Assignment - 3 :-

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{1}{Re}\frac{\partial^2 u}{\partial x^2}$$

① Explicit Scheme :-

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{1}{Re}\frac{\partial^2 u}{\partial x^2}$$

$$\therefore\; \frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{\partial}{\partial x}\left[F(u)\right] = \frac{1}{Re}\frac{\partial^2 u}{\partial x^2}$$

$$;\ \text{where } F(u) = \frac{u^2}{2}$$

$$\therefore\; \frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{F_{j+1}^n - F_{j-1}^n}{2\Delta x} = \frac{1}{Re}\left[\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}\right]$$

$$\therefore\; \frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{(u_{j+1}^n)^2 - (u_{j-1}^n)^2}{2\Delta x} = \frac{1}{Re}\left[\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}\right]$$

$$\therefore\; \frac{(u_{j+1}^n)^2}{4\Delta x} + \frac{u_j^{n+1}}{\Delta t} = \frac{u_j^n}{\Delta t} + \frac{(u_{j-1}^n)^2}{4\Delta x} + \frac{1}{Re}\left[\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}\right]$$

$$\therefore\; + \left(\frac{4\Delta x}{\Delta t}\right)u_j^{n+1} = \underbrace{\left[\left(\frac{4\Delta x}{\Delta t}\right)u_j^n + (u_{j-1}^n)^2 - (u_{j+1}^n)^2 + \frac{4}{Re\cdot\Delta x}\left[u_{j+1}^n - 2u_j^n + u_{j-1}^n\right]\right]}_{\text{RHS is known \& [let it be some constant K]}}$$

$$\left(4\frac{\Delta x}{\Delta t}\right)u_j^{n+1} = \left[\left(4\frac{\Delta x}{\Delta t}\right)u_j^n + (u_{j-1}^n)^2 - (u_{j+1}^n)^2 + \frac{4}{Re\cdot\Delta x}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)\right]$$

$$\therefore\; u_j^{n+1} = \frac{\Delta t}{4\,\Delta x}\left[4\left(\frac{\Delta x}{\Delta t}\right)u_j^n + (u_{j-1}^n)^2 - (u_{j+1}^n)^2 + \frac{4}{Re\cdot\Delta x}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)\right]$$

② Implicit Scheme :-

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{1}{Re}\frac{\partial^2 u}{\partial x^2}$$

OR

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}[F(u)] = \frac{1}{Re}\frac{\partial^2 u}{\partial x^2}$$

$$\therefore \frac{u_j^{n+1} - u_j^n}{\Delta t} + L_x(F_j^{n+1}) = \frac{1}{Re} L_{xx}(u_j^{n+1})$$

$\therefore$ using taylor series expansion :-

$$F_j^{n+1} = F_j^n + \Delta t \left.\frac{\partial F}{\partial t}\right]_j^n + \underbrace{\frac{(\Delta t)^2}{2!}\left.\frac{\partial^2 F}{\partial t^2}\right]_j^n}_{\text{HOT}}$$

[Higher order terms]

using the result ;

$$\frac{\partial F}{\partial t} = \frac{\partial(\frac{1}{2}u^2)}{\partial t} = u\frac{\partial u}{\partial t}$$

$$\therefore \left.\frac{\partial F}{\partial t}\right]_j^n = u_j^n \left.\frac{\partial u_j}{\partial t}\right|_j^n$$

$$\therefore F_j^{n+1} = F_j^n + u_j^n\left(\frac{u_j^{n+1} - u_j^n}{\cancel{\Delta t}}\right)\cdot\cancel{\Delta t}$$

$$\therefore F_j^{n+1} = F_j^n + u_j^n u_j^{n+1} - (u_j^n)^2$$

we already know that $F_j^n = \dfrac{(u_j^n)^2}{2}$

$$\therefore F_j^{n+1} = F_j^n + u_j^n u_j^{n+1} - 2F_j^n$$

$$= -F_j^n + u_j^n u_j^{n+1}$$

$$\therefore L_x(F_j^{n+1}) = L_x(-F_j^n + u_j^n u_j^{n+1}).$$

$$= L_x(-F_j^n) + L_x(u_j^n u_j^{n+1})$$

Now ; $L_x(-F_j) = -L_x(F_j) = -\left[\dfrac{F_{j+1}^n - F_{j-1}^n}{2\Delta x}\right]$

$$L_x(u_j^n u_j^{n+1}) = \left[\frac{u_{j+1}^n u_{j+1}^{n+1} - u_{j-1}^n u_{j-1}^{n+1}}{2\Delta x}\right]$$

Putting these expressions back into our original eq^n ; we get :-

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} - \left[\frac{F_{j+1}^n - F_{j-1}^n}{2\Delta x}\right] + \left[\frac{u_{j+1}^n u_{j+1}^{n+1} - u_{j-1}^n u_{j-1}^{n+1}}{2\Delta x}\right]$$

$$= \frac{1}{Re}\left[\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{(\Delta x)^2}\right]$$

PAGE NO.
DATE : / /

$$\therefore \frac{U_j^{n+1} - U_j^n}{\Delta t} - \left[\frac{(U_{j+1}^n)^2 - (U_{j-1}^n)^2}{4\,\Delta x}\right]$$

$$= -\left[\frac{U_{j+1}^n\, U_{j+1}^{n+1} - U_{j-1}^n\, U_{j-1}^{n+1}}{2\,\Delta x}\right]$$

$$+ \frac{1}{Re\cdot(\Delta x)^2}\left[U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}\right]$$

$$\therefore \frac{U_j^{n+1}}{\Delta t} + \frac{1}{2\,\Delta x}\left[U_{j+1}^n\, U_{j+1}^{n+1} - U_{j-1}^n\, U_{j-1}^{n+1}\right] - \frac{1}{Re\cdot(\Delta x)^2}\left[U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}\right]$$

$$= \underbrace{\frac{U_j^n}{\Delta t} + \left[\frac{(U_{j+1}^n)^2 - (U_{j-1}^n)^2}{4\,\Delta x}\right]}_{\text{RHS is known.}}$$

Let's focus on the LHS :-

$$\therefore U_j^{n+1}\left[\frac{1}{\Delta t} + \frac{2}{Re\cdot(\Delta x)^2}\right] + U_{j+1}^{n+1}\left[\frac{U_{j+1}^n}{2\,\Delta x} - \frac{1}{Re\cdot(\Delta x)^2}\right]$$

$$+ U_{j-1}^{n+1}\left[-\frac{U_{j-1}^n}{2\,\Delta x} - \frac{1}{Re\cdot(\Delta x)^2}\right]$$

$\therefore$ taking $b = \dfrac{1}{\Delta t} + \dfrac{2}{Re \cdot (\Delta x)^2}$

$$c = \frac{u_{j+1}^{n}}{2\Delta x} - \frac{1}{Re \cdot (\Delta x)^2}$$

$$a = -\left(\frac{u_{j-1}^{n}}{2\Delta x} + \frac{1}{Re \cdot (\Delta x)^2} \right)$$

$$\therefore \quad a\, u_{j-1}^{n+1} + b\, u_j^{n+1} + c\, u_{j+1}^{n+1} = d.$$

$$d = RHS = \frac{u_j^n}{\Delta t} + \frac{(u_{j+1}^n)^2 - (u_{j-1}^n)^2}{4\Delta x}$$

Now TDMA can be applied.

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③ Crank - Nicolson :-

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left[F(u)\right] = \frac{1}{Re}\frac{\partial^2 u}{\partial x^2}$$

$$\therefore \frac{u_j^{n+1} - u_j^n}{\Delta t} + L_x\left(\frac{F_j^n + F_j^{n+1}}{2}\right)$$

$$= \frac{1}{Re} L_{xx}\left(\frac{u_j^{n+1} + u_j^n}{2}\right)$$

using the previous result used in implicit ;

$$F_j + F_j^{n+1} = F_j^n + u_j^n \, \Delta u_j^{n+1} + F_j$$

$$= F_j^n + u_j^n\left(u_j^{n+1} - u_j^n\right) + F_j$$

$$= \cancel{2F_j^n} + u_j^n u_j^{n+1} - \cancel{(u_j^n)^2}$$

$$= u_j^n u_j^{n+1}$$

$$\therefore \frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{1}{2} L_x\left(u_j^n u_j^{n+1}\right)$$

$$= \frac{1}{2Re} L_{xx}\left(u_j^n + u_j^{n+1}\right)$$

$$\therefore \frac{u_j^{n+1}}{\Delta t} + \frac{1}{2} L_x\left(u_j^n u_j^{n+1}\right) - \frac{1}{2Re} L_{xx}\left(u_j^{n+1}\right)$$

$$= \frac{u_j^n}{\Delta t} + \frac{1}{2Re} L_{xx}\left(u_j^n\right)$$

$$\therefore \quad \frac{U_j^{n+1}}{\Delta t} + \frac{1}{2}\left(\frac{U_{j+1}^{n}\,U_{j+1}^{n+1} - U_{j-1}^{n}\,U_{j-1}^{n+1}}{2\,\Delta x}\right) - \frac{1}{2Re}\left(\frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{(\Delta x)^2}\right)$$

$$= \underbrace{\frac{U_j^{n}}{\Delta t} + \frac{1}{2Re}\left(\frac{U_{j+1}^{n} - 2U_j^{n} + U_{j-1}^{n}}{(\Delta x)^2}\right)}_{\text{RHS is known.}}$$

Focusing on the LHS :-

$$U_j^{n+1}\left(\frac{1}{\Delta t} + \frac{1}{Re\,(\Delta x)^2}\right) + U_{j+1}^{n+1}\left(\frac{U_{j+1}^{n}}{4\,\Delta x} - \frac{1}{2Re\,(\Delta x)^2}\right)$$

$$+ U_{j-1}^{n+1}\left(-\frac{U_{j-1}^{n}}{4\,\Delta x} - \frac{1}{2Re\,(\Delta x)^2}\right) =$$

$\therefore$ using ;

$$a = -\left(\frac{U_{j-1}^{n}}{4\,\Delta x} + \frac{1}{2Re(\Delta x)^2}\right)$$

$$b = \left(\frac{1}{\Delta t} + \frac{1}{Re\,(\Delta x)^2}\right)$$

$$c = \left(\frac{U_{j+1}^{n}}{4\,\Delta x} - \frac{1}{2Re\,(\Delta x)^2}\right)$$

$$\underset{(RHS)}{d} = \frac{U_j^{n}}{\Delta t} + \frac{1}{2Re}\left(\frac{U_{j+1}^{n} - 2U_j^{n} + U_{j-1}^{n}}{(\Delta x)^2}\right)$$

* Now all a, b, c, d are known.
$\therefore$ TDMA can be applied.

Analytical solⁿ :-

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⇒ Integrated using MATLAB !!