A Framework for Smoothed Analysis of Social Choice

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We propose a new model of smoothed analysis for the social choice setting. In the social choice setting, the goal is to use a voting rule to select one of m candidates based on the preferences of n agents, each of whom expresses their preferences as a ranking over the candidates. Often there are instances of these preferences, known as profiles, in which voting rules are inconsistent with desirable axioms. From the standpoint of *worst-case analysis*, the existence of any such profile draws us to conclude that the rule is simply inconsistent with the axiom. However, this mode of analysis is too coarse to distinguish between cases where such profiles are common versus where they may be rare and thus unlikely to occur in practice. While some social choice research aims to move beyond the coarseness of worst-case guarantees, most existing models make unrealistic assumptions about inputs. This paper presents an approach for applying the broadly successful framework of *smoothed analysis* to the social choice setting, which requires only the weak assumption that the world is slightly noisy. We propose a simple yet general model closely following the smoothed analysis framework. We establish several useful properties of this model, prove a dichotomy result showing that (essentially) all inconsistencies between voting rules and axioms are asymptotically (as $n \to \infty$) either brittle or robust, give state-of-the-art convergence rates for both directions of this result, and finally, apply these theoretical bounds to classify pairs of voting rules and axioms by their brittleness or robustness. A notable result we give here, which arises from an additional condition we establish for brittleness, is that the widespread worst-case inconsistencies between a strong notion of group-strategy-proofness and *almost any voting rule* are brittle. This result suggests that the worst-case impossibility proven by Gibbard and Satterthwaite is unlikely to be a concern in practice, particularly when elections are large.

1 INTRODUCTION

Since its introduction in 2001, *smoothed analysis* has proved a powerful tool in finding meaningful positive results in settings with stark worst-case impossibilities [12, 13]. In smoothed analysis, an adversary supplies an instance, that instance is perturbed by a small amount of noise, and then one analyzes the probability of performance being "good" on the new instance post-perturbation. In other words, in doing smoothed analysis, one proposes to show in a given setting that all worst-case instances are *brittle* — closely surrounded by "good" instances — and thus the only assumption needed to good performance is that the world is slightly, passively noisy.

This type of analysis may seem only very slightly beyond the worst case — and for small amounts of noise, it is — but, as illustrated by Spielman and Teng in the context of the Simplex algorithm, sometimes looking only so slightly beyond the worst-case is sufficient to ensure performance far exceeding that guaranteed in the worst case [14]. Such results are highly valuable when attainable: they demonstrate that worst-case instances are extremely unlikely to occur in practice, meaning that prohibitive worst-case bounds are much less relevant to practical considerations.

One setting ripe for smoothed analysis is *social choice* because it presents many practically burdensome worst-case impossibilities. In the social choice setting, the canonical task is to facilitate a collective decision by n agents to select one of m candidates. Agents' preferences are expressed as rankings over the candidates, and collectively, these rankings are called a *profile*. Aggregation over a profile is performed by a *voting rule*, which takes in a profile and outputs a set of winning candidates. Also central to social choice impossibilities are *axioms*, which are natural desiderata on how voting rules *should* work. A wide class of axioms in social choice can be phrased as specifying a set of allowable winners for a given profile. For example, the unanimity axiom specifies that if an candidate is ranked first by all agents in a given profile, then that candidate must be the sole winner. We say a voting rule R *satisfies* an axiom A if on all profiles, R's set of winners is among the sets deemed allowable by A. R and A are *inconsistent* when there exists a profile on which R's set of winners is *not* among those allowed by A. We call such a profile a *counterexample* with respect to (R, A).

Although some of the most famous worst-case impossibilities in social choice are axiomatic paradoxes (claims of the form, "no voting rule can satisfy this set of axioms simultaneously"), here we consider inconsistencies between voting rules and axioms. Such inconsistencies are widespread, to the point that most voting rules are inconsistent with at least one highly desirable axiom [16], and conversely, there are highly desirable axioms such as strategy-proofness with which basically *all* reasonable voting rules are inconsistent.[1] Knowing whether any of these inconsistencies are brittle would enable more informed choices between voting rules. It would also help us identify axioms that we can expect to be satisfied in practice by almost all voting rules, and thus which are less important to prioritize when they come at the cost of other axioms. As we describe in the related work, others have applied random [1, 6, 8] and semi-random [17] models of social choice to analyze social choice impossibilities beyond the worst-case. However, we feel our model and approach have several advantages over existing models in their ability to achieve the benefits of smoothed analysis in a simple and generalizable way.

As such, our goals in this paper are (1) to introduce a smoothed model of social choice that follows the described paradigm of smoothed analysis, and (2) to use this model to identify brittle inconsistencies in pairs of voting rules and axioms. In the social choice setting, we assume that the noise from the world is introduced to profiles (rather than e.g., the axiom or voting rule definition). Profiles are the inputs, so from a smoothed analysis standpoint, they are the most natural component

[1]The difficulty of satisfying strategyproofness follows from the Gibbard-Satterthwaite theorem, which says that any onto voting rule choosing a single winner that is non-dictatorial and permits $m > 2$, must not be strategyproof [7].

of the system to perturb. Assuming that noise occurs on profiles also makes sense from a practical standpoint, as people's preferences are likely susceptible to small perturbations in their daily lives, and even to the process by which their preferences are solicited [5, 9, 10], whereas the definitions of voting rules and axioms are controlled by the election designer.

1.1 Our noise model (formally defined in Section 2)

To ensure the broadest possible applicability of our results, our main goal in defining our noise model is to assume as little as possible about the form of the noise it adds to profiles. It is also important that our model allows simple measurement of "how much" noise we are adding. Based on these goals, we define a highly general noise model, which accepts an adversarially-selected profile $\boldsymbol{\pi}$, and outputs a "noisy" distribution over profiles. The quantity of noise added is measured by $\phi \in [0, 1]$, the sole parameter of our model. In this process of adding noise, we make one main assumption (Assumption 1 (Independence)), and then establish several other regularity conditions. We state these assumptions here informally and then formalize them in Section 2. We note for intuition that our model, under these assumptions, (substantially) generalizes the popular Mallows noise model (e.g., see [11]).

(1) Independence: Noise is applied independently to agents' individual rankings.
(2) Neutrality: The noise model is symmetric across candidates (i.e., if the candidates are renamed, the effect of the noise does not change).

By (1)-(2), we can subsequently define the behavior of our model in terms of individual rankings, rather than entire profiles. Applying our model on an individual ranking can be seen as equivalent to applying it to a 1-agent profile, and we will do this throughout the paper. (3)-(4) are weak regularity conditions, and (5)-(6) serve to establish ϕ as a weakly constrained tool for measuring how much noise is added.

(3) Continuity in ϕ: The distribution over profiles resulting from the noise changes continuously with ϕ.
(4) Positivity: When any nonzero amount of noise is applied to a ranking (i.e., $\phi > 0$), the resulting distribution over rankings assigns positive probability to all rankings.
(5) Weak Monotonicity in ϕ: The positive probability in the previous assumption must weakly increase with ϕ.
(6) Extremal values of ϕ: When $\phi = 0$, no noise is added (the distribution is a point mass on the original ranking); when $\phi = 1$, maximal noise is added (the distribution is equivalent to the impartial culture model—see, e.g., [6]).

We note that our results don't centrally depend on assumption (5)— (3)-(4) are sufficient to give all results; we assume (5) because it is not prohibitive, it simplifies the exposition, it allows us to give more usefully parameterized bounds, and it gives intuition for how ϕ can be used meaningfully as a measurement.

1.2 Technical challenges and approach

Given an inconsistent voting rule-axiom pair (R, A), we seek to prove that this inconsistency is *brittle*—that, after adding a small amount of noise via our model to a profile, the resulting profile will be a counterexample with probability converging to zero as the number of agents tends to infinity. Otherwise, we aim to prove that (R, A)'s inconsistency is *robust*, meaning that there exists a profile to which we can add a larger amount of noise, and the resulting profile will be a counterexample with probability converging to 1.

To prove these kinds of claims for a given voting rule-axiom pair (R, A), we must be able to answer the question: *are the profiles from our noise model likely to be counterexamples with respect* *to R and A?* The first challenge in answering this question is posed by the generality of our noise model: given an arbitrary profile and value of ϕ as input, our noise model has so few restrictions that we can *a priori* say little about the distribution over profiles it will induce. Moreover, we do not want to artificially restrict the support of our noise model to the specific R or A we are considering, since, by assumption, we do not control this noise. To make matters more difficult, we know that even slight changes across profiles can change the outcomes of voting rules and axioms, and while these changes do have some regularity, the combinatorial nature of the set of all profiles makes it difficult to capture this regularity across profiles and use it to our benefit. It is particularly difficult to do so in the way that is general enough to study a wide array of axioms and voting rules at once, which we aim to do.

The first step we take in addressing these challenges is to use a more succinct representation of profiles. In particular, we represent profiles as *proportions*, where a profile's proportion is $m! - 1$-length vector whose entries are the *fraction* of agents in that profile with each ranking. We think of these profiles as laid out over the $m! - 1$-dimensional rational simplex containing all proportions for any n (with fixed m). Then, we reason about the space of counterexamples for a general pair (R, A) as subsets of this simplex.

Next, we would like to determine how the distribution given by our noise model is overlaid on the space of profiles, and in particular, how much of its probability mass is placed on counterexamples for a given (R, A) as $n \to \infty$. One natural condition under which almost *any* distribution would place little probability mass over the set of counterexamples is that set having small "small" volume. Note that such a condition would be general across all pairs (R, A), exactly as we hoped for. We formalize this notion of small sets via measure theory, sometimes making distinctions based only on whether sets have positive measure, and in other cases using more sophisticated measure-theoretic properties. The key to enabling the use of measure here is we do not directly study the rational simplex described above, which itself has measure zero; we instead study the *closure* of that rational simplex and ensure our results hold for realizable profiles.

Finally, to get a handle on the distribution over profiles induced by our noise model, we apply our assumption of independence. This assumption allows us to think of profiles, at a high level, as the sum of independent indicators (rankings). We use this idea to formally show that, although the distribution from which we draw each agent's ranking *individually* has little structure, the distribution over the *sums of rankings*, i.e., over profiles, should concentrate to a Gaussian n gets large. We formalize this argument in Section 3.

Finally, having established our model (Section 2), our simplex representation (Section 3), and the convergence of our noise model's distribution over profiles (Section 4), we then bound the mass our converging noise distribution will place on counterexample regions. We take this approach to prove conditions for both brittleness and robustness but in different ways. In the positive direction (showing brittleness), we apply the intuitive fact that Gaussian distributions place low probability mass on regions of support with small measure to show that the probability mass placed on counterexamples will approach 0. In the negative direction (showing robustness), we do not directly use the normality of our noise model; we simply show that the expectation of the distribution over profiles is within a sufficiently large ball of counterexamples, and thus by simple concentration, the probability mass placed on counterexamples will approach 1.

1.3 Results and contributions

Although this approach is simple, it yields powerful results.

1.3.1 Results Part 1: General results across voting rules and axioms. Our first set of results, in Section 5, are general across voting rules and axioms. Our first main result (Theorem 6) is an equivalence showing that voting rule-axiom pairs (R, A) (except when A is of a very specific type) fall into one of two classes: they are either *brittle* or *robust*, as described above. The ability to broadly classify (R, A) this way is not obvious, especially given that our notion of robustness is stronger than the negation of brittleness. The condition distinguishing these classes is also simple: it is whether the set of counterexamples with respect to (R, A) is measure zero. Included in these results are explicit rates of convergence to both brittleness and robustness, including constants, which match the convergence rates achieved by other work in comparable settings [17, 18]. We then extend these general results to express convergence rates within the more restricted Mallows noise model (Corollary 7), as this particular perturbation model may be of independent interest.

Not covered by the theorem above are (R, A) in which A is parameterized by n. One important such example, for which we will provide results in this paper, is $\rho(n)$-Group-Strategy-Proofness, in which some portion $\rho(n)$ of the agents in the profile can deviate from reporting their rankings truthfully. We use new ideas in this theorem to prove an additional condition for brittleness that applies to axioms of this type, beyond just $\rho(n)$-Group-Strategy-Proofness (Theorem 8).

1.3.2 Results Part 2: Application of general results to voting rule-axiom pairs. Finally, in Section 6 we show how to apply the results in the previous to classify inconsistencies between voting rules and axioms as brittle or robust. The most notable among these results is one due to an application of Theorem 8, by which we show that one of the strongest notions of group strategy-proofness, up to groups of size $\sqrt{n}$, is satisfied with high probability for large n *by essentially all voting rules*. In the worst case, this axiom is satisfied by essentially no reasonable voting rules. Additionally, the simplicity of our condition in Theorem 6 allows us quickly classify many other pairs (R, A) via simple arguments. For instance, we find that inconsistencies between the axiom Resolvability and most—but not all—voting rules are brittle (one we study is robust!). We additionally study the inconsistency between the popular axiom Condorcet Consistency and the entire class of positional scoring rules (including Plurality, Borda Count, and Veto). As we prove these results, corollaries arise about other voting rule-axiom pairs and other social choice impossibilities, which we note throughout the section.

1.3.3 Contributions. The contributions of this paper are fourfold. **(1) We introduce a new model of smoothed analysis in social choice**, which adheres closely to the smoothed analysis framework. This model has several advantages: It is simple to work with, is general in the form of noise it assumes, allows easy measurement of the quantity of noise being added, and as we demonstrate, it requires no context-specific restrictions on the nature of the noise added in order to permit positive results. **(2) We establish several useful properties of this model**, including the normality of the noise over profiles, and clean behavior of the covariance matrix. We take care in our exposition to ensure these properties are written clearly and can be easily extended to other contexts. **(3) We prove simple conditions (including one that is necessary and sufficient) for convergence to brittleness**, and give state-of-the-art convergence rates. The conditions we find are simple to work with, and yet general across a vast array of voting rules and axioms, hopefully facilitating the future smoothed analysis of inconsistencies of other such pairs. In the discussion, we address how these results can also be extended to study other kinds of social choice impossibilities. **(4) We prove results about the brittleness and robustness of popular voting-rule axiom pairs**, in the process illustrating how to apply our general bounds to study specific pairs. These results have real implications: for instance, we find that strategy-proofness is almost always satisfied, making way for considering pursuing stronger guarantees on the truthfulness of voting mechanisms.

1.4 Related work

This paper is fundamentally built on the concept of smoothed analysis, which was introduced by Spielman and Teng in 2001 to provide a theoretical basis for Simplex's fast runtime in practice, despite its poor worst-case guarantees [14]. Since their work, smoothed analysis has been applied to show positive results in the domains of heuristic algorithms [12], machine learning, optimization, and many others [15]. The idea of applying smoothed analysis to the domain of social choice has been proposed in only two cases that we know of: in work by Lirong Xia [17], which proposes a model distinct from ours, and in a 2020 blue-sky paper [2], which did not contain technical results. The blue-sky paper does, however, explicitly suggest the use of Mallows noise as a perturbation model for settings in which agents express their preferences as rankings, an approach which our results encompass but substantively generalize. Given that our work is far closer in proximity to Xia's work than other work in social choice, we primarily dedicate this section to discussing the similarities and differences between our work and theirs.

Comparison of models. In Xia's model, first introduced in 2020 [17], noisy distributions from which profiles are drawn are implied by the following process: first, define an allowable set of distributions over rankings; then, draw agents' rankings from the worst possible assignment of these allowable distributions to agents. In defining their model, Xia makes many assumptions in common with ours, including independence, neutrality, and positivity. From a technical standpoint, Xia's model generalizes ours in the sense that any distribution over profiles realizable in our model can be realized in theirs. One main source of additional structure in our model comes with our introduction of a parameter $\phi \in [0, 1]$, which serves to measure *how much* noise we are adding in our model. We impose the most basic restrictions on this parameter possible while requiring that it functions as such a measurement. This parameter, while adding some restrictions, has benefits too, enabling more intuitively parameterized bounds and convergence rates, the straightforward study of "how much" noise is required for certain outcomes, and the direct application of our results to more restricted parameterized models, such as the popular Mallows model. Even with these restrictions, like Xia's, our model permits high generality in the form of the noise added to input rankings.

Comparison of methodology. As Xia does in [17], our work uses proportional representations of profiles. Moreover, at a high level, the main results of both our work and Xia's are achieved by showing where our respective noise distributions converge to, relative to regions of counterexamples. The geometric objects we study, however, are somewhat different—for instance, Xia does not apply measure theoretic techniques on the closure of the rational simplex of proportions, as we do. In fact, despite this approach being quite useful in this context, to our knowledge it is thus far uncommon in social choice research. The only work we know of applying measure theory to characterize sets of social choice inputs is work by Dasgupta and Maskin [4], but they use measure to quantify subsets of the space defined by different conditions and do so in pursuit of substantially different goals. One other key methodological difference between our work and Xia's is that, in getting some of their positive results, Xia imposes external restrictions on the allowable distributions over rankings specified in their model, in particular in ways that restrict how the resulting profiles can intersect with the space of counterexamples for the current claim they are proving [17]. We explicitly avoid this kind of assumption in our work, as we feel it deviates from the core smoothed analysis assumption that the only noise we have access to is added passively by the world, agnostic to our axiomatic goals.

Comparison of results. Xia's model has been applied to consider various social choice impossibilities, including the Condorcet paradox and other related paradoxes [17, 19], Condorcet inconsistency of positional scoring rules [20], and the brittleness of ties [18]. Our models overlap in the types of questions they can be used to answer, and on points of overlap we draw the similar conceptual conclusions that ties are unlikely, the ANR paradox is unlikely to be an issue, and that positional scoring rules' inconsistency with Condorcet is robust to noise. However, we also introduce a conceptually new kind of result, which does not appear in any of Xia's work, and handles axioms parameterized by n. This allows us to get positive results on the satisfiability of group-strategy-proofness, which to our knowledge, it yet unstudied in smoothed frameworks.

Beyond specific impossibilities, many of both our results and Xia's take the form of bounds on the convergence rates (as $n \to \infty$) of our respective noise distributions over profiles, in particular to regions with either few or many counterexamples. Although our respective approaches to getting these bounds are in some cases different, we get comparable convergence rates: in particular, we get the same exponential convergence rates when converging to robustness [17], and for the positive direction in applications where our studied applications overlap (e.g., Resolvability), we get the same convergence rates to brittleness of $O(1/\sqrt{n})$ [18]. In our work, we additionally give convergence rate bounds that include constants and are parameterized intuitively by ϕ.

2 MODEL

In the standard social choice setting, we have a set of m candidates M and a set of n agents N. These n agents express their preferences over the m candidates as complete rankings. We let $\mathcal{L}(M)$ be the set of $m!$ possible rankings over the candidates, which we will express as $\mathcal{L}$ when M is clear from context. We will fix an arbitrary order over rankings in $\mathcal{L}$, so that we can talk about the j-th ranking, and write vectors with the π-th component. We express a single ranking as π, and a profile as a vector $\boldsymbol{\pi}$ of n rankings, one per agent, where π_i is agent i's ranking: $\boldsymbol{\pi} = (\pi_i | i \in [n])$. We define *addition* over profiles in the natural way, so that the profile $(\boldsymbol{\pi} + \boldsymbol{\pi}') = (\pi_i | i \in [n]) \cup (\pi'_i | i \in [n'])$.[2] We extend this operation to permit scalar multiples of profiles, such that adding a profile together z times is expressed as $z\boldsymbol{\pi}$. We let Π be the set of all profiles for any n and m, and let Π_n be the set of profiles with a fixed n.

2.1 Voting rules and axioms

Voting rules and axioms. Define a *voting rule* $R : \Pi \mapsto 2^M$ as a function that maps any given profile to the set of winners, where 2^S is, in general, the power set of S. Then, $R(\boldsymbol{\pi})$ is the set of winners chosen by the voting rule on $\boldsymbol{\pi}$. Define an *axiom* A as a mapping from the space of profiles to all allowable sets of winners. When formally defining axioms, we distinguish between *absolute axioms*, which are defined over a single profile, and *relative axioms*, which are defined over multiple profiles at once—for example, an absolute axiom is Condorcet Consistency, as it specifies a property of a single profile, whereas Monotonicity is a relative axiom, because it specifies a property across two different profiles. Formally, let an absolute axiom be a mapping $A : \boldsymbol{\pi} \to 2^{2^M}$, and let a relative axiom be a mapping $A : \boldsymbol{\pi}^2 \to \left(2^{2^M}\right)^2$. In our definition of relative axioms, we restrict the two input profiles to have the same value of m, and Π_n to denote the set of all profiles with fixed number of agents, n. We consider m to be a constant in all of our results.

In this paper, we will give results pertaining to the many popular voting rules and axioms. Notable exceptions are the axioms Clone-Proofness and Independence of Irrelevant Alternatives (IIA), which are not encapsulated in our definition of relative axioms because they are defined across profiles with different m values. All voting rules and axioms studied in this paper are defined

[2]How the agents are precisely indexed in the resulting profile will not be relevant to our results, but for formality, we can assume that the agents in profile $(\boldsymbol{\pi} + \boldsymbol{\pi}')$ will be indexed such that agent i in $\boldsymbol{\pi}$ is at index i in $(\boldsymbol{\pi} + \boldsymbol{\pi}')$, and agent i in $\boldsymbol{\pi}'$ is at index $i + n$ in $(\boldsymbol{\pi} + \boldsymbol{\pi}')$.

in Appendix A according to standard definitions. We do not define any tie-breaking mechanisms for our voting rules as our results hold for arbitrary tie-breaking mechanisms (when analyzing the axiom Resolvability, we allow ties and explicitly consider their brittleness). Of the many existing definitions of strategyproofness, we consider what we call ρ-group-strategyproofness, where ρ is the maximum fraction of agents who can lie. Within the possible definitions of ρ-group-strategyproofness, we take one of the strongest.

2.2 Noise Model

Our smoothed noise model $\mathcal{S}$ accepts an arbitrary profile, applies a perturbation whose magnitude is measured by the parameter $\phi \in [0, 1]$, and outputs a distribution over profiles. We will henceforth use the notation $\mathcal{D}(S)$ to denote the collection of all possible distributions over a set S, and $D_{\text{eq}}(S)$ to denote the uniform distribution over a set S. Then, for a given ϕ, our $\mathcal{S}$ is formally defined as a function $\mathcal{S}_\phi : \Pi \to \mathcal{D}(\Pi)$. The main assumption we make about this model is the following:

Assumption 1 (Independence). *Applying $\mathcal{S}$ to a profile $\boldsymbol{\pi}$ is equivalent to the Cartesian product of the distributions yielded by applying $\mathcal{S}$ to all agents' rankings in that profile independently:*

$$\mathcal{S}_\phi(\boldsymbol{\pi}) = \prod_{i=1}^{n} \mathcal{S}_\phi(\pi_i)$$

Throughout the paper, we will abuse notation and allow $\mathcal{S}_\phi$ to operate directly on an individual ranking in addition to entire profiles, where when the input to $\mathcal{S}_\phi$ is a ranking, the output is the ranking equivalent to its behavior on the corresponding one-agent profile.

In light of this assumption, we can reason about $\mathcal{S}$ with respect to its behavior on single agents' rankings. Assumptions 2-4 define weak regularity conditions on this behavior, and Assumptions 5-6 establish some weak conditions to ensure ϕ is a reasonable measure of "how much" noise is added. Some of these assumptions concern the smallest probability $\mathcal{S}_\phi$ places on any ranking, denoted and defined as $\text{min-prob}(\mathcal{S}_\phi) = \min_{\pi,\pi' \in \mathcal{L}} \Pr[\mathcal{S}_\phi(\pi) = \pi']$.[3]

Assumption 2 (Neutrality). *$\mathcal{S}$ is symmetric (i.e.g, cannot a priori discriminate) across candidates. That is, for all permutations of the candidates σ, $\mathcal{S}(\sigma(\boldsymbol{\pi})) = \sigma(\mathcal{S}(\boldsymbol{\pi}))$, where $\sigma(\boldsymbol{\pi})$ is the same profile with candidates permuted by σ and $\sigma(\mathcal{S}(\boldsymbol{\pi}))$ is the same distribution with candidates in the outputted profiles permuted by σ.*

Assumption 3 (Continuity in ϕ). *Across $\phi \in [0, 1]$ the distributions $\mathcal{S}_\phi$ are continuous in ϕ.*

Assumption 4 (Positivity). *For all $\phi \in (0, 1]$, $\text{min-prob}(\mathcal{S}_\phi) > 0$.*

Assumption 5 (Weak Monotonicity over ϕ). *The value $\text{min-prob}(S_\phi)$ is non-decreasing in ϕ.*

Assumption 6 (Extremal values of ϕ). *Let the behavior of $\mathcal{S}_\phi$ at the endpoints of ϕ be defined to mean the minimum and maximum amount of noise added, as follows:*

- *$\phi = 0$ (no noise): $\mathcal{S}_0(\pi)$ is the point distribution at π.*
- *$\phi = 1$ (maximum noise): for all π, $\mathcal{S}_1(\pi) = \mathcal{D}_{eq}(\mathcal{L})$.*[4]

2.2.1 *Mallows Noise Model.*

As mentioned in the introduction, our model generalizes the popular Mallows noise model, which like ours is defined to apply independent noise to each ranking in a profile. Given that this model is possibly of special interest, we define it Mallows model here (based on [11]) and will provide explicit convergence rates in this model later on. Due to independence, we can define this model, which adds noise to a *profile*, in terms of how it perturbs a single ranking

[3]For intuition, by neutrality (Assumption 2), π can be treated as arbitrary, so we could have defined this min over just π'.
[4]Note that this implies that for all $\boldsymbol{\pi}$, $\mathcal{S}_1(\boldsymbol{\pi})$ is is equivalent to the impartial culture model.

(as it is traditionally defined). Let $d : \mathcal{L} \times \mathcal{L} \to \mathbb{Z}^+$ be the Kendall-Tau distance. Then, the Mallows model $\mathcal{S}^{Mallows}$, is defined as

$$Pr\left[\mathcal{S}_\phi^{Mallows}(\pi) = \pi'\right] = \frac{1}{Z}\phi^{d(\pi,\pi')},$$

where $Z = \sum_{\pi' \in \mathcal{L}} \phi^{d(\pi',\pi)}$, which can be shown to be simple polynomial in ϕ. As is visible form the functional form of the probability above, the distribution over rankings given by $\mathcal{S}_\phi^{Mallows}(\pi)$ is symmetrically concentrated around the ranking π, giving substantial regularity to its output. One way in which our model meaningfully generalizes Mallows is that we make no such requirement.

2.3 *Brittle* and *robust* inconsistencies

We express a voting rule-axiom pair as (R, A). First thinking in terms of worst-case analysis (i.e., in the absence of any noise), we say that R *satisfies* A, an absolute axiom, if $R(\boldsymbol{\pi}) \in A(\boldsymbol{\pi})$ for all $\boldsymbol{\pi} \in \Pi$. Likewise, R satisfies a relative axiom A if $(R(\boldsymbol{\pi}), R(\boldsymbol{\pi}')) \in A(\boldsymbol{\pi}, \boldsymbol{\pi}')$ for all $\boldsymbol{\pi}, \boldsymbol{\pi}' \in \Pi$. If R does not satisfy A, then there must exist a profile $\boldsymbol{\pi}$ which is a *counterexample* with respect to (R, A). Counterexamples with respect to (R, A) are just the negations the definitions of R satisfying A.[5] We say that (R, A) have an *inconsistency* if there exists a counterexample with respect to (R, A). We define the set of counterexamples with respect to (R, A) as $\Pi^{\neg(R,A)}$. We have defined formally what positive and negative results look like in the worst-case setting. Now, we define positive and negative results in the *smoothed* setting.

In the positive direction, we say that an inconsistency between (R, A) is $(\mathcal{S}, \phi)$*-brittle* if, as n gets large, the probability of the perturbed instance being a counterexample after adding at least ϕ noise *converges uniformly (over profiles)* to zero:

$$\lim_{n\to\infty} \sup_{\phi' \in [\phi,1]} \sup_{\boldsymbol{\pi} \in \Pi_n} \Pr\left[\mathcal{S}_{\phi'}(\boldsymbol{\pi}) \in \Pi^{\neg(R,A)}\right] = 0.$$

We say that an inconsistency between (R, A) is (S, ϕ)*-brittle at a rate* $f(n, \phi)$ if

$$\sup_{\phi' \in [\phi,1]} \sup_{\boldsymbol{\pi} \in \Pi_n} \Pr\left[\mathcal{S}_{\phi'}(\boldsymbol{\pi}) \in \Pi^{\neg(R,A)}\right] \le f(n, \phi).$$

We say that an inconsistency between (R, A) is $\mathcal{S}$*-brittle* if it is $(\mathcal{S}, \phi)$-brittle for all $\phi \in (0, 1]$. Similarly, it is $\mathcal{S}$*-brittle* at a rate of $f(n, \phi)$ if it is $(\mathcal{S}, \phi)$-brittle at a rate of $f(n, \phi)$ for all $\phi \in (0, 1]$.

In the negative direction, we say that an inconsistency between (R, A) is $\mathcal{S}$*-robust* if there exists $\phi \in (0, 1]$ and a profile $\boldsymbol{\pi}$ such that for all the following sequence $z\boldsymbol{\pi}$ for $z \in \mathbb{Z}^+$ converges as follows:

$$\lim_{z\to\infty} \sup_{\phi' \in [0,\phi]} \Pr\left[\mathcal{S}_{\phi'}(z\boldsymbol{\pi}) \in \Pi^{\neg(R,A)}\right] = 1.$$

Further, we say that an inconsistency between (R, A) is $\mathcal{S}$*-robust* at a rate of $f(n)$ if there exists $\phi \in (0, 1]$ and a profile $\boldsymbol{\pi}$ such that for $z \in \mathbb{Z}^+$:

$$\sup_{\phi' \in [0,\phi]} \Pr\left[\mathcal{S}_{\phi'}(z\boldsymbol{\pi}) \in \Pi^{\neg(R,A)}\right] \le f(zn).$$

We note that *robust*, as defined above, is stronger than the negation of brittle.

[5]Formally, if A is an absolute axiom, $\boldsymbol{\pi}$ is a counterexample with respect to (R, A) if $R(\boldsymbol{\pi}) \notin A(\boldsymbol{\pi})$. Similarly, if A is a relative axiom, $\boldsymbol{\pi}$ is a counterexample if there exists some other profile $\boldsymbol{\pi}'$ such that $(R(\boldsymbol{\pi}), R(\boldsymbol{\pi}')) \notin A(\boldsymbol{\pi}, \boldsymbol{\pi}')$.

3 OUR APPROACH: PROFILE PROPORTIONS IN THE CLOSED SIMPLEX

3.1 Profile proportions

When studying how probabilities of inconsistencies change with growing n, working with profiles tends to be inconvenient, because they are defined to be of a fixed size. Instead, we henceforth work primarily with *profile proportions.* The *profile proportion* for $\boldsymbol{\pi}$, denoted as $\mathcal{P}(\boldsymbol{\pi})$, is the $|\mathcal{L}|-1$-length vector whose entries describe the *proportion* of agents in $\boldsymbol{\pi}$ with each ranking. We omit the $|\mathcal{L}|$-th index because it is redundant, given that the proportions must add to 1. Formally, we think of $\mathcal{P}(\boldsymbol{\pi})$ as being indexed by π (excluding one), and the π-th index of this vector is defined as

$$\mathcal{P}(\boldsymbol{\pi})_\pi = {}^1\!/\!_n\, |\{i|\pi_i = \pi\}|.$$

When we talk about a proportion unrelated to any specific profile, we will denote it as $\mathbf{p}$.

Because this notion of proportions is so natural, we will slightly abuse notation and also use the operator $\mathcal{P}$ to translate other kinds of profile-based objects into proportions-based objects. For example, $\mathcal{P}(\Pi)$ is the set of all possible proportions (i.e., the set of all rational vectors of length $m! - 1$ to ≤ 1). Likewise, $\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi}))$ represents the distribution over *proportions* resulting from applying our noise model to the profile $\boldsymbol{\pi}$. We will occasionally also use the distribution $\mathcal{P}(\mathcal{S}_\phi(\pi))$, which is the distribution over proportions implied by a distribution over single rankings, i.e., this is a distribution over basis vectors (or all 0-entries in the case of the excluded profile), as the proportion-vector representation of a single ranking is just such a basis vector.

In order to reason about proportions instead of rankings for our purposes, we must make sure R and A can operate directly on proportions, rather than just profiles. For this to be valid for a given voting rule R, there must exist a rule $R' : \mathcal{P}(\Pi) \to 2^M$ defined on proportions such that for all profiles $\boldsymbol{\pi} \in \Pi$, it holds that $R'(\mathcal{P}(\boldsymbol{\pi})) = R(\boldsymbol{\pi})$. As discussed in Appendix A.3, this holds for all voting rules and axioms we study. This will become important in the subsequent sections, when we consider whether the distribution over *proportions* induced by our noise model places much mass on counterexamples.

3.2 H, the closure of the simplex of proportions

In addition to cleanly dealing with growing n, the proportions-based representation of profiles has an added benefit that is even more central to our results: in space, the set of all proportions forms a rational simplex, i.e., a dense subset of a closed simplex. Formally, we represent all possible proportions $\mathcal{P}(\Pi)$ as points in $|\mathcal{L}| - 1$-dimensional space where each axis corresponds to a ranking. We will formally define simplex we consider in a moment.

As described in the introduction, our goal is to distinguish pairs (R, A) with brittle versus robust inconsistencies by whether the set $\mathcal{P}\left(\Pi^{\neg(R,A)}\right)$ is "small". One way we define a set being small is it having *measure zero* (per the Lebesgue measure), where we define a measure-zero set in the standard way below.

Definition 1 (Lebesgue measure zero set). A set $S \subseteq \mathbb{R}^d$ has measure 0 if and only if, for all $\epsilon > 0$, S can be covered by the union of open sets $S_i \subseteq \mathbb{R}^d$ such that $Vol\left(\bigcup_i S_i\right) < \epsilon$.

The need to distinguish between measure-zero and positive-measure subsets of profile proportions guides our definition of the simplex we consider H, which in particular must be defined such that is it not itself measure zero. This requirement eliminates rational simplex containing just the points in $\mathcal{P}(\Pi)$, which is by definition measure zero due to its rationality. Thus, we instead define $H = cl(\mathcal{P}(\Pi))$ to be the *closure* of this rational simplex.[6] Working with measures of sets, a tool

[6]Note: the need for H not to be measure zero also motivates our definition of $p(\boldsymbol{\pi})$ as $|\mathcal{L}| - 1$ rather than $|\mathcal{L}|$-dimensional: the closure of the set of all profiles expressed in $|\mathcal{L}|$ dimensions is also measure zero, because it will like on a low-dimensional hyperplane due to all $|\mathcal{L}|$ proportions necessarily adding to 1

enabled by the use of this closed set, will allow us to apply the results in the next section to quickly prove the unlikelihood that noise added by our model will land us in a set of counterexamples, provided that set of counterexamples is measure 0.

4 CONVERGENCE PROPERTIES OF THE DISTRIBUTION $\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi}))$

4.1 The expectation and covariance of $\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi}))$

First, we give intuition for the expectation and variance of the distribution that we will work with in this section, namely $\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi}))$. We think of $\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi}))$ as the random vector of profile proportions that results from perturbing $\boldsymbol{\pi}$ with ϕ-noise. Intuitively, since all agents have noise added independently, we expect this distribution to behave well. To reason about this behavior more directly, we will use the objects $\mathcal{P}(\mathcal{S}_\phi(\pi_i))$, the distribution of noise being added to a single ranking. Recall the meaning of these two objects here:

- For all agents i, $\mathcal{P}(\mathcal{S}_\phi(\pi_i))$ is a random vector of length $m! - 1$, which takes on the value of the basis vector $e_{\pi'}$ with probability $\Pr[\mathcal{S}_\phi(\pi_i) = \pi']$ (and the all 0s vector with remaining probability when the missing ranking is chosen).
- For a profile $\boldsymbol{\pi}$, $\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi})) = 1/n \sum_{i=1}^n \mathcal{P}(\mathcal{S}_\phi(\pi_i))$ where each summand is independent.

Using standard properties of expectation and covariance, we immediately see that $\mathbf{E}[\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi}))] = 1/n \sum_{i=1}^n \mathbf{E}[\mathcal{S}_\phi(\pi_i)]$ and $\mathbf{C}[\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi}))] = 1/n^2 \sum_{i=1}^n \mathbf{C}[\mathcal{S}_\phi(\pi_i)]$. So now let us analyze each $\mathbf{E}[\mathcal{S}_\phi(\pi_i)]$ and $\mathbf{C}[\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi}))]$ more closely.

The expectation. For a agent i, the π'-th component of $\mathbf{E}[\mathcal{S}_\phi(\pi_i)]$ is simply $\Pr[\mathcal{S}_\phi(\pi_i) = \pi']$. Hence, $\mathbf{E}[\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi}))]$ is the average of these over all agents, so the π'-th component is the expected proportion of agents we expect to end up with ranking π', $1/n \sum_{i=1}^n \Pr[\mathcal{S}_\phi(\pi_i) = \pi']$.

The covariance. Likewise, the covariance matrix $\mathbf{C}[\mathcal{P}(\mathcal{S}_\phi(\pi_i))]$ is an $(m! - 1) \times (m! - 1)$ matrix whose entries each correspond to a pair of rankings. We will use the fact that $\mathrm{Cov}(X, Y) = \mathbf{E}[XY] - \mathbf{E}[X]\mathbf{E}[Y]$. For distinct rankings $\pi \neq \pi'$, note that in the support, at most one of these entries can be nonzero at a time, so the expectation of their product is always 0. Using our knowledge of the expectation, we get that this entry is $-\Pr[\mathcal{S}_\phi(\pi_i) = \pi] \cdot Pr[\mathcal{S}_\phi(\pi_i) = \pi']$. For diagonal entries, since values are always 0 or 1, the π, π entry is exactly $\Pr[\mathcal{S}_\phi(\pi_i) = \pi] - \Pr[\mathcal{S}_\phi(\pi_i) = \pi]^2$. We recover the covariance $\mathbf{C}[\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi}))]$ using $\mathbf{C}[\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi}))] = 1/n^2 \sum_{i=1}^n \mathbf{C}[\mathcal{S}_\phi(\pi_i)]$ stated earlier.

4.2 Convergence Lemmas

In proving our first condition for the brittleness of inconsistencies, we will apply the idea that, if the set of counterexamples is small, then the distribution over profile proportions output by our noise model, $\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi}))$, will place little probability mass on these counterexamples. To show this, we will apply the following intuitive fact:

FACT 1. *As long as its covariance matrix is invertible, a normal distribution in arbitrary dimensions places zero probability mass on measure zero sets in its support.*

Or course, in order to apply this fact, we must show that $\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi}))$ converges to a normal distribution at a reasonable rate. Despite the generality of the behavior of our noise model on individual rankings, this is true because a profile can be seen as the sum of independent indicators (rankings). We use this idea below to prove two lemmas about the convergence of $\mathcal{P}(S_\phi(\boldsymbol{\pi}))$ as the number of agents grows large. The first is a strong notion of convergence around the expectation:

LEMMA 2. *Let $\mathcal{S}$ be a noise models, $\phi \in [0, 1]$ a parameter, and $\boldsymbol{\pi} \in \Pi_n$ a profile on n agents. Then,*

$$\Pr[d(\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi})), \boldsymbol{E}[\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi}))]) < \varepsilon] > 1 - 2m! \exp(-2\varepsilon^2 n/m!)$$

The second lemma we prove shows the convergence of $\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi}))$ to a normal distribution. We note that this convergence to the normal distribution is *uniform over profiles*, meaning that for all profiles $\boldsymbol{\pi}$ of any fixed n, $\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi}))$ is at most $O(1/\sqrt{n})$ "distance away" from the normal distribution with expectation and variance corresponding to that of $\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi}))$. As written formally below, we measure "distance away" by comparing the probability mass our distribution and the corresponding normal distribution place on arbitrary convex sets, per the Berry-Esseen bound.

LEMMA 3. *Let $\mathcal{S}$ be a noise models, $\phi \in [0,1]$ a parameter, and $\boldsymbol{\pi} \in \Pi_n$ a profile on n agents. Then, for all convex sets $X \subseteq \mathbb{R}^{m!-1}$,*

$$\left|\Pr\left[\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi})) \in X\right] - \Pr\left[\mathcal{N}\left(\boldsymbol{E}[\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi}))], \boldsymbol{C}[\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi}))]\right) \in X\right]\right| \leq \frac{O((m!)^{7/4})}{\sqrt{n} \cdot \textit{MIN-PROB}(\mathcal{S}_\phi)^{3/2}}.$$

The former is a straightforward application of Hoeffding's inequality. The latter, however, needs incricate use of a relatively general form of the Berry Esseen bound, stated here.

LEMMA 4 (RESTATEMENT OF BERRY-ESSEEN AS IN [3]). *Let $Y_1, \ldots, Y_n$ be independent, mean-zero, $\mathbb{R}^{m!-1}$-valued random variables. Let $S = Y_1 + \cdots + Y_n$, and let C^2 be the covariance matrix of S, assumed invertible. Let $\mathcal{N}(0, C^2)$ be a $m! - 1$-dimensional Gaussian with mean zero and covariance C^2. Then for any convex subset $X \subseteq \mathbb{R}^{m!-1}$,*

$$|Pr[S \in X] - Pr[\mathcal{N}(0, C^2) \in X]| \leq O((m! - 1)^{1/4}) \cdot \left(\sum_{i=1}^{n} \boldsymbol{E}[|C^{-1}Y_i|^3]\right)$$

In order to apply the Berry-Esseen bound and for later technical results, we will need the following properties of the covariance matrix $\mathbf{C}[\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi}))]$, proved in Appendix B.3.

LEMMA 5. *For all noise models $\mathcal{S}$, parameters $\phi \in (0,1]$, and rankings $\boldsymbol{\pi} \in \Pi_n$, the covariance matrix $\boldsymbol{C}[\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi})]$ is invertible and has all positive real eigenvalues lower bounded by MIN-PROB$(S_\phi)/(m!n)$.*

The proofs of Lemmas 2 and 3 can be found in Appendices B.1 and B.2, respectively.

5 GENERAL RESULTS ACROSS VOTING RULES AND AXIOMS

We now present our first main result in Theorem 6: we show that the inconsistency between a broad class of voting rule-axiom pair (R, A) satisfying certain properties is either $\mathcal{S}$-*brittle* or $\mathcal{S}$-*robust* (we will drop the $\mathcal{S}$s henceforth). This is striking because these categories are separated: as we remarked before, *robust* is stronger than the negation of *brittle*. Moreover, the necessary and sufficient condition we get for brittleness is simple: that the closure of $\mathcal{P}(\Pi^{\neg(R,A)})$ has measure zero. This division holds for any noise model in our class, suggesting that this dichotomy might be a fundamental property of voting rule-axiom pairs, rather than a result due to a specific choice of noise model.

For each direction of this theorem, we give explicit convergence rates: under the condition for brittleness, the probability mass placed on $\mathcal{P}\left(\Pi^{\neg(R,A)}\right)$ by our model converges to 0 at a rate of $O(1/\sqrt{n})$, and under the condition for robustness, the probability mass placed on $\mathcal{P}\left(\Pi^{\neg(R,A)}\right)$ by our model converges to 1 at a rate of $\exp(-\Theta(n))$. As discussed in the related work, these match those attained by existing work in both directions; however, unlike existing work, we give explicit constants.

At a high level, the $O(1/\sqrt{n})$ rate in the positive direction comes from the Berry-Esseen bound on the rate of convergence of the distribution $\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi}))$ to the normal distribution, followed by the application of Fact 1. In the negative direction, we get exponential convergence by showing that (1) there exists $\boldsymbol{\pi}$ such that $\mathbf{E}[\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi}))] \in \Pi^{\neg(R,A)}$, (2) $\mathbf{E}[\mathcal{P}(\mathcal{S}_\phi(z\boldsymbol{\pi}))]$ is the same for all $z \in \mathbb{Z}^+$

(so our distribution is converging to a fixed expectation over growing n), and then (3) by a simple concentration bound, the (series of) distribution(s) $\mathcal{P}(\mathcal{S}_\phi(s\boldsymbol{\pi}))$ concentrates around its mean as $z \to \infty$. The fact that the mean is contained within $\mathcal{P}\left(\Pi^{\neg(R,A)}\right)$, which under the conditions of robustness has positive measure, gives the result.

Theorem 6. *Fix a noise model $\mathcal{S}$. Let $X = \bigcup_{j=1}^{\ell} X_j$ where each $X_j \subseteq H$ is convex. Note that X is necessarily measurable.*

- *If X has measure zero, then for all $\phi \in (0,1]$ and profiles $\boldsymbol{\pi} \in \Pi_n$, for all $\phi' \in [\phi, 1]$,*
$$\Pr[\mathcal{P}(\mathcal{S}_{\phi'}(\boldsymbol{\pi})) \in X] \leq \frac{\ell \cdot O((m!)^{7/4})}{\sqrt{n} \cdot \textsc{min-prob}(\mathcal{S}_\phi)^{3/2}}.$$
- *Otherwise, suppose X has positive measure. Then, there exists a ball with positive radius fully contained in X. Let $c_X > 0$ be the radius of the largest ball contained in X. There exists a number of agents n, profile $\boldsymbol{\pi} \in \Pi_n$, and a constant $\phi^* \in (0,1)$ such that for all $\phi \in [0,\phi^*]$, and $z \in \mathbb{Z}^+$ $\Pr[\mathcal{P}(\mathcal{S}_{\phi'}(z\boldsymbol{\pi})) \in X] \geq 1 - 2m!\exp(-c_X^2 zn/(8m!))$*

Proof. Fix $\mathcal{S}$, $X = \bigcup_{j=1}^{\ell} X_j$ where each X_j is convex.

To handle the first bullet, suppose X has measure 0. This implies each X_j has measure 0. Fix $\phi \in (0,1]$, a profile $\boldsymbol{\pi} \in \Pi_n$, and $\phi' \in [\phi, 1]$.

Fix an arbitrary X_j. Note that since X_j has measure zero, the probability mass placed on X_j by a Gaussian with invertible covariance matrix is 0. Hence, Lemmas 3 and 5 immediately imply that $\Pr[\mathcal{P}(\mathcal{S}_{\phi'}(\boldsymbol{\pi})) \in X_j] \leq \frac{O((m!)^{7/4})}{\sqrt{n}\cdot\textsc{min-prob}(\mathcal{S}_{\phi'})^{3/2}}$. Using the monotonicity of $\textsc{min-prob}$, we get that this is at most $\frac{O((m!)^{7/4})}{\sqrt{n}\cdot\textsc{min-prob}(\mathcal{S}_{\phi'})^{3/2}}$. Union bounding over all ℓ sets X_j tells us that
$$\Pr[\mathcal{P}(\mathcal{S}_{\phi'}(\boldsymbol{\pi})) \in X_j] \leq \frac{\ell \cdot O((m!)^{7/4})}{\sqrt{n} \cdot \textsc{min-prob}(\mathcal{S}_\phi)^{3/2}},$$
as needed.

For the second bullet, suppose X has positive measure. This implies at least one X_j has positive measure. Since X_j is convex, this implies it has an interior point, and thus has a ball contained in it. As in the theorem statement, let c_X be the radius of the largest ball contained in X. Note that although the center of such largest balls need not be rational, since the rational points are dense in H, there is a rational proportion $\mathbf{p} \in X$, such that $B_{c_X/2}(\mathbf{p}) \subseteq X$. Let $\boldsymbol{\pi}$ be a profile corresponding to $\mathbf{p}$. This will be our chosen profile.

Next, we must choose ϕ. Intuitively, we will choose it such that even after adding ϕ noise, in expectation, we stay well within $c_X/2$ (say $c_X/4$) of $\mathbf{p}$. To that end, define $g(\phi) = \sup_{\phi' \in [0,\phi]} d(\mathbf{p}, \mathbf{E}[\mathcal{P}(\mathcal{S}_{\phi'}(\boldsymbol{\pi})])$. Note that by the extremal value assumption of $\mathcal{S}_0$, $g(0) = 0$. Further, g is a continuous function of ϕ. Hence, there exists $\phi > 0$ such that $g(\phi) \leq c_X/4$. We will use this as our chosen ϕ.

Fix an arbitrary $\phi' \in [0,\phi]$ and $z \in \mathbb{Z}^+$. Note that $\mathcal{P}(z\boldsymbol{\pi}) = \mathcal{P}(\boldsymbol{\pi}) = proportions$. By choice of ϕ and the properties of g, $d(\mathbf{p}, \mathbf{E}[\mathcal{P}(\mathcal{S}_{\phi'}(z\boldsymbol{\pi})]) \leq c_X/4$. Using Lemma 2 with $\varepsilon = c_X/4$, we immediately get
$$\Pr[d(\mathcal{P}(\mathcal{S}_\phi(z\boldsymbol{\pi})), \mathbf{E}[\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi}))]) < c_X/4] > 1 - 2m!\exp(-c_X^2 zn/(8m!)).$$
Conditioned on this occurring, by the triangle inequality, we have $\mathcal{P}(\mathcal{S}_\phi(z\boldsymbol{\pi})) \in B_{c_X/2}(\mathbf{p}) \subseteq X$, as needed. □

Convergence rates for Mallows noise model. As a corollary of the above result, we now present the convergence rates specifically for the popular Mallows noise model, as defined in Section 2 and as proposed in the blue-sky paper discussed int he related work. Extending the above result to Mallows simply requires lower-bounding the smallest probability of reaching any ranking from any other ranking in terms of ϕ. For Mallows, this lower bound is $\phi^{\binom{m}{2}/m!}$. In terms of this value, we get the following bound:

Corollary 7. *Fix (R, A) and an $m \geq 3$. Then, the first bound of Theorem 6 becomes*

$$\Pr[\mathcal{P}(\mathcal{S}_{\phi'}(\boldsymbol{\pi})) \in X] \leq \frac{\ell \cdot O(m!^{13/4})}{\sqrt{n} \cdot \phi^{3/2 \cdot \binom{m}{2}}}.$$

5.1 Generalized brittleness for counterexample dependent on n

There are times when you would want to define a voting rule in a way that the counterexample set changes with n; for example, consider the axiom of Group Strategy-Proofness, parameterized by the maximum group size that is permitted to deviate (defined in detail in the next section). Unfortunately, Theorem 6 does not apply to this case. In order to give guarantees about group-strategy-proofness, we prove the following theorem, which implies that it is unlikely in our model for a post-noise profile to be within some bounded-size tolerance of a hyperplane. From this, upper bounds will follow on the extent of the impact on the outcome of an election that can result from groups of bounded size changing their behavior.

Theorem 8. *Let G be a hyerplane. Then, for all noise models $\mathcal{S}$, parameters $\phi \in [0, 1]$, and $\delta(n) \in o(\sqrt{n})$, we have that*

$$\sup_{\phi' \in [\phi, 1]} \sup_{\boldsymbol{\pi} \in \Pi_n} \Pr[d^{L_1}(\mathcal{P}(\mathcal{S}_{\phi'}(\boldsymbol{\pi})), G) \leq \delta(n)] \in o(1),$$

where d^{L_1} is the L_1 distance.

Proof. Fix $G, \mathcal{S}$, and ϕ. Fix a number n, let $\phi' \in [\phi, 1]$ and $\boldsymbol{\pi} \in \Pi_n$. We will upper bound $\Pr[d^{L_1}(\mathcal{P}(\mathcal{S}_{\phi'}(\boldsymbol{\pi})), G) \leq \delta(n)]$ by an $o(1)$ function of n (allowing ϕ, m, G, and $\mathcal{S}$ to be treated as constants in the bound, but not ϕ' or $\boldsymbol{\pi}$).

To that end, let $p = \textsc{min-prob}(S_\phi)$. Note that by monotonicity, $\textsc{min-prob}(S_{\phi'}) \geq p$ (and since it depends on ϕ, we can treat is as a constant asymptotically). Fix two arbitrary rankings $\pi, \pi' \in \mathcal{L}$.

Let $\mathcal{E}^{few}$ be the event that fewer than np agents end up voting either π or π' in $\mathcal{P}(\mathcal{S}_{\phi'}(\boldsymbol{\pi})$.

Let $\mathcal{V}^{\geq np}$ be all sets of agents of size at least np. For a specific set of agents $V \in \mathcal{V}^{\geq np}$, let (slightly abusing notation) $\Pi^{\overline{V}} = \prod_{i \in N \setminus V} \mathcal{L} \setminus \{\pi, \pi'\}$ (where by product we mean Cartesian product) be the set of all sub-profiles for agents not in V having a ranking other than π or π'. For $V \in \mathcal{V}^{\geq np}$ and $\boldsymbol{\pi} \in \Pi^{\overline{V}}$, let $\mathcal{E}^{V, \boldsymbol{\pi}}$ be the event that agents in V either vote π or π', and all other agents vote as in $\boldsymbol{\pi}$. Note that along with $\left\{\mathcal{E}^{V, \boldsymbol{\pi}}\right\}_{V \in \mathcal{V}^{\geq np}, \boldsymbol{\pi} \in \Pi^{\overline{V}}} \cup \left\{\mathcal{E}^{few}\right\}$ form a partition of the full sample space.

We will show two claims:

(1) There is some $f(n) \in o(1)$ such that

$$\Pr[\mathcal{E}^{few}] \leq f(n).$$

(2) There is some $g(n) \in o(1)$ such that for all $\mathcal{E}^{V, \boldsymbol{\pi}}$,

$$\Pr[d^{L_1}(\mathcal{P}(\mathcal{S}_{\phi'}(\boldsymbol{\pi})), G) \leq \delta(n) \mid \mathcal{E}^{V, \boldsymbol{\pi}}] \leq g(n).$$

We first show that together these are sufficient to prove the bound. Indeed, by the law of total probability, as these events form a partition

$$\begin{aligned}
&\Pr[[d^{L_1}(\mathcal{P}(\mathcal{S}_{\phi'}(\boldsymbol{\pi})), G) \leq \delta(n)] \\
=&\sum_{\mathcal{E}^{V,\boldsymbol{\pi}}} \Pr[d^{L_1}(\mathcal{P}(\mathcal{S}_{\phi'}(\boldsymbol{\pi})), G) \leq \delta(n) \mid \mathcal{E}^{V,\boldsymbol{\pi}}] \cdot \Pr[\mathcal{E}^{V,\boldsymbol{\pi}}] \\
&\qquad + \Pr[d^{L_1}(\mathcal{P}(\mathcal{S}_{\phi'}(\boldsymbol{\pi})), G) \leq \delta(n) \mid \mathcal{E}^{few}] \cdot \Pr[\mathcal{E}^{few}] \\
\leq&\sum_{\mathcal{E}^{V,\boldsymbol{\pi}}} g(n) \cdot \Pr[\mathcal{E}^{V,\boldsymbol{\pi}}] + \Pr[d^{L_1}(\mathcal{P}(\mathcal{S}_{\phi'}(\boldsymbol{\pi})), G) \leq \delta(n) \mid \mathcal{E}^{few}] \cdot f(n) \\
=&g(n) \sum_{\mathcal{E}^{V,\boldsymbol{\pi}}} \Pr[\mathcal{E}^{V,\boldsymbol{\pi}}] + \Pr[d^{L_1}(\mathcal{P}(\mathcal{S}_{\phi'}(\boldsymbol{\pi}))Yea, G) \leq \delta(n) \mid \mathcal{E}^{few}] \cdot f(n) \\
\leq&g(n) \cdot 1 + 1 \cdot f(n) \in o(1).
\end{aligned}$$

We now show the claims. The first follows from a straightforward Chernoff bound, each of the n agents places probability at least $2p$ on ending up in either π or π', hence the mean is $2np$. The probability of being at most half of these is exponentially small in n.

We now show the second. Fix an arbitrary $V \in \mathcal{V}^{\geq np}$ and $\boldsymbol{\pi} \in \Pi^{\overline{V}}$, and let us condition on $\mathcal{E}^{V,\boldsymbol{\pi}}$. Note that the only uncertainty left in this distribution is whether each agent $i \in V$ ends up at π or π'. Let I_i be the indicator random variable that agent i chose π. Note that even after conditioning, the minimum probability can only increase, so $p \leq \Pr[I_i = 1] \leq 1 - p$. Let $S_\pi = \sum_{i \in V} I_i$ be the random variable indicating the total number of agents in V that vote for π. Note that once S_π is determined, the entire profile is determined as the remaining $|V| - S_\pi$ agents choose π'. The key fact we will use is there is only one value of S_π (possibly fractional) that leads to the entire profile along with $\boldsymbol{\pi}$ to be on the hyperplane G. Further, this implies that there are at most $2n \cdot \delta(n) + 1$ values of S_π being within $\delta(n)$ L_1 distance of this value. If S_π is more than $\delta(n)$ away, then the L_1 distance from G will be at least this much. The key fact we will use is that $2n \cdot \delta(n) + 1 \in o(\sqrt{n})$. We will show that the probability of S_π being equal to *any* $(\sqrt{n})$ values is $o(1)$.

To that end, we will again apply the Berry-Esseen bound, Lemma 4, although this time to single-variable values. Note that the variance of each I_i is at least $p(1-p)$. This implies that the variance of S_π is at least $np^2(1-p) \in \Theta(n)$. Hence, the standard deviation is $\Theta(\sqrt{n})$. This further implies an error bound of $\Theta(n) \cdot \Theta(1/\sqrt{n}^3) = (1/\sqrt{n})$. Since a range of $2n \cdot \delta(n) + 1 \in o(\sqrt{n})$, assymptotically smaller than the standard deviation, the proabability the normal distribution from the Berry Esseen bound lies in this convex interval is of the form $\Phi(c + o(1)) - \Phi(c)$, where Φ is the standard normal CDF. Since the standard normal PDF has value at most 1, this difference (and associated probability) is $o(1)$. Hence, the probability S_π lies in this range is $o(1) + \Theta(1/\sqrt{n}) = o(1)$, as needed. □

6 APPLICATION OF GENERAL RESULTS TO VOTING RULE-AXIOM PAIRS

Here, we now show how to quickly apply the general results from the previous section to classify specific voting rule-axiom pairs (R, A) as brittle or robust. Definitions of axioms and voting rules can be found in Appendix A.

The classifications we study are based on three main axioms, and these classifications are summarized in Table 1. Throughout the section, we also establish additional corollaries pertaining to other voting rule-axiom pairs where possible. We highlight that the results we get on each of the three main axioms we consider here apply a different one of our theoretical bounds: Section 6.1 applies Theorem 8 show the brittleness of inconsistencies with $o(1/\sqrt{n})$-Group-Strategy-Proofness, Section 6.2 applies the positive direction of Theorem 6 to show the brittleness of inconsistencies

	Axioms		
Voting Rules	$o(\sqrt{n})$-Group-Strategy-Proofness	Resolvability	Condorcet
PSRs	brittle	brittle	robust
Minimax	brittle	brittle	satisfied
Kemeny-Young	brittle	brittle	satisfied
Copeland	brittle	robust	satisfied

Table 1. s = satisfied, b = brittle, r = robust.

between several voting rules and the axiom Resolvability, and Section 6.3 applies the negative direction of Theorem 6 to show the robustness of inconsistencies between all positional scoring rules (PSRs) and Condorcet Consistency.

6.1 Group strategy-proofness and all finite voting rules

We begin by defining some classes of profiles and voting rules, which will be helpful in analyzing these axioms. The first describes a class of profiles for which the outcome of a voting rule cannot easily be changed.

Definition 9 ($\rho(n)$-robust profile). A profile $\boldsymbol{\pi}$ on n agents is $\rho(n)$-robust with respect to a voting rule R if, for all profiles $\boldsymbol{\pi}'$ reachable by changing the rankings of up to $n \cdot \rho(n)$ agents, $R(\boldsymbol{\pi}) = R(\boldsymbol{\pi}')$.

We next define a simple property of a voting rule to help us analyze $\rho(n)$-robust profiles. All the voting rules we consider satisfy the notion.

Definition 10 (finite-hyperplane-boundary voting rule). A voting rule R has *finite-hyperplane-boundaries* if there are a finite number of hyperplanes in $\mathbb{R}$ for which if a profile $\boldsymbol{\pi} \in \Pi_n$ is such that $\mathcal{P}(\boldsymbol{\pi})$ is at least $\rho(n)$ L_1 distance from any hyperplane, it is ρ-robust.

These hyperplanes define the boundaries where the outcome of an election switch. For example, in positional scoring rules, we can use the $\binom{m}{2}$ hyperplanes where any pair of candidates tie based on score. As long as a profile is $\rho(n)$ L_1 distance away from such a tie, this proportion of agents cannot affect the outcome. In a sense, we are far in the interior of this outcome's region. It can easily be checked that all of the voting rules we consider are finite-hyerplane-boundary.

We are interested in applying Definition 9 particularly because a profile that is $\rho(n)$-robust with respect to R cannot be a counterexample to $\rho(n)$-group-strategy-proofness, as defined below:

Definition 11 ($\rho(n)$-group-strategy-proofness). A voting rule R satisfies $\rho(n)$-group-strategy-proofness if, for all profiles $\boldsymbol{\pi}$, there exists no fraction of agents of size at most $\rho(n)$ such that if they change their votes (say, resulting in some profile $\boldsymbol{\pi}'$), they are all at least as well off in the resulting outcome and at least one is strictly *better off*. A agent is at least as well (resp. better) off if their favorite candidate in the set $R(\boldsymbol{\pi}')$ is weakly (resp. strictly) preferred to their favorite candidate in $R(\boldsymbol{\pi})$.

Of course, for any non-constant voting rule, we cannot expect *all* profiles to be $\rho(n)$-robust. However, we can hope that this worst-case inconsistency is brittle — i.e., if we add noise via our model, we are likely to end up at such a profile. Fortunately, by Theorem 8, the answer is yes for almost all voting rules R, and for $\rho(n) \in o(1/\sqrt{n})$. This result is stated in the following corollary, whose implication is in an election of size n, even if up to $o(\sqrt{n})$ agents can collude to lie about their votes, as n grows large, the probability that they will be able to influence the election converges to 0.

Corollary 12. *Let R be a voting rule with finite-hyperplane boundaries, and $\rho(n) \in o(1/\sqrt{n})$. By Theorem 8, the inconsistency in $(R, o(1/\sqrt{n})$ − Group-Strategy-Proofness) is brittle.*

The proof is simply invoking Theorem 8 and union bounding over the finite number of hyperplanes. We note that in addition to $o(1/\sqrt{n})$-Group-Strategy-Proofness, the concept that $\rho(n)$-robustness defines also applies to comparably parameterized versions of the axioms of Monotonicity and Participation[7], in which the $\rho(n)$ parameter would indicate how many people can rank a candidate higher or lower, and join or leave the election, respectively. Although these extensions are perhaps less interesting because the latter two axioms tend to be easier to satisfy, the fundamental similarity that $\rho(n)$-robustness illuminates between axiom of strategy-proofness (considered a hard axiom to satisfy in the worst case) with the axioms of monotonicity and participation (generally considered easier to satisfy in the worst case) illustrates the conceptual relationship that, to our knowledge, has thus far gone unnoticed. In other words, the strong worst-case differences in satisfiability of these axioms obscured a conceptual similarity that leads to all three axioms being fundamentally easy to satisfy in the smoothed setting— a commonality that may have been previously obscured by their different behavior from the perspective of worst-case analysis.

6.2 The Resolvability Axiom

For A = Resolvability, here we will prove that inconsistencies in pairs of the form (R, A) are brittle, for all

$$R \in \textbf{Ruleset1} = \{\text{Minimax, Kemeny-Young}\} \cup \text{PSRs}$$

Recall that the set of positional scoring rules, PSRs, is defined to encompass any rule that places monotonic weights across positions, which includes Borda Count, Veto, and Plurality. A corollary of this result will be that the ANR paradox is asymptotically resolved (Corollary 14). Notably, Copeland is not among these rules; this is because the inconsistency in (Copeland, Resolvability) is actually robust (Proposition 15).

Before we begin, we illustrate that for all $R \in \textbf{Ruleset1}$, $\mathcal{P}\left(\Pi^{\neg(R,A)}\right)$ is non-empty. To see this, consider the following profile: $\boldsymbol{\pi} := \{\pi_1 = 1 \succ 2 \succ 3, \pi_2 = 2 \succ 1 \succ 3\}$. This profile is symmetric across 1 and 2, so any anonymous rule must treat 1 and 2 identically; therefore, given all rules in **Ruleset1** are anonymous that that we do not specify tie-breaking rules (in order to explicitly allow ties when they might arise), 1 and 2 must tie according to all $R \in \textbf{Ruleset1}$.

*Brittleness of inconsistencies between $R \in$ **Ruleset1** and Resolvability.*

Proposition 13. *For any voting rule $R \in$ **Ruleset1**, $(R$, Resolvability$)$ is brittle.*

We defer the proof to Appendix C.1. To show this brittleness for each $R \in \textbf{Ruleset1}$ and A = Resolvability, we apply the same high-level proof approach for each rule:

(1) Fix a profile proportion $\mathbf{p}$ assumed to lie in $\mathcal{P}\left(\Pi^{\neg(R,A)}\right)$
(2) Construct another profile proportion $\mathbf{p}' \notin \mathcal{P}\left(\Pi^{\neg(R,A)}\right)$ such that $\|\mathbf{p} - \mathbf{p}'\|_2 \leq \epsilon$, for $\epsilon > 0$ arbitrarily small

Such an argument is sufficient to demonstrate that $\mathcal{P}\left(\Pi^{\neg(R,A)}\right)$ has measure zero, and so by the positive direction of Theorem 6, we conclude the brittleness of the inconsistency in the current pair (R, A). In these proofs, we will apply the trivial property of euclidean norms that the existence of such an ϵ is implied by the existence of an $\epsilon' > 0$, also arbitrarily small, such that $\|\mathbf{p} - \mathbf{p}'\|_1 \leq \epsilon'$. In particular, this will allow us to reason more intuitively about the total quantity of mass shifted

[7]The definition of $\rho(n)$-Robustness needs to be slightly modified to allow adding or removing agents, but since we always use a sublinear $\rho(n)$, the details do not change.

between proportions of rankings (i.e., the sum across proportions of how much each proportion changes from $\mathbf{p} \to \mathbf{p}'$, which is equal to the L1 norm.

Smoothed resolution of the ANR paradox. This result also has implications for the existence of the ANR paradox, which as defined in [17], states that no voting rule can (in the worst case) simultaneously satisfy Anonymity, Neutrality, and Resolvability. However, above, we have identified several neutral and anonymous voting rules whose inconsistencies with Resolvability are brittle in our smoothed model. Then, by Proposition 13, we conclude the following corollary, which states that the ANR paradox is resolved with high probability in large elections. When we say that this paradox is *resolved* under some condition, it means we found a rule which satisfies all three axioms under that condition.

Corollary 14. *The ANR paradox is resolved after the application of our noise model with probability converging to 1 as $n \to \infty$.*

The robust inconsistency between Copeland and Resolvability. Finally, we show the inconsistency between (Copeland, Resolvability) is robust, which involves the application of the negative direction of Theorem 6. We defer this proof to Appendix C.2.

Proposition 15. *The inconsistency of (Copeland, Resolvability) is robust.*

6.3 Condorcet Consistency and PSRs

Condorcet Consistency is a highly sought-after axiom, and one of the most prominent classes of voting rules inconsistent with this axiom are positional scoring rules (PSRs), i.e., rules in which each ranking position is associated with a different number of points. As stated above, PSRs include the popular rules Borda Count, Plurality, and Veto. Here, we show that the inconsistency between any PSR and Concorcet Consistency is robust (Proposition 16). A corollary of this proof is that the inconsistency between all PSRs except Plurality and the axiom Majority is also robust, noting that Plurality satisfies majority (Corollary 17). We also apply this same theorem to show a much stronger impossibility about the potential of *any* amount of noise in $\phi \in [0, 1)$ to resolve this inconsistency, specifically in the Mallows noise model. We defer proofs of Propositions 16 and 18 to Appendices C.3 and C.4, respectively.

Proposition 16. *For all PSRs R, the inconsistency of $(R$, Condorcet Consistency$)$ is robust.*

Extension to show robustness with respect to the Majority Axiom. The instance used to prove Proposition 16 (in the $R \neq$ Plurality case) directly implies[8] the following corollary:

Corollary 17. *For any $R \in PSR \setminus \{$Plurality$\}$ $(PSR, Majority)$ is robust.*

A stronger impossibility in the Mallows noise model. Here, we prove a stronger impossibility on the ability for Mallows noise to resolve inconsistencies between PSRs and Condorcet Consistency. Note that the following lemma is stronger than the negative direction of Theorem 6, because it provides a single profile proportion *for all $\phi \in [0, 1)$, and for all positional scoring rules* of which perturbations with ϕ Mallows noise will result in a counterexample with probability converging to 1.

Proposition 18. *Let A = Condorcet Consistency. Then, there exists a $\boldsymbol{\pi}$ such that, for all $R \in PSR$ and for all $\phi \in [0, 1)$,*

$$\Pr\left[\mathcal{P}(\mathcal{S}_{\phi}^{Mallows}(z\boldsymbol{\pi})) \in \mathcal{P}(\Pi^{\neg(R,A)})\right] \to 1$$

[8]To show this, fix $\mathbf{p}$ as they are in the proof Proposition 16 for the $R \neq$ Plurality case. In constructing $\mathbf{p}'$, set δ, ϵ the same way.

7 DISCUSSION

The key takeaway of this paper is that all voting rule-axiom pairs (except those dealing with growing groups of agents, as covered in Theorem 8) can be classified into three groups: satisfied, brittle, or robust. A priori, the ability to so cleanly classify these pairs is not clear, because, given the separation of these notions, there could hypothetically be pairs in between. Although we apply this classification framework to only some specific voting rule-axiom pairs, there are several others that remain unstudied, such as the axiom Consistency. Given the simplicity of our conditions separating brittleness and robustness, it seems likely that our model could be used to classify a much wider variety of voting rules and axioms using fairly simple arguments, opening the door to study more intricate voting rules and axioms. The ability to make such classifications about voting rule-axiom pairs can help focus the study of social choice on axioms that are truly unlikely to be satisfied in practice, and in addition, motivate the formulation of stronger versions of axioms whose current versions pose only brittle impossibilities.

One area that this paper leaves fairly unexplored is the application of our model and approaches to central axiomatic impossibilities, such as Arrow's theorem. However, our analysis naturally extends to such questions, as we illustrate in our discussion of the ANR paradox in Section 6.2. One other type of impossibility that has garnered interest from the smoothed analysis perspective is the existence of Condorcet cycles [17]. In our model, it is simple to show via our an argument following that in Proposition 18 that there exist simple instances in which this paradox is robust.[9]

[9]E.g., consider the instance $p = \{p_{123} = 1/3, p_{231} = 1/3, p_{312} = 1/3\}$.

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A PROOFS AND DEFINITIONS FROM SECTION 2

A.1 Definitions of voting rules

Here we give definitions of the voting rules we study. For many rules R, we will define *score* of candidate c on a given profile $\boldsymbol{\pi}$, and then define the winners on this profile $R(\boldsymbol{\pi})$ to be those with the highest score, or formally, $R(\boldsymbol{\pi}) = c^* \in \arg\max_{c \in C} R_{scores}(\boldsymbol{\pi})_c$.

We do not specify tie-breaking mechanisms, as in some cases we explicitly study them, and in the rest of cases, how ties are broken is irrelevant to our results. To simplify this exposition, we define several indicators: $\mathbb{I}[\pi_i(j) = c]$ indicates whether agent i ranks alternative c in the jth position, $\mathbb{I}[(c \succ c')_\pi]$ indicates whether c is ranked ahead of c' in ranking π; $\mathbb{I}[(c \succ c')_{\boldsymbol{\pi}}]$ indicates whether c pairwise beats c' in $\boldsymbol{\pi}$, and $\mathbb{I}[(c \sim c')_{\boldsymbol{\pi}}]$ indicates whether c and c' are ranked ahead of each other an equal number of times across the rankings in $\boldsymbol{\pi}$.

Definition 19 (Positional scoring rules). For fixed m, a positional scoring rule is associated with a vector of weights $(\alpha_c | c \in [m])$, where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m$, $a_1 > a_m$. Without loss of generality, we let these weights be scaled such that $\alpha_1 = 1$ and $\alpha_m = 0$. Then, fixing m and a profile $\boldsymbol{\pi}$, now define a candidate c's *PSR score* as below. Then, we define the set of winners by R in $\boldsymbol{\pi}$ is defined as the alternative(s) with the highest PSR score.

$$R_{scores}(\boldsymbol{\pi})_c = \sum_{j \in [m]} \alpha_j \left(\sum_{i \in [n]} \mathbb{I}[\pi_i(j) = c] \right)$$

Definition 20 (Copeland). Fix a profile $\boldsymbol{\pi}$ and let R be Copeland. Then, then define a candidate c's *Copeland score* on $\boldsymbol{\pi}$ as the number of candidates they beat pairwise, plus those they tie with pairwise scaled by half, expressed formally below. Then, the set of winners by R in $\boldsymbol{\pi}$ is defined as the alternative(s) with the highest Copeland score.

$$R_{scores}(\boldsymbol{\pi})_c = \sum_{c' \neq c} 1/2 \cdot \mathbb{I}[(c \sim c')_{\boldsymbol{\pi}}] + \mathbb{I}[(c \succ c')_{\boldsymbol{\pi}}]$$

Definition 21 (Minimax). Fix a profile $\boldsymbol{\pi}$ and let R be Minimax. We define a candidate c's *Minimax score* on $\boldsymbol{\pi}$ as the magnitude of their maximum pairwise defeat by any other candidate, as below. Then, we define the winners by R on $\boldsymbol{\pi}$ to be the alternative(s) with the *lowest* Minimax score.

$$R_{scores}(\boldsymbol{\pi})_c = \max_{c' \neq c} \sum_{i \in [n]} \mathbb{I}[(c' \succ c)_{\pi_i}]$$

Definition 22 (Kemeny-Young). Fix an m and a profile $\boldsymbol{\pi}$ and let R be Kemeny-Young. We define a candidate c's Kemeny-Young score on $\boldsymbol{\pi}$ as follows:

$$R_{scores}(\boldsymbol{\pi})_c = \sum_{c' \neq c} \sum_{i \in [n]} \mathbb{I}[(c \succ c')_{\pi_i}]$$

A.2 Definitions of axioms

Definition 23 (Condorcet Consistency). A *condorcet winner* is defined as an alternative that would win a head-to-head plurality election against any other candidate individually. That is, c is a condorcet winner in $\boldsymbol{\pi}$ iff

$$\sum_{i \in [n]} \mathbb{I}[(c \succ c')_{\pi_i}] > n/2 \quad \forall c' \neq c$$

Then, a voting rule R satisfies Condorcet Consistency on a given profile $\boldsymbol{\pi}$ if one of two conditions hold: (1) $R(\boldsymbol{\pi})$ contains the Condorcet winner, or (2) there is no condorcet winner in $\boldsymbol{\pi}$.

Definition 24 (Resolvability). A voting rule R satisfies Resolvability on profile $\boldsymbol{\pi}$ iff $|R(\boldsymbol{\pi})| = 1$ (i.e., there are no ties).

Definition 25 (Majority). A *majority winner* is defined as an alternative that is ranked first by a majority of agents. That is, c is a majority winner in $\boldsymbol{\pi}$ iff

$$\sum_{i \in [n]} \mathbb{I}[\pi_i(1) = c] > n/2$$

A voting rule R satisfies the Majority axiom on profile $\boldsymbol{\pi}$ if it satisfies one of two conditions: (1) $R(\boldsymbol{\pi})$ contains the majority winner, or (2) there is no majority winner in $\boldsymbol{\pi}$.

A.3 Extension of voting rules to be defined on proportions

A copy-proof voting rule R satisfies $R(\boldsymbol{\pi}) = R(\boldsymbol{\pi} + \boldsymbol{\pi})$, a copy-proof absolute axiom satisfies $A(\boldsymbol{\pi}) = A(\boldsymbol{\pi} + \boldsymbol{\pi})$, and a copy-proof relative axiom satisfies $A((\boldsymbol{\pi}, \boldsymbol{\pi}')) = A((\boldsymbol{\pi} + \boldsymbol{\pi}, \boldsymbol{\pi}' + \boldsymbol{\pi}'))$. The voting rules and axioms defined above are anonymous and copy-proof, and we will show that this means that they can be applied in the natural way on proportions instead of profiles. We show this below for voting rules; the argument for axioms is essentially the same.

Observation 1. *If a voting rule R is copy-proof and anonymous, then there exists a rule R' such that $R'(\mathcal{P}(\boldsymbol{\pi})) = R(\boldsymbol{\pi})$ for all $\boldsymbol{\pi} \in \Pi$.*

Proof. We can partition the space of profiles into disjoint sets, each corresponding to a different proportion in $\mathcal{P}(\Pi)$. Fix an arbitrary $\mathbf{p} \in \mathcal{P}(\Pi)$ and let $\mathcal{P}^{-1}(\mathbf{p})$ be the set of all profiles $\boldsymbol{\pi}$ such that $\mathcal{P}(\boldsymbol{\pi}) = \mathbf{p}$ (we can think of this set of profiles as the *preimage* of $\mathbf{p}$). Now, it remains to show that we can define a voting rule R' such that $R'(\mathbf{p}) = R(\boldsymbol{\pi})$ for all $\boldsymbol{\pi} \in \mathcal{P}^{-1}(\mathbf{p})$.

Let $\boldsymbol{\pi}^*$ be the profile in $\mathcal{P}^{-1}(\mathbf{p})$ containing the fewest voters, and define $R'(\mathbf{p}) = R(\boldsymbol{\pi}^*)$. Then, given theanonymity of our voting rule, it must be that every profile in $\mathcal{P}^{-1}(\mathbf{p})$ is without loss of generality in $\mathbb{Z}\boldsymbol{\pi}^*$, as defined in the previous section. Thus, by copy-proofness, it must be that $R'(\mathbf{p}) = R(\boldsymbol{\pi}^*) = R(\boldsymbol{\pi})\ \forall \boldsymbol{\pi} \in \mathcal{P}^{-1}(\mathbf{p})$, concluding the proof. □

B PROOFS FROM SECTION 4

B.1 Proof of Lemma 2

Fix $\mathcal{S}, \phi$, and $\boldsymbol{\pi}$. Recall that $\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi})) = \sum_{i=1}^n {}^1\!/_n \mathcal{P}(\mathcal{S}_\phi(\pi_i))$.

We will show that with probability at least $1 - 2\exp(-2\varepsilon^2 n/m!)$, each of the $m! - 1$ components is within $\varepsilon/m!$ of its expected value. Union bounding over all $m! - 1$ components shows this will hold for all of them with probability at least $1 - 2m!\exp(-2\varepsilon^2 n/m!)$. Conditioned on this, note that the L_1 distance between the realized vector and the expected value vector is at most ε. Since L_2 distances are upper bounded by L_1 distances, this implies the desired result.

Finally, note that each component ${}^1\!/_n \mathcal{P}(\mathcal{S}_\phi(\pi_i))$ is bounded in the range $[0, 1/n]$. Hence, a direct application of Hoeffding's inequality shows that this occurs with probability $1 - 2\exp(-2\varepsilon^2 n/m!)$, as needed.

B.2 Proof of Lemma 3

Fix $\mathcal{S}, \phi$, and $\boldsymbol{\pi} \in \Pi_n$. Since $\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi})) = {}^1\!/_n \sum_{i=1}^n \mathcal{P}(\mathcal{S}_\phi(\pi_i))$ where each of these summands is independent, our goal will be to apply Lemma 4. However, there are some adjustments we must make to satisfy the conditions. First, we will translate everything by its expectation to make all terms mean-zero. That is, we will show the equivalent statement about $\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi})) - \mathbf{E}[\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi}))]$

approaching $\mathcal{N}(0, \mathbf{C}[\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi}))])$. Note that

$$\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi})) - \mathbf{E}[\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi}))] = \sum_{i=1}^{n} \frac{1}{n}(\mathcal{P}(\mathcal{S}_\phi(\pi_i)) - \mathbf{E}[\mathcal{P}(\mathcal{S}_\phi(\pi_i))]).$$

We let Y_i be the random variable distributed as $\frac{1}{n}(\mathcal{P}(\mathcal{S}_\phi(\pi_i)) - \mathbf{E}[\mathcal{P}(\mathcal{S}_\phi(\pi_i))])$. Note that Y_i has mean zero, and covariance $1/n^2 \cdot \mathbf{C}[\mathcal{S}_\phi(\pi_i)]$. Hence, the sum of these variances is still $\mathbf{C}[\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi}))]$, which corresponds to the lemma statement. All that remains to be shown then is the error bound, that

$$O((m!-1)^{1/4}) \cdot \left(\sum_{i=1}^{n} \mathbf{E}[|\mathbf{C}[\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi}))]^{-1/2} Y_i|^3]\right) \leq \frac{O((m!-1)^{1/4})}{(\lambda^{min,\mathcal{S},\phi})^{3/2}} \cdot \frac{1}{\sqrt{n}}.$$

(The exponentiated $\mathbf{C}[\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi}))]^{-1/2}$ is well defined because the matrix is symmetric.) To that end, recall that $\mathcal{P}(\mathcal{S}_\phi(\pi_i)$ is always either a basis vector or the all 0s vector. $\mathbf{E}[\mathcal{P}(\mathcal{S}_\phi(\pi_i)]$ is the vector whose entry corresponding to π' is $\Pr[\mathcal{S}_\phi(\pi) = \pi']$. We claim this implies that this implies $|Y_i| \leq 2/n$ with probability 1 for all i (where $|Y_i|$ denotes the L_2 norm of Y_i). Indeed, note that after subtracting the expectation, the negative entries can sum in magnitude to at most the sum of these probabilities which is at most 1. On the other hand, the positive entries can also sum to at most 1. hence, the L_1 norm before scaling by $1/n$ is at most 2. Using the fact that L_2 norms are at most L_1 norms, we get that this continues to hold for the L_2 norm. After dividing by n, we get that $|Y_i| \leq 2/n$, as needed.

We also use the fact that since $\mathbf{C}[\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi}))]$ has minimum eigenvalue at least $\textsc{min-prob}(S)/(m!n)$, as per Lemma 5. It therefore holds that $\mathbf{C}[\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi}))]^{-1/2}$ has maximum eigenvalue at most $(\textsc{min-prob}(S)/(m!n))^{-1/2} = \sqrt{n} \cdot \frac{\sqrt{m!}}{\sqrt{\textsc{min-prob}(\mathcal{S}_\phi)}}$. Combined with our previous observation about Y_i,

$$|\mathbf{C}[\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi}))]^{-1/2} Y_i|^3 \leq \frac{8 \cdot (m!)^{3/2}}{n^{3/2} \cdot \textsc{min-prob}(\mathcal{S}_\phi)^{3/2}}$$

with probability 1. Hence, this continues to hold for the expectation, and by summing over all i, we get that

$$\sum_{i=1}^{n} \mathbf{E}[|\mathbf{C}[\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi}))]^{-1/2} Y_i|^3] \leq \frac{8 \cdot (m!)^{3/2}}{\textsc{min-prob}(\mathcal{S}_\phi)^{3/2}} \cdot \frac{1}{\sqrt{n}}.$$

Multiplying by the $O((m!-1)^{1/4})$ which absorbs the 8 and adds $3/2$ to the $m!$ to turn into $O(m!^{7/4})$ yields the lemma statement. □

B.3 Proof of Lemma 5

Fix $\mathcal{S}$, ϕ, and $\boldsymbol{\pi}$. Fix an arbitrary ranking π. We will prove the following two claims about $\mathbf{C}[\mathcal{P}(\mathcal{S}_\phi(\pi))]$:

(1) All eigenvalues of $\mathbf{C}[\mathcal{P}(\mathcal{S}_\phi(\pi))]$ are positive and the minimum is at least $\textsc{min-prob}(\mathcal{S}_\phi)/m!$.
(2) The determinant $\det(\mathbf{C}[\mathcal{P}(\mathcal{S}_\phi(\pi))]) \geq \textsc{min-prob}(S_\phi)^{m!}$

We first show that these two claims imply the lemma statement, as this is relatively straightforward. Indeed, the minimum eigenvalue of the sum of matrices is at least the sum of the minimum eigenvalues of each matrix. Further, since each $\mathbf{C}[\mathcal{P}(\mathcal{S}_\phi(\pi))]$ is positive semi-definite, the determinant of the sum is at least the sum of the determinants. Scaling by n^2 scales the minimum eigenvalue by this amount and determinant by $n^{2(m!-1)}$. Finally, note that this implies $\det(\mathbf{C}[\mathcal{P}(\mathcal{S}_\phi(\boldsymbol{\pi}))]$ is non-zero, so it is invertible.

We now show the claims. Fix a ranking π. To simplify notation in the subsequent computations, for the i'th ranking π', let $q_i = \Pr[\mathcal{S}_\phi(\pi) = \pi']$. Recall that in $\mathbf{C}[\mathcal{S}_\phi(\pi)]$, the entry corresponding

i, j when $i = j$ (a diagonal entry) has value $q_i(1-q_i)$ and for $i \neq j$, the entry has value $-q_i \cdot q_j$. We can then write the covariance matrix as

$$\mathbf{C}[\mathcal{S}_\phi(\pi)] = \begin{pmatrix} q_1(1-q_1) & -q_1 \cdot q_2 & \cdots & -q_1 \cdot q_{m!-1} \\ -q_2 \cdot q_1 & q_2(1-q_2) & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ -q_{m!-1} \cdot q_1 & \cdots & \cdots & q_{m!-1}(1-q_{m!-1}) \end{pmatrix}.$$

Note that $q_{m!}$ is the probability of the "missing" ranking, and that $\sum_{j=1}^{m!} q_j = 1$.

Let us begin with the eigenvalue claim. We first demonstrate an inverse of $\mathbf{C}[\mathcal{S}_\phi(\pi)]$. Consider the matrix:

$$M = \begin{pmatrix} \frac{1}{q_1} + \frac{1}{q_{m!}} & \frac{1}{q_{m!}} & \cdots & \frac{1}{q_{m!}} \\ \frac{1}{q_{m!}} & \frac{1}{q_2} + \frac{1}{q_{m!}} & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \frac{1}{q_{m!}} & \cdots & \cdots & \frac{1}{q_{m!-1}} + \frac{1}{q_{m!}} \end{pmatrix}.$$

More formally, the ith diagnal entry is $1/q_i - 1/q_{m!}$ and all off-diagonal entries are simply $1/q_{m!}$.

We now show that $\mathbf{C}[\mathcal{S}_\phi(\pi)] \cdot M = I_{m!-1}$ where $I_{m!-1}$ is the identity matrix. To that end, let us consider the i'th diagonal entry of the product. It is precisely

$$\begin{aligned}
(\mathbf{C}[\mathcal{S}_\phi(\pi)] \cdot M)_{ii} &= \sum_{j=1, j\neq i}^{m!-1} -\frac{q_i q_j}{q_{m!}} + q_i(1-q_i)\left(\frac{1}{q_i} + \frac{1}{q_{m!}}\right) \\
&= \sum_{j=1, j\neq i}^{m!-1} -\frac{q_i q_j}{q_{m!}} + (1-q_i) + \frac{q_i}{q_{m!}}(1-q_i) \\
&= \sum_{j=1, j\neq i}^{m!-1} -\frac{q_i q_j}{q_{m!}} + 1 - q_i + \frac{q_i}{q_{m!}} - \frac{q_i \cdot q_i}{q_{m!}} \\
&= \sum_{j=1}^{m!-1} -\frac{q_i q_j}{q_{m!}} + 1 - q_i + \frac{q_i}{q_{m!}} \\
&= \frac{-q_i(1-q_{m!})}{q_{m!}} + 1 - q_i + \frac{q_i}{q_{m!}} \\
&= \frac{-q_i + q_i \cdot q_{m!}}{q_{m!}} + 1 - \frac{q_i \cdot q_{m!}}{q_{m!}} + \frac{q_i}{q_{m!}} \\
&= 1.
\end{aligned}$$

For a non-diagonal entry i, j with $i \neq j$, we have

$$
\begin{aligned}
(\mathbf{C}[\mathcal{S}_\phi(\pi)] \cdot M)_{ij} &= \sum_{k=1, k\neq i,j}^{m!} -\frac{q_i q_k}{q_{m!}} + \frac{q_i(1-q_i)}{q_{m!}} - q_i q_j \cdot \left(\frac{1}{q_j} + \frac{1}{q_{m!}}\right) \\
&= \sum_{k=1, k\neq i,j}^{m!} -\frac{q_i q_k}{q_{m!}} + \frac{q_i}{q_{m!}} - \frac{q_i q_i}{q_{m!}} - q_i - \frac{q_i q_j}{q_{m!}} \\
&= \sum_{k=1}^{m!} -\frac{q_i q_k}{q_{m!}} + \frac{q_i}{q_{m!}} - q_i \\
&= \frac{q_i(1-q_{m!})}{q_{m!}} + \frac{q_i}{q_{m!}} - \frac{q_i q_{m!}}{q_{m!}} \\
&= 0.
\end{aligned}
$$

Using this, we now consider the eigenvalues of $\mathbf{C}[\mathcal{S}_\phi(\pi)]$. Since it is a covariance matrix, it is symmetric, and therefore positive semi-definite. Since we now know it is invertible, it is in fact positive definite. This implies all of its eigenvalues exist and are positive. Further, since the eigenvalues of $\mathbf{C}[\mathcal{S}_\phi(\pi)]^{-1}$ are the reciprocals of the eigenvalues of $\mathbf{C}[\mathcal{S}_\phi(\pi)]$, we can lower bound the eigenvalues of $\mathbf{C}[\mathcal{S}_\phi(\pi)]$ upper bounding the the eigenvalues of the inverse.

To upper bound the maximum eigenvalue of M, we can upperbound the maximum absolute row sum. Note that the sum of row i is

$$
\frac{1}{q_i} + \frac{m! - 1}{q_{m!}} \leq \frac{m!}{\textsc{min-prob}(\mathcal{S}_\phi)}.
$$

This lower bounds the minimum eigenvalue of $\mathbf{C}[\mathcal{S}_\phi(\pi)]$ by $\frac{\textsc{min-prob}(\mathcal{S}_\phi)}{m!}$, as needed.

Next, we show that its determinant is exactly $\prod_{j=1}^{m!} q_j$. The claim follows because $\prod_{j=1}^{m!} q_j \geq \textsc{min-prob}(\mathcal{S}_\phi)^{m!}$. To that end, note that by dividing each row j by q_j, we get that the claim will be true if and only if the determinant of

$$
\begin{pmatrix}
(1-q_1) & -q_2 & \cdots & -q_{m!-1} \\
-q_1 & (1-q_2) & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
-q_1 & \cdots & \cdots & (1-q_{m!-1})
\end{pmatrix}.
$$

is $q_{m!}$. Next, note that by adding a single copy of all of the not-first columns to the first makes every term in the first column $1 - \sum_{j=1}^{m!-1} q_j$, i.e., $q_{m!}$. This does not affect the determinant and gives us the mtarix

$$
\begin{pmatrix}
q_{m!} & -q_2 & \cdots & -q_{m!-1} \\
q_{m!} & (1-q_2) & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
q_{m!} & \cdots & \cdots & (1-q_{m!-1})
\end{pmatrix}.
$$

Dividing the first column by $q_{m!}$, we get that our desired determinant holds even iff the following matrix has determinant 1

$$\begin{pmatrix} 1 & -q_2 & \cdots & -q_{m!-1} \\ 1 & (1-q_2) & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & \cdots & (1-q_{m!-1}) \end{pmatrix}.$$

Adding q_j multiples of the first column to each column i does not affect the determinant and results in the following matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & \cdots & 1 \end{pmatrix}.$$

This is a lower triangular matrix with trace 1, and hence its determinant is 1, as needed. □

C PROOFS FROM SECTION 6

C.1 Proof of Proposition 13

Kemeny-Young. Let R be Kemeny-Young. Recall that the Kemeny-Young winner a in a set of proportions $\mathbf{p}$ is an alternative with the largest *kemeny score*, which we define as the fraction of voters in which a pairwise beats another alternative a', summed over all a' (for formal definitions, see Appendix A).

Fix $m \geq 3$, and fix a profile proportion $\mathbf{p} \in \mathcal{P}\left(\Pi^{\neg(R,A)}\right)$. Define the set $Ties \subseteq M$ to include the set of all Kemeny-Young winners; by assumption, $|Ties| > 1$. Now, choose two alternatives $a^*, a \in Ties$. Because rankings are defined to be strict orderings, there must exist some ranking π in which $a \succ a^*$ and for which $\mathbf{p}_\pi > 0$. We express π below, along with a closely-related ranking π' below it, where M_i is just a subset of alternatives, $M_1 \cup M_2 \cup M_3 \cup \{a\} \cup \{a^*\} = M$, and these the rankings over these subsets are the same across the π, π':

$$\pi := M_1 \succ a \succ M_2 \succ a^* \succ M_3$$
$$\pi' := a^* \succ M_1 \succ a \succ M_2 \succ M_3$$

Let $\epsilon > 0$ arbitrarily small such that $\epsilon < \mathbf{p}_\pi$ (this is possible, per above). Now, we define $\mathbf{p}'$ in terms of this ϵ as follows: first let $\mathbf{p}'_{\pi''} = \mathbf{p}_{\pi''}$ for all $\pi'' \in \mathcal{L} \setminus \{\pi, \pi'\}$. Then, for $\delta \in (0, \epsilon/2)$, let $\mathbf{p}'_\pi = \mathbf{p}_\pi - \delta$, and let $\mathbf{p}'_{\pi'} = \mathbf{p}_{\pi'} + \delta$. By definition, $\|\mathbf{p} - \mathbf{p}'\| < \epsilon$.

Now, it remains to prove that a^* is the unique Kemeny-Young winner in $\mathbf{p}'$, which we will do by showing that from $\mathbf{p} \to \mathbf{p}'$, a^*'s Kemeny score strictly increased, and all other alternatives' kemeny scores weakly decreased. First, a^* pairwise beats strictly more alternatives more often in $\mathbf{p}'$, so $R_{scores}(\mathbf{p}')_{a^*} > R_{scores}(\mathbf{p})_{a^*}$. For all alternatives $a' \in M_3$, $R_{scores}(\mathbf{p}')_{a'} = R_{scores}(\mathbf{p})_{a'}$, because a' is pairwise defeated by the exact same alternatives across all rankings in $\mathbf{p}$ and $\mathbf{p}'$. Finally, for all alternatives $a' \in M_1 \cup M_2 \cup \{a\}$, we have that $R_{scores}(\mathbf{p}')_{a^*} < R_{scores}(\mathbf{p})_{a^*}$, because a' pairwise beats fewer alternatives across rankings in $\mathbf{p}'$ than in $\mathbf{p}$. a^* must therefore be the unique Kemeny-Young winner. We conclude that $\mathbf{p}' \notin \mathcal{P}\left(\Pi^{\neg(R,A)}\right)$. We have found a $\mathbf{p}'$ that is arbitrarily close to $\mathbf{p}$ such that $\mathbf{p}' \notin \mathcal{P}\left(\Pi^{\neg(R,A)}\right)$, and thus $\mathcal{P}\left(\Pi^{\neg(R,A)}\right)$ has measure zero. Applying the first part of Theorem 6, the result follows.

Minimax. Let R be Minimax. Recall that a Minimax winner a in a set of proportions $\mathbf{p}$ is an alternative with the smallest *minimax score*, which measures the size (measured as a proportion of voters) of a's greatest pairwise defeat by any other alternative. We will write this score for alternative a as $R_{scores}(\mathbf{p})_a$. In this proof, we will refer to the alternative that beats a pairwise with the greatest margin as a's *greatest defeater*. For formal definitions, see Appendix A.

Fix $m \geq 3$, and fix a profile proportion $\mathbf{p} \in \mathcal{P}\left(\Pi^{\neg(R,A)}\right)$. Define the set $Ties \subseteq M$ to include the set of all minimax winners (Definition X); by assumption, $|Ties| > 1$. Now, choose an alternative $a^* \in Ties$, and suppose a is a^*'s greatest pairwise defeater.[10] Because rankings are defined to be strict orderings, there must exist some ranking π such that $\mathbf{p}_\pi > 0$ and in which $a \succ a^*$. We write π as follows, and then based on this expression define a closely-related ranking π' below it, where M_i is just a subset of alternatives, $M_1 \cup M_2 \cup M_3 \cup \{a\} \cup \{a^*\} = M$, and these the rankings over these subsets are the same across the π, π':

$$\pi := M_1 \succ a \succ M_2 \succ a^* \succ M_3$$
$$\pi' := a^* \succ M_1 \succ a \succ M_2 \succ M_3$$

Let $\epsilon > 0$ arbitrarily small such that $\epsilon < \mathbf{p}_\pi$ (this is possible, per above). Now, we define $\mathbf{p}'$ as follows: first let $\mathbf{p}'_{\pi''} = \mathbf{p}_{\pi''}$ for all $\pi'' \in \mathcal{L} \setminus \{\pi, \pi'\}$. Then, for $\delta \in (0, \epsilon/2)$, let $\mathbf{p}'_\pi = \mathbf{p}_\pi - \delta$, and let $\mathbf{p}'_{\pi'} = \mathbf{p}_{\pi'} + \delta$. By definition, $||\mathbf{p} - \mathbf{p}'|| < \epsilon$. Now, it remains to prove that a^* is the unique Minimax winner in $\mathbf{p}'$.

First, because a was wlog (as discussed above) the unique greatest pairwise defeater of a^*, it must be that $R_{scores}(\mathbf{p}')_{a^*} < R_{scores}(\mathbf{p})_{a^*}$, i.e., from $\mathbf{p} \to \mathbf{p}'$, a^*'s Minimax score *decreased.* Now, we will show that all other alternatives' minimax scores weakly *increase*: for all alternatives $a' \in M_3$, $R_{scores}(\mathbf{p}')_{a'} = R_{scores}(\mathbf{p})_{a'}$, because a' is pairwise defeated by the exact same alternatives across all rankings in $\mathbf{p}$ and $\mathbf{p}'$. For all alternatives $a' \in M_1 \cup \cup M_2 \cup a$: first, suppose a^* was a''s greatest defeater in $\mathbf{p}$. In $\mathbf{p}'$, a^* will still be a''s greatest defeater, and so $R_{scores}(\mathbf{p}')_{a'} = R_{scores}(\mathbf{p})_{a'} + 2\epsilon$, i.e., their minimax score *increased.* If a^* is not a''s greatest defeater in $\mathbf{p}$ but is in $\mathbf{p}'$, it must again necessarily be that $R_{scores}(\mathbf{p}')_{a'} > R_{scores}(\mathbf{p})_{a'}$, because the frequency of a' being defeated by any alternative other than a^* did not change across the proportions. Finally, if a^* not a''s greatest defeater in $\mathbf{p}$ *and* $\mathbf{p}'$, for the same reason as just articulated, $R_{scores}(\mathbf{p}')_{a'} = R_{scores}(\mathbf{p})_{a'}$.

We have shown that between $\mathbf{p}$ and $\mathbf{p}'$, a's minimax score strictly decreases, and all other alternatives' either increase or remain the same. Because a by assumption had a minimal minimax score in $\mathbf{p}$, it now must have the *strictly* smallest minimax score in $\mathbf{p}'$,and thus it is the unique winner. We conclude that $\mathbf{p}' \notin \mathcal{P}\left(\Pi^{\neg(R,A)}\right)$, and thus $\mathcal{P}\left(\Pi^{\neg(R,A)}\right)$ has measure zero. Applying the first part of Theorem 6, the result follows.

PSRs. Let $R \in$PSR, i.e., R is an arbitrary positional scoring rule as defined in Appendix A. We will note the PSR score of alternative a in a proportion $\mathbf{p}$ as $R_{scores}(\mathbf{p})_a$. *Some intuition about how this compares to analogous arguments for condorcet-consistent rules:* We will again take the approach of switching alternatives, but in dealing with arbitrary positional scoring rules, we have to be a bit more careful in doing so than in the previous arguments, because we are no longer looking at binary indicators of whether an alternative beats another; we are now looking at alternatives' positional scores, which can be nonlinear across positions (although they must be weakly monotonic, by our definition), and so when switching alternatives, we must take care not to introduce new alternatives into the set $Ties$. Moreover, we must work harder to find appropriate alternatives to switch, because in rules like Veto, scores can be the same across positions, meaning that switching alternatives may have no effect.

[10]In the corner case where a^*'s greatest pairwise defeaters are multiple, the process outlined next should be applied to all a^*'s pairwise defeaters.

We will first prove a simpler claim — that we can reduce a k-way tie to a $k-1$-way tie — and then show that the claim follows. Fix m, a profile proportion $\mathbf{p} \in \mathcal{P}\left(\Pi^{\neg(R,A)}\right)$, and let $Ties \subseteq M$ be the set of k candidates with the maximum score. By assumption, we know $|Ties| > 1$. Choose an alternative $a \in Ties$ and an $a' \notin Ties$, and let $R_{score}(a) - R_{score}(a') = \epsilon' > 0$. Now, there must exist a ranking π and positions $i < j \in [m]$ such that $\mathbf{p}_\pi > 0$ and in which (1) a is at position i and a' is at position j and (2) $\alpha_i > \alpha_j$. Now, we will express π as follows, and define a closely-related ranking π', using M_i previously:

$$\pi := M_1 \succ a \succ M_2 \succ a' \succ M_3$$
$$\pi' := M_1 \succ a' \succ M_2 \succ a \succ M_3$$

Take $\epsilon < \min\{\mathbf{p}_\pi, \epsilon'\}$. Now, we define $\mathbf{p}'$ in terms of this ϵ as follows: first let $\mathbf{p}'_{\pi''} = \mathbf{p}_{\pi''}$ for all $\pi'' \in \mathcal{L} \setminus \{\pi, \pi'\}$. Then, for $\delta \in (0, \epsilon/2)$, let $\mathbf{p}'_\pi = \mathbf{p}_\pi - \delta$, and let $\mathbf{p}'_{\pi'} = \mathbf{p}_{\pi'} + \delta$. By definition, $\|\mathbf{p} - \mathbf{p}'\| < \epsilon$. Notably, the resulting profile is valid because $\epsilon/2 < \mathbf{p}_\pi$, and secondly, Now, let $Ties'$ be the set of ties in $\mathbf{p}'$. It remains to show that $|Ties'| = k - 1$. First, observe that for all alternatives $a'' \in M \setminus \{a, a'\}$, $R_{scores}(\mathbf{p})_{a''} = R_{scores}(\mathbf{p}')_{a''}$, because none of these alternatives' positions changed from $\mathbf{p} \longrightarrow \mathbf{p}'$. Second, it must be that a's score strictly decreased by $\delta(a_i - a_j)$, because it was moved to a lower position in a δ fraction of profiles. Third, a''s score strictly increased by $\delta(a_i - a_j)$ for the same reason. However, note that $R_{score}(\mathbf{p}')_{a'}$ remains less than $R_{score}(\mathbf{p})_a$, the original score of those alternatives that tied, because $\delta < \epsilon'$, the original gap. Therefore, $R(\mathbf{p}) = Ties' = Ties \setminus \{a\}$, and we conclude that $|Ties'| = k - 1$.

Notice that we can apply the above process iteratively to bring the number of tying alternatives to 1, i.e., to the scenario where there is a unique winner. If the process of going from a i-way tie to a $i-1$-way tie uses an $\epsilon_i > 0$, then the ϵ we will need to create a profile proportion with a unique winner is $\epsilon < c\sum_{i=2}^{k} \epsilon_i$ (the c is a constant for safety, to deal with translating between norms), which is arbitrarily small. Thus, given $\mathbf{p}$ we can construct a profile that is arbitrarily close by and not contained in $\mathcal{P}\left(\Pi^{\neg(R,A)}\right)$, and thus $\mathcal{P}\left(\Pi^{\neg(R,A)}\right)$ has measure zero. Applying the first part of Theorem 6, the result follows. □

C.2 Proof of Proposition 15

Fix m, let $R =$ Copeland, and let $R_{score}(\mathbf{p})$ be the vector of candidates' copeland scores for a given proportion, as defined in Appendix A. Now, define $\mathbf{p}$ such that

$$\mathbf{p}_{1 \succ 2 \succ 3 \succ \cdots \succ m} = \mathbf{p}_{2 \succ 3 \succ 1 \succ \cdots \succ m} = \mathbf{p}_{3 \succ 1 \succ 2 \succ \cdots \succ m} = 1/3$$

and by construction, all other entries of $\mathbf{p} = 0$. By definition, $\mathbf{p} \in \mathcal{P}\left(\Pi^{\neg(R,A)}\right)$, because the alternatives 1, 2, and 3 each pairwise beat one of the other two, and they all three pairwise beat alternatives $4 \ldots m$. Therefore, $R_{score}(\mathbf{p})_1 = R_{score}(\mathbf{p})_3 = R_{score}(\mathbf{p})_2 = m - 4 + 1$, and all other alternatives' Copeland scores are 0.

Now, let $\epsilon > 0$. Now, we will construct $\mathbf{p}'$ from $\mathbf{p}$, noting that in doing so, we are not permitted to move any one proportion weakly more than ϵ (as this would result in $\|\mathbf{p} - \mathbf{p}'\| \geq \epsilon$), so it must be that

$$\mathbf{p}'_{1 \succ 2 \succ 3 \succ \cdots \succ m} \in (1/3 - \epsilon, 1/3 + \epsilon)$$
$$\mathbf{p}'_{2 \succ 3 \succ 1 \succ \cdots \succ m} \in (1/3 - \epsilon, 1/3 + \epsilon)$$
$$\mathbf{p}'_{3 \succ 1 \succ 2 \succ \cdots \succ m} \in (1/3 - \epsilon, 1/3 + \epsilon)$$

Recall that in $\mathbf{p}$, 1 pairwise beat 2 but was beaten by 3. Here, if $\epsilon < 1/12$, in $\mathbf{p}'$ 1 cannot pairwise beat 3:

$$\mathbf{p}'_{1 \succ 2 \succ 3 \succ \cdots \succ m} < 1/3 + \epsilon < 1/2$$

Nor can we make any alternative $4 \dots m$ beat any alternative 1, 2, or 3. By symmetry, this means that the precise pairwise winners from $\mathbf{p}$ must be preserved in $\mathbf{p}'$, from which it follows that the Copeland scores must be the same in this new profile proportion. We conclude that for all $\mathbf{p}' \in B_\epsilon(\mathbf{p})$, $\mathbf{p}' \in \mathcal{P}\left(\Pi^{\neg(R,A)}\right)$, which implies that $\mathcal{P}\left(\Pi^{\neg(R,A)}\right)$ has positive measure. By applying the second part of Theorem 6, the result follows. □

C.3 Proof of Proposition 16

For consistency of notation throughout our proofs, here we let A = Condorcet Consistency. We first prove robustness, which amounts to proving that the set $\mathcal{P}\left(\Pi^{\neg(R,A)}\right)$ has positive measure.
Case 1: $R \neq$Plurality.
Fix m and a positional scoring rule $(b_1, b_2, \dots, b_m)$, where wlog, $b_1 = 1$ and $b_m = 0$. Because $R \neq$Plurality, we have that $b_2 > 0$. Define our profile proportion $\mathbf{p}$ such that all its entries are 0 except those corresponding to the rankings $1 \succ 2 \succ \dots \succ m$ and $2 \succ 3 \succ \dots \succ m \succ 1$; for these, let

$$\mathbf{p}_{1\succ 2\succ\cdots\succ m} = 1/2 + \delta \quad \text{and} \quad \mathbf{p}_{2\succ 3\succ\cdots\succ m\succ 1} = 1/2 - \delta,$$

Note that by definition, $\mathbf{p} \in \mathcal{P}\left(\Pi^{\neg(R,A)}\right)$, because 1 must be the condorcet winner and 2 must be the PSR winner.

Now, let $\delta = b/8$. Fix $\epsilon < \delta/2$. We will construct $\mathbf{p}'$ from $\mathbf{p}$, noting that in doing so, we are not permitted to move any one proportion more than ϵ (as this would result in $\|\mathbf{p} - \mathbf{p}'\| > \epsilon$), so it must be that

$$\mathbf{p}'_{1\succ 2\succ 3\succ\cdots\succ m} \in (1/2 + \delta/2, 1/2 + 3\delta/2)$$
$$\mathbf{p}'_{2\succ 3\succ\cdots\succ m\succ 1} \in (1/2 - 3\delta/2, 1/2 - \delta/2)$$

It is clear from these bounds that within the ϵ ball, 1 must remain the condorcet winner. 2 will get minimum $1/2 - 3\delta/2 + b(1/2 + \delta/2)$ points, based on the lower bounds. The best thing that can be done for 1 is to put maximum mass on 1 being first, and then transfer as much mass as we are allowed—$\delta/2$—from $\mathbf{p}_{2\succ 3\succ\cdots\succ m\succ 1}$ to a different ranking that puts 1 first. Thus, and 1 will get at most $1/2 + 3\delta/2 + \delta/2$. One can easily confirm that with $\delta = 8$, it must be that anywhere in the ϵ-ball, 2 receives a higher R score than 1. By inspection of the above bounds, 2 must also dominate all other alternatives by R score everywhere in the ϵ ball. We conclude that $B_\epsilon(\mathbf{p}) \subseteq \mathcal{P}\left(\Pi^{\neg(R,A)}\right)$, and therefore $\mathcal{P}\left(\Pi^{\neg(R,A)}\right)$ is not measure zero. Applying the second part of Theorem 6, the result follows.

Case 2: R =Plurality. Fix m, and define $\mathbf{p}$ such that

$$\mathbf{p}_{1\succ 2\succ 3\succ\cdots\succ m} = 1/3, \qquad \mathbf{p}_{2\succ 1\succ 3\succ\cdots\succ m} = 1/3 + \delta \qquad \mathbf{p}_{3\succ 1\succ 2\succ\cdots\succ m} = 1/3 - \delta$$

We're assuming $\delta > 0$ is arbitrarily small. Observe that 1 is the condorcet winner, and 2 is the plurality winner, so $\mathbf{p} \in \mathcal{P}\left(\Pi^{\neg(R,A)}\right)$.

Set $\epsilon < \delta/2$. Now, we construct $\mathbf{p}'$ from $\mathbf{p}$, noting that in doing so, we are not permitted to move any one proportion more than ϵ (as this would result in $\|\mathbf{p} - \mathbf{p}'\| > \epsilon$), so it must be that

$$\mathbf{p}'_{1\succ 2\succ 3\succ\cdots\succ m} \in (1/3 - \delta/2, 1/3 + \delta/2)$$
$$\mathbf{p}'_{2\succ 1\succ 3\succ\cdots\succ m} \in (1/3 + \delta/2, 1/3 + 3\delta/2)$$
$$\mathbf{p}'_{3\succ 1\succ 2\succ\cdots\succ m} \in (1/3 - 3\delta/2, 1/3 - \delta/2).$$

By these bounds, it is clear that 2 must remain the plurality winner in the ϵ ball. Moreover, 1 must remain the Condorcet winner, because 1 must beat 2 pairwise in at least $2/3 - 2\delta$ of rankings (for δ sufficiently small, this is larger than $1/2$), and 1 must beat 3 pairwise in at least $2/3$ of rankings, and 1 beats all else in all rankings. It follows that in all profiles in the ϵ ball, R is inconsistent with A.

Thus, we have found $B_\epsilon(\mathbf{p}) \subseteq \mathcal{P}\left(\Pi^{\neg(R,A)}\right)$, and we conclude that $\mathcal{P}\left(\Pi^{\neg(R,A)}\right)$ has positive measure. By applying the second part of Theorem 6, the result follows. □

C.4 Proof of Proposition 18

Define the following profile proportions, which can be realized with a profile as small as $n = 300$:

$$\mathbf{p} = \{p_{123} = 3/25, p_{132} = 4/15, p_{213} = 23/60, p_{231} = 0, p_{312} = 0, p_{321} = 23/100\}$$

Let $\boldsymbol{\pi}$ be the corresponding profile. Fix an arbitrary $R \in \text{PSR}$, which we can wlog express as $(1, b, 0)$ (these are the positional weights). Now, we will show that for all $\phi \in [0, 1)$ that for some $\epsilon > 0$,

$$\mathbf{E}\left[\mathcal{P}\left(\mathcal{S}_\phi^{Mallows}(\boldsymbol{\pi})\right)\right] \in B_\epsilon\left(\mathbf{E}[\mathcal{P}(\mathcal{S}_\phi^{Mallows}(\boldsymbol{\pi}))]\right) \subseteq \mathcal{P}\left(\Pi^{\neg(R,A)}\right)$$

I.e., the expected proportions after mallows noise fall inside a positive-measure ball contained within the set of counterexamples. From this point, we can apply the argument in Lemma 2 to conclude the convergence we want (which, as extends from that theorem, converges at an exponential rate).

Given that we are working within a specific instance with $m = 3$, we will represent an arbitrary ranking as ijk to indicate the ranking $i \succ j \succ k$. We will also use $Y_{ijk} = \mathcal{P}\left(\mathcal{S}_\phi^{Mallows}(\boldsymbol{\pi})\right)_{ijk}$ as shorthand for the random variable describing the proportion of rankings that are $i \succ j \succ k$ post ϕ Mallows noise. Then, by the definition of the Mallows model, we can write the expectation of an arbitrary proportion as

$$\mathbf{E}\left[Y_{ijk}\right] = \frac{p_{ijk}\phi^0 + (p_{jik} + p_{ikj})\phi^1 + (p_{jki} + p_{kij})\phi^2 + p_{kji}\phi^3}{1 + 2\phi + 2\phi^2 + \phi^3} \tag{1}$$

Then, we can express the set of constraints we must prove are satisfied in order for this to be the counterexample we want, meaning (1) 2 pairwise beats 1, (2) 2 pairwise beats 3, (3) $R_{score}(\mathbf{p})_1 > R_{score}(\mathbf{p})_2$, and (4) $R_{score}(\mathbf{p})_1 > R_{score}(\mathbf{p})_3$. When these constraints are satisfied (along with nonnegativity constraints and adding up constraints, which we will ensure are satisfied in the definition of our instance), 2 is the unique condorcet winner in expectation, and 1 is the unique R winner in expectation.

$$\mathbf{E}[(Y_{213} + Y_{321} + Y_{231}) - (Y_{123} + Y_{132} + Y_{312})] > 0 \qquad \forall\phi \in [0, 1) \tag{2}$$

$$\mathbf{E}[(Y_{213} + Y_{123} + Y_{231}) - (Y_{321} + Y_{132} + Y_{312})] > 0 \qquad \forall\phi \in [0, 1) \tag{3}$$

$$\mathbf{E}[(Y_{123} + Y_{132}) + b(Y_{213} + Y_{312}) - (Y_{213} + Y_{231}) - b(Y_{123} + Y_{321})] > 0 \qquad \forall\phi \in [0, 1) \tag{4}$$

$$\mathbf{E}[(Y_{123} + Y_{132}) + b(Y_{213} + Y_{312}) - (Y_{312} + Y_{321}) - b(Y_{132} + Y_{231})] > 0 \qquad \forall\phi \in [0, 1) \tag{5}$$

We plug in our instance and simplify the expectations above via Mathematica per Equation 1, which yields the following functions of ϕ:

$$1/75(1 - \phi)(17 - 23\phi + 17\phi^2) \overset{?}{>} 0 \qquad \forall\phi \in [0, 1) \tag{6}$$

$$1/150(1 - \phi)(1 + 116\phi + \phi^2) \overset{?}{>} 0 \qquad \forall\phi \in [0, 1) \tag{7}$$

$$1/300(1 - \phi)(1 + \phi)(1 + 11\phi) \overset{?}{>} 0 \qquad \forall\phi \in [0, 1) \tag{8}$$

$$1/300(1 - \phi)(1 + \phi)(47 + 82\phi) \overset{?}{>} 0 \qquad \forall\phi \in [0, 1) \tag{9}$$

The only one of these inequalities that is not obviously true for all $\phi \in (0, 1]$ is the first, but an application of the quadratic equation shows that the polynomial $17 - 23\phi + 17\phi^2$ has no real roots. Given that it is convex, this means that $17 - 23\phi + 17\phi^2 > 0 \forall\phi \in [0, 1)$.

We have now shown that $\mathbf{E}\left[\mathcal{P}\left(\mathcal{S}_{\phi}^{Mallows}(\boldsymbol{\pi})\right)\right] \in \mathcal{P}(\Pi^{\neg(R,A)})$; however, we must show that not only this expectation point, but an ϵ-ball around it, is contained within the set of counterexamples. Take the minimum of lefthand sides of the inequalities above. By inspection, neither the second nor fourth is the smallest, so we set δ the minimum of the first and third, minimized over all ϕ:

$$\delta = \min\left\{\min_{\phi\in[0,1)} {}^{1}\!/_{75}(1-\phi)(17-23\phi+17\phi^2), \min_{\phi\in[0,1)} {}^{1}\!/_{300}(1-\phi)(1+\phi)(1+11\phi)\right\} > 0$$

Our ability to find such a δ concludes the proof, but we spell out the small constants here. Set $\epsilon \in (0, \delta/48)$. Now we want to show that there exists an $\epsilon > 0$ ball around the *expected* instance, composed of the proportions $\mathbf{E}[Y_{ijk}]$ for all $i \neq j \neq k$. Call this $\mathbf{p}_{exp}$. Fix an arbitrary $\mathbf{p}'_{exp} \in B_{\epsilon}(\mathbf{p}_{exp})$. The distance between these points is $< \delta/48$, which means that from $\mathbf{p}_{exp} \longrightarrow \mathbf{p}'_{exp}$, we cannot have changed any entry by more $\delta/48$. By the functional form of the expectations in (2)-(5), the most these expectations can be affected by changing any single $\mathbf{E}[Y_{ijk}]$ is $< 8 \cdot \delta/48 = \delta/6$. There are 6 terms, so the maximum amount any of the expectations in (2)-(5) changes is $< \delta$. By how we defined δ, the constraints in (2)-(5) remain satisfied, and $\mathbf{p}' \in \mathcal{P}(\Pi^{\neg(R,A)})$.

This proves the existence of a positive-measure ball around our expectation within the space of counterexamples. Applying Lemma 2 shows concentration of probability mass over this ball as $n \longrightarrow \infty$, concluding the proof. □