However, W is not a subspace since the example $(1,0,0,0)\ (0,1,0,0) \in W$ but $(1,0,0,0) + (0,1,0,0) = (1,1,0,0) \notin W$

Linear combination

If 'β' is a vector in the vector space V, then β is said to be a linear combination of the vectors $\alpha_1, \alpha_2, \dots \alpha_n$ in V, so if β can be expressed in the form $\beta = a_1\alpha_1 + a_2\alpha_2 + \dots a_n\alpha_n$, where $a_1, a_2 \dots a_n$ are scalars.

Prob1:
Write the vector $(1,7,-4)$ as a linear combination of the vector $(1,-3,2)$ and $(2,-1,1)$ in vectorspace $V(R^3)$ - 3d.

Sol: Let $\beta = (1,7,-4)$, & $\alpha_1 = (1,-3,2)$ & $\alpha_2 = (2,-1,1)$.

$\therefore \beta = a_1(1,-3,2) + a_2(2,-1,1)$

$(1,7,-4) = a_1(1,-3,2) + a_2(2,-1,1)$

$(1,7,-4) = (a_1, -3a_1, 2a_1) + (2a_2, -a_2, a_2)$

$(1,7,-4) = (a_1 + 2a_2, -3a_1 - a_2, 2a_1 + a_2)$

$\therefore a_1 + 2a_2 = 1 \rightarrow (1)$

$-3a_1 - a_2 = 7 \rightarrow (2)$

$2a_1 + a_2 = -4 \rightarrow (3)$

Let (2) & (3) $\Rightarrow$ Multiply (1) by $\cancel{a_1}$,

$$\begin{array}{r} -3a_1 - \cancel{a_2} = 7 \\ 2a_1 + \cancel{a_2} = -4 \\ \hline -a_1 = 3 \end{array}$$

(2) + (3) $\Rightarrow$

$\boxed{a_1 = -3}$

Sub $a_1 = -3$ in eq (1) $\Rightarrow -3 + 2a_2 = 1$

$2a_2 = 4$

$\boxed{a_2 = 2}$

$\therefore \beta = (1,7,-4) = -3(1,-3,2) + 2(2,-1,1)$ is expressed as a linear combination of the given vectors.

Prob-2:

Show that $(2,-5,3)$ can't be expressed as linear combination of the vectors $(1,-3,2)$, $(2,-4,-1)$ and $(1,-5,7)$.

soln:

Let $\beta = (2,-5,3)$ and $\alpha_1 = (1,-3,2)$, $\alpha_2 = (2,-4,-1)$, $\alpha_3 = (1,-5,7)$

let $\beta = (2,-5,3) = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3$

The system of equations are $\Rightarrow$

$$\begin{aligned} 2 &= a_1 + 2a_2 + a_3 \\ -5 &= -3a_1 - 4a_2 - 5a_3 \\ 3 &= 2a_1 - a_2 + 7a_3 \end{aligned}$$

$\therefore$ The matrix form of the above system

$$\begin{bmatrix} 1 & 2 & 1 \\ -3 & -4 & -5 \\ 2 & -1 & 7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}.$$

By using Rank method, we can find the Solution of the System of the equations.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -3 & -4 & -5 \\ 2 & -1 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$$

To find: $\rho(A|B)$; $\Rightarrow$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ -3 & -4 & -5 & -5 \\ 2 & -1 & 7 & 3 \end{array}\right]$$

A

Row echelon form: pivot entry 1

Reduced row echelon: pivot ele = 1, each column p. 1.

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 1 \\ 0 & -5 & 5 & -1 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$-5+6$

$-4+6$

$-5+3$

$-1-4$

$7-2$

$3-4$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right] \quad R_3 \rightarrow 2R_3 + 5R_2$$

$$\underbrace{}_{A}$$

$10 - 10$

$-2 + 5$

$\therefore \rho(A) = $ no. of non-zero rows

$= 2.$

$\rho(A|B) = 3.$

$\rho(A) \neq \rho(A|B).$

$\therefore$ The system is inconsistent and has no solution.

$\therefore$ $(2, -5, 3)$ cannot be expressed as a linear combination of $(1, -3, 2)$, $(2, -4, -1)$ and $(1, -5, 7)$.

→ **Linearly dependent:**

Let V be the vector space. The vectors $\alpha_1, \alpha_2 \ldots \alpha_n$ belongs to V are said to be linearly dependent, if there exist a scalar $a_1, a_2, \ldots a_n \in \mathbb{R}$ (not all of them zero) such that

$$a_1\alpha_1 + a_2\alpha_2 + \ldots a_n\alpha_n = 0.$$

→ **Linearly independent:**

If $a_1\alpha_1 + a_2\alpha_2 + \ldots a_n\alpha_n = 0$, where $a_1 = a_2 \ldots = a_n = 0$, then the vectors are said to be linearly independent.

Note:

1) The two vectors α_1, α_2 are dependent iff one of them is multiple of the other.

Let $a_1\alpha_1 + a_2\alpha_2 = 0$ for $a_1, a_2 \neq 0$

$$\alpha_1 = -\left(\frac{a_2}{a_1}\right)\alpha_2$$

conversely, if $\alpha_1 = k\alpha_2$, $\alpha_1 - k\alpha_2 = 0 \Rightarrow a_1 = 1$ and $a_2 = -k$

Hence α_1 and α_2 are dependent.

2) If 0 is one of the vectors in $\alpha_1, \alpha_2 \ldots \alpha_m$ say $\alpha_1 = 0$ then the vectors must be linearly dependent.

3) If any 2 vectors are equal, say $\alpha_1 = \alpha_2$ then the vectors are linearly dependent (Take $a_1 = 1$, $a_2 = -1$, $a_3 = a_4 = \ldots a_n = 0$).

Prove that (2,1,4), (-3,2,-1) and (1,-3,-2) are linearly independent.

Sol: To prove the given vectors are independent, $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = 0$, we have to verify $a_1 = a_2 = a_3 = 0$.

$$a_1(2,1,4) + a_2(-3,2,-1) + a_3(1,-3,-2) = (0,0,0)$$

~~$2a_1 + a$~~ $2a_1 - 3a_2 + a_3 = 0$

$$a_1 + 2a_2 - 3a_3 = 0$$

$$4a_1 - a_2 - 2a_3 = 0$$

Solve the above the system / Gauss elimination method

$$[A|B] = \left[\begin{array}{ccc|c} 2 & -3 & 1 & 0 \\ 1 & 2 & -3 & 0 \\ 4 & -1 & -2 & 0 \end{array}\right]$$

Row echelon form

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 2 & -3 & 1 & 0 \\ 4 & -1 & -2 & 0 \end{array}\right] R_1 \leftrightarrow R_2$$

1 +6

-3 -4

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & -7 & 7 & 0 \\ 0 & -9 & 10 & 0 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \end{array}$$

-1 -8

-2 +12

10

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & -7 & 7 & 0 \\ 0 & 0 & 7 & 0 \end{array}\right] R_3 \to 7R_3 - 9R_2$$

7(-9) - 9(-7)

7(10) - 9(7)

70 - 63

$$a_1 + 2a_2 - 3a_3 = 0$$

$$-7a_2 + 7a_3 = 0$$

$$7a_3 = 0$$

$$\boxed{a_3 = 0}$$

$$\boxed{a_2 = 0}$$

$$\boxed{a_1 = 0}$$

$\therefore$ The given vectors are linearly independent.

20AI401

→ **Basis and dimensions:**

Prob-1:

Prove that the vectors $(2,2,-3)$, $(0,-4,1)$ and $(3,1,-4)$ are linearly dependent.

Soln:

Let $a_1(2,2,-3)+a_2(0,-4,1)+a_3(3,1,-4)=(0,0,0)$

$$2a_1+0a_2+3a_3=0$$
$$2a_1-4a_2+a_3=0$$
$$-3a_1+a_2-4a_3=0$$

matrix form:

$$\left[\begin{array}{ccc|c} 2 & 0 & 3 & 0 \\ 2 & -4 & 1 & 0 \\ -3 & 1 & -4 & 0 \end{array}\right]$$

reduce the above matrix into row-echelon form,

$R_2 \to R_2 - R_1$
$R_3 \to 2R_3 + 3R_1$

$$\left[\begin{array}{ccc|c} 2 & 0 & 3 & 0 \\ 0 & -4 & -2 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right]$$

$R_3 \to 2R_3 + R_2$

$$\left[\begin{array}{ccc|c} 2 & 0 & 3 & 0 \\ 0 & -4 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$2a_1+3a_3=0$$
$$-4a_2-2a_3=0$$
$$4a_2=2a_3$$
$$a_2=\frac{a_3}{2}$$

a_3 – free variable

Since the scalar a_2 depends upon a_3. Therefore the gn vectors are linearly dependent.

Hw

Prove that the following vectors are linearly dependent.

1) $(3,-1,+7)$, $(0,0,0)$ and $(1,2,-4)$
2) $(4,1,-1)$, $(1,-2,1)$ and $(7,-4,1)$

Linear span:

Let 'S' be a non-empty subset of vector space 'V'. The set of all linear combinations of any finite no. of elts of 'S' is said to be linear span of 'V'.

↓

denoted by $L(S)$