1. *[For this problem, I took help of internet resources to get the idea of gauss divergence]*

(NSE)$_i$ :-

$$\frac{\partial u_i}{\partial t} + u_j \partial_j u_i = -\partial_i\left(\frac{p}{\rho}\right) + \nu \partial_j \partial_j u_i$$

Multiplying the eq$^n$ by $u_i$ on both sides :-

$$\therefore\; u_i \frac{\partial u_i}{\partial t} + u_i u_j \partial_j u_i = -u_i \partial_i\left(\frac{p}{\rho}\right) + u_i(\nu \partial_j \partial_j u_i)$$

$$\therefore\; \underbrace{\frac{\partial}{\partial t}\left(\frac{1}{2} u_i u_i\right) + u_j \partial_j\left(\frac{1}{2} u_i u_i\right)}_{} = \left[-\partial_i\left(\frac{p u_i}{\rho}\right) + \nu u_i \partial_j \partial_j u_i\right]$$

$$= \frac{D}{Dt}\left(\frac{1}{2} u_i u_i\right)$$

$$\therefore\; \frac{D}{Dt}\left(\frac{1}{2} u_i u_i\right) = -\partial_i\left(\frac{p u_i}{\rho}\right) + \nu u_i \partial_j \partial_j u_i$$

$$\nu u_i \partial_j \partial_j u_i = \nu\left[\underbrace{\partial_j(u_i \partial_j u_i)}_{} - (\partial_j u_i)^2\right]$$

$$= \partial_j\left(\partial_j\left(\frac{1}{2} u_i u_i\right)\right)$$

$$\therefore\; \nu u_i \partial_j \partial_j u_i = \nu\left[\partial_j \partial_j\left(\frac{1}{2} u_i u_i\right) - (\partial_j u_i)^2\right]$$

let $\frac{1}{2} u_i u_i = k$ :-

$$\therefore\; \frac{D}{Dt}(k) = -\partial_i\left(\frac{p u_i}{\rho}\right) + \nu \partial_j \partial_j k - \nu(\partial_j u_i)^2$$

Now; $$\partial_j u_i = \underbrace{\frac{1}{2}(\partial_j u_i + \partial_i u_j)}_{S_{ij}} + \underbrace{\frac{1}{2}(\partial_j u_i - \partial_i u_j)}_{\Omega_{ij}}$$

$$\Rightarrow \frac{Dk}{Dt} = -\partial_i\left(\frac{pu_i}{\rho}\right) + \nu\,\partial_j\partial_j k - \nu\underbrace{(\partial_j u_i)(\partial_j u_i)}$$

$$\nu(S_{ij}+\Omega_{ij})(S_{ij}+\Omega_{ij})$$

$$= \nu\left(S_{ij}S_{ij} + \cancel{S_{ij}\Omega_{ij}}^{\,0} + \cancel{\Omega_{ij}S_{ij}}^{\,0} + \Omega_{ij}\Omega_{ij}\right)$$

$$\therefore \frac{Dk}{Dt} = -\partial_i\left(\frac{pu_i}{\rho}\right) + \nu\,\partial_j\partial_j k - \nu\left(S_{ij}S_{ij} + \Omega_{ij}\Omega_{ij}\right)$$

OR

$$\frac{Dk}{Dt} = -\partial_i\left(\frac{pu_i}{\rho}\right) + \nu\,\partial_j\partial_j k - \nu(\partial_j u_i)(\partial_j u_i)$$

Integrating this eq$^n$ over the volume V :-

$$\int_V \frac{Dk}{Dt}\,dV = \underbrace{-\int_V \partial_i\left(\frac{pu_i}{\rho}\right)dV}_{\Downarrow} + \nu\int_V(\partial_j\partial_j k)\,dV - \nu\int_V(\partial_j u_i)(\partial_j u_i)\,dV$$

using gauss - divg. theorem :-

$$\boxed{\int_V \partial_i\left(\frac{pu_i}{\rho}\right)dV = \int_{\partial\Omega} \frac{pu_i n_i}{\rho}\,dS.}$$

also :-

$$\frac{DK}{Dt} = \frac{\partial K}{\partial t} + u_j \frac{\partial}{\partial x_j}(K)$$

$$\therefore \int_V \frac{DK}{Dt} dV = \int_V \frac{\partial K}{\partial t} dV + \underbrace{\int_V u_j \frac{\partial}{\partial x_j}(K)\, dV}$$

$$= \int_V \frac{\partial}{\partial x_j}(K u_j)\, dV$$

$\Downarrow$ (gauss* divg.)

$$\therefore \int_V \frac{DK}{Dt} dV = \int_V \frac{\partial K}{\partial t} dV + \int_{d\Omega} K u_j n_j\, dS$$

$$\therefore \frac{\partial}{\partial t}\int_V K\, dV + \int_{d\Omega} K u_j n_j\, dS = -\int_{d\Omega} \frac{p u_i n_i}{\rho}\, dS + \nu \int_V \partial_j \partial_j K\, dV - \nu \int_V (\partial_j u_i)^2 dV$$

Deep in the flow, the surface contributions will die out.

$$\therefore \frac{\partial}{\partial t}\int_V K\, dV = \nu \int_V \underbrace{\partial_j \partial_j K}_{\frac{\partial^2 K}{\partial x_j^2}}\, dV - \nu \int_V \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} dV$$

final eq$^n$

$$\therefore \boxed{\frac{\partial}{\partial t}\int_V K\, dV = \nu \int_V \frac{\partial^2 K}{\partial x_j^2} dV - \nu \int_V \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} dV}$$

2. <u>Entrosphy Eq$^n$</u> :-

$$\frac{\partial u_i}{\partial t} + \underbrace{u_j \partial_j u_i}_{\Downarrow} = -\partial_i \left(\frac{P}{\rho}\right) + \nu \partial_j \partial_j u_i$$

$$u_j \partial_j u_i = u_j [\partial_j u_i - \partial_i u_j] + \underbrace{u_j \partial_i u_j}$$

$$= \partial_i \left(\underbrace{\tfrac{1}{2} u_j u_j}_{KE}\right)$$

$$\therefore \frac{\partial u_i}{\partial t} + u_j \underbrace{[\partial_j u_i - \partial_i u_j]}_{= \Omega_{ij}} = -\partial_i \left(\frac{P}{\rho} + \frac{1}{2} u_j u_j\right) + \nu \partial_j \partial_j u_i$$

$$\therefore \frac{\partial u_i}{\partial t} + \underbrace{u_j \Omega_{ij}}_{\Downarrow\; -\varepsilon_{ijk} u_j \omega_k} = -\partial_i \left(\frac{P}{\rho} + \frac{1}{2} u_j u_j\right) + \nu \partial_j \partial_j u_i$$

$$\therefore \frac{\partial u_i}{\partial t} - \varepsilon_{ijk} u_j \omega_k = -\partial_i \left(\frac{P}{\rho} + \frac{1}{2} u_j u_j\right) + \nu \partial_j \partial_j u_i$$

Multiplying this equation by $-\epsilon_{iqp} \partial_q$ on both the sides of the above equation :- $= \epsilon_{ipq} \partial_q$

$$+ \epsilon_{ipq} \partial_q \left(\frac{\partial u_i}{\partial t}\right) = \frac{\partial \omega_p}{\partial t}$$

$$+ \epsilon_{ipq} \partial_q (-\varepsilon_{ijk} u_j \omega_k) = -\varepsilon_{ipq} \varepsilon_{ijk} \partial_q (\omega_j u_k)$$

$$= -[\delta_{pj} \delta_{qk} - \delta_{pk} \delta_{qj}] \partial_q (\omega_j u_k)$$

$$= -\delta_{pj} \delta_{qk} \partial_q (\omega_j u_k) + \delta_{pk} \delta_{qj} \partial_q (\omega_j u_k)$$

$$= \left[-\partial_k (u_p \omega_k) + \partial_j (u_j \omega_p)\right]$$

$$\Rightarrow \varepsilon_{ipq} \partial_q \underbrace{\left( \frac{p}{\rho} + \frac{1}{2} u_j u_j \right)}_{\text{scalar}} = 0$$

$$\Rightarrow \varepsilon_{ipq} \partial_q \left( \nu \partial_j \partial_j u_i \right) = \nu \partial_j \partial_j \omega_p$$

$\therefore$ Our final eq$^n$ is :-

$$\frac{\partial \omega_p}{\partial t} + \underbrace{\partial_j (\omega_p u_j) - \partial_k (u_p \omega_k)} = \nu \partial_j \partial_j \omega_p$$

$$\Downarrow$$

$$\cancel{(\partial_j u_j)}^{0} \omega_p + u_j (\partial_j \omega_p) - u_p (\cancel{\partial_k \omega_k})^{0} - \omega_k (\partial_k u_p)$$

$$= u_j (\partial_j \omega_p) - \omega_k (\partial_k u_p)$$

$$\therefore \frac{\partial \omega_p}{\partial t} + u_j \partial_j \omega_p = \underbrace{\omega_k \partial_k u_p} + \nu \partial_j \partial_j \omega_p$$

$$\Rightarrow \boxed{\omega_j \partial_j u_p = \omega_k \partial_k u_p}$$

$$\therefore \boxed{\frac{\partial \omega_p}{\partial t} + u_j \partial_j \omega_p = \omega_j \partial_j u_p + \nu \partial_j \partial_j \omega_p}$$

$$\Downarrow$$

Replacing $p$ by $i$ :-

$$\frac{\partial \omega_i}{\partial t} + u_j \partial_j \omega_i = \omega_j \partial_j u_i + \nu \partial_j \partial_j \omega_i$$

Now multiplying by $\omega_i$ on both the sides :-

$$\omega_i \frac{\partial \omega_i}{\partial t} + \omega_i u_j \partial_j \omega_i = \omega_i \omega_j \partial_j u_i + \nu\, \omega_i \partial_j \partial_j \omega_i$$

$$\therefore \frac{\partial}{\partial t}\left(\frac{1}{2}\omega_i \omega_i\right) + u_j \partial_j \left(\frac{1}{2}\omega_i \omega_i\right)$$

$$= \omega_i \omega_j \partial_j u_i + \underbrace{\nu\, \omega_i \partial_j \partial_j \omega_i}$$

$$\nu\left[\partial_j(\omega_i \partial_j \omega_i) - (\partial_j \omega_i)^2\right]$$

$$= \nu\left[\partial_j \partial_j\left(\frac{1}{2}\omega_i \omega_i\right) - (\partial_j \omega_i)^2\right]$$

Also;

$$\therefore \frac{\partial}{\partial t}\left(\frac{1}{2}\omega_i \omega_i\right) + u_j \partial_j \left(\frac{1}{2}\omega_i \omega_i\right) = \frac{D}{Dt}\left(\frac{1}{2}\omega_i \omega_i\right)$$

$$\therefore \boxed{\frac{D}{Dt}\left(\frac{1}{2}\omega_i \omega_i\right) = \omega_i \omega_j \partial_j u_i + \nu\, \partial_j \partial_j \left(\frac{1}{2}\omega_i \omega_i\right) - \nu(\partial_j \omega_i)^2}$$

Replacing $\frac{1}{2}\omega_i \omega_i$ with $\mathcal{E}$ :-

Entrosphy eq$^n$ :-

$$\frac{D}{Dt}(\mathcal{E}) = \omega_i \omega_j \partial_j u_i + \nu\, \partial_j \partial_j (\mathcal{E}) - \nu(\partial_j \omega_i)^2 \quad (1)$$

Kinetic energy eq$^n$ :-

$$\frac{D}{Dt}(k) = -\partial_i (u_i p/\rho) + \nu\, \partial_j \partial_j (k) - \nu(\partial_j u_i)^2 \quad (2)$$

$$\frac{DE}{Dt} = \underline{\underline{\omega_i \omega_j \partial_j u_i}} + \nu \partial_j \partial_j (E) - \nu (\partial_j \omega_i)^2 \qquad (1)$$

$$\frac{DK}{Dt} = \underline{\underline{-\partial_i \left( p u_i / \rho \right)}} + \nu \partial_j \partial_j (K) - \nu (\partial_j u_i)^2 \qquad (2)$$

All the terms look similar except the term (1) - (2) in both the equations (underlined).

Entrosphy eq$^n$

$$\underbrace{\omega_i \omega_j \boxed{\partial_j u_i}}$$

$\Downarrow$

This term is responsible for vortex streching

Kinetic energy eq$^n$

$$\underbrace{-\partial_i \left( \frac{p u_i}{\rho} \right)}$$

$\Downarrow$

Pressure work

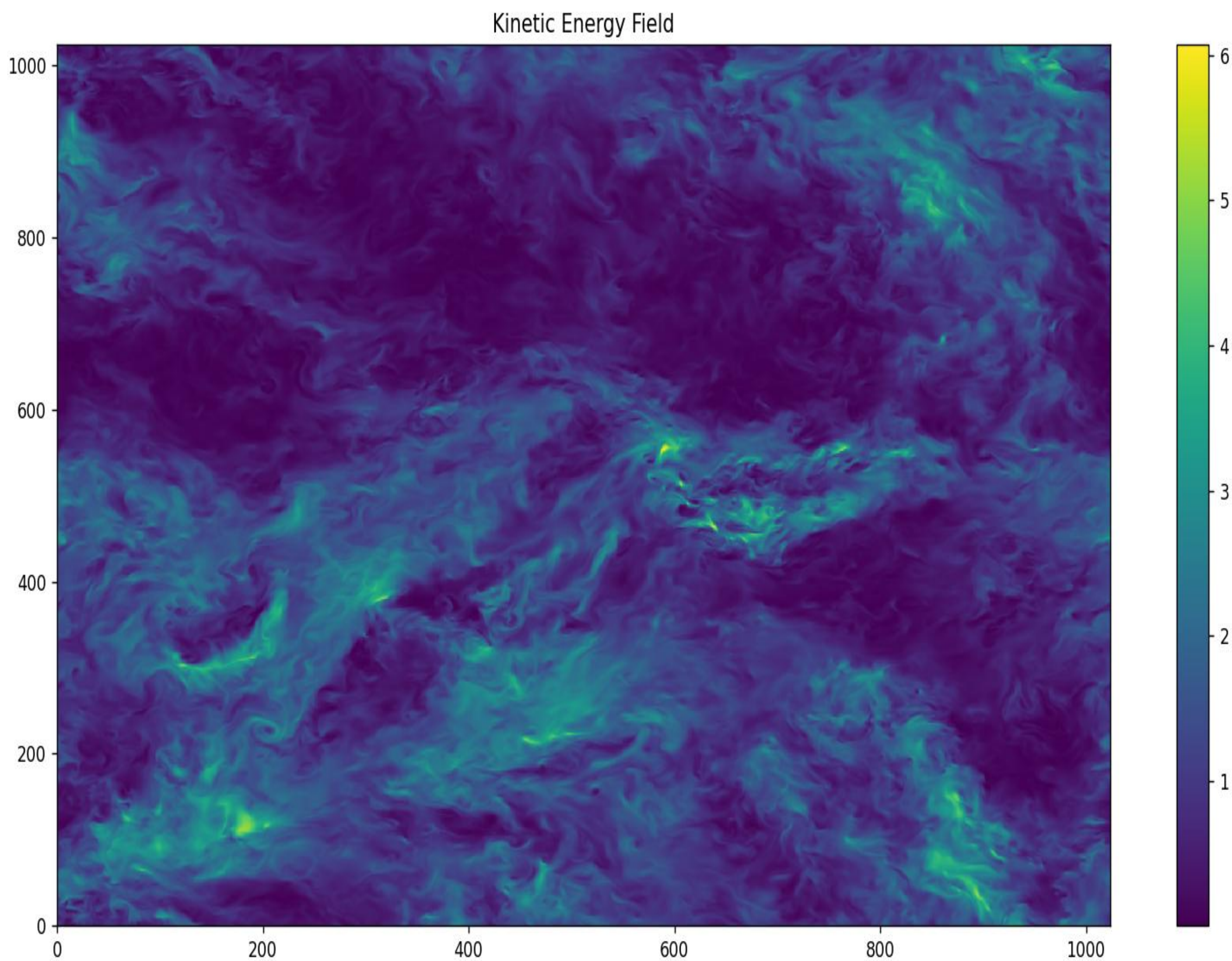
* Other terms have the similar form.

# Turbulence Data Analysis

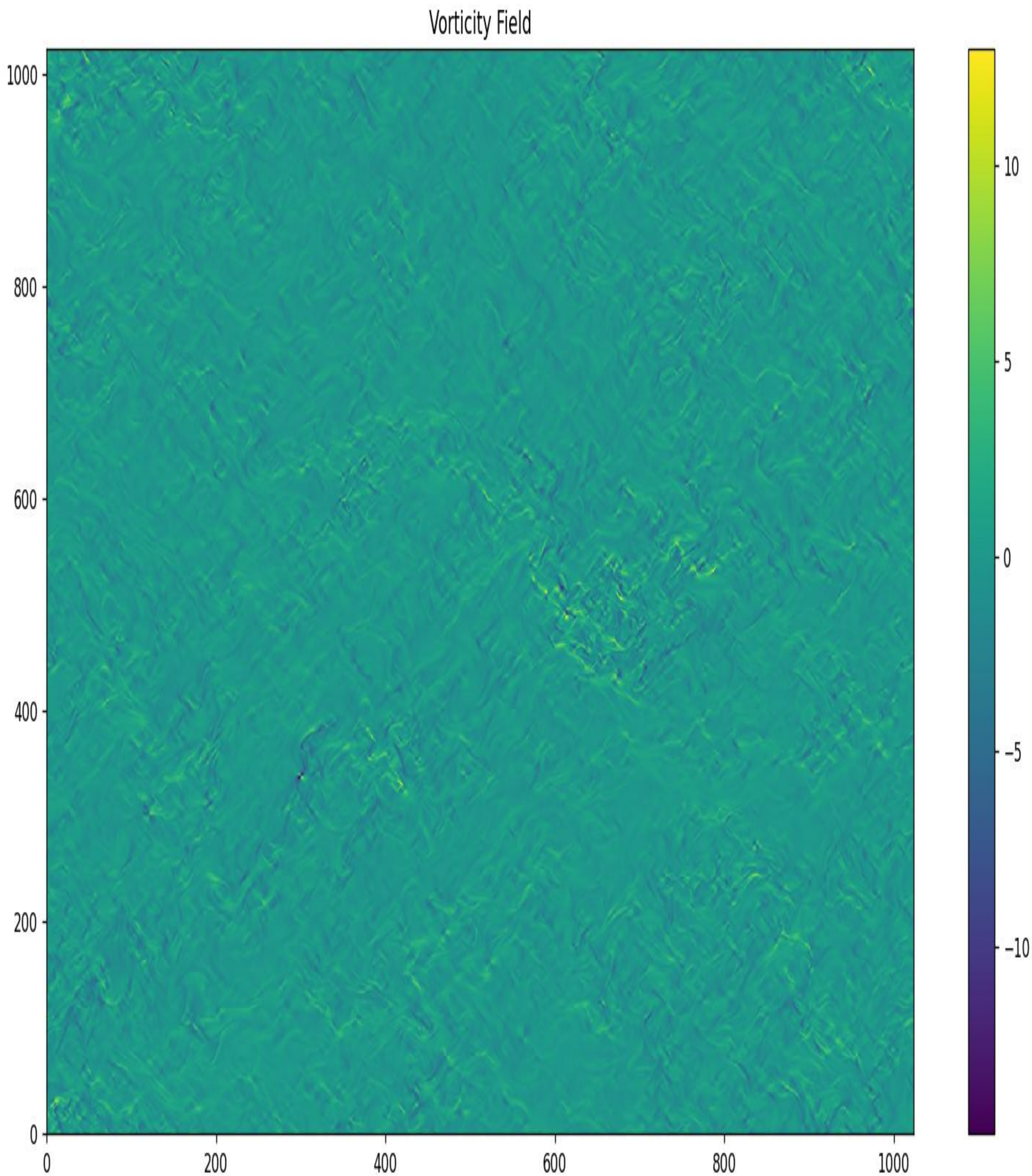
## Kolmogorov Statistics

- Reynolds Number: 4477.861008221851
- Kolmogorov Length Scale: 0.0024917898561977753
- Kolmogorov Time Scale: 0.03356225236459529
- Kolmogorov Velocity Scale: 0.0742438209786646
- Ratio of grid cell size and the Kolmogorov length: 2.4624561081187544

## Plot of Kinetic Energy Field:



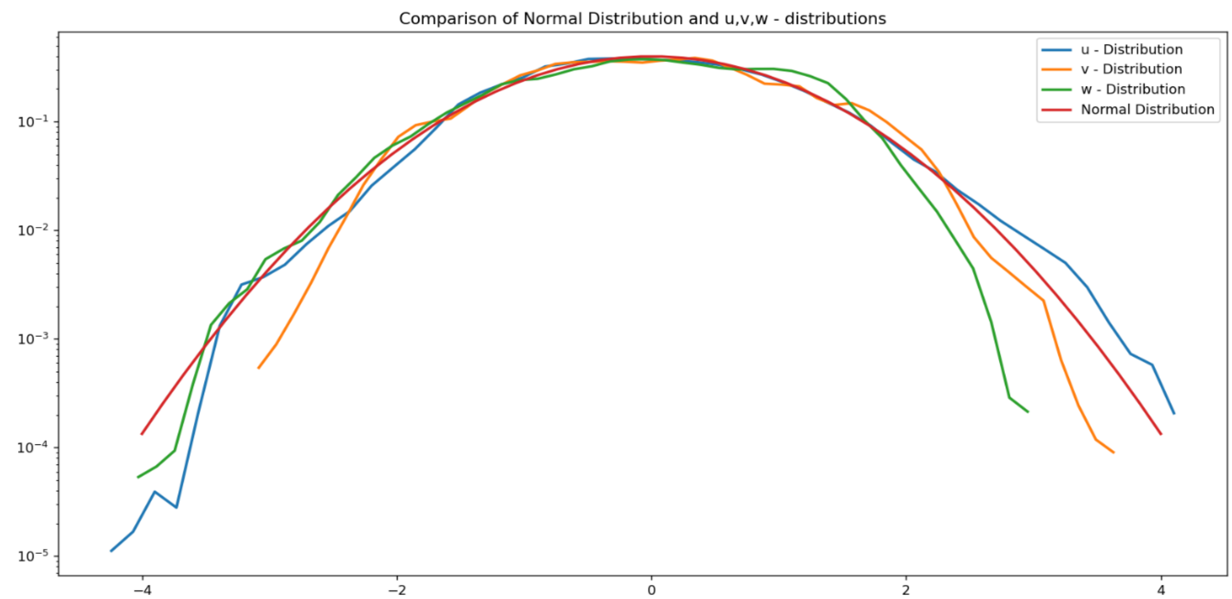
## Plot of Vorticity Field:

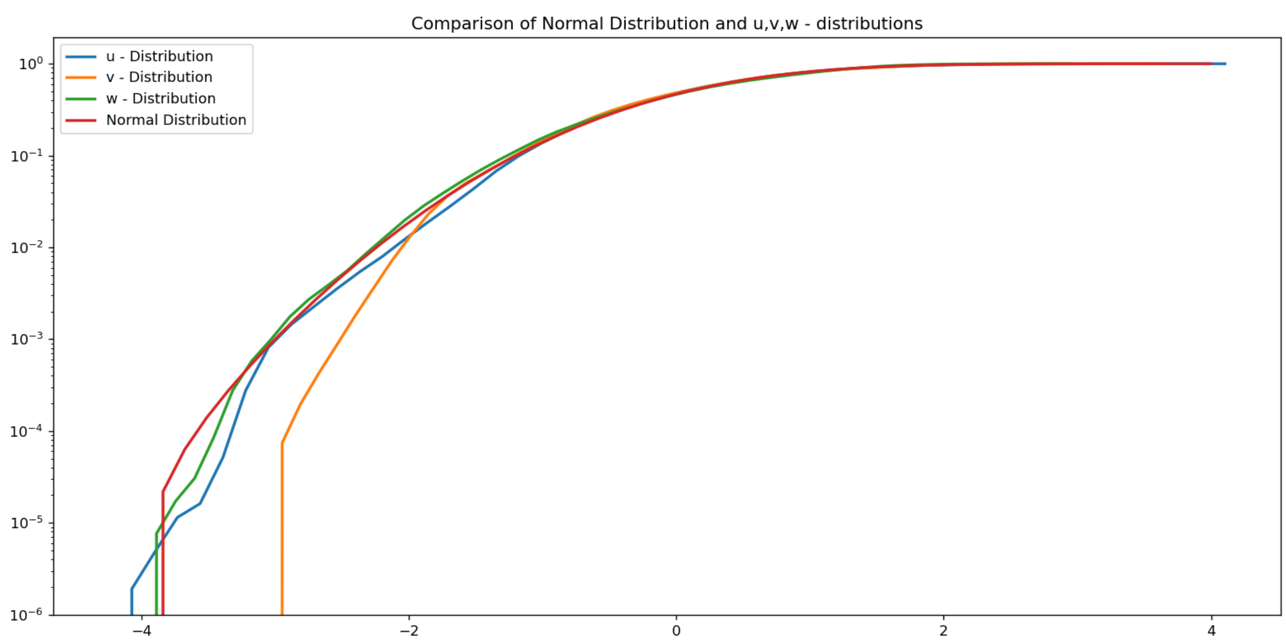


## PDFs, CDFs, Skewness and Kurtosis of quantities:

**(For all the parts the pdf is followed by the cdf)**

### Velocity components:



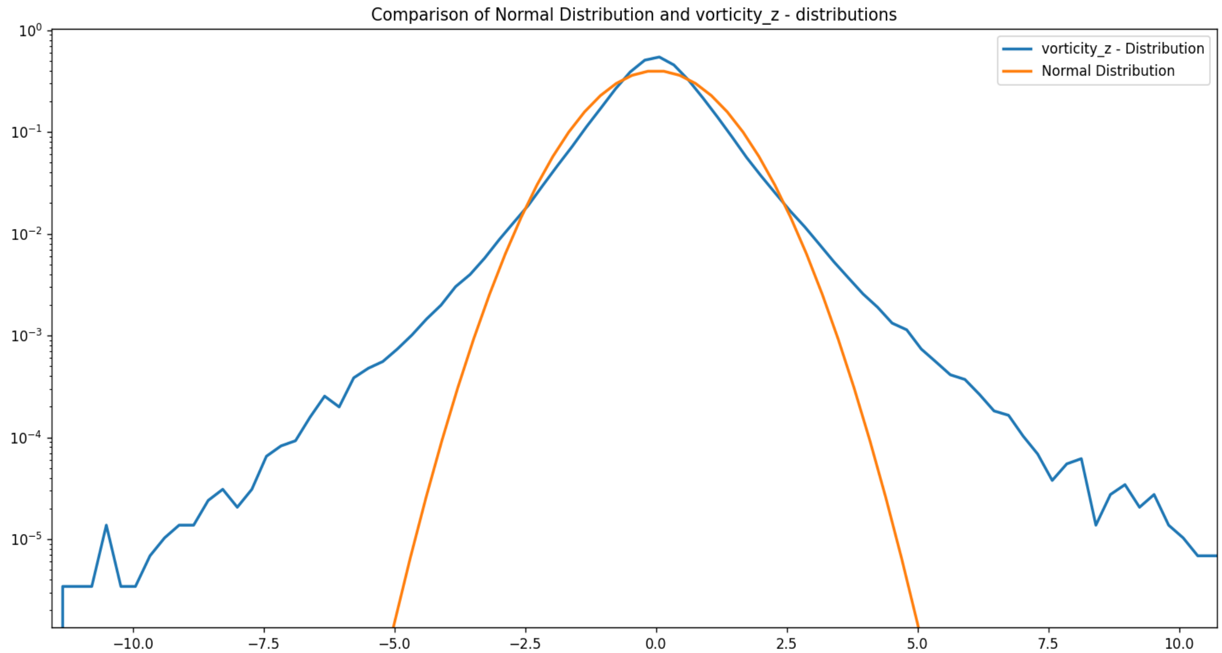
Kurtosis of velocity quantities:

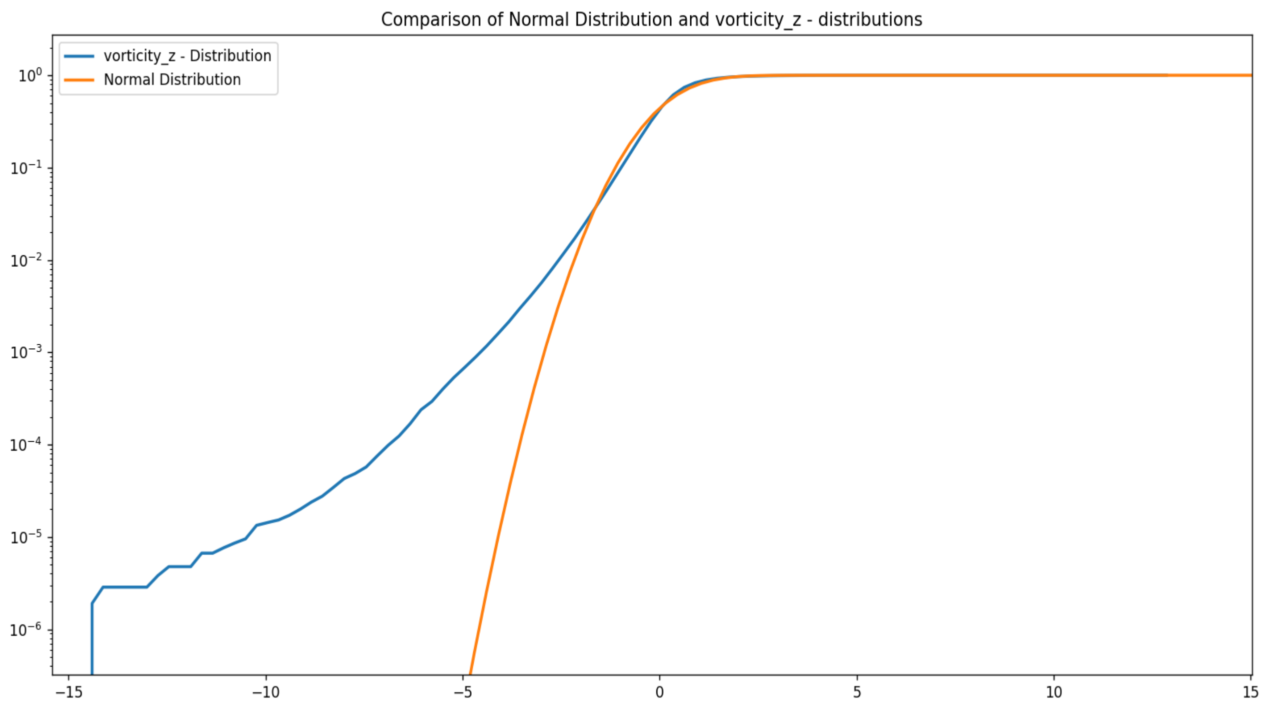
- u - component => 3.04540666483723
- v - component => 2.5911054085216922
- w - component => 2.582971778164235

Skewness of velocity quantities:

- u - component => 0.17190901669220843
- v - component => 0.09136925992119718
- w - component => - 0.2201799985538492

## Vorticity Component ($\omega_z$)



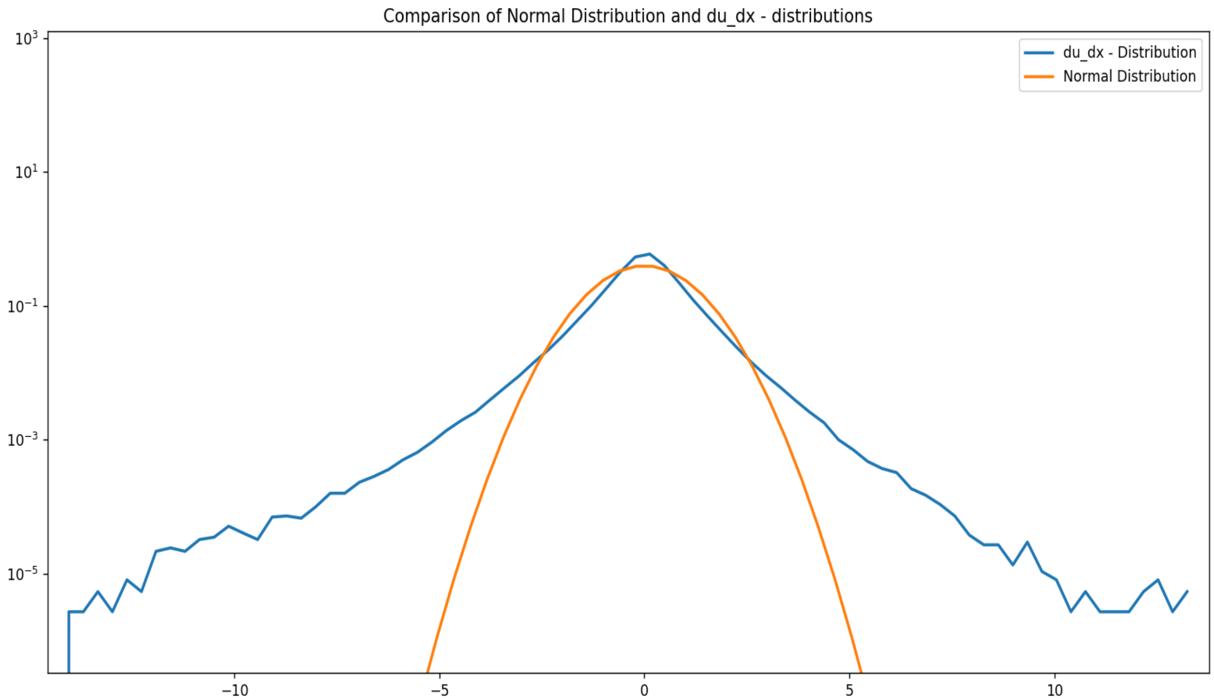



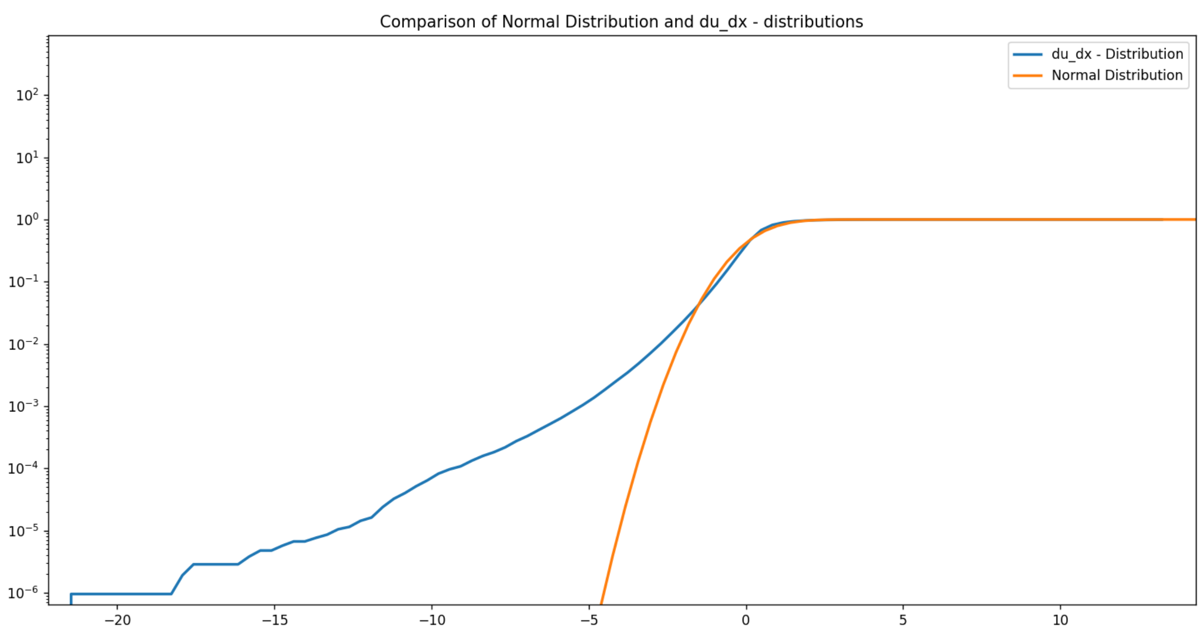

Kurtosis of the vorticity component: 7.60047927102915

Skewness of the vorticity component: 0.05660021464815684

## Velocity Gradients

### *Gradient of u with respect to x*

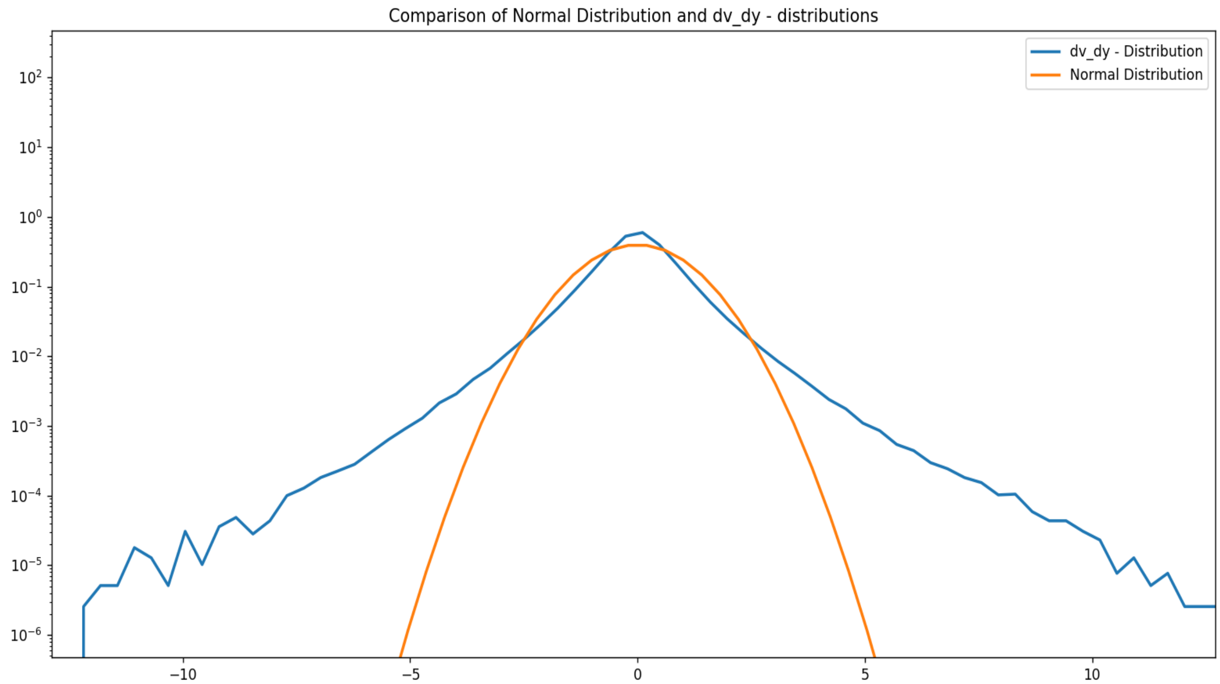


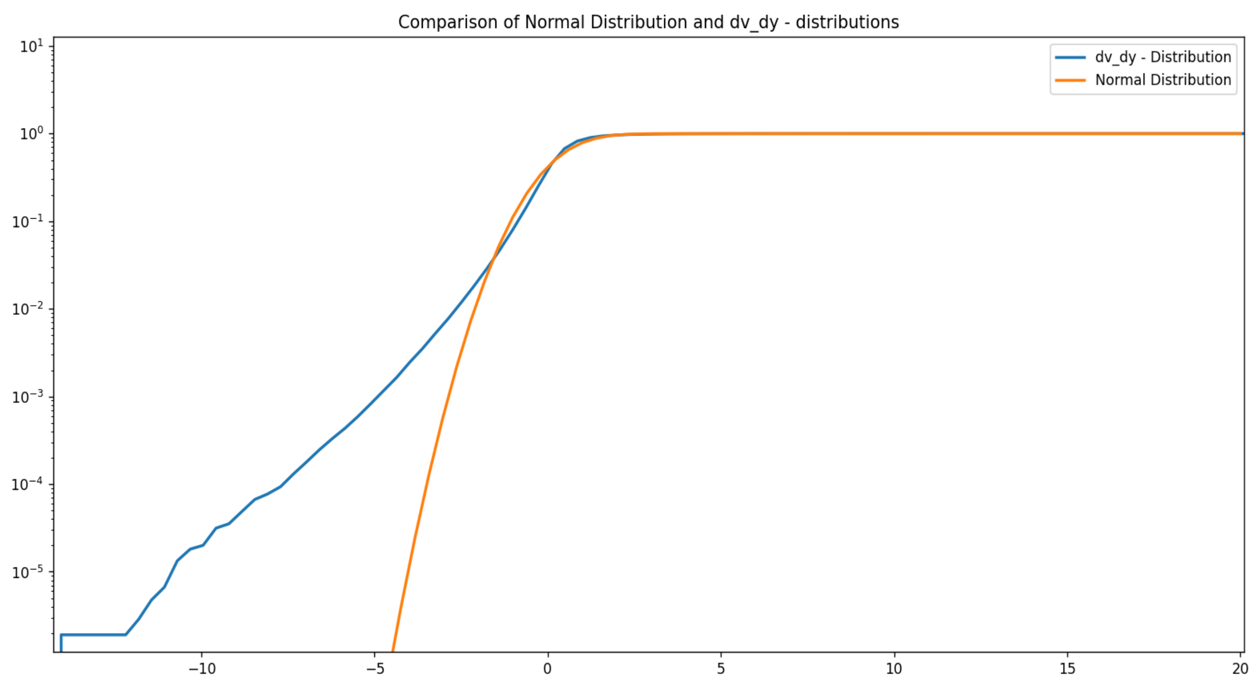




Kurtosis of the u-component gradient with respect to x: 10.806434588569548

Skewness of the u-component gradient with respect to x: - 0.33346105810277066

*Gradient of v with respect to y*

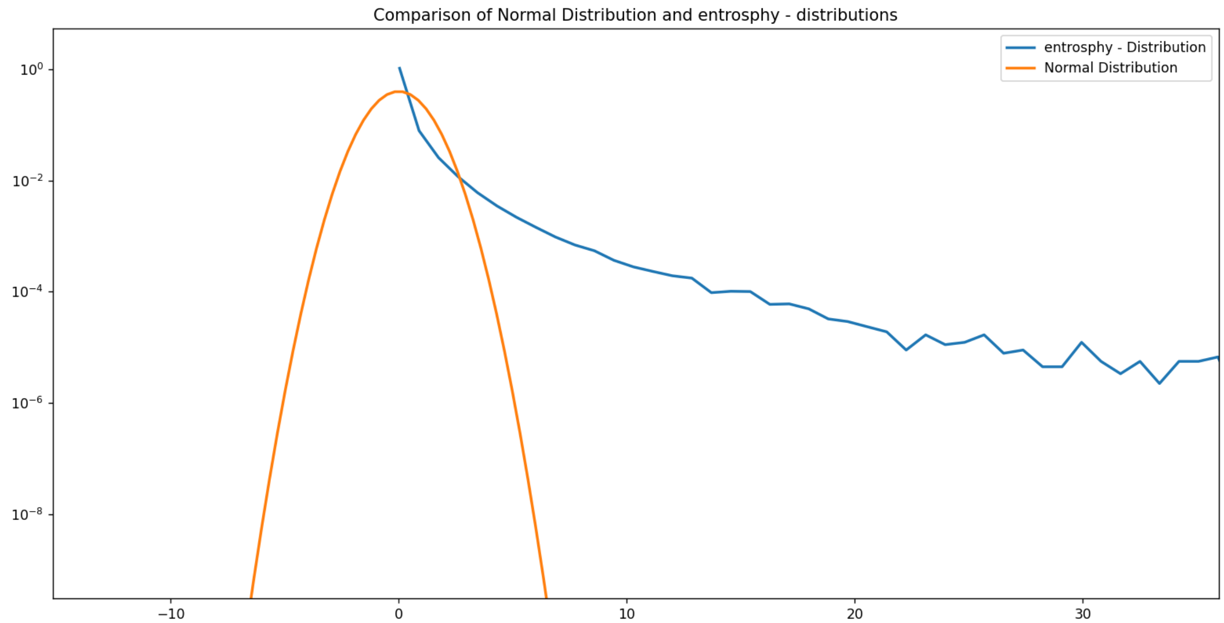


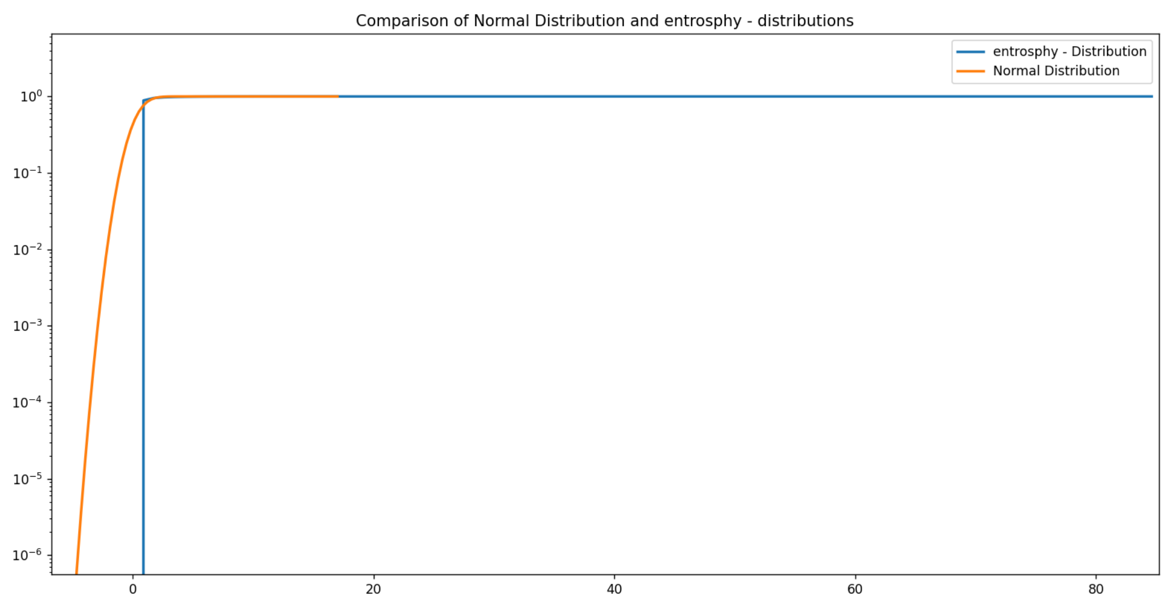




Kurtosis of the v-component gradient with respect to y: 10.578385110472853

Skewness of the v-component gradient with respect to y: 0.11653157769132681

## Entrosphy




Kurtosis of the entrosphy: 12.962020450273679

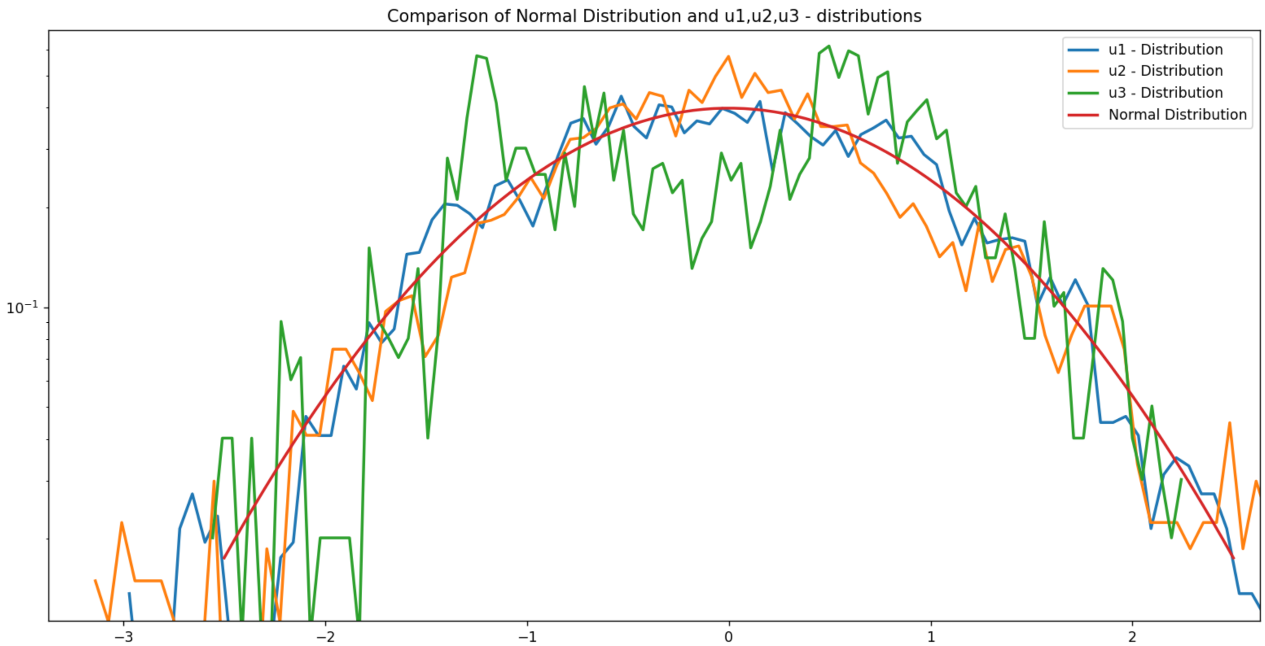
Skewness of the entrosphy: 435.07139695472176

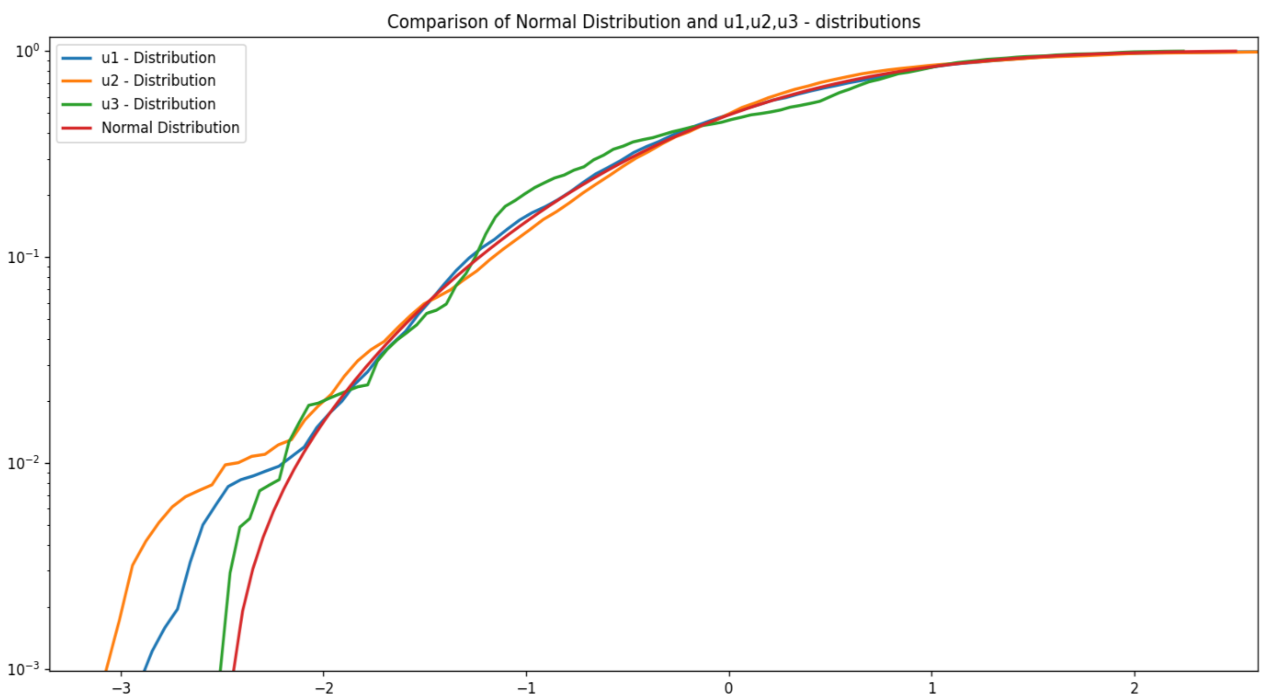
## U – component in three different sections of our domain

u1 – component => u-component of the velocity in the domain [0,128]

u2 – component => u-component of the velocity in the domain [129,256]

u3 – component => u-component of the velocity in the domain [257,512]



Kurtosis of velocity quantities:

- u1 - component => 0.022610235662881676
- u2 - component => 0.01449052620623452
- u3 - component => 0.004277477313540519

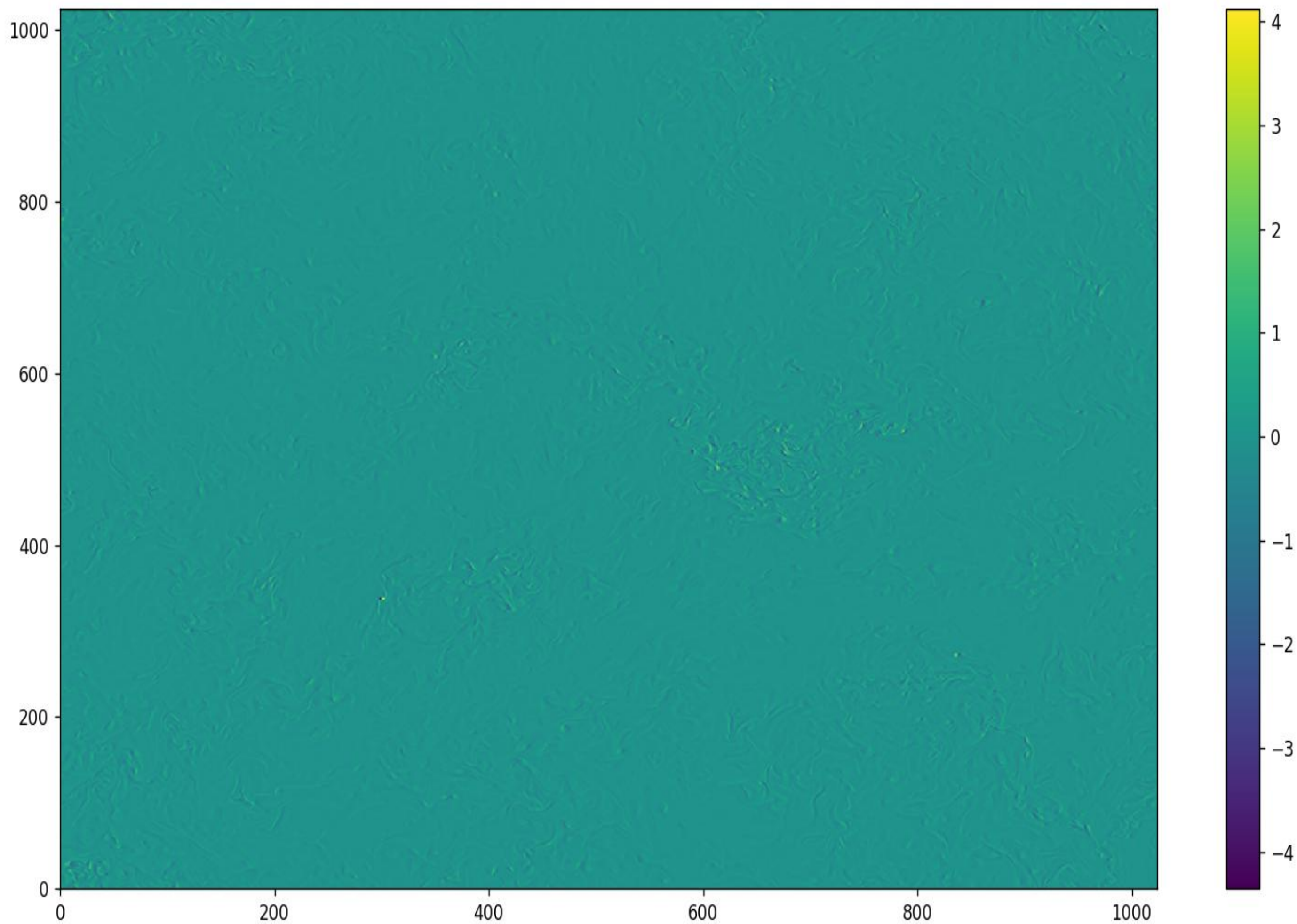
Skewness of velocity quantities:

- u1 - component => 0.0005713057618805946
- u2 - component => 0.000830360526160265
- u3 - component => -0.00028853409546430135

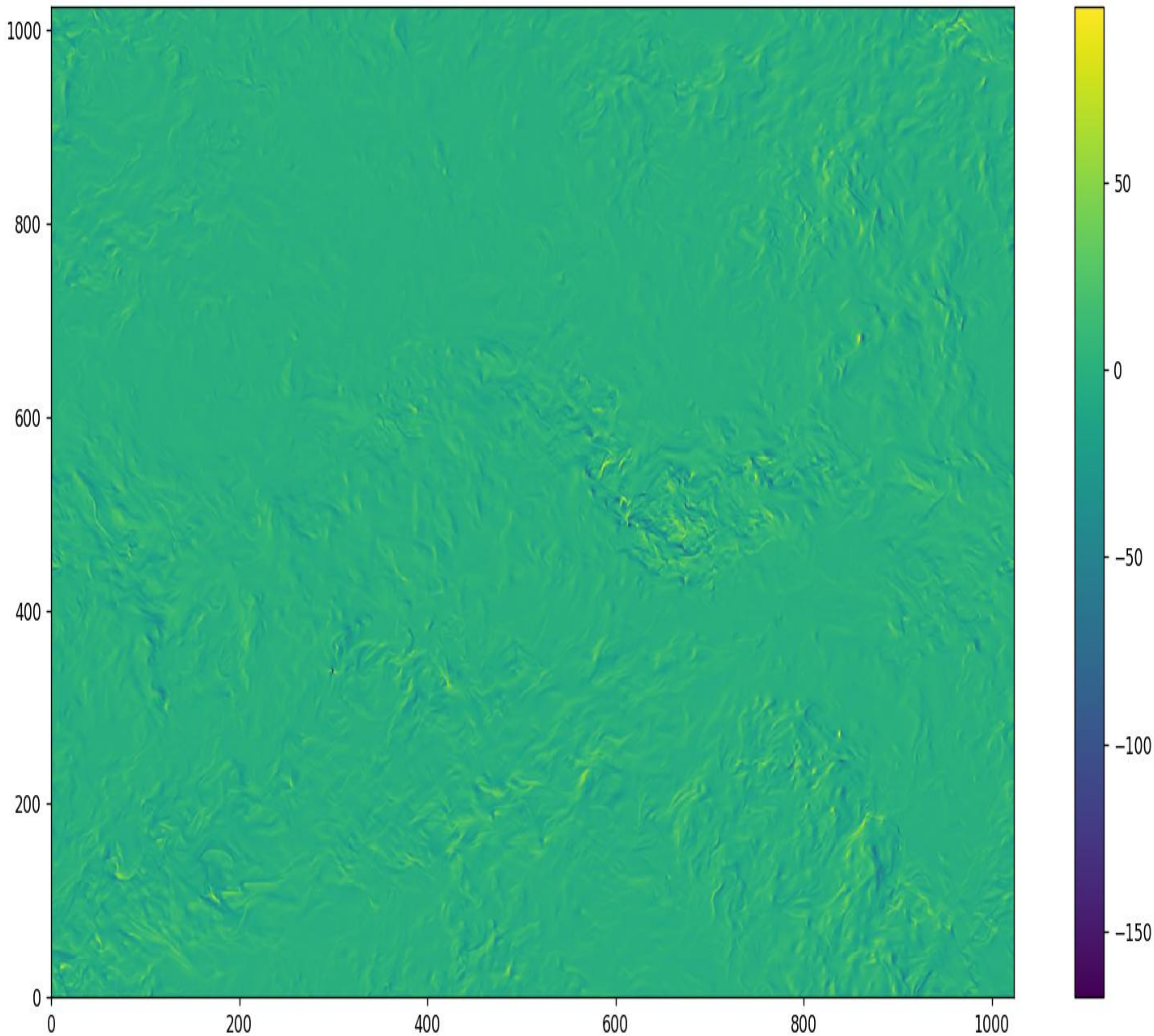
In the data corresponding to the domain [257,512], there is a lot of disturbance or randomness. There are a lot of sharp peaks in this section of the velocity data. On the other hand, the data corresponding to [0,128] and [129,256] lie near the normal distribution compared to [257,512].

## Field of the viscous term, non-linear term and local Reynolds Number

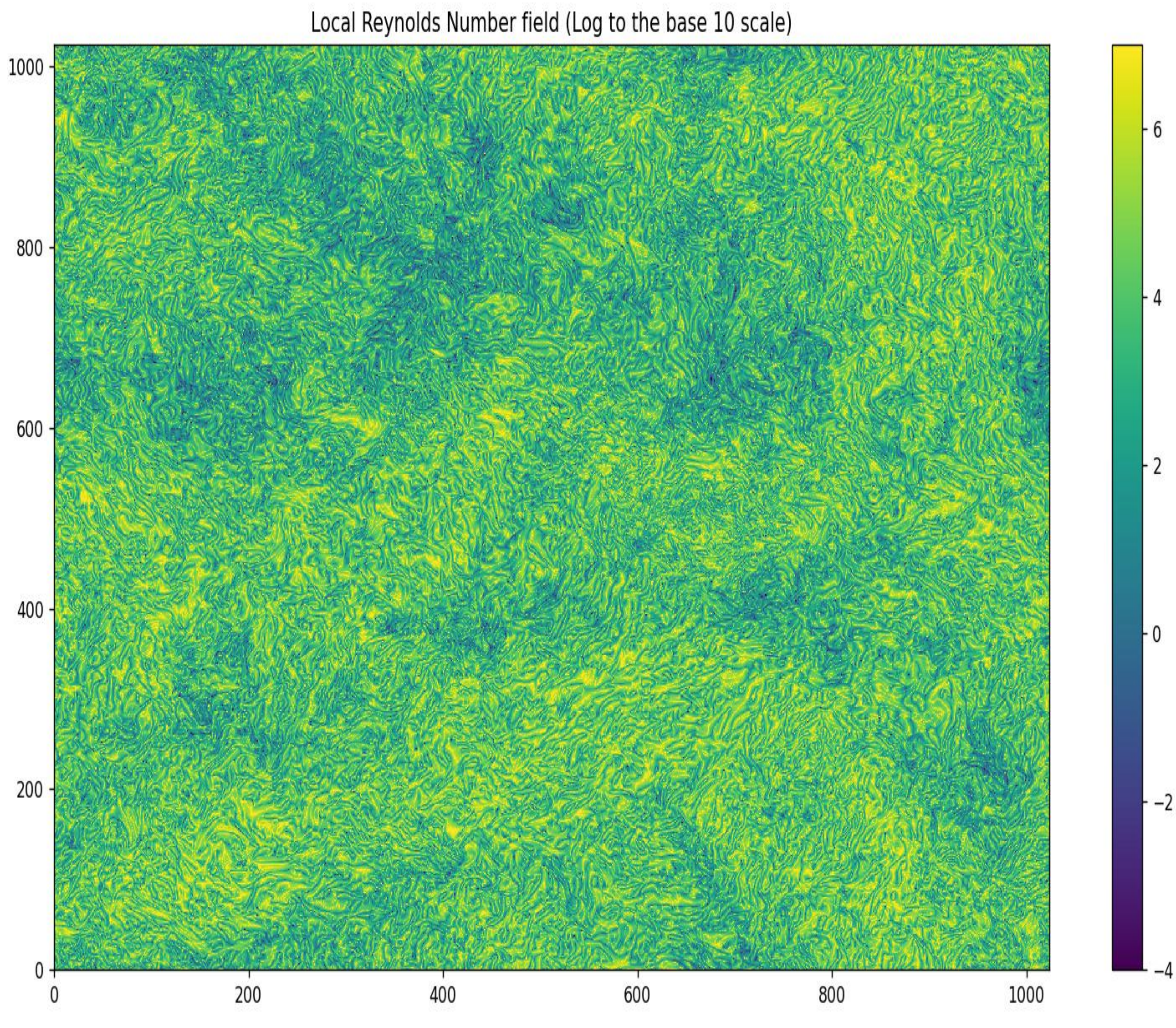
### Viscous Term



Non – Linear Term



## Local Reynolds Number



The regions where the Kinetic energy is small, the Re value there is also small. This can be seen by looking at the dark blue colored patches in the Kinetic energy plot. If we check the same areas in the local Reynolds number flow, we see cluster of bluish - green cells. This corresponds to the Re value between 1 and 100.

I don't find any correlations between the Local Reynolds Number plot and the vorticity plot as the vorticity plot in the special regions (bluish - green) discussed above shows no special features.