CS1200: Intro. to Algorithms and their Limitations Anshu & Vadhan

Sender–Receiver Exercise 5: Reading for Senders

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The goals of this exercise are:

- to develop your skills at understanding, distilling, and communicating proofs and the conceptual ideas in them,
- to practice reductions to SAT, and in particular how logic is useful for modelling problems

Section 1 is also in the reading for receivers. Your goal will be to communicate the *proof* of Theorem 1.1 (i.e. the content of Section 2) to the receivers.

1 The Result

In class, we saw how the (seemingly hard) problem of Graph k-Coloring can be efficiently reduced to CNF-Satisfiability (SAT). Although SAT also seems to be a hard problem (as we'll formalize in the last part of the course), this allows all the effort put into SAT Solvers to solve many large k-coloring instances in practice.

In this exercise, you'll see a similar reduction for the *Longest Path* problem. Recall that a *path* is a walk with no repeated vertices.

Input	**:** A digraph $G = (V, E)$ and two vertices $s, t \in V$
Output	**:** A *longest path* from s to t in G, if one exists

Computational Problem LongestPath

Actually, it will be more convenient to consider a version where the desired path length is specified in the input.

Input	**:** A digraph $G = (V, E)$, two vertices $s, t \in V$, and a path-length $k \in \mathbb{N}$
Output	**:** A path from s to t in G of length k, if one exists

Computational Problem LongPath

Since a path has no repeated vertices, it suffices to consider $k \leq n$. If we have an efficient algorithm for LongPath, then we can solve LongestPath by trying $k = n, n-1, \ldots, 0$ until we succeed in finding a path. The $k = n$ case is essentially the same as the *Hamiltonian Path* problem,[1] which is a special case of the notorious *Travelling Salesperson Problem (TSP)*. In the TSP, we have a salesperson who wishes to visit n cities (to sell their goods) in the shortest travel time possible. If we model the possible travel between cities as a directed graph, then Hamiltonian Path corresponds to the special case where all pairs u, v of cities either have a travel time of 1 (edge (u, v) present) or a very large travel time (edge (u, v) not present). In such a case, the only way to visit all cities in travel time at most $n - 1$ is via a Hamiltonian Path.[2]

[1] In the HamiltonianPath problem, we don't specify the start and end vertex; any path of length n suffices. But the two problems can be efficiently reduced to each other (exercise).

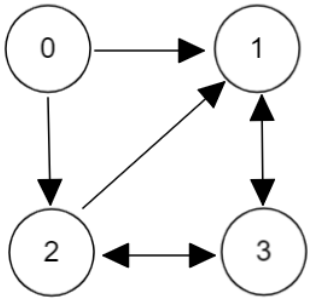
[2] Often in the TSP, it is also required that the salesperson return back to their starting city s. If we add edges of travel time 1 from all cities to the starting city, then we see that Hamiltonian Path is also a special case of this variant of the TSP.

The reduction from LongPath to SAT is given as follows.

Theorem 1.1. *LongPath on a digraph with n vertices, m edges, and a path length k reduces to SAT in time $O(n^2k)$.*

2 The Proof

Let $G = (V, E)$, s, t, k be our instance of LongPath. We will use the following graph G with $s = 0$, $t = 1$, and $k = 3$ as a running example:



The proof will be divided in two steps: constructing a SAT instance and then using the SAT instance in the reduction algorith,.

2.1 Constructing the SAT instance

The construction of the SAT instance φ, from the LongPath instance, will proceed by introducing a set of boolean 'indicator' variables (similar in spirit to what we did for k-Coloring in Lecture 15). We will have $(k + 1)n$ variables $x_{i,v}$ for each $i \in [k + 1]$ and vertex $v \in V$. Intuitively, $x_{i,v} = 1$ will mean that the i^{th} vertex in the path is v. There are multiple ways to choose the indicator variables,[3] but we will stick with this choice to achieve the runtime claimed in Theorem 1.1.

For example, the path $0 \to 2 \to 3 \to 1$ in the example graph would be represented as the following assignment (written as an $n \times (k+1)$ two dimensional array, for the ease of presentation):

$$\begin{bmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,0} & x_{3,1} & x_{3,2} & x_{3,3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The indicator variables can't arbitrarily take values in $\{0, 1\}$ - our interpretation of $x_{i,v}$ means that we need to enforce some constraints among the variables. For example, the path should start at s and end at t. Thinking about these constraints is a crucial step in the reduction and the clauses help to enforce these constraints. The CNF φ is the AND of all the clauses introduced below.

[3] Another way to define indicator variables is as follows. For each u, v with $(u, v) \in E$, let $x_{u,v}$ be a boolean variable, with the interpretation that $x_{u,v} = 1$ implies that the edge (u, v) is in the path. It is possible to construct a SAT instance using these variables. We avoid this approach at this stage since is quite non-trivial to enforce that exactly k edges be in the path.

1. **At least one vertex is assigned to each step of the path:** This constraint is enforced by the clause $\left(\bigvee_{v \in V} x_{i,v}\right)$, for each $i = 0, \ldots, k$. In our example, this would yield the following clauses:

$$(x_{0,0} \vee x_{0,1} \vee x_{0,2} \vee x_{0,3}),$$
$$(x_{1,0} \vee x_{1,1} \vee x_{1,2} \vee x_{1,3}),$$
$$(x_{2,0} \vee x_{2,1} \vee x_{2,2} \vee x_{2,3}),$$
$$(x_{3,0} \vee x_{3,1} \vee x_{3,2} \vee x_{3,3})$$

2. **Two distinct vertices are not assigned to the same step of the path:** For each pair $v, w \in V$ such that $v \neq w$ and $i \in [k+1]$, we introduce the clause $(\neg x_{i,v} \vee \neg x_{i,w})$. In our example, this would yield the following clauses:

$$(\neg x_{0,0} \vee \neg x_{0,1}), (\neg x_{0,0} \vee \neg x_{0,2}), (\neg x_{0,0} \vee \neg x_{0,3}), (\neg x_{0,1} \vee \neg x_{0,2}), (\neg x_{0,1} \vee \neg x_{0,3}), (\neg x_{0,2} \vee \neg x_{0,3}),$$
$$(\neg x_{1,0} \vee \neg x_{1,1}), (\neg x_{1,0} \vee \neg x_{1,2}), (\neg x_{1,0} \vee \neg x_{1,3}), (\neg x_{1,1} \vee \neg x_{1,2}), (\neg x_{1,1} \vee \neg x_{1,3}), (\neg x_{1,2} \vee \neg x_{1,3}),$$
$$(\neg x_{2,0} \vee \neg x_{2,1}), (\neg x_{2,0} \vee \neg x_{2,2}), (\neg x_{2,0} \vee \neg x_{2,3}), (\neg x_{2,1} \vee \neg x_{2,2}), (\neg x_{2,1} \vee \neg x_{2,3}), (\neg x_{2,2} \vee \neg x_{2,3}),$$
$$(\neg x_{3,0} \vee \neg x_{3,1}), (\neg x_{3,0} \vee \neg x_{3,2}), (\neg x_{3,0} \vee \neg x_{3,3}), (\neg x_{3,1} \vee \neg x_{3,2}), (\neg x_{3,1} \vee \neg x_{3,3}), (\neg x_{3,2} \vee \neg x_{3,3})$$

3. **The path starts with s:** This is enforced by the simple clause $(x_{0,s})$ In our example, this would yield the clause $(x_{0,0})$.

4. **The path ends with t:** This is enforced by the clause $(x_{k,t})$. In our example, this would yield the clause $(x_{3,1})$.

5. **The same vertex is not assigned to two different steps in the path:** We enforce this by introducing this clause for each $v \in V$ and $i \neq j \in [k+1]$: $(\neg x_{i,v} \vee \neg x_{j,v})$.

   In our example, the clauses would look as the follows. For vertex 0, we would have the following clauses:

   $$(\neg x_{0,0} \vee \neg x_{1,0}), (\neg x_{0,0} \vee \neg x_{2,0}), (\neg x_{0,0} \vee \neg x_{3,0}), (\neg x_{1,0} \vee \neg x_{2,0}), (\neg x_{1,0} \vee \neg x_{3,0}), (\neg x_{2,0} \vee \neg x_{3,0})$$

   We can do the same for vertices 1,2,3:

   $$(\neg x_{0,1} \vee \neg x_{1,1}), (\neg x_{0,1} \vee \neg x_{2,1}), (\neg x_{0,1} \vee \neg x_{3,1}), (\neg x_{1,1} \vee \neg x_{2,1}), (\neg x_{1,1} \vee \neg x_{3,1}), (\neg x_{2,1} \vee \neg x_{3,1}),$$
   $$(\neg x_{0,2} \vee \neg x_{1,2}), (\neg x_{0,2} \vee \neg x_{2,2}), (\neg x_{0,2} \vee \neg x_{3,2}), (\neg x_{1,2} \vee \neg x_{2,2}), (\neg x_{1,2} \vee \neg x_{3,2}), (\neg x_{2,2} \vee \neg x_{3,2}),$$
   $$(\neg x_{0,3} \vee \neg x_{1,3}), (\neg x_{0,3} \vee \neg x_{2,3}), (\neg x_{0,3} \vee \neg x_{3,3}), (\neg x_{1,3} \vee \neg x_{2,3}), (\neg x_{1,3} \vee \neg x_{3,3}), (\neg x_{2,3} \vee \neg x_{3,3}),$$

6. **For any two adjacent steps in the path, the vertices should form an edge in the graph:** We enforce this using the following clauses. For each step $i \in [k]$, and any pair of vertices $(u, v) \in (V \times V) \setminus E$ (that is, (u, v) *do not* form an edge): $(\neg x_{i,u} \vee \neg x_{i+1,v})$. Thus, we include edges by excluding non-edges.

   In our example, this would yield the following clauses. For the non-edges leaving vertex 0:

$$(\neg x_{0,0} \vee \neg x_{1,3}),$$
$$(\neg x_{1,0} \vee \neg x_{2,3}),$$
$$(\neg x_{2,0} \vee \neg x_{3,3})$$

   Looking at the non-edges of vertex 1:

$$(\neg x_{0,1} \vee \neg x_{1,0}), (\neg x_{0,1} \vee \neg x_{1,2}),$$
$$(\neg x_{1,1} \vee \neg x_{2,0}), (\neg x_{1,1} \vee \neg x_{2,2}),$$
$$(\neg x_{2,1} \vee \neg x_{3,0}), (\neg x_{2,1} \vee \neg x_{3,2})$$

   Looking at the non-edges leaving vertex 2:

$$(\neg x_{0,2} \vee \neg x_{1,0}),$$
$$(\neg x_{1,2} \vee \neg x_{2,0}),$$
$$(\neg x_{2,2} \vee \neg x_{3,0})$$

   Looking at the non-edges leaving vertex 3:

$$(\neg x_{0,3} \vee \neg x_{1,0}),$$
$$(\neg x_{1,3} \vee \neg x_{2,0}),$$
$$(\neg x_{2,3} \vee \neg x_{3,0})$$

Number of clauses in ϕ: Overall, we have 2 clauses of size 1, $k+1$ clauses of size n, and $O(n^2k + nk^2 + (n^2 - m)k) = O(n^2k)$ clauses of size 2. (Recall that $k \leq n$.)

2.2 Reduction algorithm

The reduction from LongPath to SAT is as follows.

```
1 LongPathViaSAT(G, s, t, k)
  Input  : A digraph G = (V, E), vertices s, t ∈ V, a path-length k
  Output : A longest path from s to t in G with no repeated vertices, or ⊥ if no path
           from s to t exists
2 Construct φ from G, s, t, k as described in subsection 2.1;
3 Call the SAT Oracle on φ, obtaining α, which is either a satisfying assignment to φ or ⊥ if
  φ is unsatisfiable;
4 if α = ⊥ then return ⊥;
5 else
6 |  Convert α to a LongPath solution P via Claim 2.2;
7 |  return P below;
```

Algorithm 1: LongPath by reduction to SAT

In Line 3, we call a SAT oracle on φ to find out whether φ is satisfiable. The following claim shows that - crucially - the satisfiability of φ is directly related to the existence of a path of length k from s to t in G.

Claim 2.1. *If there a LongPath solution on instance (G, s, t, k), then φ is satisfiable.*

Further, if φ is satisfiable, we also obtain a satisfying assignment α. The following Claim, used in Line 6 of the reduction algorithm, shows how to efficiently construct a path of length k from s to t in G from α.

Claim 2.2. *We can transform any satisfying assignment α to φ into a solution to LongPath on instance (G, s, t, k) in time $O(nk)$.*

Before proving the claims, lets see why Theorem 1.1 follows. First, the reduction algorithm indeed runs in time $O(n^2k)$. The construction of φ in Line 2, which has $O(n^2k)$ clauses, takes time $O(n^2k)$ using the adjacency list representation of G. The oracle call in Line 3 is—by definition—taken to be 1 step. By Claim 2.2, Line 6 takes time $O(nk)$. For the correctness, Claim 2.1 implies that we only return $\perp$ when there is no solution to the LongPath instance. Claim 2.2 implies that whenever we return a path P, it is a valid solution to the LongPath instance.

2.3 Proofs of the claims

Proof of Claim 2.1. Suppose there is a LongPath solution on instance (G, s, t, k). That is, there is a path P of length k, starting at s and ending at t. Let $(v_0, v_1, \ldots, v_k)$ be the vertices on the path P. Consider the following assignment α to the variables of φ:

$$\alpha_{i,v} = \begin{cases} 1 & \text{if } v_i = v \\ 0 & \text{otherwise.} \end{cases}$$

We verify by inspection that α satisfies all of the clauses of φ. It satisfies all clauses of Types 1 and 2 because for each $i = 0, \ldots, k$, we set $\alpha_{i,v} = 1$ for exactly one value of v (namely $v = v_i$). It satisfies clauses 3 and 4 and because P starts with s and ends with t. It satisfies all clauses of Type 5 because P has no repeated vertices. It satisfies all clauses of Type 6 because P only uses edges of G (i.e. $(v_i, v_{i+1}) \in E$ for $i = 0, \ldots, k$). □

Proof of Claim 2.2. Let α be any satisfying assignment to φ. Because α satisfies the clauses of Types 1 and 2, for each $i = 0, \ldots, k$, we have that $\alpha_{i,v} = 1$ for exactly one value of v, call this value v_i. Our LongPath solution will be $P = (v_0, v_1, \ldots, v_k)$, which can be constructed from α in time $O(nk)$.

Since α satisfies Clauses 3 and 4, we have $v_0 = s$ and $v_k = t$. Since α satisfies all clauses of Type 5, P has no repeated vertices. Since α satisfies all clauses of Type 6, we have that $(v_i, v_{i+1}) \in E$ for $i = 0, \ldots, k$. Thus, P is a path of length k from s to t, as desired. □

While the proofs of the two claims are very similar, it is important to note that they are proving different directions. In particular, Claim 2.1 would still hold even if we accidentally forgot to include some of the clause types in constructing φ; it is only by proving Claim 2.2 that we are sure that we haven't missed anything essential for ensuring that we get a solution to LongPath.