# CSE 551 Assignment 4 Solutions

April 14, 2021

**Submission Instructions:** Deadline is **11:59pm on 04/08**. Late sub-missions will be penalized, therefore please ensure that you submit (file upload is completed) before the deadline. Additionally, you can download the submitted file to verify if the file was uploaded correctly. Submit your answers electronically, in a single PDF, via *Canvas*. **Please type up the answers and keep in mind that we'll be checking for plagiarism**.
Furthermore, please note that the graders will grade 2 out of the 4 questions randomly. Therefore, if the grader decides to check questions 1 and 4, and you haven't answered question 4, you'll lose points for question 4. Hence, please answer all the questions.

1. Consider an undirected graph $G = (V, E)$ with nonnegative edge weights $w_e \geq 0$. Suppose that you have computed a minimum spanning tree of $G$, and that you have also computed the shortest paths to all nodes from a particular node $s \in V$. Now suppose, each edge weight is increased by 1: the new weights are $w'_e = w_e + 1$. **[25]**

   - Does the minimum spanning tree change? Give an example where it changes or prove it cannot change.

     **Solution:** It does not change. For positive edge weight, if $a < b$, then $a+1 < b+1$. Since the MST algorithm determines the MST based on the order of the edge weight, and since the order of the edge weights do not change when we increment them by 1, the MST does not change.

   - Do the shortest paths change? Give an example where they change or prove they cannot change.

     **Solution:** The shortest path changes. Consider a complete graph with 4 nodes $A, B, C, D$ and edge weights $AB = 5$, $BC = 5$, $CD = 5$ and $AD = 16$. The shortest path from $A$ to $D$ before increment is via nodes $B$ and $C$ for a total path length of 15. However, when the edge weights are increased by 1, the new edge weights are $AB = 6$, $BC = 6$, $CD = 6$ and $AD = 17$. The shortest path from $A$ to $D$ is now the edge $AD$ for a path length of 17.

2. Let $G = (V, E)$ be an undirected graph. A node cover of G is a subset $U$ of the vertex set $V$ such that every edge in $E$ is incident to at least one vertex in $U$. A minimum node cover is one with the fewest number of vertices. Consider the following greedy algorithm for this problem:

   (a) Procedure COVER(V, E)
   (b) $U \leftarrow \phi$
   (c) loop
   (d) Let $v \in V$ be a vertex of maximum degree
   (e) $U \leftarrow U \cup \{v\}; V \leftarrow V - \{v\}$
   (f) $E \leftarrow E - \{(u, w)\}$ such that $u = v$ or $w = v$
   (g) until $E = \phi$ go to (c)
   (h) return $(U)$
   (i) end COVER

   Does this algorithm always generate a minimum node cover? If yes, provide some arguments as to why you think so. Else, then provide a counter example where this algorithm fails. **[25]**

   **Solution:** This algorithm will NOT always generate a minimum node cover. Consider the following example: A tree with the root node denoted as 1, it's children are 2, 3 and 4. 2 has 5 as its child, 3 has 6 as its child and 4 has 7 as its child. The algorithm above will determine that node 1 is the vertex of maximum degree (= 3) and add it to the solution set. Next, note that all the remaining nodes have maximum degree = 1 and there are three edges left in the tree. Therefore, three nodes must be added to the solution set to cover all the nodes in the original tree. Thus, the cardinality of the solution set following this algorithm will be 4. However, the optimal cardinality is 3, when we just select nodes (2, 3, 4). Therefore, this algorithm will not always produce a minimum node cover.

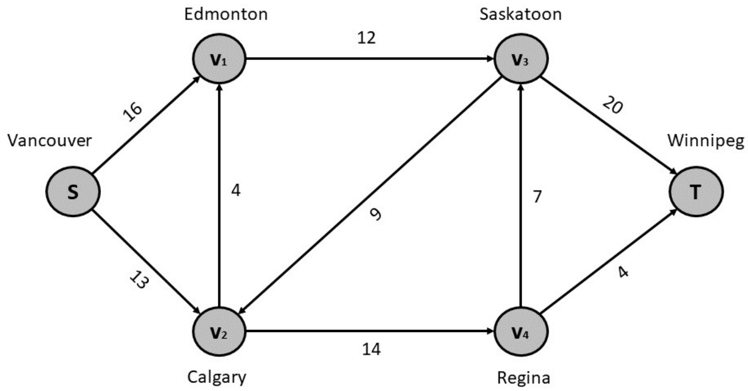


Figure 1: Network for Q3

3. For the network shown in Figure 1, compute the maximum flow from Vancouver to Winnipeg. **Show all your work**. **[25]**

**Solution:** The maximum flow in this network is 23. The minimum cut $C(S:\bar{S})$ is across $S = \{S, v_1, v_2, v_4\}$ and $\bar{S} = \{v_3, T\}$.

4. Will the maximum flow from the source to the destination node in the Ford-Fulkerson Algorithm will be equal to the capacity of the minimum cut, if the capacity of a cut is defined in the following manner? For each definition, if you agree, then please provide arguments as to why. Else, provide a counter example to show that the definition does not hold:

(a) $C(S:\overline{S}) = \sum_{e\in(S:\overline{S})} C(e) + \sum_{e\in(\overline{S}:S)} C(e)$ **[6]**

**Solution:** Consider the following network with three nodes $S, A, T$ and directed edges $SA$ with capacity 10, $ST$ with a capacity 7, $AT$ with capacity 5 and $TS$ with capacity 4. Maximum flow in this network is 12. We can have two cuts here:

- $S = \{S, A\}, \bar{S} = \{T\}$, Here, $C(S:\overline{S}) = \sum_{e\in(S:\overline{S})} C(e) + \sum_{e\in(\overline{S}:S)} C(e) = 12 + 4 = 16 \neq 12$, **False**.
- $S = \{S\}, \bar{S} = \{A, T\}$, Here, $C(S:\overline{S}) = \sum_{e\in(S:\overline{S})} C(e) + \sum_{e\in(\overline{S}:S)} C(e) = 17 + 4 = 21 \neq 12$, **False**.

Therefore, maximum flow is not equal to the capacity of the minimum cut for this definition.

(b) $C(S:\overline{S}) = \sum_{e\in(S:\overline{S})} C(e) - \sum_{e\in(\overline{S}:S)} C(e)$ **[6]**

**Solution:** Consider the same network as above. We can have two cuts here:

- $S = \{S, A\}, \bar{S} = \{T\}$, Here, $C(S:\overline{S}) = \sum_{e\in(S:\overline{S})} C(e) - \sum_{e\in(\overline{S}:S)} C(e) = 12 - 4 = 8 \neq 12$, **False**.
- $S = \{S\}, \bar{S} = \{A, T\}$, Here, $C(S:\overline{S}) = \sum_{e\in(S:\overline{S})} C(e) - \sum_{e\in(\overline{S}:S)} C(e) = 17 - 4 = 13 \neq 12$, **False**.

Therefore, maximum flow is not equal to the capacity of the minimum cut for this definition.

(c) $C(S:\overline{S}) = \sum_{e\in(S:\overline{S})} C(e)$ **[6]**

**Solution:** True. This has been shown to be true in class. Refer class slides.

(d) $C(S:\overline{S}) = min\{\sum_{e\in(S:\overline{S})} C(e), \sum_{e\in(\overline{S}:S)} C(e)\}$ **[7]**

**Solution:** Consider the same network as above. We can have two cuts here:

- $S = \{S, A\}, \bar{S} = \{T\}$, Here, $C(S:\overline{S}) = min\{\sum_{e\in(S:\overline{S})} C(e), \sum_{e\in(\overline{S}:S)} C(e)\} = min(12, 4) = 4 \neq 12$, **False**.
- $S = \{S\}, \bar{S} = \{A, T\}$, Here, $C(S:\overline{S}) = min\{\sum_{e\in(S:\overline{S})} C(e), \sum_{e\in(\overline{S}:S)} C(e)\} = min(17, 4) = 4 \neq 12$, **False**.

Therefore, maximum flow is not equal to the capacity of the minimum cut for this definition.