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EEE 554 - Home Work 5

Q.1 (a) We wish to develope a hypothesis test of the form

$P[|K - E[K]| > c] = 0.05$

to determine if coin we've been flipping is indeed a fair one. Find value of c, to determine upper and lower limits. Under fair coin hypothesis, expected number of heads, and standard deviation

$E[K] = 50 \qquad \sigma_K = \sqrt{100 \cdot 1/2 \cdot 1/2} = 5$

Using central limit theorem to find c, divide by σ_K.

$$P\left[\frac{|K - E[K]|}{\sigma_K} > \frac{c}{\sigma_K}\right] = 0.05$$

$$P\left[-\frac{c}{\sigma_K} \leq \frac{K - E[K]}{\sigma_K} \leq \frac{c}{\sigma_K}\right] = 0.95$$

Using central limit theorem,

$$\Phi\left(\frac{c}{\sigma_K}\right) - \Phi\left(\frac{-c}{\sigma_K}\right) = 2\Phi\left(\frac{c}{\sigma_K}\right) - 1 = 0.95$$

$\therefore \Phi\left(\frac{c}{\sigma_K}\right) = 0.975$ or $c/5 = 1.96$. $\therefore c = 9.8$ flips.

So with $50+10=60$ or $50-10=40$ heads, with significance level $\alpha \approx 0.05$ we should reject the hypothesis that the coin is fair.

(6) Now for $P[K>c]=0.01$

To find value of c we look to evaluate the CDF

$$F_K(k)=\sum_{i=0}^{k}\binom{100}{i}(1/2)^{100}$$

i.e $c\approx 62$ flips.

So if we observe 62 or greater heads, then significance level of 0.01 we should reject the fair coin hypothesis.

Another way to obtain result is to use Central Limit Theorem approximation

$$P[K>c]=1-P[K\leq c]$$

$$=1-P\left[\frac{K-E[K]}{\sigma_K}\leq\frac{c-E[K]}{\sigma_K}\right]=0.01$$

Since $E[K]=50$ and $\sigma_K=5$, the CLT approximation is

$$P[K>c]\approx 1-\Phi\left(\frac{c-50}{5}\right)=0.01$$

from we have $(c-50)/5=2.35$ or $c=61.75$

Once again, we see that we reject the hypothesis if we observe 62 or more heads.

Q2 The solution to this is calculate various quantities, required for the optimal linear estimator given

Calculate moments of X and Y.

$E[X] = -1(1/4) + 0(1/2) + 1(1/4) = 0$
$E[X^2] = (-1)^2(1/4) + 0^2(1/2) + 1^2(1/4) = 1/2$
$E[Y] = -1(17/48) + 0(17/48) + 1(14/48) = -1/16$
$E[Y^2] = (-1)^2(17/48) + 0^2(17/48) + 1^2(14/48) = 31/48$
$E[XY] = 3/16 - 0 - 0 + 1/8 = 5/16$

The variance and covariance are

$Var[X] = E[X^2] - (E[X])^2 = 1/2$
$Var[Y] = E[Y^2] - (E[Y])^2 = 493/768$
$Cov[X,Y] = E[XY] - E[X]E[Y] = 5/16$

$$\rho_{X,Y} = \frac{Cov[X,Y]}{\sqrt{Var[X]Var[Y]}} = \frac{5\sqrt{6}}{\sqrt{493}}$$

By reversing the labels of X and Y, we find that the optimal linear estimator of Y given X is

$$\hat{Y}_L(X) = \rho_{X,Y}\frac{\sigma_Y}{\sigma_X}(X - E[X]) + E[Y] = \frac{5}{8}x - \frac{1}{16}$$

The mean square estimation error is

$$e_L^* = Var[Y](1 - \rho_{X,Y}^2) = 343/768.$$

Q.3 First we calculate the marginal PDFs.

~~E[X], E[Y], Var[X], Var[Y] and $\rho_{X,Y}$ first~~

$$f_X(x) = \int_x^1 2(y+x)\,dy = y^2 + 2xy \Big|_{y=x}^{y=1} = 1 + 2x - 3x^2 \quad (1)$$

$$f_Y(y) = \int_0^y 2(y+x)\,dx = 2xy + x^2 \Big|_{x=0}^{x=y} = 3y^2 \quad (2)$$

The first and second moments of X are,

$$E[X] = \int_0^1 (x + 2x^2 - 3x^3)\,dx = \frac{x^2}{2} + \frac{2x^3}{3} - \frac{3x^4}{4}\Big]_0^1 = 5/12$$

$$E[X^2] = \int_0^1 (x^2 + 2x^3 - 3x^4)\,dx = \frac{x^3}{3} + \frac{x^4}{2} - \frac{3x^5}{5}\Big|_0^1 = 7/30$$

First and second moments of Y are,

$$E[Y] = \int_0^1 3y^3\,dy = 3/4 \quad , E[Y^2] = \int_0^1 3y^4\,dy = 3/5$$

Thus X and Y each have variance.

$$Var[X] = E[X^2] - (E[X])^2 = 129/2160$$
$$Var[Y] = E[Y^2] - (E[Y])^2 = 3/80.$$

To calculate correlation coefficient, we first must calculate correlation

$$E[XY] = \int_0^1 \int_0^y 2xy(x+y)\,dx\,dy$$

$$= \int_0^1 \left[2x^3y/3 + x^2y^2 \right] \Big|_{x=0}^{x=y} dy = \int_0^1 \frac{5y^4}{3} dy = 1/3$$

Here correlation coefficient is,

$$\rho_{X,Y} = \frac{\mathrm{Cov}[X,Y]}{\sqrt{\mathrm{Var}[X]\,\mathrm{Var}[Y]}} \quad \frac{E[XY] - E[X]\,E[Y]}{\sqrt{\mathrm{Var}[X]\,\mathrm{Var}[Y]}} = \frac{5}{\sqrt{129}}$$

Finally,

$$\hat{X}_L(Y) = \rho_{X,Y}\frac{\sigma_X}{\sigma_Y}\left(Y - E[Y]\right) + E[X]$$

$$= \frac{5}{\sqrt{129}} \cdot \frac{\sqrt{129}}{9}\left(Y - \frac{3}{4}\right) + \frac{5}{12} = \frac{5}{9}Y.$$