5.1 REAL VECTOR SPACES

In this section we shall extend the concept of a vector by extracting the most important properties of familiar vectors and turning them into axioms. Thus, when a set of objects satisfies these axioms, they will automatically have the most important properties of familiar vectors, thereby making it reasonable to regard these objects as new kinds of vectors.

Vector Space Axioms

The following definition consists of ten axioms. As you read each axiom, keep in mind that you have already seen each of them as parts of various definitions and theorems in the preceding two chapters (for instance, see Theorem 4.1.1). Remember, too, that you do not prove axioms; they are simply the "rules of the game."

DEFINITION

Let V be an arbitrary nonempty set of objects on which two operations are defined: addition, and multiplication by scalars (numbers). By ***addition*** we mean a rule for associating with each pair of objects $\mathbf{u}$ and $\mathbf{v}$ in V an object $\mathbf{u}+\mathbf{v}$, called the ***sum*** of $\mathbf{u}$ and $\mathbf{v}$; by ***scalar multiplication*** we mean a rule for associating with each scalar k and each object $\mathbf{u}$ in V an object $k\mathbf{u}$, called the ***scalar multiple*** of $\mathbf{u}$ by k. If the following axioms are satisfied by all objects $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in V and all scalars k and m, then we call V a ***vector space*** and we call the objects in V ***vectors***.

1. If $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u}+\mathbf{v}$ is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
3. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$
4. There is an object $\mathbf{0}$ in V, called a ***zero vector*** for V, such that $\mathbf{0}+\mathbf{u}=\mathbf{u}+\mathbf{0}=\mathbf{u}$ for all $\mathbf{u}$ in V.
5. For each $\mathbf{u}$ in V, there is an object $-\mathbf{u}$ in V, called a ***negative*** of $\mathbf{u}$, such that $\mathbf{u}+(-\mathbf{u})=(-\mathbf{u})+\mathbf{u}=\mathbf{0}$.
6. If k is any scalar and $\mathbf{u}$ is any object in V, then $k\mathbf{u}$ is in V.
7. $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$
8. $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$
9. $k(m\mathbf{u})=(km)(\mathbf{u})$
10. $1\mathbf{u}=\mathbf{u}$

REMARK Depending on the application, scalars may be real numbers or complex numbers. Vector spaces in which the scalars are complex numbers are called ***complex vector spaces***, and those in which the scalars must be real are called ***real vector spaces***. In Chapter 10 we shall discuss complex vector spaces; until then, ***all of our scalars will be real numbers***.

The reader should keep in mind that the definition of a vector space specifies neither the nature of the vectors nor the operations. Any kind of object can be a vector, and the operations of addition and scalar multiplication may not have any relationship or similarity to the standard vector operations on R^n. The only requirement is that the ten vector space axioms be satisfied. Some authors use the notations $\oplus$ and $\odot$ for vector addition and scalar multiplication to distinguish these operations from addition and multiplication of real numbers; we will not use this convention, however.

Examples of Vector Spaces

The following examples will illustrate the variety of possible vector spaces. In each example we will specify a nonempty set V and two operations, addition and scalar multiplication; then we shall verify that the ten vector space axioms are satisfied, thereby entitling V, with the specified operations, to be called a vector space.

EXAMPLE 1 R^n Is a Vector Space

The set $V = R^n$ with the standard operations of addition and scalar multiplication defined in Section 4.1 is a vector space. Axioms 1 and 6 follow from the definitions of the standard operations on R^n; the remaining axioms follow from Theorem 4.1.1. ◆

The three most important special cases of R^n are R (the real numbers), R^2 (the vectors in the plane), and R^3 (the vectors in 3-space).

EXAMPLE 2 A Vector Space of 2 × 2 Matrices

Show that the set V of all 2×2 matrices with real entries is a vector space if addition is defined to be matrix addition and scalar multiplication is defined to be matrix scalar multiplication.

Solution

In this example we will find it convenient to verify the axioms in the following order: 1, 6, 2, 3, 7, 8, 9, 4, 5, and 10. Let

$$\mathbf{u} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$$

To prove Axiom 1, we must show that $\mathbf{u} + \mathbf{v}$ is an object in V; that is, we must show that $\mathbf{u} + \mathbf{v}$ is a 2×2 matrix. But this follows from the definition of matrix addition, since

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{bmatrix}$$

Similarly, Axiom 6 holds because for any real number k, we have

$$k\mathbf{u} = k \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$

so $k\mathbf{u}$ is a 2×2 matrix and consequently is an object in V.

Axiom 2 follows from Theorem 1.4.1*a* since

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbf{v} + \mathbf{u}$$

Similarly, Axiom 3 follows from part (b) of that theorem; and Axioms 7, 8, and 9 follow from parts (h), (j), and (l), respectively.

To prove Axiom 4, we must find an object $\mathbf{0}$ in V such that $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u}$ in V. This can be done by defining $\mathbf{0}$ to be

$$\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

With this definition,

$$\mathbf{0}+\mathbf{u}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}+\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}=\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}=\mathbf{u}$$

and similarly $\mathbf{u}+\mathbf{0}=\mathbf{u}$. To prove Axiom 5, we must show that each object $\mathbf{u}$ in V has a negative $-\mathbf{u}$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$ and $(-\mathbf{u})+\mathbf{u}=\mathbf{0}$. This can be done by defining the negative of $\mathbf{u}$ to be

$$-\mathbf{u}=\begin{bmatrix}-u_{11} & -u_{12}\\ -u_{21} & -u_{22}\end{bmatrix}$$

With this definition,

$$\mathbf{u}+(-\mathbf{u})=\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}+\begin{bmatrix}-u_{11} & -u_{12}\\ -u_{21} & -u_{22}\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=\mathbf{0}$$

and similarly $(-\mathbf{u})+\mathbf{u}=\mathbf{0}$. Finally, Axiom 10 is a simple computation:

$$1\mathbf{u}=1\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}=\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}=\mathbf{u} \quad \blacklozenge$$

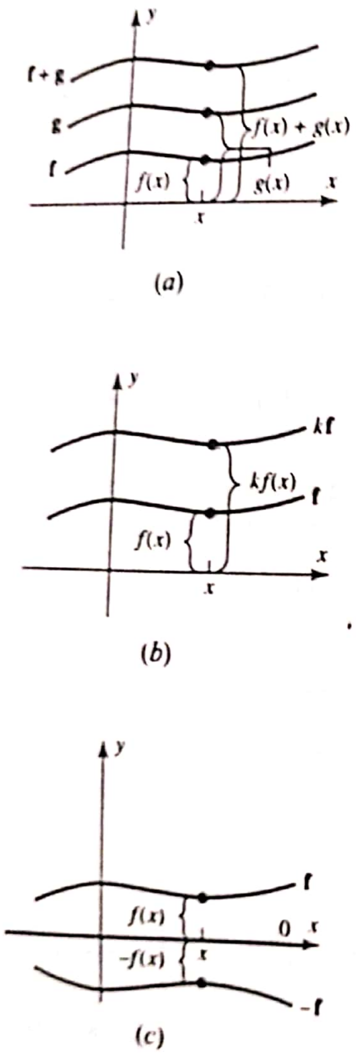

Figure 5.1.1

EXAMPLE 3 A Vector Space of $m \times n$ Matrices

Example 2 is a special case of a more general class of vector spaces. The argument in that example can be adapted to show that the set V of all $m \times n$ matrices with real entries, together with the operations of matrix addition and scalar multiplication, is a vector space. The $m \times n$ zero matrix is the zero vector $\mathbf{0}$, and if $\mathbf{u}$ is the $m \times n$ matrix U, then the matrix $-U$ is the negative $-\mathbf{u}$ of the vector $\mathbf{u}$. We shall denote this vector space by the symbol M_{mn}. ◆

EXAMPLE 4 A Vector Space of Real-Valued Functions

Let V be the set of real-valued functions defined on the entire real line $(-\infty, \infty)$. If $\mathbf{f}=f(x)$ and $\mathbf{g}=g(x)$ are two such functions and k is any real number, define the sum function $\mathbf{f}+\mathbf{g}$ and the scalar multiple $k\mathbf{f}$, respectively, by

$$(\mathbf{f}+\mathbf{g})(x)=f(x)+g(x) \quad \text{and} \quad (k\mathbf{f})(x)=kf(x)$$

In other words, the value of the function $\mathbf{f}+\mathbf{g}$ at x is obtained by adding together the values of $\mathbf{f}$ and $\mathbf{g}$ at x (Figure 5.1.1a). Similarly, the value of $k\mathbf{f}$ at x is k times the value of $\mathbf{f}$ at x (Figure 5.1.1b). In the exercises we shall ask you to show that V is a vector space with respect to these operations. This vector space is denoted by $F(-\infty, \infty)$. If $\mathbf{f}$ and $\mathbf{g}$ are vectors in this space, then to say that $\mathbf{f}=\mathbf{g}$ is equivalent to saying that $f(x)=g(x)$ for all x in the interval $(-\infty, \infty)$.

The vector $\mathbf{0}$ in $F(-\infty, \infty)$ is the constant function that is identically zero for all values of x. The graph of this function is the line that coincides with the x-axis. The negative of a vector $\mathbf{f}$ is the function $-\mathbf{f}=-f(x)$. Geometrically, the graph of $-\mathbf{f}$ is the reflection of the graph of $\mathbf{f}$ across the x-axis (Figure 5.1.1c). ◆

Remark In the preceding example we focused on the interval $(-\infty, \infty)$. Had we restricted our attention to some closed interval $[a, b]$ or some open interval (a, b), the functions defined on those intervals with the operations stated in the example would also have produced vector spaces. Those vector spaces are denoted by $F[a, b]$ and $F(a, b)$, respectively.

EXAMPLE 5 A Set That Is Not a Vector Space

Let $V = R^2$ and define addition and scalar multiplication operations as follows: If $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$, then define

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$$

and if k is any real number, then define

$$k\mathbf{u} = (ku_1, 0)$$

For example, if $\mathbf{u} = (2, 4)$, $\mathbf{v} = (-3, 5)$, and $k = 7$, then

$$\mathbf{u} + \mathbf{v} = (2 + (-3), 4 + 5) = (-1, 9)$$
$$k\mathbf{u} = 7\mathbf{u} = (7 \cdot 2, 0) = (14, 0)$$

The addition operation is the standard addition operation on R^2, but the scalar multiplication operation is not the standard scalar multiplication. In the exercises we will ask you to show that the first nine vector space axioms are satisfied; however, there are values of $\mathbf{u}$ for which Axiom 10 fails to hold. For example, if $\mathbf{u} = (u_1, u_2)$ is such that $u_2 \neq 0$, then

$$1\mathbf{u} = 1(u_1, u_2) = (1 \cdot u_1, 0) = (u_1, 0) \neq \mathbf{u}$$

Thus V is not a vector space with the stated operations. ◆

EXAMPLE 6 Every Plane through the Origin Is a Vector Space

Let V be any plane through the origin in R^3. We shall show that the points in V form a vector space under the standard addition and scalar multiplication operations for vectors in R^3. From Example 1, we know that R^3 itself is a vector space under these operations. Thus Axioms 2, 3, 7, 8, 9, and 10 hold for all points in R^3 and consequently for all points in the plane V. We therefore need only show that Axioms 1, 4, 5, and 6 are satisfied.

Since the plane V passes through the origin, it has an equation of the form

$$ax + by + cz = 0 \tag{1}$$

(Theorem 3.5.1). Thus, if $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are points in V, then $au_1 + bu_2 + cu_3 = 0$ and $av_1 + bv_2 + cv_3 = 0$. Adding these equations gives

$$a(u_1 + v_1) + b(u_2 + v_2) + c(u_3 + v_3) = 0$$

This equality tells us that the coordinates of the point

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$

satisfy (1); thus $\mathbf{u} + \mathbf{v}$ lies in the plane V. This proves that Axiom 1 is satisfied. The verifications of Axioms 4 and 6 are left as exercises; however, we shall prove that Axiom 5 is satisfied. Multiplying $au_1 + bu_2 + cu_3 = 0$ through by -1 gives

$$a(-u_1) + b(-u_2) + c(-u_3) = 0$$

Thus $-\mathbf{u} = (-u_1, -u_2, -u_3)$ lies in V. This establishes Axiom 5. ◆

Subspace

A subset W of a vector space V is called a subspace of V if W is itself a vector space under the addition and scalar multiplication defined on V.

Theorem

If W is a set of one or more vectors in a vector space V, then W is a subspace of V if and only if the following conditions are satisfied.

(a) If u and v are vectors in W, then u + v is in W.

(b) If k is a scalar and u is a vector in W, then ku is in W

Example 1

1. Let U consists of all vectors in R^3, whose entries are equal, that is U={(a, b, c) / a=b=c}is a subspace of R^3.

Example 2

Let $V = \mathbb{R}^4$, where addition and scalar multiplication are given by

$$(a,b,c,d)+(e,f,g,h) = (a+e,b+f,c+g,d+h)$$

and

$$\alpha \cdot (a,b,c,d) = (\alpha a, \alpha b, \alpha c, \alpha d)$$

respectively. Then V is a vector space, with zero vector $(0,0,0,0)$. Let

$$U = \{(a,b,c,d) \in \mathbb{R}^4; a+b+c+d = 0\}$$

and

$$W = \{(a,b,c,d) \in \mathbb{R}^4; ab = cd\}.$$

Then U is a subspace of V, since:

1. $0+0+0+0 = 0$, so $(0,0,0,0) \in U$
2. If $a+b+c+d = 0 = e+f+g+h$, then $(a+e)+(b+f)+(c+g)+(d+h) = 0$
3. If $a+b+c+d = 0$ and $\alpha \in \mathbb{R}$, then $\alpha a + \alpha b + \alpha c + \alpha d = 0$

However, W is not a subspace, since for example $(1,0,0,0),(0,1,0,0) \in W$ but $(1,0,0,0)+(0,1,0,0) = (1,1,0,0) \notin U$.

Unit 4

Vector Spaces - collection of input & output signals

Applications:

→ Space flight
→ Control
→ Space shuttle control system
→ Commands
→ Input signals & output signals → fns

A Vector space is a non-empty set 'V' of objects called vectors, on which we are defined 2 operations - addition & multiplication by scalars.
↳ real numbers

10 axioms:-

→ Axioms must hold for all vectors u, v and w in V and for all scalars $c, d \in R$.

→ (i) $u+v \in V$, $\forall\ u, v \in V$ (addition axiom) ✓
(ii) $u+v = v+u$ (commutative axiom) ✓
iii) $(u+v)+w = u+(v+w)$ (Associative axiom)
(iv) There is a zero vector, 0 in V such that,
$u+0 = u$ (zero vectors → identity) ✓
(v) For each u in V, There ~~exist~~ $-u$ in V such that
$u+(-u) = 0$ (inverse) ✓
(vi) For u in V, cu is in V (scalar multiplication) ✓
vii) $c(u+v) = cu+cv$, $\forall\ c \in R$, $u, v \in V$ (distributive axiom)
viii) $(c+d)u = cu+du$
ix) $c(du) = (cd)u$, $\forall\ c, d \in R$, $u \in V$
x) $1 \cdot u = u$

Prob-1:

Verify V = set of all 2×2 matrices with real entries where addition is defined by matrix addition and scalar multiplication of a matrix.

Soln:

Let $u = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$, $v = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$, $w = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$ be the matrices in V.

To prove:

V is a vector space

Axiom 1:

$$u+v = \begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix} \in V$$

Axiom 2:

$$u+v = \begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix} = v+u$$

$\therefore$ Commutative property is satisfied.

Axiom 4: Identity

$$u+0 = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = u \in V$$

Axiom 5: Inverse

$$u - u' = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in V$$

Axiom 6:

For $c \in R$

$$cu = c\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} cu_{11} & cu_{12} \\ cu_{21} & cu_{22} \end{bmatrix} \in V$$

By the properties of matrices, axiom 3, 7, 8, 9, 10 are satisfied.

V = set of all 2×2 matrices in a vector space

Result:

R^n is a vector space with standard operations on addition of multiplication

Prob-2:

Verify whether $V = R^2$ is a vector space in which the addition & scalar multiplication is defined by

If $u=(u_1, u_2)$ & $v=(v_1, v_2)$ then $u+v = (u_1+v_1, u_2+v_2)$ of 'k' is scalar. Show that $ku = (ku, 0)$

Soln:

Let $u = (2,4)$, $v = (-3, 5)$

Axiom 10:

If $u = (u_1, u_2)$ if $u_2 \neq 0$

$1 \cdot u = 1\cdot(u_1, u_2) = (u_1, 0) \notin V$

Axioms fail

$\therefore V$ is not in vector space.

→ Verify whether the following non-empty sets are vector space (or) not.

1. Let $D = \{$ set of real numbers $\}$ In which the operation of addition and scalar multiplication is usual. If $f(t) = 1 + \sin 2t$, $g(t) = 2 + 0.5t \in D$.

2. $y(t) = c_1 \cos \omega t + c_2 \sin \omega t \Rightarrow H = \{$ set of all output values of $y(t)\}$ where 'ω' is fixed $c_1, c_2 \in R$.

Defn: **Subspace**

A subset W of a vector space V is called a subspace of V if W is itself a vector space under the addition and scalar multiplication defined on V.

i.e) i) If $u, v \in W$, $u + v \in W$

ii) If α is a scalar then $\alpha u \in W$

$u, v, \alpha \in R$

$\alpha(u+v) = \alpha u + \alpha v$

Result:-

1) If V is a vector space then $V \subseteq V$ & V is the largest subspace of V.

2) The set $\{0\} \subseteq V$ which is the zero space and also it is the smallest subspace of V.

Pbm 1:-

Let V be the vector space of all $n \times n$ matrices and W be the set of all symmetric matrices in V. Show that W is the subspace of V.

Soln:

$W = \{A \in V \mid A^T = A\}$

Let $A, B \in W \to A = A^T;\ B = B^T$

$\alpha, \beta \in R$ (scalar)

$(\alpha A + \beta B)^T = \alpha A^T + \beta B^T = \alpha A + \beta B \in W$.

Pbm 2:

Let $V = R^4$ be the vector space with zero vector $= (0,0,0,0)$ In which addition and scalar multiplication is defined by $(a,b,c,d) + (e,f,g,h) = (a+e, b+f, c+g, d+h)$ and $\alpha.(a,b,c,d) = (\alpha a, \alpha b, \alpha c, \alpha d)$ resp'ly. Verify whether the following 2 sets are subspaces of V (or) not.

The sets $U = \{(a,b,c,d) \in R^4;\ a+b+c+d = 0\}$

$W = \{(a,b,c,d) \in R^4,\ ab = cd\}$

Soln:

1) $\vec{0}$ - zero vector:- $a=0, b=0, c=0, d=0$

$0+0+0+0 = 0$, so, $(0,0,0,0) \in U$

2) If $a+b+c+d = 0 = e+f+g+h$ then $(a+e)+(b+f)+(c+g)+(d+h) = 0 \in U$

3) If $a+b+c+d = 0$ and $\alpha \in R$ then $\alpha a + \alpha b + \alpha c + \alpha d = 0$