Solutions to HW1

Note: These solutions were generated by R. D. Yates and D. J. Goodman, the authors of our textbook. I have added comments in italics where I thought more detail was appropriate.

Problem 1.2.1 •

A fax transmission can take place at any of three speeds depending on the condition of the phone connection between the two fax machines. The speeds are high (h) at $14,400$ b/s, medium (m) at 9600 b/s, and low (l) at 4800 b/s. In response to requests for information, a company sends either short faxes of two (t) pages, or long faxes of four (f) pages. Consider the experiment of monitoring a fax transmission and observing the transmission speed and length. An observation is a two-letter word, for example, a high-speed, two-page fax is ht.

(a) What is the sample space of the experiment?

(b) Let A_1 be the event "medium-speed fax." What are the outcomes in A_1?

(c) Let A_2 be the event "short (two-page) fax." What are the outcomes in A_2?

(d) Let A_3 be the event "high-speed fax or low-speed fax." What are the outcomes in A_3?

(e) Are A_1, A_2, and A_3 mutually exclusive?

(f) Are A_1, A_2, and A_3 collectively exhaustive?

Problem 1.2.1 Solution

(a) An outcome specifies whether the fax is high (h), medium (m), or low (l) speed, and whether the fax has two (t) pages or four (f) pages. The sample space is

$$S = \{ht, hf, mt, mf, lt, lf\}. \tag{1}$$

(b) The event that the fax is medium speed is $A_1 = \{mt, mf\}$.

(c) The event that a fax has two pages is $A_2 = \{ht, mt, lt\}$.

(d) The event that a fax is either high speed or low speed is $A_3 = \{ht, hf, lt, lf\}$.

(e) Since $A_1 \cap A_2 = \{mt\}$ and is not empty, A_1, A_2, and A_3 are not mutually exclusive. *(Equivalently, since $A_2 \cap A_3 \neq \emptyset$, A_1, A_2, and A_3 are not mutually exclusive.)*

(f) Since

$$A_1 \cup A_2 \cup A_3 = \{ht, hf, mt, mf, lt, lf\} = S, \tag{2}$$

the collection A_1, A_2, A_3 is collectively exhaustive.

Problem 1.2.2 •

An integrated circuit factory has three machines X, Y, and Z. Test one integrated circuit produced by each machine. Either a circuit is acceptable (a) or it fails (f). An observation is a sequence of three test results corresponding to the circuits from machines X, Y, and Z, respectively. For example, aaf is the observation that the circuits from X and Y pass the test and the circuit from Z fails the test.

(a) What are the elements of the sample space of this experiment?

(b) What are the elements of the sets

$$Z_F = \{\text{circuit from } Z \text{ fails}\},$$
$$X_A = \{\text{circuit from } X \text{ is acceptable}\}.$$

(c) Are Z_F and X_A mutually exclusive?

(d) Are Z_F and X_A collectively exhaustive?

(e) What are the elements of the sets

$$C = \{\text{more than one circuit acceptable}\},$$
$$D = \{\text{at least two circuits fail}\}.$$

(f) Are C and D mutually exclusive?

(g) Are C and D collectively exhaustive?

Problem 1.2.2 Solution

(a) The sample space of the experiment is

$$S = \{aaa, aaf, afa, faa, ffa, faf, aff, fff\}. \tag{1}$$

(b) The event that the circuit from Z fails is

$$Z_F = \{aaf, aff, faf, fff\}. \tag{2}$$

The event that the circuit from X is acceptable is

$$X_A = \{aaa, aaf, afa, aff\}. \tag{3}$$

(c) Since $Z_F \cap X_A = \{aaf, aff\} \neq \phi$, Z_F and X_A are not mutually exclusive.

(d) Since $Z_F \cup X_A = \{aaa, aaf, afa, aff, faf, fff\} \neq S$, Z_F and X_A are not collectively exhaustive.

(e) The event that more than one circuit is acceptable is

$$C = \{aaa, aaf, afa, faa\}. \tag{4}$$

The event that at least two circuits fail is

$$D = \{ffa, faf, aff, fff\}. \tag{5}$$

(f) Inspection shows that $C \cap D = \phi$ so C and D are mutually exclusive.

(g) Since $C \cup D = S$, C and D are collectively exhaustive.

Problem 1.2.3 •

Shuffle a deck of cards and turn over the first card. What is the sample space of this experiment? How many outcomes are in the event that the first card is a heart?

Problem 1.2.3 Solution

The sample space is

$$S = \{A\clubsuit, \ldots, K\clubsuit, A\diamondsuit, \ldots, K\diamondsuit, A\heartsuit, \ldots, K\heartsuit, A\spadesuit, \ldots, K\spadesuit\}. \tag{1}$$

The event H is the set

$$H = \{A\heartsuit, \ldots, K\heartsuit\}. \tag{2}$$

Problem 1.2.6 •

Let the sample space of the experiment consist of the measured resistances of two resistors. Give four examples of event spaces.

Problem 1.2.6 Solution

Let R_1 and R_2 denote the measured resistances. The pair (R_1, R_2) is an outcome of the experiment. Some event spaces include

1. If we need to check that neither resistance is too high, an event space is

$$A_1 = \{R_1 < 100, R_2 < 100\}, \qquad A_2 = \{\text{either } R_1 \geq 100 \text{ or } R_2 \geq 100\}. \tag{1}$$

2. If we need to check whether the first resistance exceeds the second resistance, an event space is

$$B_1 = \{R_1 > R_2\} \qquad B_2 = \{R_1 \leq R_2\}. \tag{2}$$

3. If we need to check whether each resistance doesn't fall below a minimum value (in this case 50 ohms for R_1 and 100 ohms for R_2), an event space is

$$C_1 = \{R_1 < 50, R_2 < 100\}, \qquad C_2 = \{R_1 < 50, R_2 \geq 100\}, \tag{3}$$

$$C_3 = \{R_1 \geq 50, R_2 < 100\}, \qquad C_4 = \{R_1 \geq 50, R_2 \geq 100\}. \tag{4}$$

4. If we want to check whether the resistors in parallel are within an acceptable range of 90 to 110 ohms, an event space is

$$D_1 = \left\{(1/R_1 + 1/R_2)^{-1} < 90\right\}, \tag{5}$$
$$D_2 = \left\{90 \leq (1/R_1 + 1/R_2)^{-1} \leq 110\right\}, \tag{6}$$
$$D_2 = \left\{110 < (1/R_1 + 1/R_2)^{-1}\right\}. \tag{7}$$

Problem 1.3.1 •

Computer programs are classified by the length of the source code and by the execution time. Programs with more than 150 lines in the source code are big (B). Programs with ≤ 150 lines are little (L). Fast programs (F) run in less than 0.1 seconds. Slow programs (W) require at least 0.1 seconds. Monitor a program executed by a computer. Observe the length of the source code and the run time. The probability model for this experiment contains the following information: $P[LF] = 0.5$, $P[BF] = 0.2$, and $P[BW] = 0.2$. What is the sample space of the experiment? Calculate the following probabilities:

(a) $P[W]$

(b) $P[B]$

(c) $P[W \cup B]$

Problem 1.3.1 Solution

The sample space of the experiment is

$$S = \{LF, BF, LW, BW\}. \tag{1}$$

From the problem statement, we know that $P[LF] = 0.5$, $P[BF] = 0.2$ and $P[BW] = 0.2$. This implies $P[LW] = 1 - 0.5 - 0.2 - 0.2 = 0.1$ *(because BF, BW, and LF are mutually exclusive)*. The questions can be answered using Theorem 1.5.

(a) *(Because $LW \cap BW = \emptyset$)* [t]he probability that a program is slow is

$$P[W] = P[LW] + P[BW] = 0.1 + 0.2 = 0.3. \tag{2}$$

(b) *(Because $BF \cap BW = \emptyset$)* [t]he probability that a program is big is

$$P[B] = P[BF] + P[BW] = 0.2 + 0.2 = 0.4. \tag{3}$$

(c) The probability that a program is slow or big is

$$P[W \cup B] = P[W] + P[B] - P[BW] = 0.3 + 0.4 - 0.2 = 0.5. \tag{4}$$

Problem 1.3.2 •

There are two types of cellular phones, handheld phones (H) that you carry and mobile phones (M) that are mounted in vehicles. Phone calls can be classified by the traveling speed of the user as fast (F) or slow (W). Monitor a cellular phone call and observe the type of telephone and the speed of the user. The probability model for this experiment has the following information: $P[F] = 0.5$, $P[HF] = 0.2$, $P[MW] = 0.1$. What is the sample space of the experiment? Calculate the following probabilities:

(a) $P[W]$

(b) $P[MF]$

(c) $P[H]$

Problem 1.3.2 Solution

A sample outcome indicates whether the cell phone is handheld (H) or mobile (M) and whether the speed is fast (F) or slow (W). The sample space is

$$S = \{HF, HW, MF, MW\}. \tag{1}$$

The problem statement tells us that $P[HF] = 0.2$, $P[MW] = 0.1$ and $P[F] = 0.5$. We can use these facts to find the probabilities of the other outcomes. In particular,

$$P[F] = P[HF] + P[MF], \tag{2}$$

(because $HF \cap MF = \emptyset$.)
This implies

$$P[MF] = P[F] - P[HF] = 0.5 - 0.2 = 0.3. \tag{3}$$

Also, since the probabilities must sum to 1,

$$P[HW] = 1 - P[HF] - P[MF] - P[MW] = 1 - 0.2 - 0.3 - 0.1 = 0.4. \tag{4}$$

Now that we have found the probabilities of the outcomes, finding any other probability is easy.

(a) The probability a cell phone is slow is

$$P[W] = P[HW] + P[MW] = 0.4 + 0.1 = 0.5. \tag{5}$$

(Equivalently, the probability that the cell phone is slow is one minus the probability that it is fast, since these are mutually exclusive, so

$$P[W] = 1 - P[F] = 1 - 0.5 = 0.5 \tag{6}$$

which gives the same answer.)

(b) The probability that a cell hpone is mobile and fast is $P[MF] = 0.3$.

(c) The probability that a cell phone is handheld is

$$P[H] = P[HF] + P[HW] = 0.2 + 0.4 = 0.6. \tag{7}$$

Problem 1.3.3 •
Shuffle a deck of cards and turn over the first card. What is the probability that the first card is a heart?

Problem 1.3.3 Solution
A reasonable probability model that is consistent with the notion of a shuffled deck is that each card in the deck is equally likely to be the first card. Let H_i denote the event that the first card drawn is the ith heart where the first heart is the ace, the second heart is the deuce and so on. In that case, $P[H_i] = 1/52$ for $1 \leq i \leq 13$. The event H that the first card is a heart can be written as the disjoint union

$$H = H_1 \cup H_2 \cup \cdots \cup H_{13}. \tag{1}$$

Using Theorem 1.1, we have

$$P[H] = \sum_{i=1}^{13} P[H_i] = 13/52. \tag{2}$$

This is the answer you would expect since 13 out of 52 cards are hearts. The point to keep in mind is that this is not just the common sense answer but is the result of a probability model for a shuffled deck and the axioms of probability.

Problem 1.3.4 •
You have a six-sided die that you roll once and observe the number of dots facing upwards. What is the sample space? What is the probability of each sample outcome? What is the probability of E, the event that the roll is even?

Problem 1.3.4 Solution
Let s_i denote the outcome that the down face has i dots. The sample space is $S = \{s_1, \ldots, s_6\}$. The probability of each sample outcome is $P[s_i] = 1/6$. From Theorem 1.1, the probability of the event E that the roll is even is

$$P[E] = P[s_2] + P[s_4] + P[s_6] = 3/6. \tag{1}$$

Problem 1.3.5 •
A student's score on a 10-point quiz is equally likely to be any integer between 0 and 10. What is the probability of an A, which requires the student to get a score of 9 or more? What is the probability the student gets an F by getting less than 4?

Problem 1.3.5 Solution
Let s_i equal the outcome of the student's quiz. The sample space is then composed of all the possible grades that she can receive.

$$S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}. \tag{1}$$

Since each of the 11 possible outcomes is equally likely, the probability of receiving a grade of i, for each $i = 0, 1, \ldots, 10$ is $P[s_i] = 1/11$. The probability that the student gets an A is the probability that she gets a score of 9 or higher. That is

$$P[\text{Grade of A}] = P[9] + P[10] = 1/11 + 1/11 = 2/11. \tag{2}$$

The probability of failing requires the student to get a grade less than 4.

$$P[\text{Failing}] = P[3] + P[2] + P[1] + P[0] = 1/11 + 1/11 + 1/11 + 1/11 = 4/11. \tag{3}$$

Problem 1.4.1 •

Mobile telephones perform *handoffs* as they move from cell to cell. During a call, a telephone either performs zero handoffs (H_0), one handoff (H_1), or more than one handoff (H_2). In addition, each call is either long (L), if it lasts more than three minutes, or brief (B). The following table describes the probabilities of the possible types of calls.

	H_0	H_1	H_2
L	0.1	0.1	0.2
B	0.4	0.1	0.1

What is the probability $P[H_0]$ that a phone makes no handoffs? What is the probability a call is brief? What is the probability a call is long or there are at least two handoffs?

Problem 1.4.1 Solution

From the table we look to add all the disjoint events that contain H_0 to express the probability that a caller makes no hand-offs as

$$P[H_0] = P[LH_0] + P[BH_0] = 0.1 + 0.4 = 0.5. \tag{1}$$

In a similar fashion we can express the probability that a call is brief by

$$P[B] = P[BH_0] + P[BH_1] + P[BH_2] = 0.4 + 0.1 + 0.1 = 0.6. \tag{2}$$

The probability that a call is long or makes at least two hand-offs is

$$P[L \cup H_2] = P[LH_0] + P[LH_1] + P[LH_2] + P[BH_2] \tag{3}$$
$$= 0.1 + 0.1 + 0.2 + 0.1 = 0.5. \tag{4}$$

Problem 1.4.2 •

For the telephone usage model of Example 1.14, let B_m denote the event that a call is billed for m minutes. To generate a phone bill, observe the duration of the call in integer minutes (rounding up). Charge for M minutes $M = 1, 2, 3, \ldots$ if the exact duration T is $M - 1 < t \leq M$. A more complete probability model shows that for $m = 1, 2, \ldots$ the probability of each event B_m is

$$P[B_m] = \alpha(1 - \alpha)^{m-1}$$

where $\alpha = 1 - (0.57)^{1/3} = 0.171$.

(a) Classify a call as long, L, if the call lasts more than three minutes. What is $P[L]$?

(b) What is the probability that a call will be billed for nine minutes or less?

Problem 1.4.2 Solution

Note: (4) is not "obvious", at least to me, from (3). However, if you do the math, you'll find that it is the correct answer. Similarly, the final expression on the right-hand side of (5) is not "obvious" but can be obtained either by summing the terms directly or by using the fact that

$$1 + \beta + \beta^2 + \cdots + \beta^{p-1} = \frac{1-\beta^p}{1-\beta}.$$

(a) From the given probability distribution of billed minutes, M, the probability that a call is billed for more than 3 minutes is

$$\begin{aligned} P[L] &= 1 - P[3 \text{ or fewer billed minutes}] &(1)\\ &= 1 - P[B_1] - P[B_2] - P[B_3] &(2)\\ &= 1 - \alpha - \alpha(1-\alpha) - \alpha(1-\alpha)^2 &(3)\\ &= (1-\alpha)^3 = 0.57. &(4) \end{aligned}$$

(b) The probability that a call will billed for 9 minutes or less is

$$P[9 \text{ minutes or less}] = \sum_{i=1}^{9} \alpha(1-\alpha)^{i-1} = 1 - (0.57)^3. \qquad (5)$$

Problem 1.5.1 •

Given the model of handoffs and call lengths in Problem 1.4.1,

(a) What is the probability that a brief call will have no handoffs?

(b) What is the probability that a call with one handoff will be long?

(c) What is the probability that a long call will have one or more handoffs?

Problem 1.5.1 Solution

Each question requests a conditional probability.

(a) Note that the probability a call is brief is

$$P[B] = P[H_0B] + P[H_1B] + P[H_2B] = 0.6. \qquad (1)$$

The probability a brief call will have no handoffs is

$$P[H_0|B] = \frac{P[H_0B]}{P[B]} = \frac{0.4}{0.6} = \frac{2}{3}. \qquad (2)$$

(b) The probability of one handoff is $P[H_1] = P[H_1B] + P[H_1L] = 0.2$. The probability that a call with one handoff will be long is

$$P[L|H_1] = \frac{P[H_1L]}{P[H_1]} = \frac{0.1}{0.2} = \frac{1}{2}. \tag{3}$$

(c) The probability a call is long is $P[L] = 1 - P[B] = 0.4$. The probability that a long call will have one or more handoffs is

$$P[H_1 \cup H_2|L] = \frac{P[H_1L \cup H_2L]}{P[L]} = \frac{P[H_1L] + P[H_2L]}{P[L]} = \frac{0.1 + 0.2}{0.4} = \frac{3}{4}. \tag{4}$$

Problem 1.5.2 •

You have a six-sided die that you roll once. Let R_i denote the event that the roll is i. Let G_j denote the event that the roll is greater than j. Let E denote the event that the roll of the die is even-numbered.

(a) What is $P[R_3|G_1]$, the conditional probability that 3 is rolled given that the roll is greater than 1?

(b) What is the conditional probability that 6 is rolled given that the roll is greater than 3?

(c) What is $P[G_3|E]$, the conditional probability that the roll is greater than 3 given that the roll is even?

(d) Given that the roll is greater than 3, what is the conditional probability that the roll is even?

Problem 1.5.2 Solution

Let s_i denote the outcome that the roll is i. So, for $1 \le i \le 6$, $R_i = \{s_i\}$. Similarly, $G_j = \{s_{j+1}, \ldots, s_6\}$.

(a) Since $G_1 = \{s_2, s_3, s_4, s_5, s_6\}$ and all outcomes have probability 1/6, $P[G_1] = 5/6$. The event $R_3G_1 = \{s_3\}$ and $P[R_3G_1] = 1/6$ so that

$$P[R_3|G_1] = \frac{P[R_3G_1]}{P[G_1]} = \frac{1}{5}. \tag{1}$$

(b) The conditional probability that 6 is rolled given that the roll is greater than 3 is

$$P[R_6|G_3] = \frac{P[R_6G_3]}{P[G_3]} = \frac{P[s_6]}{P[s_4, s_5, s_6]} = \frac{1/6}{3/6}. \tag{2}$$

(c) The event E that the roll is even is $E = \{s_2, s_4, s_6\}$ and has probability 3/6. The joint probability of G_3 and E is

$$P[G_3E] = P[s_4, s_6] = 1/3. \tag{3}$$

The conditional probabilities of G_3 given E is

$$P[G_3|E] = \frac{P[G_3E]}{P[E]} = \frac{1/3}{1/2} = \frac{2}{3}. \tag{4}$$

(d) The conditional probability that the roll is even given that it's greater than 3 is

$$P[E|G_3] = \frac{P[EG_3]}{P[G_3]} = \frac{1/3}{1/2} = \frac{2}{3}. \tag{5}$$

Problem 1.5.3 •

You have a shuffled deck of three cards: 2, 3, and 4. You draw one card. Let C_i denote the event that card i is picked. Let E denote the event that card chosen is a even-numbered card.

(a) What is $P[C_2|E]$, the probability that the 2 is picked given that an even-numbered card is chosen?

(b) What is the conditional probability that an even-numbered card is picked given that the 2 is picked?

Problem 1.5.3 Solution

Since the 2 of clubs is an even numbered card, $C_2 \subset E$ so that $P[C_2E] = P[C_2] = 1/3$. Since $P[E] = 2/3$,

$$P[C_2|E] = \frac{P[C_2E]}{P[E]} = \frac{1/3}{2/3} = 1/2. \tag{1}$$

The probability that an even numbered card is picked given that the 2 is picked is

$$P[E|C_2] = \frac{P[C_2E]}{P[C_2]} = \frac{1/3}{1/3} = 1. \tag{2}$$

Problem 1.6.1 •

Is it possible for A and B to be independent events yet satisfy $A = B$?

Problem 1.6.1 Solution

This problems asks whether A and B can be independent events yet satisfy $A = B$? By definition, events A and B are independent if and only if $P[AB] = P[A]P[B]$. We can see that if $A = B$, that is they are the same set, then

$$P[AB] = P[AA] = P[A] = P[B]. \tag{1}$$

Thus, for A and B to be the same set and also independent,

$$P[A] = P[AB] = P[A]\,P[B] = (P[A])^2 . \tag{2}$$

There are two ways that this requirement can be satisfied:

- $P[A] = 1$ implying $A = B = S$.
- $P[A] = 0$ implying $A = B = \phi$.

Problem 1.7.1 •

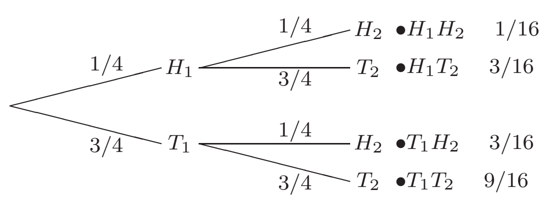
Suppose you flip a coin twice. On any flip, the coin comes up heads with probability 1/4. Use H_i and T_i to denote the result of flip i.

(a) What is the probability, $P[H_1|H_2]$, that the first flip is heads given that the second flip is heads?

(b) What is the probability that the first flip is heads and the second flip is tails?

Problem 1.7.1 Solution

A sequential sample space for this experiment is



(a) From the tree, we observe

$$P[H_2] = P[H_1H_2] + P[T_1H_2] = 1/4. \tag{1}$$

This implies

$$P[H_1|H_2] = \frac{P[H_1H_2]}{P[H_2]} = \frac{1/16}{1/4} = 1/4. \tag{2}$$

Note: If you have taken a course in probability before, you may also have used the fact that since the results of the two flips are independent, $P[H_1|H_2] = P[H_1] = 1/4$. However, while this is true, it hasn't been stated and proved a theorem yet in the text; so unless you prove it as part of your answer, you should not use it.

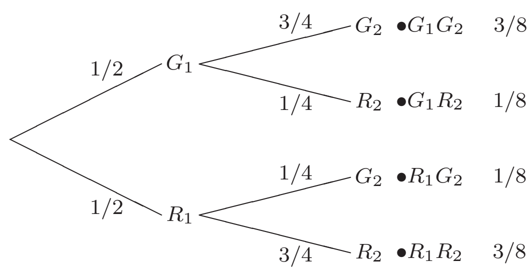
(b) The probability that the first flip is heads and the second flip is tails is $P[H_1T_2] = 3/16$ *because the results of the flips are independent.*

Problem 1.7.2 •

For Example 1.25, suppose $P[G_1] = 1/2$, $P[G_2|G_1] = 3/4$, and $P[G_2|R_1] = 1/4$. Find $P[G_2]$, $P[G_2|G_1]$, and $P[G_1|G_2]$.

Problem 1.7.2 Solution

The tree with adjusted probabilities is



From the tree, the probability the second light is green is

$$P[G_2] = P[G_1G_2] + P[R_1G_2] = 3/8 + 1/8 = 1/2. \quad (1)$$

The conditional probability that the first light was green given the second light was green is

$$P[G_1|G_2] = \frac{P[G_1G_2]}{P[G_2]} = \frac{P[G_2|G_1]\,P[G_1]}{P[G_2]} = 3/4. \quad (2)$$

Finally, from the tree diagram, we can directly read that $P[G_2|G_1] = 3/4$.

Alternatively, we can simply apply the Law of Total Probability (Thm 1.10, p. 19 of the textbook). First, since $\{R_1, G_1\}$ is an event space,

$$P[G_2] = P[G_2|G_1]\,P[G_1] + P[G_2|R_1]\,P[R_1] = \left(\frac{3}{4}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{2}\right) = \frac{1}{2}. \quad (3)$$

Next, $P[G_2|G_1] = 3/4$ is given. Finally, applying Bayes' Theorem (Thm 1.11, p. 20 of the textbook) yields

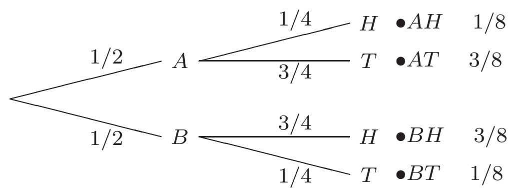
$$P[G_1|G_2] = \frac{P[G_2|G_1]\,P[G_1]}{P[G_2]} = \frac{\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)}{\frac{1}{2}} = \frac{3}{4}. \quad (4)$$

Problem 1.7.4 •

You have two biased coins. Coin A comes up heads with probability 1/4. Coin B comes up heads with probability 3/4. However, you are not sure which is which so you choose a coin randomly and you flip it. If the flip is heads, you guess that the flipped coin is B; otherwise, you guess that the flipped coin is A. Let events A and B designate which coin was picked. What is the probability $P[C]$ that your guess is correct?

Problem 1.7.4 Solution

The tree for this experiment is



The probability that you guess correctly is

$$P[C] = P[AT] + P[BH] = 3/8 + 3/8 = 3/4. \tag{1}$$

Alternatively, using the Law of Total Probability (Thm. 1.10, page 19 of the text),

$$P[C] = P[T|A]\,P[A] + P[H|B]\,P[B] = (3/4)(1/2) + (3/4)(1/2) = 3/4. \tag{2}$$

Problem 1.8.1 •

Consider a binary code with 5 bits (0 or 1) in each code word. An example of a code word is 01010. How many different code words are there? How many code words have exactly three 0's?

Problem 1.8.1 Solution

There are $2^5 = 32$ different binary codes with 5 bits. The number of codes with exactly 3 zeros equals the number of ways of choosing the bits in which those zeros occur. Therefore there are $\binom{5}{3} = 10$ codes with exactly 3 zeros.

Problem 1.8.2 •

Consider a language containing four letters: A, B, C, D. How many three-letter words can you form in this language? How many four-letter words can you form if each letter appears only once in each word?

Problem 1.8.2 Solution

Since each letter can take on any one of the 4 possible letters in the alphabet, the number of 3 letter words that can be formed is $4^3 = 64$. If we allow each letter to appear only once then we have 4 choices for the first letter and 3 choices for the second and two choices for the third letter. Therefore, there are a total of $4 \cdot 3 \cdot 2 = 24$ possible codes.

Problem 1.8.3 ■

Shuffle a deck of cards and pick two cards at random. Observe the sequence of the two cards in the order in which they were chosen.

(a) How many outcomes are in the sample space?

(b) How many outcomes are in the event that the two cards are the same type but different suits?

(c) What is the probability that the two cards are the same type but different suits?

(d) Suppose the experiment specifies observing the set of two cards without considering the order in which they are selected, and redo parts (a)–(c).

Problem 1.8.3 Solution

(a) The experiment of picking two cards and recording them in the order in which they were selected can be modeled by two sub-experiments. The first is to pick the first card and record it, the second sub-experiment is to pick the second card without replacing the first and recording it. For the first sub-experiment we can have any one of the possible 52 cards for a total of 52 possibilities. The second experiment consists of all the cards minus the one that was picked first(because we are sampling without replacement) for a total of 51 possible outcomes. So the total number of outcomes is the product of the number of outcomes for each sub-experiment.

$$52 \cdot 51 = 2652 \text{ outcomes.} \tag{1}$$

(b) To have the same card but different suit we can make the following sub-experiments. First we need to pick one of the 52 cards. Then we need to pick one of the 3 remaining cards that are of the same type but different suit out of the remaining 51 cards. So the total number outcomes is

$$52 \cdot 3 = 156 \text{ outcomes.} \tag{2}$$

(c) The probability that the two cards are of the same type but different suit is the number of outcomes that are of the same type but different suit divided by the total number of outcomes involved in picking two cards at random from a deck of 52 cards.

$$P\left[\text{same type, different suit}\right] = \frac{156}{2652} = \frac{1}{17}. \tag{3}$$

(d) Now we are not concerned with the ordering of the cards. So before, the outcomes $(K\heartsuit, 8\diamondsuit)$ and $(8\diamondsuit, K\heartsuit)$ were distinct. Now, those two outcomes are not distinct and are only considered to be the single outcome that a King of hearts and 8 of diamonds were selected. So every pair of outcomes before collapses to a single outcome when we disregard ordering. So we can redo parts (a) and (b) above by halving the corresponding values found in parts (a) and (b). The probability however, does not change because both the numerator and the denominator have been reduced by an equal factor of 2, which does not change their ratio.

Solutions to HW2

Note: These solutions were generated by R. D. Yates and D. J. Goodman, the authors of our textbook. I have added comments in italics where I thought more detail was appropriate.

Problem 1.4.3 ■

The basic rules of genetics were discovered in mid-1800s by Mendel, who found that each characteristic of a pea plant, such as whether the seeds were green or yellow, is determined by two genes, one from each parent. Each gene is either dominant d or recessive r. Mendel's experiment is to select a plant and observe whether the genes are both dominant d, both recessive r, or one of each (hybrid) h. In his pea plants, Mendel found that yellow seeds were a dominant trait over green seeds. A yy pea with two yellow genes has yellow seeds; a gg pea with two recessive genes has green seeds; a hybrid gy or yg pea has yellow seeds. In one of Mendel's experiments, he started with a parental generation in which half the pea plants were yy and half the plants were gg. The two groups were crossbred so that each pea plant in the first generation was gy. In the second generation, each pea plant was equally likely to inherit a y or a g gene from each first generation parent. What is the probability $P[Y]$ that a randomly chosen pea plant in the second generation has yellow seeds?

Problem 1.4.3 Solution

The first generation consists of two plants each with genotype yg or gy. They are crossed to produce the following second generation genotypes, $S = \{yy, yg, gy, gg\}$. Each genotype is just as likely as any other so the probability of each genotype is consequently 1/4. A pea plant has yellow seeds if it possesses at least one dominant y gene. The set of pea plants with yellow seeds is

$$Y = \{yy, yg, gy\}. \tag{1}$$

So the probability of a pea plant with yellow seeds is

$$P[Y] = P[yy] + P[yg] + P[gy] = 3/4. \tag{2}$$

Problem 1.5.5 ■

You have a shuffled deck of three cards: 2, 3, and 4 and you deal out the three cards. Let E_i denote the event that ith card dealt is even numbered.

(a) What is $P[E_2|E_1]$, the probability the second card is even given that the first card is even?

(b) What is the conditional probability that the first two cards are even given that the third card is even?

(c) Let O_i represent the event that the ith card dealt is odd numbered. What is $P[E_2|O_1]$, the conditional probability that the second card is even given that the first card is odd?

(d) What is the conditional probability that the second card is odd given that the first card is odd?

Problem 1.5.5 Solution

The sample outcomes can be written ijk where the first card drawn is i, the second is j and the third is k. The sample space is

$$S = \{234, 243, 324, 342, 423, 432\}. \tag{1}$$

and each of the six outcomes has probability 1/6. The events $E_1, E_2, E_3, O_1, O_2, O_3$ are

$$E_1 = \{234, 243, 423, 432\}, \qquad O_1 = \{324, 342\}, \tag{2}$$
$$E_2 = \{243, 324, 342, 423\}, \qquad O_2 = \{234, 432\}, \tag{3}$$
$$E_3 = \{234, 324, 342, 432\}, \qquad O_3 = \{243, 423\}. \tag{4}$$

(a) The conditional probability the second card is even given that the first card is even is

$$P[E_2|E_1] = \frac{P[E_2E_1]}{P[E_1]} = \frac{P[243, 423]}{P[234, 243, 423, 432]} = \frac{2/6}{4/6} = 1/2. \tag{5}$$

(b) The probability the first two cards are even given the third card is even is

$$P[E_1E_2|E_3] = \frac{P[E_1E_2E_3]}{P[E_3]} = 0. \tag{6}$$

(c) The conditional probabilities the second card is even given that the first card is odd is

$$P[E_2|O_1] = \frac{P[O_1E_2]}{P[O_1]} = \frac{P[O_1]}{P[O_1]} = 1. \tag{7}$$

(d) The conditional probability the second card is odd given that the first card is odd is

$$P[O_2|O_1] = \frac{P[O_1O_2]}{P[O_1]} = 0. \tag{8}$$

Problem 1.5.6 ♦

Deer ticks can carry both Lyme disease and human granulocytic ehrlichiosis (HGE). In a study of ticks in the Midwest, it was found that 16% carried Lyme disease, 10% had HGE, and that 10% of the ticks that had either Lyme disease or HGE carried both diseases.

(a) What is the probability $P[LH]$ that a tick carries both Lyme disease (L) and HGE (H)?

(b) What is the conditional probability that a tick has HGE given that it has Lyme disease?

Problem 1.5.6 Solution

The problem statement yields the obvious facts that $P[L] = 0.16$ and $P[H] = 0.10$. The words "10% of the ticks that had either Lyme disease or HGE carried both diseases" can be written as

$$P[LH|L \cup H] = 0.10. \quad (1)$$

(a) Since $LH \subset L \cup H$,

$$P[LH|L \cup H] = \frac{P[LH \cap (L \cup H)]}{P[L \cup H]} = \frac{P[LH]}{P[L \cup H]} = 0.10. \quad (2)$$

Thus,

$$P[LH] = 0.10P[L \cup H] = 0.10(P[L] + P[H] - P[LH]). \quad (3)$$

Since $P[L] = 0.16$ and $P[H] = 0.10$,

$$P[LH] = \frac{0.10(0.16 + 0.10)}{1.1} = 0.0236. \quad (4)$$

(b) The conditional probability that a tick has HGE given that it has Lyme disease is

$$P[H|L] = \frac{P[LH]}{P[L]} = \frac{0.0236}{0.16} = 0.1475. \quad (5)$$

Problem 1.6.3 ■

In an experiment, A, B, C, and D are events with probabilities $P[A] = 1/4$, $P[B] = 1/8$, $P[C] = 5/8$, and $P[D] = 3/8$. Furthermore, A and B are disjoint, while C and D are independent.

(a) Find $P[A \cap B]$, $P[A \cup B]$, $P[A \cap B^c]$, and $P[A \cup B^c]$.

(b) Are A and B independent?

(c) Find $P[C \cap D]$, $P[C \cap D^c]$, and $P[C^c \cap D^c]$.

(d) Are C^c and D^c independent?

Problem 1.6.3 Solution

(a) Since A and B are disjoint, $P[A \cap B] = 0$. Since $P[A \cap B] = 0$,

$$P[A \cup B] = P[A] + P[B] - P[A \cap B] = 3/8. \quad (1)$$

A Venn diagram should convince you that $A \subset B^c$ so that $A \cap B^c = A$. This implies

$$P[A \cap B^c] = P[A] = 1/4. \quad (2)$$

It also follows that $P[A \cup B^c] = P[B^c] = 1 - 1/8 = 7/8$.

(b) Events A and B are dependent since $P[AB] \neq P[A]P[B]$.

(c) Since C and D are independent,

$$P[C \cap D] = P[C]\, P[D] = 15/64. \tag{3}$$

The next few items are a little trickier. From Venn diagrams, we see

$$P[C \cap D^c] = P[C] - P[C \cap D] = 5/8 - 15/64 = 25/64. \tag{4}$$

It follows that

$$\begin{aligned} P[C \cup D^c] &= P[C] + P[D^c] - P[C \cap D^c] & (5)\\ &= 5/8 + (1 - 3/8) - 25/64 = 55/64. & (6) \end{aligned}$$

Using DeMorgan's law, we have

$$P[C^c \cap D^c] = \left(\frac{3}{8}\right)\left(\frac{5}{8}\right) = 15/64 = 1 - P[C \cup D] = P[(C \cup D)^c]. \tag{7}$$

(d) Since $P[C^cD^c] = P[C^c]P[D^c]$, C^c and D^c are independent.

Problem 1.6.4 ■

In an experiment, A, B, C, and D are events with probabilities $P[A \cup B] = 5/8$, $P[A] = 3/8$, $P[C \cap D] = 1/3$, and $P[C] = 1/2$. Furthermore, A and B are disjoint, while C and D are independent.

(a) Find $P[A \cap B]$, $P[B]$, $P[A \cap B^c]$, and $P[A \cup B^c]$.

(b) Are A and B independent?

(c) Find $P[D]$, $P[C \cap D^c]$, $P[C^c \cap D^c]$, and $P[C|D]$.

(d) Find $P[C \cup D]$ and $P[C \cup D^c]$.

(e) Are C and D^c independent?

Problem 1.6.4 Solution

(a) Since $A \cap B = \emptyset$, $P[A \cap B] = 0$. To find $P[B]$, we can write

$$\begin{aligned} P[A \cup B] &= P[A] + P[B] - P[A \cap B] & (1)\\ 5/8 &= 3/8 + P[B] - 0. & (2) \end{aligned}$$

Thus, $P[B] = 1/4$. Since A is a subset of B^c, $P[A \cap B^c] = P[A] = 3/8$. Furthermore, since A is a subset of B^c, $P[A \cup B^c] = P[B^c] = 3/4$.

(b) The events A and B are dependent because

$$P[AB] = 0 \neq 3/32 = P[A]P[B]. \quad (3)$$

(c) Since C and D are independent $P[CD] = P[C]P[D]$. So

$$P[D] = \frac{P[CD]}{P[C]} = \frac{1/3}{1/2} = 2/3. \quad (4)$$

In addition, $P[C \cap D^c] = P[C] - P[C \cap D] = 1/2 - 1/3 = 1/6$. To find $P[C^c \cap D^c]$, we first observe that

$$P[C \cup D] = P[C] + P[D] - P[C \cap D] = 1/2 + 2/3 - 1/3 = 5/6. \quad (5)$$

By De Morgan's Law, $C^c \cap D^c = (C \cup D)^c$. This implies

$$P[C^c \cap D^c] = P[(C \cup D)^c] = 1 - P[C \cup D] = 1/6. \quad (6)$$

Note that a second way to find $P[C^c \cap D^c]$ is to use the fact that if C and D are independent, then C^c and D^c are independent. Thus

$$P[C^c \cap D^c] = P[C^c]P[D^c] = (1 - P[C])(1 - P[D]) = 1/6. \quad (7)$$

Finally, since C and D are independent events, $P[C|D] = P[C] = 1/2$.

(d) Note that we found $P[C \cup D] = 5/6$. We can also use the earlier results to show

$$P[C \cup D^c] = P[C] + P[D] - P[C \cap D^c] = 1/2 + (1 - 2/3) - 1/6 = 2/3. \quad (8)$$

(e) By Definition 1.7, events C and D^c are independent because

$$P[C \cap D^c] = 1/6 = (1/2)(1/3) = P[C]P[D^c]. \quad (9)$$

Problem 1.6.5 ■

In an experiment with equiprobable outcomes, the event space *(I would have said "the sample space")* is $S = \{1, 2, 3, 4\}$ and $P[s] = 1/4$ for all $s \in S$. Find three events in S that are pairwise independent but are not independent. (Note: Pairwise independent events meet the first three conditions of Definition 1.8).

Problem 1.6.5 Solution

For a sample space $S = \{1, 2, 3, 4\}$ with equiprobable outcomes, consider the events

$$A_1 = \{1, 2\} \qquad A_2 = \{2, 3\} \qquad A_3 = \{3, 1\}. \quad (1)$$

Each event A_i has probability 1/2. Moreover, each pair of events is independent since

$$P[A_1A_2] = P[A_2A_3] = P[A_3A_1] = 1/4. \quad (2)$$

However, the three events A_1, A_2, A_3 are not independent since

$$P[A_1A_2A_3] = 0 \neq P[A_1]P[A_2]P[A_3]. \quad (3)$$

Problem 1.6.6 ■

(Continuation of Problem 1.4.3) One of Mendel's most significant results was the conclusion that genes determining different characteristics are transmitted independently. In pea plants, Mendel found that round peas are a dominant trait over wrinkled peas. Mendel crossbred a group of (rr, yy) peas with a group of (ww, gg) peas. In this notation, rr denotes a pea with two "round" genes and ww denotes a pea with two "wrinkled" genes. The first generation were either (rw, yg), (rw, gy), (wr, yg), or (wr, gy) plants with both hybrid shape and hybrid color. Breeding among the first generation yielded second-generation plants in which genes for each characteristic were equally likely to be either dominant or recessive. What is the probability $P[Y]$ that a second-generation pea plant has yellow seeds? What is the probability $P[R]$ that a second-generation plant has round peas? Are R and Y independent events? How many visibly different kinds of pea plants would Mendel observe in the second generation? What are the probabilities of each of these kinds?

Problem 1.6.6 Solution

There are 16 distinct equally likely outcomes for the second generation of pea plants based on a first generation of $\{rwyg, rwgy, wryg, wrgy\}$. They are listed below

$$\begin{array}{cccc} rryy & rryg & rrgy & rrgg \\ rwyy & rwyg & rwgy & rwgg \\ wryy & wryg & wrgy & wrgg \\ wwyy & wwyg & wwgy & wwgg \end{array} \tag{1}$$

A plant has yellow seeds, that is event Y occurs, if a plant has at least one dominant y gene. Except for the four outcomes with a pair of recessive g genes, the remaining 12 outcomes have yellow seeds. From the above, we see that

$$P[Y] = 12/16 = 3/4 \tag{2}$$

and

$$P[R] = 12/16 = 3/4. \tag{3}$$

To find the conditional probabilities $P[R|Y]$ and $P[Y|R]$, we first must find $P[RY]$. Note that RY, the event that a plant has rounded yellow seeds, is the set of outcomes

$$RY = \{rryy, rryg, rrgy, rwyy, rwyg, rwgy, wryy, wryg, wrgy\}. \tag{4}$$

Since $P[RY] = 9/16$,

$$P[Y|R] = \frac{P[RY]}{P[R]} = \frac{9/16}{3/4} = 3/4 \tag{5}$$

and

$$P[R|Y] = \frac{P[RY]}{P[Y]} = \frac{9/16}{3/4} = 3/4. \tag{6}$$

Thus $P[R|Y] = P[R]$ and $P[Y|R] = P[Y]$ and R and Y are independent events. There are four visibly different pea plants, corresponding to whether the peas are round (R) or not (R^c), or yellow (Y) or not (Y^c). These four visible events have probabilities

$$P[RY] = 9/16 \qquad P[RY^c] = 3/16, \tag{7}$$

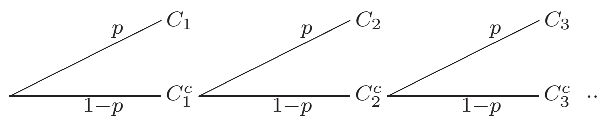
$$P[R^cY] = 3/16 \qquad P[R^cY^c] = 1/16. \tag{8}$$

Problem 1.7.10 ■

Each time a fisherman casts his line, a fish is caught with probability p, independent of whether a fish is caught on any other cast of the line. The fisherman will fish all day until a fish is caught and then he will quit and go home. Let C_i denote the event that on cast i the fisherman catches a fish. Draw the tree for this experiment and find $P[C_1]$, $P[C_2]$, and $P[C_n]$.

Problem 1.7.10 Solution

The experiment ends as soon as a fish is caught. The tree resembles



From the tree, $P[C_1] = p$ and $P[C_2] = (1-p)p$. Finally, a fish is caught on the nth cast if no fish were caught on the previous $n-1$ casts. Thus,

$$P\left[C_n\right] = (1-p)^{n-1}p. \tag{1}$$

Problem 1.9.1 •

Consider a binary code with 5 bits (0 or 1) in each code word. An example of a code word is 01010. In each code word, a bit is a zero with probability 0.8, independent of any other bit.

(a) What is the probability of the code word 00111?

(b) What is the probability that a code word contains exactly three ones?

Problem 1.9.1 Solution

(a) Since the probability of a zero is 0.8, we can express the probability of the code word 00111 as 2 occurrences of a 0 and three occurrences of a 1. Therefore

$$P\left[00111\right] = (0.8)^2(0.2)^3 = 0.00512. \tag{1}$$

(b) The probability that a code word has exactly three 1's is

$$P\left[\text{three 1's}\right] = \binom{5}{3}(0.8)^2(0.2)^3 = 0.0512. \tag{2}$$

Problem 1.9.3 •

Suppose each day that you drive to work a traffic light that you encounter is either green with probability 7/16, red with probability 7/16, or yellow with probability 1/8, independent of the status of the light on any other day. If over the course of five days, G, Y, and R denote the number of times the light is found to be green, yellow, or red, respectively, what is the probability that $P[G = 2, Y = 1, R = 2]$? Also, what is the probability $P[G = R]$?

Problem 1.9.3 Solution

We know that the probability of a green and red light is 7/16, and that of a yellow light is 1/8. Since there are always 5 lights, G, Y, and R obey the multinomial probability law:

$$P[G = 2, Y = 1, R = 2] = \frac{5!}{2!1!2!}\left(\frac{7}{16}\right)^2\left(\frac{1}{8}\right)\left(\frac{7}{16}\right)^2. \tag{1}$$

The probability that the number of green lights equals the number of red lights

$$P[G = R] = P[G = 1, R = 1, Y = 3] + P[G = 2, R = 2, Y = 1] + P[G = 0, R = 0, Y = 5] \tag{2}$$

$$= \frac{5!}{1!1!3!}\left(\frac{7}{16}\right)\left(\frac{7}{16}\right)\left(\frac{1}{8}\right)^3 + \frac{5!}{2!1!2!}\left(\frac{7}{16}\right)^2\left(\frac{7}{16}\right)^2\left(\frac{1}{8}\right) + \frac{5!}{0!0!5!}\left(\frac{1}{8}\right)^5 \tag{3}$$

$$\approx 0.1449. \tag{4}$$

Problem 1.10.2 ■

We wish to modify the cellular telephone coding system in Example 1.41 in order to reduce the number of errors. In particular, if there are two or three zeroes in the received sequence of 5 bits, we will say that a deletion (event D) occurs. Otherwise, if at least 4 zeroes are received, then the receiver decides a zero was sent. Similarly, if at least 4 ones are received, then the receiver decides a one was sent. We say that an error occurs if either a one was sent and the receiver decides zero was sent or if a zero was sent and the receiver decides a one was sent. For this modified protocol, what is the probability $P[E]$ of an error? What is the probability $P[D]$ of a deletion?

Problem 1.10.2 Solution

Suppose that the transmitted bit was a 1. We can view each repeated transmission as an independent trial. We call each repeated bit the receiver decodes as 1 a success. Using $S_{k,5}$ to denote the event of k successes in the five trials, then the probability k 1's are decoded at the receiver is

$$P[S_{k,5}] = \binom{5}{k} p^k (1-p)^{5-k}, \quad k = 0, 1, \ldots, 5. \tag{1}$$

The probability that *[the transmitted]* bit is decoded correctly is

$$P[C] = P[S_{5,5}] + P[S_{4,5}] = p^5 + 5p^4(1-p) = 0.91854. \tag{2}$$

The probability a deletion occurs is

$$P[D] = P[S_{3,5}] + P[S_{2,5}] = 10p^3(1-p)^2 + 10p^2(1-p)^3 = 0.081. \tag{3}$$

The probability of an error is

$$P[E] = P[S_{1,5}] + P[S_{0,5}] = 5p(1-p)^4 + (1-p)^5 = 0.00046. \tag{4}$$

Note that if a 0 is transmitted, then 0 is sent five times and we call decoding a 0 a success. You should convince yourself that this a symmetric situation with the same deletion and error probabilities. Introducing deletions reduces the probability of an error by roughly a factor of 20. However, the probability of successful decoding is also reduced.

Problem 1.10.3 ■

Suppose a 10-digit phone number is transmitted by a cellular phone using four binary symbols for each digit, using the model of binary symbol errors and deletions given in Problem 1.10.2. If C denotes the number of bits sent correctly, D the number of deletions, and E the number of errors, what is $P[C = c, D = d, E = e]$? Your answer should be correct for any choice of c, d, and e.

Problem 1.10.3 Solution

Note that each digit 0 through 9 is mapped to the 4 bit binary representation of the digit. That is, 0 corresponds to 0000, 1 to 0001, up to 9 which corresponds to 1001. Of course, the 4 bit binary numbers corresponding to numbers 10 through 15 go unused, however this is unimportant to our problem. the 10 digit number results in the transmission of 40 bits. For each bit, an independent trial determines whether the bit was correct, a deletion, or an error. In Problem 1.10.2, we found the probabilities of these events to be

$$P[C] = \gamma = 0.91854, \qquad P[D] = \delta = 0.081, \qquad P[E] = \epsilon = 0.00046. \tag{1}$$

Since each of the 40 bit transmissions is an independent trial, the joint probability of c correct bits, d deletions, and e erasures has the multinomial probability

$$P[C = c, D = d, E = d] = \begin{cases} \frac{40!}{c!d!e!}\gamma^c\delta^d\epsilon^e & c + d + e = 40; c, d, e \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Problem 1.11.2 ■

Build a Matlab simulation of 50 trials of the experiment of Example 1.27. Your output should be a pair of 50×1 vectors $\mathbf{C}$ and $\mathbf{H}$. For the ith trial, H_i will record whether it was heads ($H_i = 1$) or tails ($H_i = 0$), and $C_i \in \{1, 2\}$ will record which coin was picked.

Problem 1.11.2 Solution

Rather than just solve the problem for 50 trials, we can write a function that generates vectors $\mathbf{C}$ and $\mathbf{H}$ for an arbitrary number of trials n. The code for this task is

```
function [C,H]=twocoin(n);
C=ceil(2*rand(n,1));
P=1-(C/4);
H=(rand(n,1)< P);
```

The first line produces the $n \times 1$ vector `C` such that `C(i)` indicates whether coin 1 or coin 2 is chosen for trial i. Next, we generate the vector `P` such that `P(i)=0.75` if `C(i)=1`; otherwise, if `C(i)=2`, then `P(i)=0.5`. As a result, `H(i)` is the simulated result of a coin flip with heads, corresponding to `H(i)=1`, occurring with probability `P(i)`.

Problem 1.11.3 ■

Following Quiz 1.9, suppose the communication link has different error probabilities for transmitting 0 and 1. When a 1 is sent, it is received as a 0 with probability 0.01. When a 0 is sent, it is received as a 1 with probability 0.03. Each bit in a packet is still equally likely to be a 0 or 1. Packets have been coded such that if five or fewer bits are received in error, then the packet can be decoded. Simulate the transmission of 100 packets, each containing 100 bits. Count the number of packets decoded correctly.

Problem 1.11.3 Solution

Rather than just solve the problem for 100 trials, we can write a function that generates n packets for an arbitrary number of trials n. The code for this task is

```
function C=bit100(n);
% n is the number of 100 bit packets sent
B=floor(2*rand(n,100));
P=0.03-0.02*B;
E=(rand(n,100)< P);
C=sum((sum(E,2)<=5));
```

First, `B` is an $n \times 100$ matrix such that `B(i,j)` indicates whether bit `i` of packet `j` is zero or one. Next, we generate the $n \times 100$ matrix `P` such that `P(i,j)=0.03` if `B(i,j)=0`; otherwise, if `B(i,j)=1`, then `P(i,j)=0.01`. As a result, `E(i,j)` is the simulated error indicator for bit `i` of packet `j`. That is, `E(i,j)=1` if bit `i` of packet `j` is in error; otherwise `E(i,j)=0`. Next we sum across the rows of `E` to obtain the number of errors in each packet. Finally, we count the number of packets with 5 or more errors.

For $n = 100$ packets, the packet success probability is inconclusive. Experimentation will show that `C=97`, `C=98`, `C=99` and `C=100` correct packets are typical values that might be observed. By increasing n, more consistent results are obtained. For example, repeated trials with $n = 100,000$ packets typically produces around $C = 98,400$ correct packets. Thus 0.984 is a reasonable estimate for the probability of a packet being transmitted correctly.

Solutions to HW3

Note: Most of these solutions were generated by R. D. Yates and D. J. Goodman, the authors of our textbook. I have added comments in italics where I thought more detail was appropriate. I have also largely rewritten the solutions to problems 2.10.1, 2.10.2, and 2.10.3.

Problem 2.2.1 •

The random variable N has PMF

$$P_N(n) = \begin{cases} c(1/2)^n & n = 0, 1, 2, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the value of the constant c?

(b) What is $P[N \leq 1]$?

Problem 2.2.1 Solution

(a) We wish to find the value of c that makes the PMF sum up to one.

$$P_N(n) = \begin{cases} c(1/2)^n & n = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Therefore, $\sum_{n=0}^{2} P_N(n) = c + c/2 + c/4 = 1$, implying $c = 4/7$.

(b) The probability that $N \leq 1$ is

$$P[N \leq 1] = P[N = 0] + P[N = 1] = 4/7 + 2/7 = 6/7 \tag{2}$$

Problem 2.2.2 •

For random variables X and R defined in Example 2.5, find $P_X(x)$ and $P_R(r)$. In addition, find the following probabilities:

(a) $P[X = 0]$

(b) $P[X < 3]$

(c) $P[R > 1]$

Problem 2.2.2 Solution

From Example 2.5, we can write the PMF of X and the PMF of R as

$$P_X(x) = \begin{cases} 1/8 & x = 0 \\ 3/8 & x = 1 \\ 3/8 & x = 2 \\ 1/8 & x = 3 \\ 0 & \text{otherwise} \end{cases} \qquad P_R(r) = \begin{cases} 1/4 & r = 0 \\ 3/4 & r = 2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

From the PMFs $P_X(x)$ and $P_R(r)$, we can calculate the requested probabilities

(a) $P[X = 0] = P_X(0) = 1/8$.

(b) $P[X < 3] = P_X(0) + P_X(1) + P_X(2) = 7/8$.

(c) $P[R > 1] = P_R(2) = 3/4$.

Problem 2.2.3 •
The random variable V has PMF

$$P_V(v) = \begin{cases} cv^2 & v = 1, 2, 3, 4, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the value of the constant c.

(b) Find $P[V \in \{u^2 | u = 1, 2, 3, \cdots\}]$.

(c) Find the probability that V is an even number.

(d) Find $P[V > 2]$.

Problem 2.2.3 Solution

(a) We must choose c to make the PMF of V sum to one.

$$\sum_{v=1}^{4} P_V(v) = c(1^2 + 2^2 + 3^2 + 4^2) = 30c = 1 \tag{1}$$

Hence $c = 1/30$.

(b) Let $U = \{u^2 | u = 1, 2, \ldots\}$ so that

$$P[V \in U] = P_V(1) + P_V(4) = \frac{1}{30} + \frac{4^2}{30} = \frac{17}{30} \tag{2}$$

(c) The probability that V is even is

$$P[V \text{ is even}] = P_V(2) + P_V(4) = \frac{2^2}{30} + \frac{4^2}{30} = \frac{2}{3} \tag{3}$$

(d) The probability that $V > 2$ is

$$P[V > 2] = P_V(3) + P_V(4) = \frac{3^2}{30} + \frac{4^2}{30} = \frac{5}{6} \tag{4}$$

Problem 2.2.4 •
The random variable X has PMF

$$P_X(x) = \begin{cases} c/x & x = 2, 4, 8, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the value of the constant c?

(b) What is $P[X = 4]$?

(c) What is $P[X < 4]$?

(d) What is $P[3 \leq X \leq 9]$?

Problem 2.2.4 Solution

(a) We choose c so that the PMF sums to one.

$$\sum_x P_X(x) = \frac{c}{2} + \frac{c}{4} + \frac{c}{8} = \frac{7c}{8} = 1 \tag{1}$$

Thus $c = 8/7$.

(b)

$$P[X = 4] = P_X(4) = \frac{8}{7 \cdot 4} = \frac{2}{7} \tag{2}$$

(c)

$$P[X < 4] = P_X(2) = \frac{8}{7 \cdot 2} = \frac{4}{7} \tag{3}$$

(d)

$$P[3 \leq X \leq 9] = P_X(4) + P_X(8) = \frac{8}{7 \cdot 4} + \frac{8}{7 \cdot 8} = \frac{3}{7} \tag{4}$$

Problem 2.3.1 •
In a package of M&Ms, Y, the number of yellow M&Ms, is uniformly distributed between 5 and 15.

(a) What is the PMF of Y?

(b) What is $P[Y < 10]$?

(c) What is $P[Y > 12]$?

(d) What is $P[8 \leq Y \leq 12]$?

Problem 2.3.1 Solution

(a) If it is indeed true that Y, the number of yellow M&M's in a package, is uniformly distributed between 5 and 15, then the PMF of Y, is

$$P_Y(y) = \begin{cases} 1/11 & y = 5, 6, 7, \ldots, 15 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(b)

$$P[Y < 10] = P_Y(5) + P_Y(6) + \cdots + P_Y(9) = 5/11 \tag{2}$$

(c)

$$P[Y > 12] = P_Y(13) + P_Y(14) + P_Y(15) = 3/11 \tag{3}$$

(d)

$$P[8 \le Y \le 12] = P_Y(8) + P_Y(9) + \cdots + P_Y(12) = 5/11 \tag{4}$$

Problem 2.3.4 •

Anytime a child throws a Frisbee, the child's dog catches the Frisbee with probability p, independent of whether the Frisbee is caught on any previous throw. When the dog catches the Frisbee, it runs away with the Frisbee, never to be seen again. The child continues to throw the Frisbee until the dog catches it. Let X denote the number of times the Frisbee is thrown.

(a) What is the PMF $P_X(x)$?

(b) If $p = 0.2$, what is the probability that the child will throw the Frisbee more than four times?

Problem 2.3.4 Solution

(a) Let X be the number of times the frisbee is thrown until the dog catches it and runs away. Each throw of the frisbee can be viewed as a Bernoulli trial in which a success occurs if the dog catches the frisbee an runs away. Thus, the experiment ends on the first success and X has the geometric PMF

$$P_X(x) = \begin{cases} (1-p)^{x-1}p & x = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(b) The child will throw the frisbee more than four times iff there are failures on the first 4 trials which has probability $(1-p)^4$. If $p = 0.2$, the probability of more than four throws is $(0.8)^4 = 0.4096$.

Note: There is a less elegant but equally effective way to solve the problem, which I show below is equivalent.

$$\begin{aligned} P[X > 4] &= 1 - (P[X = 4] + P[X = 3] + P[X = 2] + P[X = 1]) && (2)\\ &= 1 - \left(p(1-p)^3 + p(1-p)^2 + p(1-p) + p\right) && (3)\\ &= 1 - 4p + 6p^2 - 4p^3 + p^4 && (4)\\ &= (1-p)^4 && (5)\end{aligned}$$

Problem 2.3.6 •

The number of bits B in a fax transmission is a geometric $(p = 2.5 \cdot 10^{-5})$ random variable. What is the probability $P[B > 500{,}000]$ that a fax has over 500,000 bits?

Problem 2.3.6 Solution

The probability of more than 500,000 bits is

$$\begin{aligned} P[B > 500{,}000] &= 1 - \sum_{b=1}^{500{,}000} P_B(b) && (1)\\ &= 1 - p \sum_{b=1}^{500{,}000} (1-p)^{b-1} && (2)\end{aligned}$$

Math Fact B.4 implies that $(1-x)\sum_{b=1}^{500{,}000} x^{b-1} = 1 - x^{500{,}000}$. Substituting, $x = 1 - p$, we obtain:

$$\begin{aligned} P[B > 500{,}000] &= 1 - (1 - (1-p)^{500{,}000}) && (3)\\ &= (1 - 2.5 \times 10^{-5})^{500{,}000} \approx 0.3726 \times 10^{-5} && (4)\end{aligned}$$

Problem 2.3.7 •

The number of buses that arrive at a bus stop in T minutes is a Poisson random variable B with expected value $T/5$.

(a) What is the PMF of B, the number of buses that arrive in T minutes?

(b) What is the probability that in a two-minute interval, three buses will arrive?

(c) What is the probability of no buses arriving in a 10-minute interval?

(d) How much time should you allow so that with probability 0.99 at least one bus arrives?

Problem 2.3.7 Solution

Since an average of $T/5$ buses arrive in an interval of T minutes, buses arrive at the bus stop at a rate of $1/5$ buses per minute.

(a) From the definition of the Poisson PMF, the PMF of B, the number of buses in T minutes, is

$$P_B(b) = \begin{cases} (T/5)^b e^{-T/5}/b! & b = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(b) Choosing $T = 2$ minutes, the probability that three buses arrive in a two minute interval is

$$P_B(3) = (2/5)^3 e^{-2/5}/3! \approx 0.0072 \tag{2}$$

(c) By choosing $T = 10$ minutes, the probability of zero buses arriving in a ten minute interval is

$$P_B(0) = e^{-10/5}/0! = e^{-2} \approx 0.135 \tag{3}$$

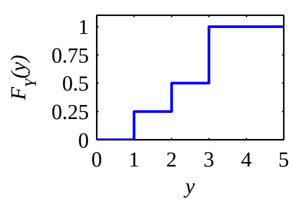
(d) The probability that at least one bus arrives in T minutes is

$$P[B \geq 1] = 1 - P[B = 0] = 1 - e^{-T/5} \geq 0.99 \tag{4}$$

Rearranging yields $T \geq 5 \ln 100 \approx 23.0$ minutes.

Problem 2.4.1 •

Discrete random variable Y has the CDF $F_Y(y)$ as shown:



Use the CDF to find the following probabilities:

(a) $P[Y < 1]$

(b) $P[Y \leq 1]$

(c) $P[Y > 2]$

(d) $P[Y \geq 2]$

(e) $P[Y = 1]$

(f) $P[Y = 3]$

(g) $P_Y(y)$

Problem 2.4.1 Solution

Using the CDF given in the problem statement we find that

(a) $P[Y < 1] = 0$

(b) $P[Y \leq 1] = 1/4$

(c) $P[Y > 2] = 1 - P[Y \leq 2] = 1 - 1/2 = 1/2$

(d) $P[Y \geq 2] = 1 - P[Y < 2] = 1 - 1/4 = 3/4$

(e) $P[Y = 1] = 1/4$

(f) $P[Y = 3] = 1/2$

(g) From the staircase CDF of Problem 2.4.1, we see that Y is a discrete random variable. The jumps in the CDF occur at at the values that Y can take on. The height of each jump equals the probability of that value. The PMF of Y is

$$P_Y(y) = \begin{cases} 1/4 & y = 1 \\ 1/4 & y = 2 \\ 1/2 & y = 3 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Problem 2.4.3 •

The random variable X has CDF

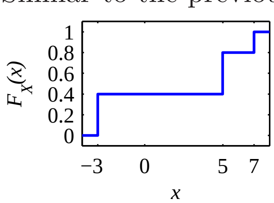
$$F_X(x) = \begin{cases} 0 & x < -3, \\ 0.4 & -3 \leq x < 5, \\ 0.8 & 5 \leq x < 7, \\ 1 & x \geq 7. \end{cases}$$

(a) Draw a graph of the CDF.

(b) Write $P_X(x)$, the PMF of X.

Problem 2.4.3 Solution

(a) Similar to the previous problem, the graph of the CDF is shown below.



$$F_X(x) = \begin{cases} 0 & x < -3 \\ 0.4 & -3 \leq x < 5 \\ 0.8 & 5 \leq x < 7 \\ 1 & x \geq 7 \end{cases} \tag{1}$$

(b) The corresponding PMF of X is

$$P_X(x) = \begin{cases} 0.4 & x = -3 \\ 0.4 & x = 5 \\ 0.2 & x = 7 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Problem 2.5.2 •
Voice calls cost 20 cents each and data calls cost 30 cents each. C is the cost of one telephone call. The probability that a call is a voice call is $P[V] = 0.6$. The probability of a data call is $P[D] = 0.4$.

(a) Find $P_C(c)$, the PMF of C.

(b) What is $E[C]$, the expected value of C?

Problem 2.5.2 Solution
Voice calls and data calls each cost 20 cents and 30 cents respectively. Furthermore the respective probabilities of each type of call are 0.6 and 0.4.

(a) Since each call is either a voice or data call, the cost of one call can only take the two values associated with the cost of each type of call. Therefore the PMF of X is

$$P_X(x) = \begin{cases} 0.6 & x = 20 \\ 0.4 & x = 30 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(b) The expected cost, $E[C]$, is simply the sum of the cost of each type of call multiplied by the probability of such a call occurring.

$$E[C] = 20(0.6) + 30(0.4) = 24\,\text{cents} \tag{2}$$

Problem 2.5.7 •
Find the expected value of a binomial $(n = 5, p = 1/2)$ random variable X.

Problem 2.5.7 Solution
From Definition 2.7, random variable X has PMF

$$P_X(x) = \begin{cases} \binom{5}{x}(1/2)^5 & x = 0,1,2,3,4,5 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The expected value of X is

$$\begin{aligned} E[X] &= \sum_{x=0}^{5} x P_X(x) && (2) \\ &= 0\binom{5}{0}\frac{1}{2^5} + 1\binom{5}{1}\frac{1}{2^5} + 2\binom{5}{2}\frac{1}{2^5} + 3\binom{5}{3}\frac{1}{2^5} + 4\binom{5}{4}\frac{1}{2^5} + 5\binom{5}{5}\frac{1}{2^5} && (3) \\ &= [5 + 20 + 30 + 20 + 5]/2^5 = 2.5 && (4) \end{aligned}$$

Problem 2.6.1 •
Given the random variable Y in Problem 2.4.1, let $U = g(Y) = Y^2$.

(a) Find $P_U(u)$.

(b) Find $F_U(u)$.

(c) Find $E[U]$.

Problem 2.6.1 Solution
From the solution to Problem 2.4.1, the PMF of Y is

$$P_Y(y) = \begin{cases} 1/4 & y = 1 \\ 1/4 & y = 2 \\ 1/2 & y = 3 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) Since Y has range $S_Y = \{1, 2, 3\}$, the range of $U = Y^2$ is $S_U = \{1, 4, 9\}$. The PMF of U can be found by observing that

$$P[U = u] = P\left[Y^2 = u\right] = P\left[Y = \sqrt{u}\right] + P\left[Y = -\sqrt{u}\right] \tag{2}$$

Since Y is never negative, $P_U(u) = P_Y(\sqrt{u})$. Hence,

$$P_U(1) = P_Y(1) = 1/4 \qquad P_U(4) = P_Y(2) = 1/4 \qquad P_U(9) = P_Y(3) = 1/2 \tag{3}$$

For all other values of u, $P_U(u) = 0$. The complete expression for the PMF of U is

$$P_U(u) = \begin{cases} 1/4 & u = 1 \\ 1/4 & u = 4 \\ 1/2 & u = 9 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

(b) From the PMF, it is straighforward to write down the CDF.

$$F_U(u) = \begin{cases} 0 & u < 1 \\ 1/4 & 1 \le u < 4 \\ 1/2 & 4 \le u < 9 \\ 1 & u \ge 9 \end{cases} \tag{5}$$

(c) From Definition 2.14, the expected value of U is

$$E[U] = \sum_u u P_U(u) = 1(1/4) + 4(1/4) + 9(1/2) = 5.75 \tag{6}$$

From Theorem 2.10, we can calculate the expected value of U as

$$E[U] = E\left[Y^2\right] = \sum_y y^2 P_Y(y) = 1^2(1/4) + 2^2(1/4) + 3^2(1/2) = 5.75 \tag{7}$$

As we expect, both methods yield the same answer.

Problem 2.6.3 •
Given the random variable X in Problem 2.4.3, let $W = g(X) = -X$.

(a) Find $P_W(w)$.

(b) Find $F_W(w)$.

(c) Find $E[W]$.

Problem 2.6.3 Solution
From the solution to Problem 2.4.3, the PMF of X is

$$P_X(x) = \begin{cases} 0.4 & x = -3 \\ 0.4 & x = 5 \\ 0.2 & x = 7 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) The PMF of $W = -X$ satisfies

$$P_W(w) = P[-X = w] = P_X(-w) \tag{2}$$

This implies

$$P_W(-7) = P_X(7) = 0.2 \qquad P_W(-5) = P_X(5) = 0.4 \qquad P_W(3) = P_X(-3) = 0.4 \tag{3}$$

The complete PMF for W is

$$P_W(w) = \begin{cases} 0.2 & w = -7 \\ 0.4 & w = -5 \\ 0.4 & w = 3 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

(b) From the PMF, the CDF of W is

$$F_W(w) = \begin{cases} 0 & w < -7 \\ 0.2 & -7 \le w < -5 \\ 0.6 & -5 \le w < 3 \\ 1 & w \ge 3 \end{cases} \tag{5}$$

(c) From the PMF, W has expected value

$$E[W] = \sum_w w P_W(w) = -7(0.2) + -5(0.4) + 3(0.4) = -2.2 \tag{6}$$

Problem 2.6.4 •

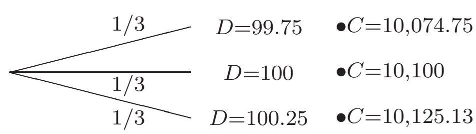
At a discount brokerage, a stock purchase or sale worth less than \$10,000 incurs a brokerage fee of 1% of the value of the transaction. A transaction worth more than \$10,000 incurs a fee of \$100 plus 0.5% of the amount exceeding \$10,000. Note that for a fraction of a cent, the brokerage always charges the customer a full penny. You wish to buy 100 shares of a stock whose price D in dollars has PMF

$$P_D(d) = \begin{cases} 1/3 & d = 99.75, 100, 100.25, \\ 0 & \text{otherwise.} \end{cases}$$

What is the PMF of C, the cost of buying the stock (including the brokerage fee).

Problem 2.6.4 Solution

A tree for the experiment is



Thus C has three equally likely outcomes. The PMF of C is

$$P_C(c) = \begin{cases} 1/3 & c = 10{,}074.75, \quad 10{,}100, \quad 10{,}125.13 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Problem 2.7.1 •

For random variable T in Quiz 2.6, first find the expected value $E[T]$ using Theorem 2.10. Next, find $E[T]$ using Definition 2.14.

Problem 2.7.1 Solution

From the solution to Quiz 2.6, we found that $T = 120 - 15N$. By Theorem 2.10,

$$\begin{aligned} E[T] &= \sum_{n \in S_N} (120 - 15n) P_N(n) && (1) \\ &= 0.1(120) + 0.3(120 - 15) + 0.3(120 - 30) + 0.3(120 - 45) = 93 && (2) \end{aligned}$$

Also from the solution to Quiz 2.6, we found that

$$P_T(t) = \begin{cases} 0.3 & t = 75, 90, 105 \\ 0.1 & t = 120 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Using Definition 2.14,

$$E[T] = \sum_{t \in S_T} t P_T(t) = 0.3(75) + 0.3(90) + 0.3(105) + 0.1(120) = 93 \tag{4}$$

As expected, the two calculations give the same answer.

Problem 2.7.2 •

In a certain lottery game, the chance of getting a winning ticket is exactly one in a thousand. Suppose a person buys one ticket each day (except on the leap year day February 29) over a period of fifty years. What is the expected number $E[T]$ of winning tickets in fifty years? If each winning ticket is worth \$1000, what is the expected amount $E[R]$ collected on these winning tickets? Lastly, if each ticket costs \$2, what is your expected net profit $E[Q]$?

Problem 2.7.2 Solution

Whether a lottery ticket is a winner is a Bernoulli trial with a success probability of 0.001. If we buy one every day for 50 years for a total of $50 \cdot 365 = 18250$ tickets, then the number of winning tickets T is a binomial random variable with mean

$$E[T] = 18250(0.001) = 18.25 \tag{1}$$

Since each winning ticket grosses \$1000, the revenue we collect over 50 years is $R = 1000T$ dollars. The expected revenue is

$$E[R] = 1000E[T] = 18250 \tag{2}$$

But buying a lottery ticket everyday for 50 years, at \$2.00 a pop isn't cheap and will cost us a total of $18250 \cdot 2 = \$36500$. Our net profit is then $Q = R - 36500$ and the result of our loyal 50 year patronage of the lottery system, is disappointing expected loss of

$$E[Q] = E[R] - 36500 = -18250 \tag{3}$$

Problem 2.7.4 •

It can take up to four days after you call for service to get your computer repaired. The computer company charges for repairs according to how long you have to wait. The number of days D until the service technician arrives and the service charge C, in dollars, are described by

$$P_D(d) = \begin{cases} 0.2 & d = 1, \\ 0.4 & d = 2, \\ 0.3 & d = 3, \\ 0.1 & d = 4, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$C = \begin{cases} 90 & \text{for 1-day service,} \\ 70 & \text{for 2-day service,} \\ 40 & \text{for 3-day service,} \\ 40 & \text{for 4-day service.} \end{cases}$$

(a) What is the expected waiting time $\mu_D = E[D]$?

(b) What is the expected deviation $E[D-\mu_D]$?

(c) Express C as a function of D.

(d) What is the expected value $E[C]$?

Problem 2.7.4 Solution

Given the distributions of D, the waiting time in days and the resulting cost, C, we can answer the following questions.

(a) The expected waiting time is simply the expected value of D.

$$E[D] = \sum_{d=1}^{4} d \cdot P_D(d) = 1(0.2) + 2(0.4) + 3(0.3) + 4(0.1) = 2.3 \tag{1}$$

(b) The expected deviation from the waiting time is

$$E[D-\mu_D] = E[D] - E[\mu_d] = \mu_D - \mu_D = 0 \tag{2}$$

(c) C can be expressed as a function of D in the following manner.

$$C(D) = \begin{cases} 90 & D=1 \\ 70 & D=2 \\ 40 & D=3 \\ 40 & D=4 \end{cases} \tag{3}$$

(d) The expected service charge is

$$E[C] = 90(0.2) + 70(0.4) + 40(0.3) + 40(0.1) = 62 \text{ dollars} \tag{4}$$

Problem 2.8.1 •

In an experiment to monitor two calls, the PMF of N, the number of voice calls, is

$$P_N(n) = \begin{cases} 0.2 & n=0, \\ 0.7 & n=1, \\ 0.1 & n=2, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $E[N]$, the expected number of voice calls.

(b) Find $E[N^2]$, the second moment of N.

(c) Find $\mathrm{Var}[N]$, the variance of N.

(d) Find σ_N, the standard deviation of N.

Problem 2.8.1 Solution
Given the following PMF

$$P_N(n) = \begin{cases} 0.2 & n = 0 \\ 0.7 & n = 1 \\ 0.1 & n = 2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) $E[N] = (0.2)0 + (0.7)1 + (0.1)2 = 0.9$

(b) $E[N^2] = (0.2)0^2 + (0.7)1^2 + (0.1)2^2 = 1.1$

(c) $\mathrm{Var}[N] = E[N^2] - E[N]^2 = 1.1 - (0.9)^2 = 0.29$

(d) $\sigma_N = \sqrt{\mathrm{Var}[N]} = \sqrt{0.29}$

Problem 2.8.12 •
For the delay D in Problem 2.7.4, what is the standard deviation σ_D of the waiting time?

Problem 2.8.12 Solution
The standard deviation can be expressed as

$$\sigma_D = \sqrt{\mathrm{Var}[D]} = \sqrt{E\left[D^2\right] - E\left[D\right]^2} \tag{1}$$

where

$$E\left[D^2\right] = \sum_{d=1}^{4} d^2 P_D(d) = 0.2 + 1.6 + 2.7 + 1.6 = 6.1 \tag{2}$$

So finally we have

$$\sigma_D = \sqrt{6.1 - 2.3^2} = \sqrt{0.81} = 0.9 \tag{3}$$

Problem 2.9.1 •
In Problem 2.4.1, find $P_{Y|B}(y)$, where the condition $B = \{Y < 3\}$. What are $E[Y|B]$ and $\mathrm{Var}[Y|B]$?

Problem 2.9.1 Solution
From the solution to Problem 2.4.1, the PMF of Y is

$$P_Y(y) = \begin{cases} 1/4 & y = 1 \\ 1/4 & y = 2 \\ 1/2 & y = 3 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The probability of the event $B = \{Y < 3\}$ is $P[B] = 1 - P[Y = 3] = 1/2$. From Theorem 2.17, the conditional PMF of Y given B is

$$P_{Y|B}(y) = \begin{cases} \frac{P_Y(y)}{P[B]} & y \in B \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1/2 & y = 1 \\ 1/2 & y = 2 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

The conditional first and second moments of Y are

$$E[Y|B] = \sum_y y P_{Y|B}(y) = 1(1/2) + 2(1/2) = 3/2 \tag{3}$$

$$E\left[Y^2|B\right] = \sum_y y^2 P_{Y|B}(y) = 1^2(1/2) + 2^2(1/2) = 5/2 \tag{4}$$

The conditional variance of Y is

$$\text{Var}[Y|B] = E\left[Y^2|B\right] - (E[Y|B])^2 = 5/2 - 9/4 = 1/4 \tag{5}$$

Problem 2.9.3 •

In Problem 2.4.3, find $P_{X|B}(x)$, where the condition $B = \{X > 0\}$. What are $E[X|B]$ and $\text{Var}[X|B]$?

Problem 2.9.3 Solution

From the solution to Problem 2.4.3, the PMF of X is

$$P_X(x) = \begin{cases} 0.4 & x = -3 \\ 0.4 & x = 5 \\ 0.2 & x = 7 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The event $B = \{X > 0\}$ has probability $P[B] = P_X(5) + P_X(7) = 0.6$. From Theorem 2.17, the conditional PMF of X given B is

$$P_{X|B}(x) = \begin{cases} \frac{P_X(x)}{P[B]} & x \in B \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 2/3 & x = 5 \\ 1/3 & x = 7 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

The conditional first and second moments of X are

$$E[X|B] = \sum_x x P_{X|B}(x) = 5(2/3) + 7(1/3) = 17/3 \tag{3}$$

$$E\left[X^2|B\right] = \sum_x x^2 P_{X|B}(x) = 5^2(2/3) + 7^2(1/3) = 33 \tag{4}$$

The conditional variance of X is

$$\text{Var}[X|B] = E\left[X^2|B\right] - (E[X|B])^2 = 33 - (17/3)^2 = 8/9 \tag{5}$$

Problem 2.10.1 •

Let X be a binomial (n, p) random variable with $n = 100$ and $p = 0.5$. Let E_2 denote the event that X is a perfect square. Calculate $P[E_2]$.

Problem 2.10.1 Solution

For a binomial (n, p) random variable X, the probability of the event that X is a perfect square is

$$P[E_2] = \sum_{k=0}^{\lfloor \sqrt{n} \rfloor} P_X\left(k^2\right), \tag{1}$$

where

$$P_X(x) = \binom{n}{x} p^x (1-p)^{n-x}, \tag{2}$$

n is a positive integer, and $p \in (0, 1)$.

Here's a matlab function that can be used to calculate this value.

```
%
% function f2_10_1 determines the probability that a binomial
% random variable having parameters n and p takes a value that
% is a perfect square.                                  2/01/06 sk
%

function pe2 = f2_10_1(n,p);
% first determine values of P_x(x) for the squares
for index = 0:10,
   pxx(index+1) = choose(100,index^2)*p^(index^2)*(1-p)^(n-index^2);
end;
% then sum them (events are disjoint)
pe2 = sum(pxx);

%------------------------------------------------------------------

function [num_com] =  choose(n,x);
num_com = factorial(n)/(factorial(x)*factorial(n-x));
```

The output is

```
>> f2_10_1(100,1/2)

ans =

    0.0811
```

Problem 2.10.2 •

Write a MATLAB function `x=faxlength8(m)` that produces m random sample values of the fax length X with PMF given in Example 2.29.

Problem 2.10.2 Solution

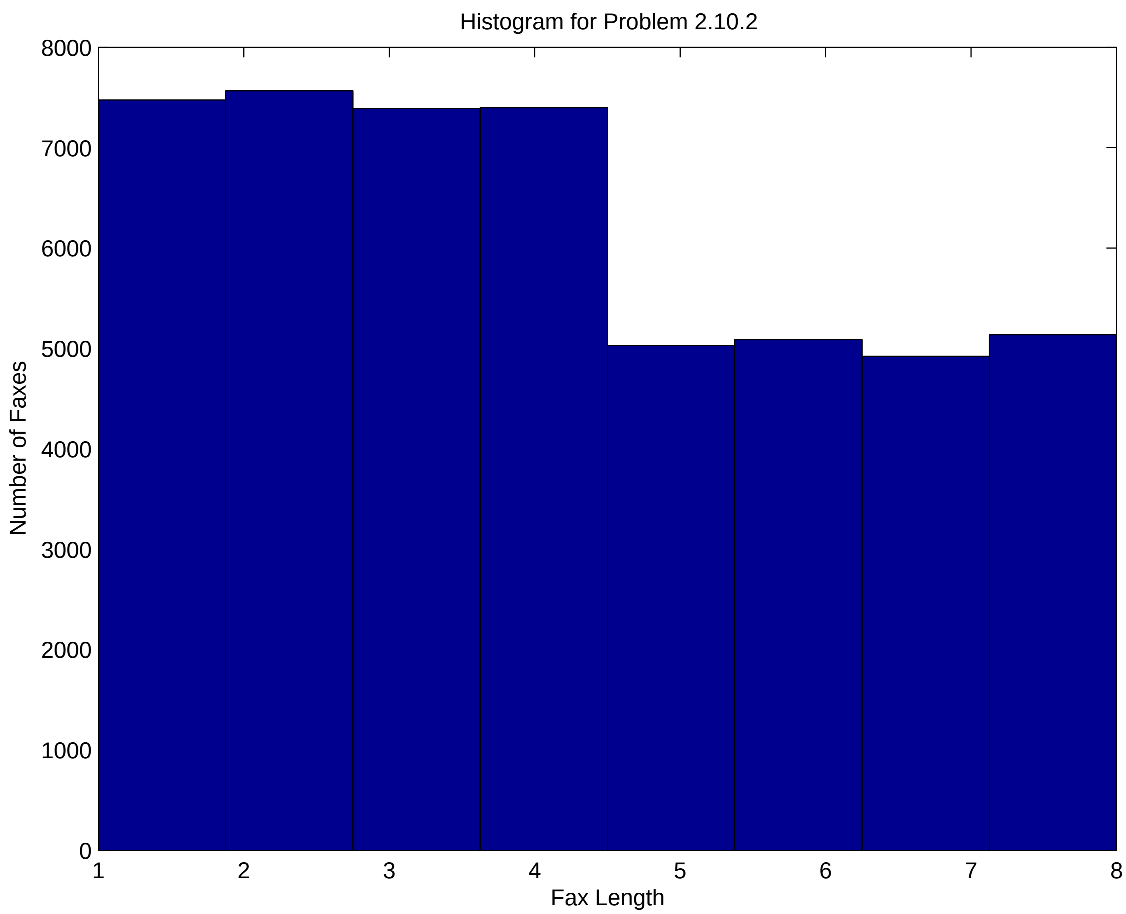
The random variable X given in Example 2.29 is a finite random variable. We can generate random samples using the following code.

```
%
% generates m random samples of the "number of fax pages" random
% variable defined in Example 2.29 on p. 73 of Y&G. 1/31/06
%

function [x] = my_faxlength8(m);
t = rand(m,1);
for index = 1:m,
    if t(index) < 0.15
         x(index) = 1;
    elseif t(index) < 0.3
         x(index) = 2;
    elseif t(index) < 0.45
         x(index) = 3;
    elseif t(index) < 0.6
         x(index) = 4;
    elseif t(index) < 0.7
         x(index) = 5;
    elseif t(index) < 0.8
         x(index) = 6;
    elseif t(index) < 0.9
         x(index) = 7;
    else
         x(index) = 8;
    end;
end;
hist(x,8)
title('Histogram for Problem 2.10.2')
xlabel('Fax Length')
ylabel('Number of Faxes')
disp(['Mean:  ',num2str(mean(x))])
disp(['Variance:  ',num2str(var(x))])
```

Here's the output, including a calculation of the mean and variance of the given PMF. We see that our mean and variance of the experimental data is reasonably close to the predicted values.

```
>> x=my_faxlength8(50);
Mean:  4.06
Variance:  4.9555

>> truemean = .15*(1+2+3+4)+.1*(5+6+7+8)

truemean =

    4.1000

>> truevar = .15*(1^2+2^2+3^2+4^2)+.1*(5^2+6^2+7^2+8^2) - truemean^2

truevar =

    5.0900
```

Here's a histogram for 50,000 samples. (Truth in advertising: It didn't match the distribution so nicely when I tried 50, 500, and 5000 samples!)



Problem 2.10.3 •

For $m = 10$, $m = 100$, and $m = 1000$, use MATLAB to find the average cost of sending m faxes using the model of Example 2.29. Your program input should have the number of trials m as the input. The output should be $\overline{Y} = \frac{1}{m}\sum_{i=1}^{m} Y_i$ where Y_i is the cost of the ith fax. As m becomes large, what do you observe?

Problem 2.10.3 Solution

First we use `faxlength8` from Problem 2.10.2 to generate m samples of the faqx length X. Next we convert that to m samples of the fax cost Y. Summing these samples and dividing by m, we obtain the average cost of m samples. Here is the code:

```
%
% calculate fax costs for problem YG 2.10.3
% 2/01/06 sk
%

for index = 1:max(size(x)),
    if x(index) < 6,
        cost(index) = 10.5*x(index)-0.5*x(index)^2; %%% using (2.65) page 71
    else
        cost(index) = 50;
    end;
end;
mean_cost = sum(cost)/max(size(cost))
```

```
>> clear x; x = my_faxlength8(10); calcost
Mean:  4.3
Variance:  4.6778

mean_cost =

   32.5997

>> clear x; x = my_faxlength8(100); calcost
Mean:  4.09
Variance:  5.2948

mean_cost =

   32.5963

>> clear x; x = my_faxlength8(1000); calcost
Mean:  4.177
Variance:  5.0727

mean_cost =

   32.6132

>> clear x; x = my_faxlength8(10000); calcost
Mean:  4.0698
Variance:  5.063

mean_cost =

   32.5195
```

As the number of samples increases, we should see the mean approach the true mean of the distribution. However, we must remember that these are random samples. While it is unlikely that we will generate an "unusual" sample, it is not impossible. Hence, we should not be too alarmed that the approach to the true mean was not monotone in the number of samples in our test. (By monotone I mean strictly increasing numbers of samples corresponding to strictly decreasing differences between the sample mean and the true mean.)

In a later chapter, we will develop techniques to show how $\overline{Y}$ converges to $E[Y]$ as $m \to \infty$.

Solutions to HW4

Note: Most of these solutions were generated by R. D. Yates and D. J. Goodman, the authors of our textbook. I have added comments in italics where I thought more detail was appropriate.

Problem 2.2.6 ■

You are manager of a ticket agency that sells concert tickets. You assume that people will call three times in an attempt to buy tickets and then give up. You want to make sure that you are able to serve at least 95% of the people who want tickets. Let p be the probability that a caller gets through to your ticket agency. What is the minimum value of p necessary to meet your goal.

Problem 2.2.6 Solution

The probability that a caller fails to get through in three tries is $(1-p)^3$. To be sure that at least 95% of all callers get through, we need $(1-p)^3 \leq 0.05$. This implies $p = 0.6316$.

Problem 2.2.7 ■

In the ticket agency of Problem 2.2.6, each telephone ticket agent is available to receive a call with probability 0.2. If all agents are busy when someone calls, the caller hears a busy signal. What is the minimum number of agents that you have to hire to meet your goal of serving 95% of the customers who want tickets?

Problem 2.2.7 Solution

In Problem 2.2.6, each caller is willing to make 3 attempts to get through. An attempt is a failure if all n operators are busy, which occurs with probability $q = (0.8)^n$. Assuming call attempts are independent, a caller will suffer three failed attempts with probability $q^3 = (0.8)^{3n}$. The problem statement requires that $(0.8)^{3n} \leq 0.05$. This implies $n \geq 4.48$ and so we need 5 operators.

Problem 2.2.9 ♦

When someone presses "SEND" on a cellular phone, the phone attempts to set up a call by transmitting a "SETUP" message to a nearby base station. The phone waits for a response and if none arrives within 0.5 seconds it tries again. If it doesn't get a response after $n = 6$ tries the phone stops transmitting messages and generates a busy signal.

(a) Draw a tree diagram that describes the call setup procedure.

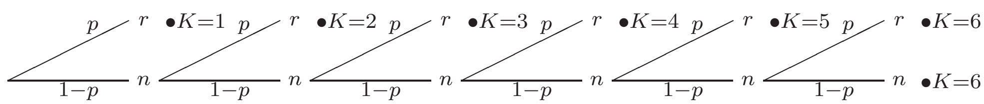
(b) If all transmissions are independent and the probability is p that a "SETUP" message will get through, what is the PMF of K, the number of messages transmitted in a call attempt?

(c) What is the probability that the phone will generate a busy signal?

(d) As manager of a cellular phone system, you want the probability of a busy signal to be less than 0.02 If $p = 0.9$, what is the minimum value of n necessary to achieve your goal?

Problem 2.2.9 Solution

(a) In the setup of a mobile call, the phone will send the "SETUP" message up to six times. Each time the setup message is sent, we have a Bernoulli trial with success probability p. Of course, the phone stops trying as soon as there is a success. Using r to denote a successful response, and n a non-response, the sample tree is



(b) We can write the PMF of K, the number of "SETUP" messages sent as

$$P_K(k) = \begin{cases} (1-p)^{k-1}p & k = 1, 2, \ldots, 5 \\ (1-p)^5 p + (1-p)^6 = (1-p)^5 & k = 6 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Note that the expression for $P_K(6)$ is different because $K = 6$ if either there was a success or a failure on the sixth attempt. In fact, $K = 6$ whenever there were failures on the first five attempts which is why $P_K(6)$ simplifies to $(1-p)^5$.

(c) Let B denote the event that a busy signal is given after six failed setup attempts. The probability of six consecutive failures is $P[B] = (1-p)^6$.

(d) To be sure that $P[B] \leq 0.02$, we need $p \geq 1 - (0.02)^{1/6} = 0.479$.

Problem 2.3.10 ■

A radio station gives a pair of concert tickets to the sixth caller who knows the birthday of the performer. For each person who calls, the probability is 0.75 of knowing the performer's birthday. All calls are independent.

(a) What is the PMF of L, the number of calls necessary to find the winner?

(b) What is the probability of finding the winner on the tenth call?

(c) What is the probability that the station will need nine or more calls to find a winner?

Problem 2.3.10 Solution

(a) We can view whether each caller knows the birthdate as a Bernoulli trial. As a result, L is the number of trials needed for 6 successes. That is, L has a Pascal PMF with parameters $p = 0.75$ and $k = 6$ as defined by Definition 2.8. In particular,

$$P_L(l) = \begin{cases} \binom{l-1}{5}(0.75)^6(0.25)^{l-6} & l = 6, 7, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(b) The probability of finding the winner on the tenth call is

$$P_L(10) = \binom{9}{5}(0.75)^6(0.25)^4 \approx 0.0876 \tag{2}$$

(c) The probability that the station will need nine or more calls to find a winner is

$$\begin{aligned} P[L \geq 9] &= 1 - P[L < 9] && (3)\\ &= 1 - P_L(6) - P_L(7) - P_L(8) && (4)\\ &= 1 - (0.75)^6[1 + 6(0.25) + 21(0.25)^2] \approx 0.321 && (5) \end{aligned}$$

Problem 2.3.11 ■

In a packet voice communications system, a source transmits packets containing digitized speech to a receiver. Because transmission errors occasionally occur, an acknowledgment (ACK) or a nonacknowledgment (NAK) is transmitted back to the source to indicate the status of each received packet. When the transmitter gets a NAK, the packet is retransmitted. Voice packets are delay sensitive and a packet can be transmitted a maximum of d times. If a packet transmission is an independent Bernoulli trial with success probability p, what is the PMF of T, the number of times a packet is transmitted?

Problem 2.3.11 Solution

The packets are delay sensitive and can only be retransmitted d times. For $t < d$, a packet is transmitted t times if the first $t-1$ attempts fail followed by a successful transmission on attempt t. Further, the packet is transmitted d times if there are failures on the first $d-1$ transmissions, no matter what the outcome of attempt d. So the random variable T, the number of times that a packet is transmitted, can be represented by the following PMF.

$$P_T(t) = \begin{cases} p(1-p)^{t-1} & t = 1, 2, \ldots, d-1 \\ (1-p)^{d-1} & t = d \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Problem 2.4.5 ■

At the One Top Pizza Shop, a pizza sold has mushrooms with probability $p = 2/3$. On a day in which 100 pizzas are sold, let N equal the number of pizzas sold before the first pizza with mushrooms is sold. What is the PMF of N? What is the CDF of N?

Problem 2.4.5 Solution

Since mushrooms occur with probability 2/3, the number of pizzas sold before the first mushroom pizza is $N = n < 100$ if the first n pizzas do not have mushrooms followed by mushrooms on pizza $n+1$. Also, it is possible that $N = 100$ if all 100 pizzas are sold without mushrooms. the resulting PMF is

$$P_N(n) = \begin{cases} (1/3)^n(2/3) & n = 0, 1, \ldots, 99 \\ (1/3)^{100} & n = 100 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

For integers $n < 100$, the CDF of N obeys

$$F_N(n) = \sum_{i=0}^{n} P_N(i) = \sum_{i=0}^{n} (1/3)^i(2/3) = 1 - (1/3)^{n+1} \tag{2}$$

A complete expression for $F_N(n)$ must give a valid answer for every value of n, including non-integer values. We can write the CDF using the floor function $\lfloor x \rfloor$ which denote the largest integer less than or equal to X. The complete expression for the CDF is

$$F_N(x) = \begin{cases} 0 & x < 0 \\ 1 - (1/3)^{\lfloor x \rfloor + 1} & 0 \le x < 100 \\ 1 & x \ge 100 \end{cases} \tag{3}$$

Problem 2.4.8 ■

In Problem 2.2.9, find and sketch the CDF of N, the number of attempts made by the cellular phone for $p = 1/2$.

Problem 2.4.8 Solution

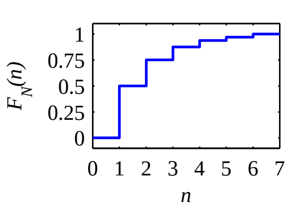
From Problem 2.2.9, the PMF of the number of call attempts is

$$P_N(n) = \begin{cases} (1-p)^{k-1}p & k = 1, 2, \ldots, 5 \\ (1-p)^5 p + (1-p)^6 = (1-p)^5 & k = 6 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

For $p = 1/2$, the PMF can be simplified to

$$P_N(n) = \begin{cases} (1/2)^n & n = 1, 2, \ldots, 5 \\ (1/2)^5 & n = 6 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

The corresponding CDF of N is



$$F_N(n) = \begin{cases} 0 & n < 1 \\ 1/2 & 1 \le n < 2 \\ 3/4 & 2 \le n < 3 \\ 7/8 & 3 \le n < 4 \\ 15/16 & 4 \le n < 5 \\ 31/32 & 5 \le n < 6 \\ 1 & n \ge 6 \end{cases} \tag{3}$$

Problem 2.6.5 ■

A source wishes to transmit data packets to a receiver over a radio link. The receiver uses error detection to identify packets that have been corrupted by radio noise. When a packet is received error-free, the receiver sends an acknowledgment (ACK) back to the source. When the receiver gets a packet with errors, a negative acknowledgment (NAK) message is sent back to the source. Each time the source receives a NAK, the packet is retransmitted. We assume that each packet transmission is independently corrupted by errors with probability q.

(a) Find the PMF of X, the number of times that a packet is transmitted by the source.

(b) Suppose each packet takes 1 millisecond to transmit and that the source waits an additional millisecond to receive the acknowledgment message (ACK or NAK) before retransmitting. Let T equal the time required until the packet is successfully received. What is the relationship between T and X? What is the PMF of T?

Problem 2.6.5 Solution

(a) The source continues to transmit packets until one is received correctly. Hence, the total number of times that a packet is transmitted is $X = x$ if the first $x - 1$ transmissions were in error. Therefore the PMF of X is

$$P_X(x) = \begin{cases} q^{x-1}(1-q) & x = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(b) The time required to send a packet is a millisecond and the time required to send an acknowledgment back to the source takes another millisecond. Thus, if X transmissions of a packet are needed to send the packet correctly, then the packet is correctly received after $T = 2X - 1$ milliseconds. Therefore, for an odd integer $t > 0$, $T = t$ iff $X = (t + 1)/2$. Thus,

$$P_T(t) = P_X((t+1)/2) = \begin{cases} q^{(t-1)/2}(1-q) & t = 1, 3, 5, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Problem 2.6.6 ■

Suppose that a cellular phone costs \$20 per month with 30 minutes of use included and that each additional minute of use costs \$0.50. If the number of minutes you use in a month is a geometric random variable M with expected value of $E[M] = 1/p = 30$ minutes, what is the PMF of C, the cost of the phone for one month?

Problem 2.6.6 Solution

The cellular calling plan charges a flat rate of \$20 per month up to and including the 30th minute, and an additional 50 cents for each minute over 30 minutes. Knowing that the time you spend on the phone is a geometric random variable M with mean $1/p = 30$, the PMF of M is

$$P_M(m) = \begin{cases} (1-p)^{m-1}p & m = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The monthly cost, C obeys

$$P_C(20) = P[M \le 30] = \sum_{m=1}^{30} (1-p)^{m-1}p = 1 - (1-p)^{30} \tag{2}$$

When $M \ge 30$, $C = 20 + (M-30)/2$ or $M = 2C - 10$. Thus,

$$P_C(c) = P_M(2c - 10) \qquad c = 20.5, 21, 21.5, \ldots \tag{3}$$

The complete PMF of C is

$$P_C(c) = \begin{cases} 1 - (1-p)^{30} & c = 20 \\ (1-p)^{2c-10-1}p & c = 20.5, 21, 21.5, \ldots \end{cases} \tag{4}$$

Problem 2.7.5 ■

For the cellular phone in Problem 2.6.6, express the monthly cost C as a function of M, the number of minutes used. What is the expected monthly cost $E[C]$?

Problem 2.7.5 Solution

As a function of the number of minutes used, M, the monthly cost is

$$C(M) = \begin{cases} 20 & M \le 30 \\ 20 + (M-30)/2 & M \ge 30 \end{cases} \tag{1}$$

The expected cost per month is

$$E[C] = \sum_{m=1}^{\infty} C(m)P_M(m) = \sum_{m=1}^{30} 20P_M(m) + \sum_{m=31}^{\infty} (20 + (m-30)/2)P_M(m) \tag{2}$$

$$= 20 \sum_{m=1}^{\infty} P_M(m) + \frac{1}{2} \sum_{m=31}^{\infty} (m-30)P_M(m) \tag{3}$$

Since $\sum_{m=1}^{\infty} P_M(m) = 1$ and since $P_M(m) = (1-p)^{m-1}p$ for $m \ge 1$, we have

$$E[C] = 20 + \frac{(1-p)^{30}}{2} \sum_{m=31}^{\infty} (m-30)(1-p)^{m-31}p \tag{4}$$

Making the substitution $j = m - 30$ yields

$$E[C] = 20 + \frac{(1-p)^{30}}{2} \sum_{j=1}^{\infty} j(1-p)^{j-1}p = 20 + \frac{(1-p)^{30}}{2p} \tag{5}$$

Problem 2.7.6 ■

A new cellular phone billing plan costs \$15 per month plus \$1 for each minute of use. If the number of minutes you use the phone in a month is a geometric random variable with mean $1/p$, what is the expected monthly cost $E[C]$ of the phone? For what values of p is this billing plan preferable to the billing plan of Problem 2.6.6 and Problem 2.7.5?

Problem 2.7.6 Solution

Since our phone use is a geometric random variable M with mean value $1/p$,

$$P_M(m) = \begin{cases} (1-p)^{m-1}p & m = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

For this cellular billing plan, we are given no free minutes, but are charged half the flat fee. That is, we are going to pay 15 dollars regardless and \$1 for each minute we use the phone. Hence $C = 15 + M$ and for $c \geq 16$, $P[C = c] = P[M = c - 15]$. Thus we can construct the PMF of the cost C

$$P_C(c) = \begin{cases} (1-p)^{c-16}p & c = 16, 17, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Since $C = 15 + M$, the expected cost per month of the plan is

$$E[C] = E[15 + M] = 15 + E[M] = 15 + 1/p \tag{3}$$

In Problem 2.7.5, we found that that the expected cost of the plan was

$$E[C] = 20 + [(1-p)^{30}]/(2p) \tag{4}$$

In comparing the expected costs of the two plans, we see that the new plan is better (i.e. cheaper) if

$$15 + 1/p \leq 20 + [(1-p)^{30}]/(2p) \tag{5}$$

A simple plot will show that the new plan is better if $p \leq p_0 \approx 0.2$.

Problem 2.8.5 ■

Let X have the binomial PMF

$$P_X(x) = \binom{4}{x}(1/2)^4.$$

(a) Find the standard deviation of the random variable X.

(b) What is $P[\mu_X - \sigma_X \leq X \leq \mu_X + \sigma_X]$, the probability that X is within one standard deviation of the expected value?

Problem 2.8.5 Solution

(a) The expected value of X is

$$E[X] = \sum_{x=0}^{4} x P_X(x) = 0\binom{4}{0}\frac{1}{2^4} + 1\binom{4}{1}\frac{1}{2^4} + 2\binom{4}{2}\frac{1}{2^4} + 3\binom{4}{3}\frac{1}{2^4} + 4\binom{4}{4}\frac{1}{2^4} \quad (1)$$

$$= [4 + 12 + 12 + 4]/2^4 = 2 \quad (2)$$

The expected value of X^2 is

$$E\left[X^2\right] = \sum_{x=0}^{4} x^2 P_X(x) = 0^2\binom{4}{0}\frac{1}{2^4} + 1^2\binom{4}{1}\frac{1}{2^4} + 2^2\binom{4}{2}\frac{1}{2^4} + 3^2\binom{4}{3}\frac{1}{2^4} + 4^2\binom{4}{4}\frac{1}{2^4} \quad (3)$$

$$= [4 + 24 + 36 + 16]/2^4 = 5 \quad (4)$$

The variance of X is

$$\text{Var}[X] = E\left[X^2\right] - (E[X])^2 = 5 - 2^2 = 1 \quad (5)$$

Thus, X has standard deviation $\sigma_X = \sqrt{\text{Var}[X]} = 1$.

(b) The probability that X is within one standard deviation of its expected value is

$$P[\mu_X - \sigma_X \le X \le \mu_X + \sigma_X] = P[2 - 1 \le X \le 2 + 1] = P[1 \le X \le 3] \quad (6)$$

This calculation is easy using the PMF of X.

$$P[1 \le X \le 3] = P_X(1) + P_X(2) + P_X(3) = 7/8 \quad (7)$$

Problem 2.8.6 ■

The binomial random variable X has PMF

$$P_X(x) = \binom{5}{x}(1/2)^5.$$

(a) Find the standard deviation of X.

(b) Find $P[\mu_X - \sigma_X \le X \le \mu_X + \sigma_X]$, the probability that X is within one standard deviation of the expected value.

Problem 2.8.6 Solution

(a) The expected value of X is

$$E[X] = \sum_{x=0}^{5} x P_X(x) \quad (1)$$

$$= 0\binom{5}{0}\frac{1}{2^5} + 1\binom{5}{1}\frac{1}{2^5} + 2\binom{5}{2}\frac{1}{2^5} + 3\binom{5}{3}\frac{1}{2^5} + 4\binom{5}{4}\frac{1}{2^5} + 5\binom{5}{5}\frac{1}{2^5} \quad (2)$$

$$= [5 + 20 + 30 + 20 + 5]/2^5 = 5/2 \quad (3)$$

The expected value of X^2 is

$$E\left[X^2\right] = \sum_{x=0}^{5} x^2 P_X(x) \tag{4}$$

$$= 0^2\binom{5}{0}\frac{1}{2^5} + 1^2\binom{5}{1}\frac{1}{2^5} + 2^2\binom{5}{2}\frac{1}{2^5} + 3^2\binom{5}{3}\frac{1}{2^5} + 4^2\binom{5}{4}\frac{1}{2^5} + 5^2\binom{5}{5}\frac{1}{2^5} \tag{5}$$

$$= [5 + 40 + 90 + 80 + 25]/2^5 = 240/32 = 15/2 \tag{6}$$

The variance of X is

$$\text{Var}[X] = E\left[X^2\right] - (E[X])^2 = 15/2 - 25/4 = 5/4 \tag{7}$$

By taking the square root of the variance, the standard deviation of X is $\sigma_X = \sqrt{5/4} \approx 1.12$.

(b) The probability that X is within one standard deviation of its mean is

$$P[\mu_X - \sigma_X \le X \le \mu_X + \sigma_X] = P[2.5 - 1.12 \le X \le 2.5 + 1.12] \tag{8}$$

$$= P[1.38 \le X \le 3.62] \tag{9}$$

$$= P[2 \le X \le 3] \tag{10}$$

By summing the PMF over the desired range, we obtain

$$P[2 \le X \le 3] = P_X(2) + P_X(3) = 10/32 + 10/32 = 5/8 \tag{11}$$

Problem 2.9.5 •

In Problem 2.8.6, find $P_{X|B}(x)$, where the condition $B = \{X \ge \mu_X\}$. What are $E[X|B]$ and $\text{Var}[X|B]$?

Problem 2.9.5 Solution

The probability of the event B is

$$P[B] = P[X \ge \mu_X] = P[X \ge 3] = P_X(3) + P_X(4) + P_X(5) \tag{1}$$

$$= \frac{\binom{5}{3} + \binom{5}{4} + \binom{5}{5}}{32} = 21/32 \tag{2}$$

The conditional PMF of X given B is

$$P_{X|B}(x) = \begin{cases} \frac{P_X(x)}{P[B]} & x \in B \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \binom{5}{x}\frac{1}{21} & x = 3, 4, 5 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

The conditional first and second moments of X are

$$E[X|B] = \sum_{x=3}^{5} x P_{X|B}(x) = 3\binom{5}{3}\frac{1}{21} + 4\binom{5}{4}\frac{1}{21} + 5\binom{5}{5}\frac{1}{21} \quad (4)$$

$$= [30 + 20 + 5]/21 = 55/21 \quad (5)$$

$$E\left[X^2|B\right] = \sum_{x=3}^{5} x^2 P_{X|B}(x) = 3^2\binom{5}{3}\frac{1}{21} + 4^2\binom{5}{4}\frac{1}{21} + 5^2\binom{5}{5}\frac{1}{21} \quad (6)$$

$$= [90 + 80 + 25]/21 = 195/21 = 65/7 \quad (7)$$

The conditional variance of X is

$$\text{Var}[X|B] = E\left[X^2|B\right] - (E[X|B])^2 = 65/7 - (55/21)^2 = 1070/441 = 2.43 \quad (8)$$

Problem 2.9.6 ■

Select integrated circuits, test them in sequence until you find the first failure, and then stop. Let N be the number of tests. All tests are independent with probability of failure $p = 0.1$. Consider the condition $B = \{N \geq 20\}$.

(a) Find the PMF $P_N(n)$.

(b) Find $P_{N|B}(n)$, the conditional PMF of N given that there have been 20 consecutive tests without a failure.

(c) What is $E[N|B]$, the expected number of tests given that there have been 20 consecutive tests without a failure?

Problem 2.9.6 Solution

(a) Consider each circuit test as a Bernoulli trial such that a failed circuit is called a success. The number of trials until the first success (i.e. a failed circuit) has the geometric PMF

$$P_N(n) = \begin{cases} (1-p)^{n-1}p & n = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

(b) The probability there are at least 20 tests is

$$P[B] = P[N \geq 20] = \sum_{n=20}^{\infty} P_N(n) = (1-p)^{19} \quad (2)$$

Note that $(1-p)^{19}$ is just the probability that the first 19 circuits pass the test, which is what we would expect since there must be at least 20 tests if the first 19 circuits pass. The conditional PMF of N given B is

$$P_{N|B}(n) = \begin{cases} \frac{P_N(n)}{P[B]} & n \in B \\ 0 & \text{otherwise} \end{cases} = \begin{cases} (1-p)^{n-20}p & n = 20, 21, \ldots \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

(c) Given the event B, the conditional expectation of N is

$$E[N|B] = \sum_n n P_{N|B}(n) = \sum_{n=20}^{\infty} n(1-p)^{n-20} p \tag{4}$$

Making the substitution $j = n - 19$ yields

$$E[N|B] = \sum_{j=1}^{\infty} (j+19)(1-p)^{j-1} p = 1/p + 19 \tag{5}$$

We see that in the above sum, we effectively have the expected value of $J + 19$ where J is geometric random variable with parameter p. This is not surprising since the $N \geq 20$ iff we observed 19 successful tests. After 19 successful tests, the number of additional tests needed to find the first failure is still a geometric random variable with mean $1/p$.

Problem 2.9.7 ■

Every day you consider going jogging. Before each mile, including the first, you will quit with probability q, independent of the number of miles you have already run. However, you are sufficiently decisive that you never run a fraction of a mile. Also, we say you have run a marathon whenever you run at least 26 miles.

(a) Let M equal the number of miles that you run on an arbitrary day. What is $P[M > 0]$? Find the PMF $P_M(m)$.

(b) Let r be the probability that you run a marathon on an arbitrary day. Find r.

(c) Let J be the number of days in one year (not a leap year) in which you run a marathon. Find the PMF $P_J(j)$. This answer may be expressed in terms of r found in part (b).

(d) Define $K = M - 26$. Let A be the event that you have run a marathon. Find $P_{K|A}(k)$.

Problem 2.9.7 Solution

(a) The PMF of M, the number of miles run on an arbitrary day is

$$P_M(m) = \begin{cases} q(1-q)^m & m = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

And we can see that the probability that $M > 0$, is

$$P[M > 0] = 1 - P[M = 0] = 1 - q \tag{2}$$

(b) The probability that we run a marathon on any particular day is the probability that $M \geq 26$.

$$r = P[M \geq 26] = \sum_{m=26}^{\infty} q(1-q)^m = (1-q)^{26} \tag{3}$$

(c) We run a marathon on each day with probability equal to r, and we do not run a marathon with probability $1-r$. Therefore in a year we have 365 tests of our jogging resolve, and thus 365 chances to run a marathon. So the PMF of the number of marathons run in a year, J, can be expressed as

$$P_J(j) = \begin{cases} \binom{365}{j} r^j (1-r)^{365-j} & j = 0, 1, \ldots, 365 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

(d) The random variable K is defined as the number of miles we run above that required for a marathon, $K = M - 26$. Given the event, A, that we have run a marathon, we wish to know how many miles in excess of 26 we in fact ran. So we want to know the conditional PMF $P_{K|A}(k)$.

$$P_{K|A}(k) = \frac{P[K = k, A]}{P[A]} = \frac{P[M = 26 + k]}{P[A]} \tag{5}$$

Since $P[A] = r$, for $k = 0, 1, \ldots,$

$$P_{K|A}(k) = \frac{(1-q)^{26+k} q}{(1-q)^{26}} = (1-q)^k q \tag{6}$$

The complete expression of for the conditional PMF of K is

$$P_{K|A}(k) = \begin{cases} (1-q)^k q & k = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Problem 2.9.8 ■

In the situation described in Example 2.29, the firm sends all faxes with an even number of pages to fax machine A and all faxes with an odd number of pages to fax machine B.

(a) Find the conditional PMF of the length X of a fax, given the fax was sent to A. What are the conditional expected length and standard deviation?

(b) Find the conditional PMF of the length X of a fax, given the fax was sent to B and had no more than six pages. What are the conditional expected length and standard deviation?

Problem 2.9.8 Solution

Recall that the PMF of the number of pages in a fax is

$$P_X(x) = \begin{cases} 0.15 & x = 1, 2, 3, 4 \\ 0.1 & x = 5, 6, 7, 8 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) The event that a fax was sent to machine A can be expressed mathematically as the event that the number of pages X is an even number. Similarly, the event that a fax was sent to B is the event that X is an odd number. Since $S_X = \{1, 2, \ldots, 8\}$, we define the set $A = \{2, 4, 6, 8\}$. Using this definition for A, we have that the event that a fax is sent to A is equivalent to the event $X \in A$. The event A has probability

$$P[A] = P_X(2) + P_X(4) + P_X(6) + P_X(8) = 0.5 \tag{2}$$

Given the event A, the conditional PMF of X is

$$P_{X|A}(x) = \begin{cases} \frac{P_X(x)}{P[A]} & x \in A \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 0.3 & x = 2,4 \\ 0.2 & x = 6,8 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

The conditional first and second moments of X given A is

$$E[X|A] = \sum_x x P_{X|A}(x) = 2(0.3) + 4(0.3) + 6(0.2) + 8(0.2) = 4.6 \tag{4}$$

$$E\left[X^2|A\right] = \sum_x x^2 P_{X|A}(x) = 4(0.3) + 16(0.3) + 36(0.2) + 64(0.2) = 26 \tag{5}$$

The conditional variance and standard deviation are

$$\text{Var}[X|A] = E\left[X^2|A\right] - (E[X|A])^2 = 26 - (4.6)^2 = 4.84 \tag{6}$$

$$\sigma_{X|A} = \sqrt{\text{Var}[X|A]} = 2.2 \tag{7}$$

(b) Let the event B' denote the event that the fax was sent to B and that the fax had no more than 6 pages. Hence, the event $B' = \{1, 3, 5\}$ has probability

$$P\left[B'\right] = P_X(1) + P_X(3) + P_X(5) = 0.4 \tag{8}$$

The conditional PMF of X given B' is

$$P_{X|B'}(x) = \begin{cases} \frac{P_X(x)}{P[B']} & x \in B' \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 3/8 & x = 1,3 \\ 1/4 & x = 5 \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

Given the event B', the conditional first and second moments are

$$E\left[X|B'\right] = \sum_x x P_{X|B'}(x) = 1(3/8) + 3(3/8) + 5(1/4) + = 11/4 \tag{10}$$

$$E\left[X^2|B'\right] = \sum_x x^2 P_{X|B'}(x) = 1(3/8) + 9(3/8) + 25(1/4) = 10 \tag{11}$$

The conditional variance and standard deviation are

$$\text{Var}[X|B'] = E\left[X^2|B'\right] - (E\left[X|B'\right])^2 = 10 - (11/4)^2 = 39/16 \tag{12}$$

$$\sigma_{X|B'} = \sqrt{\text{Var}[X|B']} = \sqrt{39}/4 \approx 1.56 \tag{13}$$

Problem 2.10.6 ■

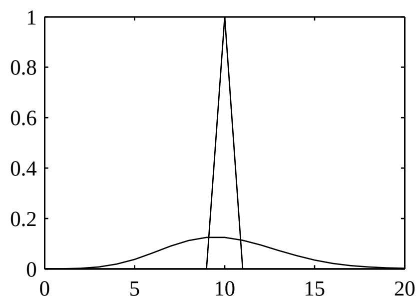
Test the convergence of Theorem 2.8. For $\alpha = 10$, plot the PMF of K_n for $(n,p) = (10,1)$, $(n,p) = (100,0.1)$, and $(n,p) = (1000,0.01)$ and compare against the Poisson (α) PMF.

Problem 2.10.6 Solution

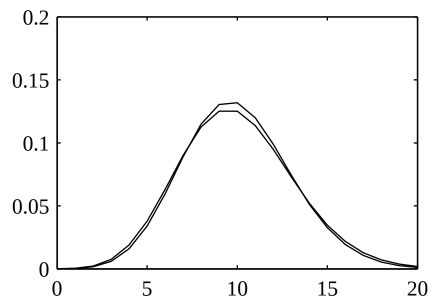
We can compare the binomial and Poisson PMFs for $(n,p) = (100,0.1)$ using the following MATLAB code:

```
x=0:20;
p=poissonpmf(100,x);
b=binomialpmf(100,0.1,x);
plot(x,p,x,b);
```

For $(n,p) = (10,1)$, the binomial PMF has no randomness. For $(n,p) = (100,0.1)$, the approximation is reasonable:

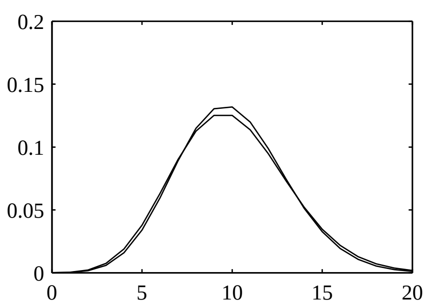


(a) $n = 10, p = 1$

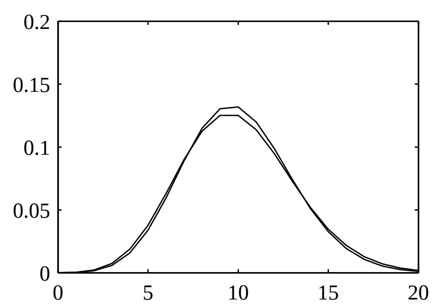


(b) $n = 100, p = 0.1$

Finally, for $(n,p) = (1000,0.01)$, and $(n,p) = (10000,0.001)$, the approximation is very good:



(a) $n = 1000, p = 0.01$



(b) $n = 10000, p = 0.001$

Solutions to HW5

Note: Most of these solutions were generated by R. D. Yates and D. J. Goodman, the authors of our textbook. I have added comments in italics where I thought more detail was appropriate. I have made corrections where needed. The solution to problem 3.9.2 is my own.

Problem 3.1.1 •

The cumulative distribution function of random variable X is

$$F_X(x) = \begin{cases} 0 & x < -1, \\ (x+1)/2 & -1 \le x < 1, \\ 1 & x \ge 1. \end{cases}$$

(a) What is $P[X > 1/2]$?

(b) What is $P[-1/2 < X \le 3/4]$?

(c) What is $P[|X| \le 1/2]$?

(d) What is the value of a such that $P[X \le a] = 0.8$?

Problem 3.1.1 Solution

The CDF of X is

$$F_X(x) = \begin{cases} 0 & x < -1 \\ (x+1)/2 & -1 \le x < 1 \\ 1 & x \ge 1 \end{cases} \tag{1}$$

Each question can be answered by expressing the requested probability in terms of $F_X(x)$.

(a)

$$P[X > 1/2] = 1 - P[X \le 1/2] = 1 - F_X(1/2) = 1 - 3/4 = 1/4 \tag{2}$$

(b) This is a little trickier than it should be. Being careful, we can write

$$P[-1/2 \le X < 3/4] = P[-1/2 < X \le 3/4] + P[X = -1/2] - P[X = 3/4] \tag{3}$$

Since the CDF of X is a continuous function, the probability that X takes on any specific value is zero. This implies $P[X = 3/4] = 0$ and $P[X = -1/2] = 0$. (If this is not clear at this point, it will become clear in Section 3.6.) Thus,

$$P[-1/2 \le X < 3/4] = P[-1/2 < X \le 3/4] = F_X(3/4) - F_X(-1/2) = 5/8 \tag{4}$$

(c)

$$P[|X| \le 1/2] = P[-1/2 \le X \le 1/2] = P[X \le 1/2] - P[X < -1/2] \tag{5}$$

Note that $P[X \le 1/2] = F_X(1/2) = 3/4$. Since the probability that $P[X = -1/2] = 0$, $P[X < -1/2] = P[X \le 1/2]$. Hence $P[X < -1/2] = F_X(-1/2) = 1/4$. This implies

$$P[|X| \le 1/2] = P[X \le 1/2] - P[X < -1/2] = 3/4 - 1/4 = 1/2 \tag{6}$$

(d) Since $F_X(1) = 1$, we must have $a \leq 1$. For $a \leq 1$, we need to satisfy

$$P[X \leq a] = F_X(a) = \frac{a+1}{2} = 0.8 \tag{7}$$

Thus $a = 0.6$.

Problem 3.1.2 •

The cumulative distribution function of the continuous random variable V is

$$F_V(v) = \begin{cases} 0 & v < -5, \\ c(v+5)^2 & -5 \leq v < 7, \\ 1 & v \geq 7. \end{cases}$$

(a) What is c?

(b) What is $P[V > 4]$?

(c) $P[-3 < V \leq 0]$?

(d) What is the value of a such that $P[V > a] = 2/3$?

Problem 3.1.2 Solution

The CDF of V was given to be

$$F_V(v) = \begin{cases} 0 & v < -5 \\ c(v+5)^2 & -5 \leq v < 7 \\ 1 & v \geq 7 \end{cases} \tag{1}$$

(a) For V to be a continuous random variable, $F_V(v)$ must be a continuous function. This occurs if we choose c such that $F_V(v)$ doesn't have a discontinuity at $v = 7$. We meet this requirement if $c(7+5)^2 = 1$. This implies $c = 1/144$.

(b)

$$P[V > 4] = 1 - P[V \leq 4] = 1 - F_V(4) = 1 - 81/144 = 63/144 = 7/16 \tag{2}$$

(c)

$$P[-3 < V \leq 0] = F_V(0) - F_V(-3) = 25/144 - 4/144 = 21/144 = 7/48 \tag{3}$$

(d) Since $0 \leq F_V(v) \leq 1$ and since $F_V(v)$ is a nondecreasing function, it must be that $-5 \leq a \leq 7$. In this range,

$$P[V > a] = 1 - F_V(a) = 1 - (a+5)^2/144 = 2/3 \tag{4}$$

The unique solution in the range $-5 \leq a \leq 7$ is $a = 4\sqrt{3} - 5 = 1.928$.

Problem 3.2.1 •
The random variable X has probability density function

$$f_X(x) = \begin{cases} cx & 0 \le x \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Use the PDF to find

(a) the constant c,

(b) $P[0 \le X \le 1]$,

(c) $P[-1/2 \le X \le 1/2]$,

(d) the CDF $F_X(x)$.

Problem 3.2.1 Solution

$$f_X(x) = \begin{cases} cx & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) From the above PDF we can determine the value of c by integrating the PDF and setting it equal to 1.

$$\int_0^2 cx\, dx = 2c = 1 \tag{2}$$

Therefore $c = 1/2$.

(b) $P[0 \le X \le 1] = \int_0^1 \frac{x}{2}\, dx = 1/4$

(c) $P[-1/2 \le X \le 1/2] = \int_0^{1/2} \frac{x}{2}\, dx = 1/16$

(d) The CDF of X is found by integrating the PDF from 0 to x.

$$F_X(x) = \int_0^x f_X(x')\, dx' = \begin{cases} 0 & x < 0 \\ x^2/4 & 0 \le x \le 2 \\ 1 & x > 2 \end{cases} \tag{3}$$

Problem 3.2.2 •
The cumulative distribution function of random variable X is

$$F_X(x) = \begin{cases} 0 & x < -1, \\ (x+1)/2 & -1 \le x < 1, \\ 1 & x \ge 1. \end{cases}$$

Find the PDF $f_X(x)$ of X.

Problem 3.2.2 Solution

From the CDF, we can find the PDF by direct differentiation. The CDF and correponding PDF are

$$F_X(x) = \begin{cases} 0 & x < -1 \\ (x+1)/2 & -1 \leq x < 1 \\ 1 & x \geq 1 \end{cases} \qquad f_X(x) = \begin{cases} 1/2 & -1 \leq x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Problem 3.3.3 •

Random variable X has CDF

$$F_X(x) = \begin{cases} 0 & x < 0, \\ x/2 & 0 \leq x \leq 2, \\ 1 & x > 2. \end{cases}$$

(a) What is $E[X]$?

(b) What is $\text{Var}[X]$?

Problem 3.3.3 Solution

The CDF of X is

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x/2 & 0 \leq x < 2 \\ 1 & x \geq 2 \end{cases} \tag{1}$$

(a) To find $E[X]$, we first find the PDF by differentiating the above CDF.

$$f_X(x) = \begin{cases} 1/2 & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

The expected value is then

$$E[X] = \int_0^2 \frac{x}{2}\, dx = 1 \tag{3}$$

(b)

$$E\left[X^2\right] = \int_0^2 \frac{x^2}{2}\, dx = 8/6 = 4/3 \tag{4}$$

$$\text{Var}[X] = E\left[X^2\right] - E[X]^2 = 4/3 - 1 = 1/3 \tag{5}$$

Problem 3.3.4 •

The probability density function of random variable Y is

$$f_Y(y) = \begin{cases} y/2 & 0 \leq y < 2, \\ 0 & \text{otherwise.} \end{cases}$$

What are $E[Y]$ and $\text{Var}[Y]$?

Problem 3.3.4 Solution

We can find the expected value of X by direct integration of the given PDF.

$$f_Y(y) = \begin{cases} y/2 & 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The expectation is

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y) dy = \int_0^2 \frac{y^2}{2} dy = 4/3 \tag{2}$$

To find the variance, we first find the second moment

$$E\left[Y^2\right] = \int_{-\infty}^{\infty} y^2 f_Y(y) dy = \int_0^2 \frac{y^3}{2} dy = 2. \tag{3}$$

The variance is then $\text{Var}[Y] = E[Y^2] - E[Y]^2 = 2 - (4/3)^2 = 2/9$.

Problem 3.4.2 •

Y is an exponential random variable with variance $\text{Var}[Y] = 25$.

(a) What is the PDF of Y?

(b) What is $E[Y^2]$?

(c) What is $P[Y > 5]$?

Problem 3.4.2 Solution

(a) From Appendix A, we observe that an exponential PDF Y with parameter $\lambda > 0$ has PDF

$$f_Y(y) = \begin{cases} \lambda e^{-\lambda y} & y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

In addition, the mean and variance of Y are

$$E[Y] = \frac{1}{\lambda} \qquad \text{Var}[Y] = \frac{1}{\lambda^2} \tag{2}$$

Since $\text{Var}[Y] = 25$, we must have $\lambda = 1/5$.

(b) The expected value of Y is $E[Y] = 1/\lambda = 5$, so

$$E\left[Y^2\right] = \text{Var}[Y] + (E[Y])^2 = 50 \tag{3}$$

(c)

$$P[Y > 5] = \int_5^{\infty} f_Y(y)\, dy = -e^{-y/5}\Big|_5^{\infty} = e^{-1} \tag{4}$$

Problem 3.4.3 •
X is an Erlang (n, λ) random variable with parameter $\lambda = 1/3$ and expected value $E[X] = 15$.

(a) What is the value of the parameter n?

(b) What is the PDF of X?

(c) What is $\text{Var}[X]$?

Problem 3.4.3 Solution
From Appendix A, an Erlang random variable X with parameters $\lambda > 0$ *a postive real number* and n `a positive integer` has PDF

$$f_X(x) = \begin{cases} \lambda^n x^{n-1} e^{-\lambda x}/(n-1)! & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

In addition, the mean and variance of X are

$$E[X] = \frac{n}{\lambda} \qquad \text{Var}[X] = \frac{n}{\lambda^2} \tag{2}$$

(a) Since $\lambda = 1/3$ and $E[X] = n/\lambda = 15$, we must have $n = 5$.

(b) Substituting the parameters $n = 5$ and $\lambda = 1/3$ into the given PDF, we obtain

$$f_X(x) = \begin{cases} (1/3)^5 x^4 e^{-x/3}/24 & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

(c) From above, we know that $\text{Var}[X] = n/\lambda^2 = 45$.

Note: we need not use the definitions in Appendix A to solve these problems. We can obtain the expressions for the expected value and the variance by applying the definitions. This will require using integration by parts and induction, but is not otherwise difficult.

Problem 3.5.1 •
The peak temperature T, as measured in degrees Fahrenheit, on a July day in New Jersey is the Gaussian $(85, 10)$ random variable. What is $P[T > 100]$, $P[T < 60]$, and $P[70 \leq T \leq 100]$?

Problem 3.5.1 Solution
Given that the peak temperature, T, is a Gaussian random variable with mean 85 and standard deviation 10 we can use the fact that $F_T(t) = \Phi((t - \mu_T)/\sigma_T)$ and Table 3.1 on

page 123 to evaluate the following

$$\begin{aligned} P[T > 100] &= 1 - P[T \leq 100] = 1 - F_T(100) = 1 - \Phi\left(\frac{100-85}{10}\right) \\ &= 1 - \Phi(1.5) = 1 - 0.9332 = 0.0668 \quad (1) \\ P[T < 60] &= \Phi\left(\frac{60-85}{10}\right) = \Phi(-2.5) \\ &= 1 - \Phi(2.5) = 1 - .9938 = 0.0062 \quad (2) \\ P[70 \leq T \leq 100] &= F_T(100) - F_T(70) \\ &= \Phi(1.5) - \Phi(-1.5) = 2\Phi(1.5) - 1 = .8664 \quad (3) \end{aligned}$$

Problem 3.5.3 •

X is a Gaussian random variable with $E[X] = 0$ and $P[|X| \leq 10] = 0.1$. What is the standard deviation σ_X?

Problem 3.5.3 Solution

X is a Gaussian random variable with zero mean but unknown variance. We do know, however, that

$$P[|X| \leq 10] = 0.1 \quad (1)$$

We can find the variance Var$[X]$ by expanding the above probability in terms of the $\Phi(\cdot)$ function.

$$P[-10 \leq X \leq 10] = F_X(10) - F_X(-10) = \Phi\left(\frac{10}{\sigma_X}\right) - \left(1 - \Phi\left(\frac{10}{\sigma_X}\right)\right) = 2\Phi\left(\frac{10}{\sigma_X}\right) - 1 \quad (2)$$

This implies $\Phi(10/\sigma_X) = 0.55$. Using Table 3.1 for the Gaussian CDF, we find that $10/\sigma_X \approx 0.125$ or $\sigma_X \approx 80$.

Problem 3.6.2 •

Let X be a random variable with CDF

$$F_X(x) = \begin{cases} 0 & x < -1, \\ x/4 + 1/2 & -1 \leq x < 1, \\ 1 & 1 \leq x. \end{cases}$$

Sketch the CDF and find

(a) $P[X < -1]$ and $P[X \leq -1]$,

(b) $P[X < 0]$ and $P[X \leq 0]$,

(c) $P[X > 1]$ and $P[X \geq 1]$.

Problem 3.6.2 Solution

Here the authors use the notation

$$F_X\left(a^-\right) := \lim_{x \to a^-} F_X(a) \tag{1}$$

$$F_X\left(a^+\right) := \lim_{x \to a^+} F_X(a) \tag{2}$$

where a is any value in the range of the CDF.

[As in] the previous problem we find

(a)

$$P[X < -1] = F_X\left(-1^-\right) = 0 \qquad P[X \leq -1] = F_X(-1) = 1/4 \tag{3}$$

Here we notice the discontinuity of value $1/4$ at $x = -1$.

(b)

$$P[X < 0] = F_X\left(0^-\right) = 1/2 \qquad P[X \leq 0] = F_X(0) = 1/2 \tag{4}$$

Since there is no discontinuity at $x = 0$, $F_X(0^-) = F_X(0^+) = F_X(0)$.

(c)

$$P[X > 1] = 1 - P[X \leq 1] = 1 - F_X(1) = 0 \tag{5}$$

$$P[X \geq 1] = 1 - P[X < 1] = 1 - F_X\left(1^-\right) = 1 - 3/4 = 1/4 \tag{6}$$

Again we notice a discontinuity of size $1/4$, here occurring at $x = 1$.

Problem 3.6.3 •

For random variable X of Problem 3.6.2, find

(a) $f_X(x)$

(b) $E[X]$

(c) $\mathrm{Var}[X]$

Problem 3.6.3 Solution

(a) By taking the derivative of the CDF $F_X(x)$ given in Problem 3.6.2, we obtain the PDF

$$f_X(x) = \begin{cases} \frac{\delta(x+1)}{4} + 1/4 + \frac{\delta(x-1)}{4} & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The reason for the factor of 1/4 multiplying the impulses can be seen by graphing the CDF and determining the magnitude of the jumps in the CDF that occur at ± 1. (You can calculate this without drawing the graph but it can be helpful to visualize the behavior of the function.)

(b) The first moment of X is

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx \tag{2}$$

$$= x/4|_{x=-1} + x^2/8\Big|_{-1}^{1} + x/4|_{x=1} = -1/4 + 0 + 1/4 = 0. \tag{3}$$

(c) The second moment of X is

$$E\left[X^2\right] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx \tag{4}$$

$$= x^2/4\Big|_{x=-1} + x^3/12\Big|_{-1}^{1} + x^2/4\Big|_{x=1} = 1/4 + 1/6 + 1/4 = 2/3. \tag{5}$$

Since $E[X] = 0$, $\mathrm{Var}[X] = E[X^2] = 2/3$.

Problem 3.7.2 •

Let X have an exponential (λ) PDF. Find the CDF and PDF of $Y = \sqrt{X}$. Show that Y is a Rayleigh random variable (see Appendix A.2). Express the Rayleigh parameter a in terms of the exponential parameter λ.

Problem 3.7.2 Solution

Since $Y = \sqrt{X}$, the fact that X is nonegative and that we asume the square root is always positive implies $F_Y(y) = 0$ for $y < 0$. In addition, for $y \geq 0$, we can find the CDF of Y by writing

$$F_Y(y) = P[Y \leq y] = P\left[\sqrt{X} \leq y\right] = P\left[X \leq y^2\right] = F_X\left(y^2\right) \tag{1}$$

For $x \geq 0$, $F_X(x) = 1 - e^{-\lambda x}$. Thus,

$$F_Y(y) = \begin{cases} 1 - e^{-\lambda y^2} & y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

By taking the derivative with respect to y, it follows that the PDF of Y is

$$f_Y(y) = \begin{cases} 2\lambda y e^{-\lambda y^2} & y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

In comparing this result to the Rayleigh PDF given in Appendix A, we observe that Y is a Rayleigh (a) random variable with $a = \sqrt{2\lambda}$.

Problem 3.7.3 •
If X has an exponential (λ) PDF, what is the PDF of $W = X^2$?

Problem 3.7.3 Solution
Since X is non-negative, $W = X^2$ is also non-negative. Hence for $w < 0$, $f_W(w) = 0$. For $w \geq 0$,

$$\begin{aligned} F_W(w) = P[W \leq w] &= P\left[X^2 \leq w\right] && (1)\\ &= P[X \leq w] && (2)\\ &= 1 - e^{-\lambda\sqrt{w}} && (3) \end{aligned}$$

Taking the derivative with respect to w yields, for $w \geq 0$, $f_W(w) = \lambda e^{-\lambda\sqrt{w}}/(2\sqrt{w})$. The complete expression for the PDF is

$$f_W(w) = \begin{cases} \frac{\lambda e^{-\lambda\sqrt{w}}}{2\sqrt{w}} & w > 0 \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

where we note that we cannot allow $w = 0$ *to be part of the first case when we have* $\sqrt{w}$ *in the denominator.*

Problem 3.8.1 •
X is a uniform random variable with parameters -5 and 5. Given the event $B = \{|X| \leq 3\}$,

(a) Find the conditional PDF, $f_{X|B}(x)$.

(b) Find the conditional expected value, $E[X|B]$.

(c) What is the conditional variance, $\text{Var}[X|B]$?

Problem 3.8.1 Solution
The PDF of X is

$$f_X(x) = \begin{cases} 1/10 & -5 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

(a) The event B has probability

$$P[B] = P[-3 \leq X \leq 3] = \int_{-3}^{3} \frac{1}{10}\, dx = \frac{3}{5} \quad (2)$$

From Definition 3.15, the conditional PDF of X given B is

$$f_{X|B}(x) = \begin{cases} f_X(x)/P[B] & x \in B \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1/6 & |x| \leq 3 \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

(b) Given B, we see that X has a uniform PDF over $[a, b]$ with $a = -3$ and $b = 3$. From Theorem 3.6, the conditional expected value of X is $E[X|B] = (a + b)/2 = 0$.

(c) From Theorem 3.6, the conditional variance of X is $\text{Var}[X|B] = (b-a)^2/12 = 3$. *Of course we do not have to use Theorem 3.6. We can instead use the definitions and write*

$$\text{Var}[X|B] = E\left[X^2|B\right] - (E\left[X|B\right])^2 = \int_{-3}^{3} x^2 \left(\frac{1}{6}\right) dx - 0^2 = \cdots = 3. \tag{4}$$

Problem 3.8.2 •

Y is an exponential random variable with parameter $\lambda = 0.2$. Given the event $A = \{Y < 2\}$,

(a) What is the conditional PDF, $f_{Y|A}(y)$?

(b) Find the conditional expected value, $E[Y|A]$.

Problem 3.8.2 Solution

From Definition 3.6, the PDF of Y is

$$f_Y(y) = \begin{cases} (1/5)e^{-y/5} & y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) The event A has probability

$$P[A] = P[Y < 2] = \int_0^2 (1/5)e^{-y/5}\, dy = -e^{-y/5}\Big|_0^2 = 1 - e^{-2/5} \tag{2}$$

From Definition 3.15, the conditional PDF of Y given A is

$$f_{Y|A}(y) = \begin{cases} f_Y(y)/P[A] & x \in A \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

$$= \begin{cases} (1/5)e^{-y/5}/(1-e^{-2/5}) & 0 \leq y < 2 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

(b) The conditional expected value of Y given A is

$$E[Y|A] = \int_{-\infty}^{\infty} y f_{Y|A}(y)\, dy = \frac{1/5}{1-e^{-2/5}} \int_0^2 y e^{-y/5}\, dy \tag{5}$$

Using the integration by parts formula $\int u\, dv = uv - \int v\, du$ with $u = y$ and $dv = e^{-y/5}\, dy$ yields

$$E[Y|A] = \frac{1/5}{1-e^{-2/5}} \left(-5ye^{-y/5}\Big|_0^2 + \int_0^2 5e^{-y/5}\, dy \right) \tag{6}$$

$$= \frac{1/5}{1-e^{-2/5}} \left(-10e^{-2/5} - 25e^{-y/5}\Big|_0^2 \right) \tag{7}$$

$$= \frac{5-7e^{-2/5}}{1-e^{-2/5}} \tag{8}$$

Problem 3.9.2 •

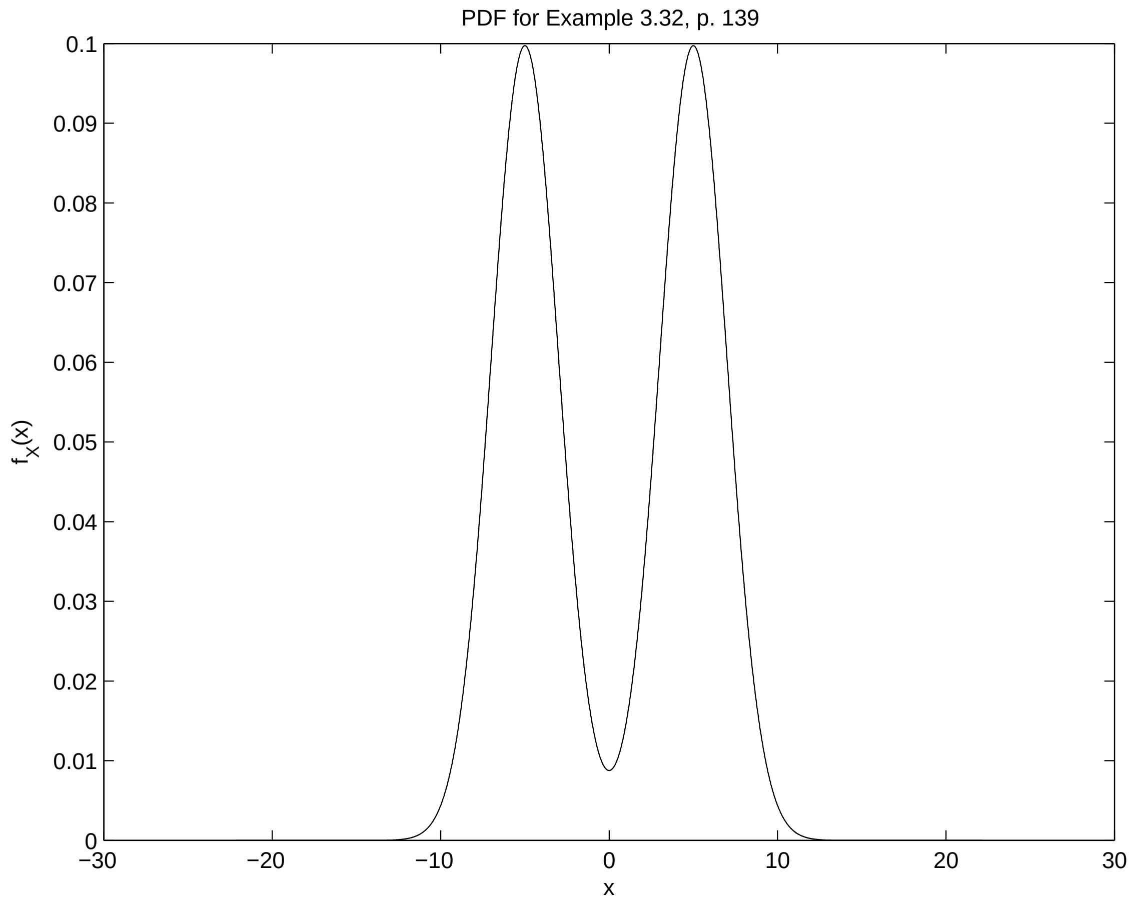
For the modem receiver voltage X with PDF given in Example 3.32, use MATLAB to plot the PDF and CDF of random variable X. Write a MATLAB function `x=modemrv(m)` that produces `m` samples of the modem voltage X.

Problem 3.9.2 Solution

I generated the PDF and CDF using the Matlab commands `normpdf` and `normcdf`. I generated a histogram of the random samples from the PDF generated by my Matlab function `modemrv`. The code for generating the required plots is given below.

```
%
% Yates and Goodman 3.9.2 2/14/06 --sk
%
x1 = (normpdf([-30:.01:30],-5,2)+normpdf([-30:0.01:30],5,2))/2;
figure(1)
plot([-30:.01:30],x1)
title('PDF for Example 3.32, p. 139');
xlabel('x');
ylabel('f_X(x)');
print -deps pdf_3_9_2
figure(2)
x2 = (normcdf([-30:.01:30],-5,2)+normcdf([-30:0.01:30],5,2))/2;
plot([-30:.01:30],x2)
title('CDF for Example 3.32, p. 139');
xlabel('x');
ylabel('F_X(x)');
print -deps cdf_3_9_2

%  some checks:

% height of local minimum of PDF at x = 0
disp('height of local minimum of PDF at x = 0')
fx0 = 2*exp(-25/8)/sqrt(32*pi)
% height of local maximum of PDF at x = +/- 5
disp('height of local maximum of PDF at x = +/- 5')
fx5 = (exp(-100/8)+1)/sqrt(32*pi)

x = modemrv(10000);
figure(3)
hist(x,100);
title('Histogram of samples from the PDF of Example 3.32')
xlabel('x (Volts)')
print -deps hist_3_9_2
```

Here is the plot of the PDF.

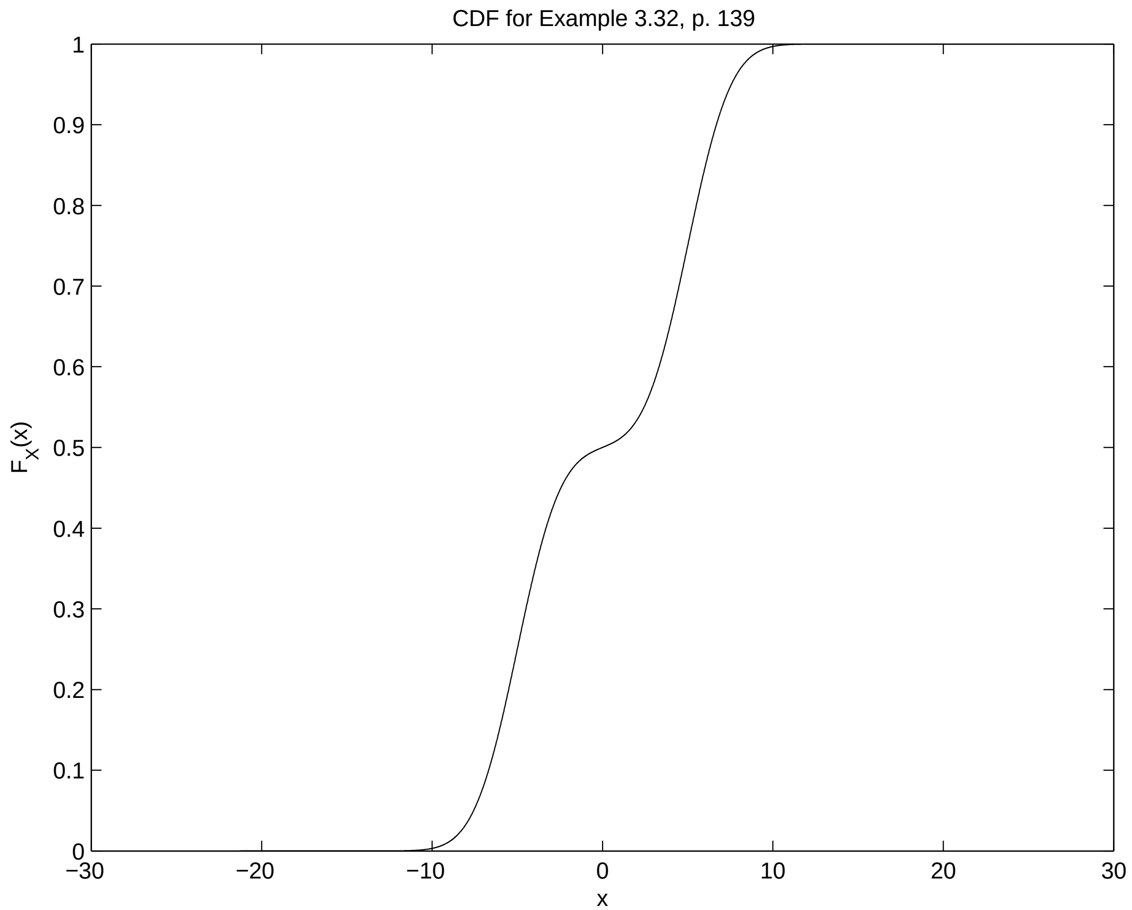


Note that in order to verify that I was using the function `normpdf` correctly I plugged in the values of 0 and 5 to see what the local maxima and minima should be. The results were

```
>> p3_9_2
height of local minimum of PDF at x = 0

fx0 =

    0.0088

height of local maximum of PDF at x = +/- 5

fx5 =

    0.0997
```

which match the values seen in the plot of the PDF.

Here is the plot of the CDF.

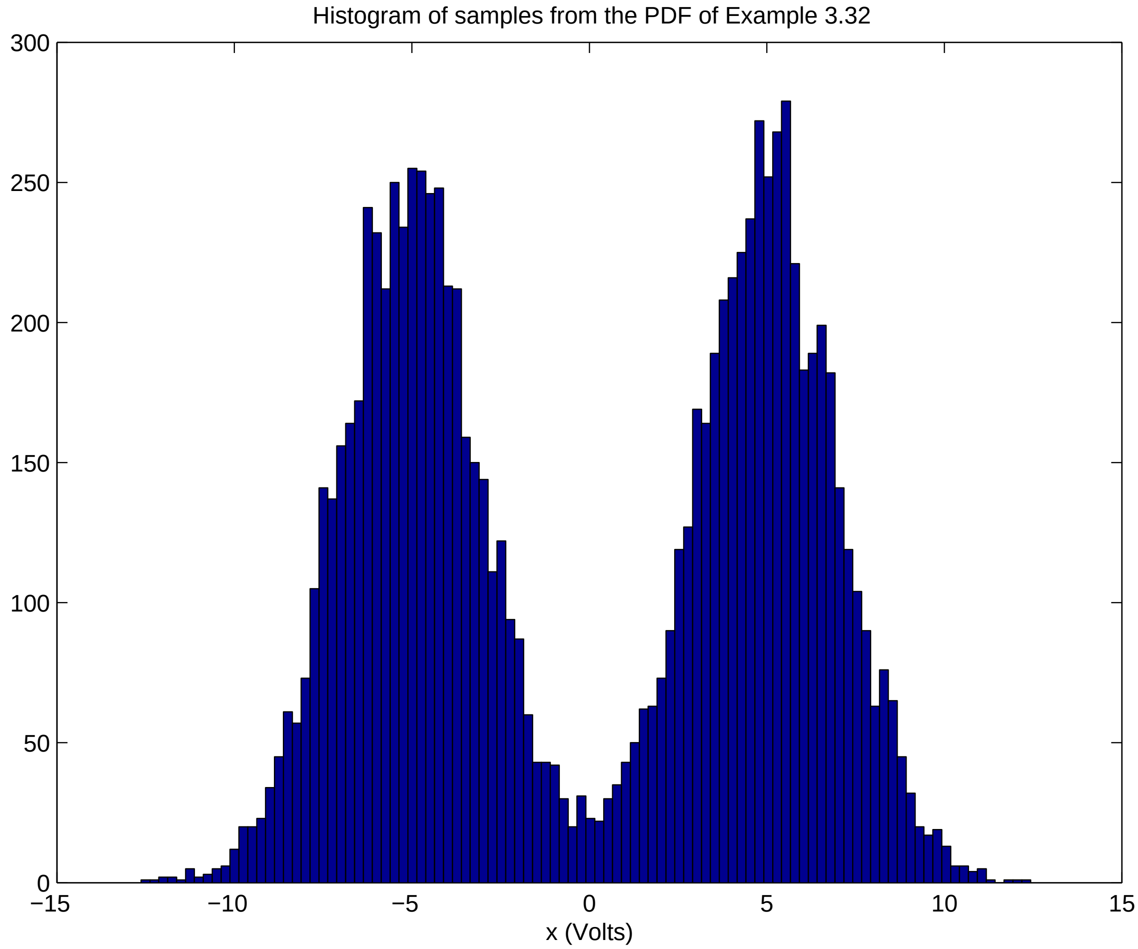


Without doing any calculations, we know that the CDF should have value $F_X(0) = 1/2$ and that $F_X(x)$ should approach 1 as x becomes large. Our plot meets both of these criteria.

Finally, here is the function `modemrv`,

```
function x=modemrv(m);
%Usage: x=modemrv(m)
%generates m samples of X, the modem
%receiver voltage in Exampe 3.32.
%X=+-5 + N where N is Gaussian (0,2)
sb=[-5; 5]; pb=[0.5; 0.5];
b=finiterv(sb,pb,m);
noise=gaussrv(0,2,m);
x=b+noise;
```

and the histogram of the data generated by the function `modemrv`. The shape of the histogram matches the shape of the PDF as it should.


Histogram of samples from the PDF of Example 3.32
300
250
200
150
100
50
0
−15
−10
−5
0
5
10
15
x (Volts)

Solutions to HW6

Note: Most of these solutions were generated by R. D. Yates and D. J. Goodman, the authors of our textbook. I have added comments in italics where I thought more detail was appropriate.

Problem 3.2.5 ♦♦

For constants a and b, random variable X has PDF

$$f_X(x) = \begin{cases} ax^2 + bx & 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

What conditions on a and b are necessary and sufficient to guarantee that $f_X(x)$ is a valid PDF?

Problem 3.2.5 Solution

$$f_X(x) = \begin{cases} ax^2 + bx & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

First, we note that a and b must be chosen such that the above PDF integrates to 1.

$$\int_0^1 (ax^2 + bx)\, dx = a/3 + b/2 = 1 \tag{2}$$

Hence, $b = 2 - 2a/3$ and our PDF becomes

$$f_X(x) = x(ax + 2 - 2a/3) \tag{3}$$

PDF's must take only nonnegative values. For the PDF to be non-negative for $x \in [0,1]$, we must have $ax + 2 - 2a/3 \ge 0$ for all $x \in [0,1]$. *We want to obtain constraints on a and b so first we must isolate a.* [The] requirement can be written as

$$a(2/3 - x) \le 2 \qquad (0 \le x \le 1) \tag{4}$$

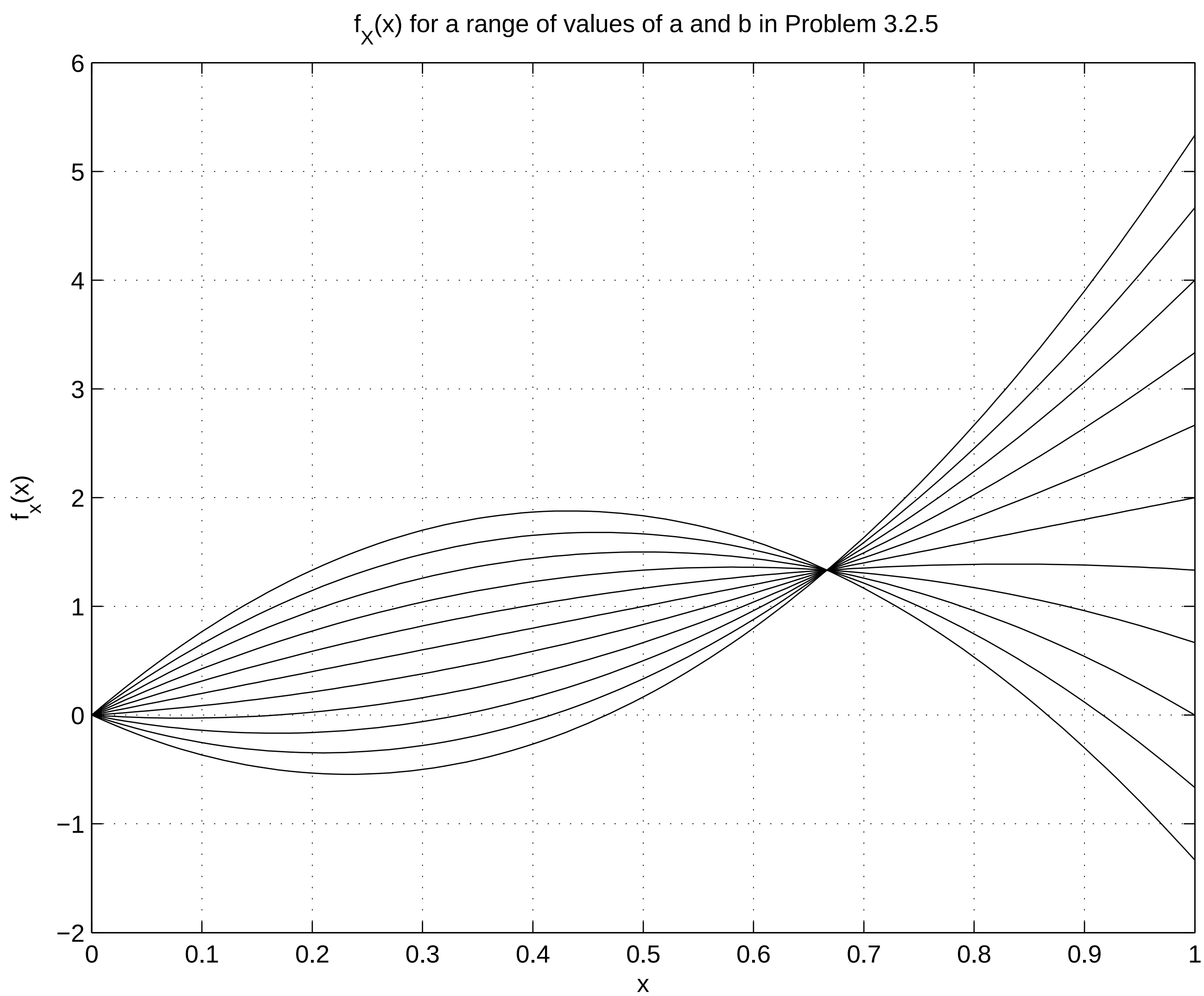
For $x = 2/3$, the requirement holds for all a. However, the problem is tricky because we must consider the cases $0 \le x < 2/3$ and $2/3 < x \le 1$ separately because of the sign change of the inequality. When $0 \le x < 2/3$, we have $2/3 - x > 0$ and the requirement is most stringent at $x = 0$ where we require $2a/3 \le 2$ or $a \le 3$. When $2/3 < x \le 1$, we can write the constraint as $a(x - 2/3) \ge -2$. In this case, the constraint is most stringent at $x = 1$, where we must have $a/3 \ge -2$ or $a \ge -6$. Thus a complete expression for our requirements are

$$-6 \le a \le 3 \qquad b = 2 - 2a/3 \tag{5}$$

Now, if the above reasoning was hard to follow, and even if it wasn't, I suggest that we use the tools we have available to help us gain insight into the behavior of the function $f_X(x)$ as a varies. All of the values of a that were of potential significance above were in the range $[-10, 10]$ so let's plot $f_X(x)$ for values of a in that range. A Matlab script that generates the values is given below.

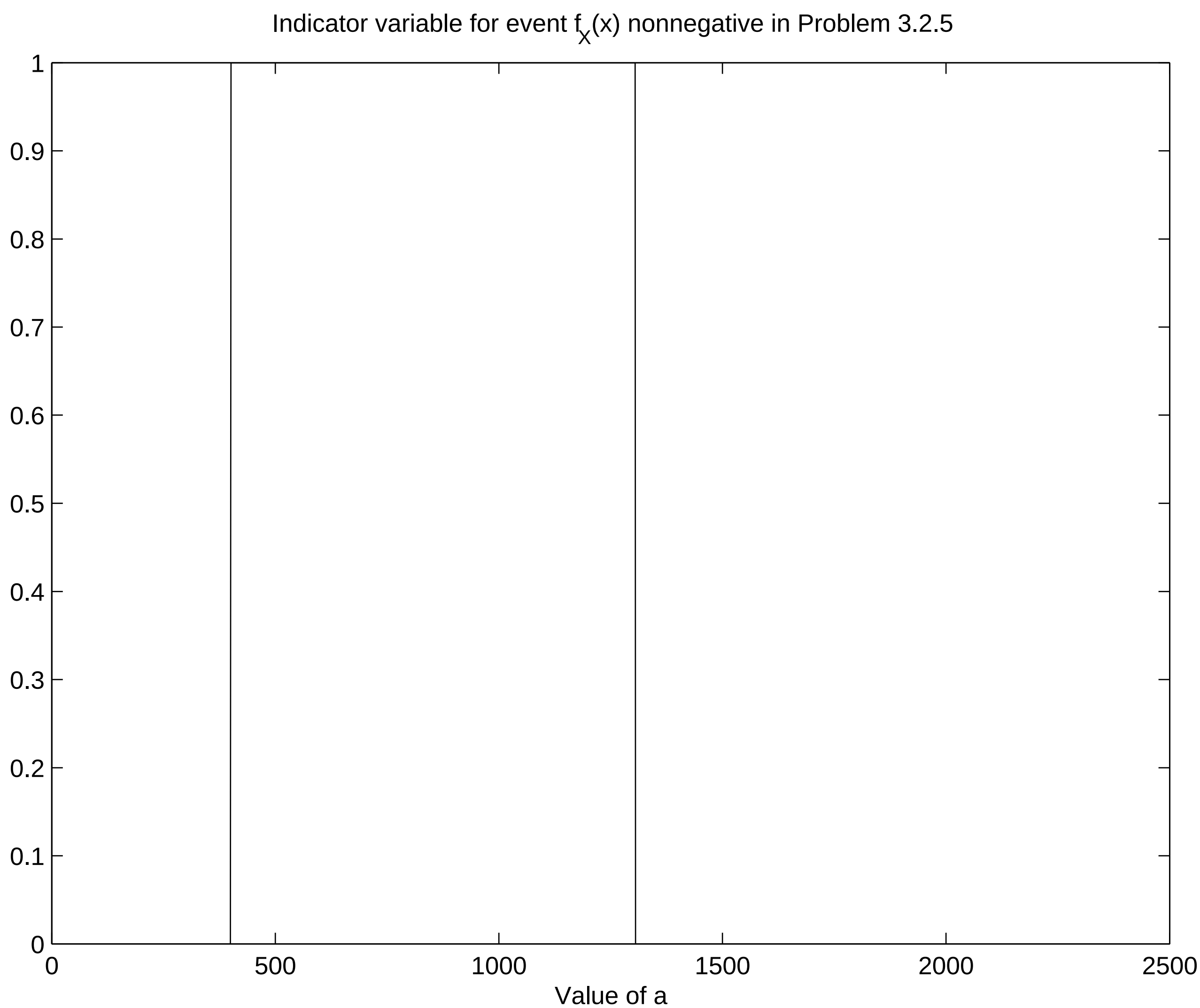
```
%%% 3.2.5                                                  2/19/06 sk
a = [-10:.01:10];b = 2*ones(size(a))-2*a/3;
x = [0:.01:10];
for index = 1:200:2001,plot(x,y(index,:)),hold on,end;
xlabel('x')
ylabel('f_X(x)')
title('f_X(x) for a range of values of a and b in Problem 3.2.5')
print -deps p3_2_5
```

The resulting figure clearly shows that there is a maximum and a minimum value of a.



Next, I defined an indicator random variable which takes the value 1 if the value of a results in an acceptable PDF and 0 otherwise. Here's the Matlab code and a plot of the value of the indicator random variable.

```
for index = 1:2001,if min(y(index,:))>=0, ok(index)=1;else ok(index)=0;end;end;
plot(ok)
figure(2)
title('Indicator variable for event f_X(x) nonnegative in Problem 3.2.5')
xlabel('Value of a')
print -deps p3_2_5b
```



Checking individual values to find the transitions in the plot of the indicator variable, we find values $a = -6$ and $a = 3 + \delta$ for a very small positive δ. We suspect that numerical error (the values plotted are generated for points at discrete intervals) is the source of the δ. Now that we see what is going on – for $a < -6$ the value of the function goes negative as x increases from zero and for $a > 3$, the value of the function goes negative at some x less than 1 – the analysis is more straightforward.

Problem 3.3.6 ■

The cumulative distribution function of random variable V is

$$F_V(v) = \begin{cases} 0 & v < -5, \\ (v+5)^2/144 & -5 \le v < 7, \\ 1 & v \ge 7. \end{cases}$$

(a) What is $E[V]$?

(b) What is $\text{Var}[V]$?

(c) What is $E[V^3]$?

Problem 3.3.6 Solution

To evaluate the moments of V, we need the PDF $f_V(v)$, which we find by taking the derivative of the CDF $F_V(v)$. The CDF and corresponding PDF of V are

$$F_V(v) = \begin{cases} 0 & v < -5 \\ (v+5)^2/144 & -5 \le v < 7 \\ 1 & v \ge 7 \end{cases} \qquad f_V(v) = \begin{cases} 0 & v < -5 \\ (v+5)/72 & -5 \le v < 7 \\ 0 & v \ge 7 \end{cases} \tag{1}$$

We must check to see that there are no discontinuities in the CDF to determine whether we need any impulses in the PDF. We find that $F_V(-5) = 0$ *and that if we let* $(7+5)^2/144 = 1$ *so there are no discontinuities and we have correctly determined the PDF.*

(a) The expected value of V is

$$E[V] = \int_{-\infty}^{\infty} v f_V(v)\, dv = \frac{1}{72}\int_{-5}^{7} (v^2 + 5v)\, dv \tag{2}$$

$$= \frac{1}{72}\left(\frac{v^3}{3} + \frac{5v^2}{2}\right)\Bigg|_{-5}^{7} = \frac{1}{72}\left(\frac{343}{3} + \frac{245}{2} + \frac{125}{3} - \frac{125}{2}\right) = 3 \tag{3}$$

(b) To find the variance, we first find the second moment

$$E\left[V^2\right] = \int_{-\infty}^{\infty} v^2 f_V(v)\, dv = \frac{1}{72}\int_{-5}^{7} (v^3 + 5v^2)\, dv \tag{4}$$

$$= \frac{1}{72}\left(\frac{v^4}{4} + \frac{5v^3}{3}\right)\Bigg|_{-5}^{7} = 17 \tag{5}$$

The variance is $\text{Var}[V] = E[V^2] - (E[V])^2 = 17 - 9 = 8$.

(c) The third moment of V is

$$E\left[V^3\right] = \int_{-\infty}^{\infty} v^3 f_V(v)\, dv = \frac{1}{72}\int_{-5}^{7} (v^4 + 5v^3)\, dv \tag{6}$$

$$= \frac{1}{72}\left(\frac{v^5}{5} + \frac{5v^4}{4}\right)\Bigg|_{-5}^{7} = 86.2 \tag{7}$$

Problem 3.3.7 ■

The cumulative distribution function of random variable U is

$$F_U(u) = \begin{cases} 0 & u < -5, \\ (u+5)/8 & -5 \le u < -3m, \\ 1/4 & -3 \le u < 3, \\ 1/4 + 3(u-3)/8 & 3 \le u < 5, \\ 1 & u \ge 5. \end{cases}$$

(a) What is $E[U]$?

(b) What is $\mathrm{Var}[U]$?

(c) What is $E[2^U]$?

Problem 3.3.7 Solution

To find the moments, we first find the PDF of U by taking the derivative of $F_U(u)$. The CDF and corresponding PDF are

$$F_U(u) = \begin{cases} 0 & u < -5 \\ (u+5)/8 & -5 \le u < -3 \\ 1/4 & -3 \le u < 3 \\ 1/4 + 3(u-3)/8 & 3 \le u < 5 \\ 1 & u \ge 5. \end{cases} \qquad f_U(u) = \begin{cases} 0 & u < -5 \\ 1/8 & -5 \le u < -3 \\ 0 & -3 \le u < 3 \\ 3/8 & 3 \le u < 5 \\ 0 & u \ge 5. \end{cases} \tag{1}$$

Note that we should also verify that there are no jumps in the CDF. If there are jumps, we will need impulses of those magnitudes in the PDF. There are four values of u that we must check, as follows:

$$\left.\frac{u+5}{8}\right|_{u=-5} = 0 \qquad \left.\frac{u+5}{8}\right|_{u=-3} = \frac{1}{4} \qquad \left.\frac{1}{4} + \frac{3(u-8)}{8}\right|_{u=3} = \frac{1}{4} \qquad \left.\frac{1}{4} + \frac{3(u-8)}{8}\right|_{u=5} = 1 \tag{2}$$

So we have shown that there are no discontiuities (jumps) in the CDF.

(a) The expected value of U is

$$E[U] = \int_{-\infty}^{\infty} u f_U(u)\, du = \int_{-5}^{-3} \frac{u}{8}\, du + \int_{3}^{5} \frac{3u}{8}\, du \tag{3}$$

$$= \left.\frac{u^2}{16}\right|_{-5}^{-3} + \left.\frac{3u^2}{16}\right|_{3}^{5} = 2 \tag{4}$$

(b) The second moment of U is

$$E\left[U^2\right] \int_{-\infty}^{\infty} u^2 f_U(u)\, du = \int_{-5}^{-3} \frac{u^2}{8}\, du + \int_{3}^{5} \frac{3u^2}{8}\, du \tag{5}$$

$$= \left.\frac{u^3}{24}\right|_{-5}^{-3} + \left.\frac{u^3}{8}\right|_{3}^{5} = 49/3 \tag{6}$$

The variance of U is $\mathrm{Var}[U] = E[U^2] - (E[U])^2 = 37/3$.

(c) Note that $2^U = e^{(\ln 2)U}$. This implies that

$$\int 2^u \, du = \int e^{(\ln 2)u} \, du = \frac{1}{\ln 2} e^{(\ln 2)u} = \frac{2^u}{\ln 2} \tag{7}$$

The expected value of 2^U is then

$$E\left[2^U\right] = \int_{-\infty}^{\infty} 2^u f_U(u) \, du = \int_{-5}^{-3} \frac{2^u}{8} \, du + \int_{3}^{5} \frac{3 \cdot 2^u}{8} \, du \tag{8}$$

$$= \left.\frac{2^u}{8\ln 2}\right|_{-5}^{-3} + \left.\frac{3 \cdot 2^u}{8\ln 2}\right|_{3}^{5} = \frac{2307}{256\ln 2} = 13.001 \tag{9}$$

Problem 3.4.5 ■

X is a continuous uniform $(-5, 5)$ random variable.

(a) What is the PDF $f_X(x)$?

(b) What is the CDF $F_X(x)$?

(c) What is $E[X]$?

(d) What is $E[X^5]$?

(e) What is $E[e^X]$?

Problem 3.4.5 Solution

(a) The PDF of a continuous uniform $(-5, 5)$ random variable is

$$f_X(x) = \begin{cases} 1/10 & -5 \le x \le 5 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(b) For $x < -5$, $F_X(x) = 0$. For $x \ge 5$, $F_X(x) = 1$. For $-5 \le x \le 5$, the CDF is

$$F_X(x) = \int_{-5}^{x} f_X(\tau) \, d\tau = \frac{x+5}{10} \tag{2}$$

The complete expression for the CDF of X is

$$F_X(x) = \begin{cases} 0 & x < -5 \\ (x+5)/10 & 5 \le x \le 5 \\ 1 & x > 5 \end{cases} \tag{3}$$

(c) The expected value of X is

$$\int_{-5}^{5} \frac{x}{10}\,dx = \left.\frac{x^2}{20}\right|_{-5}^{5} = 0 \tag{4}$$

Another way to obtain this answer is to use Theorem 3.6 which says the expected value of X is $E[X] = (5 + -5)/2 = 0$.

(d) The fifth moment of X is

$$\int_{-5}^{5} \frac{x^5}{10}\,dx = \left.\frac{x^6}{60}\right|_{-5}^{5} = 0 \tag{5}$$

(e) The expected value of e^X is

$$\int_{-5}^{5} \frac{e^x}{10}\,dx = \left.\frac{e^x}{10}\right|_{-5}^{5} = \frac{e^5 - e^{-5}}{10} = 14.84 \tag{6}$$

Problem 3.4.7 ■

The probability density function of random variable X is

$$f_X(x) = \begin{cases} (1/2)e^{-x/2} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is $P[1 \leq X \leq 2]$?

(b) What is $F_X(x)$, the cumulative distribution function of X?

(c) What is $E[X]$, the expected value of X?

(d) What is $\text{Var}[X]$, the variance of X?

Problem 3.4.7 Solution

Given that

$$f_X(x) = \begin{cases} (1/2)e^{-x/2} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a)

$$P[1 \leq X \leq 2] = \int_1^2 (1/2)e^{-x/2}\,dx = e^{-1/2} - e^{-1} = 0.2387 \tag{2}$$

(b) The CDF of X may be be expressed as

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \int_0^x (1/2)e^{-\tau/2}\,d\tau & x \geq 0 \end{cases} = \begin{cases} 0 & x < 0 \\ 1 - e^{-x/2} & x \geq 0 \end{cases} \tag{3}$$

(c) X is an exponential random variable with parameter $a = 1/2$. By Theorem 3.8, the expected value of X is $E[X] = 1/a = 2$. *It is also simple to compute it directly.*

(d) By Theorem 3.8, the variance of X is $\text{Var}[X] = 1/a^2 = 4$.

Problem 3.5.5 ■

The peak temperature T, in degrees Fahrenheit, on a July day in Antarctica is a Gaussian random variable with a variance of 225. With probability 1/2, the temperature T exceeds 10 degrees. What is $P[T > 32]$, the probability the temperature is above freezing? What is $P[T < 0]$? What is $P[T > 60]$?

Problem 3.5.5 Solution

Moving to Antarctica, we find that the temperature, T is still Gaussian but with variance $\sigma^2 = 225$. We also know that with probability 1/2, T exceeds 10 degrees. First we would like to find the mean temperature, and we do so by looking at the second fact.

$$P[T > 10] = 1 - P[T \leq 10] = 1 - \Phi\left(\frac{10 - \mu_T}{15}\right) = 1/2 \tag{1}$$

By looking at the table we find that if $\Phi(\Gamma) = 1/2$, then $\Gamma = 0$. Therefore,

$$\Phi\left(\frac{10 - \mu_T}{15}\right) = 1/2 \tag{2}$$

implies that $(10 - \mu_T)/15 = 0$ or $\mu_T = 10$. Now we have a Gaussian T with mean 10 and standard deviation 15. So we are prepared to answer the following problems.

$$P[T > 32] = 1 - P[T \leq 32] = 1 - \Phi\left(\frac{32 - 10}{15}\right) \tag{3}$$

$$= 1 - \Phi(1.47) = 1 - 0.929 = 0.071 \tag{4}$$

$$P[T < 0] = F_T(0) = \Phi\left(\frac{0 - 10}{15}\right) \tag{5}$$

$$= \Phi(-2/3) = 1 - \Phi(2/3) \tag{6}$$

$$= 1 - \Phi(0.67) = 1 - 0.749 = 0.251 \tag{7}$$

$$P[T > 60] = 1 - P[T \leq 60] = 1 - F_T(60) \tag{8}$$

$$= 1 - \Phi\left(\frac{60 - 10}{15}\right) = 1 - \Phi(10/3) \tag{9}$$

$$= Q(3.33) = 4.34 \cdot 10^{-4} \tag{10}$$

Problem 3.6.6 ■

When you make a phone call, the line is busy with probability 0.2 and no one answers with probability 0.3. The random variable X describes the conversation time (in minutes) of a phone call that is answered. X is an exponential random variable with $E[X] = 3$ minutes. Let the random variable W denote the conversation time (in seconds) of all calls ($W = 0$ when the line is busy or there is no answer.)

(a) What is $F_W(w)$?

(b) What is $f_W(w)$?

(c) What are $E[W]$ and $\text{Var}[W]$?

Problem 3.6.6 Solution

We are given that random variable X (conversation time in minutes of answered calls) has an exponential distribution with expected value $E = [X] = 3$. We define an event space $\{A, A^c\}$, where A is the event that the call is answered and, of course, A^c is the event that either the line was busy or the call was not answered. We define a new random variable W (conversation time in seconds of all calls) which is thus defined as

$$W = \begin{cases} 60X & \text{if the call is answered, } i.e. \text{ if } A \text{ occurs,} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

in terms of the random variable X. Now, we can work through the entire problem in terms of $F_X(x)$ and probably make a lot of mistakes in accounting for the scale factor, or we can let $V = 60X$, determine $F_V(v)$ from $F_X(x)$, and work with $F_V(v)$ instead. I recommend the latter approach, especially after having attempted the former a couple of times. X being exponential with parameter $1/3 = 1/E[X]$, we can apply Theorem 3.20 to show that

$$f_V(v) = \begin{cases} \frac{1}{180} e^{-v/180} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

so integrating we have

$$F_V(v) = \begin{cases} 1 - e^{-v/180} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

and

$$W = \begin{cases} V & \text{if the call is answered, } i.e. \text{ if } A \text{ occurs,} \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

We will use Theorem 3.23 in determining the CDF. We note that this theorem gives a result for the PDF's not the CDF's so we will have to extend it as follows. Theorem 3.23 applied to our problem says that

$$f_W(w) = f_{W|A}(w)\, P[A] + f_{W|A^c}(w)\, P[A^c]. \tag{5}$$

To obtain the CDF from the PDF we integrate. This yields

$$\begin{aligned} f_W(w) &= \int_{-\infty}^{v} f_W(y) dy = \int_{-\infty}^{v} \left(f_{W|A}(y)\, P[A] + f_{W|A^c}(y)\, P[A^c] \right) dy & (6) \\ &= \int_{-\infty}^{v} f_{W|A}(y)\, P[A] dy + \int_{-\infty}^{v} f_{W|A^c}(y)\, P[A^c] dy & (7) \\ &= P[A] \int_{-\infty}^{v} f_{W|A}(y)\, dy + P[A^c] \int_{-\infty}^{v} f_{W|A^c}(y)\, dy & (8) \\ &= P[A] F_{W|A}(y) + P[A^c] F_{W|A^c}(y) = F_{W|A}(w)\, P[A] + F_{W|A^c}(w)\, P[A^c]. & (9) \end{aligned}$$

Thus we have shown that a similar relation to that given for the PDF in Theorem 3.23 holds for the CDF.

(a) We determine the CDF of W separately on the intervals $(-\infty, 0)$ and $[0, \infty]$. Since the conversation time cannot be negative, we know that $F_W(w) = 0$ for $w < 0$. Also, the event A^c implies $W = 0$, whereas the event A implies $w > 0$. Using (6), we have, for $w \geq 0$,

$$F_W(w) = P[A^c] F_{W|A^c}(w) + P[A] F_{W|A}(w) = (1/2) + (1/2)F_V(w) \tag{10}$$

so

$$F_W(w) = \begin{cases} 0 & w < 0 \\ 1/2 + (1/2)F_V(w) & w \geq 0 \end{cases} \tag{11}$$

so

$$F_W(w) = \begin{cases} 0 & w < 0 \\ 1/2 + 1/2\left(1 - e^{-(w/180)}\right) & w \geq 0. \end{cases} \tag{12}$$

(b) Taking the derivative, we obtain that the PDF of W is

$$f_W(w) = \begin{cases} 0 & w < 0 \\ \frac{1}{2}\delta(w) + (1/360)e^{-w/180} & w \geq 0 \end{cases} \tag{13}$$

(c) From the PDF $f_W(w)$, calculating the moments is straightforward.

$$E[W] = \int_{-\infty}^{\infty} w f_W(w)\, dw = (1/2)\int_{-\infty}^{\infty} v f_V(v)\, dv = 1/2E[V] = 90 \tag{14}$$

where we note that the contribution at zero is zero because we have $v f_V(v) = 0\delta(0)$. The second moment is

$$E\left[W^2\right] = \int_{-\infty}^{\infty} w^2 f_W(w)\, dw = (1/2)\int_{-\infty}^{\infty} v^2 f_V(v)\, dw = 1/2E\left[V^2\right] = 16{,}200 \tag{15}$$

The variance of W is

$$\text{Var}[W] = E\left[W^2\right] - (E[W])^2 = 16{,}200 - 90^2 = 8100 \tag{16}$$

Problem 3.7.5 ■

U is a uniform $(0, 1)$ random variable and $X = -\ln(1 - U)$.

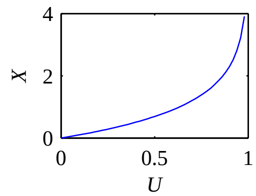
(a) What is $F_X(x)$?

(b) What is $f_X(x)$?

(c) What is $E[X]$?

Problem 3.7.5 Solution

Before solving for the PDF, it is helpful to have a sketch of the function $X = -\ln(1 - U)$.



(a) From the sketch, we observe that X will be nonnegative. Hence $F_X(x) = 0$ for $x < 0$. Since U has a uniform distribution on $[0,1]$, for $0 \leq u \leq 1$, $P[U \leq u] = u$. We use this fact to find the CDF of X. For $x \geq 0$,

$$F_X(x) = P[-\ln(1-U) \leq x] = P\left[1 - U \geq e^{-x}\right] = P\left[U \leq 1 - e^{-x}\right] \tag{1}$$

For $x \geq 0$, $0 \leq 1 - e^{-x} \leq 1$ and so

$$F_X(x) = F_U\left(1 - e^{-x}\right) = 1 - e^{-x} \tag{2}$$

The complete CDF can be written as

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & x \geq 0 \end{cases} \tag{3}$$

(b) By taking the derivative, the PDF is

$$f_X(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Thus, X has an exponential PDF. In fact, since most computer languages provide uniform $[0,1]$ random numbers, the procedure outlined in this problem provides a way to generate exponential random variables from uniform random variables.

(c) Since X is an exponential random variable with parameter $a = 1$, $E[X] = 1$ *by the table in Appendix A.*

Note that it is not necessary to have the textbook handy to determine the expected value. Letting $u = x$ and $dv = e^{-x}$ so that $du = dx$ and $v = -e^{-x}$ and performing integration by parts we obtain

$$\int_{-\infty}^{\infty} x f_X(x)dx = \int_0^{\infty} xe^{-x}dx = -xe^{-x}\big|_0^1 - \int_0^{\infty}(-e^{-x})dx = 0 - e^{-x}\big|_0^{\infty} = 1 \tag{5}$$

Problem 3.7.9 ■

U is a uniform random variable with parameters 0 and 2. The random variable W is the output of the clipper:

$$W = g(U) = \begin{cases} U & U \leq 1, \\ 1 & U > 1. \end{cases}$$

Find the CDF $F_W(w)$, the PDF $f_W(w)$, and the expected value $E[W]$.

Problem 3.7.9 Solution
The uniform $(0, 2)$ random variable U has PDF and CDF

$$f_U(u) = \begin{cases} 1/2 & 0 \le u \le 2, \\ 0 & \text{otherwise}, \end{cases} \qquad F_U(u) = \begin{cases} 0 & u < 0, \\ u/2 & 0 \le u \le 2, \\ 1 & u > 2. \end{cases} \tag{1}$$

The uniform random variable U is subjected to the following clipper.

$$W = g(U) = \begin{cases} U & U \le 1 \\ 1 & U > 1 \end{cases} \tag{2}$$

To find the CDF of the output of the clipper, W, we remember that $W = U$ for $0 \le U \le 1$ while $W = 1$ for $1 \le U \le 2$. First, this implies W is nonnegative, i.e., $F_W(w) = 0$ *and* $f_W(w) = 0$ for $w < 0$. Furthermore, for $0 \le w \le 1$,

$$F_W(w) = P[W \le w] = P[U \le w] = F_U(w) = w/2 \tag{3}$$

Lastly, we observe that it is always true that $W \le 1$. This implies $F_W(w) = 1$ for $w \ge 1$. Therefore the CDF of W is

$$F_W(w) = \begin{cases} 0 & w < 0 \\ w/2 & 0 \le w < 1 \\ 1 & w \ge 1 \end{cases} \tag{4}$$

From the jump in the CDF at $w = 1$, we see that $P[W = 1] = 1/2$. The corresponding PDF can be found by taking the derivative and using the delta function to model the discontinuity.

$$f_W(w) = \begin{cases} 1/2 + (1/2)\delta(w-1) & 0 \le w \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

The expected value of W is

$$E[W] = \int_{-\infty}^{\infty} w f_W(w)\, dw = \int_0^1 w[1/2 + (1/2)\delta(w-1)]\, dw \tag{6}$$

$$= 1/4 + 1/2 = 3/4. \tag{7}$$

where $1/4$ is the integral of w over the interval $[0, 1]$ and $1/2$ is obtained using the sifting property of the impulse function:

$$\int_0^1 \frac{1}{2}\delta(w-1)dw = \frac{1}{2}. \tag{8}$$

(Recall that the sifting property holds because the impulse function makes the integrand zero except at the value $w = 1$.)

Problem 3.7.10 ■
X is a random variable with CDF $F_X(x)$. Let $Y = g(X)$ where

$$g(x) = \begin{cases} 10 & x < 0, \\ -10 & x \geq 0. \end{cases}$$

Express $F_Y(y)$ in terms of $F_X(x)$.

Problem 3.7.10 Solution
Given the following function of random variable X,

$$Y = g(X) = \begin{cases} 10 & X < 0 \\ -10 & X \geq 0 \end{cases} \tag{1}$$

we follow the same procedure as in Problem 3.7.4. We attempt to express the CDF of Y in terms of the CDF of X. We know that Y is *never* less than -10 *so for* $y < -10$, $F_Y(y) = 0$. We also know that $-10 \leq Y < 10$ when $X \geq 0$, and finally, that $Y = 10$ when $X < 0$. Therefore

$$F_Y(y) = P[Y \leq y] = \begin{cases} 0 & y < -10 \\ P[X \geq 0] = 1 - F_X(0) & -10 \leq y < 10 \\ 1 & y \geq 10 \end{cases} \tag{2}$$

Problem 3.7.13 ■
For a uniform $(0, 1)$ random variable U, find the CDF and PDF of $Y = a + (b - a)U$ with $a < b$. Show that Y is a uniform (a, b) random variable.

Problem 3.7.13 Solution
If X has a uniform distribution from 0 to 1 then the PDF and corresponding CDF of X are

$$f_X(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \qquad F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases} \tag{1}$$

We can determine the CDF interval by interval. First, let's consider the interval corresponding to $b - a > 0$*, i.e. when* y *is between* a *and* b. For $b - a > 0$, we can find the CDF of the function $Y = a + (b - a)X$

$$\begin{aligned} F_Y(y) = P[Y \leq y] &= P[a + (b - a)X \leq y] && (2) \\ &= P\left[X \leq \frac{y - a}{b - a}\right] && (3) \\ &= F_X\left(\frac{y - a}{b - a}\right) = \frac{y - a}{b - a} && (4) \end{aligned}$$

Now we note that since the value of the random variable Y never exceeds b, $F_Y(y) = 1$ for all $y > b$. Similarly, since the value of the random variable Y is never less than a, $F_Y(y) = 0$ for all $y < a$. Therefore the CDF of Y is

$$F_Y(y) = \begin{cases} 0 & y < a \\ \frac{y-a}{b-a} & a \leq y \leq b \\ 1 & y > b \end{cases} \tag{5}$$

By differentiating with respect to y we arrive at the PDF

$$f_Y(y) = \begin{cases} 1/(b-a) & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

which we recognize as the PDF of a uniform (a, b) random variable.

Problem 3.8.4 ■

W is a Gaussian random variable with expected value $\mu = 0$, and variance $\sigma^2 = 16$. Given the event $C = \{W > 0\}$,

(a) What is the conditional PDF, $f_{W|C}(w)$?

(b) Find the conditional expected value, $E[W|C]$.

(c) Find the conditional variance, $\mathrm{Var}[W|C]$.

Problem 3.8.4 Solution

From Definition 3.8, $W \sim \mathcal{N}(0, 4)$ *imlpies that* the PDF of W is

$$f_W(w) = \frac{1}{\sqrt{32\pi}} e^{-w^2/32} \tag{1}$$

(a) Since W has expected value $\mu = 0$, $f_W(w)$ is symmetric about $w = 0$. Hence $P[C] = P[W > 0] = 1/2$. From Definition 3.15, the conditional PDF of W given C is

$$f_{W|C}(w) = \begin{cases} f_W(w)/P[C] & w \in C \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 2e^{-w^2/32}/\sqrt{32\pi} & w > 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

(b) The conditional expected value of W given C is

$$E[W|C] = \int_{-\infty}^{\infty} w f_{W|C}(w)\, dw = \frac{2}{4\sqrt{2\pi}} \int_0^{\infty} we^{-w^2/32}\, dw \tag{3}$$

Making the substitution $v = w^2/32$, we obtain *($dv = 2w\,dw/32 = w\,dw/16$, so $w\,dw = 16dv$ and)*

$$E[W|C] = \frac{32}{\sqrt{32\pi}} \int_0^{\infty} e^{-v}\, dv = \frac{32}{\sqrt{32\pi}} \tag{4}$$

(c) The conditional second moment of W is

$$E\left[W^2|C\right] = \int_{-\infty}^{\infty} w^2 f_{W|C}(w)\, dw = \int_{0}^{\infty} w^2 2 f_W(w)\, dw \tag{5}$$

We observe that $w^2 f_W(w)$ is an even function. Hence

$$E\left[W^2|C\right] = 2\int_{0}^{\infty} w^2 f_W(w)\, dw \tag{6}$$

$$= \int_{-\infty}^{\infty} w^2 f_W(w)\, dw = E\left[W^2\right] = \sigma^2 = 16 \tag{7}$$

Lastly, the conditional variance of W given C is

$$\mathrm{Var}[W|C] = E\left[W^2|C\right] - (E[W|C])^2 = 16 - 32/\pi = 5.8141 \tag{8}$$

Problem 3.8.5 ■

The time between telephone calls at a telephone switch is an exponential random variable T with expected value 0.01. Given $T > 0.02$,

(a) What is $E[T|T > 0.02]$, the conditional expected value of T?

(b) What is $\mathrm{Var}[T|T > 0.02]$, the conditional variance of T?

Problem 3.8.5 Solution

(a) We first find the conditional PDF of T. The PDF of T is

$$f_T(t) = \begin{cases} 100e^{-100t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The conditioning event has probability

$$P[T > 0.02] = \int_{0.02}^{\infty} f_T(t)\, dt = -e^{-100t}\Big|_{0.02}^{\infty} = e^{-2} \tag{2}$$

From Definition 3.15, the conditional PDF of T is

$$f_{T|T>0.02}(t) = \begin{cases} \frac{f_T(t)}{P[T>0.02]} & t \geq 0.02 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 100e^{-100(t-0.02)} & t \geq 0.02 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

The conditional expected value of T is

$$E[T|T > 0.02] = \int_{0.02}^{\infty} t(100)e^{-100(t-0.02)}\, dt \tag{4}$$

The substitution $\tau = t - 0.02$ yields $d\tau = dt$ *and*

$$E[T|T > 0.02] = \int_0^\infty (\tau + 0.02)(100)e^{-100\tau}\, d\tau \tag{5}$$
$$= \int_0^\infty (\tau + 0.02) f_T(\tau)\, d\tau = E[T + 0.02] = 0.03 \tag{6}$$

where the second to last equality is a result of both linearity of integration and linearity of the expected value.

(b) The conditional second moment of T is

$$E\left[T^2|T > 0.02\right] = \int_{0.02}^\infty t^2(100)e^{-100(t-0.02)}\, dt \tag{7}$$

The substitution $\tau = t - 0.02$ yields

$$E\left[T^2|T > 0.02\right] = \int_0^\infty (\tau + 0.02)^2(100)e^{-100\tau}\, d\tau \tag{8}$$
$$= \int_0^\infty (\tau + 0.02)^2 f_T(\tau)\, d\tau \tag{9}$$
$$= E\left[(T + 0.02)^2\right] \tag{10}$$

Now we can calculate the conditional variance.

$$\mathrm{Var}[T|T > 0.02] = E\left[T^2|T > 0.02\right] - (E[T|T > 0.02])^2 \tag{11}$$
$$= E\left[(T + 0.02)^2\right] - (E[T + 0.02])^2 \tag{12}$$
$$= \mathrm{Var}[T + 0.02] \tag{13}$$
$$= \mathrm{Var}[T] = 0.01 \tag{14}$$

where the second to last equality can be justified by citing Theorem 3.5(d) or by direct calculation.

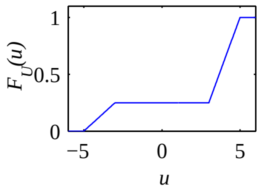
Problem 3.9.8 ■

Write a MATLAB function `u=urv(m)` that generates m samples of random variable U defined in Problem 3.3.7.

Problem 3.9.8 Solution

To solve this problem, we want to use Theorem 3.22. One complication is that in the theorem, U denotes the uniform random variable while X is the derived random variable. In this problem, we are using U for the random variable we want to derive. As a result, we will use Theorem 3.22 with the roles of X and U reversed. Given U with CDF $F_U(u) = F(u)$, we need to find the inverse functon $F^{-1}(x) = F_U^{-1}(x)$ so that for a uniform $(0, 1)$ random variable X, $U = F^{-1}(X)$.

Recall that random variable U defined in Problem 3.3.7 has CDF



$$F_U(u) = \begin{cases} 0 & u < -5 \\ (u+5)/8 & -5 \le u < -3 \\ 1/4 & -3 \le u < 3 \\ 1/4 + 3(u-3)/8 & 3 \le u < 5 \\ 1 & u \ge 5. \end{cases} \tag{1}$$

At $x = 1/4$, there are multiple values of u such that $F_U(u) = 1/4$. However, except for $x = 1/4$, the inverse $F_U^{-1}(x)$ is well defined over $0 < x < 1$. At $x = 1/4$, we can arbitrarily define a value for $F_U^{-1}(1/4)$ because when we produce sample values of $F_U^{-1}(X)$, the event $X = 1/4$ has probability zero. To generate the inverse CDF, given a value of x, $0 < x < 1$, we ave to find the value of u such that $x = F_U(u)$. From the CDF we see that

$$0 \le x \le \frac{1}{4} \qquad \Rightarrow x = \frac{u+5}{8} \tag{2}$$

$$\frac{1}{4} < x \le 1 \qquad \Rightarrow x = \frac{1}{4} + \frac{3}{8}(u-3) \tag{3}$$

$$\tag{4}$$

These conditions can be inverted to express u as a function of x.

$$u = F^{-1}(x) = \begin{cases} 8x-5 & 0 \le x \le 1/4 \\ (8x+7)/3 & 1/4 < x \le 1 \end{cases} \tag{5}$$

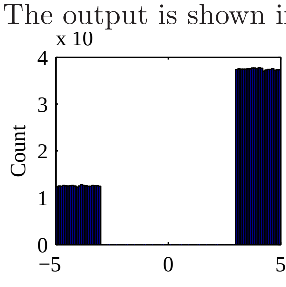
In particular, when X is a uniform $(0,1)$ random variable, $U = F^{-1}(X)$ will generate samples of the rndom variable U. A Matlab program to implement this solution is now straightforward:

```
function u=urv(m)
%Usage: u=urv(m)
%Generates m samples of the random
%variable U defined in Problem 3.3.7
x=rand(m,1);
u=(x<=1/4).*(8*x-5);
u=u+(x>1/4).*(8*x+7)/3;
```

To see that this generates the correct output, we can generate a histogram of a million sample values of U using the commands

```
u=urv(1000000); hist(u,100);
```

The output is shown in the following graph, alongside the corresponding PDF of U.



$$f_U(u) = \begin{cases} 0 & u < -5 \\ 1/8 & -5 \le u < -3 \\ 0 & -3 \le u < 3 \\ 3/8 & 3 \le u < 5 \\ 0 & u \ge 5. \end{cases} \tag{6}$$

Note that the scaling constant 10^4 on the histogram plot comes from the fact that the histogram was generated using 10^6 sample points and 100 bins. The width of each bin is $\Delta = 10/100 = 0.1$. Consider a bin of idth Δ centered at u_0. A sample value of U would fall in that bin with probability $f_U(u_0)\Delta$. Given that we generate $m = 10^6$ samples, we would expect about $mf_U(u_0)\Delta = 10^5 f_U(u_0)$ samples in each bin. For $-5 < u_0 < -3$, we would expect to see about 1.25×10^4 samples in each bin. For $3 < u_0 < 5$, we would expect to see about 3.75×10^4 samples in each bin. As can be seen, these conclusions are consistent with the histogam data.

Finally, we comment that if you generate histograms for a range of values of m, the number of samples, you will see that the histograms will become more and more similar to a scaled version of the PDF. This gives the (false) impression that any bin centered on u_0 has a number of samples increasingly close to $mf_U(u_0)\Delta$. Because the histpgram is always the same height, what is actually happening is that the vertical axis is effectively scaled by $1/m$ and the height of a histogram bar is proportional to *the fraction* of m samples that land in that bin. We will see in Chapter 7 that the fraction of samples in a bin does converge to the probability of a sample being in that bin as the number of samples m goes to infinity.

Solutions to HW7

Note: Most of these solutions were generated by R. D. Yates and D. J. Goodman, the authors of our textbook. I have added comments in italics where I thought more detail was appropriate. The solution to problem 5.8.3 is mine.

Problem 4.1.1 •

Random variables X and Y have the joint CDF

$$F_{X,Y}(x,y) = \begin{cases} (1-e^{-x})(1-e^{-y}) & x \geq 0; \\ & y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is $P[X \leq 2, Y \leq 3]$?

(b) What is the marginal CDF, $F_X(x)$?

(c) What is the marginal CDF, $F_Y(y)$?

Problem 4.1.1 Solution

(a) *Using Definition 4.1* The probability $P[X \leq 2, Y \leq 3]$ can be found be evaluating the joint CDF $F_{X,Y}(x,y)$ at $x = 2$ and $y = 3$. This yields

$$P[X \leq 2, Y \leq 3] = F_{X,Y}(2,3) = (1-e^{-2})(1-e^{-3}) \quad (1)$$

(b) *By Theorem 4.1* To find the marginal CDF of X, $F_X(x)$, we simply evaluate the joint CDF at $y = \infty$.

$$F_X(x) = F_{X,Y}(x,\infty) = \begin{cases} 1-e^{-x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

(c) Likewise *by Theorem 4.1* for the marginal CDF of Y, we evaluate the joint CDF at $X = \infty$.

$$F_Y(y) = F_{X,Y}(\infty,y) = \begin{cases} 1-e^{-y} & y \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

Problem 4.1.2 •

Express the following extreme values of $F_{X,Y}(x,y)$ in terms of the marginal cumulative distribution functions $F_X(x)$ and $F_Y(y)$.

(a) $F_{X,Y}(x,-\infty)$

(b) $F_{X,Y}(x,\infty)$

(c) $F_{X,Y}(-\infty,\infty)$

(d) $F_{X,Y}(-\infty, y)$

(e) $F_{X,Y}(\infty, y)$

Problem 4.1.2 Solution

(a) Because the probability that any random variable is less than $-\infty$ is zero, we have *(also by Theorem 4.1d)*

$$F_{X,Y}(x, -\infty) = P[X \leq x, Y \leq -\infty] \leq P[Y \leq -\infty] = 0 \tag{1}$$

(b) The probability that any random variable is less than infinity is always one. *See also Theorem 4.1.b.*

$$F_{X,Y}(x, \infty) = P[X \leq x, Y \leq \infty] = P[X \leq x] = F_X(x) \tag{2}$$

(c) Although $P[Y \leq \infty] = 1$, $P[X \leq -\infty] = 0$. Therefore the following is true. *(Theorem 4.1b and the definition of the CDF)*

$$F_{X,Y}(-\infty, \infty) = P[X \leq -\infty, Y \leq \infty] \leq P[X \leq -\infty] = 0 \tag{3}$$

(d) Part (d) follows the same logic as that of part (a). *Theorem 4.1d*

$$F_{X,Y}(-\infty, y) = P[X \leq -\infty, Y \leq y] \leq P[X \leq -\infty] = 0 \tag{4}$$

(e) Analogous to Part (b), we find that *(See also Theorem 4.1c.)*

$$F_{X,Y}(\infty, y) = P[X \leq \infty, Y \leq y] = P[Y \leq y] = F_Y(y) \tag{5}$$

Problem 4.2.1 •

Random variables X and Y have the joint PMF

$$P_{X,Y}(x, y) = \begin{cases} cxy & x = 1, 2, 4; \quad y = 1, 3, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the value of the constant c?

(b) What is $P[Y < X]$?

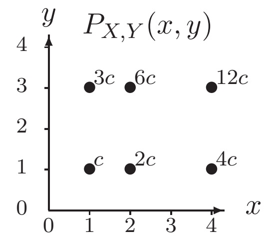
(c) What is $P[Y > X]$?

(d) What is $P[Y = X]$?

(e) What is $P[Y = 3]$?

Problem 4.2.1 Solution

In this problem, it is helpful to label points with nonzero probability on the X, Y plane:



(a) We must choose c so the PMF sums to one:

$$\sum_{x=1,2,4} \sum_{y=1,3} P_{X,Y}(x,y) = c \sum_{x=1,2,4} x \sum_{y=1,3} y \tag{1}$$

$$= c\,[1(1+3) + 2(1+3) + 4(1+3)] = 28c \tag{2}$$

Thus $c = 1/28$.

(b) The event $\{Y < X\}$ has probability

$$P[Y < X] = \sum_{x=1,2,4} \sum_{y<x} P_{X,Y}(x,y) = \frac{1(0) + 2(1) + 4(1+3)}{28} = \frac{18}{28} \tag{3}$$

(c) The event $\{Y > X\}$ has probability

$$P[Y > X] = \sum_{x=1,2,4} \sum_{y>x} P_{X,Y}(x,y) = \frac{1(3) + 2(3) + 4(0)}{28} = \frac{9}{28} \tag{4}$$

(d) There are two ways to solve this part. The direct way is to calculate

$$P[Y = X] = \sum_{x=1,2,4} \sum_{y=x} P_{X,Y}(x,y) = \frac{1(1) + 2(0)}{28} = \frac{1}{28} \tag{5}$$

The indirect way is to use the previous results and the observation that

$$P[Y = X] = 1 - P[Y < X] - P[Y > X] = (1 - 18/28 - 9/28) = 1/28 \tag{6}$$

(e)

$$P[Y = 3] = \sum_{x=1,2,4} P_{X,Y}(x,3) = \frac{(1)(3) + (2)(3) + (4)(3)}{28} = \frac{21}{28} = \frac{3}{4} \tag{7}$$

Problem 4.4.1 •
Random variables X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} c & x+y \leq 1, x \geq 0, y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

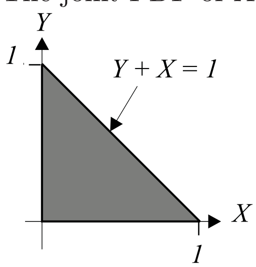
(a) What is the value of the constant c?

(b) What is $P[X \leq Y]$?

(c) What is $P[X + Y \leq 1/2]$?

Problem 4.4.1 Solution

(a) The joint PDF of X and Y is



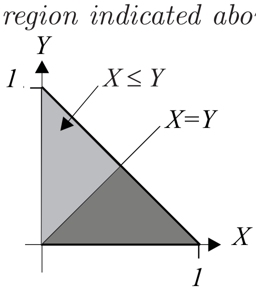
$$f_{X,Y}(x,y) = \begin{cases} c & x+y \leq 1, x,y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

To find the constant c we integrate over the region shown. *Note that we are integrating a constant over a region of area 1/2, so we should expect the result of the integration to be $c/2$.* This gives

$$\int_0^1 \int_0^{1-x} c\, dy\, dx = cx - \frac{cx}{2}\Big|_0^1 = \frac{c}{2} = 1 \tag{2}$$

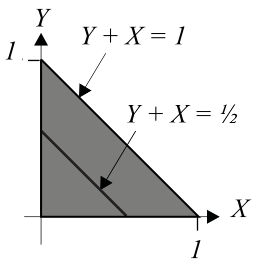
Therefore $c = 2$.

(b) To find the $P[X \leq Y]$ we look to integrate over the area indicated by the graph *This time we're determining the probability by integrating the constant PDF over 1/2 the region indicated above, so we should expect the probability to be 1/2.*



$$\begin{aligned} P[X \leq Y] &= \int_0^{1/2} \int_x^{1-x} dy\, dx & (3) \\ &= \int_0^{1/2} (2 - 4x)\, dx & (4) \\ &= 1/2 & (5) \end{aligned}$$

(c) The probability $P[X + Y \leq 1/2]$ can be seen in the figure. *Here we integrate the constant PDF over 1/4 of the original region so we should expect the probability to be 1/4.* Here we can set up the following integrals



$$P[X+Y \leq 1/2] = \int_0^{1/2} \int_0^{1/2-x} 2\,dy\,dx \tag{6}$$
$$= \int_0^{1/2} (1-2x)\,dx \tag{7}$$
$$= 1/2 - 1/4 = 1/4 \tag{8}$$

Problem 5.1.1 •

Every laptop returned to a repair center is classified according its needed repairs: (1) LCD screen, (2) motherboard, (3) keyboard, or (4) other. A random broken laptop needs a type i repair with probability $p_i = 2^{4-i}/15$. Let N_i equal the number of type i broken laptops returned on a day in which four laptops are returned.

(a) Find the joint PMF

$$P_{N_1,N_2,N_3,N_4}(n_1, n_2, n_3, n_4)$$

(b) What is the probability that two laptops require LCD repairs?

(c) What is the probability that more laptops require motherboard repairs than keyboard repairs?

Problem 5.1.1 Solution

The repair of each laptop can be viewed as an independent trial with four possible outcomes corresponding to the four types of needed repairs. *My first question here was whether a laptop could need two repairs. I resolved this by summing the probabilities assigned to each of the individual pairs and determining that the sum was one, so there were no laptops requiring two repairs. The problem would be much more complicated if this were not the case.*

(a) Since the four types of repairs are mutually exclusive choices and since 4 laptops are returned for repair, the joint distribution of $N_1, \ldots, N_4$ is the multinomial PMF

$$P_{N_1,\ldots,N_4}(n_1, \ldots, n_4) = \binom{4}{n_1, n_2, n_3, n_4} p_1^{n_1} p_2^{n_2} p_3^{n_3} p_4^{n_4} \tag{1}$$
$$= \begin{cases} \frac{4!}{n_1!n_2!n_3!n_4!} \left(\frac{8}{15}\right)^{n_1} \left(\frac{4}{15}\right)^{n_2} \left(\frac{2}{15}\right)^{n_3} \left(\frac{1}{15}\right)^{n_4} & n_1 + \cdots + n_4 = 4; n_i \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

(b) Let L_2 denote the event that exactly two laptops need LCD repairs. Thus $P[L_2] = P_{N_1}(2)$. Since each laptop requires an LCD repair with probability $p_1 = 8/15$, the number of LCD repairs, N_1, is a binomial $(4, 8/15)$ random variable with PMF

$$P_{N_1}(n_1) = \binom{4}{n_1} (8/15)^{n_1} (7/15)^{4-n_1} = \binom{4}{n_1} (8/7)^{n_1} (7/15)^4 \tag{3}$$

The probability that two laptops need LCD repairs is

$$P_{N_1}(2) = \binom{4}{2}(8/15)^2(7/15)^2 = 0.3717 \tag{4}$$

(c) A repair is type (2) with probability $p_2 = 4/15$. A repair is type (3) with probability $p_3 = 2/15$; otherwise a repair is type "other" *[Note that this is not the same "other" as the fourth type of repairs in the problem statement. This "other" includes both types 1 and types 4.]* with probability $p_o = 9/15$. Define X as the number of "other" repairs needed. The joint PMF of X, N_2, N_3 is the multinomial PMF

$$P_{N_2,N_3,X}(n_2,n_3,x) = \binom{4}{n_2,n_3,x}\left(\frac{4}{15}\right)^{n_2}\left(\frac{2}{15}\right)^{n_3}\left(\frac{9}{15}\right)^{x} \tag{5}$$

However, Since $X + 4 - N_2 - N_3$, we observe that

$$P_{N_2,N_3}(n_2,n_3) = P_{N_2,N_3,X}(n_2,n_3,4-n_2-n_3) \tag{6}$$

$$= \binom{4}{n_2,n_3,4-n_2-n_3}\left(\frac{4}{15}\right)^{n_2}\left(\frac{2}{15}\right)^{n_3}\left(\frac{9}{15}\right)^{4-n_2-n_3} \tag{7}$$

$$= \left(\frac{9}{15}\right)^{4}\binom{4}{n_2,n_3,4-n_2-n_3}\left(\frac{4}{9}\right)^{n_2}\left(\frac{2}{9}\right)^{n_3} \tag{8}$$

Similarly, since each repair is a motherboard repair with probability $p_2 = 4/15$, the number of motherboard repairs has binomial PMF

$$P_{N_2}(n_2) = \binom{4}{n_2}\left(\frac{4}{15}\right)^{n_2}\left(\frac{11}{15}\right)^{4-n_2} = \binom{4}{n_2}\left(\frac{4}{11}\right)^{n_2}\left(\frac{11}{15}\right)^{4} \tag{9}$$

Finally, the probability that more laptops require motherboard repairs than keyboard repairs is

$$P[N_2 > N_3] = P_{N_2,N_3}(1,0) + P_{N_2,N_3}(2,0) + P_{N_2,N_3}(2,1) + P_{N_2}(3) + P_{N_2}(4) \tag{10}$$

where we use the fact that if $N_2 = 3$ or $N_2 = 4$, then we must have $N_2 > N_3$. Inserting the various probabilities, we obtain

$$P[N_2 > N_3] = P_{N_2,N_3}(1,0) + P_{N_2,N_3}(2,0) + P_{N_2,N_3}(2,1) + P_{N_2}(3) + P_{N_2}(4) \tag{11}$$

Plugging in the various probabilities yields $P[N_2 > N_3] = 8{,}656/16{,}875 \approx 0.5129$.

Problem 4.3.1 •

Given the random variables X and Y in Problem 4.2.1, find

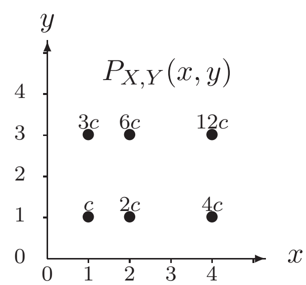
(a) The marginal PMFs $P_X(x)$ and $P_Y(y)$,

(b) The expected values $E[X]$ and $E[Y]$,

(c) The standard deviations σ_X and σ_Y.

Problem 4.3.1 Solution

On the X, Y plane, the joint PMF $P_{X,Y}(x, y)$ is



By choosing $c = 1/28$, the PMF sums to one.

(a) The marginal PMFs of X and Y are

$$P_X(x) = \sum_{y=1,3} P_{X,Y}(x,y) = \begin{cases} 4/28 & x = 1 \\ 8/28 & x = 2 \\ 16/28 & x = 4 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$P_Y(y) = \sum_{x=1,2,4} P_{X,Y}(x,y) = \begin{cases} 7/28 & y = 1 \\ 21/28 & y = 3 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

(b) The expected values of X and Y are

$$E[X] = \sum_{x=1,2,4} x P_X(x) = (4/28) + 2(8/28) + 4(16/28) = 3 \tag{3}$$

$$E[Y] = \sum_{y=1,3} y P_Y(y) = 7/28 + 3(21/28) = 5/2 \tag{4}$$

(c) The second moments are

$$E\left[X^2\right] = \sum_{x=1,2,4} x P_X(x) = 1^2(4/28) + 2^2(8/28) + 4^2(16/28) = 73/7 \tag{5}$$

$$E\left[Y^2\right] = \sum_{y=1,3} y P_Y(y) = 1^2(7/28) + 3^2(21/28) = 7 \tag{6}$$

The variances are

$$\mathrm{Var}[X] = E\left[X^2\right] - (E[X])^2 = 10/7 \qquad \mathrm{Var}[Y] = E\left[Y^2\right] - (E[Y])^2 = 3/4 \tag{7}$$

The standard deviations are $\sigma_X = \sqrt{10/7}$ and $\sigma_Y = \sqrt{3/4}$.

Problem 4.3.3 •
For $n = 0, 1, \ldots$ and $0 \le k \le 100$, the joint PMF of random variables N and K is

$$P_{N,K}(n,k) = \frac{100^n e^{-100}}{n!} \binom{100}{k} p^k (1-p)^{100-k}.$$

Otherwise, $P_{N,K}(n,k) = 0$. Find the marginal PMFs $P_N(n)$ and $P_K(k)$.

Problem 4.3.3 Solution
We recognize that the given joint PMF is written as the product of two marginal PMFs $P_N(n)$ and $P_K(k)$ where

$$P_N(n) = \sum_{k=0}^{100} P_{N,K}(n,k) = \begin{cases} \frac{100^n e^{-100}}{n!} & n = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases} \qquad (1)$$

$$P_K(k) = \sum_{n=0}^{\infty} P_{N,K}(n,k) = \begin{cases} \binom{100}{k} p^k (1-p)^{100-k} & k = 0, 1, \ldots, 100 \\ 0 & \text{otherwise} \end{cases} \qquad (2)$$

Problem 4.5.1 •
Random variables X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 1/2 & -1 \le x \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Sketch the region of nonzero probability.

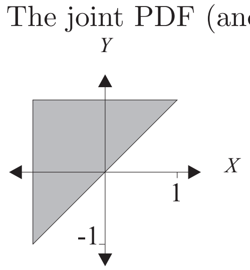
(b) What is $P[X > 0]$?

(c) What is $f_X(x)$?

(d) What is $E[X]$?

Problem 4.5.1 Solution

(a) The joint PDF (and the corresponding region of nonzero probability) are



$$f_{X,Y}(x,y) = \begin{cases} 1/2 & -1 \le x \le y \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad (1)$$

(b)

$$P[X > 0] = \int_0^1 \int_x^1 \frac{1}{2}\,dy\,dx = \int_0^1 \frac{1-x}{2}\,dx = 1/4 \tag{2}$$

This result can be deduced by geometry. The shaded triangle of the X, Y plane corresponding to the event $X > 0$ is 1/4 of the total shaded area.

(c) For $x > 1$ or $x < -1$, $f_X(x) = 0$. For $-1 \le x \le 1$,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\; dy = \int_x^1 \frac{1}{2}\,dy = (1-x)/2. \tag{3}$$

The complete expression for the marginal PDF is

$$f_X(x) = \begin{cases} (1-x)/2 & -1 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

(d) From the marginal PDF $f_X(x)$, the expected value of X is

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\; dx = \frac{1}{2}\int_{-1}^{1} x(1-x)\,dx \tag{5}$$

$$= \frac{x^2}{4} - \frac{x^3}{6}\bigg|_{-1}^{1} = -\frac{1}{3}. \tag{6}$$

Problem 5.3.2 ■

A wireless data terminal has three messages waiting for transmission. After sending a message, it expects an acknowledgement from the receiver. When it receives the acknowledgement, it transmits the next message. If the acknowledgement does not arrive, it sends the message again. The probability of successful transmission of a message is p independent of other transmissions. Let $\mathbf{K} = \begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix}'$ be the 3-dimensional random vector in which K_i is the total number of transmissions when message i is received successfully. (K_3 is the total number of transmissions used to send all three messages.) Show that

$$P_{\mathbf{K}}(\mathbf{k}) = \begin{cases} p^3(1-p)^{k_3-3} & k_1 < k_2 < k_3; \\ & k_i \in \{1, 2 \ldots\}, \\ 0 & \text{otherwise.} \end{cases}$$

Problem 5.3.2 Solution

Since J_1, J_2 and J_3 are independent, we can write

$$P_{\mathbf{K}}(\mathbf{k}) = P_{J_1}(k_1)\, P_{J_2}(k_2 - k_1)\, P_{J_3}(k_3 - k_2) \tag{1}$$

Since $P_{J_i}(j) > 0$ only for integers $j > 0$, we have that $P_{\mathbf{K}}(\mathbf{k}) > 0$ only for $0 < k_1 < k_2 < k_3$; otherwise $P_{\mathbf{K}}(\mathbf{k}) = 0$. Finally, for $0 < k_1 < k_2 < k_3$,

$$P_{\mathbf{K}}(\mathbf{k}) = (1-p)^{k_1-1}p(1-p)^{k_2-k_1-1}p(1-p)^{k_3-k_2-1}p \tag{2}$$

$$= (1-p)^{k_3-3}p^3 \tag{3}$$

I solved this differently. I first defined new random variables N_1, N_2, and N_3 to be the number of transmissions needed for successful receipt of message i, $i \in \{1,2,3\}$. Then for each N_i,

$$P_{N_i}(n_i) = (1-p)^{n_i}\, p \tag{4}$$

where we have no $\binom{x}{y}$ because we know which of the transmissions is the successful one. It is the last. Now the successes or failures of the transmissions being independent, we have for n_1, n_2, and n_3 positive integers,

$$P_{N_1,N_2,N_3}(n_1,n_2,n_3) = (1-p)^{n_1-1}\, p\, (1-p)^{n_2-1}\, p\, (1-p)^{n_3-1}\, p = (1-p)^{n_1+n_2+n_3-3}\, p^3 \tag{5}$$

so with $K_1 = N_1$, $K_2 = N_1 + N_2$, and $K_3 = N_1 + N_2 + N_3$, we have

$$P_{\mathbf{K}}(\mathbf{k}) = \begin{cases} (1-p)^{k_3-3}\, p^3 & k_1 < k_2 < k_3, \quad \text{integers} \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Problem 4.10.2 •

X and Y are independent, identically distributed random variables with PMF

$$P_X(k) = P_Y(k) = \begin{cases} 3/4 & k = 0, \\ 1/4 & k = 20, \\ 0 & \text{otherwise.} \end{cases}$$

Find the following quantities:

$$\begin{array}{lll} E[X], & \text{Var}[X], & \\ E[X+Y], & \text{Var}[X+Y], & E\left[XY2^{XY}\right]. \end{array}$$

Problem 4.10.2 Solution

Using the following probability model

$$P_X(k) = P_Y(k) = \begin{cases} 3/4 & k = 0 \\ 1/4 & k = 20 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

We can calculate the requested moments.

$$E[X] = 3/4 \cdot 0 + 1/4 \cdot 20 = 5 \tag{2}$$

$$\text{Var}[X] = 3/4 \cdot (0-5)^2 + 1/4 \cdot (20-5)^2 = 75 \tag{3}$$

$$E[X+Y] = E[X] + E[X] = 2E[X] = 10 \tag{4}$$

Since X and Y are independent, Theorem 4.27 yields

$$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y] = 2\,\text{Var}[X] = 150 \tag{5}$$

Since X and Y are independent, $P_{X,Y}(x,y) = P_X(x)P_Y(y)$ and

$$E\left[XY2^{XY}\right] = \sum_{x=0,20}\sum_{y=0,20} XY2^{XY} P_{X,Y}(x,y) = (20)(20)2^{20(20)} P_X(20)\, P_Y(20) \tag{6}$$

$$= 6.46 \times 10^{121} \tag{7}$$

Problem 4.10.7 •
X and Y are independent random variables with PDFs

$$f_X(x) = \begin{cases} \frac{1}{3}e^{-x/3} & x \geq 0, \\ 0 & \text{otherwise}, \end{cases}$$
$$f_Y(y) = \begin{cases} \frac{1}{2}e^{-y/2} & y \geq 0, \\ 0 & \text{otherwise}. \end{cases}$$

(a) What is $P[X > Y]$?

(b) What is $E[XY]$?

(c) What is $\text{Cov}[X, Y]$?

Problem 4.10.7 Solution
X and Y are independent random variables with PDFs

$$f_X(x) = \begin{cases} \frac{1}{3}e^{-x/3} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} \frac{1}{2}e^{-y/2} & y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) To calculate $P[X > Y]$, we use the joint PDF $f_{X,Y}(x, y) = f_X(x)f_Y(y)$.

$$P[X > Y] = \iint_{x>y} f_X(x) f_Y(y)\, dx\, dy \tag{2}$$
$$= \int_0^\infty \frac{1}{2}e^{-y/2} \int_y^\infty \frac{1}{3}e^{-x/3}\, dx\, dy \tag{3}$$
$$= \int_0^\infty \frac{1}{2}e^{-y/2}e^{-y/3}\, dy \tag{4}$$
$$= \int_0^\infty \frac{1}{2}e^{-(1/2+1/3)y}\, dy = \frac{1/2}{1/2 + 2/3} = \frac{3}{7} \tag{5}$$

(b) Since X and Y are exponential random variables with parameters $\lambda_X = 1/3$ and $\lambda_Y = 1/2$, Appendix A tells us that $E[X] = 1/\lambda_X = 3$ and $E[Y] = 1/\lambda_Y = 2$. Since X and Y are independent, the correlation is $E[XY] = E[X]E[Y] = 6$.

(c) Since X and Y are independent, $\text{Cov}[X, Y] = 0$.

Problem 5.4.2 •
In Problem 5.1.1, are N_1, N_2, N_3, N_4 independent?

Problem 5.4.2 Solution
The random variables N_1, N_2, N_3 and N_4 are *dependent*. To see this we observe that $P_{N_i}(4) = p_i^4$. However,

$$P_{N_1,N_2,N_3,N_4}(4, 4, 4, 4) = 0 \neq p_1^4p_2^4p_3^4p_4^4 = P_{N_1}(4)\, P_{N_2}(4)\, P_{N_3}(4)\, P_{N_4}(4). \tag{1}$$

Note that this can also be determined by calculating the marginals and checking to see whether the product of the marginals equals the original function (more work, but just as effective).

Problem 5.4.5 •
The PDF of the 3-dimensional random vector $\mathbf{X}$ is

$$f_{\mathbf{X}}(\mathbf{x}) = \begin{cases} e^{-x_3} & 0 \le x_1 \le x_2 \le x_3, \\ 0 & \text{otherwise.} \end{cases}$$

Are the components of $\mathbf{X}$ independent random variables?

Problem 5.4.5 Solution
This problem can be solved without any real math. Some thought should convince you that for any $x_i > 0$, $f_{X_i}(x_i) > 0$. Thus, $f_{X_1}(10) > 0$, $f_{X_2}(9) > 0$, and $f_{X_3}(8) > 0$. Thus $f_{X_1}(10)f_{X_2}(9)f_{X_3}(8) > 0$. However, from the definition of the joint PDF

$$f_{X_1,X_2,X_3}(10, 9, 8) = 0 \neq f_{X_1}(10) f_{X_2}(9) f_{X_3}(8). \tag{1}$$

It follows that X_1, X_2 and X_3 are dependent. Readers who find this quick answer dissatisfying are invited to confirm this conclusions by solving Problem 5.4.6 for the exact expressions for the marginal PDFs $f_{X_1}(x_1)$, $f_{X_2}(x_2)$, and $f_{X_3}(x_3)$. *Solution to 5.4.6. is included below.*

Problem 5.4.6 ■
The random vector $\mathbf{X}$ has PDF

$$f_{\mathbf{X}}(\mathbf{x}) = \begin{cases} e^{-x_3} & 0 \le x_1 \le x_2 \le x_3, \\ 0 & \text{otherwise.} \end{cases}$$

Find the marginal PDFs $f_{X_1}(x_1)$, $f_{X_2}(x_2)$, and $f_{X_3}(x_3)$.

Problem 5.4.6 Solution

This problem was not assigned but contains the straightforward solution to problem 5.4.5, hence is included here in the solutions.

We find the marginal PDFs using Theorem 5.5. First we note that for $x < 0$, $f_{X_i}(x) = 0$. For $x_1 \ge 0$,

$$f_{X_1}(x_1) = \int_{x_1}^{\infty} \left(\int_{x_2}^{\infty} e^{-x_3}\, dx_3 \right) dx_2 = \int_{x_1}^{\infty} e^{-x_2}\, dx_2 = e^{-x_1} \tag{1}$$

Similarly, for $x_2 \ge 0$, X_2 has marginal PDF

$$f_{X_2}(x_2) = \int_{0}^{x_2} \left(\int_{x_2}^{\infty} e^{-x_3}\, dx_3 \right) dx_1 = \int_{0}^{x_2} e^{-x_2}\, dx_1 = x_2 e^{-x_2} \tag{2}$$

Lastly,

$$f_{X_3}(x_3) = \int_{0}^{x_3} \left(\int_{x_1}^{x_3} e^{-x_3}\, dx_2 \right) dx_1 = \int_{0}^{x_3} (x_3 - x_1) e^{-x_3}\, dx_1 \tag{3}$$

$$= -\frac{1}{2}(x_3 - x_1)^2 e^{-x_3} \Big|_{x_1=0}^{x_1=x_3} = \frac{1}{2} x_3^2 e^{-x_3} \tag{4}$$

The complete expressions for the three marginal PDFs are

$$f_{X_1}(x_1) = \begin{cases} e^{-x_1} & x_1 \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

$$f_{X_2}(x_2) = \begin{cases} x_2 e^{-x_2} & x_2 \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

$$f_{X_3}(x_3) = \begin{cases} (1/2)x_3^2 e^{-x_3} & x_3 \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

In fact, each X_i is an Erlang $(n, \lambda) = (i, 1)$ random variable.

Problem 4.6.1 •

Given random variables X and Y in Problem 4.2.1 and the function $W = X - Y$, find

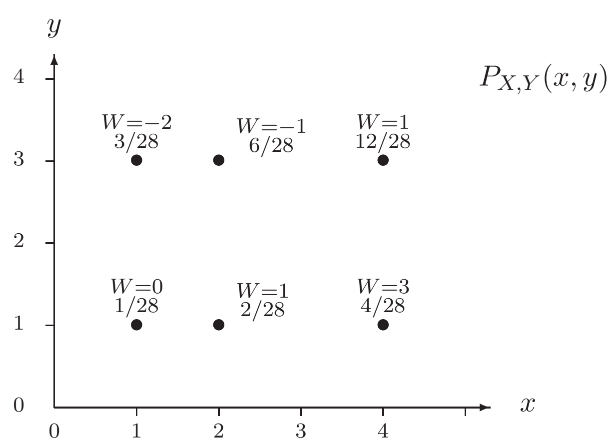
(a) The probability mass function $P_W(w)$,

(b) The expected value $E[W]$,

(c) $P[W > 0]$.

Problem 4.6.1 Solution

In this problem, it is helpful to label possible points X, Y along with the corresponding values of $W = X - Y$. From the statement of Problem 4.6.1,



(a) To find the PMF of W, we simply add the probabilities associated with each possible value of W:

$$P_W(-2) = P_{X,Y}(1,3) = 3/28 \qquad P_W(-1) = P_{X,Y}(2,3) = 6/28 \tag{1}$$

$$P_W(0) = P_{X,Y}(1,1) = 1/28 \qquad P_W(1) = P_{X,Y}(2,1) + P_{X,Y}(4,3) \tag{2}$$

$$P_W(3) = P_{X,Y}(4,1) = 4/28 \qquad = 14/28 \tag{3}$$

For all other values of w, $P_W(w) = 0$.

(b) The expected value of W is

$$E[W] = \sum_w wP_W(w) \tag{4}$$
$$= -2(3/28) + -1(6/28) + 0(1/28) + 1(14/28) + 3(4/28) = 1/2 \tag{5}$$

(c) $P[W > 0] = P_W(1) + P_W(3) = 18/28$.

Problem 5.5.2 ■

In the message transmission problem, Problem 5.3.2, the PMF for the number of transmissions when message i is received successfully is

$$P_{\mathbf{K}}(\mathbf{k}) = \begin{cases} p^3(1-p)^{k_3-3} & k_1 < k_2 < k_3; \\ & k_i \in \{1, 2 \ldots\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $J_3 = K_3 - K_2$, the number of transmissions of message 3; $J_2 = K_2 - K_1$, the number of transmissions of message 2; and $J_1 = K_1$, the number of transmissions of message 1. Derive a formula for $P_{\mathbf{J}}(\mathbf{j})$, the PMF of the number of transmissions of individual messages.

Problem 5.5.2 Solution

The random variable J_n is the number of times that message n is transmitted. Since each transmission is a success with probability p, independent of any other transmission, the number of transmissions of message n is independent of the number of transmissions of message m. That is, for $m \neq n$, J_m and J_n are independent random variables. Moreover, because each message is transmitted over and over until it is transmitted succesfully, each J_m is a geometric (p) random variable with PMF

$$P_{J_m}(j) = \begin{cases} (1-p)^{j-1}p & j = 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Thus the PMF of $\mathbf{J} = \begin{bmatrix} J_1 & J_2 & J_3 \end{bmatrix}'$ is

$$P_{\mathbf{J}}(\mathbf{j}) = P_{J_1}(j_1) P_{J_2}(j_2) P_{J_3}(j_3) = \begin{cases} p^3(1-p)^{j_1+j_2+j_3-3} & j_i = 1, 2, \ldots; \\ & i = 1, 2, 3 \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Problem 4.7.1 •

For the random variables X and Y in Problem 4.2.1, find

(a) The expected value of $W = Y/X$,

(b) The correlation, $E[XY]$,

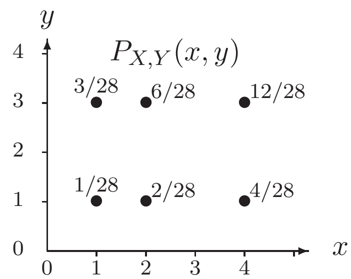
(c) The covariance, $\text{Cov}[X, Y]$,

(d) The correlation coefficient, $\rho_{X,Y}$,

(e) The variance of $X + Y$, $\text{Var}[X + Y]$.

(Refer to the results of Problem 4.3.1 to answer some of these questions.)

Problem 4.7.1 Solution



In Problem 4.2.1, we found the joint PMF $P_{X,Y}(x, y)$ as shown. Also the expected values and variances were

$$E[X] = 3 \qquad \text{Var}[X] = 10/7 \tag{1}$$
$$E[Y] = 5/2 \qquad \text{Var}[Y] = 3/4 \tag{2}$$

We use these results now to solve this problem.

(a) Random variable $W = Y/X$ has expected value

$$E[Y/X] = \sum_{x=1,2,4} \sum_{y=1,3} \frac{y}{x} P_{X,Y}(x, y) \tag{3}$$
$$= \frac{1}{1}\frac{1}{28} + \frac{3}{1}\frac{3}{28} + \frac{1}{2}\frac{2}{28} + \frac{3}{2}\frac{6}{28} + \frac{1}{4}\frac{4}{28} + \frac{3}{4}\frac{12}{28} = 15/14 \tag{4}$$

(b) The correlation of X and Y is

$$r_{X,Y} = \sum_{x=1,2,4} \sum_{y=1,3} xy P_{X,Y}(x, y) \tag{5}$$
$$= \frac{1 \cdot 1 \cdot 1}{28} + \frac{1 \cdot 3 \cdot 3}{28} + \frac{2 \cdot 1 \cdot 2}{28} + \frac{2 \cdot 3 \cdot 6}{28} + \frac{4 \cdot 1 \cdot 4}{28} + \frac{4 \cdot 3 \cdot 12}{28} \tag{6}$$
$$= 210/28 = 105/14 \tag{7}$$

Recognizing that $P_{X,Y}(x, y) = xy/28$ yields the faster calculation

$$r_{X,Y} = E[XY] = \sum_{x=1,2,4} \sum_{y=1,3} \frac{(xy)^2}{28} \tag{8}$$
$$= \frac{1}{28} \sum_{x=1,2,4} x^2 \sum_{y=1,3} y^2 \tag{9}$$
$$= \frac{1}{28}(1 + 2^2 + 4^2)(1^2 + 3^2) = 210/28 = 105/14 \tag{10}$$

(c) The covariance of X and Y is

$$\text{Cov}[X, Y] = E[XY] - E[X] E[Y] = \frac{15}{2} - 3\frac{5}{2} = 0 \tag{11}$$

(d) Since X and Y have zero covariance, the correlation coefficient is

$$\rho_{X,Y} = \frac{\text{Cov}[X,Y]}{\sqrt{\text{Var}[X]\,\text{Var}[Y]}} = 0. \tag{12}$$

(e) Since X and Y are uncorrelated, the variance of $X+Y$ is

$$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y] = \frac{61}{28}. \tag{13}$$

Problem 5.6.1 •

Random variables X_1 and X_2 have zero expected value and variances $\text{Var}[X_1] = 4$ and $\text{Var}[X_2] = 9$. Their covariance is $\text{Cov}[X_1, X_2] = 3$.

(a) Find the covariance matrix of $\mathbf{X} = \begin{bmatrix} X_1 & X_2 \end{bmatrix}'$.

(b) X_1 and X_2 are transformed to new variables Y_1 and Y_2 according to

$$\begin{aligned} Y_1 &= X_1 - 2X_2 \\ Y_2 &= 3X_1 + 4X_2 \end{aligned}$$

Find the covariance matrix of $\mathbf{Y} = \begin{bmatrix} Y_1 & Y_2 \end{bmatrix}'$.

Problem 5.6.1 Solution

(a) The coavariance matrix of $\mathbf{X} = \begin{bmatrix} X_1 & X_2 \end{bmatrix}'$ is

$$\mathbf{C_X} = \begin{bmatrix} \text{Var}[X_1] & \text{Cov}[X_1, X_2] \\ \text{Cov}[X_1, X_2] & \text{Var}[X_2] \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix}. \tag{1}$$

(b) From the problem statement,

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \mathbf{X} = \mathbf{AX}. \tag{2}$$

By Theorem 5.13, $\mathbf{Y}$ has covariance matrix

$$\mathbf{C_Y} = \mathbf{AC_XA}' = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 28 & -66 \\ -66 & 252 \end{bmatrix}. \tag{3}$$

Problem 4.11.1 •

Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = ce^{-(x^2/8)-(y^2/18)}.$$

What is the constant c? Are X and Y independent?

Problem 4.11.1 Solution

$$f_{X,Y}(x,y) = ce^{-(x^2/8)-(y^2/18)} \tag{1}$$

The omission of any limits for the PDF indicates that it is defined over all x and y. We know that $f_{X,Y}(x,y)$ is in the form of the bivariate Gaussian distribution so we look to Definition 4.17 and attempt to find values for σ_Y, σ_X, $E[X]$, $E[Y]$ and ρ. First, we know that the constant is

$$c = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \tag{2}$$

Because the exponent of $f_{X,Y}(x,y)$ doesn't contain any cross terms we know that ρ must be zero, and we are left to solve the following for $E[X]$, $E[Y]$, σ_X, and σ_Y:

$$\left(\frac{x-E[X]}{\sigma_X}\right)^2 = \frac{x^2}{8} \qquad \left(\frac{y-E[Y]}{\sigma_Y}\right)^2 = \frac{y^2}{18} \tag{3}$$

From which we can conclude that

$$E[X] = E[Y] = 0 \tag{4}$$

$$\sigma_X = \sqrt{8} \tag{5}$$

$$\sigma_Y = \sqrt{18} \tag{6}$$

Putting all the pieces together, we find that $c = \frac{1}{24\pi}$. Since $\rho = 0$, we also find that X and Y are independent.

Problem 5.7.1 •

$\mathbf{X}$ is the 3-dimensional Gaussian random vector with expected value $\boldsymbol{\mu}_\mathbf{X} = \begin{bmatrix}4 & 8 & 6\end{bmatrix}'$ and covariance

$$\mathbf{C_X} = \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix}.$$

Calculate

(a) the correlation matrix, $\mathbf{R_X}$,

(b) the PDF of the first two components of $\mathbf{X}$, $f_{X_1,X_2}(x_1,x_2)$,

(c) the probability that $X_1 > 8$.

Problem 5.7.1 Solution

(a) From Theorem 5.12, the correlation matrix of $\mathbf{X}$ is

$$\mathbf{R}_X = \mathbf{C}_X + \boldsymbol{\mu}_X \boldsymbol{\mu}_X' \tag{1}$$

$$= \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix} + \begin{bmatrix} 4 \\ 8 \\ 6 \end{bmatrix} \begin{bmatrix} 4 & 8 & 6 \end{bmatrix} \tag{2}$$

$$= \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix} + \begin{bmatrix} 16 & 32 & 24 \\ 32 & 64 & 48 \\ 24 & 48 & 36 \end{bmatrix} = \begin{bmatrix} 20 & 30 & 25 \\ 30 & 68 & 46 \\ 25 & 46 & 40 \end{bmatrix} \tag{3}$$

(b) Let $\mathbf{Y} = \begin{bmatrix} X_1 & X_2 \end{bmatrix}'$. Since $\mathbf{Y}$ is a subset of the components of $\mathbf{X}$, it is a Gaussian random vector with expected velue vector

$$\boldsymbol{\mu}_Y = \begin{bmatrix} E[X_1] & E[X_2] \end{bmatrix}' = \begin{bmatrix} 4 & 8 \end{bmatrix}'. \tag{4}$$

and covariance matrix

$$\mathbf{C}_Y = \begin{bmatrix} \mathrm{Var}[X_1] & \mathrm{Cov}[X_1, X_2] \\ \mathbf{C}_{X_1} X_2 & \mathrm{Var}[X_2] \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix} \tag{5}$$

We note that $\det(\mathbf{C}_Y) = 12$ and that

$$\mathbf{C}_Y^{-1} = \frac{1}{12} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix}. \tag{6}$$

This implies that

$$(\mathbf{y} - \boldsymbol{\mu}_Y)' \mathbf{C}_Y^{-1} (\mathbf{y} - \boldsymbol{\mu}_Y) = \begin{bmatrix} y_1 - 4 & y_2 - 8 \end{bmatrix} \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} \begin{bmatrix} y_1 - 4 \\ y_2 - 8 \end{bmatrix} \tag{7}$$

$$= \begin{bmatrix} y_1 - 4 & y_2 - 8 \end{bmatrix} \begin{bmatrix} y_1/3 + y_2/6 - 8/3 \\ y_1/6 + y_2/3 - 10/3 \end{bmatrix} \tag{8}$$

$$= \frac{y_1^2}{3} + \frac{y_1 y_2}{3} - \frac{16 y_1}{3} - \frac{20 y_2}{3} + \frac{y_2^2}{3} + \frac{112}{3} \tag{9}$$

The PDF of $\mathbf{Y}$ is

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{2\pi\sqrt{12}} e^{-(\mathbf{y}-\boldsymbol{\mu}_Y)' \mathbf{C}_Y^{-1} (\mathbf{y}-\boldsymbol{\mu}_Y)/2} \tag{10}$$

$$= \frac{1}{\sqrt{48\pi^2}} e^{-(y_1^2 + y_1 y_2 - 16 y_1 - 20 y_2 + y_2^2 + 112)/6} \tag{11}$$

Since $\mathbf{Y} = \begin{bmatrix} X_1, X_2 \end{bmatrix}'$, the PDF of X_1 and X_2 is simply

$$f_{X_1,X_2}(x_1, x_2) = f_{Y_1,Y_2}(x_1, x_2) = \frac{1}{\sqrt{48\pi^2}} e^{-(x_1^2 + x_1 x_2 - 16 x_1 - 20 x_2 + x_2^2 + 112)/6} \tag{12}$$

(c) We can observe directly from $\boldsymbol{\mu}_X$ and $\mathbf{C}_X$ that X_1 is a Gaussian $(4, 2)$ random variable. Thus,

$$P[X_1 > 8] = P\left[\frac{X_1 - 4}{2} > \frac{8 - 4}{2}\right] = Q(2) = 0.0228 \tag{13}$$

Problem 5.8.1 •

Consider the vector $\mathbf{X}$ in Problem 5.7.1 and define the average to be $Y = (X_1+X_2+X_3)/3$. What is the probability that $Y > 4$?

Problem 5.8.1 Solution

We can use Theorem 5.16 since the scalar Y is also a 1-dimensional vector. To do so, we write

$$Y = \begin{bmatrix} 1/3 & 1/3 & 1/3 \end{bmatrix} \mathbf{X} = \mathbf{AX}. \tag{1}$$

By Theorem 5.16, Y is a Gaussian vector with expected value

$$E[Y] = \mathbf{A}\boldsymbol{\mu}_X = (E[X_1] + E[X_2] + E[X_3])/3 = (4+8+6)/3 = 6 \tag{2}$$

and covariance matrix

$$\begin{aligned} \mathbf{C}_Y &= \text{Var}[Y] = \mathbf{AC}_X\mathbf{A}' & (3)\\ &= \begin{bmatrix} 1/3 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \frac{2}{3} & (4) \end{aligned}$$

Thus Y is a Gaussian $(6, \sqrt{2/3})$ random variable, implying

$$P[Y > 4] = P\left[\frac{Y-6}{\sqrt{2/3}} > \frac{4-6}{\sqrt{2/3}}\right] = 1 - \Phi(-\sqrt{6}) = \Phi(\sqrt{6}) = 0.9928 \tag{5}$$

Problem 5.8.3 ■

For the vector of daily temperatures $\begin{bmatrix} T_1 & \cdots & T_{31} \end{bmatrix}'$ and average temperature Y modeled in Quiz 5.8, we wish to estimate the probability of the event

$$A = \left\{ Y \leq 82, \min_i T_i \geq 72 \right\}$$

To form an estimate of A, generate 10,000 independent samples of the vector $\mathbf{T}$ and calculate the relative frequency of A in those trials.

Problem 5.8.3 Solution

We are given that the random temperature vector has covariance

$$\text{Cov}[T_i, T_j] = C_T[i-j] = \frac{36}{1+|i-j|} \tag{1}$$

where i and j indicate the ith and jth days of the month and corresponding elements of the temperature vector. Thus

$$\mathbf{C_T} = \begin{bmatrix} C_T[0] & C_T[1] & \cdots & C_T[30] \\ C_T[1] & C_T[0] & \ddots & \vdots \\ \vdots & \ddots & \ddots & C_T[1] \\ C_T[30] & \cdots & C_T[1] & C_T[0] \end{bmatrix}. \tag{2}$$

Applying the method discussed on p. 236 of the textbook, we use the matlab script below to generate the 10,000 samples and estimate the relative frequency of occurance of the event A.

```
%
% Solves problem 5.8.3 of Yates and Goodman.  3/09/06 sk
%
% 5.8.3  Generate 10,000 random samples of the vector T of length 31
% having E[T_i]=80 and Cov[T_i,T_j]=36/(1+|i-j|).  Let Y = sum_i(T_i)/31.
% Let A be the event that Y \leq 82 AND min_i(T_i) \geq 72
%
   muT = 80*ones(31,1);
% First generate covariance matrix C.
   for iindex=1:31,
       for jindex = 1:31,
   C(iindex,jindex)=36/(1+abs(iindex-jindex));
       end;
   end;
% Then generate samples.  To avoid having a 10,000x31 variable in our
% workspace, after generating each sample, let's keep only Y and min(T).
   [u,d,v]=svd(C); % as shown on page 236.
   for index = 1:10000,
       T = v*(d^(0.5))*randn(31,1) + muT;
       Y(index) = sum(T)/31;
       minT(index) = min(T);
   end;
% Finally count the number of times A occurs out of the 10,000 samples.
   for index = 1:10000,
       if and((Y(index)<= 82),(minT(index) >= 72)),
          A(index) = 1;
       else
          A(index) = 0;
       end;
   end;
% Now estimate P[A].

disp(['On the basis of this experiment, we estimate P[A] to be  ',...
       num2str((sum(A)/10000))]);
```

The output obtained is

```
>> p5_8_3
On the basis of this experiment, we estimate P[A] to be  0.0719
```

Solutions to HW8

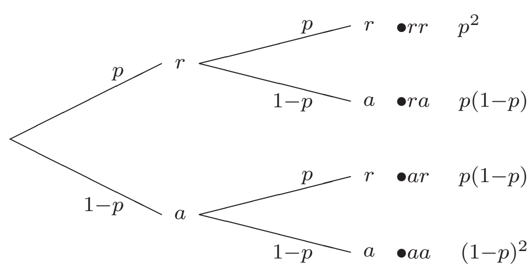
Note: Most of these solutions were generated by R. D. Yates and D. J. Goodman, the authors of our textbook. I have added comments in italics where I thought more detail was appropriate.

Problem 4.2.3 •

Test two integrated circuits. In each test, the probability of rejecting the circuit is p. Let X be the number of rejects (either 0 or 1) in the first test and let Y be the number of rejects in the second test. Find the joint PMF $P_{X,Y}(x, y)$.

Problem 4.2.3 Solution

Let r (reject) and a (accept) denote the result of each test. There are four possible outcomes: rr, ra, ar, aa. The sample tree is



Now we construct a table that maps the sample outcomes to values of X and Y.

outcome	$P[\cdot]$	X	Y
rr	p^2	1	1
ra	$p(1-p)$	1	0
ar	$p(1-p)$	0	1
aa	$(1-p)^2$	0	0

(1)

This table is esentially the joint PMF $P_{X,Y}(x, y)$.

$$P_{X,Y}(x,y) = \begin{cases} p^2 & x=1, y=1 \\ p(1-p) & x=0, y=1 \\ p(1-p) & x=1, y=0 \\ (1-p)^2 & x=0, y=0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Problem 4.4.3 ■

Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 6e^{-(2x+3y)} & x \geq 0, y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $P[X > Y]$ and $P[X + Y \leq 1]$.

(b) Find $P[\min(X, Y) \geq 1]$.

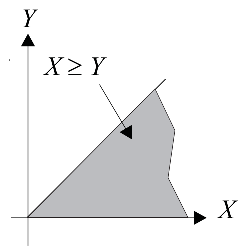
(c) Find $P[\max(X, Y) \leq 1]$.

Problem 4.4.3 Solution

The joint PDF of X and Y is

$$f_{X,Y}(x, y) = \begin{cases} 6e^{-(2x+3y)} & x \geq 0, y \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

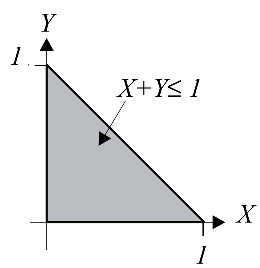
(a) The probability that $X \geq Y$ is:



$$P[X \geq Y] = \int_0^\infty \int_0^x 6e^{-(2x+3y)}\, dy\, dx \tag{2}$$
$$= \int_0^\infty 2e^{-2x} \left(-e^{-3y}\Big|_{y=0}^{y=x} \right) dx \tag{3}$$
$$= \int_0^\infty [2e^{-2x} - 2e^{-5x}]\, dx = 3/5 \tag{4}$$

The $P[X + Y \leq 1]$ is found by integrating over the region where $X + Y \leq 1$

$$P[X + Y \leq 1] = \int_0^1 \int_0^{1-x} 6e^{-(2x+3y)}\, dy\, dx \tag{5}$$
$$= \int_0^1 2e^{-2x} \left[-e^{-3y}\Big|_{y=0}^{y=1-x} \right] dx \tag{6}$$
$$= \int_0^1 2e^{-2x} \left[1 - e^{-3(1-x)} \right] dx \tag{7}$$
$$= -e^{-2x} - 2e^{x-3}\Big|_0^1 \tag{8}$$
$$= 1 + 2e^{-3} - 3e^{-2} \tag{9}$$



We should check here to make sure that our answer is not obviously invalid. A probability must not exceed one so, not knowing offhand whether $2e^{-3}$ *is less than or greater than* $3e^{-2}$*, we consult Matlab. Matlab says that* $1 + 2e^{-3} - 3e^{-2} = 0.6936$ *which is less than one so we have no cause for alarm.*

(b) The event $\min(X, Y) \geq 1$ is the same as the event $\{X \geq 1, Y \geq 1\}$. Thus,

$$P[\min(X, Y) \geq 1] = \int_1^\infty \int_1^\infty 6e^{-(2x+3y)}\, dy\, dx = e^{-(2+3)} \tag{10}$$

(c) The event $\max(X, Y) \leq 1$ is the same as the event $\{X \leq 1, Y \leq 1\}$ so that

$$P[\max(X, Y) \leq 1] = \int_0^1 \int_0^1 6e^{-(2x+3y)}\, dy\, dx = (1 - e^{-2})(1 - e^{-3}) \tag{11}$$

Problem 5.1.3 •
The random variables $X_1, \ldots, X_n$ have the joint PDF

$$f_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \begin{cases} 1 & 0 \le x_i \le 1; \\ & i = 1,\ldots,n, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the joint CDF, $F_{X_1,\ldots,X_n}(x_1,\ldots,x_n)$?

(b) For $n = 3$, what is the probability that $\min_i X_i \le 3/4$?

Problem 5.1.3 Solution

(a) In terms of the joint PDF, we can write joint CDF as

$$F_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f_{X_1,\ldots,X_n}(y_1,\ldots,y_n)\, dy_1 \cdots dy_n \tag{1}$$

However, simplifying the above integral depends on the values of each x_i. In particular, $f_{X_1,\ldots,X_n}(y_1,\ldots,y_n) = 1$ if and only if $0 \le y_i \le 1$ for each i. Since $F_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = 0$ if any $x_i < 0$, we limit, for the moment, our attention to the case where $x_i \ge 0$ for all i. In this case, some thought will show that we can write the limits in the following way:

$$\begin{align} F_{X_1,\ldots,X_n}(x_1,\ldots,x_n) &= \int_0^{\max(1,x_1)} \cdots \int_0^{\min(1,x_n)} dy_1 \cdots dy_n \tag{2} \\ &= \min(1,x_1)\min(1,x_2)\cdots\min(1,x_n) \tag{3} \end{align}$$

A complete expression for the CDF of $X_1,\ldots,X_n$ is

$$F_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \begin{cases} \prod_{i=1}^n \min(1,x_i) & 0 \le x_i, i = 1,2,\ldots,n \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

(b) For $n = 3$,

$$\begin{align} 1 - P\left[\min_i X_i \le 3/4\right] &= P\left[\min_i X_i > 3/4\right] \tag{5} \\ &= P\left[X_1 > 3/4, X_2 > 3/4, X_3 > 3/4\right] \tag{6} \\ &= \int_{3/4}^1 \int_{3/4}^1 \int_{3/4}^1 dx_1\, dx_2\, dx_3 \tag{7} \\ &= (1 - 3/4)^3 = 1/64 \tag{8} \end{align}$$

Thus $P[\min_i X_i \le 3/4] = 63/64$.

Problem 5.2.1 •
For random variables $X_1, \ldots, X_n$ in Problem 5.1.3, let $\mathbf{X} = \begin{bmatrix} X_1 & \cdots & X_n \end{bmatrix}'$. What is $f_{\mathbf{X}}(\mathbf{x})$?

Problem 5.2.1 Solution
This problem is very simple. In terms of the vector $\mathbf{X}$, the PDF is

$$f_{\mathbf{X}}(\mathbf{x}) = \begin{cases} 1 & \mathbf{0} \le \mathbf{x} \le \mathbf{1} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

However, just keep in mind that the inequalities $\mathbf{0} \le \mathbf{x}$ and $\mathbf{x} \le \mathbf{1}$ are vector inequalities that must hold for every component x_i. *As noted in class, we should say* $x_i \in [0,1]$, $\forall i \in \{1, 2, \ldots, n\}$ *rather than* $\mathbf{0} \le \mathbf{x} \le \mathbf{1}$.

Problem 4.3.4 ■
Random variables N and K have the joint PMF

$$P_{N,K}(n,k) = \begin{cases} (1-p)^{n-1}p/n & k = 1, \ldots, n; \\ & n = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Find the marginal PMFs $P_N(n)$ and $P_K(k)$.

Problem 4.3.4 Solution
The joint PMF of N, K is

$$P_{N,K}(n,k) = \begin{cases} (1-p)^{n-1}p/n & k = 1, 2, \ldots, n \\ & n = 1, 2 \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

For $n \ge 1$, the marginal PMF of N is

$$P_N(n) = \sum_{k=1}^{n} P_{N,K}(n,k) = \sum_{k=1}^{n} (1-p)^{n-1}p/n = (1-p)^{n-1}p \tag{2}$$

The marginal PMF of K is found by summing $P_{N,K}(n,k)$ over all possible N. Note that if $K = k$, then $N \ge k$. Thus,

$$P_K(k) = \sum_{n=k}^{\infty} \frac{1}{n}(1-p)^{n-1}p \tag{3}$$

Unfortunately, this sum cannot be simplified.

Problem 4.5.3 ■
Over the circle $X^2 + Y^2 \leq r^2$, random variables X and Y have the uniform PDF

$$f_{X,Y}(x, y) = \begin{cases} 1/(\pi r^2) & x^2 + y^2 \leq r^2, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the marginal PDF $f_X(x)$?

(b) What is the marginal PDF $f_Y(y)$?

Problem 4.5.3 Solution
Random variables X and Y have joint PDF

$$f_{X,Y}(x, y) = \begin{cases} 1/(\pi r^2) & 0 \leq x^2 + y^2 \leq r^2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) The marginal PDF of X is

$$f_X(x) = 2\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \frac{1}{\pi r^2}\, dy = \begin{cases} \frac{2\sqrt{r^2-x^2}}{\pi r^2} & -r \leq x \leq r, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

(b) Similarly, for random variable Y,

$$f_Y(y) = 2\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{1}{\pi r^2}\, dx = \begin{cases} \frac{2\sqrt{r^2-y^2}}{\pi r^2} & -r \leq y \leq r, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Problem 5.3.7 ■
For Example 5.2, we derived the joint PMF of three types of fax transmissions:

$$P_{X,Y,Z}(x, y, z) = \binom{4}{x, y, z} \frac{1}{3^x} \frac{1}{2^y} \frac{1}{6^z}.$$

(a) In a group of four faxes, what is the PMF of the number of 3-page faxes?

(b) In a group of four faxes, what is the expected number of 3-page faxes?

(c) Given that there are two 3-page faxes in a group of four, what is the joint PMF of the number of 1-page faxes and the number of 2-page faxes?

(d) Given that there are two 3-page faxes in a group of four, what is the expected number of 1-page faxes?

(e) In a group of four faxes, what is the joint PMF of the number of 1-page faxes and the number of 2-page faxes?

Problem 5.3.7 Solution

(a) Note that Z is the number of three-page faxes. In principle, we can sum the joint PMF $P_{X,Y,Z}(x,y,z)$ over all x,y to find $P_Z(z)$. However, it is better to realize that each fax has 3 pages with probability 1/6, independent of any other fax. Thus, Z has the binomial PMF

$$P_Z(z) = \begin{cases} \binom{4}{z}(1/6)^z(5/6)^{4-z} & z = 0,1,\ldots,4 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(b) From the properties of the binomial distribution given in Appendix A, we know that $E[Z] = 4(1/6)$.

(c) We want to find the conditional PMF of the number X of 1-page faxes and number Y of 2-page faxes given $Z = 2$ 3-page faxes. Note that given $Z = 2$, $X + Y = 2$. Hence for non-negative integers x, y satisfying $x + y = 2$,

$$P_{X,Y|Z}(x,y|2) = \frac{P_{X,Y,Z}(x,y,2)}{P_Z(2)} = \frac{\frac{4!}{x!y!2!}(1/3)^x(1/2)^y(1/6)^2}{\binom{4}{2}(1/6)^2(5/6)^2} \tag{2}$$

With some algebra, the complete expression of the conditional PMF is

$$P_{X,Y|Z}(x,y|2) = \begin{cases} \frac{2!}{x!y!}(2/5)^x(3/5)^y & x+y=2, x \geq 0, y \geq 0; x,y \text{ integer} \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

In the above expression, we note that if $Z = 2$, then $Y = 2 - X$ and

$$P_{X|Z}(x|2) = P_{X,Y|Z}(x, 2-x|2) = \begin{cases} \binom{2}{x}(2/5)^x(3/5)^{2-x} & x = 0,1,2,3 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

That is, given $Z = 2$, there are 2 faxes left, each of which independently could be a 1-page fax. The conditonal PMF of the number of 1-page faxes is binomial where 2/5 is the conditional probability that a fax has 1 page given that it either has 1 page or 2 pages. Moreover given $X = x$ and $Z = 2$ we must have $Y = 2 - x$.

(d) Given $Z = 2$, the conditional PMF of X is binomial for 3 trials and success probability 2/5. The conditional expectation of X given $Z = 2$ is $E[X|Z = 2] = 2(2/5) = 4/5$.

(e) There are several ways to solve this problem. The most straightforward approach is to realize that for integers $0 \leq x \leq 4$ and $0 \leq y \leq 4$, the event $\{X = x, Y = y\}$ occurs iff $\{X = x, Y = y, Z = 4 - (x + y)\}$. For the rest of this problem, we assume x and y are non-negative integers so that

$$\begin{aligned} P_{X,Y}(x,y) &= P_{X,Y,Z}(x,y,4-(x+y)) && (5) \\ &= \begin{cases} \frac{4!}{x!y!(4-x-y)!}\left(\frac{1}{3}\right)^x\left(\frac{1}{2}\right)^y\left(\frac{1}{6}\right)^{4-x-y} & 0 \leq x+y \leq 4, x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases} && (6) \end{aligned}$$

The above expression may seem unwieldy and it isn't even clear that it will sum to 1. To simplify the expression, we observe that

$$P_{X,Y}(x,y) = P_{X,Y,Z}(x,y,4-x-y) = P_{X,Y|Z}(x,y|4-x+y)\,P_Z(4-x-y) \quad (7)$$

Using $P_Z(z)$ found in part (c), we can calculate $P_{X,Y|Z}(x,y|4-x-y)$ for $0 \leq x+y \leq 4$, integer valued.

$$\begin{aligned} P_{X,Y|Z}(x,y|4-x+y) &= \frac{P_{X,Y,Z}(x,y,4-x-y)}{P_Z(4-x-y)} && (8) \\ &= \binom{x+y}{x}\left(\frac{1/3}{1/2+1/3}\right)^x \left(\frac{1/2}{1/2+1/3}\right)^y && (9) \\ &= \binom{x+y}{x}\left(\frac{2}{5}\right)^x \left(\frac{3}{5}\right)^{(x+y)-x} && (10) \end{aligned}$$

In the above expression, it is wise to think of $x+y$ as some fixed value. In that case, we see that given $x+y$ is a fixed value, X and Y have a joint PMF given by a binomial distribution in x. This should not be surprising since it is just a generalization of the case when $Z=2$. That is, given that there were a fixed number of faxes that had either one or two pages, each of those faxes is a one page fax with probability $(1/3)/(1/2+1/3)$ and so the number of one page faxes should have a binomial distribution, Moreover, given the number X of one page faxes, the number Y of two page faxes is completely specified.

Finally, by rewriting $P_{X,Y}(x,y)$ given above, the complete expression for the joint PMF of X and Y is

$$P_{X,Y}(x,y) = \begin{cases} \binom{4}{4-x-y}\left(\frac{1}{6}\right)^{4-x-y}\left(\frac{5}{6}\right)^{x+y}\binom{x+y}{x}\left(\frac{2}{5}\right)^x\left(\frac{3}{5}\right)^y & x,y \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (11)$$

Problem 4.10.10 ■

An ice cream company orders supplies by fax. Depending on the size of the order, a fax can be either

1 page for a short order,
2 pages for a long order.

The company has three different suppliers:

The vanilla supplier is 20 miles away.
The chocolate supplier is 100 miles away.
The strawberry supplier is 300 miles away.

An experiment consists of monitoring an order and observing N, the number of pages, and D, the distance the order is transmitted. The following probability model describes the experiment:

	van.	choc.	straw.
short	0.2	0.2	0.2
long	0.1	0.2	0.1

(a) What is the joint PMF $P_{N,D}(n,d)$ of the number of pages and the distance?

(b) What is $E[D]$, the expected distance of an order?

(c) Find $P_{D|N}(d|2)$, the conditional PMF of the distance when the order requires 2 pages.

(d) Write $E[D|N=2]$, the expected distance given that the order requires 2 pages.

(e) Are the random variables D and N independent?

(f) The price per page of sending a fax is one cent per mile transmitted. C cents is the price of one fax. What is $E[C]$, the expected price of one fax?

Problem 4.10.10 Solution

The key to this problem is understanding that "short order" and "long order" are synonyms for $N=1$ and $N=2$. Similarly, "vanilla", "chocolate", and "strawberry" correspond to the events $D=20$, $D=100$ and $D=300$.

(a) The following table is given in the problem statement.

	vanilla	choc.	strawberry
short order	0.2	0.2	0.2
long order	0.1	0.2	0.1

This table can be translated directly into the joint PMF of N and D.

$P_{N,D}(n,d)$	$d=20$	$d=100$	$d=300$
$n=1$	0.2	0.2	0.2
$n=2$	0.1	0.2	0.1

(1)

(b) We find the marginal PMF $P_D(d)$ by summing the columns of the joint PMF. This yields

$$P_D(d) = \begin{cases} 0.3 & d=20, \\ 0.4 & d=100, \\ 0.3 & d=300, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

(c) To find the conditional PMF $P_{D|N}(d|2)$, we first need to find the probability of the conditioning event

$$P_N(2) = P_{N,D}(2,20) + P_{N,D}(2,100) + P_{N,D}(2,300) = 0.4 \tag{3}$$

The conditional PMF of N D given $N = 2$ is

$$P_{D|N}(d|2) = \frac{P_{N,D}(2,d)}{P_N(2)} = \begin{cases} 1/4 & d = 20 \\ 1/2 & d = 100 \\ 1/4 & d = 300 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

(d) The conditional expectation of D given $N = 2$ is

$$E[D|N = 2] = \sum_d dP_{D|N}(d|2) = 20(1/4) + 100(1/2) + 300(1/4) = 130 \tag{5}$$

(e) To check independence, we could calculate the marginal PMFs of N and D. In this case, however, it is simpler to observe that $P_D(d) \neq P_{D|N}(d|2)$. Hence N and D are dependent.

(f) In terms of N and D, the cost (in cents) of a fax is $C = ND$. The expected value of C is

$$E[C] = \sum_{n,d} ndP_{N,D}(n,d) \tag{6}$$

$$= 1(20)(0.2) + 1(100)(0.2) + 1(300)(0.2) \tag{7}$$

$$+ 2(20)(0.1) + 2(100)(0.2) + 2(300)(0.1) = 188 \tag{8}$$

Problem 5.4.6 ■

The random vector $\mathbf{X}$ has PDF

$$f_{\mathbf{X}}(\mathbf{x}) = \begin{cases} e^{-x_3} & 0 \leq x_1 \leq x_2 \leq x_3, \\ 0 & \text{otherwise.} \end{cases}$$

Find the marginal PDFs $f_{X_1}(x_1)$, $f_{X_2}(x_2)$, and $f_{X_3}(x_3)$.

Problem 5.4.6 Solution

This problem was not assigned but contains the straightforward solution to problem 5.4.5, hence is included here in the solutions.

We find the marginal PDFs using Theorem 5.5. First we note that for $x < 0$, $f_{X_i}(x) = 0$. For $x_1 \geq 0$,

$$f_{X_1}(x_1) = \int_{x_1}^{\infty} \left(\int_{x_2}^{\infty} e^{-x_3}\, dx_3 \right) dx_2 = \int_{x_1}^{\infty} e^{-x_2}\, dx_2 = e^{-x_1} \tag{1}$$

Similarly, for $x_2 \geq 0$, X_2 has marginal PDF

$$f_{X_2}(x_2) = \int_0^{x_2} \left(\int_{x_2}^{\infty} e^{-x_3}\, dx_3 \right) dx_1 = \int_0^{x_2} e^{-x_2}\, dx_1 = x_2 e^{-x_2} \tag{2}$$

Lastly,

$$f_{X_3}(x_3) = \int_0^{x_3} \left(\int_{x_1}^{x_3} e^{-x_3}\, dx_2 \right) dx_1 = \int_0^{x_3} (x_3 - x_1) e^{-x_3}\, dx_1 \tag{3}$$

$$= -\frac{1}{2}(x_3 - x_1)^2 e^{-x_3} \Big|_{x_1=0}^{x_1=x_3} = \frac{1}{2} x_3^2 e^{-x_3} \tag{4}$$

The complete expressions for the three marginal PDFs are

$$f_{X_1}(x_1) = \begin{cases} e^{-x_1} & x_1 \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

$$f_{X_2}(x_2) = \begin{cases} x_2 e^{-x_2} & x_2 \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

$$f_{X_3}(x_3) = \begin{cases} (1/2) x_3^2 e^{-x_3} & x_3 \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

In fact, each X_i is an Erlang $(n, \lambda) = (i, 1)$ random variable.

Problem 4.6.6 ■

Random variables X and Y have joint PDF

$$f_{X,Y}(x, y) = \begin{cases} x + y & 0 \leq x \leq 1, 0 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $W = \max(X, Y)$.

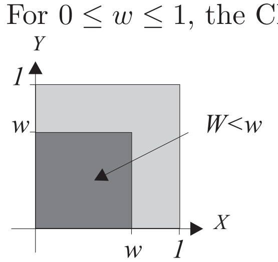
(a) What is S_W, the range of W?

(b) Find $F_W(w)$ and $f_W(w)$.

Problem 4.6.6 Solution

(a) The minimum value of W is $W = 0$, which occurs when $X = 0$ and $Y = 0$. The maximum value of W is $W = 1$, which occurs when $X = 1$ or $Y = 1$. The range of W is $S_W = \{w | 0 \leq w \leq 1\}$.

(b) For $0 \leq w \leq 1$, the CDF of W is



$$F_W(w) = P[\max(X, Y) \leq w] \tag{1}$$

$$= P[X \leq w, Y \leq w] \tag{2}$$

$$= \int_0^w \int_0^w f_{X,Y}(x, y)\, dy\, dx \tag{3}$$

Substituting $f_{X,Y}(x,y) = x + y$ yields

$$F_W(w) = \int_0^w \int_0^w (x+y)\,dy\,dx \tag{4}$$

$$= \int_0^w \left(xy + \frac{y^2}{2}\Big|_{y=0}^{y=w} \right) dx = \int_0^w (wx + w^2/2)\,dx = w^3 \tag{5}$$

The complete expression for the CDF is

$$F_W(w) = \begin{cases} 0 & w < 0 \\ w^3 & 0 \le w \le 1 \\ 1 & \text{otherwise} \end{cases} \tag{6}$$

The PDF of W is found by differentiating the CDF.

$$f_W(w) = \frac{dF_W(w)}{dw} = \begin{cases} 3w^2 & 0 \le w \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Problem 5.5.3 ■

In an automatic geolocation system, a dispatcher sends a message to six trucks in a fleet asking their locations. The waiting times for responses from the six trucks are iid exponential random variables, each with expected value 2 seconds.

(a) What is the probability that all six responses will arrive within 5 seconds?

(b) If the system has to locate all six vehicles within 3 seconds, it has to reduce the expected response time of each vehicle. What is the maximum expected response time that will produce a location time for all six vehicles of 3 seconds or less with probability of at least 0.9?

Problem 5.5.3 Solution

The response time X_i of the ith truck has PDF $f_{X_i}(x_i)$ and CDF $F_{X_i}(x_i)$ given by

$$f_{X_i}(x_i) = \begin{cases} \frac{1}{2}e^{-x/2} & x \ge 0, \\ 0 & \text{otherwise,} \end{cases} \qquad F_{X_i}(x_i) = F_X(x_i) = \begin{cases} 1 - e^{-x/2} & x \ge 0 \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Let $R = \max(X_1, X_2, \ldots, X_6)$ denote the maximum response time. From Theorem 5.7, R has PDF

$$F_R(r) = (F_X(r))^6. \tag{2}$$

(a) The probability that all six responses arrive within five seconds is

$$P[R \le 5] = F_R(5) = (F_X(5))^6 = (1 - e^{-5/2})^6 = 0.5982. \tag{3}$$

(b) This question is worded in a somewhat confusing way. The "expected response time" refers to $E[X_i]$, the response time of an individual truck, rather than $E[R]$. If the expected response time of a truck is τ, then each X_i has CDF

$$F_{X_i}(x) = F_X(x) = \begin{cases} 1 - e^{-x/\tau} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

The goal of this problem is to find the maximum permissible value of τ. When each truck has expected response time τ, the CDF of R is

$$F_R(r) = (F_X(x)\, r)^6 = \begin{cases} (1 - e^{-r/\tau})^6 & r \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

We need to find τ such that

$$P[R \leq 3] = (1 - e^{-3/\tau})^6 = 0.9. \tag{6}$$

This implies

$$\tau = \frac{-3}{\ln\left(1 - (0.9)^{1/6}\right)} = 0.7406 \text{ s.} \tag{7}$$

Problem 4.7.9 ■

Random variables X and Y have joint PDF

$$f_{X,Y}(x, y) = \begin{cases} 4xy & 0 \leq x \leq 1, 0 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What are $E[X]$ and $\text{Var}[X]$?

(b) What are $E[Y]$ and $\text{Var}[Y]$?

(c) What is $\text{Cov}[X, Y]$?

(d) What is $E[X + Y]$?

(e) What is $\text{Var}[X + Y]$?

Problem 4.7.9 Solution

(a) The first moment of X is

$$E[X] = \int_0^1 \int_0^1 4x^2 y\, dy\, dx = \int_0^1 2x^2\, dx = \frac{2}{3} \tag{1}$$

The second moment of X is

$$E\left[X^2\right] = \int_0^1 \int_0^1 4x^3 y\, dy\, dx = \int_0^1 2x^3\, dx = \frac{1}{2} \tag{2}$$

The variance of X is $\text{Var}[X] = E[X^2] - (E[X])^2 = 1/2 - (2/3)^2 = 1/18$.

(b) The mean of Y is

$$E[Y] = \int_0^1 \int_0^1 4xy^2\,dy\,dx = \int_0^1 \frac{4x}{3}\,dx = \frac{2}{3} \tag{3}$$

The second moment of Y is

$$E\left[Y^2\right] = \int_0^1 \int_0^1 4xy^3\,dy\,dx = \int_0^1 x\,dx = \frac{1}{2} \tag{4}$$

The variance of Y is $\text{Var}[Y] = E[Y^2] - (E[Y])^2 = 1/2 - (2/3)^2 = 1/18$.

(c) To find the covariance, we first find the correlation

$$E[XY] = \int_0^1 \int_0^1 4x^2y^2\,dy\,dx = \int_0^1 \frac{4x^2}{3}\,dx = \frac{4}{9} \tag{5}$$

The covariance is thus

$$\text{Cov}[X,Y] = E[XY] - E[X]E[Y] = \frac{4}{9} - \left(\frac{2}{3}\right)^2 = 0 \tag{6}$$

(d) $E[X+Y] = E[X] + E[Y] = 2/3 + 2/3 = 4/3$.

(e) By Theorem 4.15, the variance of $X+Y$ is

$$\text{Var}[X] + \text{Var}[Y] + 2\,\text{Cov}[X,Y] = 1/18 + 1/18 + 0 = 1/9 \tag{7}$$

Problem 5.6.5 •

In the message transmission system in Problem 5.3.2, the solution to Problem 5.5.2 is a formula for the PMF of $\mathbf{J}$, the number of transmissions of individual messages. For $p = 0.8$, find the expected value vector $E[\mathbf{J}]$, the correlation matrix $\mathbf{R_J}$, and the covariance matrix $\mathbf{C_J}$.

Problem 5.6.5 Solution

The random variable J_m is the number of times that message m is transmitted. Since each transmission is a success with probability p, independent of any other transmission, J_1, J_2 and J_3 are iid geometric (p) random variables with

$$E[J_m] = \frac{1}{p}, \qquad \text{Var}[J_m] = \frac{1-p}{p^2}. \tag{1}$$

Thus the vector $\mathbf{J} = \begin{bmatrix} J_1 & J_2 & J_3 \end{bmatrix}'$ has expected value

$$E[\mathbf{J}] = \begin{bmatrix} E[J_1] & E[J_2] & EJ_3 \end{bmatrix}' = \begin{bmatrix} 1/p & 1/p & 1/p \end{bmatrix}'. \tag{2}$$

For $m \neq n$, the correlation matrix $\mathbf{R_J}$ has m,nth entry

$$R_{\mathbf{J}}(m,n) = E[J_m J_n] = E[J_m]\,J_n = 1/p^2 \tag{3}$$

For $m = n$,

$$R_{\mathbf{J}}(m,m) = E\left[J_m^2\right] = \text{Var}[J_m] + (E\left[J_m^2\right])^2 = \frac{1-p}{p^2} + \frac{1}{p^2} = \frac{2-p}{p^2}. \tag{4}$$

Thus

$$\mathbf{R_J} = \frac{1}{p^2}\begin{bmatrix} 2-p & 1 & 1 \\ 1 & 2-p & 1 \\ 1 & 1 & 2-p \end{bmatrix}. \tag{5}$$

Because J_m and J_n are independent, off-diagonal terms in the covariance matrix are

$$C_{\mathbf{J}}(m,n) = \text{Cov}\left[J_m, J_n\right] = 0 \tag{6}$$

Since $C_{\mathbf{J}}(m,m) = \text{Var}[J_m]$, we have that

$$\mathbf{C_J} = \frac{1-p}{p^2}\mathbf{I} = \frac{1-p}{p^2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{7}$$

Problem 5.6.6 ■

In the message transmission system in Problem 5.3.2,

$$P_{\mathbf{K}}(\mathbf{k}) = \begin{cases} p^3(1-p)^{k_3-3}; & k_1 < k_2 < k_3; \\ & k_i \in \{1, 2, \ldots\} \\ 0 & \text{otherwise.} \end{cases}$$

For $p = 0.8$, find the expected value vector $E[\mathbf{K}]$, the covariance matrix $\mathbf{C_K}$, and the correlation matrix $\mathbf{R_K}$.

Problem 5.6.6 Solution

This problem is quite difficult unless one uses the observation that the vector $\mathbf{K}$ can be expressed in terms of the vector $\mathbf{J} = \begin{bmatrix} J_1 & J_2 & J_3 \end{bmatrix}'$ where J_i is the number of transmissions of message i. Note that we can write

$$\mathbf{K} = \mathbf{AJ} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\mathbf{J} \tag{1}$$

We also observe that since each transmission is an independent Bernoulli trial with success probability p, the components of $\mathbf{J}$ are iid geometric (p) random variables. Thus $E[J_i] = 1/p$ and $\text{Var}[J_i] = (1-p)/p^2$. Thus $\mathbf{J}$ has expected value

$$E\left[\mathbf{J}\right] = \boldsymbol{\mu}_J = \begin{bmatrix} E\left[J_1\right] & E\left[J_2\right] & E\left[J_3\right] \end{bmatrix}' = \begin{bmatrix} 1/p & 1/p & 1/p \end{bmatrix}'. \tag{2}$$

Since the components of $\mathbf{J}$ are independent, it has the diagonal covariance matrix

$$\mathbf{C}_J = \begin{bmatrix} \text{Var}[J_1] & 0 & 0 \\ 0 & \text{Var}[J_2] & 0 \\ 0 & 0 & \text{Var}[J_3] \end{bmatrix} = \frac{1-p}{p^2}\mathbf{I} \tag{3}$$

Given these properties of $\mathbf{J}$, finding the same properties of $\mathbf{K} = \mathbf{AJ}$ is simple.

(a) The expected value of $\mathbf{K}$ is

$$E[\mathbf{K}] = \mathbf{A}\boldsymbol{\mu}_J = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/p \\ 1/p \\ 1/p \end{bmatrix} = \begin{bmatrix} 1/p \\ 2/p \\ 3/p \end{bmatrix} \tag{4}$$

(b) From Theorem 5.13, the covariance matrix of $\mathbf{K}$ is

$$\begin{aligned} \mathbf{C}_K &= \mathbf{A}\mathbf{C}_J\mathbf{A}' && (5) \\ &= \frac{1-p}{p^2}\mathbf{A}\mathbf{I}\mathbf{A}' && (6) \\ &= \frac{1-p}{p^2} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1-p}{p^2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} && (7) \end{aligned}$$

(c) Given the expected value vector $\boldsymbol{\mu}_K$ and the covariance matrix $\mathbf{C}_K$, we can use Theorem 5.12 to find the correlation matrix

$$\begin{aligned} \mathbf{R}_K &= \mathbf{C}_K + \boldsymbol{\mu}_K\boldsymbol{\mu}_K' && (8) \\ &= \frac{1-p}{p^2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 1/p \\ 2/p \\ 3/p \end{bmatrix} \begin{bmatrix} 1/p & 2/p & 3/p \end{bmatrix} && (9) \\ &= \frac{1-p}{p^2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} + \frac{1}{p^2} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} && (10) \\ &= \frac{1}{p^2} \begin{bmatrix} 2-p & 3-p & 4-p \\ 3-p & 6-2p & 8-2p \\ 4-p & 8-2p & 12-3p \end{bmatrix} && (11) \end{aligned}$$

Problem 4.11.4 ■

An archer shoots an arrow at a circular target of radius 50cm. The arrow pierces the target at a random position (X, Y), measured in centimeters from the center of the disk at position $(X, Y) = (0, 0)$. The "bullseye" is a solid black circle of radius 2cm, at the center of the target. Calculate the probability $P[B]$ of the event that the archer hits the bullseye under each of the following models:

(a) X and Y are iid continuous uniform $(-50, 50)$ random variables.

(b) The PDF $f_{X,Y}(x, y)$ is uniform over the 50cm circular target.

(c) X and Y are iid Gaussian $(\mu = 0, \sigma = 10)$ random variables.

Problem 4.11.4 Solution

The event B is the set of outcomes satisfying $X^2 + Y^2 \leq 2^2$. Of ocurse, the calculation of $P[B]$ depends on the probability model for X and Y.

(a) In this instance, X and Y have the same PDF

$$f_X(x) = f_Y(x) = \begin{cases} 0.01 & -50 \leq x \leq 50 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Since X and Y are independent, their joint PDF is

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) = \begin{cases} 10^{-4} & -50 \leq x \leq 50, -50 \leq y \leq 50 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Because X and Y have a uniform PDF over the bullseye area, $P[B]$ is just the value of the joint PDF over the area times the area of the bullseye.

$$P[B] = P\left[X^2 + Y^2 \leq 2^2\right] = 10^{-4} \cdot \pi 2^2 = 4\pi \cdot 10^{-4} \approx 0.0013 \tag{3}$$

(b) In this case, the joint PDF of X and Y is inversely proportional to the area of the target.

$$f_{X,Y}(x,y) = \begin{cases} 1/[\pi 50^2] & x^2 + y^2 \leq 50^2 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

The probability of a bullseye is

$$P[B] = P\left[X^2 + Y^2 \leq 2^2\right] = \frac{\pi 2^2}{\pi 50^2} = \left(\frac{1}{25}\right)^2 \approx 0.0016. \tag{5}$$

(c) In this instance, X and Y have the identical Gaussian $(0, \sigma)$ PDF with $\sigma^2 = 100$; i.e.,

$$f_X(x) = f_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} \tag{6}$$

Since X and Y are independent, their joint PDF is

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2} \tag{7}$$

To find $P[B]$, we write

$$P[B] = P\left[X^2 + Y^2 \leq 2^2\right] = \iint_{x^2+y^2 \leq 2^2} f_{X,Y}(x,y)\, dx\, dy \tag{8}$$

$$= \frac{1}{2\pi\sigma^2} \iint_{x^2+y^2 \leq 2^2} e^{-(x^2+y^2)/2\sigma^2}\, dx\, dy \tag{9}$$

This integral is easy using polar coordinates. With the substitutions $x^2 + y^2 = r^2$, and $dx\, dy = r\, dr\, d\theta$

$$P[B] = \frac{1}{2\pi\sigma^2} \int_0^2 \int_0^{2\pi} e^{-r^2/2\sigma^2} r\, d\theta\, dr \tag{10}$$

$$= \frac{1}{\sigma^2} \int_0^2 r e^{-r^2/2\sigma^2}\, dr \tag{11}$$

$$= -e^{-r^2/2\sigma^2} \Big|_0^2 = 1 - e^{-4/200} \approx 0.0198. \tag{12}$$

Problem 5.7.2 •

Given the Gaussian random vector $\mathbf{X}$ in Problem 5.7.1, $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 1/2 & 2/3 \\ 1 & -1/2 & 2/3 \end{bmatrix}$$

and $\mathbf{b} = \begin{bmatrix} -4 & -4 \end{bmatrix}'$. Calculate

(a) the expected value $\boldsymbol{\mu}_{\mathbf{Y}}$,

(b) the covariance $\mathbf{C}_{\mathbf{Y}}$,

(c) the correlation $\mathbf{R}_{\mathbf{Y}}$,

(d) the probability that $-1 \leq Y_2 \leq 1$.

Problem 5.7.2 Solution

We are given that $\mathbf{X}$ is a Gaussian random vector with

$$\boldsymbol{\mu}_{\mathbf{X}} = \begin{bmatrix} 4 \\ 8 \\ 6 \end{bmatrix} \qquad \mathbf{C}_{\mathbf{X}} = \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix}. \tag{1}$$

We are also given that $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$ where

$$\mathbf{A} = \begin{bmatrix} 1 & 1/2 & 2/3 \\ 1 & -1/2 & 2/3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -4 \\ -4 \end{bmatrix}. \tag{2}$$

Since the two rows of $\mathbf{A}$ are linearly independent row vectors, $\mathbf{A}$ has rank 2. By Theorem 5.16, $\mathbf{Y}$ is a Gaussian random vector. Given these facts, the various parts of this problem are just straightforward calculations using Theorem 5.16.

(a) The expected value of $\mathbf{Y}$ is

$$\boldsymbol{\mu}_{\mathbf{Y}} = \mathbf{A}\boldsymbol{\mu}_{\mathbf{X}} + \mathbf{b} = \begin{bmatrix} 1 & 1/2 & 2/3 \\ 1 & -1/2 & 2/3 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ 6 \end{bmatrix} + \begin{bmatrix} -4 \\ -4 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}. \tag{3}$$

(b) The covariance matrix of $\mathbf{Y}$ is

$$\mathbf{C}_{\mathbf{Y}} = \mathbf{A}\mathbf{C}_{\mathbf{X}}\mathbf{A} \tag{4}$$

$$= \begin{bmatrix} 1 & 1/2 & 2/3 \\ 1 & -1/2 & 2/3 \end{bmatrix} \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1/2 & -1/2 \\ 2/3 & 2/3 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 43 & 55 \\ 55 & 103 \end{bmatrix}. \tag{5}$$

(c) $\mathbf{Y}$ has correlation matrix

$$\mathbf{R_Y} = \mathbf{C_Y} + \boldsymbol{\mu}_\mathbf{Y}\boldsymbol{\mu}'_\mathbf{Y} = \frac{1}{9}\begin{bmatrix} 43 & 55 \\ 55 & 103 \end{bmatrix} + \begin{bmatrix} 8 \\ 0 \end{bmatrix}\begin{bmatrix} 8 & 0 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 619 & 55 \\ 55 & 103 \end{bmatrix} \tag{6}$$

(d) From $\boldsymbol{\mu}_\mathbf{Y}$, we see that $E[Y_2] = 0$. From the covariance matrix $\mathbf{C_Y}$, we learn that Y_2 has variance $\sigma_2^2 = C_\mathbf{Y}(2,2) = 103/9$. Since Y_2 is a Gaussian random variable,

$$\begin{aligned} P[-1 \le Y_2 \le 1] &= P\left[-\frac{1}{\sigma_2} \le \frac{Y_2}{\sigma_2} \le \frac{1}{\sigma_2}\right] && (7) \\ &= \Phi\left(\frac{1}{\sigma_2}\right) - \Phi\left(\frac{-1}{\sigma_2}\right) && (8) \\ &= 2\Phi\left(\frac{1}{\sigma_2}\right) - 1 && (9) \\ &= 2\Phi\left(\frac{3}{\sqrt{103}}\right) - 1 = 0.2325. && (10) \end{aligned}$$

Problem 4.12.2 •

For random variables X and Y in Example 4.27, use MATLAB to calculate $E[X]$, $E[Y]$, the correlation $E[XY]$, and the covariance $\text{Cov}[X,Y]$.

Problem 4.12.2 Solution

In this problem, we need to calculate $E[X]$, $E[Y]$, the correlation $E[XY]$, and the covariance $\text{Cov}[X,Y]$ for random variables X and Y in Example 4.27. In this case, we can use the script `imagepmf.m` in Example 4.27 to generate the grid variables `SX`, `SY` and `PXY` that describe the joint PMF $P_{X,Y}(x,y)$.

However, for the rest of the problem, a general solution is better than a specific solution. The general problem is that given a pair of finite random variables described by the grid variables `SX`, `SY` and `PXY`, we want MATLAB to calculate an expected value $E[g(X,Y)]$. This problem is solved in a few simple steps. First we write a function that calculates the expected value of any finite random variable.

```
function ex=finiteexp(sx,px);
%Usage: ex=finiteexp(sx,px)
%returns the expected value E[X]
%of finite random variable X described
%by samples sx and probabilities px
ex=sum((sx(:)).*(px(:)));
```

Note that `finiteexp` performs its calculations on the sample values `sx` and probabilities `px` using the column vectors `sx(:)` and `px(:)`. As a result, we can use the same `finiteexp` function when the random variable is represented by grid variables. For example, we can calculate the correlation $r = E[XY]$ as

```
r=finiteexp(SX.*SY,PXY)
```

It is also convenient to define a function that returns the covariance:

```
function covxy=finitecov(SX,SY,PXY);
%Usage: cxy=finitecov(SX,SY,PXY)
%returns the covariance of
%finite random variables X and Y
%given by grids SX, SY, and PXY
ex=finiteexp(SX,PXY);
ey=finiteexp(SY,PXY);
R=finiteexp(SX.*SY,PXY);
covxy=R-ex*ey;
```

The following script calculates the desired quantities:

```
%imageavg.m
%Solution for Problem 4.12.2
imagepmf; %defines SX, SY, PXY
ex=finiteexp(SX,PXY)
ey=finiteexp(SY,PXY)
rxy=finiteexp(SX.*SY,PXY)
cxy=finitecov(SX,SY,PXY)
```

```
>> imageavg
ex =
        1180
ey =
   860
rxy =
     1064000
cxy =
       49200
>>
```

The careful reader will observe that `imagepmf` is inefficiently coded in that the correlation $E[XY]$ is calculated twice, once directly and once inside of `finitecov`. For more complex problems, it would be worthwhile to avoid this duplication.

Problem 4.12.3 •

Write a script `trianglecdfplot.m` that generates the graph of $F_{X,Y}(x,y)$ of Figure 4.4.

Problem 4.12.3 Solution

The script is just a MATLAB calculation of $F_{X,Y}(x,y)$ in Equation (4.29).

```
%trianglecdfplot.m
[X,Y]=meshgrid(0:0.05:1.5);
R=(0<=Y).*(Y<=X).*(X<=1).*(2*(X.*Y)-(Y.^2));
R=R+((0<=X).*(X<Y).*(X<=1).*(X.^2));
R=R+((0<=Y).*(Y<=1).*(1<X).*((2*Y)-(Y.^2)));
R=R+((X>1).*(Y>1));
mesh(X,Y,R);
xlabel('\it x');
ylabel('\it y');
```

For functions like $F_{X,Y}(x,y)$ that have multiple cases, we calculate the function for each case and multiply by the corresponding boolean condition so as to have a zero contribution when that case doesn't apply. Using this technique, its important to define the boundary conditions carefully to make sure that no point is included in two different boundary conditions.

Solutions to HW9

Note: Most of these solutions were generated by R. D. Yates and D. J. Goodman, the authors of our textbook. I have added comments in italics where I thought more detail was appropriate. The solution to problem 6.2.1 is mine.

Problem 6.1.2 •

Let X_1 and X_2 denote a sequence of independent samples of a random variable X with variance $\text{Var}[X]$.

(a) What is $E[X_1 - X_2]$, the expected difference between two outcomes?

(b) What is $\text{Var}[X_1 - X_2]$, the variance of the difference between two outcomes?

Problem 6.1.2 Solution

Let $Y = X_1 - X_2$.

(a) Since $Y = X_1 + (-X_2)$, Theorem 6.1 says that the expected value of the difference is

$$E[Y] = E[X_1] + E[-X_2] = E[X] - E[X] = 0 \tag{1}$$

(b) By Theorem 6.2, the variance of the difference is

$$\text{Var}[Y] = \text{Var}[X_1] + \text{Var}[-X_2] = 2\,\text{Var}[X] \tag{2}$$

Problem 6.1.4 ■

Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2 & x \geq 0, y \geq 0, x + y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

What is the variance of $W = X + Y$?

Problem 6.1.4 Solution

We can solve this problem using Theorem 6.2 which says that

$$\text{Var}[W] = \text{Var}[X] + \text{Var}[Y] + 2\,\text{Cov}[X,Y] \tag{1}$$

The first two moments of X are

$$E[X] = \int_0^1 \int_0^{1-x} 2x\,dy\,dx = \int_0^1 2x(1-x)\,dx = 1/3 \tag{2}$$

$$E[X^2] = \int_0^1 \int_0^{1-x} 2x^2\,dy\,dx = \int_0^1 2x^2(1-x)\,dx = 1/6 \tag{3}$$

$$\tag{4}$$

Thus the variance of X is $\text{Var}[X] = E[X^2] - (E[X])^2 = 1/18$. By symmetry, it should be apparent that $E[Y] = E[X] = 1/3$ and $\text{Var}[Y] = \text{Var}[X] = 1/18$. To find the covariance, we first find the correlation

$$E[XY] = \int_0^1 \int_0^{1-x} 2xy\,dy\,dx = \int_0^1 x(1-x)^2\,dx = 1/12 \tag{5}$$

The covariance is

$$\text{Cov}[X,Y] = E[XY] - E[X]E[Y] = 1/12 - (1/3)^2 = -1/36 \tag{6}$$

Finally, the variance of the sum $W = X + Y$ is

$$\text{Var}[W] = \text{Var}[X] + \text{Var}[Y] + 2\,\text{Cov}[X,Y] = 2/18 - 2/36 = 1/18 \tag{7}$$

For this specific problem, it's arguable whether it would easier to find $\text{Var}[W]$ by first deriving the CDF and PDF of W. In particular, for $0 \le w \le 1$,

$$F_W(w) = P[X + Y \le w] = \int_0^w \int_0^{w-x} 2\,dy\,dx = \int_0^w 2(w-x)\,dx = w^2 \tag{8}$$

Hence, by taking the derivative of the CDF, the PDF of W is

$$f_W(w) = \begin{cases} 2w & 0 \le w \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

From the PDF, the first and second moments of W are

$$E[W] = \int_0^1 2w^2\,dw = 2/3 \qquad E\left[W^2\right] = \int_0^1 2w^3\,dw = 1/2 \tag{10}$$

The variance of W is $\text{Var}[W] = E[W^2] - (E[W])^2 = 1/18$. Not surprisingly, we get the same answer both ways.

Problem 6.2.1 ■

Find the PDF of $W = X + Y$ when X and Y have the joint PDF

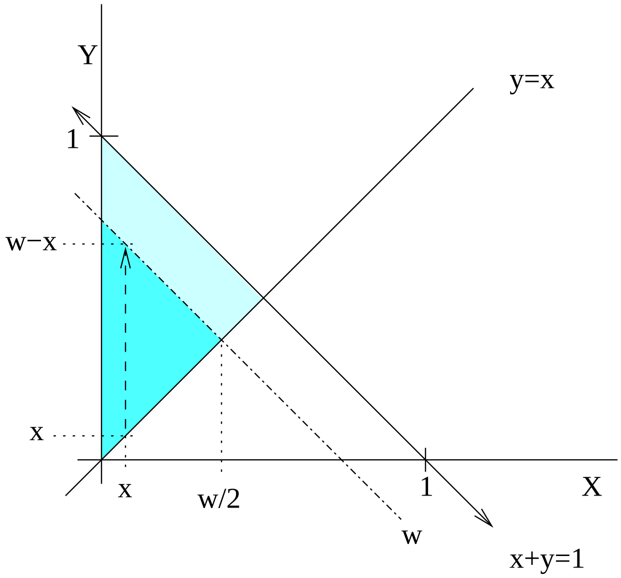
$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le x \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Problem 6.2.1 Solution

We are given that $W = X + Y$ and that the joint PDF of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le x \le y \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

We are asked to find the PDF of W. The first step is to find the CDF of W, $F_W(w)$. Note that we must integrate over different shaped areas for different values of w.

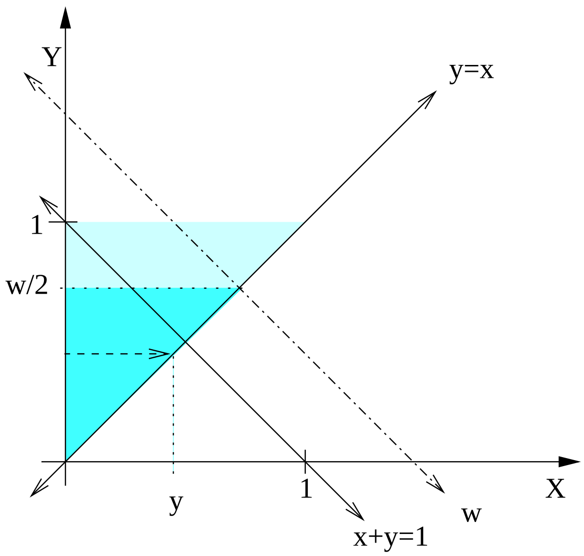


To distinguish between the random variables and their values, I have been careful here to use capital letters for the random variable names and lower case letters for the specific values they take. For values of W in the range $0 \le w \le 1$, we integrate over the shaded area in the figure to the left. A particular value w (indicated by the dotted and dashed line) of the random variable W corresponds to a set of pairs of X and Y values. For this value of w, we integrate from $Y = w - x$ to $Y = w$. To integrate over all values of the random variable W up to the value w, we then integrate with respect to X. As the value of the random variable W goes from 0 to w, the value of the random variable X goes from 0 to $w/2$. The lightly shaded area represents the region in which $0 \le w \le 1$ and $f_{X,Y}(x, y) > 0$. The darker shaded area represents the region corresponding to the limits of integration.

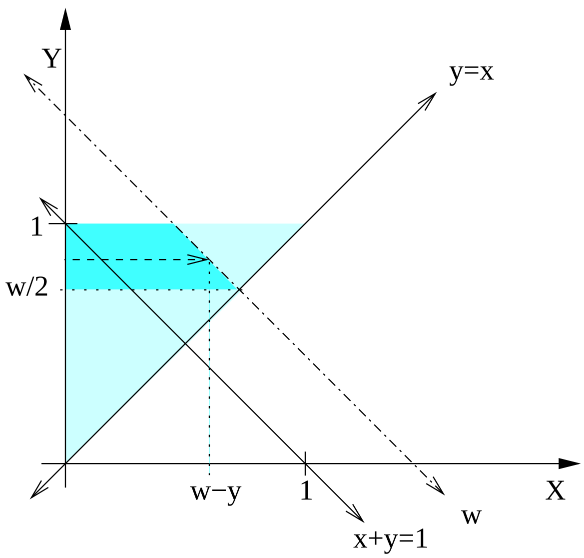
Here is the integration.

$$
\begin{aligned}
F_W(w) &= \int_0^{\frac{w}{2}} \int_x^{w-x} 2\, dy\, dx && (2)\\
&= \int_0^{\frac{w}{2}} 2(w - x - x)\, dx && (3)\\
&= \int_0^{\frac{w}{2}} (2w - 4x)\, dx && (4)\\
&= 2wx - \frac{4x^2}{2}\Big|_0^{\frac{w}{2}} && (5)\\
&= w^2 - w^2/2 && (6)\\
&= \frac{w^2}{2} && (7)
\end{aligned}
$$

continued next page...



For values of w in the region $1 \leq w \leq 2$ we must compute the CDF as the sum of the integrals over two regions. The first region is shown at left and the second below. The first region is triangular but this time we integrate first with respect to X and then with respect to Y. We need to integrate with respect to X from 0 to y so that as Y goes from 0 to $w/2$, we cover the darker shaded triangular region. As before, the lightly shaded area represents the portion of the region in which $0 \leq w \leq 1$ and $f_{X,Y}(x, y) > 0$.



Next we consider the remainder of the region over which we must integrate to find the CDF of W. The extent of the remaining region in the X direction is limited by the value of w. The integral over this remaining region is captured in the second integral below, corresponding to the darkly shaded trapezoid at left. In the inner integration, the random variable X takes values from 0 to $w - y$ as indicated by the dashed line. Then in the outer integration, the random variable Y takes values from $w/2$ to 1.

Here's the integration.

$$
\begin{aligned}
F_W(w) &= \int_0^{\frac{w}{2}} \int_0^{y} 2\,dx\,dy + \int_{\frac{w}{2}}^{1} \int_0^{w-y} 2\,dx\,dy && (8)\\
&= \int_0^{\frac{w}{2}} 2y\,dy + \int_{\frac{w}{2}}^{1} 2(w-y)\,dy && (9)\\
&= \frac{w^2}{4} + \left(2wy - y^2\right)\Big|_{\frac{w}{2}}^{1} && (10)\\
&= \frac{w^2}{4} + 2w - 1 - \left(w^2 - \frac{w^2}{4}\right) && (11)\\
&= 2w - 1 - \frac{w^2}{2} && (12)
\end{aligned}
$$

Calculating the CDF of W over the various ranges of values w was the first step. The second step is to assemble the parts of the CDF $F_W(w)$ calculated above, and, by taking the derivative, calculate the PDF $f_W(w)$.

$$F_W(w) = \begin{cases} 0 & w < 0 \\ \frac{w^2}{2} & 0 \leq w \leq 1 \\ 2w - 1 - \frac{w^2}{2} & 1 \leq w \leq 2 \\ 1 & w > 2 \end{cases} \qquad f_W(w) = \begin{cases} w & 0 \leq w \leq 1 \\ 2 - w & 1 \leq w \leq 2 \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

Problem 6.2.4 ■

Random variables X and Y have joint PDF

$$f_{X,Y}(x, y) = \begin{cases} 8xy & 0 \leq y \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

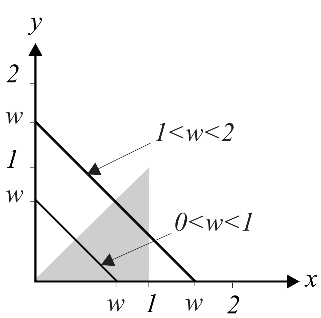
What is the PDF of $W = X + Y$?

Problem 6.2.4 Solution

In this problem, X and Y have joint PDF

$$f_{X,Y}(x, y) = \begin{cases} 8xy & 0 \leq y \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

We can find the PDF of W using Theorem 6.4: $f_W(w) = \int_{-\infty}^{\infty} f_{X,Y}(x, w - x)\, dx$. The only tricky part remaining is to determine the limits of the integration. First, for $w < 0$, $f_W(w) = 0$. The two remaining cases are shown in the accompanying figure. The shaded area shows where the joint PDF $f_{X,Y}(x, y)$ is nonzero. The diagonal lines depict $y = w - x$ as a function of x. The intersection of the diagonal line and the shaded area define our limits of integration.



For $0 \leq w \leq 1$,

$$f_W(w) = \int_{w/2}^{w} 8x(w - x)\, dx \tag{2}$$

$$= 4wx^2 - 8x^3/3\Big|_{w/2}^{w} = 2w^3/3 \tag{3}$$

For $1 \leq w \leq 2$,

$$f_W(w) = \int_{w/2}^{1} 8x(w - x)\, dx \tag{4}$$

$$= 4wx^2 - 8x^3/3\Big|_{w/2}^{1} \tag{5}$$

$$= 4w - 8/3 - 2w^3/3 \tag{6}$$

Since $X + Y \leq 2$, $f_W(w) = 0$ for $w > 2$. Hence the complete expression for the PDF of W is

$$f_W(w) = \begin{cases} 2w^3/3 & 0 \leq w \leq 1 \\ 4w - 8/3 - 2w^3/3 & 1 \leq w \leq 2 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Problem 6.3.1 •

For a constant $a > 0$, a Laplace random variable X has PDF

$$f_X(x) = \frac{a}{2} e^{-a|x|}, \quad -\infty < x < \infty.$$

Calculate the moment generating function $\phi_X(s)$.

Problem 6.3.1 Solution

For a constant $a > 0$, a zero mean Laplace random variable X has PDF

$$f_X(x) = \frac{a}{2} e^{-a|x|} \quad -\infty < x < \infty \tag{1}$$

The moment generating function of X is

$$\begin{aligned}
\phi_X(s) = E\left[e^{sX}\right] &= \frac{a}{2}\int_{-\infty}^{0} e^{sx} e^{ax}\,dx + \frac{a}{2}\int_{0}^{\infty} e^{sx} e^{-ax}\,dx && (2)\\
&= \frac{a}{2}\frac{e^{(s+a)x}}{s+a}\bigg|_{-\infty}^{0} + \frac{a}{2}\frac{e^{(s-a)x}}{s-a}\bigg|_{0}^{\infty} && (3)\\
&= \frac{a}{2}\left(\frac{1}{s+a} - \frac{1}{s-a}\right) && (4)\\
&= \frac{a^2}{a^2 - s^2} && (5)
\end{aligned}$$

Problem 6.3.3 ■

Continuous random variable X has a uniform distribution over $[a, b]$. Find the MGF $\phi_X(s)$. Use the MGF to calculate the first and second moments of X.

Problem 6.3.3 Solution

We find the MGF by calculating $E[e^{sX}]$ from the PDF $f_X(x)$.

$$\phi_X(s) = E\left[e^{sX}\right] = \int_a^b e^{sX}\frac{1}{b-a}\,dx = \frac{e^{bs} - e^{as}}{s(b-a)} \tag{1}$$

Now to find the first moment, we evaluate the derivative of $\phi_X(s)$ at $s = 0$.

$$E[X] = \frac{d\phi_X(s)}{ds}\bigg|_{s=0} = \frac{s\left[be^{bs} - ae^{as}\right] - \left[e^{bs} - e^{as}\right]}{(b-a)s^2}\Bigg|_{s=0} \tag{2}$$

Direct evaluation of the above expression at $s = 0$ yields $0/0$ so we must apply l'Hôpital's rule and differentiate the numerator and denominator.

$$\begin{aligned}
E[X] &= \lim_{s\to 0} \frac{be^{bs} - ae^{as} + s\left[b^2e^{bs} - a^2e^{as}\right] - \left[be^{bs} - ae^{as}\right]}{2(b-a)s} && (3)\\
&= \lim_{s\to 0} \frac{b^2e^{bs} - a^2e^{as}}{2(b-a)} = \frac{b+a}{2} && (4)
\end{aligned}$$

To find the second moment of X, we first find that the second derivative of $\phi_X(s)$ is

$$\frac{d^2\phi_X(s)}{ds^2} = \frac{s^2\left[b^2e^{bs} - a^2e^{as}\right] - 2s\left[be^{bs} - ae^{as}\right] + 2\left[be^{bs} - ae^{as}\right]}{(b-a)s^3} \tag{5}$$

Substituting $s = 0$ will yield $0/0$ so once again we apply l'Hôpital's rule and differentiate the numerator and denominator.

$$E\left[X^2\right] = \lim_{s\to 0} \frac{d^2\phi_X(s)}{ds^2} = \lim_{s\to 0} \frac{s^2\left[b^3e^{bs} - a^3e^{as}\right]}{3(b-a)s^2} \tag{6}$$

$$= \frac{b^3 - a^3}{3(b-a)} = (b^2 + ab + a^2)/3 \tag{7}$$

In this case, it is probably simpler to find these moments without using the MGF.

Problem 6.4.1 •

N is a binomial $(n = 100, p = 0.4)$ random variable. M is a binomial $(n = 50, p = 0.4)$ random variable. Given that M and N are independent, what is the PMF of $L = M + N$?

Problem 6.4.1 Solution

N is a binomial $(n = 100, p = 0.4)$ random variable. M is a binomial $(n = 50, p = 0.4)$ random variable. Thus N is the sum of 100 independent Bernoulli $(p = 0.4)$ and M is the sum of 50 independent Bernoulli $(p = 0.4)$ random variables. Since M and N are independent, $L = M + N$ is the sum of 150 independent Bernoulli $(p = 0.4)$ random variables. Hence L is a binomial $n = 150, p = 0.4)$ and has PMF

$$P_L(l) = \binom{150}{l}(0.4)^l(0.6)^{150-l}. \tag{1}$$

Problem 6.4.2 •

Random variable Y has the moment generating function $\phi_Y(s) = 1/(1-s)$. Random variable V has the moment generating function $\phi_V(s) = 1/(1-s)^4$. Y and V are independent. $W = Y + V$.

(a) What are $E[Y]$, $E[Y^2]$, and $E[Y^3]$?

(b) What is $E[W^2]$?

Problem 6.4.2 Solution

Random variable Y has the moment generating function $\phi_Y(s) = 1/(1-s)$. Random variable V has the moment generating function $\phi_V(s) = 1/(1-s)^4$. Y and V are independent. $W = Y + V$.

(a) From Table 6.1, Y is an exponential $(\lambda = 1)$ random variable. For an exponential (λ) random variable, Example 6.5 derives the moments of the exponential random variable. For $\lambda = 1$, the moments of Y are

$$E[Y] = 1, \qquad E\left[Y^2\right] = 2, \qquad E\left[Y^3\right] = 3! = 6. \tag{1}$$

(b) Since Y and V are independent, $W = Y + V$ has MGF

$$\phi_W(s) = \phi_Y(s)\phi_V(s) = \left(\frac{1}{1-s}\right)\left(\frac{1}{1-s}\right)^4 = \left(\frac{1}{1-s}\right)^5. \tag{2}$$

W is the sum of five independent exponential ($\lambda = 1$) random variables $X_1, \ldots, X_5$. (That is, W is an Erlang ($n = 5, \lambda = 1$) random variable.) Each X_i has expected value $E[X] = 1$ and variance $\text{Var}[X] = 1$. From Theorem 6.1 and Theorem 6.3,

$$E[W] = 5E[X] = 5, \qquad \text{Var}[W] = 5\,\text{Var}[X] = 5. \tag{3}$$

It follows that

$$E\left[W^2\right] = \text{Var}[W] + (E[W])^2 = 5 + 25 = 30. \tag{4}$$

Problem 6.4.5 •

At time $t = 0$, you begin counting the arrivals of buses at a depot. The number of buses K_i that arrive between time $i-1$ minutes and time i minutes, has the Poisson PMF

$$P_{K_i}(k) = \begin{cases} 2^k e^{-2}/k! & k = 0, 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

and $K_1, K_2, \ldots$ are an iid random sequence. Let $R_i = K_1 + K_2 + \cdots + K_i$ denote the number of buses arriving in the first i minutes.

(a) What is the moment generating function $\phi_{K_i}(s)$?

(b) Find the MGF $\phi_{R_i}(s)$.

(c) Find the PMF $P_{R_i}(r)$. Hint: Compare $\phi_{R_i}(s)$ and $\phi_{K_i}(s)$.

(d) Find $E[R_i]$ and $\text{Var}[R_i]$.

Problem 6.4.5 Solution

$$P_{K_i}(k) = \begin{cases} 2^k e^{-2}/k! & k = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

And let $R_i = K_1 + K_2 + \ldots + K_i$

(a) From Table 6.1, we find that the Poisson ($\alpha = 2$) random variable K has MGF $\phi_K(s) = e^{2(e^s-1)}$.

(b) The MGF of R_i is the product of the MGFs of the K_i's.

$$\phi_{R_i}(s) = \prod_{n=1}^{i} \phi_K(s) = e^{2i(e^s-1)} \tag{2}$$

(c) Since the MGF of R_i is of the same form as that of the Poisson with parameter, $\alpha = 2i$. Therefore we can conclude that R_i is in fact a Poisson random variable with parameter $\alpha = 2i$. That is,

$$P_{R_i}(r) = \begin{cases} (2i)^r e^{-2i}/r! & r = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

(d) Because R_i is a Poisson random variable with parameter $\alpha = 2i$, the mean and variance of R_i are then both $2i$.

Problem 6.5.1 ■

Let $X_1, X_2, \ldots$ be a sequence of iid random variables each with exponential PDF

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $\phi_X(s)$.

(b) Let K be a geometric random variable with PMF

$$P_K(k) = \begin{cases} (1-q)q^{k-1} & k = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Find the MGF and PDF of $V = X_1 + \cdots + X_K$.

Problem 6.5.1 Solution

(a) From Table 6.1, we see that the exponential random variable X has MGF

$$\phi_X(s) = \frac{\lambda}{\lambda - s} \tag{1}$$

(b) Note that K is a geometric random variable identical to the geometric random variable X in Table 6.1 with parameter $p = 1 - q$. From Table 6.1, we know that random variable K has MGF

$$\phi_K(s) = \frac{(1-q)e^s}{1 - qe^s} \tag{2}$$

Since K is independent of each X_i, $V = X_1 + \cdots + X_K$ is a random sum of random variables. From Theorem 6.12,

$$\phi_V(s) = \phi_K(\ln \phi_X(s)) = \frac{(1-q)\frac{\lambda}{\lambda - s}}{1 - q\frac{\lambda}{\lambda - s}} = \frac{(1-q)\lambda}{(1-q)\lambda - s} \tag{3}$$

We see that the MGF of V is that of an exponential random variable with parameter $(1-q)\lambda$. The PDF of V is

$$f_V(v) = \begin{cases} (1-q)\lambda e^{-(1-q)\lambda v} & v \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Problem 6.6.1 •

The waiting time W for accessing one record from a computer database is a random variable uniformly distributed between 0 and 10 milliseconds. The read time R (for moving the information from the disk to main memory) is 3 milliseconds. The random variable X milliseconds is the total access time (waiting time + read time) to get one block of information from the disk. Before performing a certain task, the computer must access 12 different blocks of information from the disk. (Access times for different blocks are independent of one another.) The total access time for all the information is a random variable A milliseconds.

(a) What is $E[X]$, the expected value of the access time?

(b) What is $\text{Var}[X]$, the variance of the access time?

(c) What is $E[A]$, the expected value of the total access time?

(d) What is σ_A, the standard deviation of the total access time?

(e) Use the central limit theorem to estimate $P[A > 116\text{ms}]$, the probability that the total access time exceeds 116 ms.

(f) Use the central limit theorem to estimate $P[A < 86\text{ms}]$, the probability that the total access time is less than 86 ms.

Problem 6.6.1 Solution

We know that the waiting time, W is uniformly distributed on [0,10] and therefore has the following PDF.

$$f_W(w) = \begin{cases} 1/10 & 0 \le w \le 10 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

We also know that the total time is 3 milliseconds plus the waiting time, that is $X = W + 3$, *and that the random variable A represents the total access time for accessing 12 blocks of information, also that the access times for different blocks are independent, in other words that the* X_n *for consecutive blocks are iid.*

(a) *By Theorem 3.5b,* the expected value of X is $E[X] = E[W + 3] = E[W] + 3 = 5 + 3 = 8$.

(b) *By Theorem 3.5d,* the variance of X is $\text{Var}[X] = \text{Var}[W + 3] = \text{Var}[W] = 25/3$.

(c) $A = \sum_{n=1}^{12} X_n$ *and the* X_n *are iid so* $\text{Var}[X_n] = \text{Var}[X]$ *for all* n*, and by Theorem 6.1,* the expected value of A is $E[A] = 12E[X] = 96$.

(d) *Independence implies, by Theorem 4.27c uncorrelatedness, then by Theorem 6.3, the variance of A is the sum of the variances of the* X_i*. Thus* the standard deviation of A is $\sigma_A = \sqrt{\text{Var}[A]} = \sqrt{12(25/3)} = 10$.

(e) $P[A > 116] = 1 - \Phi(\frac{116-96}{10}) = 1 - \Phi(2) = 0.02275$.

(f) $P[A < 86] = \Phi(\frac{86-96}{10}) = \Phi(-1) = 1 - \Phi(1) = 0.1587$

Problem 6.6.2 •

Telephone calls can be classified as voice (V) if someone is speaking, or data (D) if there is a modem or fax transmission. Based on a lot of observations taken by the telephone company, we have the following probability model: $P[V] = 0.8$, $P[D] = 0.2$. Data calls and voice calls occur independently of one another. The random variable K_n is the number of data calls in a collection of n phone calls.

(a) What is $E[K_{100}]$, the expected number of voice calls in a set of 100 calls?

(b) What is $\sigma_{K_{100}}$, the standard deviation of the number of voice calls in a set of 100 calls?

(c) Use the central limit theorem to estimate $P[K_{100} \geq 18]$, the probability of at least 18 voice calls in a set of 100 calls.

(d) Use the central limit theorem to estimate $P[16 \leq K_{100} \leq 24]$, the probability of between 16 and 24 voice calls in a set of 100 calls.

Problem 6.6.2 Solution

In the preamble to the problem statement, K_n is defined to be the number of data calls in a collection of n phone calls. In (a) K_{100} is defined as the number of voice calls in a set of 100 phone calls. Obviously these are inconsistent. The textbook authors give the solution for the data option. I will give the solution below for the voice option. I will use L_{100} for the number of voice calls in 100 phone calls.

Knowing that the probability that voice call occurs is 0.8 and the probability that a data call occurs is 0.2 we can define the random variable D_i as the number of data calls in a single telephone call. It is obvious that for any i there are only two possible values for D_i, namely 0 and 1. Furthermore for all i the D_i's are independent and identically distributed with the following PMF.

$$P_D(d) = \begin{cases} 0.8 & d = 0 \\ 0.2 & d = 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

From the above we can determine that

$$E[D] = 0.2 \qquad \text{Var}[D] = 0.2 - 0.04 = 0.16 \tag{2}$$

With these facts, we can answer the questions posed by the problem.

(a) $E[K_{100}] = 100E[D] = 20$

(b) $\text{Var}[K_{100}] = \sqrt{100\,\text{Var}[D]} = \sqrt{16} = 4$

(c) $P[K_{100} \geq 18] = 1 - \Phi\left(\frac{18-20}{4}\right) = 1 - \Phi(-1/2) = \Phi(1/2) = 0.6915$

(d) $P[16 \leq K_{100} \leq 24] = \Phi(\frac{24-20}{4}) - \Phi(\frac{16-20}{4}) = \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 0.6826$

Knowing that the probability that voice call occurs is 0.8 and the probability that a data call occurs is 0.2 we can define the random variable V_i as the number of voice calls in a single telephone call. It is obvious that for any i there are only two possible values for V_i, namely 0 and 1. Furthermore for all i the V_i's are independent and identically distributed with the following PMF.

$$P_V(v) = \begin{cases} 0.8 & v = 0 \\ 0.2 & v = 1 \\ 0 & \textit{otherwise} \end{cases} \tag{3}$$

From the above we can determine that

$$E[V] = 0.8 \qquad \mathrm{Var}[D] = 0.8 - 0.64 = 0.16 \tag{4}$$

With these facts, we can answer the questions posed by the problem. Let L_n be the number of voice calls in n phone calls.

(a) $E[L_{100}] = 100E[V] = 80$

(b) $\mathrm{Var}[L_{100}] = \sqrt{100\,\mathrm{Var}[V]} = \sqrt{16} = 4$

(c) $P[L_{100} \geq 18] = 1 - \Phi\left(\frac{18-80}{4}\right) = 1 - \Phi(-31/2) = \Phi(31/2) \cong 1$ *(Our tables don't include* $\Phi(31/2)$*. We'd consult a book of mathematical or statistical tables if we needed a precise value.)*

(d) $P[16 \leq L_{100} \leq 24] = \Phi(\frac{24-80}{4}) - \Phi(\frac{16-80}{4}) = \Phi(-14) - \Phi(-16) = (1 - \Phi(14)) - (1 - \Phi(16)) = \Phi(16) - \Phi(14) =$ *a very small number. (Again, our tables don't include such values. We'd consult a book of mathematical or statistical tables if we needed a precise value.)*

Solutions to HW10

Note: These solutions are based on those generated by R. D. Yates and D. J. Goodman, the authors of our textbook. I have extensively rewritten them.

Problem 7.1.1 •

$X_1, \ldots, X_n$ is an iid sequence of exponential random variables, each with expected value 5.

(a) What is $\text{Var}[M_9(X)]$, the variance of the sample mean based on nine trials?

(b) What is $P[X_1 > 7]$, the probability that one outcome exceeds 7?

(c) Estimate $P[M_9(X) > 7]$, the probability that the sample mean of nine trials exceeds 7? Hint: Use the central limit theorem.

Problem 7.1.1 Solution

We are given that $X_1, X_2 \ldots X_n$ are independent exponential random variables with mean value $\mu_X = 5 = 1/\lambda$ so that for $x \geq 0$, $F_X(x) = 1 - e^{-\lambda x} = 1 - e^{-x/5}$ and $\sigma_X^2 = 1/\lambda^2 = 25$.

(a) By Theorem 7.1, $\sigma^2_{M_n(x)} = \sigma_X^2/n$, so

$$\text{Var}[M_9(X)] = \frac{\sigma_X^2}{9} = \frac{25}{9}. \tag{1}$$

(b) A comment is in order here. The question asks "What is the value of $P[X_1 > 7]$ the probability that one outcome exceeds 7". The probability that X_1 exceeds 7 is not, in general, the same as the probability that some X_i exceeds 7, however in this problem, the X_i are iid, so these two quantities are equal.

$$\begin{aligned} P[X_1 \geq 7] &= 1 - P[X_1 \leq 7] & (2)\\ &= 1 - F_X(7) = 1 - (1 - e^{-7/5}) = e^{-7/5} \approx 0.247 & (3) \end{aligned}$$

(c) First we express $P[M_9(X) > 7]$ in terms of $X_1, \ldots, X_9$.

$$P[M_9(X) > 7] = 1 - P[M_9(X) \leq 7] = 1 - P[(X_1 + \ldots + X_9) \leq 63] \tag{4}$$

Now the probability that $M_9(X) > 7$ can be approximated using the Central Limit Theorem (CLT).

$$\begin{aligned} P[M_9(X) > 7] &= 1 - P[(X_1 + \ldots + X_9) \leq 63] & (5)\\ &\approx 1 - \Phi\left(\frac{63 - 9\mu_X}{\sqrt{9}\sigma_X}\right) = 1 - \Phi(6/5) & (6) \end{aligned}$$

Consulting Table 3.1 to obtain a value for $\Phi(6/5)$ and substituting into the expression above yields $P[M_9(X) > 7] \approx 0.1151$.

Problem 7.1.2 •

$X_1, \ldots, X_n$ are independent uniform random variables, all with expected value $\mu_X = 7$ and variance $\mathrm{Var}[X] = 3$.

(a) What is the PDF of X_1?

(b) What is $\mathrm{Var}[M_{16}(X)]$, the variance of the sample mean based on 16 trials?

(c) What is $P[X_1 > 9]$, the probability that one outcome exceeds 9?

(d) Would you expect $P[M_{16}(X) > 9]$ to be bigger or smaller than $P[X_1 > 9]$? To check your intuition, use the central limit theorem to estimate $P[M_{16}(X) > 9]$.

Problem 7.1.2 Solution

$X_1, X_2 \ldots X_n$ are independent uniform random variables with mean value $\mu_X = 7$ and $\sigma_X^2 = 3$

(a) Since X_1 is a uniform random variable, it must have a uniform PDF over an interval $[a, b]$. From Appendix A, we have that for a uniform random variable on the interval $[a, b]$, the mean and variance are $\mu_X = (a + b)/2$ and that $\mathrm{Var}[X] = (b - a)^2/12$. Hence, given the mean and variance, we obtain the following equations for a and b.

$$(b - a)^2/12 = 3 \qquad (a + b)/2 = 7 \tag{1}$$

Solving the first of these equations yields $|b - a| = 6$. For a nonempty interval, b must be greater than a so we have that $b = a + 6$. Then $(2a + 6)/2 = 7$ implies that $a = 4$ and thus $b = 10$ so the distribution of X is

$$f_X(x) = \begin{cases} 1/6 & 4 \le x \le 10 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

(b) By Theorem 7.1, we have

$$\mathrm{Var}[M_{16}(X)] = \frac{\mathrm{Var}[X]}{16} = \frac{3}{16} \tag{3}$$

(c) Since $f_X(x) = 0 \; \forall \, x > 10$,

$$P[X_1 \ge 9] = \int_9^\infty f_{X_1}(x)\, dx = \int_9^{10} (1/6)\, dx = 1/6 \tag{4}$$

(d) The variance of $M_{16}(X)$ is much less than $\mathrm{Var}[X_1]$. Hence, the PDF of $M_{16}(X)$ should be much more concentrated about $E[X]$ than the PDF of X_1. Thus we should expect $P[M_{16}(X) > 9]$ to be much less than $P[X_1 > 9]$.

$$P[M_{16}(X) > 9] = 1 - P[M_{16}(X) \le 9] = 1 - P[(X_1 + \cdots + X_{16}) \le 16(9)] \tag{5}$$

Applying the Central Limit Theorem to obtain an approximation of this probability yields

$$P[M_{16}(X) > 9] \approx 1 - \Phi\left(\frac{144 - 16\mu_X}{\sqrt{16}\sigma_X}\right) \approx 1 - \Phi(2.67) \approx 1 - 0.9962 \approx 0.0038 \quad (6)$$

As predicted, $P[M_{16}(X) > 9] \ll P[X_1 > 9]$.

Problem 7.2.1 •

The weight of a randomly chosen Maine black bear has expected value $E[W] = 500$ pounds and standard deviation $\sigma_W = 100$ pounds. Use the Chebyshev inequality to upper bound the probability that the weight of a randomly chosen bear is more than 200 pounds from the expected value of the weight.

Problem 7.2.1 Solution

If the average weight of a Maine black bear is 500 pounds with standard deviation equal to 100 pounds, we can use the Chebyshev inequality to upper bound the probability that a randomly chosen bear will be more then 200 pounds away from the average.

$$P[|W - E[W]| \geq 200] \leq \frac{\text{Var}[W]}{200^2} \leq \frac{100^2}{200^2} = 0.25 \quad (1)$$

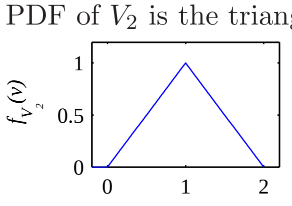
Problem 7.2.3 ■

Let X equal the arrival time of the third elevator in Quiz 7.2. Find the exact value of $P[W \geq 75]$. Compare your answer to the upper bounds derived in Quiz 7.2.

Problem 7.2.3 Solution

First we derive the PDF of the sum $W = X_1 + X_2 + X_3$ of iid uniform $(0, 30)$ random variables, using the techniques of Chapter 6. To simplify our calculations, we find the PDF of $V = Y_1 + Y_2 + Y_3$ where the Y_i are iid uniform $(0, 1)$ random variables, then apply Theorem 3.20 to conclude that $W = 30V$ represents the sum of three iid uniform $(0, 30)$ random variables.

To start, let $V_2 = Y_1 + Y_2$. Since each Y_1 has a PDF shaped like a unit area pulse, the PDF of V_2 is the triangular function



$$f_{V_2}(v) = \begin{cases} v & 0 \leq v \leq 1 \\ 2 - v & 1 < v \leq 2 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

Then the PDF of $V = V_2 + Y_3$ is the convolution integral

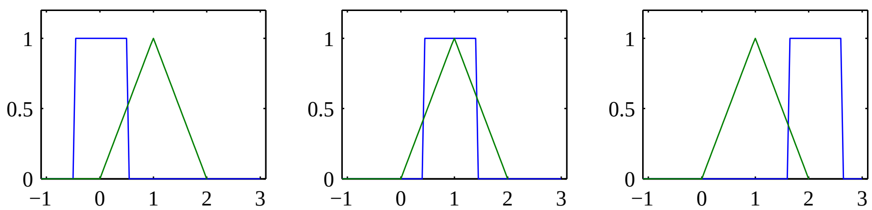
$$f_V(v) = \int_{-\infty}^{\infty} f_{V_2}(y) f_{Y_3}(v - y)\, dy \quad (2)$$

$$= \int_0^1 y f_{Y_3}(v - y)\, dy + \int_1^2 (2 - y) f_{Y_3}(v - y)\, dy. \quad (3)$$

Evaluation of these integrals depends on v through the function

$$f_{Y_3}(v-y) = \begin{cases} 1 & v-1 < v < 1 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

To compute the convolution, it is helpful to depict the three distinct cases. In each case, the square "pulse" is $f_{Y_3}(v-y)$ and the triangular pulse is $f_{V_2}(y)$.



$0 \le v < 1$ $1 \le v < 2$ $2 \le v < 3$

From the graphs, we can compute the convolution for each case:

$$0 \le v < 1: \qquad f_{V_3}(v) = \int_0^v y\,dy = \frac{1}{2}v^2 \tag{5}$$

$$1 \le v < 2: \qquad f_{V_3}(v) = \int_{v-1}^1 y\,dy + \int_1^v (2-y)\,dy = -\frac{v^2}{2} + 3v - 2 \tag{6}$$

$$2 \le v < 3: \qquad f_{V_3}(v) = \int_{v-1}^2 (2-y)\,dy = \frac{(3-v)^2}{2} \tag{7}$$

To complete the problem, we use Theorem 3.20 to observe that $W = 30V_3$ is the sum of three iid uniform $(0,30)$ random variables. From Theorem 3.19,

$$f_W(w) = \frac{1}{30} f_{V_3}(v_3)\, v/30 = \begin{cases} (w/30)^2/60 & 0 \le w < 30, \\ [-(w/30)^2/2 + 3(w/30) - 2]/30 & 30 \le w < 60, \\ [3-(w/30)]^2/60 & 60 \le w < 90, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Finally, we can compute the exact probability

$$P[W \ge 75] = \frac{1}{60}\int_{75}^{90} [3-(w/30)]^2\,dw = -\left.\frac{(3-w/30)^3}{6}\right|_{75}^{90} = \frac{1}{48} \tag{9}$$

For comparison, the Markov inequality indicated that $P[W < 75] \le 3/5$ and the Chebyshev inequality showed that $P[W < 75] \le 1/4$. As we see, both inequalities are quite weak in this case.

Problem 7.3.1 •

When X is Gaussian, verify the claim of Equation (7.16) that the sample mean is within one standard error of the expected value with probability 0.68.

Problem 7.3.1 Solution

For an an arbitrary Gaussian (μ, σ) random variable Y,

$$\begin{aligned} P[\mu - \sigma \leq Y \leq \mu + \sigma] &= P[-\sigma \leq Y - \mu \leq \sigma] && (1) \\ &= P\left[-1 \leq \frac{Y-\mu}{\sigma} \leq 1\right] && (2) \\ &= \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 0.6827. && (3) \end{aligned}$$

Note that Y can be any Gaussian random variable, including, for example, $M_n(X)$ when X is Gaussian. When X is not Gaussian, the same claim holds to the extent that the central limit theorem promises that $M_n(X)$ is nearly Gaussian for large n.

Problem 7.4.1 •

$X_1, \ldots, X_n$ are n independent identically distributed samples of random variable X with PMF

$$P_X(x) = \begin{cases} 0.1 & x = 0, \\ 0.9 & x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) How is $E[X]$ related to $P_X(1)$?

(b) Use Chebyshev's inequality to find the confidence level α such that $M_{90}(X)$, the estimate based on 90 observations, is within 0.05 of $P_X(1)$. In other words, find α such that

$$P[|M_{90}(X) - P_X(1)| \geq 0.05] \leq \alpha.$$

(c) Use Chebyshev's inequality to find out how many samples n are necessary to have $M_n(X)$ within 0.03 of $P_X(1)$ with confidence level 0.1. In other words, find n such that

$$P[|M_n(X) - P_X(1)| \geq 0.03] \leq 0.1.$$

Problem 7.4.1 Solution

We are given that $X_1, \ldots, X_n$ are n independent identically distributed samples of the random variable X having PMF

$$P_X(x) = \begin{cases} 0.1 & x = 0 \\ 0.9 & x = 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) $E[X]$ is in fact the same as $P_X(1)$ because X is a Bernoulli random variable.

(b) By Chebyshev's inequality,

$$P[|M_{90}(X) - P_X(1)| \geq .05] = P[|M_{90}(X) - E[X]| \geq .05] \leq \frac{\text{Var}[Y]}{(0.5)^2} = \alpha \tag{2}$$

so

$$\alpha = \frac{\sigma_X^2}{90(.05)^2} = \frac{.09}{90(.05)^2} = 0.4 \tag{3}$$

(c) Now we wish to find the value of n such that $P[|M_n(X) - P_X(1)| \geq .03] \leq .1$. From the Chebyshev inequality, we write

$$0.1 = \frac{\sigma_X^2}{n(.03)^2}. \tag{4}$$

Since $\sigma_X^2 = 0.09$, solving for n yields $n = 100$.

Problem 7.4.2 •

Let $X_1, X_2, \ldots$ denote an iid sequence of random variables, each with expected value 75 and standard deviation 15.

(a) How many samples n do we need to guarantee that the sample mean $M_n(X)$ is between 74 and 76 with probability 0.99?

(b) If each X_i has a Gaussian distribution, how many samples n' would we need to guarantee $M_{n'}(X)$ is between 74 and 76 with probability 0.99?

Problem 7.4.2 Solution

$X_1, X_2, \ldots$ are iid random variables each with mean 75 and standard deviation 15.

(a) We would like to find the value of n such that

$$P[74 \leq M_n(X) \leq 76] = 0.99 \tag{1}$$

When we know only the mean and variance of X_i, our only real tool is the Chebyshev inequality which says that

$$\begin{aligned} P[74 \leq M_n(X) \leq 76] &= 1 - P[|M_n(X) - E[X]| \geq 1] &(2)\\ &\geq 1 - \frac{\operatorname{Var}[X]}{n} = 1 - \frac{225}{n} \geq 0.99 &(3) \end{aligned}$$

This yields $n \geq 22{,}500$.

(b) If each X_i is a Gaussian, the sample mean, $M_n(X)$ will also be Gaussian with mean and variance

$$\begin{aligned} E[M_{n'}(X)] &= E[X] = 75 &(4)\\ \operatorname{Var}[M_{n'}(X)] &= \operatorname{Var}[X]/n' = 225/n' &(5) \end{aligned}$$

In this case,

$$\begin{aligned} P[74 \leq M_{n'}(X) \leq 76] &= \Phi\left(\frac{76-\mu}{\sigma}\right) - \Phi\left(\frac{74-\mu}{\sigma}\right) &(6)\\ &= \Phi(\sqrt{n'}/15) - \Phi(-\sqrt{n'}/15) &(7)\\ &= 2\Phi(\sqrt{n'}/15) - 1 = 0.99 &(8) \end{aligned}$$

so $\Phi(\sqrt{n'}/15) = 1.99/2 = .995$. Then from the table, $\sqrt{n'}/15 \approx 2.58$ so $n' \approx 1{,}498$.

Since even under the Gaussian assumption, the number of samples n' is so large that even if the X_i are not Gaussian, the sample mean may be approximated by a Gaussian. Hence, about 1500 samples probably is about right. However, in the absence of any information about the PDF of X_i beyond the mean and variance, we cannot make any guarantees stronger than that given by the Chebyshev inequality.

Problem 7.4.3 •

Let X_A be the indicator random variable for event A with probability $P[A] = 0.8$. Let $\hat{P}_n(A)$ denote the relative frequency of event A in n independent trials.

(a) Find $E[X_A]$ and $\mathrm{Var}[X_A]$.

(b) What is $\mathrm{Var}[\hat{P}_n(A)]$?

(c) Use the Chebyshev inequality to find the confidence coefficient $1-\alpha$ such that $\hat{P}_{100}(A)$ is within 0.1 of $P[A]$. In other words, find α such that

$$P\left[\left|\hat{P}_{100}(A) - P[A]\right| \leq 0.1\right] \geq 1 - \alpha.$$

(d) Use the Chebyshev inequality to find out how many samples n are necessary to have $\hat{P}_n(A)$ within 0.1 of $P[A]$ with confidence coefficient 0.95. In other words, find n such that

$$P\left[\left|\hat{P}_n(A) - P[A]\right| \leq 0.1\right] \geq 0.95.$$

Problem 7.4.3 Solution

(a) Since X_A is a Bernoulli ($p = P[A]$) random variable,

$$E[X_A] = P[A] = 0.8, \qquad \mathrm{Var}[X_A] = P[A](1 - P[A]) = 0.16. \tag{1}$$

(b) Let $X_{A,i}$ denote X_A for the ith trial. Since $\hat{P}_n(A) = M_n(X_A) = \frac{1}{n}\sum_{i=1}^{n} X_{A,i}$,

$$\mathrm{Var}[\hat{P}_n(A)] = \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{Var}[X_{A,i}] = \frac{P[A](1 - P[A])}{n}. \tag{2}$$

(c) Since $\hat{P}_{100}(A) = M_{100}(X_A)$, we can use Theorem 7.12(b) to write

$$P\left[\left|\hat{P}_{100}(A) - P[A]\right| < c\right] \geq 1 - \frac{\mathrm{Var}[X_A]}{100c^2} = 1 - \frac{0.16}{100c^2} = 1 - \alpha. \tag{3}$$

For $c = 0.1$, $\alpha = 0.16/[100(0.1)^2] = 0.16$. Thus, with 100 samples, our confidence coefficient is $1 - \alpha = 0.84$.

(d) In this case, the number of samples n is unknown. Once again, we use Theorem 7.12(b) to write

$$P\left[\left|\hat{P}_n(A) - P[A]\right| < c\right] \geq 1 - \frac{\mathrm{Var}[X_A]}{nc^2} = 1 - \frac{0.16}{nc^2} = 1 - \alpha. \tag{4}$$

For $c = 0.1$, we have confidence coefficient $1 - \alpha = 0.95$ if $\alpha = 0.16/[n(0.1)^2] = 0.05$, or $n = 320$.

Problem 7.4.5 •

In n independent experimental trials, the relative frequency of event A is $\hat{P}_n(A)$. How large should n be to ensure that the confidence interval estimate

$$\hat{P}_n(A) - 0.05 \leq P[A] \leq \hat{P}_n(A) + 0.05$$

has confidence coefficient 0.9?

Problem 7.4.5 Solution

First we observe that the interval estimate can be expressed as

$$\left|\hat{P}_n(A) - P[A]\right| < 0.05. \quad (1)$$

Since $\hat{P}_n(A) = M_n(X_A)$ and $E[M_n(X_A)] = P[A]$, we can use Theorem 7.12(b) to write

$$P\left[\left|\hat{P}_n(A) - P[A]\right| < 0.05\right] \geq 1 - \frac{\text{Var}[X_A]}{n(0.05)^2}. \quad (2)$$

Note that $\text{Var}[X_A] = P[A](1 - P[A]) \leq \max_{x\in(0,1)} x(1-x) = 0.25$. Thus for confidence coefficient 0.9, we require that

$$1 - \frac{\text{Var}[X_A]}{n(0.05)^2} \geq 1 - \frac{0.25}{n(0.05)^2} \geq 0.9. \quad (3)$$

This implies $n \geq 1{,}000$ samples are needed.

Problem 8.1.1 •

Let L equal the number of flips of a coin up to and including the first flip of heads. Devise a significance test for L at level $\alpha = 0.05$ to test the hypothesis H that the coin is fair. What are the limitations of the test?

Problem 8.1.1 Solution

To test the hypothesis H that the coin is fair. Then, we must choose a rejection region R such that, given that H is true, the probability that the outcome s is in the rejection region R is0.05, *i.e* $\alpha = P[s \in R|H] = 0.05$. Our outcome in this experiment is the value of the random variable L. We will define the rejection region by picking a threshold l^* such that rejection region $R = \{l > l^*\}$. What remains is to choose l^* so that $P[L > l^*|H] = 0.05$. Note that $L > l$ if we have observed l tails in a row before observing the first heads. Under the hypothesis that the coin is fair, l tails in a row occurs with probability

$$P[L > l] = (1/2)^l \quad (1)$$

Thus, we need

$$P[R] = P[L > l^*] = 2^{-l^*} = 0.05 \quad (2)$$

Thus, $l^* = -\log_2(0.05) = \log_2(20) = 4.32$. In this case, we reject the hypothesis that the coin is fair if $L \geq 5$. The significance level of the test is $\alpha = P[L > 4] = 2^{-4} = 0.0625$ which close to but not exactly 0.05.

The shortcoming of this test is that we always accept the hypothesis that the coin is fair whenever heads occurs on the first, second, third or fourth flip. If the coin was biased such that the probability of heads was much higher than 1/2, say 0.8 or 0.9, we would hardly ever reject the hypothesis that the coin is fair. In that sense, our test cannot identify that kind of biased coin. This means that this particular test is only suited to the case that we know that the coin is fair or has a higher probability of tails than heads. (If we wanted to design a test that considered deviations in either direction from fairness, we'd want to have both lower and upper threshold values for our rejection region. However, we'd also want to change the structure of our experiment so that we observed at least a minimum number of flips. You can calculate the probabilities of error for different approaches to see why.)

Problem 8.1.4 •

The duration of a voice telephone call is an exponential random variable T with expected value $E[T] = 3$ minutes. Data calls tend to be longer than voice calls on average. Observe a call and reject the null hypothesis that the call is a voice call if the duration of the call is greater than t_0 minutes.

(a) Write a formula for α, the significance of the test as a function of t_0.

(b) What is the value of t_0 that produces a significance level $\alpha = 0.05$?

Problem 8.1.4 Solution

(a) The rejection region is $R = \{T > t_0\}$. The duration of a voice call has exponential PDF

$$f_T(t) = \begin{cases} (1/3)e^{-t/3} & t \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The significance level of the test is

$$\alpha = P[T > t_0] = \int_{t_0}^{\infty} f_T(t)\, dt = e^{-t_0/3}. \tag{2}$$

(b) The significance level is $\alpha = 0.05$ if $t_0 = -3\ln\alpha = 8.99$ minutes.

Problem 8.2.1 •

In a random hour, the number of call attempts N at a telephone switch has a Poisson distribution with a mean of either α_0 (hypothesis H_0) or α_1 (hypothesis H_1). For a priori probabilities $P[H_i]$, find the MAP and ML hypothesis testing rules given the observation of N.

Problem 8.2.1 Solution

For the MAP test, we must choose acceptance regions A_0 and A_1 for the two hypotheses H_0 and H_1. From Theorem 8.2, the MAP rule is

$$n \in A_0 \text{ if } \frac{P_{N|H_0}(n)}{P_{N|H_1}(n)} \geq \frac{P[H_1]}{P[H_0]}; \qquad n \in A_1 \text{ otherwise.} \tag{1}$$

Since $P_{N|H_i}(n) = \lambda_i^n e^{-\lambda_i}/n!$, where $\lambda_i = 1/\alpha_i$, the MAP rule becomes

$$n \in A_0 \text{ if } \left(\frac{\lambda_0}{\lambda_1}\right)^n e^{-(\lambda_0-\lambda_1)} \geq \frac{P[H_1]}{P[H_0]}; \qquad n \in A_1 \text{ otherwise.} \tag{2}$$

We obtain the threshold n^* by substituting n^* for n in (2) and isolating n^*. Taking logarithms we obtain

$$n^*(\ln\lambda_0 - \ln\lambda_1) - (\lambda_0 - \lambda_1) \geq \ln(P[H_1]/P[H_2]). \tag{3}$$

Rearranging yields

$$n \geq \frac{\ln(P[H_1]/P[H_2]) + \lambda_0 - \lambda_1}{\ln\lambda_0 - \ln\lambda_1}. \tag{4}$$

Now, in order to determine whether n^* should be a lower bound or an upper bound for our rejection region, we need to know which is larger, α_0 or α_1. Suppose that $\alpha_0 > \alpha_1$. Then $\lambda_0 < \lambda_1$ and we state the MAP rule as

$$n \in A_0 \text{ if } n \leq n^* = \frac{\lambda_0 - \lambda_1 + \ln(P[H_0]/P[H_1])}{\ln(\lambda_0/\lambda_1)}; \qquad n \in A_1 \text{ otherwise.} \tag{5}$$

From the MAP rule, we can get the ML rule by setting the a priori probabilities to be equal. This yields the ML rule

$$n \in A_0 \text{ if } n \leq n^* = \frac{\lambda_0 - \lambda_1}{\ln(\lambda_0/\lambda_1)}; \qquad n \in A_1 \text{ otherwise.} \tag{6}$$

Solutions to HW11

Note: These solutions are D. J. Goodman, the authors of our textbook. I have annotated and corrected them as necessary. Text in italics is mine.

Problem 10.2.1 •

For the random processes of Examples 10.3, 10.4, 10.5, and 10.6, identify whether the process is discrete-time or continuous-time, discrete-value or continuous-value.

Problem 10.2.1 Solution

- In Example 10.3, the daily noontime temperature at Newark Airport is a discrete time, continuous value random process. However, if the temperature is recorded only in units of one degree, then the process was would be discrete value.
- In Example 10.4, the number of active telephone calls is discrete time and discrete value.
- The dice rolling experiment of Example 10.5 yields a discrete time, discrete value random process.
- The QPSK system of Example 10.6 is a continuous time and continuous value random process.

Problem 10.2.2 ■

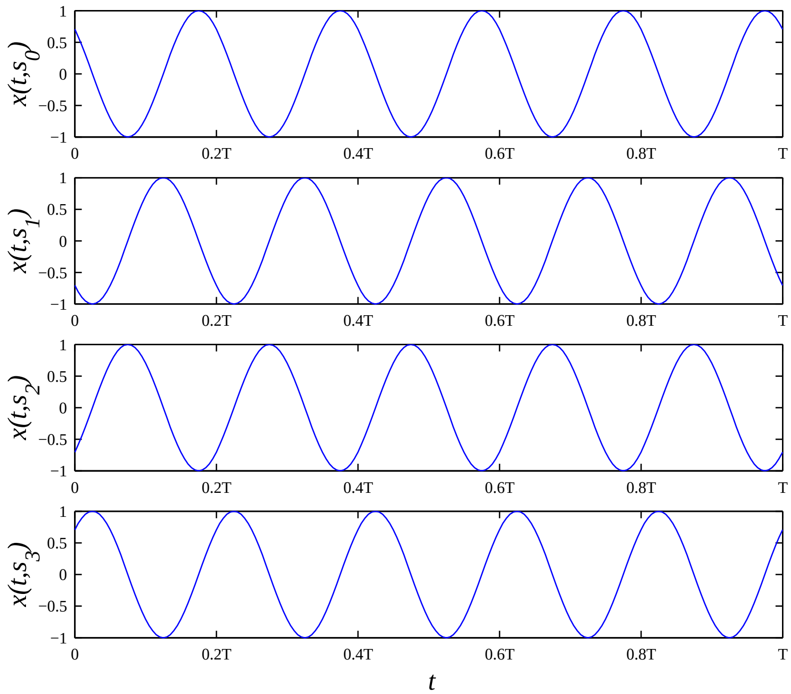
Let $Y(t)$ denote the random process corresponding to the transmission of one symbol over the QPSK communications system of Example 10.6. What is the sample space of the underlying experiment? Sketch the ensemble of sample functions.

Problem 10.2.2 Solution

The sample space of the underlying experiment is $S = \{s_0, s_1, s_2, s_3\}$. The four elements in the sample space are equally likely. The ensemble of sample functions is $\{x(t, s_i)|i = 0, 1, 2, 3\}$ where

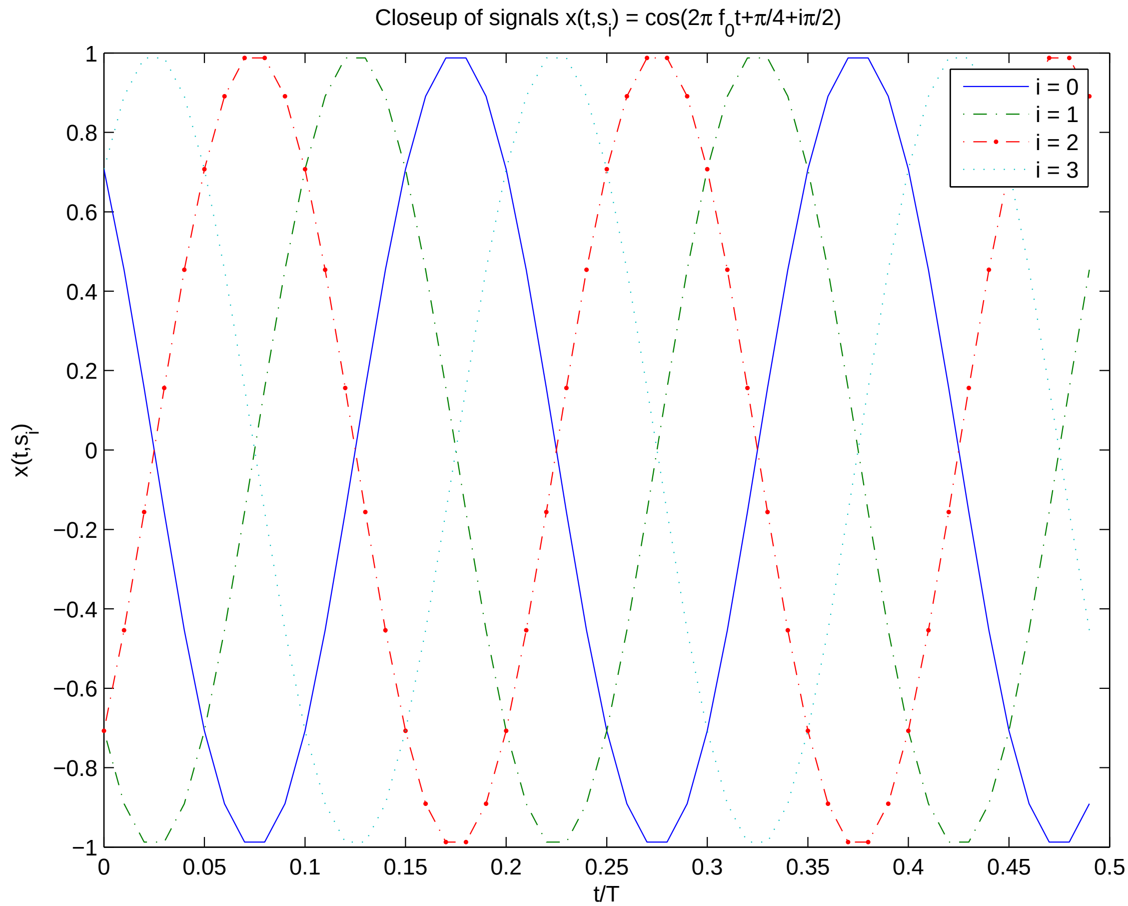
$$x(t, s_i) = \cos(2\pi f_0 t + \pi/4 + i\pi/2) \qquad (0 \leq t \leq T) \tag{1}$$

For $f_0 = 5/T$, this ensemble is shown below.



I don't think the above figure gives a very good picture of the relationship among the signals, so I've used the matlab script below to generate a single figure (shown following the script) clearly showing the effect of changing the phase by multiples of $\pi/4$.

```
%
% alternate plot for solution of 10.2.2 sk
%
x0 = cos(2*pi*f0*t+pi/4*ones(size(t)));
x1 = cos(2*pi*f0*t+pi/4*ones(size(t))+pi/2);
x2 = cos(2*pi*f0*t+pi/4*ones(size(t))+2*pi/2);
x3 = cos(2*pi*f0*t+pi/4*ones(size(t))+3*pi/2);
clf
plot(t(1:50),x0(1:50),'-',t(1:50),x1(1:50),'-.',t(1:50),...
x2(1:50),'-..',t(1:50),x3(1:50),':')
legend('i = 0','i = 1','i = 2','i = 3')
xlabel('t/T')
ylabel('x(t,s_i)')
title('Closeup of signals x(t,s_i) = cos(2\pi f_0t+\pi/4+i\pi/2)')
print -depsc p10.2.2a.eps
```



Problem 10.4.1 •

Suppose that at the equator, we can model the noontime temperature in degrees Celsius, X_n, on day n by a sequence of iid Gaussian random variables with a mean of 30 degrees and standard deviation of 5 degrees. A new random process $Y_k = [X_{2k-1} + X_{2k}]/2$ is obtained by averaging the temperature over two days. Is Y_k an iid random sequence?

Problem 10.4.1 Solution

Each Y_k is the sum of two identical independent Gaussian random variables. Hence, *by Thm. 6.10, p. 253,* each Y_k *is a Gaussian random variable and, since the distribution does not depend on* k*, all of the* Y_k must have the same PDF. That is, the Y_k are identically distributed. Next, we observe that the sequence of Y_k is independent. To see this, we observe that each Y_k is composed of two samples of X_k that are unused by any other Y_j for

$j \neq k$. Formally, for $k \neq l$

$$\begin{aligned}
\operatorname{Cov}[Y_k, Y_l] &= E[Y_k Y_l] - E[Y_k] E[Y_l] && (1)\\
&= \frac{1}{4} E[(X_{2k-1} + X_{2k})(X_{2l-1} + X_{2l})] - 30^2 && (2)\\
&= \frac{1}{4} E[(X_{2k-1}X_{2l-1} + X_{2k-1}X_{2l} + X_{2k}X_{2l-1} + X_{2k}X2l)] - 900 && (3)\\
&= \frac{1}{4}(E[X_{2k-1}]E[X_{2l-1}] + E[X_{2k-1}]E[X_{2l}] + E[X_{2k}]E[X_{2l-1}] + E[X_{2k}]E[X_{2l}]) - 900 && (4)\\
&= \frac{1}{4}(900 + 900 + 900 + 900) - 900 && (5)\\
&= 0, && (6)
\end{aligned}$$

where the fourth equality follows from the X_n being independent, so off-diagonal terms of the covariance are zero so the Y_k are independent.

Problem 10.4.2 ■

For the equatorial noontime temperature sequence X_n of Problem 10.4.1, a second sequence of averaged temperatures is $W_n = [X_n + X_{n-1}]/2$. Is W_n an iid random sequence?

Problem 10.4.2 Solution

To be iid, the sequence must be both independent and identically distributed. First, let's look at the distribution. Each W_n is the sum of two identical independent Gaussian random variables. Hence, each W_n must have the same PDF. That is, the W_n are identically distributed. *Next we check for independence. Informally,* since W_{n-1} and W_n both use X_{n-1} in their averaging, W_{n-1} and W_n are dependent. We verify this observation by calculating the covariance of W_{n-1} and W_n. First, we observe that for all n,

$$E[W_n] = (E[X_n] + E[X_{n-1}])/2 = 30 \qquad (1)$$

Next, we observe that W_{n-1} and W_n have covariance

$$\begin{aligned}
\operatorname{Cov}[W_{n-1}, W_n] &= E[W_{n-1}W_n] - E[W_n]E[W_{n-1}] && (2)\\
&= \frac{1}{4}E[(X_{n-1} + X_{n-2})(X_n + X_{n-1})] - 900 && (3)
\end{aligned}$$

We observe that for $n \neq m$, $E[X_n X_m] = E[X_n]E[X_m] = 900$ while

$$E\left[X_n^2\right] = \operatorname{Var}[X_n] + (E[X_n])^2 = 925 \qquad (4)$$

Thus,

$$\operatorname{Cov}[W_{n-1}, W_n] = \frac{900 + 925 + 900 + 900}{4} - 900 = 25/4 \qquad (5)$$

Since $\operatorname{Cov}[W_{n-1}, W_n] \neq 0$, *we conclude that* W_n and W_{n-1} must be dependent. *So W_n is not an iid sequence.*

Problem 10.5.1 •
The arrivals of new telephone calls at a telephone switching office is a Poisson process $N(t)$ with an arrival rate of $\lambda = 4$ calls per second. An experiment consists of monitoring the switching office and recording $N(t)$ over a 10-second interval.

(a) What is $P_{N(1)}(0)$, the probability of no phone calls in the first second of observation?

(b) What is $P_{N(1)}(4)$, the probability of exactly four calls arriving in the first second of observation?

(c) What is $P_{N(2)}(2)$, the probability of exactly two calls arriving in the first two seconds?

Problem 10.5.1 Solution
This is a very straightforward problem. The Poisson process has rate $\lambda = 4$ calls per second. When t is measured in seconds, each $N(t)$ is a Poisson random variable with mean $4t$ and thus has PMF

$$P_{N(t)}(n) = \begin{cases} \frac{(4t)^n}{n!} e^{-4t} & n = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Using the general expression for the PMF, we can write down the answer for each part.

(a) $P_{N(1)}(0) = 4^0 e^{-4}/0! = e^{-4} \approx 0.0183$.

(b) $P_{N(1)}(4) = 4^4 e^{-4}/4! = 32e^{-4}/3 \approx 0.1954$.

(c) $P_{N(2)}(2) = 8^2 e^{-8}/2! = 32e^{-8} \approx 0.0107$.

Problem 10.5.2 •
Queries presented to a computer database are a Poisson process of rate $\lambda = 6$ queries per minute. An experiment consists of monitoring the database for m minutes and recording $N(m)$, the number of queries presented. The answer to each of the following questions can be expressed in terms of the PMF $P_{N(m)}(k) = P[N(m) = k]$.

(a) What is the probability of no queries in a one minute interval?

(b) What is the probability of exactly six queries arriving in a one-minute interval?

(c) What is the probability of exactly three queries arriving in a one-half-minute interval?

Problem 10.5.2 Solution
Following the instructions given, we express each answer in terms of $N(m)$ which has PMF

$$P_{N(m)}(n) = \begin{cases} (6m)^n e^{-6m}/n! & n = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) The probability of no queries in a one minute interval is $P_{N(1)}(0) = 6^0 e^{-6}/0! = 0.0025$.

(b) The probability of exactly 6 queries arriving in a one minute interval is $P_{N(1)}(6) = 6^6e^{-6}/6! = 0.1606$.

(c) The probability of exactly three queries arriving in a one-half minute interval is $P_{N(0.5)}(3) = 3^3e^{-3}/3! = 0.2240$.

Problem 10.5.3 •

At a successful garage, there is always a backlog of cars waiting to be serviced. The service times of cars are iid exponential random variables with a mean service time of 30 minutes. Find the PMF of $N(t)$, the number of cars serviced in the first t hours of the day.

Problem 10.5.3 Solution

Since there is always a backlog an the service times are iid exponential random variables, The time between service completions are a sequence of iid exponential random variables. that is, the service completions are a Poisson process. Since the expected service time is 30 minutes, the rate of the Poisson process is $\lambda = 1/30$ per minute. Since t hours equals $60t$ minutes, the expected number serviced is $\lambda(60t)$ or $2t$. Moreover, the number serviced in the first t hours has the Poisson PMF

$$P_{N(t)}(n) = \begin{cases} \frac{(2t)^n e^{-2t}}{n!} & n = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Problem 10.6.1 •

Customers arrive at a casino as a Poisson process of rate 100 customers per hour. Upon arriving, each customer must flip a coin, and only those customers who flip heads actually enter the casino. Let $N(t)$ denote the process of customers entering the casino. Find the PMF of N, the number of customers who arrive between 5 PM and 7 PM.

Problem 10.6.1 Solution

Customers entering (or not entering) the casino is a Bernoulli decomposition of the Poisson process of arrivals at the casino doors. By Theorem 10.6, customers entering the casino are a Poisson process of rate $100/2 = 50$ customers/hour. Thus in the two hours from 5 to 7 PM, the number, N, of customers entering the casino is a Poisson random variable with expected value $\alpha = 2 \cdot 50 = 100$. The PMF of N is

$$P_N(n) = \begin{cases} 100^n e^{-100}/n! & n = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Problem 10.7.1 •

Over the course of a day, the stock price of a widely traded company can be modeled as a Brownian motion process where $X(0)$ is the opening price at the morning bell. Suppose the unit of time t is an hour, the exchange is open for eight hours, and the standard deviation of the daily price change (the difference between the opening bell and closing bell prices) is 1/2 point. What is the value of the Brownian motion parameter α?

Problem 10.7.1 Solution

From the problem statement, the change in the stock price is $X(8)-X(0)$ and the standard deviation of $X(8)-X(0)$ is 1/2 point. In other words, the variance of $X(8)-X(0)$ is $\text{Var}[X(8)-X(0)] = 1/4$. By the definition of Brownian motion. $\text{Var}[X(8)-X(0)] = 8\alpha$. Hence $\alpha = 1/32$.

Problem 10.7.2 ■

Let $X(t)$ be a Brownian motion process with variance $\text{Var}[X(t)] = \alpha t$. For a constant $c > 0$, determine whether $Y(t) = X(ct)$ is a Brownian motion process.

Problem 10.7.2 Solution

We need to verify that $Y(t) = X(ct)$ satisfies the conditions given in Definition 10.10. First we observe that $Y(0) = X(c \cdot 0) = X(0) = 0$. Second, we note that $X(t)$ being a Brownian motion process implies that $Y(t) - Y(s) = X(ct) - X(cs)$ is a Gaussian random variable. Further, $X(ct) - X(cs)$ is independent of $X(t')$ for all $t' \leq cs$. Equivalently, we can say that $X(ct) - X(cs)$ is independent of $X(c\tau)$ for all $\tau \leq s$. In other words, $Y(t) - Y(s)$ is independent of $Y(\tau)$ for all $\tau \leq s$. Thus $Y(t)$ is a Brownian motion process.

Problem 10.8.1 •

X_n is an iid random sequence with mean $E[X_n] = \mu_X$ and variance $\text{Var}[X_n] = \sigma_X^2$. What is the autocovariance $C_X[m,k]$?

Problem 10.8.1 Solution

The discrete time autocovariance function is

$$C_X[m,k] = E[(X_m - \mu_X)(X_{m+k} - \mu_X)] \tag{1}$$

for $k = 0$, $C_X[m,0] = \text{Var}[X_m] = \sigma_X^2$. For $k \neq 0$, X_m and X_{m+k} are independent so that

$$C_X[m,k] = E[(X_m - \mu_X)]\, E[(X_{m+k} - \mu_X)] = 0(0) = 0 \tag{2}$$

Thus the autocovariance of X_n is

$$C_X[m,k] = \begin{cases} \sigma_X^2 & k = 0 \\ 0 & k \neq 0 \end{cases} \tag{3}$$

We see that the autocovariance matrix of an independent sequence is diagonal.

Problem 10.8.3 ■

A simple model (in degrees Celsius) for the daily temperature process $C(t)$ of Example 10.3 is

$$C_n = 16\left[1 - \cos\left(\frac{2\pi n}{365}\right)\right] + 4X_n$$

where $X_1, X_2, \ldots$ is an iid random sequence of Gaussian $(0,1)$ random variables.

(a) What is the mean $E[C_n]$?

(b) Find the autocovariance function $C_C[m,k]$.

(c) Why is this model overly simple?

Problem 10.8.3 Solution

In this problem, the daily temperature process results from

$$C_n = 16\left[1-\cos\left(\frac{2\pi n}{365}\right)\right] + 4X_n \quad (1)$$

where X_n is an iid random sequence of $N[0,1]$ random variables. The hardest part of this problem is distinguishing between the process C_n and the covariance function $C_C[k]$.

(a) The expected value of the process is

$$E[C_n] = 16E\left[1-\cos\left(\frac{2\pi n}{365}\right)\right] + 4E[X_n] = 16\left[1-\cos\left(\frac{2\pi n}{365}\right)\right] \quad (2)$$

because we are given that X_n has zero mean.

(b) The autocovariance of C_n is

$$C_C[m,k] = E\left[\left(C_m - 16\left[1-\cos\left(\frac{2\pi m}{365}\right)\right]\right)\left(C_{m+k} - 16\left[1-\cos\left(\frac{2\pi(m+k)}{365}\right)\right]\right)\right] \quad (3)$$

$$= E[4X_m 4X_{m+k}] = 16E[X_m X_{m+k}] = \begin{cases} 16 & k=0 \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

because the Gaussian random variable X_n has variance 1 and the X_n are independent.

(c) A model of this type may be able to capture the mean and variance of the daily temperature. However, one reason this model is overly simple is that *according to this model,* day to day temperatures are uncorrelated. A more realistic model might incorporate the effects of "heat waves" or "cold spells" through correlated daily temperatures.

Problem 10.9.1 •

For an arbitrary constant a, let $Y(t) = X(t+a)$. If $X(t)$ is a stationary random process, is $Y(t)$ stationary?

Problem 10.9.1 Solution

For an arbitrary set of samples $Y(t_1),\ldots,Y(t_k)$, we observe that $Y(t_j) = X(t_j + a)$. This implies

$$f_{Y(t_1),\ldots,Y(t_k)}(y_1,\ldots,y_k) = f_{X(t_1+a),\ldots,X(t_k+a)}(y_1,\ldots,y_k) \quad (1)$$

Thus,

$$f_{Y(t_1+\tau),\ldots,Y(t_k+\tau)}(y_1,\ldots,y_k) = f_{X(t_1+\tau+a),\ldots,X(t_k+\tau+a)}(y_1,\ldots,y_k) \tag{2}$$

Since $X(t)$ is a stationary process,

$$f_{X(t_1+\tau+a),\ldots,X(t_k+\tau+a)}(y_1,\ldots,y_k) = f_{X(t_1+a),\ldots,X(t_k+a)}(y_1,\ldots,y_k) \tag{3}$$

This implies

$$\begin{aligned} f_{Y(t_1+\tau),\ldots,Y(t_k+\tau)}(y_1,\ldots,y_k) &= f_{X(t_1+a),\ldots,X(t_k+a)}(y_1,\ldots,y_k) &(4)\\ &= f_{Y(t_1),\ldots,Y(t_k)}(y_1,\ldots,y_k) &(5)\end{aligned}$$

We can conclude that $Y(t)$ is a stationary process.

Problem 10.9.2 •

For an arbitrary constant a, let $Y(t) = X(at)$. If $X(t)$ is a stationary random process, is $Y(t)$ stationary?

Problem 10.9.2 Solution

For an arbitrary set of samples $Y(t_1),\ldots,Y(t_k)$, we observe that $Y(t_j) = X(at_j)$. This implies

$$f_{Y(t_1),\ldots,Y(t_k)}(y_1,\ldots,y_k) = f_{X(at_1),\ldots,X(at_k)}(y_1,\ldots,y_k) \tag{1}$$

Thus,

$$f_{Y(t_1+\tau),\ldots,Y(t_k+\tau)}(y_1,\ldots,y_k) = f_{X(at_1+a\tau),\ldots,X(at_k+a\tau)}(y_1,\ldots,y_k) \tag{2}$$

We see that a time offset of τ for the $Y(t)$ process corresponds to an offset of time $\tau' = a\tau$ for the $X(t)$ process. Since $X(t)$ is a stationary process,

$$\begin{aligned} f_{Y(t_1+\tau),\ldots,Y(t_k+\tau)}(y_1,\ldots,y_k) &= f_{X(at_1+\tau'),\ldots,X(at_k+\tau')}(y_1,\ldots,y_k) &(3)\\ &= f_{X(at_1),\ldots,X(at_k)}(y_1,\ldots,y_k) &(4)\\ &= f_{Y(t_1),\ldots,Y(t_k)}(y_1,\ldots,y_k) &(5)\end{aligned}$$

We can conclude that $Y(t)$ is a stationary process.

Solutions to HW12

Note: These solutions are D. J. Goodman, the authors of our textbook. I have annotated and corrected them as necessary. Text in italics is mine.

Problem 10.10.2 •

Let A be a nonnegative random variable that is independent of any collection of samples $X(t_1), \ldots, X(t_k)$ of a wide sense stationary random process $X(t)$. Is $Y(t) = A + X(t)$ a wide sense stationary process?

Problem 10.10.2 Solution

To show that $Y(t)$ is wide-sense stationary we must show that it meets the two requirements of Definition 10.15, namely that its expected value and autocorelation function must be independent of t. Since $Y(t) = A + X(t)$, the mean of $Y(t)$ is

$$E[Y(t)] = E[A] + E[X(t)] = E[A] + \mu_X \tag{1}$$

The autocorrelation of $Y(t)$ is

$$\begin{aligned} R_Y(t,\tau) &= E[(A + X(t))(A + X(t+\tau))] &(2)\\ &= E\left[A^2\right] + E[A]\,E[X(t)] + AE[X(t+\tau)] + E[X(t)X(t+\tau)] &(3)\\ &= E\left[A^2\right] + 2E[A]\,\mu_X + R_X(\tau), &(4)\end{aligned}$$

where the last equality is justified by the fact that we are given that $X(t)$ is wide sense stationary. We see that neither $E[Y(t)]$ nor $R_Y(t,\tau)$ depend on t. Thus $Y(t)$ is a wide sense stationary process.

Problem 10.11.1 •

$X(t)$ and $Y(t)$ are independent wide sense stationary processes with expected values μ_X and μ_Y and autocorrelation functions $R_X(\tau)$ and $R_Y(\tau)$ respectively. Let $W(t) = X(t)Y(t)$.

(a) Find μ_W and $R_W(t,\tau)$ and show that $W(t)$ is wide sense stationary.

(b) Are $W(t)$ and $X(t)$ jointly wide sense stationary?

Problem 10.11.1 Solution

(a) Since $X(t)$ and $Y(t)$ are independent processes,

$$E[W(t)] = E[X(t)Y(t)] = E[X(t)]\,E[Y(t)] = \mu_X\mu_Y. \tag{1}$$

In addition,

$$\begin{aligned} R_W(t,\tau) &= E[W(t)W(t+\tau)] &(2)\\ &= E[X(t)Y(t)X(t+\tau)Y(t+\tau)] &(3)\\ &= E[X(t)X(t+\tau)]\,E[Y(t)Y(t+\tau)] &(4)\\ &= R_X(\tau)R_Y(\tau) &(5)\end{aligned}$$

We can conclude that $W(t)$ is wide sense stationary.

(b) To examine whether $X(t)$ and $W(t)$ are jointly wide sense stationary, we calculate

$$R_{WX}(t,\tau) = E\left[W(t)X(t+\tau)\right] = E\left[X(t)Y(t)X(t+\tau)\right]. \tag{6}$$

By independence of $X(t)$ and $Y(t)$,

$$R_{WX}(t,\tau) = E\left[X(t)X(t+\tau)\right] E\left[Y(t)\right] = \mu_Y R_X(\tau). \tag{7}$$

Since $W(t)$ and $X(t)$ are both wide sense stationary and since $R_{WX}(t,\tau)$ depends only on the time difference τ, we can conclude from Definition 10.18 that $W(t)$ and $X(t)$ are jointly wide sense stationary.

Problem 10.11.2 ■

$X(t)$ is a wide sense stationary random process. For each process $X_i(t)$ defined below, determine whether $X_i(t)$ and $X(t)$ are jointly wide sense stationary.

(a) $X_1(t) = X(t+a)$

(b) $X_2(t) = X(at)$

Problem 10.11.2 Solution

To show that $X(t)$ and $X_i(t)$ are jointly wide sense stationary, we must first show that $X_i(t)$ is wide sense stationary and then we must show that the cross correlation $R_{XX_i}(t,\tau)$ is only a function of the time difference τ. For each $X_i(t)$, we have to check whether these facts are implied by the fact that $X(t)$ is wide sense stationary.

(a) Since $E[X_1(t)] = E[X(t+a)] = \mu_X$ and

$$\begin{aligned} R_{X_1}(t,\tau) &= E\left[X_1(t)X_1(t+\tau)\right] && (1)\\ &= E\left[X(t+a)X(t+\tau+a)\right] && (2)\\ &= R_X(\tau), && (3)\end{aligned}$$

we have verified that $X_1(t)$ is wide sense stationary. Now we calculate the cross correlation

$$\begin{aligned} R_{XX_1}(t,\tau) &= E\left[X(t)X_1(t+\tau)\right] && (4)\\ &= E\left[X(t)X(t+\tau+a)\right] && (5)\\ &= R_X(\tau+a). && (6)\end{aligned}$$

Since $R_{XX_1}(t,\tau)$ depends on the time difference τ but not on the absolute time t, we conclude that $X(t)$ and $X_1(t)$ are jointly wide sense stationary.

(b) Since $E[X_2(t)] = E[X(at)] = \mu_X$ and

$$\begin{aligned} R_{X_2}(t,\tau) &= E\left[X_2(t)X_2(t+\tau)\right] && (7)\\ &= E\left[X(at)X(a(t+\tau))\right] && (8)\\ &= E\left[X(at)X(at+a\tau)\right] = R_X(a\tau), && (9)\end{aligned}$$

we have verified that $X_2(t)$ is wide sense stationary. Now we calculate the cross correlation

$$\begin{aligned} R_{XX_2}(t,\tau) &= E\left[X(t)X_2(t+\tau)\right] & (10)\\ &= E\left[X(t)X(a(t+\tau))\right] & (11)\\ &= R_X((a-1)t+\tau). & (12) \end{aligned}$$

Except for the trivial case when $a = 1$ and $X_2(t) = X(t)$, $R_{XX_2}(t,\tau)$ depends on both the absolute time t and the time difference τ, we conclude that $X(t)$ and $X_2(t)$ are not jointly wide sense stationary.

Problem 10.11.3 ■

$X(t)$ is a wide sense stationary stochastic process with autocorrelation function $R_X(\tau) = 10\sin(2\pi 1000\tau)/(2\pi 1000\tau)$. The process $Y(t)$ is a version of $X(t)$ delayed by 50 microseconds: $Y(t) = X(t-t_0)$ where $t_0 = 5\times 10^{-5}$s.

(a) Derive the autocorrelation function of $Y(t)$.

(b) Derive the cross-correlation function of $X(t)$ and $Y(t)$.

(c) Is $Y(t)$ wide sense stationary?

(d) Are $X(t)$ and $Y(t)$ jointly wide sense stationary?

Problem 10.11.3 Solution

(a) $Y(t)$ has autocorrelation function

$$\begin{aligned} R_Y(t,\tau) &= E\left[Y(t)Y(t+\tau)\right] & (1)\\ &= E\left[X(t-t_0)X(t+\tau-t_0)\right] & (2)\\ &= R_X(\tau). & (3) \end{aligned}$$

(b) The cross correlation of $X(t)$ and $Y(t)$ is

$$\begin{aligned} R_{XY}(t,\tau) &= E\left[X(t)Y(t+\tau)\right] & (4)\\ &= E\left[X(t)X(t+\tau-t_0)\right] & (5)\\ &= R_X(\tau-t_0). & (6) \end{aligned}$$

(c) We have already verified that $R_Y(t,\tau)$ depends only on the time difference τ. Since $E[Y(t)] = E[X(t-t_0)] = \mu_X$, we have verified that $Y(t)$ is wide sense stationary.

(d) Since $X(t)$ and $Y(t)$ are wide sense stationary and since we have shown that $R_{XY}(t,\tau)$ depends only on τ, we know that $X(t)$ and $Y(t)$ are jointly wide sense stationary.

Comment: This problem is badly designed since the conclusions don't depend on the specific $R_X(\tau)$ given in the problem text. (Sorry about that!)

Problem 11.2.1 •

The random sequence X_n is the input to a discrete-time filter. The output is

$$Y_n = \frac{X_{n+1} + X_n + X_{n-1}}{3}.$$

(a) What is the impulse response h_n?

(b) Find the autocorrelation of the output Y_n when X_n is a wide sense stationary random sequence with $\mu_X = 0$ and autocorrelation

$$R_X[n] = \begin{cases} 1 & n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Problem 11.2.1 Solution

(a) Note that

$$Y_i = \sum_{n=-\infty}^{\infty} h_n X_{i-n} = \frac{1}{3}X_{i+1} + \frac{1}{3}X_i + \frac{1}{3}X_{i-1} \tag{1}$$

By matching coefficients, we see that

$$h_n = \begin{cases} 1/3 & n = -1, 0, 1 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

(b) By Theorem 11.5, the output autocorrelation is

$$\begin{aligned} R_Y[n] &= \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} h_i h_j R_X[n+i-j] && (3) \\ &= \frac{1}{9} \sum_{i=-1}^{1} \sum_{j=-1}^{1} R_X[n+i-j] && (4) \\ &= \frac{1}{9} \left(R_X[n+2] + 2R_X[n+1] + 3R_X[n] + 2R_X[n-1] + R_X[n-2] \right) && (5) \end{aligned}$$

We see that the filter is linear and time invariant. Substituting in $R_X[n]$ yields

$$R_Y[n] = \begin{cases} 1/3 & n = 0 \\ 2/9 & |n| = 1 \\ 1/9 & |n| = 2 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Problem 11.2.2 •
$X(t)$ is a wide sense stationary process with autocorrelation function

$$R_X(\tau) = 10\frac{\sin(2000\pi t) + \sin(1000\pi t)}{2000\pi t}.$$

The process $X(t)$ is sampled at rate $1/T_s = 4{,}000$ Hz, yielding the discrete-time process X_n. What is the autocorrelation function $R_X[k]$ of X_n?

Problem 11.2.2 Solution
Applying Theorem 11.4 with sampling period $T_s = 1/4000$ s yields

$$R_X[k] = R_X(kT_s) = 10\frac{\sin(2000\pi kT_s) + \sin(1000\pi kT_s)}{2000\pi kT_s} \tag{1}$$
$$= 20\frac{\sin(0.5\pi k) + \sin(0.25\pi k)}{\pi k} \tag{2}$$
$$= 10\,\mathrm{sinc}(0.5k) + 5\,\mathrm{sinc}(0.25k) \tag{3}$$

Problem 11.3.1 •
X_n is a stationary Gaussian sequence with expected value $E[X_n] = 0$ and autocorrelation function $R_X[k] = 2^{-|k|}$. Find the PDF of $\mathbf{X} = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}'$.

Problem 11.3.1 Solution
Since the process X_n has expected value $E[X_n] = 0$, we know that $C_X(k) = R_X(k) = 2^{-|k|}$. Thus $\mathbf{X} = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}'$ has covariance matrix

$$\mathbf{C_X} = \begin{bmatrix} 2^0 & 2^{-1} & 2^{-2} \\ 2^{-1} & 2^0 & 2^{-1} \\ 2^{-2} & 2^{-1} & 2^0 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 1 \end{bmatrix}. \tag{1}$$

From Definition 5.17, the PDF of $\mathbf{X}$ is

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}[\det(\mathbf{C_X})]^{1/2}} \exp\left(-\frac{1}{2}\mathbf{x}'\mathbf{C_X}^{-1}\mathbf{x}\right). \tag{2}$$

Equivalently, we can write out the PDF in terms of the variables x_1, x_2 and x_3. To do so we find that the inverse covariance matrix is

$$\mathbf{C_X}^{-1} = \begin{bmatrix} 4/3 & -2/3 & 0 \\ -2/3 & 5/3 & -2/3 \\ 0 & -2/3 & 4/3 \end{bmatrix} \tag{3}$$

A little bit of algebra will show that $\det(\mathbf{C_X}) = 9/16$ and that

$$\frac{1}{2}\mathbf{x}'\mathbf{C_X}^{-1}\mathbf{x} = \frac{2x_1^2}{3} + \frac{5x_2^2}{6} + \frac{2x_3^2}{3} - \frac{2x_1x_2}{3} - \frac{2x_2x_3}{3}. \tag{4}$$

It follows that

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{4}{3(2\pi)^{3/2}} \exp\left(-\frac{2x_1^2}{3} - \frac{5x_2^2}{6} - \frac{2x_3^2}{3} + \frac{2x_1x_2}{3} + \frac{2x_2x_3}{3}\right). \tag{5}$$

Problem 11.3.2 •

X_n is a sequence of independent random variables such that $X_n = 0$ for $n < 0$ while for $n \geq 0$, each X_n is a Gaussian $(0,1)$ random variable. Passing X_n through the filter $\mathbf{h} = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}'$ yields the output Y_n. Find the PDFs of:

(a) $\mathbf{Y}_3 = \begin{bmatrix} Y_1 & Y_2 & Y_3 \end{bmatrix}'$,

(b) $\mathbf{Y}_2 = \begin{bmatrix} Y_1 & Y_2 \end{bmatrix}'$.

Problem 11.3.2 Solution

The sequence X_n is passed through the filter

$$\mathbf{h} = \begin{bmatrix} h_0 & h_1 & h_2 \end{bmatrix}' = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}' \tag{1}$$

The output sequence is Y_n.

(a) Following the approach of Equation (11.58), we can write the output $\mathbf{Y}_3 = \begin{bmatrix} Y_1 & Y_2 & Y_3 \end{bmatrix}'$ as

$$\mathbf{Y}_3 = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} h_1 & h_0 & 0 & 0 \\ h_2 & h_1 & h_0 & 0 \\ 0 & h_2 & h_1 & h_0 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix}}_{\mathbf{X}}. \tag{2}$$

We note that the components of $\mathbf{X}$ are iid Gaussian $(0,1)$ random variables. Hence $\mathbf{X}$ has covariance matrix $\mathbf{C_X} = \mathbf{I}$, the identity matrix. Since $\mathbf{Y}_3 = \mathbf{HX}$,

$$\mathbf{C}_{\mathbf{Y}_3} = \mathbf{HC_XH}' = \mathbf{HH}' = \begin{bmatrix} 2 & -2 & 1 \\ -2 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix}. \tag{3}$$

Some calculation (by hand or by MATLAB) will show that $\det(\mathbf{C}_{\mathbf{Y}_3}) = 3$ and that

$$\mathbf{C}_{\mathbf{Y}_3}^{-1} = \frac{1}{3} \begin{bmatrix} 5 & 4 & 1 \\ 4 & 5 & 2 \\ 1 & 2 & 2 \end{bmatrix}. \tag{4}$$

Some algebra will show that

$$\mathbf{y}'\mathbf{C}_{\mathbf{Y}_3}^{-1}\mathbf{y} = \frac{5y_1^2 + 5y_2^2 + 2y_3^2 + 8y_1y_2 + 2y_1y_3 + 4y_2y_3}{3}. \tag{5}$$

This implies $\mathbf{Y}_3$ has PDF

$$f_{\mathbf{Y}_3}(\mathbf{y}) = \frac{1}{(2\pi)^{3/2}[\det(\mathbf{C}_{\mathbf{Y}_3})]^{1/2}} \exp\left(-\frac{1}{2}\mathbf{y}'\mathbf{C}_{\mathbf{Y}_3}^{-1}\mathbf{y}\right) \tag{6}$$

$$= \frac{1}{(2\pi)^{3/2}\sqrt{3}} \exp\left(-\frac{5y_1^2 + 5y_2^2 + 2y_3^2 + 8y_1y_2 + 2y_1y_3 + 4y_2y_3}{6}\right). \tag{7}$$

(b) To find the PDF of $\mathbf{Y}_2 = \begin{bmatrix} Y_1 & Y_2 \end{bmatrix}'$, we start by observing that the covariance matrix of $\mathbf{Y}_2$ is just the upper left 2×2 submatrix of $\mathbf{C}_{\mathbf{Y}_3}$. That is,

$$\mathbf{C}_{\mathbf{Y}_2} = \begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{C}_{\mathbf{Y}_2}^{-1} = \begin{bmatrix} 3/2 & 1 \\ 1 & 1 \end{bmatrix}. \tag{8}$$

Since $\det(\mathbf{C}_{\mathbf{Y}_2}) = 2$, it follows that

$$f_{\mathbf{Y}_2}(\mathbf{y}) = \frac{1}{(2\pi)^{3/2}[\det(\mathbf{C}_{\mathbf{Y}_2})]^{1/2}} \exp\left(-\frac{1}{2}\mathbf{y}'\mathbf{C}_{\mathbf{Y}_2}^{-1}\mathbf{y}\right) \tag{9}$$

$$= \frac{1}{(2\pi)^{3/2}\sqrt{2}} \exp\left(-\frac{3}{2}y_1^2 - 2y_1y_2 - y_2^2\right). \tag{10}$$

Problem 11.5.1 •

$X(t)$ is a wide sense stationary process with autocorrelation function

$$R_X(\tau) = 10\frac{\sin(2000\pi\tau) + \sin(1000\pi\tau)}{2000\pi\tau}.$$

What is the power spectral density of $X(t)$?

Problem 11.5.1 Solution

To use Table 11.1, we write $R_X(\tau)$ in terms of the autocorrelation

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}. \tag{1}$$

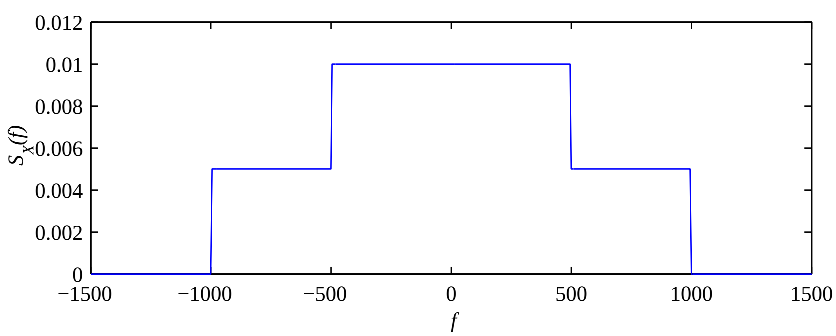
In terms of the sinc($\cdot$) function, we obtain

$$R_X(\tau) = 10\operatorname{sinc}(2000\tau) + 5\operatorname{sinc}(1000\tau). \tag{2}$$

From Table 11.1,

$$S_X(f) = \frac{10}{2{,}000}\operatorname{rect}\left(\frac{f}{2000}\right) + \frac{5}{1{,}000}\operatorname{rect}\left(\frac{f}{1{,}000}\right) \tag{3}$$

Here is a graph of the PSD.



Problem 11.6.1 •
X_n is a wide sense stationary discrete-time random sequence with autocorrelation function

$$R_X[k] = \begin{cases} \delta[k] + (0.1)^{|k|} & k = 0, \pm 1, \pm 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Find the power spectral density $S_X(f)$.

Problem 11.6.1 Solution
Since the random sequence X_n has autocorrelation function

$$R_X[k] = \delta_k + (0.1)^{|k|}, \tag{1}$$

We can find the PSD directly from Table 11.2 with $0.1^{|k|}$ corresponding to $a^{|k|}$. The table yields

$$S_X(\phi) = 1 + \frac{1 - (0.1)^2}{1 + (0.1)^2 - 2(0.1)\cos 2\pi\phi} = \frac{2 - 0.2\cos 2\pi\phi}{1.01 - 0.2\cos 2\pi\phi}. \tag{2}$$