EEE 554 HW2

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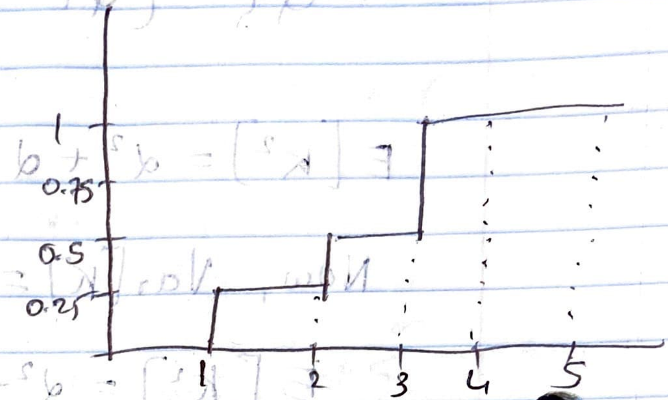
Q.1

(a) $P[Y<1]$ and $P[Y\leq 1]$

from diagram



(CDF)

$F_Y(1^-) = P[Y<1] = 0$

$F_Y(1^+) = P[Y\leq 1] = 0.25$

(b) $P[Y>2]$ and $P[Y\geq 2]$

we know,

$P[Y>2] = 1 - P[Y\leq 2] = 1 - F_Y(2^+)$

$= 1 - 0.5$ — from CDF graph

$= 0.5$

$P[Y\geq 2] = 1 - P[Y<2] = 1 - F_Y(2^-)$

$= 1 - 0.25$ — from CDF graph.

$= 0.75$

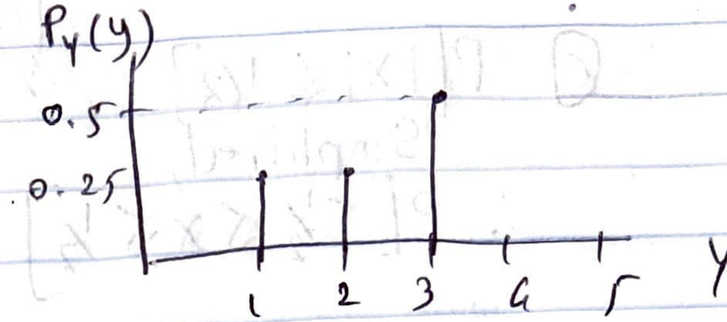
(c) $P[Y=3]$ and $P[Y>3]$

$P[Y=3] = F_Y[3^+] - F_Y[3^-]$

$= 1 - 0.5$

$= 0.5$

ⓓ $P_Y(y)$



∴ From given figure.

$$\therefore P_Y(y) = \begin{cases} 0.25 & y=1 \\ 0.25 & y=2 \\ 0.5 & y=3 \\ 0 & \text{o.w} \end{cases}$$

Q.2 Given:-

$$F_X(x) = \begin{cases} 0 & x < -1 \\ (x+1)/2 & -1 \leq x < 1, \\ 1 & x \geq 1 \end{cases} \quad \text{— CDF}$$

ⓐ $P[X > 1/2]$

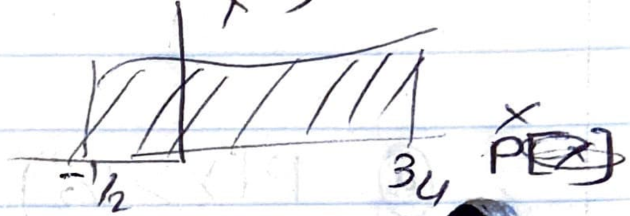
$= 1 - P[X \leq 1/2]$

$= 1 - F_X[1/2^{+}]$

$= 1 - \frac{3}{4}$ — from CDF

$= 1/4$ //

(b) $P[-\tfrac{1}{2} < X \leq \tfrac{3}{4}]$ $\longrightarrow$



$\therefore P[-\tfrac{1}{2} < X \leq \tfrac{3}{4}]$

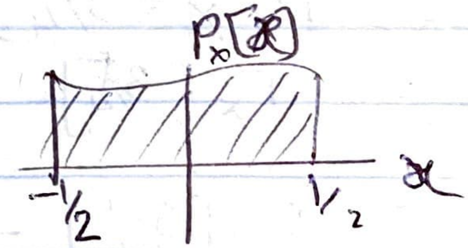
$$= P[X \leq \tfrac{3}{4}] - P[X \leq -\tfrac{1}{2}]$$

$$= \left(\frac{\tfrac{3}{4}+1}{2}\right) - \left(\frac{-\tfrac{1}{2}+1}{2}\right)$$

$$= \frac{7}{8} - \frac{1}{4}$$

$$= \tfrac{5}{8} \;//$$

(c) $P[|X| \leq \tfrac{1}{2}]$ $\longrightarrow$



$$\therefore = P[\tfrac{1}{2} \leq X] - P[-\tfrac{1}{2} < X]$$

$$= \left(\frac{\tfrac{1}{2}+1}{2}\right) - \left(\frac{-\tfrac{1}{2}+1}{2}\right)$$

$$= \frac{3}{4} - \frac{1}{4} = \frac{1}{2} \;//$$

(d) what is value of a such that $P[X \leq a] = 0.8$

$\therefore P[X \leq a] = 0.8$ — given

$\therefore P[X \leq a] = F_X(a^+)$

from CDF function

$$F_X(a^+) = \frac{a+1}{2} = 0.8 \text{ — given}$$

$$a + 1 = 1.6$$

$$\boxed{a = 0.6}$$

Q. 3 Given — PDF

$$f_X(x) = \begin{cases} ax^2 + bx & 0 \leq x \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

First integration should be '1' $\therefore \int_0^1 f_X(x)\,dx = 1$

$$\therefore \int_0^1 (ax^2 + bx)\,dx = 1$$

$$\left[\frac{ax^3}{3} + \frac{bx^2}{2}\right]_0^1 = 1$$

$$\therefore \frac{a}{3} + \frac{b}{2} = 1$$

$$\therefore \boxed{2a + 3b = 6} \text{ — (1)}$$

Second condition should be $f(x) \geq 0$

$\therefore ax^2+bx \geq 0$ where $0 \leq x \leq 1$

when $x=1$

$$\boxed{a+b \geq 0}$$

$\therefore$ $2a+3b=6$ and $a+b \geq 0$ are conditions on a and b necessary and sufficient gurantee $f_X(x)$ is a valid PDF.

Q 4

~~T minutes is expected time~~

Let x be random variabel denoting duration of phone call.

$\therefore E[X] = \tau$

Since it is exponential distribution, for an exponential distribution with parameter 'λ'

$$PDF = f_X(x) \Rightarrow \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{O.W.} \end{cases}$$

and $E[x] = \int_0^a x f_X(x) = \frac{1}{\lambda}$

but we are given that

$$E[x] = \tau \therefore \lambda = 1/\tau$$

$PDF \rightarrow f_X(x) = \left(\frac{1}{\tau}\right) e^{-x/\tau}$ for $x \geq 0$

Let the cost $= f(x_t)$

$f(x_t)$ is different for both A & B plan,

$\therefore$ Given 10 cent per min

$\therefore f_A(x_t) = 10x_t$

Plan B charges 99 cent till 20 min then 10 cent per minute

$$f_B(x_t) = \begin{cases} 99 & x_t \leq 20 \\ 99 + 10(x_t - 20) & x_t > 20 \end{cases}$$

$\therefore E[f_A(x_t)] = E[10x_t] = 10E[x_t] = 10\tau$

$\therefore E[f_B(x_t)]$

Let $p(x_t \leq 20) = 1-p$
$p(x_t > 20) = p$

$\therefore$ Using law of total Expectation

$\therefore E[f_B(x_t)] = pE[f_B(x_t) \mid x_t > 20)$
$+ (1-p) E[f_B(x_t) \mid x_t \leq 20]$

$\therefore f_B(x_t) = 99$ for $x_t \leq 20$ $(\because E[99] = 99)$

$\therefore E[f_B(x_t) \mid x_t \leq 20] = 99$

$f_B(x_t) = 99 + 10(x_t - 20)$ for $x_t > 20$
$+ (1-p)99$

~~E Ext~~
$E[f_B(x_t)] = 99 + 10pE[x_t - 20 \mid x_t > 20]$

$p = P[x_t > 20) = 1 - F_x[x \leq 20)$
$= 1 - (1 - e^{-x/\tau})$
$= 1 - (1 - e^{-20/\tau}) = e^{-20/\tau}$

$\therefore E[X_t - 20 \mid X_t \geqslant 20] = E[Y_t \mid Y_t \geqslant 0]$

where $Y_t = X_t - 20$

$\because$ The distribution is exponential and

$\therefore E[Y_t] = \tau$

So, $E[f_B(X_t)] = 99 + 10\tau e^{-20/\tau}$

$E[f_A(X_t)] = 10\tau$

Comparing these two,

$E[f_B(X_t)] \leq E[f_A(X_t)]$ for interval

$12.34 \leq x < \infty$

$\therefore$ If time τ is greater than 12.34 plan B is better.

for $\tau < 12.34$ min, plan A is better.

Q.5 Given

$V \sim U(-1,1)$

$\therefore f_V(v) = \begin{cases} \frac{1}{1-(-1)} = \frac{1}{2} & -1 \leq v \leq 1, \\ 0 & \text{o.w.} \end{cases}$

This voltage is processed by a clipper circuit

$L = \begin{cases} |v| & -0.5 \leq v \leq 0.5, \\ 0.5 & \text{o.w.} \end{cases}$

a) $P[L=0.5] = P[|V| \geq 0.5]$

$$= 1 - P[|V| < 0.5]$$

$$= 1 - P[-0.5 < V < 0.5]$$

$$= 1 - \int_{-0.5}^{0.5} f_V(v)\,dv$$

$$= 1 - \frac{1}{2}\int_{-0.5}^{0.5} dv = 1 - \frac{1}{2}(0.5 + 0.5)$$

$P[L=0.5] = 1/2$

b) $F_L(l) = P[L \leq l]$

$$P[L \leq l] = \begin{cases} 0 & \text{if } l < -0.5 \\ P[-0.5 \leq V < l] & -0.5 \leq l < 0.5 \\ & l \geq 0.5 \end{cases}$$

$$= \begin{cases} 0 & l < -0.5 \\ \int_{-0.5}^{l} f_V(v)\,dv & -0.5 \leq l < 0.5 \\ 1 & l \geq 0.5 \end{cases}$$

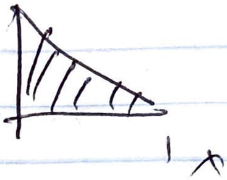
$$= \begin{cases} 0 & l < -0.5, \\ \int_{-0.5}^{l} \frac{1}{2}\,dv & -0.5 \leq l < 0.5 \\ 1 & l \geq 0.5 \end{cases}$$

$$F_L(l) = \begin{cases} 0 & l < 0.5, \\ 1/2\,(l + 0.5) & -0.5 \leq l < 0.5, \\ 1 & l \geq 0.5 \end{cases}$$

9.6 $f_{X,Y}(x,y) = \begin{cases} 2 & x \geq 0,\ y \geq 0,\ x+y \leq 1, \\ 0 & \text{o.w.} \end{cases}$

a) Are X and Y independent?

we plot the region



to check independence

we calculate marginal PDFs

$$f_X(x) = f_X(x,\infty) = \int_0^{1-x} 2\,dy = \begin{cases} 2(1-x) & 0 \leq x \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

Likewise,

$$f_Y(y) = f_Y(0,y) = \int_0^{1-y} 2\,dx = \begin{cases} 2(1-y) & 0 \leq y \leq 1 \\ 0 & \text{o.w.} \end{cases}$$

$$f_{XY}(x,y) \neq f_X(x)\, f_Y(y)$$

$$2 \neq 2(1-x) \cdot 2(1-y) = 4(1-x)(1-y)$$

$\therefore$ X and Y are not independent.

Also, the area is not a rectangle region, it can be seen that X and Y are not independent.

b) $U = \min(X, Y)$

$F_U(u) = P[U \leq u] = P[\min(X, Y) \leq u]$

$= 1 - P[X, Y > u] = 1 - \int_u^1 \int_u^{1-y} f_{XY}(x, y)\, dx\, dy$

$= 1 - \int_u^1 \int_u^{1-y} 2\, dx\, dy = 1 - \int_0^1 2(1-y-u)\, dy$

$= 1 - \left[2(1-u) - 2u(1-u) - y^2 \Big|_u^1 \right]$

$= 1 - [(1-u)(1-3u)]$

$= 4u - 3u^2$

$\therefore F_U(u) = 4u - 3u^2$

$\text{PDF} = f_U(u) = \frac{d}{du} F_U(u) = \frac{d}{du}(4u - 3u^2)$

$\therefore f_U(u) = \begin{cases} 4 - 6u & 0 < u < 1 \\ 0 & \text{o.w.} \end{cases}$

c) $V = \max(X, Y)$

$F_V(v) = P(\max(X, Y) \leq v)$

$= P(X \leq v, Y \leq v)$

$= \int_0^v \int_0^{1-y} f(v, y)\, dx\, dy$

$$= \int_0^v \int_0^{1-y} 2\, dx\, dy = \int_0^v 2(1-y)\, dy$$

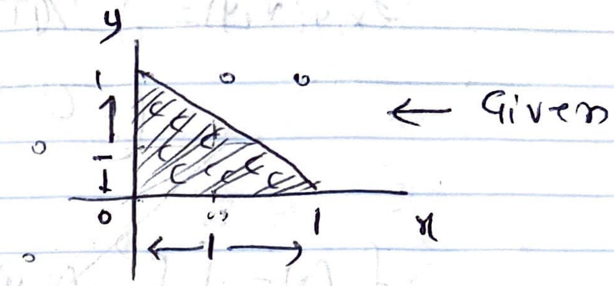
$$\therefore F_V(v) = 2v - y^2 \Big|_0^v = 2v - v^2$$

$$PDF = \frac{d}{dv} F_V(v) = \frac{d}{dv} (2v - v^2)$$

$$\therefore f_V(v) = (2 - 2v)$$

$$f_V(v) = \begin{cases} (2-2v) & 0 < v < 1 \\ 0 & \text{o.w.} \end{cases}$$

7) $f_{X,Y}(x,y) = \begin{cases} c & x \geq 0,\ y \geq 0,\ x+y \leq 1 \\ 0 & \text{o.w.} \end{cases}$



(a) ∴ total probability should be 1 in shaded area.

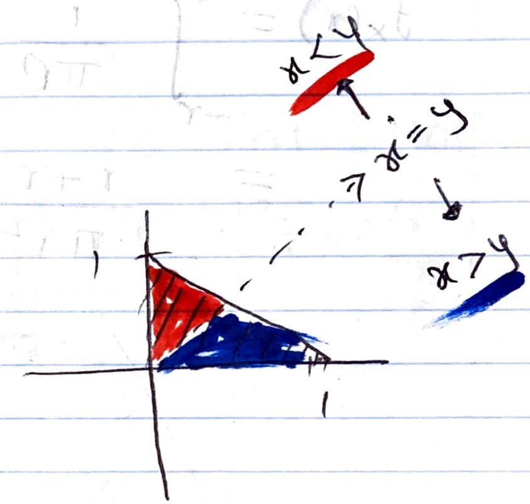
∴ $\cancel{1 = \frac{1}{2} x 1 + 1}$

$$1 = \underbrace{\frac{1}{2} \times 1 \times 1}_{\text{area of triangle}} \times c$$

F

∴ $\boxed{c = 2}$

(b) $P[X \leq Y]$ →

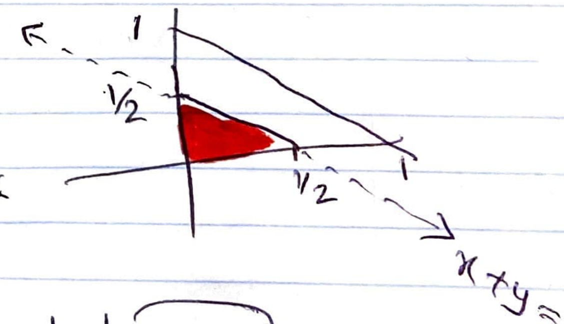


Need to find probability of 'red' area.

∴ i.e. half part.

∴ $\boxed{0.5}$

(c) $P[X+Y \leq 1/2]$



∴ Need to find probability of red area.

∴ $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times c = \frac{1}{8} \times 2 = \frac{1}{4} = \boxed{0.25}$

Q.8 . $x^2+y^2 \leq r^2$

$$f_{X,Y}(x,y) = \begin{cases} 1/(\pi r^2) & x^2+y^2 \leq r^2 \\ 0 & \text{o.w.} \end{cases}$$

(a) Marginal PDF $f_X(x)$

Limits of Y $-\sqrt{r^2-x^2} \to \sqrt{r^2-x^2}$

integrating y

$$f_X(x) = \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \frac{1}{\pi r^2}\, dy = \frac{\sqrt{r^2-x^2} - (-\sqrt{r^2-x^2})}{\pi r^2}$$

$$f_X(x) = \begin{cases} \frac{2\sqrt{r^2-x^2}}{\pi r^2} & -r \leq x \leq r \\ & \text{o.w.} \end{cases}$$

For $f_Y(y)$ = Limits of $x: -\sqrt{r^2-y^2} \to \sqrt{r^2-y^2}$

$$\boxed{f_Y(y)} = \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{1}{\pi r^2}\, dx = \begin{cases} \frac{2\sqrt{r^2-y^2}}{\pi r^2} & -r \leq y \leq r \\ 0 & \text{o.w.} \end{cases}$$

Q9 Given
$$f_{X,Y}(x,y) = Ce^{-(2x^2-4xy+4y^2)}$$

This is bivoriate Gaussian distribution.

$$f_{XY}(x,y) = Ce^{-\left[\left(\frac{x-0}{1/\sqrt{2}}\right)^2+\left(\frac{y-0}{1/\sqrt{2}}\right)^2-4\frac{1}{2}\frac{1}{2}\frac{(x-0)(y-0)}{1/\sqrt{2}\ 1/\sqrt{2}}\right]}$$

$$= Ce^{-\left[\left(\frac{x-0}{1/\sqrt{2}}\right)^2+\left(\frac{y-0}{1/\sqrt{2}}\right)^2-\frac{2}{\sqrt{2}}\frac{(x-0)(y-0)}{1/\sqrt{2}\ 1/\sqrt{2}}\right]}$$

The a Gaussion distribution ~~of~~ joint PDF

$$f_{XY}(x,y) = \frac{e^{-\left[\frac{-1}{2(1-\rho)^2}\left(\left(\frac{x-\mu_x}{\sigma_x}\right)^2+\left(\frac{y-\mu_y}{\sigma_y}\right)^2+2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\ \sigma_y}\right)\right]}}{2\sqrt{\pi}\,\sigma_x\,\sigma_y\,\sqrt{1-\rho^2}}$$

Comparing $\mu_x = 0$ $\quad \sigma_y = 1/2 \quad \mu_y = 0$

$\sigma_x = 1/\sqrt{2} \quad \rho = 1/\sqrt{2}$

$$\frac{1}{2}(1-\rho^2) = \frac{1}{2(1-(1/\sqrt{2})^2)} = 1$$

(a) E(X) and E(Y)

$E(X) = \mu_x = 0$

$E(Y) = \mu_y = 0$

(b) find ρ as $\rho = 1/\sqrt{2}$

(c) $Var[X] = \sigma_x = 1/\sqrt{2}$

$Var[Y] = \sigma_y = 1/\sqrt{2}$

d) $c = \dfrac{1}{2\pi \sigma_x \sigma_y \sqrt{1-\rho^2}} = \dfrac{1}{2\pi \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)\sqrt{1-\frac{1}{2}}}$

$c = 2/\pi$

e) x and y independent or not?

as $\rho \neq 0$

X and Y are not independent.

Q. 10

$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le y < x \le 1 \\ 0 & \text{o.w.} \end{cases}$$

Let's find marginal PDFs first,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^x 2\,dy = 2x$$

as $0 \le x \le 1$, $\therefore$ $y : 0 \le y < x$

$$\therefore f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_y^1 2\,dx = 2(1-y)$$

for $y < 0$, and $y > 1$ $f_Y(y) = 0$

$$\therefore f_Y(y) = \begin{cases} 2(1-y) & 0 \le y \le 1, \\ 0 & \text{ow} \end{cases}$$

we calculate. $E[X]$, $E[X^2]$, $E[Y]$, $E[Y^2]$

ⓐ $E[X]$ and $Var[X]$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_0^1 2x^2 dx = \frac{2x^3}{3}\Big|_0^1 = \frac{2}{3}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_0^1 2x^3 dx = \frac{2x^4}{4}\Big|_0^1 = \frac{1}{2}$$

$\therefore \underline{\underline{E[X]}} = \frac{2}{3}$ //

$\underline{\underline{Var[X]}} = E[X^2] - (E[X])^2$
$= \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}$ //

ⓑ $E[Y]$ and $var[Y]$

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \int_0^1 2y(1-y) = y^2 - \frac{2y^3}{3}\Big|_0^1 = \frac{1}{3}$$

$\therefore \underline{\underline{E[Y]}} = 1/3$ //

$$E[Y^2] = \int_{-\infty}^{\infty} y^2 f_Y(y)\, dy = \int_0^1 2y^2(1-y)\, dy = \left(\frac{2y^3}{3} - \frac{y^4}{2}\right)\Big|_0^1$$

$E[Y^2] = 2/3 - 1/2 = 1/6$

$Var[Y] = E[Y^2] - (E[Y])^2 = 1/6 - (1/3)^2$

$Var[Y] = 1/18$ //

c) $Cov[X,Y]$

$$E[X.Y] = \int_0^1 \int_0^x 2xy\,dy\,dx = \int_0^1 (y^2)_0^x \, 2x\,dx$$

$$= \int_0^1 2x^3\,dx = 1/4$$

$E[X.Y] = 1/4$

$$Cov[X,Y] = E[XY] - E[X]E[Y]$$

$$= 1/4 - \frac{2}{3} \cdot 1/3 = 1/36$$

d) $E[X+Y] = E[X] + E[Y] = \frac{2}{3} + 1/3 = 1$

e) $Var[X+Y] = Var[X] + Var[Y] + 2Cov[X,Y]$

$$= \frac{1}{18} + \frac{1}{18} + 2\left(\frac{1}{36}\right) = \frac{3}{18} = 1/6$$