Homework 1 : Computational Economics
Search Models

Dhananjay Ghei

October 31, 2018

Search Models and Labor Wedge

1. First show that there exists an equilibrium in the following transformed variables: $\tilde{c}(s^t) = c(s^t)z(s^t)^{\frac{-1}{1-\alpha}}$, $\tilde{k}(s^t) = k(s^t)z(s^t)^{\frac{-1}{1-\alpha}}$ and $\tilde{w}(s^t) = w(s^t)z(s^t)^{\frac{-1}{1-\alpha}}$. Then show that there exists a Markovian solution with k, n, s as state variables. Boil down the equations below to two functional equations in two unknown functions: $\theta(k, n, s)$ and $\tilde{c}(k, n, s)$.

Appendix A gives the calculations for the equations derived below with the appropriate transformations.
The Euler equation is given by:

$$1 = \beta \sum_{s^{t+1}|s^t} \pi(s_{t+1}|s_t) \frac{\tilde{c}(s^t)}{\tilde{c}(s^{t+1})} e^{-\frac{s_{t+1}}{1-\alpha}} \left(\alpha \left(\frac{\tilde{k}(s^{t+1})}{n(s^{t+1}) - \theta(s^{t+1})(1 - n(s^{t+1}))} \right)^{\alpha - 1} + 1 - \delta \right)$$

The condition w.r.t employment is given by:

$$(1-\alpha)\left(\frac{\tilde{k}(s^t)}{n(s^t) - \theta(s^t)(1 - n(s^t))} \right)^{\alpha} = \beta \mu(\theta(s^t)) \sum_{s^{t+1}|s^t} \pi(s_{t+1}|s_t)$$
$$\times \left((1-\alpha) \left(\frac{\tilde{k}(s^{t+1})}{n(s^{t+1}) - \theta(s^{t+1})(1 - n(s^{t+1}))} \right)^{\alpha} \left(1 + \frac{1-x}{\mu(\theta(s^{t+1}))} \right) - \tilde{w}(s^{t+1}) \right)$$

The wage equation evaluated in history s^{t+1} reduces to:

$$(1-\tau)\tilde{w}(s^{t+1}) = (1-\phi)\gamma\tilde{c}(s^{t+1}) + \phi(1-\tau)(1-\alpha)\left(\frac{\tilde{k}(s^{t+1})}{n(s^{t+1}) - \theta(s^{t+1})(1 - n(s^{t+1}))} \right)^{\alpha} (1 + \theta(s^{t+1}))$$

Eliminating wage $\tilde{w}(s^{t+1})$ between the last two equations gives:

$$(1-\alpha)\left(\frac{\tilde{k}(s^t)}{n(s^t) - \theta(s^t)(1 - n(s^t))} \right)^{\alpha} = \beta \mu(\theta(s^t)) \sum_{s^{t+1}|s^t} \pi(s_{t+1}|s_t) \frac{\tilde{c}(s^t)}{\tilde{c}(s^{t+1})}$$
$$\left(-\frac{(1-\phi)\gamma\tilde{c}(s^{t+1})}{1-\tau} + (1-\alpha)\left(\frac{\tilde{k}(s^{t+1})}{n(s^{t+1}) - \theta(s^{t+1})(1 - n(s^{t+1}))} \right)^{\alpha} \left(\frac{1-x}{\mu(\theta(s^{t+1}))} + 1 - \phi - \phi\theta(s^{t+1}) \right) \right)$$

The resource constraint is given by:

$$\tilde{k}(s^{t+1}) e^{\frac{s_{t+1}}{1-\alpha}} = (\tilde{k}(s^t))^{\alpha} (n(s^t) - \theta(s^t)(1 - n(s^t)))^{1-\alpha} + (1-\delta)\tilde{k}(s^t) - \tilde{c}(s^t)$$

The law of motion for employment is given by:

$$n(s^{t+1}) = (1-x)n(s^t) + f(\theta(s^t))(1-n(s^t))$$

In order to show that there exists a Markovian solution, we first conjecture that the solution is given as constants $\bar{c}, \bar{k}, \bar{w}, \bar{w}, \bar{\theta}$. Then, we plug in these values into the 5 equations above. If there is such an equilibrium, the equations are given by:

$$\begin{aligned}
1 &= \beta \sum \pi(s_{t+1}|s_t) e^{-\frac{s_{t+1}}{1-\alpha}} \left(\alpha \left(\frac{\bar{k}}{\bar{n} - \bar{\theta}(1-\bar{n})} \right)^{\alpha-1} + 1 - \delta \right) \\
(1-\alpha)\left(\frac{\bar{k}}{\bar{n} - \bar{\theta}(1-\bar{n})} \right)^{\alpha} &= \beta \mu(\bar{\theta}) \sum \pi(s_{t+1}|s_t) \times \left((1-\alpha)\left(\frac{\bar{k}}{\bar{n} - \bar{\theta}(1-\bar{n})} \right)^{\alpha} \left(1 + \frac{1-x}{\mu(\bar{\theta})} \right) - \bar{w} \right) \\
\bar{n} &= (1-x)\bar{n} + f(\bar{\theta})(1-\bar{n}) \\
\bar{k} e^{\frac{s_{t+1}}{1-\alpha}} &= \bar{k}^{\alpha}(\bar{n} - \bar{\theta}(1-\bar{n}))^{1-\alpha} + (1-\delta)\bar{k} - \bar{c} \\
(1-\tau)\bar{w} &= (1-\phi)\gamma\bar{c} + \phi(1-\alpha)(1-\tau)\left(\frac{\bar{k}}{\bar{n} - \bar{\theta}(1-\bar{n})} \right)^{\alpha} (1+\bar{\theta})
\end{aligned}$$

Substituting out $\bar{c}, \bar{n}, \bar{k}$, we get an equation in $\bar{\theta}$ and $\bar{w}$ as: There is an equilibrium of this form if and only if this equation can be satisfied in any history s^t. Thus, this requires that there exists a number $\bar{s}$ satisfying:

Finally, let the functions Θ and C define the recruiter-employment ratio and equilibrium consumption relative to trend as functions of the current state $(s, n, \tilde{k})$ so, $\theta(s^t) = \Theta(s_t, n(s^t), \tilde{k}(s^t))$ and $\tilde{c}(s^t) = C(s_t, n(s^t), \tilde{k}(s^t))$. Substituting in the expressions and eliminating $\tilde{k}(s^{t+1})$ and $n(s^{t+1})$, we get two nonlinear equations in Θ and C. Appendix B gives the calculations for this transformation. The functional equations are given as:

2. Compute the non stochastic steady state. Log linearise around the non stochastic steady state to get a system of linear expectational difference equations and solve for the policy rules. For example:

$$\log\theta_{t+1} = \log\bar{\theta} + \theta_s(s_{t+1} - \bar{s}) + \theta_n(\log n_t - \log\bar{n}) + \theta_k(\log k_t - \log\bar{k})$$

Appendix C uses the equations in part 1) to derive the equations for the non stochastic steady state. The steady state equations are given by:

$$\begin{aligned}
1 &= \beta e^{-\frac{\bar{s}}{1-\alpha}} \left(\alpha \left(\frac{\bar{k}}{\bar{n} - \bar{\theta}(1-\bar{n})} \right)^{\alpha-1} + 1 - \delta \right) \\
(1-\alpha)\left(\frac{\bar{k}}{\bar{n} - \bar{\theta}(1-\bar{n})} \right)^{\alpha} &= \beta \mu(\bar{\theta}) \left((1-\alpha)\left(\frac{\bar{k}}{\bar{n} - \bar{\theta}(1-\bar{n})} \right)^{\alpha} \left(1 + \frac{1-x}{\mu(\bar{\theta})} \right) - \bar{w} \right) \\
\bar{n} &= (1-x)\bar{n} + f(\bar{\theta})(1-\bar{n}) \\
\bar{k} e^{\frac{\bar{s}}{1-\alpha}} &= \bar{k}^{\alpha}(\bar{n} - \bar{\theta}(1-\bar{n}))^{1-\alpha} + (1-\delta)\bar{k} - \bar{c} \\
(1-\tau)\bar{w} &= (1-\phi)\gamma\bar{c} + \phi(1-\alpha)(1-\tau)\left(\frac{\bar{k}}{\bar{n} - \bar{\theta}(1-\bar{n})} \right)^{\alpha} (1+\bar{\theta})
\end{aligned}$$

We put these equations along with the parameter values to solve for the steady state using a non-linear solver. The steady state is given by: $\bar{n} = 0.949$, $\bar{k} = 218.239$, $\bar{c} = 4.695$, $\bar{w} = 4.016$, $\bar{\theta} = 0.077$. In order to log-linearise the equations, we convert all the variables into log deviations from steady state and then use a first order Taylor approximation of the equation around 0 to get the coefficients.

The log linearised equations are given by:

$$
\begin{aligned}
\log k_{t+1} - \log \bar{k} &= -.610(s_t - \bar{s}) + .0186(\log n_t - \log \bar{n}) + 0.991(\log k_t - \log \bar{k}) \\
\log n_{t+1} - \log \bar{n} &= 0.026(s_t - \bar{s}) + 0.312(\log n_t - \log \bar{n}) - 0.047(\log k_t - \log \bar{k}) \\
\log \tilde{y}_t - \log \bar{y} &= -0.003(s_t - \bar{s}) + 0.726(\log n_t - \log \bar{n}) + 0.337(\log k_t - \log \bar{k}) \\
\log \tilde{w}_t - \log \bar{w} &= -0.236(s_t - \bar{s}) - 0.205(\log n_t - \log \bar{n}) + 0.345(\log k_t - \log \bar{k})
\end{aligned}
$$

where the bar variables are the steady state values of these variables.

3. Using the calibration in Table 3.2 to compute the IRF and the ergodic moments of all the relevant variables. Please check your results against figure 3.2 in the book.

For most of the parameters, Shimer uses the same calibration as in Table 3.2. However, he changes some of the parameters. He sets $\bar{s} = 0.0012$, $\rho = 0.4$ and $\zeta = 0.00325$. Table 3 gives the values of the remaining parameters. The log linear solutions for the relevant variables can be calculated directly

Table 1 Calibration of parameters

These are the same parameters as in Shimer's book from Table 3.2

Parameter	Description	Values
β	Discounting	0.996
α	Share of capital	0.33
τ	Tax rate	0.4
x	Employment exit probability	0.034
ϕ	Worker's bargaining power	0.5
δ	Depreciation	0.0028

once we have the solutions for the main variables as given in the previous part. As an example, the coefficients for the log linear solution of consumption-output ratio c/y is directly given by subtracting the coefficients of output from that of consumption. We can do that same exercise for the labor share of income wn/y.

Table 2 shows the ergodic moment of all the relevant variables. In particular, one can compare this table directly with Table 3.5 from Shimer's book. Note that, the relative standard deviations and the correlations are close to the values in the book.

Table 2 Model with capital, stochastic trend. Co-movements of variables in infinite samples.

The table shows the co-movement of variables in an infinite sample. The table closely resembles to the one Shimer has in his book. The corresponding table to look at is Table 3.5 on Page 101. The variables have similar ergodic moments as in the book.

		$\tilde{y}$	$\tilde{c}$	θ	$\tilde{k}$	n	wn/y	c/y	$\hat{\tau}$	s
Relative sd.		1	2.09	9.632	3.475	0.239	0.129	1.1	1.291	0.541
Correlations	$\tilde{y}$	1	0.99468	-0.99657	0.99997	-0.99874	0.01101	0.98064	-0.97011	-0.06679
	$\tilde{c}$	-	1	-0.98281	0.99514	-0.99583	0.11375	0.9956	-0.98992	0.03617
	θ	-	-	1	-0.99621	0.99391	0.07163	-0.96122	0.94676	0.14895
	$\tilde{k}$	-	-	-	1	-0.99911	0.0154	0.98156	-0.97117	-0.06238
	n	-	-	-	-	1	-0.03118	-0.98398	0.974	0.04636
	wn/y	-	-	-	-	-	1	0.20612	-0.25329	0.99696
	c/y	-	-	-	-	-	-	1	-0.99879	0.12946
	$\hat{\tau}$	-	-	-	-	-	-	-	1	-0.17731
	s	-	-	-	-	-	-	-	-	1

TODO: Impulse response functions. The impulse response graphs do not match the figures from Shimer's book.

4. Derive an expression for labor wedge in this economy using the consumption output ratio and the hours which here is a fraction of HH that are employed. Obtain a log linear expansion of the wedge using the log linear policy rules.

The expression for labor wedge is given by:

$$\hat{\tau}(s^t) = 1 - \frac{\hat{\gamma}}{1-\alpha}(c(s^t)/y(s^t))n(s^t)$$

Considering the fact that the true tax factor is 0.4,the disutility of work is $\hat{\gamma} = .513$. Next, we log linearise the equation for wedge and the policy rule for labor wedge is given by:

$$\log \hat{\tau}_t = \log .4 - 0.572(s - \bar{s}) - 0.429(\log n_t - \log \bar{n}) - 0.396(\log k_t - \log \bar{k})$$

5. What is the correlation of the labor wedge with output and employment? You can sign it by staring at the policy rules. What is the ergodic std. of the labor wedge? How does the correlations and volatility compare to data.

We can look at Table 2 and see that the correlation of labor wedge with output is -0.9701 and the correlation with employment is 0.974. The relative standard deviation of labor wedge in an infinite sample is equal to 1.29. The labor wedge is countercyclical.

6. Now extract a TFP series such that the simulate path of $y(s^t)$ is exactly as in the data. Start with the steady state level of capital and employment. Using the policy rule for output reverse engineer a shock such that detrended output matches exactly to the value in the data. Now update the state variables using your policy rules and keep iterating this procedure. How does the TFP series extracted compare to a Solow residual in the data?

A Transforming the variables on a balanced growth path

B Reducing the set of equations to a pair of nonlinear equations

C Steady state calculations