1.

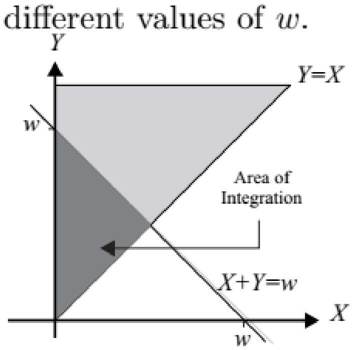
Find the PDF of $W = X + Y$ when $X$ and $Y$ have the joint PDF

$$f_{X,Y}(x, y) = \begin{cases} 2 & 0 \leq x \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

The joint PDF of $X$ and $Y$ is

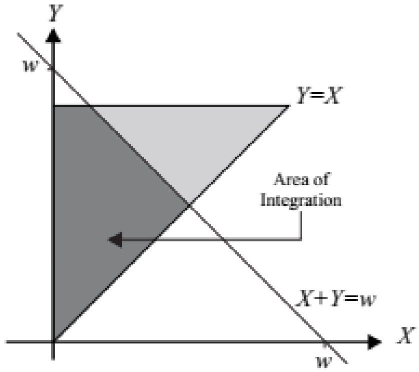
$$f_{X,Y}(x, y) = \begin{cases} 2 & 0 \leq x \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

We wish to find the PDF of $W$ where $W = X + Y$. First we find the CDF of $W$, $F_W(w)$, but we must realize that the CDF will require different integrations for different values of $w$.




For values of $0 \leq w \leq 1$ we look to integrate the shaded area in the figure to the right.

$$F_W(w) = \int_0^{\frac{w}{2}} \int_x^{w-x} 2\, dy\, dx = \frac{w^2}{2}. \tag{2}$$




For values of $w$ in the region $1 \le w \le 2$ we look to integrate over the shaded region in the graph to the right. From the graph we see that we can integrate with respect to $x$ first, ranging $y$ from 0 to $w/2$, thereby covering the lower right triangle of the shaded region and leaving the upper trapezoid, which is accounted for in the second term of the following expression:

$$\begin{aligned} F_W(w) &= \int_0^{\frac{w}{2}} \int_0^y 2\,dx\,dy + \int_{\frac{w}{2}}^1 \int_0^{w-y} 2\,dx\,dy \\ &= 2w - 1 - \frac{w^2}{2}. \end{aligned} \tag{3}$$

Putting all the parts together gives the CDF

$$F_W(w) = \begin{cases} 0 & w < 0, \\ \frac{w^2}{2} & 0 \le w \le 1, \\ 2w - 1 - \frac{w^2}{2} & 1 \le w \le 2, \\ 1 & w > 2, \end{cases} \tag{4}$$

and (by taking the derivative) the PDF

$$f_W(w) = \begin{cases} w & 0 \le w \le 1, \\ 2 - w & 1 \le w \le 2, \\ 0 & \text{otherwise}. \end{cases} \tag{5}$$

2.

Let $X$ be a Gaussian $(0, \sigma)$ random variable. Use the moment generating function to show that

$$\begin{aligned} \mathrm{E}[X] &= 0, & \mathrm{E}[X^2] &= \sigma^2, \\ \mathrm{E}[X^3] &= 0, & \mathrm{E}[X^4] &= 3\sigma^4. \end{aligned}$$

Let $Y$ be a Gaussian $(\mu, \sigma)$ random variable. Use the moments of $X$ to show that

$$\begin{aligned} \mathrm{E}\left[Y^2\right] &= \sigma^2 + \mu^2, \\ \mathrm{E}\left[Y^3\right] &= 3\mu\sigma^2 + \mu^3, \\ \mathrm{E}\left[Y^4\right] &= 3\sigma^4 + 6\mu\sigma^2 + \mu^4. \end{aligned}$$

Using the moment generating function of $X$, $\phi_X(s) = e^{\sigma^2 s^2/2}$. We can find the $n$th moment of $X$, $\mathrm{E}[X^n]$ by taking the $n$th derivative of $\phi_X(s)$ and setting $s = 0$.

$$\mathrm{E}\left[X\right] = \sigma^2 s e^{\sigma^2 s^2/2}\Big|_{s=0} = 0, \tag{1}$$

$$\mathrm{E}\left[X^2\right] = \sigma^2 e^{\sigma^2 s^2/2} + \sigma^4 s^2 e^{\sigma^2 s^2/2}\Big|_{s=0} = \sigma^2. \tag{2}$$

Continuing in this manner we find that

$$\mathrm{E}\left[X^3\right] = \left(3\sigma^4 s + \sigma^6 s^3\right) e^{\sigma^2 s^2/2}\Big|_{s=0} = 0, \tag{3}$$

$$\mathrm{E}\left[X^4\right] = \left(3\sigma^4 + 6\sigma^6 s^2 + \sigma^8 s^4\right) e^{\sigma^2 s^2/2}\Big|_{s=0} = 3\sigma^4. \tag{4}$$

To calculate the moments of $Y$, we define $Y = X + \mu$ so that $Y$ is Gaussian $(\mu, \sigma)$. In this case the second moment of $Y$ is

$$\mathrm{E}\left[Y^2\right] = \mathrm{E}\left[(X+\mu)^2\right] = \mathrm{E}\left[X^2 + 2\mu X + \mu^2\right] = \sigma^2 + \mu^2. \tag{5}$$

Similarly, the third moment of $Y$ is

$$\begin{aligned} \mathrm{E}\left[Y^3\right] &= \mathrm{E}\left[(X+\mu)^3\right] \\ &= \mathrm{E}\left[X^3 + 3\mu X^2 + 3\mu^2 X + \mu^3\right] = 3\mu\sigma^2 + \mu^3. \end{aligned} \tag{6}$$

Finally, the fourth moment of $Y$ is

$$\begin{aligned} \mathrm{E}\left[Y^4\right] &= \mathrm{E}\left[(X+\mu)^4\right] \\ &= \mathrm{E}\left[X^4 + 4\mu X^3 + 6\mu^2 X^2 + 4\mu^3 X + \mu^4\right] \\ &= 3\sigma^4 + 6\mu^2\sigma^2 + \mu^4. \end{aligned} \tag{7}$$

3.

$X$ and $Y$ are independent random variables with PDFs

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1, \\ 0 & \text{otherwise}, \end{cases}$$

$$f_Y(y) = \begin{cases} 3y^2 & 0 \le y \le 1, \\ 0 & \text{otherwise}. \end{cases}$$

Let $A = \{X > Y\}$.

(a) What are $\mathrm{E}[X]$ and $\mathrm{E}[Y]$?

(b) What are $\mathrm{E}[X|A]$ and $\mathrm{E}[Y|A]$?

$X$ and $Y$ are independent random variables with PDFs

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1, \\ 0 & \text{otherwise}, \end{cases} \qquad f_Y(y) = \begin{cases} 3y^2 & 0 \le y \le 1, \\ 0 & \text{otherwise}. \end{cases} \tag{1}$$

For the event $A = \{X > Y\}$, this problem asks us to calculate the conditional expectations $\mathrm{E}[X|A]$ and $\mathrm{E}[Y|A]$. We will do this using the conditional joint PDF $f_{X,Y|A}(x,y)$. Since $X$ and $Y$ are independent, it is tempting to argue that the event $X > Y$ does not alter the probability model for $X$ and $Y$. Unfortunately, this is not the case. When we learn that $X > Y$, it increases the probability that $X$ is large and $Y$ is small. We will see this when we compare the conditional expectations $\mathrm{E}[X|A]$ and $\mathrm{E}[Y|A]$ to $\mathrm{E}[X]$ and $\mathrm{E}[Y]$.

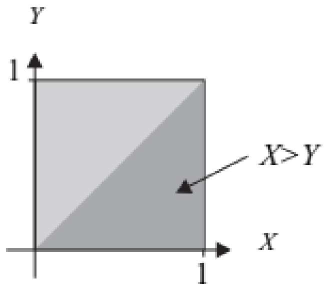
(a) We can calculate the unconditional expectations, $\mathrm{E}[X]$ and $\mathrm{E}[Y]$, using the marginal PDFs $f_X(x)$ and $f_Y(y)$.

$$\mathrm{E}[X] = \int_{-\infty}^{\infty} f_X(x)\, dx = \int_0^1 2x^2\, dx = 2/3, \tag{2}$$

$$\mathrm{E}[Y] = \int_{-\infty}^{\infty} f_Y(y)\, dy = \int_0^1 3y^3\, dy = 3/4. \tag{3}$$

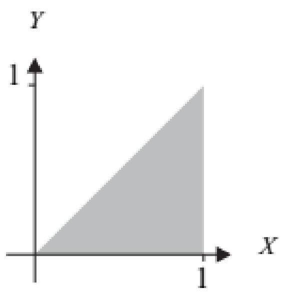
(b) First, we need to calculate the conditional joint PDF $ipdf X,Y|Ax,y$. The first step is to write down the joint PDF of $X$ and $Y$:

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y) = \begin{cases} 6xy^2 & 0 \le x \le 1, 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$




The event $A$ has probability

$$\begin{aligned} \mathrm{P}[A] &= \iint_{x>y} f_{X,Y}(x,y)\, dy\, dx \\ &= \int_0^1 \int_0^x 6xy^2\, dy\, dx \\ &= \int_0^1 2x^4\, dx = 2/5. \end{aligned} \tag{5}$$




The conditional joint PDF of $X$ and $Y$ given $A$ is

$$\begin{aligned} f_{X,Y|A}(x,y) &= \begin{cases} \frac{f_{X,Y}(x,y)}{\mathrm{P}[A]} & (x,y) \in A, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} 15xy^2 & 0 \le y \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{6}$$

The triangular region of nonzero probability is a signal that given $A$, $X$ and $Y$ are no longer independent. The conditional expected value of $X$ given $A$ is

$$\begin{aligned} \mathrm{E}[X|A] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{X,Y|A}(x,y|a)\, x,y\, dy\, dx \\ &= 15 \int_0^1 x^2 \int_0^x y^2\, dy\, dx \\ &= 5 \int_0^1 x^5\, dx = 5/6. \end{aligned} \tag{7}$$

Instructor's note: The first equality in (7) is a few typos. It should be the double integral of x times

Double integral of f_X,Y|A(x,y) dy x dx

The conditional expected value of $Y$ given $A$ is

$$\begin{aligned}
\mathrm{E}[Y|A] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f_{X,Y|A}(x,y)\ dy\, dx \\
&= 15 \int_0^1 x \int_0^x y^3\, dy\, dx \\
&= \frac{15}{4} \int_0^1 x^5\, dx = 5/8. \qquad (8)
\end{aligned}$$

We see that $\mathrm{E}[X|A] > \mathrm{E}[X]$ while $\mathrm{E}[Y|A] < \mathrm{E}[Y]$. That is, learning $X > Y$ gives us a clue that $X$ may be larger than usual while $Y$ may be smaller than usual.

4.

This problem outlines the steps needed to show that the Gaussian PDF integrates to unity. For a Gaussian $(\mu, \sigma)$ random variable $W$, we will show that

$$I = \int_{-\infty}^{\infty} f_W(w)\ dw = 1.$$

(a) Use the substitution $x = (w - \mu)/\sigma$ to show that

$$I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx.$$

(b) Show that

$$I^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\, dx\, dy.$$

(c) Change to polar coordinates to show that $I^2 = 1$.

First we note that since $W$ has an $N[\mu, \sigma^2]$ distribution, the integral we wish to evaluate is

$$I = \int_{-\infty}^{\infty} f_W(w)\, dw = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-(w-\mu)^2/2\sigma^2}\, dw. \tag{1}$$

(a) Using the substitution $x = (w - \mu)/\sigma$, we have $dx = dw/\sigma$ and

$$I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx. \tag{2}$$

(b) When we write $I^2$ as the product of integrals, we use $y$ to denote the other variable of integration so that

$$\begin{aligned} I^2 &= \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx\right) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2}\, dy\right) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\, dx\, dy. \end{aligned} \tag{3}$$

(c) By changing to polar coordinates, $x^2 + y^2 = r^2$ and $dx\, dy = r\, dr\, d\theta$ so that

$$\begin{aligned} I^2 &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} e^{-r^2/2} r\, dr\, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} -e^{-r^2/2}\Big|_0^{\infty}\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} d\theta = 1. \end{aligned} \tag{4}$$

5.

Random variables $X$ and $Y$ have joint PDF

$$f_{X,Y}(x, y) = ce^{-(x^2/8)-(y^2/18)}.$$

What is the constant $c$? Are $X$ and $Y$ independent?

$X$ and $Y$ have joint PDF

$$f_{X,Y}(x,y) = ce^{-(x^2/8)-(y^2/18)}. \tag{1}$$

The omission of any limits for the PDF indicates that it is defined over all $x$ and $y$. We know that $f_{X,Y}(x,y)$ is in the form of the bivariate Gaussian distribution so we look to Definition 5.10 and attempt to find values for $\sigma_Y$, $\sigma_X$, $E[X]$, $E[Y]$ and $\rho$. First, we know that the constant is

$$c = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}. \tag{2}$$

Equating the exponent of (1) with the bivariate Gaussian, we get

rho = 0, sigma_X = 2, sigma_Y = 3, and c=1/(12 pi).

6.

Random variables $X$ and $Y$ have joint PDF

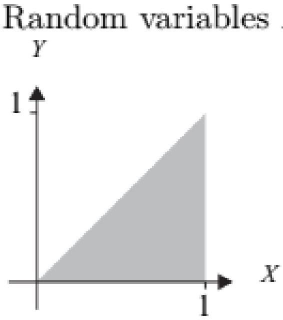
$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le y \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $W = Y/X$.

(a) What is $S_W$, the range of $W$?

(b) Find $F_W(w)$, $f_W(w)$, and $E[W]$.

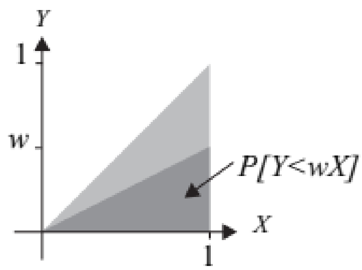
Random variables $X$ and $Y$ have joint PDF




$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le y \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a) Since $X$ and $Y$ are both nonnegative, $W = Y/X \geq 0$. Since $Y \leq X$, $W = Y/X \leq 1$. Note that $W = 0$ can occur if $Y = 0$. Thus the range of $W$ is $S_W = \{w | 0 \leq w \leq 1\}$.

(b) For $0 \leq w \leq 1$, the CDF of $W$ is




$$\begin{aligned} F_W(w) &= \mathrm{P}[Y/X \leq w] \\ &= \mathrm{P}[Y \leq wX] = w. \end{aligned} \tag{2}$$

The complete expression for the CDF is

$$F_W(w) = \begin{cases} 0 & w < 0, \\ w & 0 \leq w < 1, \\ 1 & w \geq 1. \end{cases} \tag{3}$$

By taking the derivative of the CDF, we find that the PDF of $W$ is

$$f_W(w) = \begin{cases} 1 & 0 \leq w < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

We see that $W$ has a uniform PDF over $[0, 1]$. Thus $\mathrm{E}[W] = 1/2$.

7.

$X$ and $Y$ are independent identically distributed Gaussian $(0, 1)$ random variables. Find the CDF of $W = X^2 + Y^2$.

Since $X_1$ and $X_2$ are iid Gaussian $(0, 1)$, each has PDF

$$f_{X_i}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}. \tag{1}$$

For $w < 0$, $F_W(w) = 0$. For $w \geq 0$, we define the disc

$$\mathcal{R}(w) = \left\{(x_1, x_2) | x_1^2 + x_2^2 \leq w\right\}. \tag{2}$$

and we write

$$F_W(w) = \mathrm{P}\left[X_1^2 + X_2^2 \le w\right] = \iint_{\mathcal{R}(w)} f_{X_1}(x_1)\, f_{X_2}(x_2)\; dx_1\, dx_2$$
$$= \iint_{\mathcal{R}(w)} \frac{1}{2\pi} e^{-(x_1^2+x_2^2)/2}\, dx_1\, dx_2. \tag{3}$$

Changing to polar coordinates with $r = \sqrt{x_1^2 + x_2^2}$ yields

$$F_W(w) = \int_0^{2\pi} \int_0^{\sqrt{w}} \frac{1}{2\pi} e^{-r^2/2} r\, dr\, d\theta$$
$$= \int_0^{\sqrt{w}} r e^{-r^2/2}\, dr = -e^{-r^2/2}\Big|_0^{\sqrt{w}} = 1 - e^{-w/2}. \tag{4}$$

Taking the derivative of $F_W(w)$, the complete expression for the PDF of $W$ is

$$f_W(w) = \begin{cases} 0 & w < 0, \\ \frac{1}{2} e^{-w/2} & w \ge 0. \end{cases} \tag{5}$$

Thus $W$ is an exponential $(\lambda = 1/2)$ random variable.

8.

For a constant $a > 0$, a Laplace random variable $X$ has PDF

$$f_X(x) = \frac{a}{2} e^{-a|x|}, \quad -\infty < x < \infty.$$

Calculate the MGF $\phi_X(s)$.

For a constant $a > 0$, a zero mean Laplace random variable $X$ has PDF

$$f_X(x) = \frac{a}{2} e^{-a|x|} \quad -\infty < x < \infty \tag{1}$$

The moment generating function of $X$ is

$$\begin{aligned}
\phi_X(s) = \mathrm{E}\left[e^{sX}\right] &= \frac{a}{2}\int_{-\infty}^{0} e^{sx}e^{ax}\,dx + \frac{a}{2}\int_{0}^{\infty} e^{sx}e^{-ax}\,dx \\
&= \frac{a}{2}\frac{e^{(s+a)x}}{s+a}\bigg|_{-\infty}^{0} + \frac{a}{2}\frac{e^{(s-a)x}}{s-a}\bigg|_{0}^{\infty} \\
&= \frac{a}{2}\left(\frac{1}{s+a} - \frac{1}{s-a}\right) \\
&= \frac{a^2}{a^2 - s^2}. \qquad (2)
\end{aligned}$$

9.

Random variables $X$ and $Y$ have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2 & x \geq 0, y \geq 0, x+y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

What is the variance of $W = X + Y$?

We can use the variance identity

$$\mathrm{Var}[W] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}\,[X,Y]. \qquad (1)$$

The first two moments of $X$ are

$$\mathrm{E}\,[X] = \int_0^1 \int_0^{1-x} 2x\,dy\,dx = \int_0^1 2x(1-x)\,dx = 1/3, \qquad (2)$$

$$\mathrm{E}\left[X^2\right] = \int_0^1 \int_0^{1-x} 2x^2\,dy\,dx = \int_0^1 2x^2(1-x)\,dx = 1/6. \qquad (3)$$

Thus the variance of $X$ is $\mathrm{Var}[X] = \mathrm{E}[X^2] - (\mathrm{E}[X])^2 = 1/18$. By symmetry, it should be apparent that $\mathrm{E}[Y] = \mathrm{E}[X] = 1/3$ and $\mathrm{Var}[Y] = \mathrm{Var}[X] = 1/18$. To find the covariance, we first find the correlation

$$\mathrm{E}\,[XY] = \int_0^1 \int_0^{1-x} 2xy\,dy\,dx = \int_0^1 x(1-x)^2\,dx = 1/12. \qquad (4)$$

The covariance is

$$\text{Cov}\,[X,Y] = \text{E}\,[XY] - \text{E}\,[X]\,\text{E}\,[Y] = 1/12 - (1/3)^2 = -1/36. \tag{5}$$

Finally, the variance of the sum $W = X + Y$ is

$$\begin{aligned}\text{Var}[W] &= \text{Var}[X] + \text{Var}[Y] - 2\,\text{Cov}\,[X,Y] \\ &= 2/18 - 2/36 = 1/18.\end{aligned} \tag{6}$$

For this specific problem, it's arguable whether it would easier to find $\text{Var}[W]$ by first deriving the CDF and PDF of $W$. In particular, for $0 \leq w \leq 1$,

$$\begin{aligned}F_W(w) &= \text{P}\,[X + Y \leq w] \\ &= \int_0^w \int_0^{w-x} 2\,dy\,dx \\ &= \int_0^w 2(w-x)\,dx = w^2.\end{aligned} \tag{7}$$

Hence, by taking the derivative of the CDF, the PDF of $W$ is

$$f_W(w) = \begin{cases} 2w & 0 \leq w \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

From the PDF, the first and second moments of $W$ are

$$\text{E}\,[W] = \int_0^1 2w^2\,dw = 2/3, \qquad \text{E}\left[W^2\right] = \int_0^1 2w^3\,dw = 1/2. \tag{9}$$

The variance of $W$ is $\text{Var}[W] = \text{E}[W^2] - (\text{E}[W])^2 = 1/18$. Not surprisingly, we get the same answer both ways.