou-Dalton principle.*

Proof. 2 $\implies$ 3 (and in fact 1 $\implies$ 3 as well) are known properties of such allocations. We will show 1 $\implies$ 2 and 3 $\implies$ 1.

1 $\implies$ 2: Given some allocation $\mathbf{A}$ that is a Maximum Nash Welfare allocation, we'll show it is also a Leximin allocation.

In this proof we'll making use of the graph of $\mathbf{A}$, $G(\mathbf{A})$. First we'll need to show a few facts about the graph which we will use several times throughout the proofs in this paper.

Lemma 1. *If $\mathbf{A}$ is fully allocated, and $U \subseteq \mathcal{N}$ is a subset of agents with no outgoing edges in $G(\mathbf{A})$, an agent in U can only like goods that are given to agents in U. That is, $V_U \subseteq A_U$.*

Proof. Otherwise there would be an outgoing edge to the agent with that good. □

Lemma 2. *If $\mathbf{A}$ is an MNW allocation and there is a path from agent i to agent j, then $v_j(A_j) \leq v_i(A_i) + 1$*

Proof. If not, we could pass back an item from j to i and increase the Nash Welfare as shown in [Barman *et al.*, 2018]. □

Corollary 1. *If $\mathbf{A}$ is a Maximum Nash Welfare allocation, then given some set of agents S, if $R_S := \{j \in \mathcal{N} | \text{there is a path from some } i \in S \text{ to } j\}$ is the set of agents reachable from S in $G(\mathbf{A})$, then:*

1. *All items liked by agents in R_S are given to agents in R_S. More formally, if $i \in R_S$ and $v_i(g) = 1$ for some good g, then $g \in A_j$ for some $j \in R_S$.*
2. *The largest utility received by an agent in R_S is at most 1 more than the largest utility received by an agent in S. More formally, $\max\{v_i(A_i) | i \in R_S\} \leq \max\{v_i(A_i) | i \in S\} + 1$.*

Proof. 1. R_S has no outgoing edges, as if it did, then they agent reached by that edge would clearly also be reachable from an agent in S. Since $\mathbf{A}$ is an MNW allocation, it is PO, and therefore fully allocated, so this holds by Lemma 1

2. Otherwise there would be a path from some agent $i \in S$ to the agent j receiving at least $\max\{v_i(A_i) | i \in S\} + 2$. This would contradict Lemma 2

□

Define $U_t := \{i \in \mathcal{N} | v_i(A_i) < t\}$ to be the agents with utility at most t in $\mathbf{A}$. Let R_{U_t} as defined above be the set of agents reachable from U_t in $G(\mathbf{A})$. To save space, we'll let $R_t := R_{U_t}$. We have that for each R_t, $U_t \subseteq R_t$ so it contains all agents with utilities at most $t-1$. By Lemma 1, R_t may contain agents with utility t, but cannot contain agents with utility greater than t. In addition, by the second part of Lemma 1, agents in R_t can only like goods given to agents in R_t. Since $\mathbf{A}$ is MNW and therefore PO, it is also the case that if an agent likes a good, it must be given to an agent that also likes it (otherwise we could easily get a PO improvement). Therefore, the agents in R_t collectively like $\sum_{i \in R_t} v_i(A_i)$ items in total, i.e. $|V_{R_t}| = \sum_{i \in R_t} v_i(A_i)$.

For a set of agents $S \subseteq \mathcal{N}$, let the *vector of S in $\mathbf{A}$* be the sorted utility vector of the utilities of agents in S. Let the *k-prefix* of a vector $\mathbf{u}$ be the vector of the first k values, $(u_1, \dots, u_k)$. We'll show by induction on t that for all t the following statement holds: The vector of R_t in $\mathbf{A}$ is Leximin dominant. That is, if the vector of R_t in $\mathbf{A}$ is $(u_1, \dots, u_{|R_t|})$, there is no allocation $\mathbf{A}'$ with vector of R_t $(u'_1, \dots, u'_{|R_t|})$ and a k such that $u_i = u'_i$ for $i < k$ and $u_k < u'_k$.

This is quite a strong statement as this is independent of the utilities of agents not in R_t, so $\mathbf{A}'$ is allowed to allocate *all* items to agents in R_t, but it should still be impossible to increase the sorted utility of such agents. Once we've proven this for all t, if we take some t larger than the maximum utility of any agent, then $\mathcal{N} = U_t = R_t$, so this proves that $\mathbf{A}$ is in fact a Leximin allocation.

For $t = 0$, $U_t = \emptyset$ so $R_t = \emptyset$, therefore the statement is vacuously true.

Suppose the statement is true for t and consider $t+1$. Let $\mathbf{u} = (u_1, \dots, u_{|R_{t+1}|})$ be the vector of R_{t+1} in $\mathbf{A}$. We have that R_t contains all the agents with utility at most $t-1$ and potentially some with utility t. In addition, the agents of $R_{t+1} \setminus R_t$ either have utility t or $t+1$. Therefore, the largest utility in R_t is at most the smallest utility in $R_{t+1} \setminus R_t$, so the vector of R_t in $\mathbf{A}$ in fact matches the $|R_t|$-prefix, $(u_1, \dots, u_{|R_t|})$.

Suppose for a contradiction there was some allocation $\mathbf{A}'$ with vector of R_{t+1}, $\mathbf{u}' = (u'_1, \dots, u'_{|R_{t+1}|})$ that lexicgraphically dominated $(u_1, \ldots, u_{|R_{t+1}|})$. Now, if the vector R_t of $\mathbf{A}$, which is simply the $|R_t|$-prefix of $\mathbf{u}$, dominated the sorted vector of those agents in $\mathbf{A}'$, then it surely would also dominate the $|R_t|$-prefix of $\mathbf{u}'$, which would imply $\mathbf{u}$ actually dominated $\mathbf{u}'$, a contradiction. Therefore, since the vector of R_t in $\mathbf{A}'$ cannot dominate the one on $\mathbf{A}$ by the induction hypothesis, the vector of R_t must match; in particular the prefix of $\mathbf{u}'$must match, as otherwise $\mathbf{u}$ would dominate $\mathbf{u}'$. So we have that $u_i = u'_i$ for $i \leq |R_t|$. Therefore, the only place $\mathbf{u}'$ could be different is for u_k, $k > |R_t|$. This means some u_k was turned from a t to at least a $t+1$ with all u_i for $i > k$ being at least $t+1$, or u_k was turned from a $t+1$ to at least a $t+2$ with all u_i for $i > k$ being at least $t+2$. In either case, $\sum_{i \in R_{t+1}} v_i(A'_i) > \sum_{i \in R_t} v_i(A_i) = |V_{R_t}|$, a contradiction as a set of agents cannot get larger utility than the number of goods they like.

$3 \implies 1$: We will prove the contrapositive. Suppose $\mathbf{A}$ is not a Maximum Nash Welfare allocation. Then as shown in Algorithm 2 in [Barman *et al.*, 2018], there is some allocation $\mathbf{A}'$ achieved by reallocating along some path from some agent i to agent j that increases Nash Welfare. Note that for all $k \neq i, j$, $v_k(A'_k) = v_k(A_k)$ by construction. We know that $v_j(A'_j) = v_j(A_j) + 1$. Additionally, $A'_i = A_i \setminus \{g\}$ for some good g. If i didn't like g, then this is a Pareto improvement so $\mathbf{A}$ is not PO. If i liked g, then $v_i(A'_i) = v_i(A_i) - 1$, but since $\mathbf{A}'$ had a strictly higher Nash Welfare than A_i, $v_j(A'_j) \leq v_i(A'_i)$, so $\mathbf{A}'$ is a Pigou-improvement over $\mathbf{A}$, meaning $\mathbf{A}$ does not satisfy the PDP. $\square$

From now on, we will refer to such allocations as MNW and Leximin interchangeably. The following Lemma shows a sense of equality even among different MNW allocations.

Lemma 3. *For two Maximum Nash Welfare allocations $\mathbf{A}^1$, $\mathbf{A}^2$, the values for an agent i in each differ by at most 1. That is, $|v_i(A^1_i) - v_i(A^2_i)| \leq 1$*

The purpose of this section is to show why we should care about these allocations. There are many notions in which these are the "fairest" possible allocations in the binary setting, in the sense that no other allocation can satisfy any of them. Ideally, we may hope that in a binary world, we would only use the allocations defined above.

4 Deterministic Setting

A *deterministic mechanism F* is a function that given valuations $\mathbf{v}$, outputs an allocation $\mathbf{A}$. We'll say a deterministic mechanism is EF, EF1, or PO if at always outputs an allocation that is EF, EF1, or PO respectively. As shown in the last section, it would be nice if we could always simply output some MNW allocation. However, the issue with doing this arbitrarily is that agents may not be incentivized to actually tell us what their true valuations are. For that reason, we will aim for the following properties:

Definition 7 (Strategyproofness)**.** *A deterministic mechanism F is* strategyproof *if there do not exist possible valuations $\mathbf{v}$, an agent i, and an alternative valuation for i v'_i such that $v_i(F(\mathbf{v}_{-i}, v'_i)_i) > v_i(F(\mathbf{v})_i)$.*

In words, it is impossible for an agent i to lie about their values and end up with an allocation that they strictly prefer. Similarly:

Definition 8 (Group Strategyproofness)**.** *A deterministic mechanism F is* group strategyproof *if there do not exist possible valuations $\mathbf{v}$, a group of agents $C \subset \mathcal{N}$, and alternative valuations for each $j \in C$, v'_j such that $v_j(F(\mathbf{v}_{-C}, \mathbf{v}'_C)_j) > v_j(F(\mathbf{v})_j)$ for all j.*

Similar to above, it is impossible for a group of agents to all lie about their values in a way that makes all of them strictly better off. Note that strategyproofness is a special case of group strategyproofness where the group is simply a single agent.

Consider the following mechanism we'll call MNW^{tie}. Each vector of valuations $\mathbf{v}$ is mapped to some arbitrary allocation $\mathbf{A}$ such that:

1. $\mathbf{A}$ is an MNW allocation
2. $\mathbf{A}$ does not allocate any non-valued goods $N_{\mathbf{v}}$
3. Given the first two constraints, $\mathbf{A}$ maximizes the lexicographic value of $(v_1(A_1), \ldots, v_n(A_n))$. This is simply a tie-breaking rule that prefers lower valued agents.

We'll provide a few examples now of how MNW^{tie} works. We'll represent values as a matrix, with rows representing agents, columns goods, and a specific row and column being that agent's value for that good. Consider the valuations: $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. The only possibility for MNW^{tie} is $A_1 = \{g_1\}$ and $A_2 = \emptyset$. This is because g_2 cannot be allocated as it is non-valued, and MNW^{tie} must prefer the allocation of giving g_1 to agent 1 than to agent 2. Note that possible allocations for MNW^{tie} need not be unique. Consider $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Any allocation that gives two items to agent 1 and one item to agent 2 is acceptable, and can be chosen arbitrarily (although consistently).

Theorem 2. *MNW^{tie} is group strategyproof.*

Proof. First, we'll need to prove a few more properties of MNW^{tie}.

Lemma 4. *For any allocation problem $\mathcal{A} = (\mathcal{N}, \mathcal{M}, \mathbf{v})$ and for all agents $i \in \mathcal{N}$, if $MNW^{tie}(\mathcal{A}) = (A_1, \ldots, A_n)$ then if agent i received in item, they must have liked it. That is, $A_i \subseteq V_i$. This means $v_i(A_i) = |A_i|$. Additionally, this extends to any subset $U \subseteq \mathcal{N}$, $A_U \subseteq V_U$, a group of agents can only receive items at least one of them likes.*

Proof. MNW^{tie} is non-wasteful (will not allocate a good that somebody likes to somebody that doesn't like it) and does not allocate goods that nobody likes. $\square$

Define $L^i_{\mathbf{A}} = \{j \in \mathcal{N} | v_j(A_j) < v_i(A_i)\}$ to be the set of agents who have strictly less utility than i. Define $S^i_{\mathbf{A}} = \left\{j \in \mathcal{N} | j \text{ is reachable from } L^i_{\mathbf{A}} \cup \{i\} \text{ in } G(\mathbf{A})\right\}$ to be the set of agents reachable from i and any agent with utility less than i.

Corollary 2. *The goods liked by agents in $S^i_{\mathbf{A}}$ are the exact same as the goods received by agents in $S^i_{\mathbf{A}}$, $V_{S^i_{\mathbf{A}}} = A_{S^i_{\mathbf{A}}}$.*

Proof. Combining Lemma 4 and Lemma 1, for any allocation $\mathbf{A}$ that can be output by MNW^{tie}, the agents of $S^i_{\mathbf{A}}$ only like allocated goods given to agents in $S^i_{\mathbf{A}}$, and they must like all such items. □

Lemma 5. *For any allocation problem $\mathcal{A}$, if $MNW^{tie}(\mathbf{v}) = \mathbf{A}$, then for any agent $i \in \mathcal{N}$ and all agents $j \neq i \in S^i_{\mathbf{A}}$, if $j < i$, then $v_j(A_j) \leq v_i(A_i) + 1$ otherwise if $j > i$, $v_j(A_j) \leq v_i(A_i)$*

Proof. For any allocation $\mathbf{A}$ that is an output of MNW^{tie} and for any agent i, for any j reachable from an agent $k \in L^i_{\mathbf{A}}$, $v_j(A_j) \leq v_k(A_k) + 1 v_i(A_i)$ by Lemma 1. Additionally, any agent j reachable from i must have $v_j(A_j) \leq v_i(A_i) + 1$ again by Lemma 1; if $j > i$, this is restricted to $v_j(A_j) \leq v_i(A_i)$ as if $v_j(A_j) = v_i(A_i) + 1$, then passing back from j to i would keep the same (maximal) Nash Welfare, but increase lexicographically the utility vector. □

Suppose for a contradiction MNW^{tie} was not group strategyproof. That is, there is some allocation problem $\mathcal{A}$ such that there exists a set of agents $C \subset \mathcal{N}$ and alternative valuations v'_j for $j \in C$ such that $v_j(MNW^{tie}(\mathbf{v}_{-C}, \mathbf{v}'_C)_j) > v_j(MNW^{tie}(\mathbf{v})_j)$ for all $j \in C$. Define $\mathbf{A}^{Truth} := MNW^{tie}(\mathbf{v})$ to be the allocation when agents report their true values and $\mathbf{A}^{Lie} := MNW^{tie}(\mathbf{v}_{-C}, \mathbf{v}'_C)$ be the allocation when agents misreport. Let $i \in C$ be the agent of C with minimum utility in the truthful allocation and given that constraint, has the lowest index. Formally, $i = \min(\arg\min_{j \in C} v_j(A^{Truth}_j))$. Let $S = S^i_{\mathbf{A}^{Truth}}$ to save space on notation. We have for every $j \in C$, $|V_j \cap A^{Truth}_j| < |V_j \cap A^{Lie}_j|$ and since $A^{Truth}_j \subseteq V_j$ by Lemma 4, this simplifies to $|A^{Truth}_j| < |V_j \cap A^{Lie}_j|$. If that agent also happens to be in S ($j \in S \cap C$, this is the case for i and potentially others), since $V_j \subseteq V_S$ by definition, we can simplify again to $|A^{Truth}_j| < |V_j \cap A^{Lie}_j| \leq |V_S \cap A^{Lie}_j|$. Define $R \subseteq S$ to be the set of agents in S that can reach *some* misreporting agent $j \in C$ in $G(\mathbf{A}^{Lie})$.

Claim 1. *There exists an agent $j \in R \setminus C$ such that $|A^{Lie}_j| < |A^{Truth}_j|$*

Proof. Suppose for a contradiction for all $j \in R \setminus C$, $|A^{Lie}_j| \geq |A^{Truth}_j|$. For all such j, since $j \notin C$, j is reporting their true values so by Lemma 4, $A^{Lie}_j \subseteq V_j$. Since $j \in R \subseteq S$, $V_j \subseteq V_S$ by definition. Putting that together, $A^{Lie}_j \subseteq V_j \subseteq V_S$, so $|A^{Lie}_j \cap V_S| = |A^{Lie}_j| \geq |A^{Truth}_j|$. Additionally, for each $j \in R \cap C \subseteq C$, we have that $|A^{Lie}_j \cap V_S| \geq |A^{Lie}_j \cap V_j| > |A^{Truth}_j|$. Since the bundles for a specific allocation are disjoint, we can add these inequalities together for all agents in $R \setminus C$ and $R \cap C$ (which will together form just R) to get $|A^{Lie}_R \cap V_S| > |A^{Truth}_R|$. The inequality is strict as there must be at least one agent in $R \cap C$, namely i. Since $A^{Truth}_S = V_S$ by Corollary 2, this becomes $|A^{Lie}_R \cap A^{Truth}_S| > |A^{Truth}_j|$. This implies there must be some good that is in both A^{Lie}_R and A^{Truth}_S but not A^{Truth}_R. Therefore, there exists an agent $j \in R$, and $k \in S \setminus R$ such that $g \in A^{Lie}_j$ and $g \in A^{Truth}_k$. Since $g \in A^{Truth}_k$ this means $v_k(g) = 1$ by Lemma 4. Since $k \notin R$, $k \notin C$ as every node can reach itself. Therefore, k must still like this item even in the misreported allocation problem, meaning there is an edge from k to j in $G(\mathbf{A}^{Lie})$, k can reach whatever coalition agent j could reach. This contradicts that $k \notin R$. □

Claim 2. *The allocation computed by passing back to j from an agent $k \in C$ that is reachable from j is strictly preferred to $\mathbf{A}^{Lie}$ by MNW^{tie}.*

Proof. We have

$$\begin{aligned} v'_k(A^{Lie}_k) &= |A^{Lie}_k| && \text{(By Lemma 4)} \\ &\geq |A^{Lie}_k \cap V_k| \\ &> |A^{Truth}_k| && \text{(By assumption for contradiction)} \\ &\geq |A^{Truth}_i| && \text{(By choice of } i\text{)} \end{aligned}$$

Additionally, $v_j(A^{Lie}_j) < v_j(A^{Truth}_j)$ from applications of Lemma 4 to Claim 2. After passing back, all utilities are the same except agent k's is replaced with $v'_k(A^{Lie}_k) - 1$ and agent j's is replaced with $v_j(A^{Lie}_j) + 1$. Note that the same set of goods has been allocated, it must be the case that this new allocation is fully-allocated and does not allocate goods that no agent likes (otherwise this would not be the case for $\mathbf{A}^{Lie}$).

If $v_j(A^{Lie}_j) + 1 \leq v'_k(A^{Lie}_k) - 1$, then this new allocation has a strictly higher Nash welfare. If not, note that $v_j(A^{Lie}_j) + 1 \leq v_j(A^{Truth}_j)$ and $v_j(A^{Truth}_j) \leq v_i(A^{Truth}_i) + 1$ from Lemma **??**. Additionally, $v'_k(A^{Lie}_k) \geq v_k(A^{Truth}_k) + 1 \geq v_i(A^{Truth}_i) + 1$ by choice of i. Therefore, the only way this inequality could not hold is if $v_j(A^{Lie}_j) + 1 = v_j(A^{Truth}_j) = v_i(A^{Truth}_i) + 1 = v_k(A^{Truth}_k) + 1 = v'_k(A^{Lie}_k)$. By the second equality, $j < i$ by Lemma 1. By the third equality, $k \geq i$ since their values in the truthful allocation are equal. Therefore, $k > j$. In this case, the Nash welfare remains exactly the same, but the utility vector is larger lexigraphically. □

This contradicts the fact that $MNW^{tie}(\mathcal{N}, \mathcal{M}, (\mathbf{v}_{-i}, v'_i)) = \mathbf{A}^{Lie}$. □

Proposition 1. *MNW^{tie} can be computed in polynomial time.*

5 Randomized Setting

As in many subfields of computer science, it's common to ask the question, can randomness help? In our case, as in many others, it can. First we'll extend many of the definitions from before to make sense when we talk about randomization. A *randomized mechanism F* is a function that given valuations $\mathbf{v}$, outputs a distribution over allocations $\mathcal{D}$. Many of the properties of deterministic mechanisms can be extended to the randomized world. A randomized mechanism F is called ex-ante EF or ex-ante PO if all possible outputted distributions are EF or PO in expectation,. That is, they are EF and PO if we replace the values of the agent by their expected values. Similarly, strategyproofness and group-strategyproofness can be defined in terms of the expectation of the agent. A randomized mechanism is ex-post EF, ex-post EF1, or ex-post PO if all allocations in the support of any outputted distribution is EF, EF1, or PO.

Theorem 3. *There is a randomized mechanism that is group strategyproof, ex-ante envy-free, whose support contains only Maximum Nash Welfare allocations (which implies it is ex-post Pareto Efficient and ex-post EF1) that can be computed in polynomial time.*

6 Conclusion

References

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