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HW 3

Q.1 Given $f_{XY}(x,y) = \begin{cases} 2 & 0<x<y<1 \\ 0 & o.w \end{cases}$

→ $W = X + Y$ Let's find CDF of W $F_W(w)$.

$X = W - Y$

$Z = Y$

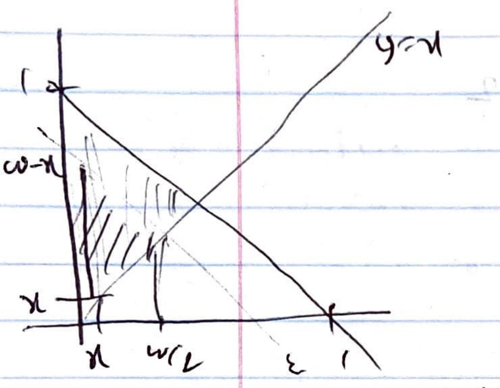


fig ①

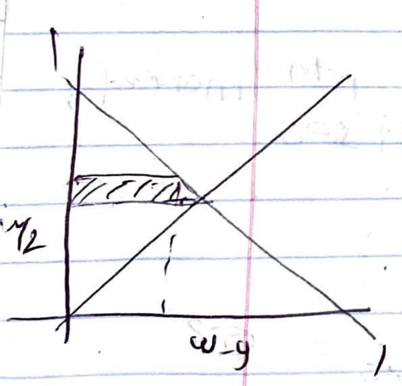
Here, from fig ①

$$F_W(w) = \int_0^{w/2}\int_x^{w-x} 2\,dy\,dx$$

$$= \int_0^{w/2} 2(w-x-x)\,dx$$

$$= \int_0^{w/2} (2w-4x)\,dx = 2wx - \frac{4x^2}{2}\Big|_0^{w/2}$$

$$= w^2 - \frac{w^2}{2} = \frac{w^2}{2}$$



for value in region $1 \le w < 2$

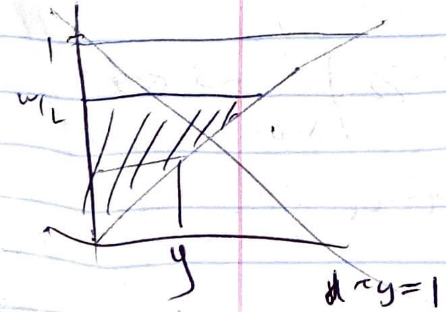
$$F_W(w) = \int_0^{w/2}\int_0^{y} 2\,dx\,dy + \int_{w/2}^{1}\int_0^{w-y} 2\,dx\,dy$$

$$= \int_0^{w/2} 2y\,dy + \int_{w/2}^{1} 2(w-y)\,dy$$

$$= \frac{w^2}{4} + (2wy - y^2)\Big|_{w/2}^{1}$$

$$= \frac{w^2}{4} + 2w - 1 - \left(w^2 - \frac{w^2}{4}\right)$$

$$= 2w - 1 - \frac{w^2}{2}$$



Calculating CDF of W over various ranges of values w was first step.

Now,

$$F_W(w) = \begin{cases} 0 & w<0 \\ w^2/2 & 0 \leq w < 1 \\ 2w-1-\frac{w^2}{2} & 1 \leq w \leq 2 \\ 1 & w>2 \end{cases}$$

$$f_W(w) = \begin{cases} w & 0 \leq w \leq 1 \\ 2-w & 1 \leq w \leq 2 \\ 0 & \text{o.w.} \end{cases}$$

Q2 X is a Gaussian$(0,\sigma)$

Using the moment generating function of X,

$\phi_X(s) = e^{\sigma^2 s^2/2}$ we can find nth moment, by taking nth derivation of $\phi_X(s)$ at $s=0$.

$$E[X] = \sigma^2 s e^{\sigma^2 s^2/2}\Big|_{s=0} = 0$$

$$E[X^2] = \sigma^2 e^{\sigma^2 s^2/2} + \sigma^4 s^2 e^{\sigma^2 s^2/2}\Big|_{s=0} = \sigma^2$$

For $$E[X^3] = (3\sigma^4 s + \sigma^6 s^3) e^{\sigma^2 s^2/2}\Big|_{s=0} = 0$$

$$E[X^4] = (3\sigma^4 + 6\sigma^6 s^2 + \sigma^8 s^4) e^{\sigma^2 s^2/2}\Big|_{s=0} = 3\sigma^4$$

To calculate Y, we define $Y = X + \mu$

so Y is Gaussian(μ, σ).

Now, $E[Y^2] = E[(X+\mu)^2] = E[X^2 + 2\mu X + \mu^2] = \sigma^2 + \mu$

$E[Y^3] = E[X^3 + 3\mu X^2 + 3\mu^2 X + \mu^3] = 3\mu\sigma^2 + \mu^3$

$E[Y^4] = E[X^4 + 4\mu X^3 + 6\mu^2 X^2 + 4\mu^3 X + \mu^4]$

$= 3\sigma^4 + 6\mu^2\sigma^2 + \mu^4$

Q 3

$$f_X(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{else,} \end{cases} \qquad f_Y(y) = \begin{cases} 3y^2 & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

a) $$E[X] = \int_{-\infty}^{\infty} x f_x(x)\, dx = \int_0^1 x \cdot 2x\, dx = \frac{2x^3}{3}\Big|_0^1$$

$$= 2/3$$

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \int_0^1 3y^3\, dy = \frac{3y^4}{4}\Big|_0^1 = 3/4$$

event $A = \{x > y\}$

$$E[X|A] = \iint_{-\infty}^{\infty} x f_{X,Y/A}(x,y/A)\, dy\, dx$$

for this we need $f_{X,Y/A}(x,y/A)$

we will first calculate the joint PDF $f_{X,Y}(x,y)$

$$f_{X,Y}(x,y) = \begin{cases} 6xy^2 & 0 \leq x, y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Now, we calculate probability of event A

$$P[A] = \iint_{x>y} f_{X,Y}(x,y)\, dy\, dx = \int_0^1 \int_0^x 6xy^2\, dy\, dx$$

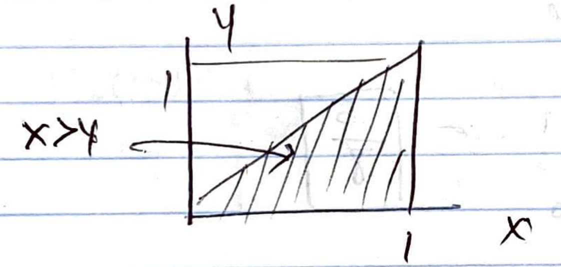
$$= \int_0^1 2x^4\, dx = 2/5$$

$\therefore P[A] = 2/5$

$$f_{X,Y|A}(x,y) = \begin{cases} \dfrac{f_{X,Y}(x,y)}{P[A]} & x,y \in A \\ 0 & \text{otherwise} \end{cases}$$

$$f_{X,Y|A}(x,y) = \begin{cases} \dfrac{6xy^2}{2/5} \Rightarrow 15xy^2 & x,y \in A \\ 0 & \text{otherwise} \end{cases}$$

Since our distribution looks like a triangle now



Therefore, X, Y are no longer independent given event A.

b) $E[X|A] = \iint x \, f_{X,Y|A}(x,y|a) \, dy \, dx$

$= \int_0^1 \int_0^x x \cdot 15xy^2 \, dy \, dx$

$= 15 \int_0^1 x^2 \int_0^x y^2 \, dy \, dx = 15 \int_0^1 x^2 \cdot \frac{x^3}{3} \, dx$

$= \frac{15x^6}{18} \Big|_0^1 = \boxed{\frac{5}{6}}$

$$E[Y|A] = \int_{-\infty}^{\infty}\int y\, f_{X,Y|A}(x,y|a)\, dy\, dx$$

$$= \int_0^1 \int_0^x y\, 15xy^2\, dy\, dx = 15\int_0^1 x \int y^3\, dy\, dx$$

$$= 15\int_0^1 x\left(\frac{y^4}{4}\Big|_0^x\right) dx$$

$$= 15\int \frac{x^5}{4}\, dx$$

$$= \frac{15x^6}{4\times 6}\Big|_0^1 = \boxed{\frac{5}{8}}$$

4.84 First we note that W has an $N[\mu, \sigma^2]$ distribution, integral we wish to evaluate is,

$$I = \int_{-\infty}^{\infty} f_W(w)\,dw = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-(w-\mu)^2/2\sigma^2}\,dw,$$

(a) Using substitution $x = (w-\mu)/\sigma$ we have ~~dx/dw~~ $dx = dw/\sigma$ and

$$I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\,dx.$$

(b) when we write I^2 as product integral, we use y to denote other variable of integration,

$$I^2 = \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\,dx\right)\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2}\,dy\right)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\,dx\,dy.$$

(c) By changing polar coordinates, $x^2+y^2 = r^2$.

and $dx\,dy = r\,dr\,d\theta$ so that,

$$I^2 = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} e^{-r^2/2}\, r\,dr\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} -e^{-r^2/2}\Big|_0^{\infty} d\theta = \frac{1}{2\pi}\int_0^{2\pi} d\theta = 1$$

Q.5 $f_{x,y}(x,y) = ce^{-(x^2/8)-y^2/18)}$

we know that,

$$c = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}$$

Because the exponent $f_{x,y}(x,y)$ doesn't contain any cross terms we know that ρ must be zero,

~~and~~

$$\left(\frac{x-E[X]}{\sigma_X}\right)^2 = \frac{x^2}{8} \qquad \left(\frac{y-E[Y]}{\sigma_Y}\right)^2 = \frac{y^2}{18}$$

$\therefore$ $E[X] = E[Y] = 0$

$\sigma_X = \sqrt{8}$ $\qquad \sigma_Y = \sqrt{18}$

$$\therefore \quad c = \frac{1}{2\pi\sqrt{8}\sqrt{18}} = \frac{1}{24\pi}$$

Since $\rho = 0$, we also find that X and Y are independent.

9.6

$$f_{X,Y}(x,y)=\begin{cases}2 & 0\leq y\leq x\leq 1\\ 0 & \text{otherwise}\end{cases}$$

$w = y/x$

a) S_W range of w

we know $x \geq 0$, $y \geq 0$

$\therefore w \geq 0$

when $y = 0$, $w = 0$

also, $y \leq x$ $\therefore w = \frac{y}{x} \leq 1$

Therefore, range of W is $S_W = \{w \mid 0 \leq w \leq 1\}$

b) CDF $F_W(w)$ for $0 \leq w \leq 1$

$F_W(w) = P[Y/X \leq w]$ $\quad P[Y \leq wX]$

$$= \int_0^1 \int_0^{wx} 2\, dy\, dx = \int_0^1 2y \Big|_0^{wx} dx = \int_0^1 2wx\, dx$$

$$= wx^2 \Big|_0^1 = w$$

$\therefore F_W(w) = 0$ for $w < 0$
$F_W(w) = 1$ for $w > 1$

$$F_W(w) = \begin{cases} 0 & w < 0 \\ w & 0 \leq w \leq 1 \\ 1 & w > 1 \end{cases}$$

We find pdf $f_W(w)$ by taking derivative of CDF $F_W(w)$

$$f_W(w) = \begin{cases} 1 & 0 \leq w \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$E[W] = \int_{-\infty}^{\infty} w f_W(w)\, dw$$

$$= \int_0^1 w\, dw = \frac{w^2}{2} \Big|_0^1 = \frac{1}{2}$$

$E[W] = 1/2$

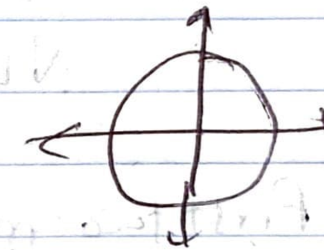
Also by $f_W(w)$ we can see that it is a normal distribution.

(Q.7) Gaussian

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

$W = X^2 + Y^2$

$P(W \le w) = P(X^2+Y^2 \le w)$



$$= \iint f_{X,Y}(x,y)\,dx\,dy$$

$$= \iint f_X(x) \cdot f_Y(y)\,dx\,dy \rightarrow \text{independent.}$$

$$= \iint \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

$$= \iint \frac{1}{(\sqrt{2\pi})^2} \cdot e^{-(x^2+y^2)/2}\, dx\, dy$$

radius of circle $= r$

$x^2+y^2=r^2$

$r=\sqrt{x^2+y^2} \Rightarrow r=\sqrt{w}$

As $x \to r$; $y \to \sqrt{r^2-x^2}$

$$F_W = \int_0^{2\pi}\int_0^{\sqrt{w}} \frac{1}{2\pi} \cdot e^{-r^2/2} \cdot r\, dr\, d\theta$$

Let r^2 be K

$\Rightarrow 2r\, dr = dk \Rightarrow K$ limits 0 to w

$$F_W = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{w} e^{-k/2} \cdot \frac{dk}{2}\, d\theta$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\left[\frac{e^{-k/2}}{(-1/2)}\right]_0^w d\theta$$

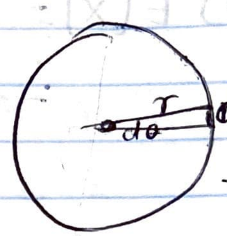
$$= \frac{1}{2\pi}\int_0^{2\pi} -e^{-w/2} + e^0\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} 1-e^{-w/2}\, d\theta = \frac{1}{2\pi}\left[\theta - e^{-w/2}\theta\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left[1-e^{-w/2}\right]-0\right] = 1-e^{-w/2}$$

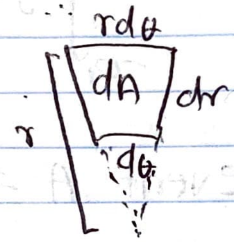
$$\therefore F_W(w) \Rightarrow \begin{cases} 1-e^{-w/2}, & w \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

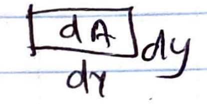
~~Use the proof~~ :



$\therefore dr = r\, d\theta$

~~$\therefore dx^2+dy^2=dr^2$~~





$\therefore dx\, dy = r\, dr\, d\theta$

8) For constant $a>0$, zero mean Laplace random variable X has PDF.

$$f_X(x) = \frac{a}{2} e^{-a|x|} \quad -\infty < x < \infty$$

The moment generating function of X is,

$$\phi_X(s) = E[e^{sX}] = \frac{a}{2}\int_{-\infty}^{0} e^{sx} e^{ax}\,dx + \frac{a}{2}\int_{0}^{\infty} e^{sx} e^{-ax}\,dx$$

$$= \frac{a}{2}\left.\frac{e^{(s+a)x}}{s+a}\right|_{-\infty}^{0} + \frac{a}{2}\left.\frac{e^{(s-a)x}}{s-a}\right|_{0}^{\infty}$$

$$= \frac{a}{2}\left(\frac{1}{s+a} - \frac{1}{s-a}\right)$$

$$= \frac{a^2}{a^2-s^2}$$

9) According to the theorem,

$$Var[W] = Var[X] + Var[Y] + 2\,Cov[X,Y]$$

First two moments of X are,

$$E[X] = \int_0^1\int_0^{1-x} 2x\,dy\,dx = \int_0^1 2x(1-x)\,dx = 1/3$$

$$E[X^2] = \int_0^1\int_0^{1-x} 2x^2\,dy\,dx = \int_0^1 2x^2(1-x)\,dx = 1/6$$

Thus the variance of X is $Var[X] = E[X^2] - (E[X])^2 = 1/18$

By symmetry it should be $E[Y] = E[X] = 1/3$
and $Var[Y] = Var[X] = 1/18$.

$\therefore E[XY] = \int_0^1 \int_0^{1-x} 2xy \, dy \, dx = \int_0^1 x(1-x)^2 \, dx = 1/12$

$Cov[X, Y] = E[XY] - E[X]E[Y] = 1/12 - (1/3)^2 = -1/36$

Finally, variance of sum, $W = X + Y$

$Var[W] = Var[X] + Var[Y] + 2Cov[X, Y] = 2/18 - 2/36$
$= 1/18$