CS1200: Intro. to Algorithms and their Limitations Anshu & Vadhan

Sender–Receiver Exercise 3: Reading for Receivers

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The goals of this exercise are:

1. To develop your skills at understanding, distilling, and communicating proofs and the conceptual ideas in them, especially for proofs in graph theory

2. To deepen your understanding of breadth-first search and its efficiency

Sections 1 and 3, as well as the statement of Theorem 2.1, are also in the reading for receivers. Your goal will be to communicate the *proof* of Theorem 2.1 to the receivers.

1 Connected Components

We begin by defining the *connected components* of an undirected graph. To gain intuition, you may find it useful to draw some pictures of graphs with multiple connected components and use them to help you follow along the prof.

Theorem 1.1. *Every* undirected *graph $G = (V, E)$ can be partitioned into connected components. That is, there are sets $V_0, \ldots, V_{c-1} \subseteq V$ of vertices such that:*

1. *$V_0, \ldots, V_{c-1}$ are disjoint, nonempty, and $V_0 \cup V_1 \cup \cdots \cup V_{c-1} = V$. (This is what it means for $V_0, \ldots, V_{c-1}$ to be a* partition *of V.)*

2. *For every two vertices $u, v \in V$, u and v are in the same component V_i if and only if there is a path from u to v.*

Moreover the sets $V_0, \ldots, V_{c-1}$ are unique (up to ordering), and are called the connected components *of V.*

In case you are interested, we include a proof of Theorem 1.1 below in Section 1, but studying that proof is not required for this exercise.

We remark that for *directed* graphs, one can consider *weakly connected components*, where we ignore the directions of edges, and *strongly connected components*, where two vertices u, v are the same component if and only if there is a directed path from u to v *and* a directed path from v to u. Strongly connected components are more useful, but more complicated. In particular, unlike in undirected graphs (or weakly connected components), there can be edges crossing between strongly connected components.

2 Finding Connected Components via BFS

The main result of this exercise is an efficient algorithm for finding connected components:

Theorem 2.1. *There is an algorithm that given an undirected graph $G = (V, E)$ with n vertices and m edges, partitions V into connected components in time $O(n + m)$.*

Proof.
Proof outline:

Modification of BFS:

Runtime of modified BFS:

Algorithm to Find Connected Components:

Correctness of Algorithm:

Runtime of Algorithm: □

3 Proof of Theorem 1.1

Proof. For every vertex u, define

$$[\![u]\!] = \{v : \text{there is a path from } u \text{ to } v \text{ in } G\}.$$

Observe that $u \in [\![u]\!]$; in particular, the set $[\![u]\!]$ is nonempty.

Now let's show that for every two vertices u and w, we have either that $[\![u]\!]$ and $[\![w]\!]$ are disjoint or equal. Suppose they are not disjoint, i.e. there is a vertex $v \in [\![u]\!] \cap [\![w]\!]$. This means that there is a path p_{uv} from u to v and a path p_{wv} from w to v. Now we argue that $[\![u]\!] \subseteq [\![w]\!]$. Let a be any vertex in $[\![u]\!]$, so there is a path p_{ua} from u to a. Then we can get a path from w to a by first following the path p_{wv} to get from w to v, then reversing the edges in p_{uv} to get from v to u, and then following the path p_{ua} to get from u to a. Thus, $a \in [\![w]\!]$. Since we showed that this holds for every $a \in [\![u]\!]$, we conclude that $[\![u]\!] \subseteq [\![w]\!]$. The reverse inclusion $[\![w]\!] \subseteq [\![u]\!]$ is proved in a similar manner.

So now we take $V_0, \ldots, V_{c-1}$ to be all of the distinct sets that occur among those of the form $[\![u]\!]$. Since every vertex $u \in V$ is in the set $[\![u]\!]$, the sets $V_0, \ldots, V_{c-1}$ will cover all of V, and by what we just showed, any two distinct sets will be disjoint from each other. This establishes Item 1 Now if a vertex u is in component V_i, this means that $[\![u]\!] = V_i$ (else $[\![u]\!]$ and V_i would be distinct but not disjoint, contradicting what we showed above). So V_i contains exactly the vertices v that are reachable from u, establishing Item 2.

We omit the proof of uniqueness of the connected components. $\square$

If you have seen equivalence relations, you may recognize some similarity with the above proof. Indeed, the above proof amounts to showing that "v is reachable from u" is an equivalence relation on V, and then taking the connected components to be the equivalence classes under that relation.