i) (a) $P(R_3) = 1/6$, $P(W_2) = 1/6$

$P(R_3 W_2) = 1/6 \times 1/6 = 1/36 = 0.028$

(b) Sum 5 $\Rightarrow$ $R_1 W_4$, $R_4 W_1$
$R_2 W_3$, $R_3 W_2$

$\therefore$ Total events $6 \times 6 = 36$

$4/36 = 1/9 = 0.11$

2)

Sample space

$$S = \{234, 243, 324, 342, 423, 432\}.$$

Events,

$E_1 = \{234, 243, 423, 432\}$ $O_1 = \{324, 342\}$

$E_2 = \{243, 324, 342, 423\}$ $O_2 = \{234, 432\}$

$E_3 = \{234, 324, 342, 432\}$ $O_3 = \{243, 432\}$

ⓐ

$$P[E_2 | E_1] = \frac{P[E_2 E_1]}{P[E_1]} = \frac{P[243, 423]}{P[234, 243, 423, 432]} = \frac{2/6}{4/6} = 1/2$$

ⓑ

$$P[E_1 E_2 | E_3] = \frac{P[E_1 E_2 E_3]}{P[E_3]} = 0$$

(c) $P[E_2 | O_1] = \dfrac{P[O_1 E_2]}{P[O_1]} = \dfrac{P[O_1]}{P[O_1]} = 1$

(d) $P[O_2 | O_1] = \dfrac{P[O_1 O_2]}{P[O_1]} = 0$

3) Experiment with equiprobable Outcomes

Sample space: $S = \{1, 2, 3, 4\}$

$P(s) = 1/4$ for $s \in S$.

Let A_i, $i = 1, 2, 3$ be 3 events in S

A, B, C

Now to show that $\cancel{A_1, A_2,} A_3$ are pairwise independent but not independent we have to show..

$\cancel{P\{A_i A}$ where $A = \{1, 2\}$ $B = \{2, 3\}$ $C = \{3, 1\}$

For pairwise independence, we have to show

$P(E_A \cap E_B) = P(E_A)\, P(E_B)$ for all pairs.

$P(A) = \frac{1}{2}$ $P(B) = \frac{1}{2}$ $P(C) = \frac{1}{2}$

$P(A \cap B) = \frac{1}{4}$ $P(B \cap C) = \frac{1}{4}$ $P(C \cap A) = \frac{1}{4}$

we can see,

$$P(A \cap B) = P(A)\,P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

Similarly

$$P(B \cap C) = P(B)\,P(C) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

and

$$P(C \cap A) = P(C)\,P(A) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

Hence events are pairwise independent.

But, these 3 event are not independend

$P(A \cap B \cap C) = P(A)\,P(B)\,P(C)$ for events to be independent.

$\therefore P(A \cap B \cap C) = 0$

But $P(A)\,P(B)\,P(C) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$

$\therefore$

$$P(A \cap B \cap C) \neq P(A)\,P(B)\,P(C)$$

$\therefore$ events are not independent

4] For independent event A & B —— (1)

(a) A and B^c are independent.
(b) A^c and B are independent } Same proof.

If we prove (a) ; (b) can also be proved by simply changing argument variables name.

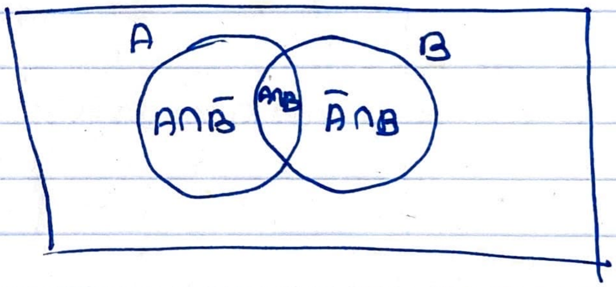
→ By defination from (1)

$$P(A \cap B) = P(A)P(B) \quad \text{i.e. } P(A|B) = P(A)$$

So we need to show

$$P(\bar{A} \cap B) = P(\bar{A})\,P(B) \quad \text{—— (I)}$$

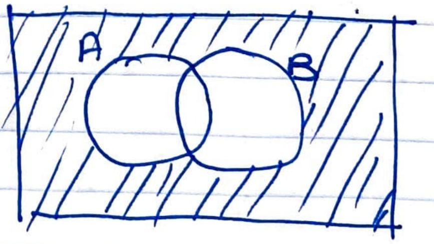
From diagram,



$$\begin{aligned} P(\bar{A} \cap B) &= P(B) - P(A \cap B) \\ &= P(B) - P(A)\,P(B) \\ &= P(B)\,(1 - P(A)) \\ &= P(B)\,P(\bar{A}) \end{aligned}$$

∴ If A & B are independent, then so are $\bar{A}$ and B.
(and so are A and $\bar{B}$)

(c) A^c and B^c are independent



we know that,

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\therefore P(\overline{A \cup B}) = P(\bar{A} \cap \bar{B}) \quad \text{—— (II)}$$

— (De Morgan's Law)

By defination, $P(A \cap B) = P(A)\,P(A)$ i.e. $P(A|B) = P(A)$

we need to prove,

$$P(\bar{A} \cap \bar{B}) = P(\bar{A})\,P(\bar{B}) \quad \text{—— (III)}$$

Q.5 Given X is non negative integer - valued random variable

prove that

$$E[X] = \sum_{K=0}^{\infty} P[X > K]$$

Let's write

$$P[X = K] = P_K \quad \text{for } K = 0, 1, 2, \dots \infty$$

$$E[X] = \sum_{K=0}^{\infty} P[X > K]$$

from definition of Expectation

$E[X] = \sum (x_1 P_1, x_2 P_2, \dots, x_n P_n)$ where x is a random variable with probability function $f(v)$, p is the probability of occurence.

$$\therefore E[X] = (0 \times P_0) + (1 \times P_1) + (2 \times P_2) \dots (\infty \times P_\infty)$$

$$= P_1 + (P_2 + P_2) + (P_3 + P_3 + P_3) + \dots$$

$$= \underbrace{(P_1 + P_2 + \dots + P_\infty)}_{P[X \geq 1]} + \underbrace{(P_2 + P_3 + \dots + P_\infty)}_{P[X \geq 2]} + \dots$$

$$= P[X \geq 1] + P[X \geq 2] + \dots + P[X \geq \infty]$$

$$E[X] = \sum_{i=1}^{\infty} P[X \geq i] \Rightarrow \sum_{K=0}^{\infty} P[X > K]$$

6. If X is binomial with parameter n, p, then variance and standard deviation of X is,

$$\sigma^2_x = np(1-p)$$

$$\sigma_x = \sqrt{np(1-p)}$$

(a) ~~σx~~ here n = 5, p = 0.5

$$\therefore \sigma_x = \sqrt{5 \times 0.5(1-0.5)}$$

$$= \sqrt{5} \times 0.5$$

$$= 1.12$$

(b) The standard deviation is 1.12 from previous question.

The expected ~~value~~ value (mean) is given by $\mu = np$

$$\mu = 5 \times 0.5 = 2.5$$

$$\therefore [\mu - \sigma, \mu + \sigma] = [1.38, 3.62]$$

$$\therefore P[\mu - \sigma_x \leq X \leq \mu + \sigma_x] \approx P[1.38 \leq X \leq 3.62]$$

$$= P\left[\frac{1.38 - 2.5}{1.12} \leq \frac{X - \mu}{\sigma} \leq \frac{3.62 - 2.5}{1.12}\right]$$

$$= P[-1 \leq Z \leq 1]$$

$$= P(-1 < Z < 0) + P(0 < Z < 1)$$

$$= 0.3413 + 0.3413$$

$$= 0.6826$$

$$= \underline{\underline{68.26\%}}$$

7) the variance of $Y = aX + b$ is $Var[Y] = a^2 Var[X]$ – to prove.

$\therefore Var(aX+b) = E[(aX+b)^2] - [E(aX+b)]^2$

$= E(a^2x^2 + 2abx + b^2) - [aE(X) + E(b)]^2$

$= a^2 E(x^2) + 2ab E(X) + b^2 - [a^2 E(X)^2 + 2ab E(X) + b^2]$

$= a^2 E(x^2) - a^2 [E(X)]^2$

$= a^2 [E(x^2) - E(X)^2]$

$Var(aX+b) = a^2 Var(X)$

$\therefore Var[Y] = a^2 Var(X)$ — hence proved.

8) random variable X

Given $E[X] = \mu_x$ $Var[X] = \sigma_x^2$

Find expected value and variance of Y where

$$Y = \frac{X - \mu_x}{\sigma_x}$$

$\rightarrow$ $E[Y] = E\left[\frac{X - \mu_x}{\sigma_x}\right]$

$$= E\left[\frac{X}{\sigma_x} - \frac{\mu_x}{\sigma_x}\right] = \frac{E[X]}{\sigma_x} - \frac{\mu_x}{\sigma_x}$$

given $E[X] = \mu_x$

$$\therefore E[Y] = \frac{\mu_x}{\sigma_x} - \frac{\mu_x}{\sigma_x} = 0$$

$$Var[Y] = E[Y^2] - (E[Y])^2$$

$$= E\left[\left(\frac{X - \mu_x}{\sigma_x}\right)^2\right] - 0$$

$$= E\left[\frac{X^2 + \mu_x^2 - 2X\mu_x}{\sigma_x^2}\right] = \frac{E[X^2]}{\sigma_x^2} + \frac{\mu_x^2}{\sigma_x^2} - \frac{2\mu_x E[X]}{\sigma_x^2}$$

given $E[X] = \mu_x$ substituting:

$$Var[Y] = \frac{E[X^2]}{\sigma^2} + \frac{\mu_x^2}{\sigma_x^2} - \frac{2\mu_x^2}{\sigma_x^2}$$

$$E[X^2] = Var[X] + (E[X])^2$$

$$Var[Y] = \frac{Var[X] + (E[X])^2}{\sigma^2} - \frac{\mu^2}{\sigma^2}$$

Since $Var[X] = \sigma_x^2$ and $E[X] = \mu_x$

$$\boxed{\therefore Var[Y] = \frac{\sigma^2 + \mu^2}{\sigma^2} - \frac{\mu^2}{\sigma^2} = 1}$$

Q. 9 Given random variable K and poisson distribution with parameter α

Poisson PMF is given by

$$P_K(x) = \begin{cases} \dfrac{\alpha^x . e^{-\alpha}}{x!} & x = 0, 1, 2, \dots \\ 0 & \text{otherwise.} \end{cases}$$

$$E[K] = \sum_{x=0}^{\infty} x P_K(x) = \sum_{x=0}^{\infty} x . \frac{e^{-\alpha} . \alpha^x}{x!}$$

$$= e^{-\alpha} \sum_{x=1}^{\infty} \frac{\alpha^x}{(x-1)!} . \frac{\alpha}{\alpha} \Rightarrow e^{-\alpha} \alpha \sum_{x=1}^{\infty} \frac{\alpha^{(x-1)}}{(x-1)!}$$

$$E[K] = e^{-\alpha} . \alpha e^{\alpha} = \alpha$$

$$Var[K] = E[K^2] - (E[K])^2$$

$$E[K^2] = E[K(K-1)] + E[K]$$

$$= \sum_{x=0}^{\infty} x^2 \frac{1}{x!} \alpha^x . e^{-\alpha}$$

$$= \alpha e^{-\alpha} . \sum_{x=1}^{\infty} \frac{x}{(x-1)!} \alpha^{x-1}$$

$$= \alpha e^{-\alpha} \left(\sum_{x=1}^{\infty} \frac{(x-1)}{(x-1)!} \alpha^{x-1} + \sum_{x=1}^{\infty} \frac{1}{(x-1)!} . \alpha^{x-1} \right)$$

$$E(K^2) = \alpha e^{-\alpha}\left(\alpha \sum_{x=2}^{\infty} \frac{\alpha^{x-2}}{(x-2)!} + \sum_{x=1}^{\infty} \frac{\alpha^{x-1}}{(x-1)!}\right)$$

$$= \alpha e^{-\alpha}\left(\alpha \sum_{i=0}^{\infty} \frac{1}{i!}\alpha^{i} + \sum_{j=0}^{\infty} \frac{\alpha^{j}}{j!}\right)$$

$$= \alpha e^{-\alpha}\left(\alpha e^{\alpha} + e^{\alpha}\right) = \alpha(\alpha+1)$$

$$E[K^2] = \alpha^2 + \alpha$$

Now, $Var[K] = E[K^2] - (E[K])^2$

$E[K^2] = \alpha^2 + \alpha \qquad E[K] = \alpha$

$$\boxed{\therefore \quad Var[K] = \alpha^2 + \alpha - \alpha^2 = \alpha}$$