The following digraph will illustrate these features.

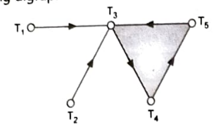


Fig. 3.42

$$T_1 \le T_3 \le T_4 \le T_5, \quad T_1 \le T_2$$

$$T_1 \le T_3 \le T_5$$

$$T_2 \le T_3 \le T_4 \le T_5$$

Let $\mu\,(T_i)$ denote the execution time of task T_i. For preparing a schedule of a set of tasks on a multiprocessor, we must specify for each task, the processor on which it is going to be executed as well as the time period of the execution. In addition we must also know the time interval during which a processor is kept idle. Idle periods are denoted by ϕ_1, ϕ_2, etc. and durations by $\mu\,(\phi_1)$, $\mu\,(\phi_2)$, etc. The total elapsed time of a schedule is the total time taken to complete the tasks, as per the schedule. Obviously this will also include the idle periods. In any problem on scheduling the objective is to minimize the total elapsed time. One way of attaining this objective is to work out a schedule in which no processor is left intentionally idle, that is a processor is idle only if in that period no task is available for execution on it.

Consider a computing system where there are two processors P_1, P_2, on which tasks $T_1, T_2, \ldots, T_n$ are to be executed, and no processor is left intentionally idle. Note that the idle period of one processor is overlapped by the execution period of tasks on the other processor. Suppose there is a chain of tasks $T_{i1} \le T_{i2} \le \ldots\ldots \le T_{ir}$ on processor P_1 which overlaps the idle period ϕ_i on P_2. Similarly let $T_{j1} \le T_{j2} \le \ldots \le T_{js}$ be a chain of tasks on P_2 which overlaps the idle period ϕ_j on P_1. Then we have the following inequalities.

$$\sum_{k=1}^{r} \mu\,(T_{ik}) \ge \mu\,(\phi_i) \qquad \text{(for } P_1\text{)}$$

and $$\sum_{k=1}^{s} \mu\,(T_{jk}) \ge \mu\,(\phi_j) \qquad \text{(for } P_2\text{)}$$

The following timer diagram illustrates the above situation.

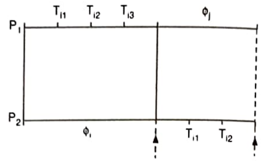


Fig. 3.43

In general, if we consider all such subsets of chains τ and Φ, the set of all idle periods ϕ_i, then

$$\sum_{T_R \in \tau} \mu\,(T_k) \ge \sum_{\phi_i \in \Phi} \mu\,(\phi_i) \qquad \ldots (3.1)$$

Let ω denote the total elapsed time when the tasks are executed according to a schedule that contains no intentional idle periods and let ω_0 denote the minimum possible total elapsed time.

Then, $$\omega = \frac{1}{2}\left[\sum_{T_j \in T} \mu\,(T_j) + \sum_{\phi_j \in \Phi} \mu\,(\phi_j)\right]$$

$$\le \frac{1}{2}\left[\sum_{T_j \in T_k} \mu\,(T_j) + \sum_{\phi_j \in \tau} \mu\,(T_R)\right] \qquad \ldots (3.2)$$

Let ω_0 denote the minimum possible total elapsed time.

Clearly, $$\omega_0 \ge \frac{1}{2}\left[\sum_{T_j \in T} \mu\,(T_j)\right] \qquad \ldots (3.3)$$

and $$\omega_0 \ge \left[\sum_{T_k \in T} \mu\,(T_k)\right] \qquad \ldots (3.4)$$

Hence it follows from equation (2) that

$$\omega \le \omega_0 + \frac{1}{2}\,\omega_0 = \frac{3}{2}\,\omega_0$$

or $$\frac{\omega}{\omega_0} \le \frac{3}{2}$$

The above bound indicates that in the case of processors, the policy of not leaving a processor intentionally idle yields good result in the sense that it consumes at the most only fifty percent more time than the best possible schedule.

The following couple of examples illustrate this point.

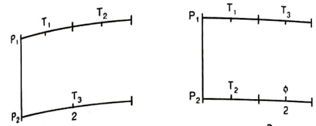
Example 1: Timing diagram:

$T_1 / 1$

$T_2 / 1$

$T_3 / 2$

- (T_i/j denotes that task T_i consumes j)
- Units of time
- Nodes or vertices indicate the tasks



$$\omega_0 = 2 \qquad \omega = 3$$

$$\frac{\omega}{\omega_0} = \frac{3}{2}$$

Fig. 3.44

Example 2:

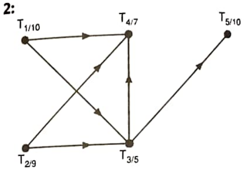
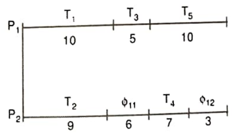


Fig. 3.45

Timing Charts:

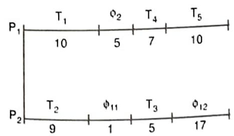
(a)



$\omega_0 = 25$

Fig. 3.46 (a)

(b)



$$\omega = 32$$

$$\frac{\omega}{\omega_0} = 1.28$$

Fig. 3.46 (b)

Remark: For a n - processor computing system, $\frac{\omega}{\omega_0} \le 2 - \frac{1}{n}$, and this bound is the best possible.

3.15 LATTICE

Let (A, ≤) be a poset with partial order ≤. We first define the following terms given in next section.

3.15.1 Maximal and Minimal Elements

An element a ∈ A is called a **maximal element** if there is no element b ∈ A such that b ≠ a and a ≤ b. An element c ∈ A is called a minimal element if there is no element d ∈ A such that d ≠ c and d ≤ a.

Example: Consider the poset whose Hasse diagram is given below.

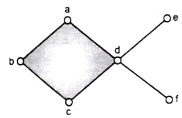


Fig. 3.47

Maximal elements are a, e. Minimal elements are c, f.

3.15.2 Upper Bound and Lower Bound

Let a, b be elements in a poset (A, ≤). An element c is said to be an **upper bound** of a and b if a ≤ c and b ≤ c. An element c is said to be a **least upper bound** (lub) of a and b if c is an upper bound of a and b and if there is no other upper bound d of a and b such that d ≤ c.

Similarly an element e is said to be a **lower bound** of a and b if e ≤ a and e ≤ b; and e is called a **greatest lower bound** (glb) of a and b if there is no other lower bound f of a, b such that e ≤ f.

Example: Consider the poset whose diagram is given below.

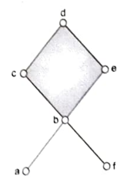


Fig. 3.48

Then

(i) upper bounds for {c, e} is d

lub {c, e} = d.

lower bounds for {c, e} are the elements b, a and f

glb {c, e} = b.

(ii) upper bounds for {a, f} are the elements b, c, e, d.

lub {a, f} = b.

lower bounds for a, f do not exist.

(iii) upper bounds for {b, d} is d.

lub {b, d} = d.

lower bounds for {b, d} are b, a and f.

glb {b, d} = b.

In this manner, one can find the bounds for various pairs of elements.

3.15.3 Lattice – Definition

A lattice is a poset (L, ≤) in which every subset {a, b} of L, has a least upper bound and a greatest lower bound.

Examples:

1. For any set A, consider its power set P(A). Then (P(A), ⊆) is a poset. It is also a lattice since for any pair of subsets B, C of A, lub {B, C} = B ∪ C and glb {B, C} = B ∩ C.
2. Let N be the set of natural numbers. For a, b ∈ N, let a ≤ b if b is divisible by a. Then (N, ≤) is a lattice since for any pair of elements a, b ∈ N, lub {a, b} = lcm of a and b and glb {a, b} = gcd of a and b.
3. Consider the poset whose diagram is given below.

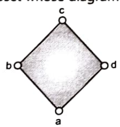


Fig. 3.49

If A = {a, b, c, d}, every pair of elements has a lub and glb. Hence (A, ≤) is a lattice.

4. Let A = {2, 3, 4, 6, 12} and define a ≤ b as a divides b. Consider the diagram of the poset (A, ≤).

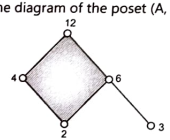


Fig. 3.50

The poset is not a lattice since the pair {2, 3} does not ha a greatest lower bound.

If the element 1 is included in A, then (A, ≤) will be a lattic

3.15.4 Lattice Operators

Let (L, ≤) be a lattice. For any pair of elements a, b ∈ L denote lub {a, b} by a ∨ b and glb {a, b} by a ∧ b. a ∨ b called as the **join** of a and b. a ∧ b is called as the mee of a and b. The meet and join are therefore binar operators.

3.15.5 Basic Properties of Lattices

Theorem 1: For any element a ∈ L, a ∨ a= a and a ∧ a = a (Idempotent property).

Proof:

Since a ∨ a is an upper bound for a, a ≤ a ∨ a ... (1)

By reflexivity, a ≤ a ... (2)

Since a ∨ a = lub {a, a}, from (1) and (2) it follows that

a ∨ a ≤ a ... (3)

But ≤ is antisymmetric.

Hence a ∨ a = a

Similarly, we can prove a ∧ a = a

Theorem 2: For any a, b ∈ L

a ∨ (a ∧ b) = a

a ∧ (a ∨ b) = a (Absorption property of join and m

Proof:

Since a ∨ (a ∧ b) is an upper bound for a and a ∧ b, it follows that

a ≤ a ∨ (a ∧ b)

Now a ∧ b is a lower bound for a and b, hence a ∧ b ≤ a.

Now a ≤ a and a ∧ b ≤ a. Hence a is an upper bound for the pair

{a, a ∧ b}.

Since a ∨ (a ∧ b) = lub {a, a ∧ b}, it follows that

a ∨ (a ∧ b) ≤ a

From (1) and (2) by antisymmetry of ≤, we have

a = a ∨ (a ∧ b)

Similarly, we can prove a ∧ (a ∨ b) = a.

Theorem 3: The meet and join operations are associative,

i.e. a ∨ (a ∨ c) = (a ∨ b) ∨ c

and a ∧ (b ∧ c) = (a ∧ b) ∧ c.

Proof: Proof is left as an exercise.

Theorem 4: The meet and join operations are commutative.

Proof: Left as an easy exercise.

In general, the distributive law is not satisfied for a lattice, as the following example demonstrates.

3.15.6 Example of a Non-Distributive Lattice

1. Consider the Hasse diagram given below of a lattice.

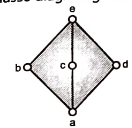


Fig. 3.51

We have b ∧ (c ∨ d) = b ∧ e = b ... (1)

On the other hand (b ∧ c) ∨ (b ∧ d) = a ∨ a = a.

Since a ≠ b, the distributive law does not hold for this lattice.

3.15.7 Universal Bounds

An element in a lattice (L, ≤) is called universal lower bound if for every a ∈ L, l ≤ a. Similarly an element u in L is called universal upper bound if for every a ∈ L, a ≤ u.

One can easily see that these universal bounds are unique. We denote the universal lower bound by 0 and the universal upper bound by 1. 0 and 1 are merely symbols and should not confuse with the numbers 0 and 1. All lattices do not have the universal bounds. The set of real numbers with the usual order (≤) has neither the universal lower bound nor the universal upper bound.

Theorem 1: Let (L, ≤) be a lattice with universal bounds 0 and 1. Then for every a ∈ L, a ∨ 1 = 1, a ∧ 1 = a, a ∨ 0 = a, a ∧ 0 = 0.

Proof: Easy exercise

3.15.8 Complement and Complemented Lattice

Definition:

Let (L, ≤) be a lattice with universal bounds 0 and 1. For an element a ∈ L, b is said to be a complement of a if a ∨ b = 1 and a ∧ b = 0.

Note by the commutativity property of meet and join, if b is the complement of a, then a is the complement of b.

In a lattice an element can have more than one complement, as demonstrated in the following example.

Example: Let A = {1, 2, 3, 5, 30} and let a ≤ b iff a divides b.

The Hasse diagram is

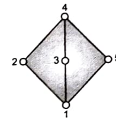


Fig. 3.52

Now 2 ∧ 3 = 1, 2 ∨ 3 = 30

2 ∧ 5 = 1, 2 ∨ 5 = 30

Hence 2 has two complements 3 and 5.

Hence complement is not unique.

However, as the following theorem will prove, there is a special type of lattice, in which every element has a unique complement.

Theorem 1: Let (L, ≤) be a complemented distributive lattice. Then every element in L has a unique complement.

Proof:

Let a ∈ L. Suppose there exist elements a_1, a_2 ∈ L

Such that $a \wedge a_1 = 0$ and $a \vee a_1 = 1$,

and $a \wedge a_2 = 0$ and $a \vee a_2 = 1$,

then, we have to prove $a_1 = a_2$.

Consider a_1 $= a_1 \wedge 1 = a_1 \wedge (a \vee a_2)$

$= (a_1 \wedge a) \vee (a_1 \wedge a_2)$ **(by distributivity)**

$= 0 \vee (a_1 \wedge a_2) = a_1 \wedge a_2$ ... (1) (Commutativity)

Similarly, $a_2 = a_2 \wedge 1 = a_2 \wedge (a \vee a_1)$

$= (a_2 \wedge a) \vee (a_2 \wedge a_1) = 0 \vee (a_2 \wedge a_1)$

$= a_1 \wedge a_2$ ... (2)

From (1) and (2), a_1 and a_2 are both equal to $a_1 \wedge a_2$.

Hence $a_1 = a_2$, which proves the uniqueness of the complement.

We denote the complement of a by a'.

The following theorem proves De Morgan's laws for a complemented distributive lattice.

Theorem 2: Let (L, ≤) be a complemented distributive lattice.

Show that (a ∨ b)' = a' ∧ b' and (a ∧ b)' = a' ∨ b'.

Proof:

We use the uniqueness of complement.

Consider $(a \vee b) \vee (a' \wedge b')$

$$= (a \vee b \vee a') \wedge (a \vee b \vee b')$$

$$= ((a \vee a') \vee b) \wedge (a \vee 1)$$

(By associativity, commutativity and distributivity)

$$= (1 \vee b) \wedge (a \vee 1) = |\wedge| = 1$$

Next consider $(a \vee b) \wedge (a' \wedge b')$

$$= ((a \wedge a') \wedge b') \vee (b \wedge b' \wedge a')$$

(Using distributivity, commutativity and associativity)

$$= (0 \wedge b) \vee (0 \wedge a')$$

$$= 0 \vee 0 = 0$$

Hence $a' \wedge b'$ satisfies the conditions for complement of $a \vee b$. By uniqueness of complement, it follows that

$$(a \vee b') = a' \wedge b'.$$

One can similarly prove that

$$(a \wedge b)' = a' \vee b'.$$

3.15.9 Principle of Duality

Any statement about lattices involving the join and meet operations and the relations $\leq$ = $\geq$ remains true if $\wedge$ is replaced by $\vee$ and $\vee$ by $\wedge$, $\leq$ by $\geq$ and $\geq$ by $\leq$. (= remains as =).

Hence if "$a \vee a = a$" is true, then so is "$a \wedge a = a$".

If "$a \wedge b \leq a$" is true, then so is "$a \vee b' \geq a$" (or "$a \leq a \vee b$") is true.

If a lattice has universal bounds 0 and 1, then in the dual statement, 0 is replaced by 1 and 1 is replaced by 0.

This concept is known as the principle of duality. Use this principle in the following theorem.

Theorem 1: If the meet operation is distributive over the join operation in a lattice, then the join operation is also distributive over the meet operation. The converse is also true.

Proof:

It is given that $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

We obtain $(a \vee b) \wedge (a \vee c) = [(a \vee b) \wedge a] \vee [(a \vee b) \wedge c]$

$$= a \vee [(a \vee b) \wedge c]$$

$$= a \vee [(a \wedge c) \vee (b \wedge c)]$$

$$= [a \vee (a \wedge c)] \vee (b \wedge c)$$

$$= a \vee (b \wedge c)$$

Hence we have proved that if the meet operation is distributive over the join operation, the join operation is also distributive over the meet operation.

The converse is obtained by the principle of duality.

SOLVED EXAMPLES

Example 1: Show that in a distributive lattice $(A, \leq)$ if $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$ for some a, then $x = y$.

Solution: By Absorption property

$$x = x \wedge (a \vee x) = (x \wedge a) \vee (x \wedge x)$$

(Distributive law)

$$= (a \wedge x) \vee x$$

(Commutativity and Idempotent)

$$= (a \wedge y) \vee x$$

$$= (a \vee x) \wedge (y \vee x)$$

$$= (a \vee y) \wedge (x \vee y)$$

$$= (a \wedge x) \vee y = (a \wedge y) \vee y = y$$

***Example 2:** Let $(A, \leq)$ be a lattice with a universal upper and lower bounds 0 and 1. For any element $a \in A$, prove*

$$a \vee 1 = a, \qquad a \wedge 1 = a$$

$$a \vee 0 = a, \qquad a \wedge 0 = 0$$

Solution: $a \vee 1 = \text{lub } \{a, 1\}$

Hence by definition $1 \leq a \vee 1$, but by definition glb, $a \vee 1 \leq 1$

Hence $a \vee 1 = 1$

$a \vee 0 = \text{lub } \{a, 0\}$

Hence $a \leq a \vee 0$

Now $0 \leq a$ for any $a \in A$

and $a \leq a$.

Hence a is an upper bound for {a, 0}. But $a \vee 0$ is the least upper bound for {a, 0}. Hence it follows that $a \vee 0 \leq a$.

By reflexivity of $\leq$, we obtain

$a \vee 0 = a$

$a \wedge 1 = a$ is the dual statement of

$a \vee 0 = a$, and

$a \wedge 0 = 0$ is the dual of

$a \vee 1 = 1$

Hence by the principle of duality, these statements are also true.

***Example 3:** Let $(A, \leq)$ be a lattice. For any $a, b \in A$, prove that $a \leq b$ iff $a \wedge b = a$ iff $a \vee b = b$. Let $a \leq b$. Then a is a lower bound for {a, b}. But $a \wedge b$ = glb {a, b}. Hence $a \leq a \wedge b$. By definition of $a \wedge b$, $a \wedge b \leq a$. Hence by reflexivity of $\leq$ $a \wedge b = a$.*

Conversely let $a \wedge b = a$.

Hence $a = \text{glb } \{a, b\}$

Therefore $a \leq b$.

Similarly, we can prove that $a \leq b$ iff $a \vee b = b$.

***Example 4:** Prove that in a complemented distributive lattice, if $b \wedge \bar{c} = 0$, then $b \leq c$.*

Solution: $b = b \wedge 1 = b \wedge (c \vee \bar{c})$

$$= (b \wedge c) \vee (b \wedge \bar{c}) = (b \wedge c) \vee 0$$

$$= b \wedge c$$

But $b \wedge c = \text{glb } \{b, c\}$

Hence $b \leq c$.

***Example 5:** Show that the set of all divisors of 70 forms a lattice.*

Solution: Let A = {1, 2, 5, 7, 10, 14, 35, 70}, and let '$\leq$' is "a divisor of ".

The join operation $\vee$ = lcm {a, b} and meet $\wedge$ = gcd {a, b}.

The universal upper bound '1' is 70 and the lower bound '0' is 1.

The Hasse diagram is

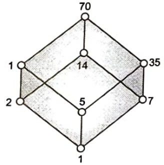


Fig. 3.53

***Example 6:** For the set X = {2, 3, 6, 12, 24, 36}, a relation $\leq$ is defined as $x \leq y$ if x divides y. Draw the Hasse diagram for $(X, \leq)$. Answer the following:*

(i) What are the maximal elements?

(ii) What are the minimal elements?

(iii) Give one example of chain and one example of antichain.

(iv) What is the maximum length of chain?

(v) Is the poset a lattice?

Solution:

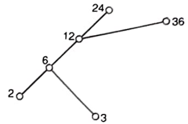


Fig. 3.54

(i) Maximal elements are 24, 36.

(ii) Minimal elements are 2, 3.

(iii) Chain: {2, 6, 12, 24}

Antichain: {2, 3} or {24, 36}.

(iv) Maximum length of chain is 3.

(v) The poset is not a lattice since the set {2, 3} has no greatest lower bound.

The set {24, 36} has no least upper bound.

***Example 7:** Let A is set of factors of positive integer m and relation is divisibility on A. i.e. R = {(x, y) | x, y $\in$ A, x divides y}.*

For m = 45, show that POSET $(A, \leq)$ is a lattice. Draw Hasse diagram and give join and meet for the lattice.

Solution: A = {1, 3, 5, 9, 15, 45}

R = {(1, 1), (1, 3), (1, 5), (1, 9), (1, 15), (1, 45), (3, 3), (3, 9), (3, 15), (3, 45), (5, 5), (5, 15), (5, 45), (9, 9), (9, 45), (15, 15), (15, 45), (45, 45)}

For any pair of elements $a, b \in A$,

lub {a, b} = lcm of a and b

and glb {a, b} = gcd of a and b

For e.g. lub {3, 5} = 15

and gcd {3, 5} = 1

Tables of operations for $\wedge$ and $\vee$ operations where $\wedge$ = gcd and $\vee$ = lcm.

$\wedge$	1	3	5	9	15	45
1	1	1	1	1	1	1
3	1	3	1	3	3	3
5	1	1	5	1	5	5
9	1	3	1	9	3	9
15	1	3	5	3	15	15
45	1	3	5	9	15	45

$\vee$	1	3	5	9	15	45
1	1	3	5	9	15	45
3	3	3	15	9	15	45
5	5	15	5	45	15	45
9	9	9	45	9	45	45
15	15	15	15	45	15	45
45	45	45	45	45	45	45

Hasse Diagram:

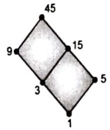


Fig. 3.55

Hence (A, ≤) is a lattice.

Example 8: *For any a, b, c, d in a lattice (A, ≤), if (a ≤ b) and (c ≤ d) then prove*

(i) $a \vee c \leq b \vee d$, (ii) $a \wedge c \leq b \wedge d$.

Solution: (i) $a \leq b$ and $b \leq b \vee d$.

Hence $\quad a \leq b \vee d$. (by transitivity of ≤)

Similarly $\quad c \leq d$ and $d \leq b \vee d$

$c \leq b \vee d$.

Hence $b \vee d$ is an upper bound for {a, c}. But by definition

$a \vee c = \text{lub}\{a, c\}$

Hence $\quad a \vee c \leq b \vee d$.

(ii) is proved on similar lines.

Example 9: *For set A = {a, b} and lattice (P(A), ⊴), construct the tables for ∨ and ∧*

P(A) = {A, ϕ {a}, {b}}

A + 1 and ϕ = 0

∨ + ∪ (union), ∧ + ∩ (intersection)

We have the following tables for ∨ and ∧ operations.

∨	0	1	{a}	{b}
0	0	1	{a}	{b}
1	1	1	1	1
{a}	{a}	1	{a}	1
{b}	{b}	1	1	{b}

∧	0	1	{a}	{b}
0	0	0	0	0
1	0	1	{a}	{b}
{a}	0	{a}	{a}	0
{b}	0	{b}	0	{b}

Example 10 : *Show that the set of all divisors of 36 forms a lattice.* **(Nov./Dec. 14)**

Solution : Let A denote the set of all divisors of 36.

Then $\quad$ A = {1, 2, 3, 4, 6, 9, 12, 18, 36}

R is the relation 'a is a divisor of b'.

R = {(1, 2), (1, 3), (1, 4), (1, 6), (1, 9,) (1, 12), (1, 18), (1, 36), (2, 2), (2, 4), (2, 6), (2, 12), (2, 18), (2, 36), (3, 3), (3, 6) (3, 9), (3, 12), (3, 18), (3, 36), (4, 4), (4, 12), (4, 36), (6, 6), (6, 12), (6, 18), (6, 36), (9, 9), (9, 18), (9, 36), (12, 12), (12, 36), (18, 18), (18, 36), (36, 36)}

Define $a \vee b$ = *l*cm {a, b} and $a \wedge b$ = g c d {a, b}.

The Hasse diagram of (A, R) is :

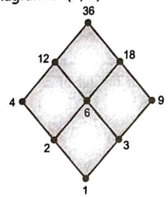


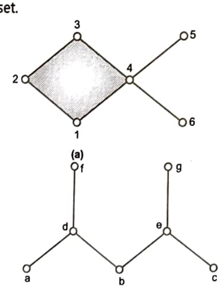
Fig. 3.56

It is clear from the Hasse diagram that every pair of elements {a, b} has a *l*ub given by $a \vee b$ and g*l*b given by $a \wedge b$.

Hence, (A, R) is a lattice.

EXERCISE 3.4

1. Determine all the maximal and minimal elements of the poset.



(a) (b)

Fig. 3.57

2. In (1) above, determine, if any, least upper bound and greatest lower bound of the following sets:
   (a) {2, 4, 6}, {3, 5}, {1, 6}, {2, 3, 5}.
   (b) {d, e}, {f, g}, {a, d, e}, {d, g}.
3. On each of the following sets, let the partial order ≤ denote 'is a divisor of '. Draw the corresponding Hasse diagrams and determine which posets are lattices.
   (i) A = {1, 2, 3, 5, 30}
   (ii) A = {1, 2, 3, 4, 6}
   (iii) A = {2, 3, 4, 16, 12, 24, 36}
   (iv) A = {1, 3, 5, 9, 15, 45}
   (v) A = {2, 3, 5, 7, 10, 14, 21}
4. Is the Cartesian product of two lattices always a lattice? Prove your claim.
5. If a and b are elements in a lattice (A, ≤), then show that $a \wedge b = b$ iff $a \vee b = a$.
6. Prove the associative laws for a lattice.
   i.e. $a \vee (b \vee c) = (a \vee b) \vee c$ and $a \wedge (b \wedge c) = (a \wedge b) \wedge c$
7. For elements a, b, c in a lattice (A, ≤), show that if a ≤ b then $a \vee (b \wedge c) \leq b \wedge (a \vee c)$.
8. Show that a lattice (A, ≤) is distributive iff for any element a, b, c in A,
   $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$
   [Hint: To show that (A, ≤) is distributive, consider the elements a, $b \vee c$ and $(a \vee b) \wedge (a \vee c)$].
9. A lattice (A, ≤) is called a modular lattice if for any a, b, c in A where a ≤ c, $a \vee (b \wedge c) = (a \vee b) \wedge c$.
   Show that a lattice is modular iff
   $a \vee (b \wedge (a \vee c)) = (a \vee b) \wedge (a \vee c)$.
10. Show that for any elements a, b, c in a modular lattice,
    $(a \vee b) \wedge c = b \wedge c$ implies $(a \vee b) \wedge a + b \vee a$.
11. Let n be a positive integer and let S_n be the set of all divisors of n. Let D denote the relation of 'division'. Draw the diagrams of lattices for (i) n = 24, (ii) n = 30, (iii) n = 6.

POINTS TO REMEMBER

- A common notion of relation is a type of association that exists between two or more objects.
- An **ordered n-tuple,** for n > 0, is a sequence of objects or elements, denoted by $(a_1, a_2, \ldots, a_n)$. If n = 2, the ordered n-tuple is called an **ordered pair.**

  If n = 3, the ordered n-tuple is called an ordered triple; and so on.

  Let A and B be non-empty sets. The **product set** or the **Cartesian product** A × B is defined as

  $$A \times B = \{ (a, b) \mid a \in A \text{ and } b \in B \}.$$

  If A = ϕ or B = ϕ, then A × B = ϕ.
- Let $\{A_1, A_2, \ldots, A_n\}$ be a finite collection of sets. A subset R of $A_1 \times A_2 \times \ldots \times A_n$ is called an **n-ary relation** on $A_1, A_2, \ldots, A_n$.
- Let A and B be non-empty sets. Then a binary relation R from A to B is a subset of A × B, i.e. R ⊆ A × B. The **domain** of R, denoted by D(R), is the set of elements in A that are related to some element in B, i.e.

  $$D(R) = \left\{ a \in A \mid \text{for some } b \in B,\ (a, b) \in R \right\}$$
- Let R be a relation from A to B. Then the **converse** of R, denoted by R^c is the relation from B to A, defined as

  $$R^c = \left\{ (b, a) \mid (a, b) \in R \right\}$$
- Let R_1 be a relation from A to B and R_2 a relation from B to C. The **composite relation** from A to C, denoted by $R_1 \bullet R_2$ (or $R_1 R_2$) is defined as

  $$R_1 \bullet R_2 = \left\{ (a, c) \mid a \in A \wedge c \in C \wedge \exists b\, [b \in B \wedge (a, b) \in R_1 \wedge (b, c) \in R_2] \right\}$$

  A relation matrix has entries which are either one or zero. Such a matrix is called a Boolean matrix.
- **Properties of Relation Matrix are:**
  1. $M_{R_1 \cdot R_2} = M_{R_1} \cdot M_{R_2}$
  2. M_{R^c} = transpose of M_R (for $R = R_1$ or $R = R_2$)
  3. $M_{(R_1 \cdot R_2)^c} = M_{R_2^c R_1^c} = M_{R_2^c} \cdot M_{R_1^c}$
- Special properties of binary relations are
  (A) Reflexive relation (B) Irreflexive relation
  (C) Symmetric relation (D) Asymmetric relation
  (E) Antisymmetric relation (F) Transitive relation
- A binary relation R on a set A is said to be compatible if it is reflexive and symmetric.

- Let π and π' be partitions of a non-empty set A. Then π' is called a refinement of π if every block (element) of π' is contained in a block of π.
- Let π_1 and π_2 be partitions of a non-empty set A. The product of π_1 and π_2, denoted $\pi_1 \cdot \pi_2$ is a partition π of A such that
  (i) π refines both π_1 and π_2.
  (ii) If π' refines both π_1 and π_2, then π' refines π.
- Let π_1 and π_2 be partitions of a non-empty set. The sum of π_1 and π_2, denoted by $\pi_1 + \pi_2$, is a partition π such that
  (i) Both π_1 and π_2 refine π.
  (ii) If π' is a partition of A such that both π_1 and π_2 refine π', then π refines π'.
- A relation R on a set A is said to be compatible if it is reflexive and symmetric.
- A binary relation R on a non-empty set A is a **partial order** if R is reflexive, antisymmetric and transitive.
- The ordered pair (A, R) is called a **partially ordered set** or **poset.**
- The transitive closure of a relation R is the smallest transitive relation containing R. We denote transitive closure of R by R^*.
- A binary relation R on a non-empty set A is a **partial order** if R is reflexive, antisymmetric and transitive.
- Let (A, $\leq$) be a poset. A subset of A is called a **chain** if every pair of elements in the subset is related.
- A subset of A is called an antichain if no two distinct elements in the subset are related.
- An element $a \in A$ is called a **maximal element** if there is no element $b \in A$ such that $b \neq a$ and $a \leq b$. An element $c \in A$ is called a minimal element if there is no element $d \in A$ such that $d \neq c$ and $d \leq a$.
- A lattice is a poset (L, $\leq$) in which every subset {a, b} of L, has a least upper bound and a greatest lower bound.
- Let (L, $\leq$) be a lattice with universal bounds 0 and 1. For an element $a \in L$, b is said to be a complement of a if $a \vee b = 1$ and $a \wedge b = 0$.

◈◈◈

CHAPTER 4
FUNCTIONS

4.1 INTRODUCTION

- Function is a special type of relation. It is basically an input-output relation. Many concepts in computer science can be conveniently stated in the language of functions. Recursive and generating functions are of special importance in software development.
- In this chapter, we discuss infinite sets and their cardinalities, where the concept of bijective function is used. We will also briefly discuss an important principle called as the Pigeonhole Principle, and use it to solve some problems related to counting.

4.2 FUNCTIONS

In this section, we deal with functions which form a special class of binary relations. We shall discuss the various properties of functions and focus our attention on some special types of functions.

4.2.1 Definitions

> Let A and B be non-empty sets. A **function** f from A to B, denoted as f: A $\to$ B, is a relation from A to B such that for every $a \in A$, there exists a **unique** $b \in B$ such that $(a, b) \in f$.
>
> Normally if $(a, b) \in f$, we write $f(a) = b$.

An important point to be re-emphasised is that f is a relation with the following special property:

If $f(a) = b$ and $f(a) = c$ then $b = c$.

This condition implies that to each element $a \in A$, a unique element $b \in B$ should be assigned by the relation f.

Consider the following relation

$$f: R^+ \to R \text{ where}$$

$$f(x) = \sqrt{x}$$

f is obviously not a function since $f(4) = +2$ as well as $-\sqrt{2}$.

Hence in general only a many-to-one relation or a one-to-one relation is a function. A one-to-many relation is not a function.

The set A is called as the **domain** of f, denoted **D(f).** The set B is called as the **codomain**, and the set $\{ f(a) \mid a \in A \}$, which is a subset of B, is called as the **range** of f, and denoted as **R(f).** The element a is called an **argument** of the function f and f(a) is called the **value** of the function for the argument a.

As t is a relation we may also express f as a set of ordered pairs, i.e.

$$f = \{ (a, f(a)) \mid a \in A, f(a) \in B \}.$$

Functions are also called as **mappings** or **transformations,** since they can be thought of as rules for assigning to each element $a \in A$, the unique element $f(a) \in B$. In this context, it is customary to refer to f(a) as **image** of a and a as the **pre-image** of b which is equal to f(a).

A typical way of representing a function graphically is given below Fig. 4.1.

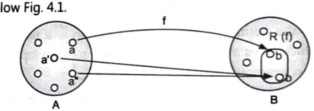


Fig. 4.1

$R(f) \subseteq B$, i.e. range of f is a proper subset of B or is equal to B.

To describe a function completely it is necessary to specify its domain, codomain and the value f(a), for each argument a.

Examples :

1. Let A be a non-empty set. Then we can always define a function f: A $\to$ A (i.e. B = A) as f(a) = a.
   f is called the **identity** function on A and is denoted by $\mathbf{1_A}$.
2. Let f: N $\to$ N be defined as f(x) = 2x.
   f is a function with R(f) being the set of even natural numbers.
3. Let f: **R** $\to$ **R** be defined as $f(x) = x^2$.
   f is a function. Geometrically, R(f) is the parabola $y = x^2$.

4. Let $A = \{1, 2, 3\}$ and $B = \{a, b, c, d\}$

Let $f: A \rightarrow B$ be defined as

$f(1) = a$

$f(2) = c$

$f(3) = a$

f is a function, which is graphically represented as

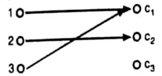


Fig. 4.2

$R(f) = \{a, c\}$.

5. Let $A = \{a, b, c\}$ and $B = \{e, f\}$

Let $R = \{(a, e), (b, e), (a, f), (c, e)\}$

The graph of R is

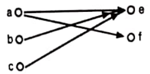


Fig. 4.3

The relation R is not a function since $f(a) = e$ as well as $f(a) = f$, which violates the definition of a function.

To the computer engineer, a function is a procedure which gives a unique output for any suitable input. The next example demonstrates this aspect.

6. *Let P be a computer program that accepts an integer as input and produces an integer as output. Let A = B = Z. Then P determines a relation f_P as follows: $(a, b) \in f_P$ implies that b is the output produced by program P when the input is a. f_P is clearly a function, since any particular input corresponds to a unique output.*

7. Let A be a finite set and let P(A) denote its power set. Define

$f: P(A) \rightarrow Z^+$ by

$f(S) = |S|$, for any $S \in P(A)$, (i.e. $S \subseteq A$),

|S| denotes the cardinality of S. f is clearly a function.

4.2.2 Partial Functions

In actual application of functions, it is often convenient to treat the domain of a function as a subset of another set known as the **source.** (In this case, the codomain is appropriately called as the **target** set). In other words, the function has the set A as its domain but is not defined for some arguments. This leads to the following definition.

Definition:

Let A and B be two sets. A partial function f with **domain A** and codomain B is any function from A' to B where $A' \subset A$. For any element $x \in A - A'$, the value of f(x) is said to be undefined

A partial function

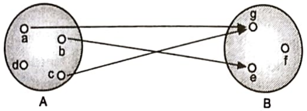


Fig. 4.4

To make the distinction more clear, the function which is not a partial function is sometimes called as a **total function**.

However, in what follows, we will use the unqualified term "function" to denote total function and the qualifier "partial" while referring to partial functions.

Examples :

1. The function $f: \mathbf{R} \rightarrow \mathbf{R}$ defined as $f(x) = 1/x$, is a partial function, as it is undefined for $x = 0$.
2. The function $f: \mathbf{R} \rightarrow \mathbf{R}$ defined as $f(x) = \sqrt{x}$ is a partial function, as $\sqrt{x}$ is not defined for $x < 0$, in **R.**
3. Computer programs represent partial functions. Let P be a program which has one natural number as its input, and which, for some input values will never terminate, or terminates abnormally (e.g. while attempting to divide by 0, an illegal operation). Then P is not defined for such arguments and hence can be regarded as a partial function from N to **R.** The following function plays an important role in computer applications.
4. **Hashing Functions:**

   A symbol table is constructed by a compiler.

   The identifiers in a program are read and inserted into a table (say) with 1000 spaces, labelled 0 to 999.

   A unique identifier is called a key.

   To determine to which space (location) in the table a particular key is assigned we create a hashing function from the set of keys to the set of locations.

Hashing functions generally use a mod function.)

Let I = Set of all possible identifiers

and $N = \{0, 1, ..., 999\}$. We may define

$f: I \rightarrow N$ as

$f(i) = |i|^3 \pmod{1000}$

For example,

$f(23) = 23^3 \pmod{1000}$

$= 12167 \pmod{1000}$

$= 167$

Hence an identifier with length 23 characters is inserted in position 167.

4.2.3 Equivalent Functions

Definition:

Let $f: A \rightarrow B$ and $g: C \rightarrow D$ be functions. Then f and g are said to be equivalent or **identical** only if $A = C$, $B = D$ and $f(a) = g(a)$ for all $a \in A$.

The function $f: Z \rightarrow Z$ given by $f(x) = x$ and $g: Z \rightarrow Z^+$ given by $g(x) = x$ are not equivalent.

4.2.4 Composite Function

Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be two functions. Then the composition of f and g denoted as **gof** is a relation from A to C, where $gof(a) = g(f(a))$. $gof: A \rightarrow C$ is also a function. This is because if there exists elements $c, d \in C$ such that $gof(a) = c$ and $gof(a) = d$, for some $a \in A$, this would imply that $f(f(a)) = c$ and $g(f(a)) = d$. But f is a function, hence f(a) is unique. Then since g is also a function, it follows that $c = d$. Hence gof is a function from A to C. Note that gof is defined only when the range of f is a subset of the domain of g. The Fig. 4.5 given below depicts a composite function.

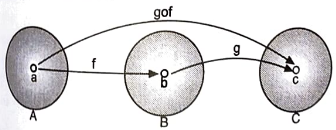


Fig. 4.5

The rule to compose two functions can be extended to a finite number of functions: $f_1: A_1 \rightarrow A_2$, $f_2: A_2 \rightarrow B_3$, ..., $f_n: A_n \rightarrow A_{n+1}$, where range of f_i = domain of f_{i+1}, for $1 \le i \le n$. Thus $f_n o\, f_{n-1}$ o ... o f_1 (denoted usually as $f_n f_{n-1} \dots f_1$) is a function from A_1 to A_{n+1}.

In particular if $A_1 = A_2 = ... = A_{n+1} = A$ and $f_1 = f_2 = ... = f_n = f$ then f o f o ... o f (n times) denoted as f^n is the composite function from A to A.

Examples :

1. Let $f: Z \rightarrow Z$ be defined as

   $f(x) = x^2 + 2x + 2$, and

   $g: Z \rightarrow Z$ be defined as

   $g(x) = x - 1$

   Then $gof: Z \rightarrow Z$ is defined as

   $gof(x) = x^2 + 2x + 2 - 1 = (x + 1)^2$

   In this case, we also have the function $fog: Z \rightarrow Z$ which is defined

   $fog(x) = (x - 1)^2 + 2(x - 1) + 2$

   $= x^2 + 1$

   $fof: Z \rightarrow Z$ is defined as

   $fof(x) = f(x^2 + 2x + 2)$

   $= (x^2 + 2x + 2)^2 + 2(x^2 + 2x + 2) + 2$

   $gog: Z \rightarrow Z$ is defined as

   $gog(x) = g(x - 1) = x - 1 - 1 = x - 2.$

2. Let $f: Z \rightarrow \mathbf{R}$ be defined as $f(x) = \frac{(x+1)}{2}$, and

   $g: \mathbf{R} \rightarrow \mathbf{R}$ be defined as

   $g(x) = x^2$

   Then $gof: Z \rightarrow \mathbf{R}$ is defined as $gof(x) = g(f(x))$

   $= g\left(\frac{x+1}{2}\right) = \frac{(x+1)^2}{4}$

3. Let A = Set of students (or their names)

   B = Set of their examination seat numbers,

   and C = Set of the students mark lists.

   $f: A \rightarrow B$ is defined as

   $f(s) = n$, where n is the seat number of the students

   $g: B \rightarrow C$ is defined as

   $g(x) = l$, where l is the mark list corresponding to the seat number n.

   Then $gof: A \rightarrow C$ is the function defined as $gof(s) = l$, l being the mark list of the students whose seat number is n.

4. Suppose a manufacturer has a list of all the parts which are supplied to him, together with the supplier's name. He has also a list of suppliers names, together with the suppliers addresses. To obtain the address from which

to order a given part, he composes two functions f: P → S and g: S → A to obtain gof: P → A, which gives the address a of the supplier of part p.

5. Let f: A → B, g: B → C and h: C → D be defined as shown in the following graphs.

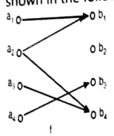


Fig. 4.6

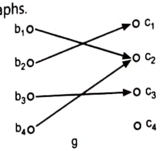


Fig. 4.7

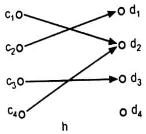


Fig. 4.8

Then we have the following composite functions

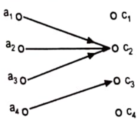


Fig. 4.9

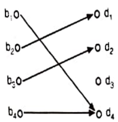


Fig. 4.10

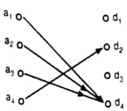


Fig. 4.11

Note that composition is associative. Hence ho (gof) = (hog) of. Hence dispersing with the brackets, we simply write hogof.

4.2.5 Special Types of Functions

Definitions:

Let f: A → B be a function.

1. f is called a **surjective** (onto) function if f(A) = B, i.e. range of f is equal to the codomain of f.
2. f is called an **injective** (one-to-one) function if for elements a, a' ∈ A, a ≠ a' implies f(a) ≠ f(a'), or equivalently if f(a) = f(a'), then a = a'.
3. f is called bijection (one-to-one and onto) function if f is both surjective and injective.

Functions with these properties are called **surjections, injections** and **bijections** respectively. The following diagrams represent these three types of functions.

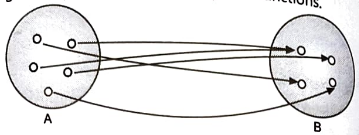


Fig. 4.12: Surjection

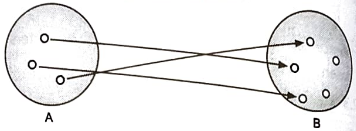


Fig. 4.13: Injection

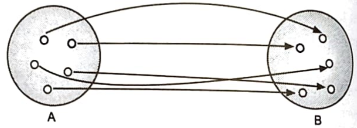


Fig. 4.14: Bijection

Examples :

1. The identity function $\mathbf{1_A}$: A → A is both surjective and injective, hence is a bijective function.
2. Let f: **R** → **R** be defined as f(x) = x + 1. Then f is an injective function. We know that geometrically, this function represents a straight line.

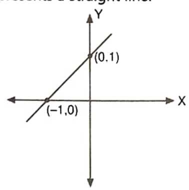


Fig. 4.15

In general, any function f defined as f(x) = mx + c is an injective function.

3. Let f: **R** → $\mathbf{R}^+$ be defined as f(x) = x^2

This is a surjective function, since for any y ∈ $\mathbf{R}^+$, (i.e. y > 0) there exists x ∈ **R** such that f(x) = y, i.e. x = $\pm\sqrt{y}$.

Geometrically this function represents the parabola y = x^2, symmetric about the y-axis.

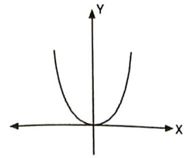


Fig. 4.16

Clearly the function is not injective.

4. The function in example (ii) is also a surjective function, since for any y ∈ **R** there exists x ∈ **R** such that f(x) = y, i.e. x + 1 = y which implies x = y – 1. This means that for every element y in the co-domain **R**, there is a pre-image x in the domain **R**, whose image is y. Hence the function f is surjective. f is therefore a bijective function.
5. Let E be the set of even integer and define a function f: E → Z as f(x) = 2x. Then f is an injective function, which is not surjective.
6. Let A be the set of students in a class, and B be the set of their roll numbers. Assign to a student his or her roll number. The assignment is then a bijective function.
7. Let A be the set of students and B be the set of their ages (in years). Assign to a student his or her age. This assignment is then a many-one function (i.e. not injective) which is surjective.
8. Let A be the set of students and B be the set of integers {0, 1, 2, ... 100}. Assign to a student an integer which is his marks (out of 100) in a particular subject, assuming that all the students in the set, appeared for this subject. Then this assignment is a function which is not necessarily injective or surjective.

The following theorems, give some important properties of the injective, surjective and bijective functions, for composite functions.

Theorem 1:

Let f: A → B and g: B → C be functions. Then

(i) If f and g are surjective functions, then gof is surjective.

Proof:

gof: A → C. We have to show gof (A) = C. Let c ∈ C; then since g is surjective, there exists an element b ∈ B such that g(b) = c.

Since f is surjective as well, for the element b ∈ B, there exists an element a ∈ A such that f(a) = b.

Then gof (a) = g (f(a)) = g(b) = c.

Hence c ∈ gof (A), i.e. gof is surjective.

Let f: A → B and g: B → C be functions. Then

(ii) If f and g are injective, then gof is injective

Proof:

Let elements a, a' ∈ A such that gof (a) = gof (a').

We have to prove a = a'.

Let f(a) = b and f(a') = b', where elements b, b' ∈ B.

Then gof (a) = gof (a') implies g(b) = g(b').

But g is injective; hence b = b'.

This implies f(a) = f(a').

Since f is injective, we have a = a'.

Hence gof is injective.

Let f: A → B and g: B → C be functions. Then

(iii) If f and g are bijective, then gof is bijective.

Proof:

(iii) Since (i) and (ii) are true, it follows that gof is bijective.

However, converse to the theorem is not true; as shown by the following examples:

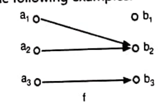


Fig. 4.17

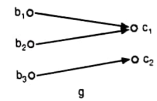


Fig. 4.18

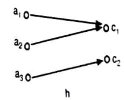


Fig. 4.19

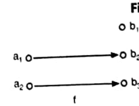


Fig. 4.20

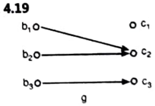


Fig. 4.21

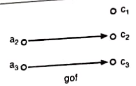


Fig. 4.22

The following theorem, however gives a "partial converse" to the above theorem.

Theorem 2:

Let $f: A \to B$ and $g: B \to C$ be functions. Then If gof is surjective, then g is surjective.

Proof:

We have to prove $g(B) = C$.

Let $c \in C$. Since gof is surjective, there exists an element $a \in A$ such that gof $(a) = c$, i.e. $g(f(a)) = c$.

Let $f(a) = b$.

Then $g(b) = c$ which implies that b is the pre-image of c in B.

Hence $g(B) = C$, i.e. g is surjective.

Let $f: A \to B$ and $g: B \to C$ be functions. Then If gof is injective, then f is injective.

Proof:

Let for elements $a, a' \in A$, $f(a) = f(a')$.

We have to prove $a = a'$. Since $f(a) = f(a')$, if follows that $g(f(a)) = g(f(a'))$ i.e. gof (a) = gof (a').

Since gof is injective, it follows that $a = a'$.

Hence f is injective

Let $f: A \to B$ and $g: B \to C$ be functions. Then If gof is bijective, then g is surjective and f is injective.

Proof:

This statement is true, as consequence of (i) and (ii).

4.2.6 Inverse Function

The concept of inverse of a function is analogous to that of the converse of a relation.

Definition:

Let $f: A \to B$ be a bijection from A to B. The inverse of f denoted by f^{-1} is the function $f^{-1}: B \to A$ such that

$f^{-1} o f = 1_A$ and

$f o f^{-1} = 1_B$

Example : Let $f: \{a_1, a_2, a_3\} \to \{b_1, b_2, b_3\}$ be defined as

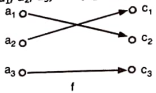


Fig. 4.23

Then f^{-1} is given by the graph

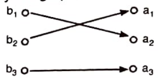


Fig. 4.24

Then $f^{-1} of = 1_A$ and $f o f^{-1} = 1_B$.

1. Properties of the Inverse Function

(i) $(f^{-1})^{-1} = f$. (Proof left as an easy exercise)

(ii) If f and g are bijective functions from A to B, and B to C respectively, then $(gof)^{-1} = f^{-1} o f^{-1}$.

Proof:

First observe that both $(gof)^{-1}$ and $f^{-1} o^{-1}$ are functions from C to A.

Hence domains of $(gof)^{-1}$ and $f^{-1} o g^{-1}$ are equal; and so are their codomains.

We have only to prove $(gof)^{-1}(c) = f^{-1} o g^{-1}(c)$ for all $c \in C$.

Let $(gof)^{-1}(c) = a$, then (gof) $(a) = c$ which means that $g(f(a)) = c$ since g^{-1} exists, we have $f(a) = g^{-1}(c)$.

Since f^{-1} also exists, $a = f^{-1}(g^{-1}(c)) = f^{-1} o g^{-1}(c)$.

Hence $(gof)^{-1}(c) = f^{-1} o g^{-1}(c)$ for all $c \in C$.

Hence $(gof)^{-1} = f^{-1} o g^{-1}$

2. One-Sided Inverse Functions

We have seen that if $f: A \to B$ is a bijective function, then f^{-1} exists and $f^{-1} o f = 1_A$ and $f o f^{-1} = 1_B$.

We then say that f has a **left inverse** as well as a **right inverse**. Only bijections have a two sided inverse. However, there are some functions which possess one-sided inverses. The existence of these one-sided inverses is determined by whether the function is injective or surjective.

Definition:

Let $f: A \to B$ and $g: B \to A$ be functions. If gof $= 1_A$ then g is a left inverse of f, while f is a right inverse of g.

We have the following theorem.

Let $f: A \to B$ be a function $(A \neq \phi, B \neq \phi)$ f has a left inverse if and only if f is injective.

Proof:

Assume first that f is injective.

We have to show that f has a left inverse, i.e. we must define a function $g: B \to A$ such that gof $= 1_A$.

Let $b \in B$.

Then either $b \in f(A)$ or $b \in B - f(A)$.

If $b \in f(A)$, then $b = f(a)$ for some $a \in A$.

In that case, define $g(b) = a$.

Otherwise, if $b \in B - f(A)$, choose any arbitrary element $a' \in A$ and define $g(b) = a'$.

The function g is well-defined since exactly one value is specified for each $b \in B$ as f is injective.

Then gof $(b) = g(b) = a$. Hence gof $= 1_A$.

Conversely, let f have a left inverse.

We have to show that f is injective.

Let $g: B \to A$ such that gof $= 1_A$. Let $f(a) = f(a')$.

We have to show $a = a'$.

Now $g(f(a)) = g(f(a'))$ since g is well defined.

This implies $1_A(a) = 1_A(a')$ i.e. $a = a'$.

Hence f is injective.

Let $f: A \to B$ be a function $(A \neq \phi, B \neq \phi)$ f has a right inverse if and only if f is surjective.

Proof:

Let f be surjective.

We have to show that f has a right inverse, i.e. we must define a function $g: B \to A$ so that fog $= 1_B$.

Let $b \in B$.

Since f is surjective, there exists an element $a \in A$ such that $f(a) = b$.

Define $g(b) = a$.

Then fog $(b) = f(a) = b$. Hence fog $= 1_B$.

Conversely, let f have a right inverse $g: B \to A$ such that fog $= 1_B$.

We have to show that f is surjective.

Let $b \in B$ and let $g(b) = a$.

Then fog $(b) = b$ which implies that $f(a) = b$.

Hence f is surjective, as there exists an element $a \in A$ such that $f(a) = b$.

The next theorem deals with bijective function.

Theorem 2: If $f: A \to B$ is bijective, then the left and right inverses of f are equal.

Proof: Left as an exercise.

SOLVED EXAMPLES

Example 1: *A function $f: N \to N$, where N is the set of natural members including 0. Comment on the type of the following functions (one-one/onto etc.)*

(i) $f(j) = j^2 + 2$

(ii) $f(j) = 1$; if j is odd

$= 0$; if j is even.

Solution: (i) f is one-one function since if $f(j) = f(k)$, then $j^2 + 2 = k^2 + 2$ which implies $j = k$.

But f is not onto since there is no natural number $j \in N$, such that $f(j) = 0$, since $j^2 + 2 = 0 \to j^2 = -2$. Hence 0 has no pre-image. Hence f is not onto.

(ii) f is not onto since $R(f) = \{0, 1\}$. f is also not one-one since all odd numbers (even numbers) have the same image.

Example 2: *Functions f, g, h are defined on a set*

$X = \{1, 2, 3\}$ as

$f = \{(1, 2), (2, 3), (3, 1)\}$

$g = \{(1, 2), (2, 1), (3, 3)\}$

$h = \{(1, 1), (2, 2), (3, 1)\}$.

(i) Find fog, gof. Are they equal?

(ii) Find fogoh and fohog.

Solution: We may depict f, g, h graphically as

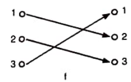


Fig. 4.25

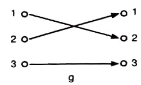


Fig. 4.26

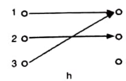


Fig. 4.27

(i) fog is depicted as

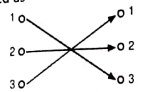


Fig. 4.28

Hence fog = {(1, 3), (3, 1), (2, 2)}

gof is depicted as

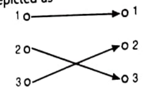


Fig. 4.29

∴ gof = {(1, 1), (2, 3), (3, 2)}

∴ fog ≠ gof

(ii) fogoh = (fog) oh, which can be depicted as

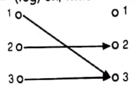


Fig. 4.30

∴ fogoh = {(1, 3), (2, 2), (3, 3)}

fohog = fo (hog)

hog can be depicted as

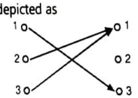


Fig. 4.31

∴ fo (hog) is

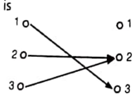


Fig. 4.32

∴ fohog = {(1, 3), (2, 2), (3, 2)}

Example 3: *Let A = {a, b, c, d}, B = {s, t, u}, C = {l, m, n}. Obtain the composition of the following functions f: A → B, g: B → C*

where f = {(a, s), (b, t), (c, u), (d, t)}

g = {(s, m), (t, l), (u, n)}.

Solution:

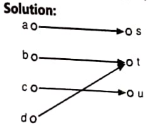


Fig. 4.33

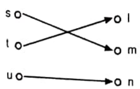


Fig. 4.34

then gof is

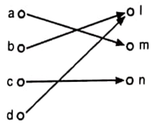


Fig. 4.35

∴ gof = {(a, m), (b, l), (c, n), (d, l)}.

Example 4: *Let A = {1, 2, 3, 4, 5}, g: A → A is as shown in the figure.*

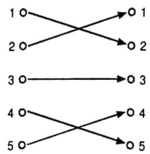


Fig. 4.36

Find the composition gog, go (gog). Determine whether each is one-to-one or onto function.

Solution: gog is

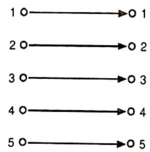


Fig. 4.37

gog is one-one and onto function.

go (gog) is

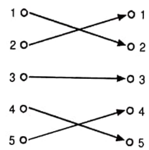


Fig. 4.38

go (gog) is also one-one and onto function.

Example 5: *The functions f: A → B, g: B → C, h: C → D are defined in the following diagram. Determine the range of each function. State which functions are into and which are onto. Draw the diagram of the composite function hogof.*

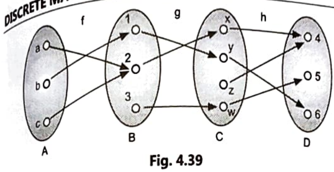


Fig. 4.39

Solution: R(f) = {1, 2}, f is onto

R(g) = {x, y, w}, g is into

R(h) = {4, 5, 6}, h is onto.

gof is

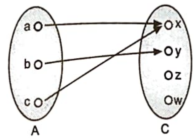


Fig. 4.40

h o (gof) is

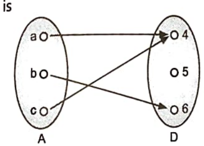


Fig. 4.41

Example 6: *State whether the following functions are one-one*

(i) To each person on the earth assign the number which corresponds to his age.

(ii) To each country assign the number of people living in the country.

(iii) To each book written by only one author, assign the author.

(iv) To each country having prime minister, assign the prime minister.

Solution:

(i) Many than one person can have the same age. f is not one-one but many-one.

(ii) More than one country may not have exactly the same population. Hence function is one-one.

(iii) The same author may have written more than one book. Hence function is many- one and not one-one.

(iv) Function is one-one.

Example 7: *Let f(x) = x + 2, g(x) = x – 2 and h(x) = 3x for x ∈ R, where R = set of real numbers. Find gof, fog, fof, gog, foh, hog, hof, fohog.* ***(May 15)***

Solution:

$$\begin{aligned}
gof(x) &= g(f(x)) = g(x+2) \\
&= (x+2) - 2 = x \\
fog(x) &= f(g(x)) = f(x-2) \\
&= (x-2) + 2 = x \\
fof(x) &= f(f(x)) = f(x+2) \\
&= (x+2) + 2 = x + 4 \\
gog(x) &= g(g(x)) = g(x-2) \\
&= (x-2) - 2 = x - 4 \\
foh(x) &= f(h(x)) = f(3x) = 3x + 2 \\
hog(x) &= h(g(x)) = h(x-2) \\
&= 3(x-2) = 3x - 6 \\
hof(x) &= h(f(x)) = h(x+2) \\
&= 3(x+2) = 3x + 6 \\
fohog(x) &= foh(g(x)) \\
&= foh(x-2) = f(h(x-2)) \\
&= f(3x-6) = (3x-6) + 2 \\
&= 3x - 4
\end{aligned}$$

Example 8: *Let f(x) = 2x + 3, g(x) = 3x + 4, h(x) = 4x for x ∈ R, where R = set of real numbers. Find gof, fog, foh, hof, goh.*

Solution:

$$\begin{aligned}
gof(x) &= g(f(x)) \\
&= g(2x+3) = 3(2x+3) + 4 \\
&= 6x + 13 \\
fog(x) &= f(g(x)) = f(3x+4) = 2(3x+4) + 3 \\
&= 6x + 11 \\
foh(x) &= f(h(x)) = f(4x) = 2(4x) + 3 = 8x + 3 \\
hof(x) &= h(2x+3) = 4(2x+3) = 8x + 12 \\
goh(x) &= g(4x) = 3(4x) + 4 = 12x + 4.
\end{aligned}$$

Example 9: *If $f(x) = x^2 + 1$ and g(x) = x + 2 are functions from R to R, where R is the set of real numbers, find fog and gof.*

Solution:

$$\begin{aligned}
fog(x) &= f(g(x)) = f(x+2) \\
&= (x+2)^2 + 1 = x^2 + 4x + 5 \\
gof(x) &= g(f(x)) = g(x^2+1) \\
&= (x^2+1) + 2 = x^2 + 3.
\end{aligned}$$

***Example 10:** Let $f(x) = ax + b$ and $g(x) = cx + d$, where a, b, c, d are constants. Determine for which constants a, b, c, d it is true that fog = gof.*

Solution:

$$fog(x) = f(g(x)) = f(cx + d)$$
$$= a(cx + d) + b = acx + ad + b$$
$$gof(x) = g(f(x)) = g(ax + b)$$
$$= c(ax + b) + d = acx + cb + d.$$
$$\therefore \quad fog = gof \Rightarrow acx + ad + b$$
$$= acx + cd + d$$
$$\therefore \quad ad + b = cb + d$$
$$\Rightarrow d(a - 1) = b(c - 1)$$
$$\text{i.e.} \quad \frac{b}{d} = \frac{a-1}{c-1}$$

is the relation between the constants if fog = gof.

***Example 11:** Let A = B = C = R and let $f: A \to B$, $g: B \to C$ be defined by $f(a) = a - 1$ and $g(b) = b^2$. Find (i) fog (2), (ii) gof (2), (iii) gof (x), (iv) fog (x), (v) fof (y), (vi) gog (y).*

Solution:

(i) $fog(2) = f(4) = 4 - 1 = 3$

(ii) $gof((2) = g(2 - 1) = 1$

(iii) $gof(x) = g(x - 1) = (x - 1)^2$

(iv) $fof(x) = f(x^2) = x^2 - 1$

(v) $fof(y) = f(y - 1) = y - 2$

(vi) $gog(y) = g(y^2) = y^4$

4.3 BIJECTION AND CARDINALITY OF FINITE SETS

Recall that cardinality of a finite set $\left(\text{denoted as } |A|\right)$ is the number of distinct elements in that set. The concept of bijection is a powerful tool to compare the cardinalities of two sets, especially for infinite sets, as we shall see later. For the present, we shall confine ourselves to finite sets. We have the following theorem.

Theorem :

Let A and B be finite sets and suppose there is a bijection from A to B. Then |A| = |B|.

Proof:

Let |A| = m and |B| = n.

Then, $A = \{a_1, a_2, a_3, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$.

Let $f: A \to B$ be the bijection.

Then since f is surjective for each $b_i \in B$ $(1 \le i \le n)$, there exists an element $a_i \in A$ such that $f(a_i) = b_i$.

This means that $m \ge n$.

But f is also injective ; hence for $a_i \ne a_j$, $f(a_i) \ne f(a_j)$, i.e. $b_i \ne b_j$, i.e. $m \le n$.

Hence |A| = |B|.

Conversely if A and B are finite sets of same cardinality, we have the following theorem.

Theorem :

If A and B are finite sets of same cardinality, and $f: A \to B$ is a function then f is injective iff f is surjective.

Proof:

Let $f: A \to B$ be injective.

Then $|\text{Range}(f)| = |A|$. As A and B have the same cardinality, and Range (f) $\subseteq$ B, it follows that Range (f) = B.

This proves that f is surjective.

Conversely, let f be surjective.

We have to prove f is injective. Let $f(a) = f(a')$ for elements a, $a' \in A$. Suppose $a \ne a'$.

Then, assuming $|A| = m \; (= |B|)$, this would imply that the remaining $(m - 2)$ elements of B.

This is possible only if at least one element is mapped onto two different elements in B, which contradicts the fact that f is a function.

Hence our supposition that $a \ne a'$ is false. Hence $a = a'$ which means that f is injective.

4.3.1 Pigeonhole Principle

In theorem we have proved that if A and B are finite sets and a bijection exists from A to B, then their cardinalities are the same.

Hence if A and B are any two sets such that $|A| > |B|$, then no bijection can exist from A to B. This fact is stated as a principle, famously known as the 'Pigeon hole principle'.

This principle states that if there are n + 1 pigeons and only n pigeonholes, then two pigeons will share the same hole.

This pigeonhole principle though self-evident (and seemingly trivial) serves as a powerful tool in solving many intricate problems in counting.

SOLVED EXAMPLES

***Example 1:** Show that if seven numbers from 1 to 12 are chosen, then two of them will add upto 13.*

Solution: We form the six different sets, each containing two numbers that add upto 13, $A_1 = \{1, 12\}$, $A_2 = \{2, 11\}$, $A_3 = \{3, 10\}$, $A_4 = \{4, 9\}$, $A_5 = \{5, 8\}$, $A_6 = \{6, 7\}$. Each of the seven numbers chosen must belong to one of these sets. Since there are only six sets, by pigeonhole principle, two of the chosen numbers must belong to the same set; hence their sum is 13.

***Example 2:** Let S be a square whose sides have length 2 units. Show that for any five points on or inside S, there must be two points whose distance apart is almost $\sqrt{2}$ units.*

Solution: Divide the square into four equal squares, as shown in Fig. 4.42. If five points are chosen in the square, we can assign each of them to a square that contains it. If a point belongs to more than one square, we assign it to one of them arbitrarily. Then the five points are assigned to four square regions, so by the pigeonhole principle at least two points must belong to the same region. These two cannot be more than $\sqrt{2}$ units apart, as the side of each square being 1 unit, the length of the diagonal is $\sqrt{2}$ units, which is the maximum distance, that the two points can be apart.

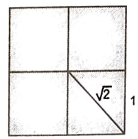


Fig. 4.42

***Example 3:** Show that if any 51 numbers are chosen from the set {1, 2, …, 100}, then one of them will be a multiple of the other.*

Solution: Every positive integer n can be written as $n = 2^k m$, where m is odd and $k \ge 0$. Since the set contains only 50 odd numbers and 51 numbers are chosen, it follows from pigeonhole principle that two of the numbers chosen will have the same odd factor. Let these numbers be n_1 and n_2. Then $n_1 = 2^{k_1}$ m and $n_2 = 2^{k_2}$ m, for some $k_1, k_2 \ge 0$. Then if $k_1 \ge k_2$, n_1 is a multiple of n_2 ; otherwise n_2 is a multiple of n_1.

***Example 4:** Show that among n + 1 arbitrarily chosen positive integers, there are two whose difference is divisible by n.*

Solution: We use Euclid's division algorithm. Given positive integers a and b, we can divide a by b and get a quotient q and remainder r, i.e.

$$a = bq + r.$$

Let $S = \{a_1, a_2, \ldots, a_{n+1}\}$ be the set of n + 1 arbitrarily chosen positive integers.

Define $f: S \to \{0, 1, 2, \ldots, n - 1\}$

by $f(a_i) = r_i$, the remainder left after dividing by n.

Here $|S| = n + 1$ and cardinality of the co-domain is n.

Hence by pigeonhole principle $f(a_i) = f(a_j)$ for $i \ne j$. This means that $r_i = r_j$. Hence $a_i - a_j = n(q_i - q_j)$. This means that there are two integers a_i and a_j in S whose difference is divisible by n.

***Example 5:** A sports tournament consisting of 45 events is spread over 30 days. There is at least one event per day. Prove that no matter how the events are arranged there will be a period of consecutive days during which exactly 14 events will take place.*

Solution: Let a_i denote the total number of events that takes place upto and including the i-th day. Hence $a_1 \ge 1$ and $a_{30} = 45$, and we have a sequence $a_1, a_2, a_3, \ldots, a_{30} = 45$, which is strictly increasing since there is at least one event per day. Adding 14 to each term in the sequence, we obtain $a_1 + 14, a_2 + 14, \ldots, a_{30} + 14 = 59$. Now consider the sequence $a_1, a_2, \ldots, a_{30}, a_1 + 14, a_2 + 14, \ldots, a_{30} + 14$, which consists of 60 numbers ranging from 1 to 59. Hence by pigeonhole principle, two of these numbers must be the same. Since $a_i \ne a_j$ for $i \ne j$, it follows that $a_j = a_i + 14$ for some $j > i$. Hence $a_j - a_i = 14$, which means that there is a period of consecutive days from the i-th day, during which exactly 14 games take place.

***Example 6:** If a set of 16 numbers is selected from {2, …, 50}, at least 2 numbers will be in the set with a common divisor greater than 1.*

Solution: In the set {2, …, 50}, there are 15 prime numbers viz. {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47}.

Suppose in any set of 16 numbers from 2 to 50, no two have a common divisor greater than one. Consider the prime factors of these numbers. By our assumption no two numbers will have common prime factor. This would mean that there should be at least 16 different prime numbers. This contradicts the fact that there are only 15 prime numbers. Hence our assumption is wrong. Therefore, in any set of 16 numbers from 2 to 50, 2 numbers will have a common divisor greater than 1.

Example 7: *Show that among any n + 1 positive integers not exceeding 2n, there must be an integer that divides one of the other integers.*

Solution: Let us denote the n + 1 positive integers as $a_1, a_2, \ldots, a_{n+1}$. Then we can write each integer a_i as $a_i = 2^{k_j} b_j$, for j = 1, 2, ... , n + 1, where k_j is a non-negative integer and b_j is odd positive integer. For example, $1 = 2^0 \cdot 1$, $2 = 2^1 \cdot 1$, $4 = 2^2 \cdot 1$, $6 = 2^1 \cdot 3$, and so on. Now $b_1, b_2, \ldots, b_{n+1}$ are all odd positive integers less than 2n. Since there are only n odd positive integers which are less than 2n, it follows from pigeon-hole principle that $b_i = b_j$ for some i and j. Then $a_i = 2^{k_i} q$ and $a_j = 2^{k_j} q$, where $q = q_i = q_j$. If $k_i < k_j$, a_i divides a_j; otherwise a_j divides a_i. Hence the result.

4.3.2 The Extended Pigeonhole Principle

Let A and B be two non-empty finite sets. If cardinality of A is greater than B, the following theorem (without proof) is a stronger version of the pigeonhole principle.

Theorem 3: Let f: A → B be a function and let |A| = m, |B| = n, m > n.

Let $$k = \left[\frac{(m-1)}{n}\right] + 1.$$

Then there exist k elements $a_1, a_2, \ldots, a_k \in A$ such that $f(a_1) = f(a_2) \ldots = f(a_k)$.

SOLVED EXAMPLES

Examples 1: *Prove that among 100,000 people, there are two who are born at exactly the same time (hour, minute and second).*

Solution: Let A be the set of people (pigeons) and B, the set of seconds (pigeonholes) of one day.

$$|A| = 100{,}000 = m, \quad |B| = 24 \times 3600 = 86400 = n.$$

Then $$k = \left[\frac{100000 - 1}{86400}\right] + 1 = 1 + 1 = 2.$$

Hence there are at least two who are born on the same day.

Example 2: *Show that there must be at least 90 ways to choose six numbers from 1 to 15 so that all the choices have the same sum.*

Solution: $m = {}^{15}C_6 = 5005$

The lowest sum of 6 numbers chosen from 1 to 15

$$= 1 + 2 + 3 + 4 + 5 + 6 = 21$$

Highest sum $= 10 + 11 + 12 + 13 + 14 + 15 = 75$

Hence $n = 75 - 21 + 1 = 55$

Hence by the pigeonhole principle

$$k = \left[\frac{m-1}{n}\right] + 1 = \left(\frac{5004}{55}\right) + 1 = 91$$

Hence in at least 90 ways, we can choose six numbers from 1 to 15 so that all the choices have the same sum.

Example 3: *There are 3000 students in a college which offers 7 distinct courses of 4 year's duration. A student who has taken a course in Discrete Mathematics learns that the largest classroom can hold only 100 students. She at once realizes there is a problem. What is the problem?*

Solution: Since there are 7 distinct classes of 4 year's duration, we have 7 × 4 = 28 different classes. Hence, by extended pigeon-hole principle, each classroom must hold at least $\left[\frac{3000-1}{28}\right] + 1 = 107 + 1 = 108$ students. But since the capacity of the largest classroom is only 100, this is obviously a problem.

Example 4: *In a group of six people at a party, each pair of individuals consists of two mutual acquaintances or two strangers. Show that there are either three mutual acquaintances or three mutual strangers in the group.*

Solution: Let A be one of the six people. Divide the remaining five into two sets, one consisting only of acquaintances of A, the other only of strangers to A. By Extended Pigeon- hole Principle, cardinality of one of the sets must be at least [5/2] = 3. Hence it follows that in the group there are either 3 or more who are acquaintances of A, or there are 3 or more who are strangers to A. Let us assume the former, i.e. say B, C, D are acquaintances of A. Since any pair of individuals are either acquaintances or strangers, if say B, C are acquaintances, then together with A, A, B, C form a group of 3 mutual acquaintances. On the other hand if B, C, D are mutual strangers, they form a group of three mutual strangers.

If we assume the latter, when B, C, D are strangers to A, the proof follows in similar manner.

Example 5: *A man hiked for 10 hours and covered a total distance of 45 miles. It is known that he hiked 6 miles in the first hour and only 3 miles in the last hour. Show that he must have hiked at least 9 miles within a certain period of two consecutive hours.*

Solution: Let a_i, $1 \le i \le 10$, denote the number of miles hiked by the man during the i^{th} hour. Then $a_1 = 6$, $a_{10} = 3$. Hence $a_2 + a_3 + \ldots + a_9 = 45 - (6 + 3) = 36$ miles. Consider the set

$$A = \{(a_2, a_3), (a_4, a_5), (a_6, a_7), (a_8, a_9)\}$$

of pairs of consecutive hours.

Apply Pigeon-hole Principle to the sum $(a_2 + a_3) + (a_4 + a_5) + (a_6 + a_7) + (a_8 + a_9) = 36$.

Since sum of 4 numbers is 36, value of one number should be at least 9.

Hence the man must have hiked at least 9 miles within a certain period of two consecutive hours.

Example 6: *Show that if 7 colours are used to paint 50 bicycles, at least 8 bicycles will be the same colour.*

Solution: Let A be the set of bicycles and B be the set of colours. Let f: A → B be a function, assigning a bicycle in A, a colour in B. Let |A| = m and |B| = n. Then m = 50 and n = 7.

Let $$K = \left[\frac{m-1}{n}\right] + 1 = \left[\frac{50-1}{7}\right] + 1 = 8.$$ Then by Extended Pigeon Hole Principle, there exist 8 elements in A such that $f(a_1) = f(a_2) = f(a_8)$, i.e. there exist at least 8 bicycles in A, which have the same colour.

EXERCISE – 4.1

1. Let A = {a, b, c, d}, B = {s, t, v}, C = {1, 2}. Define f: A → B, g: B → C, h: C → D as shown in the figures.

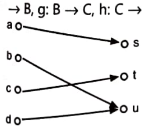


Fig. 4.43

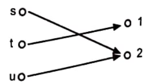


Fig. 4.44

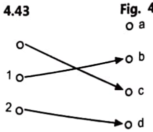


Fig. 4.45

Find the compositions.

(i) goh, (ii) foh, (iii) hog, (iv) gof.

Find in each case the domain and codomain.

2. The following diagrams define functions $f_1 g_1 h$ which map the set {1, 2, 3, 4} into itself. Find (i) range of f, (ii) range of g, (iii) range of h, (iv) fog, (v) hof, (vi) gog.

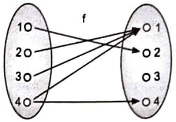


Fig. 4.46

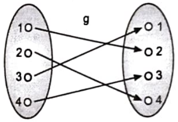


Fig. 4.47

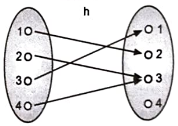


Fig. 4.48

3. Let f, g, h be defined by the following diagrams. Find which of them are injective, surjective or bijective.

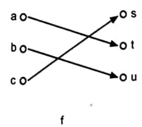


Fig. 4.49

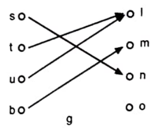


Fig. 4.50

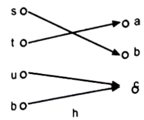


Fig. 4.51

Let ,

$X = \{1, 2, 3\}$, f, g, h, s be functions from X to X given by

$$f = \{(1, 2), (2, 3), (3, 1)\}$$
$$g = \{(1, 2), (2, 1), (3, 3)\}$$
$$h = \{(1, 1), (2, 2), (3, 1)\}$$
$$s = \{(1, 1), (2, 2), (3, 3)$$

Find fohog, sog, fos.

4. Give example one each of the following functions:
   (i) A function which is injective but not surjective.
   (ii) A function which is surjective but not injective.
   (iii) A function which is neither surjective nor injective.
   (iv) A function which is bijective.
5. Construct an example to show that for two functions f and g from A to A, fog ≠ gof.
6. Let f, g, h be functions from N to N where N, is the set of natural numbers including 0, such that

$$f(n) = n + 1$$
$$g(n) = 2n$$
$$h(n) = \begin{cases} 0 & \text{if n is even} \\ 1 & \text{if n is odd.} \end{cases}$$

Determine fof, fog, gof, goh, hog, (fog)oh.

7. *Show that for any functions $f: A \to B$, $g: B \to C$ and $h: C \to D$,*

$$ho\,(gof) = (hog)of$$

8. *Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d\}$ and let $f: A \to B$ be given by*

$$f = \{(1, a), (2, a), (3, c), (4, a)\}.$$

Determine whether f^{-1} exists. If so, find f^{-1}.

9. Let $f: \mathbf{R} \to \mathbf{R}$, $g: \mathbf{R} \to \mathbf{R}$ be defined by $f(a) = 2a + 1$, $g(b) = b/3$. Verify $(gof)^{-1} = f^{-1}og^{-1}$.
10. Let $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$. Determine whether the following relations from A to B is a function. If it is a function, find its range.
    (i) $R = \{(a, 1), (b, 2), (c, 1), (d, 2)\}$
    (ii) $R = \{(a, 1), (b, 2), (a, 2), (c, 1), (d, 2)\}$
    (iii) $R = \{(a, 3), (b, 2), (c, 1)\}$
    (iv) $R = \{(a, 1), (b, 1), (c, 1), (d, 1)\}$
11. Determine whether the following functions are one-one, onto (or both or neither).
    (i) $A = \{1, 2, 3, 4\} = B$
    $f = \{(1, 1), (2, 3), (3, 4), (4, 2)\}$
    (ii) $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$
    $f = \{(1, a), (2, a), (3, c)\}$
    (iii) $A = B = \{1, 2, 3, 4, 5\}$,
    $f = \{(1, 3), (2, 2), (3, 4), (4, 5), (5, 1)\}$
    (iv) $A = \{1, 2, 3, 4, 5\}$, $B = \{a, b, c, d\}$
    $f = \{(1, a), (2, c), (3, b), (4, d)\}$
12. Show that one of any m consecutive integers is divisible by m.
13. Suppose the figures from 1 to 12 on a clock dial are reshuffled among themselves. Prove that there exists a pair of adjacent figures which add up to at least 14.
14. Suppose that at a party, there are at least 6 persons. Prove that either there exist 3 persons, every two of whom know each other or else there exist 3 persons no two of whom know each other.
15. Suppose 18 students in a class appear at an entrance examination. Prove that there exist two among them whose seat numbers differ by a multiple of 17.
16. If 101 integers are selected from the set {1, 2, ..., 200}, prove that among the selected integers there exist two integers such that one of them is a multiple of the other.
17. Show that if seven points are chosen a region bounded by a regular hexagon of side 1 unit each, two of the points must be no further apart than 1 unit.
18. How many friends must you have to guarantee that at least five of them will have birthdays in the same month?

4.4 INFINITE SETS AND COUNTABILITY

Although many problems in counting involve only finite sets, it is the infinite set which plays a more important and significant role in computability theory. It is the infinite set which provides an insight into the limitations of that which can be computed algorithmically. One can show, with the help of infinite sets that there are tasks which cannot be performed by any computer.

This section is therefore devoted to the study of some important infinite sets and their properties.

4.4.1 Infinite Sets

We have an intuitive idea of an 'infinite' set, it is a set with an infinite number of elements, an inexhaustible storehouse of elements which do not cease to appear; one will never reach the last element in the set. We are also familiar with some infinite sets; the set of natural numbers, set of integers etc. Let us now formally define an infinite set.

Definition:

A set A is infinite if there exists an injection $f: A \to A$ such that f(A) is a proper subset of A. If no such injection exists, the set is finite.

Examples:

1. The set of natural numbers N is an infinite set.
   Consider $f: N \to N$, where $f(x) = 2x$.
   f(N) is the set of all positive even integers which is a proper subset of N.
2. The set of real numbers **R** is an infinite set.
   Define $f: \mathbf{R} \to \mathbf{R}$ as

$$f(x) = x + 1, \quad \text{if } x \geq 0$$
$$= x, \quad \text{if } x < 0.$$

Clearly f is an injective function. Note that if $y \in R$ such that $y = x + 1$, then $x = y - 1$. Hence $x \geq 0$ implies $y \geq 1$. Hence range $(f) = \{y \in \mathbf{R} \mid y < 0 \wedge y \geq 1\}$, which is a proper subset of **R**.

4.4.2 Properties of Infinite Sets

1. Let B be a subset of A. Then if B is infinite, A is infinite.

Proof:

Since B is infinite, we have an injection $f: B \to B$ such that f(B) is a proper subset of B.

Define a function $g: A \to A$ as

$$g(a) = f(a), \quad \text{if } a \in B$$
$$g(a) = a \text{ if } a \in A - B.$$

g is injective and g(A) does not include the elements in B − g(B). Hence A is infinite.

2. If A is infinite, its power set P(A) is infinite.

Proof:

The mapping $f: A \to P(A)$, where $f(a) = \{a\}$, $a \in A$ is an injective function.

3. If A and B are sets where A or B is infinite then $A \cup B$ is infinite.

Proof:

Define $f: A \to A \cup B$ as $f(a) = a$, $a \in A$

4. If A and B are sets where A (or B) is infinite and $B \neq \phi$ (or $A \neq \phi$) then $A \times B$ is infinite.

Proof:

Choose an element $b_o \in B$ (as $B \neq \phi$) and define $f: A \to A \times B$ as

$$f(a) = (a, b_o)$$

Clearly f is an injective function.

Examples

(i) The set of integers Z is infinite.

Solution: N, the set of natural numbers is a subset of Z. Hence Z is infinite by prop. 1, 4.3.2.

(ii) The set of rational numbers Q is infinite.

Solution: N (or Z) is a subset of Q. Hence, Q is infinite by pro. 1, 4.3.2.

(iii) The set $N \times N$ is infinite.

Solution: N is infinite and product of two infinite sets is infinite.

(iv) The intersection of two infinite sets is not infinite.

Solution: (i) Let

$$E = \text{set of even integers}$$
and $$O = \text{set of odd integers.}$$
Then $E \cap O = \phi$ which is a finite set

(ii) Let $A = [0, 1]$ and $B = [1, 2]$
then $A \cap B = \{1\}$.

(v) Let A and B be infinite sets such that $B \subseteq A$. Is the set A − B necessarily infinite? Give example to support your assertion.

Solution: No, A – B need not be infinite

Let $A = [0, 1]$ and $B = [0, 1)$

Then $A - B = \{1\}$.

(vi) The set of all strings in {a, b} of prime length.

Solution: The set of prime numbers is infinite. Hence the set of all strings in {a, b} of prime length is also infinite.

4.5 COUNTABILITY

If A is a finite set with cardinality n, we can describe A as $\{a_1, a_2, \dots, a_n\}$, that is, we can list or **enumerate** the elements in the set. In other words, there exists a bijection $f: A \to \{1, 2, \dots, n\}$, $\{1, 2, \dots, n\}$ being a subset of N. This property enables us to "count" the elements in the set. Hence we say that a finite set is **countable.**

We extend the concept of countability to infinite sets as follows.

Definition:

An infinite set A is said to be **countable** if there exists a bijection $f: N \to A$.

A countably infinite set is also called a **denumerable** set.

This definition does not mean that we can actually 'count' the elements in a denumerable set and say how many there are ! All it means is that it is theoretically possible to put the elements into an infinite list or sequence.

If A is a denumerable set then we list the elements as f(1), f(2), f(3), or $a_1, a_2, a_3, \dots$, where $a_i = f(i)$.

One can also compare the 'sizes' of two sets (finite or infinite) by the following definition.

Definition:

If A and B are sets and there exists a bijection $f: A \to B$, then A and B have the same cardinality.

- We denote the cardinality of N by No (aleph nought). Hence if A is countably infinite then $|A|$ = No.
- If A and b are sets and there exists a bijection $f: A \to B$, then if A is countable, so is B and vice versa.

Example 1: *The set of integers Z is countable.*

Solution: Define $f: N \to Z$ as

$$f(n) = \frac{(n+1)}{2}, \quad n = 1, 3, 5, \dots\dots$$

$$= -\frac{n}{2}, \quad n = 0, 2, 4 \dots$$

Clearly f is a bijection and hence Z is countable.

4.6 PROPERTIES OF COUNTABLE SETS

Theorem :

A subset of a countable set is countable.

Proof:

Let B be a subset of A. If B is finite, then it is countable. Assume therefore that B is infinite, in which case A is also infinite.

Since A is denumerable, there is a bijection $f: N \to A$.

Hence the elements in A can be sequentially arranged as f(1), f(2), f(3) .... Let $b \in B$.

Since B is a subset of A, $b = f(i)$ for some $i \in N$.

Designate this element as $f(i_1)$.

Since B is an infinite set, there exists $b' \in B$, where $b' \neq b$.

Since $b' \in A$, we denote this element as $f(i_2)$.

Proceeding in this manner, the elements in B can be sequentially arranged as $f(i_1), f(i_2), f(i_3), \dots\dots$

Define a mapping $g: N \to B$ as

$$g(n) = f(i_n)$$

Clearly g is a bijection, and hence B is countable.

This theorem gives us more examples of denumerable sets.

Examples:

1. The set of all prime numbers is denumerable.
2. The sets of odd integers and even integers are denumerable.
3. For a fixed integer, the set of all integral powers is denumerable.

Theorem :

Let A and B be countable sets. Then $A \cup B$ is countable.

Proof:

We shall first assume that A and B are **disjoint** sets.

If A and B are finite sets, then $A \cup B$ is countable.

If one of them is finite then also $A \cup B$ is countable.

We consider the more important case when A and B are both infinite.

Since A and B are denumerable sets, there exists bijections $f: N \to A$ and $g: N \to B$.

Hence we can express the elements of A as $A = \{a_1, a_2, a_3, \dots\dots\}$ where $f(i) = a_i$ and $B = \{b_1, b_2, b_3, \dots\}$, where $g(i) = b_i$.

Now define a reverse bijection

$f: A \cup B \to N$ as

$$f(a_i) = 2i - 1$$

and $f(b_i) = 2i$

We have thus put the elements of A in one-one correspondence with the odd numbers and elements of B with the even numbers.

As $A \cap B = \phi$, this is really a bijection.

Thus we have proved $A \cup B$ is countable where A and B are disjoint.

The theorem is also true if A and B are not necessarily disjoint.

This is because we can express

$A \cup B = (A - B) \cup (A \cap B) \cup (B - A)$, mutually disjoint union of three sets.

Theorem :

If A and B are countable, then $A \times B$ is also countable.

Proof:

We prove the theorem for the case when A and B are both infinite, as the remaining cases are easily proved.

Let $A = \{a_1, a_2, \dots\}$ and $B = \{b_1, b_2, \dots\}$. Construct a bijection $f: N \to A \times B$ inductively as

$$f(1) = (a_1, b_1)$$

Suppose $f(p) = (a_m, b_n)$.

If $m \neq 1$, $f(p + 1) = (a_{m-1}, b_{n+1})$

If $m = 1$, $f(p + 1) = (a_{n+1}, b_1)$

Hence $f(1) = (a_1, b_1)$, $f(2) = (a_2, b_1)$,

$f(3) = (a_1, b_2)$, $f(4) = (a_3, b_1)$, ...

Since f is a bijection, the elements of $A \times B$ are arranged sequentially as

(a_1, b_1)	(a_1, b_2)	(a_1, a_3)
(a_2, b_1)	(a_2, b_2)	(a_2, a_3)
(a_3, b_1)	(a_3, a_2)	(a_3, a_3)
(a_4, a_1)	(a_4, a_2)	(a_4, a_3)

As corollary to the above, theorem if $A_1, A_2, \dots, A_n$ is a finite collection of countably infinite sets, then $A_1 \times A_2 \times \dots \times A_n$ is also countably infinite.

Examples:

(i) *The sets $N \times N$, $Z \times Z$, $Z \times N$, $Z \times Z \times Z$ are all countably infinite.*

(ii) *Let Q denote the set of rational numbers. Then Q is countable.*

Solution: We first make the observation that the set of rational numbers can be described as the set of all quotients p/q of integers p and q, where

(i) $q > 0$

(ii) p and q have no positive common factor other than 1.

Define a function $f: Q \to Z \times N$ as $f(p/q) = (p, q)$.

f is an injection. Hence f (Q) is an infinite subset of $Z \times N$ and hence countably infinite. Since f(Q) is countably infinite and f is an injection, it follows that Q is also countably infinite.

Theorem :

The union of a countable collection of countable sets is countable.

Proof:

We prove the theorem only for a countably infinite collection of mutually disjoint countably infinite sets, from which the general result can be easily deduced.

For each set A_i, enumerate its elements as a (i, 1), a (i, 2), a (i, 3), ...

Define a function $f: N \times N \to \bigcup_{i=1}^{\times} A_i$ as $f(i, j) = a(i, j)$.

As the sets are mutually disjoint, f is a bijection and hence it follows that $\bigcup_{i=1}^{\times} A_i$ is countably infinite.

The following is an important example from 'formal language'.

Example: *Let W = {a, b} and let W^* be the set of all words, formed from a, b of any length k. Then the set W^* is countably infinite.*

Solution: We can write W^* as the disjoint union of the following countably infinite collection of countable sets. For each integer $k \geq 0$, let W_k be the set of all words over {a, b} of length k. Then W_k is a finite set and every word in W is an element of W_k for some k. Hence $W = \bigcup_{k=1}^{\infty} W_k$ and therefore countably infinite.

4.7 CANTOR'S DIAGONAL ARGUMENT

The following theorem demonstrates an important technique to prove that the infinite set of positive rational numbers is countable.

Theorem The set of positive rational numbers is countable

Proof We have to show that all positive rational numbers can be enumerated, i.e. listed in a particular order. Any rational number is expressible in the form p/q, (q>0) where p and q have no common divisor except 1. For each such rational number, find the sum $n = p + q$

We have, $\frac{0}{1}$, $0 + 1 = 1$

$\frac{1}{1}$, $1 + 1 = 2$

$\frac{1}{2}$, $1 + 2 = 3$

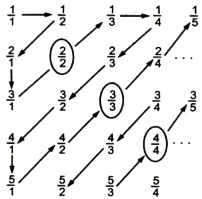
$\frac{2}{1}$, $2 + 1 = 3$

$\frac{1}{3}$, $1 + 3 = 4$

$\frac{3}{1}$, $3 + 1 = 4$, omitting 2/2

and so on.

The positive rational numbers can then be displayed as follows.



From the arrangement, it is clear why this method is called a diagonal argument. Following the sequence of *expanding diagonals* from the first row to the first column, and omitting the encircled elements (which are repeated), we have the enumeration as :

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{3}{1}, \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \frac{4}{1}, \frac{5}{1}, \ldots$$

Now define a mapping $f : N \to Q^+$, where f (n) is the n^{th} rational number in the list. f is clearly a one-one function. To show f is onto, consider p/q, where $p + q = n$, $q > 0$.

Then there are at most $1 + 2 + 3 + \ldots + n-1 = \frac{(n-2)(n-1)}{2}$ rational numbers that could be listed in front of p/q. Hence every positive rational number does appear on the list.

Hence Q^+ is countable.

Remarks The function $f: N \to Q^+$ can be extended to function $f^* : N \to Q$, which is also one-to-one and onto. This proves that the set of all rational numbers is countable.

4.8 NON-DENUMERABLE SETS

Thus for all the examples of infinite sets we have seen are countable. One should not however be misled in assuming that every infinite set is countable. We shall now deal with some important sets that are not countable.

Theorem :

The set of real numbers **R** is non-denumerable.

Proof:

By theorem it is sufficient to show that there exists a subset of **R** is uncountable.

We shall show in fact, that the set of real numbers between 0 and 1, i.e. (0, 1) is uncountable.

Assume that the set is denumerable (i.e. countably infinite).

Then we can arrange the elements as an infinite sequence, each element in the sequence being represented as a unique decimal, without an infinite string of 9's at the end.

Thus 0.1239999 will be simply represented as 0.123000.

Let the infinite sequence be given by:

$$x_1 = \cdot a_{11}\, a_{12}\, a_{13} \ldots$$
$$x_2 = \cdot a_{21}\, a_{22}\, a_{23} \ldots$$
$$x_3 = \cdot a_{31}\, a_{32}\, a_{33}$$
$$\vdots$$
$$x_k = \cdot a_{k_1}\, a_{k_2}\, a_{k_3}$$
$$\vdots$$
$$\vdots$$

Considering the diagonal elements of the array, construct a number

$$y = \cdot b_1 b_2 b_3 \ldots\ldots \text{ where,}$$
$$b_i = 0 \quad \text{if } a_{ii} \neq 0$$
$$= 1 \quad \text{if } a_{ii} = 0.$$

This means that $b_1 \neq a_{11}$, $b_2 \neq a_{22}$, and so on. Hence $b_i \neq a_{ii}$ for any i.

Hence $y \neq x_1$ since it differs from x_1 at the first decimal place,

$y \neq x_2$ since it differs from x_2 at the second place.

and hence in general

$y \neq x_i$ since it differs from x_i at the i^{th} place

Thus we have constructed a number $y \in (0, 1)$ which does not belong to the sequence. Hence the set (0, 1) is non-denumerable, from which it follows that **R** is also non-denumerable.

The cardinality of **R** is denoted by **c.** The choice of **c** is based on the fact that the set (0, 1) is called a **continuum.**

In fact a set A is of cardinality c if there is a bijection from (0, 1) to A.

Corollary

The set of irrational numbers is non-denumerable.

Proof:

The set of irrational numbers is $\mathbf{R} - Q = \bar{Q}$.

Now $\mathbf{R} = Q \cup \bar{Q}$. Q is countable.

Hence if $\bar{Q}$ is also countable, then so will be **R**, since union of two countable sets is countable.

Hence the set of irrational numbers is non-denumerable.

The following is another important example of uncountable set.

***Example:** The power set of N, P(N) is non-denumerable.*

Solution: For any subset A of N, let us represent A by a sequence $(a_1, a_2, a_3, \ldots)$ where $a_k = 1$ if k is an element of A and $a_k = 0$ if k is not an element of A.

Hence {1} is represented by (1, 0, 0, ...), {1, 3} is represented by (1, 0, 1, ...), {3, 5, 6} is represented by (0, 0, 1, 0, 1, 1, ...) and so on.

Now, suppose P(N) is denumerable, and let us consider a countably infinite collection of subsets $\{S_1, S_2, \ldots\}$ of N. We shall construct a subset T of N, that is not included in this collection.

Let $k \in T$ iff $k \notin S_k$.

So if $S_3 = (0, 1, 0, \ldots)$, then $3 \notin S_3$ (since if $3 \in S_3$, 1 should appear at the third position of the sequence). Hence $3 \in T$ on the other hand if $S_2 = (1, 1, 0, 0, \ldots)$ then $2 \in S_2$ and hence $2 \notin T$. In this manner, the set T is different from each of the S_i' s and hence not a member of the collection $\{S_1, S_2, \ldots\}$.

Thus any countable collection of subsets of N excludes some subset of N. Hence the set P(n) is non-denumerable.

Note that we have used similar argument to show that the set of real numbers is uncountable. This technique called as 'diagonalisation argument' is used frequently in computability theory.

This technique proves that there are elements which cannot be computed by any computer program, even if a computer of unlimited storage and speed is assumed to exist.

SOLVED EXAMPLES

***Example 1:** What is the cardinality of the following sets*

(i) $I = \{\ldots -4, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

(ii) $N \times N$, N is the set of natural numbers.

(iii) Union of finite number of countable sets.

Solution:

(i) I is countably infinite $\therefore\ |I| = \aleph_0$.

(ii) $N \times N$ is also countably infinite

$\therefore\ |N \times N| = \aleph_0$.

(iii) Countably infinite. Hence cardinality is $\aleph_0$.

***Example 2:** If A and B are denumerable, then prove $A \times B$ is also denumerable.*

Solution: See Theorem in Article 4.8 for solution.

***Example 3:** Classify the following into finite, denumerable and non-denumerable.*

(i) Number of trees in India.

(ii) Set of real numbers between 2 and 3.

(iii) Number of songs sung by Lata Mangeshkar.

(iv) Power set of a countably infinite set.

Solution:

(i) Since the number of trees is not static but continues to increase, the set is denumerable.

(ii) The set of real number between (2, 3) is not denumerable. We know that (0, 1) is non-denumerable. Define

$f: (0, 1) \to (2, 3)$ as

$f(a) = a + 1$

f is clearly a bijection and since (0, 1) is non-denumerable, (2, 3) must also be non-denumerable.

(iii) The set is finite.

(iv) We know that power set of N is non-denumerable. Hence power set of a countably infinite set is non-denumerable.

***Example 4:** Define countably infinite and uncountably infinite sets. Define their cardinality numbers. Show that the cardinalities of two open intervals of a real line are equal.*

Solution: For definitions, refer to the theory.

Any open interval of the real line is non-denumerable. Hence the cardinalities of two open intervals on the real line are equal, each being the continuum $\subseteq$.

***Example 5:** Define countably infinite set and prove that the set of positive rational numbers is countably infinite.*

Solution: See Theorem in Article 4.8 for solution.

***Example 6:** Classify the following into countable or uncountable sets.*

(i) $Q \times Q$, Q is the set of rational numbers.

(ii) The power set of N, where N is the set of natural numbers.

(iii) All books that can ever be written in English.

Solution: (i) Countably infinite (denumerable).

(ii) P(N) is uncountable.

(iii) Countably infinite: A book is composed of words of varying but finite length.

Hence the set of all words in a book is a finite union of finite sets, and hence is finite. Hence all the books that could ever be written in the English language can be thought of as a countably infinite union of countable sets, and is therefore countably infinite.

Example 7: *Show that union of a countably infinite number of countably infinite sets is a countably infinite set.*

Solution: See Theorem in Article 4.4.2 for solution.

EXERCISE – 4.2

1. Give an example to show that the cardinality of a set which is the intersection of two countably infinite sets may also be countably infinite.
2. Give an example to show that the cardinality of a set that is the intersection of two countably infinite sets may be finite.
3. Classify the following sets into finite, countably infinite or uncountably infinite.
   (i) Set of all types of trees in India.
   (ii) Set of all prime numbers.
4. State if the following sets are finite, denumerable or non-denumerable.
   (i) Class of all possible programs that can be written in any given programming language.
   (ii) Number of fish in the Pacific ocean.
   (iii) All possible books written in the English language.
   (iv) Set of real numbers between 0 and 1.
5. Classify the following sets as finite, denumerable or non-denumerable, giving reason
   (i) The set of all strings (words) in {a, b} of prime length.
   (ii) The set of all strings in {a, b} of length no greater than k.
   (iii) The set of all m × n matrices with entries from {0, 1, 2, ... k}.
   (iv) The set of all prepositional forms over the prepositional variables p, q, r and s.
   (v) The set of all points in the plane.
   (vi) The set of all points in the plane with positive integer co-ordinates.
6. Determine the cardinalities of the sets
   (a) $A = \left\{ n^7 \,\middle|\, n \text{ is a positive integer} \right\}$
   (b) $B = \left\{ n^{109} \,\middle|\, n \text{ is a positive integer} \right\}$
   (c) $A \cup B$ (d) $A \cap B$.
7. Let N denote the set of all natural numbers. Let S denote the set of all finite subsets of N. What is the cardinality of the set S? Justify your answer.

POINTS TO REMEMBER

- Let A and B be non-empty sets. A **function** f from A to B, denoted as f: A → B, is a relation from A to B such that for every a ∈ A, there exists a **unique** b ∈ B such that (a, b) ∈ f. If (a, b) ∈ f, we write f(a) = b.
- The set A is called as the **domain** of f, denoted **D(f)**. The set B is called as the **codomain**, and the set $\{ f(a) \mid a \in A \}$, which is a subset of B, is called as the **range** of f, and denoted as **R(f).** The element a is called an **argument** of the function f and f(a) is called the **value** of the function for the argument a.
- Let A and B be two sets. A partial function f with **domain A** and codomain B is any function from A' to B where A' ⊂ A. For any element x ∈ A – A', the value of f(x) is said to be undefined.
- Let f: A → B and g: C → D be functions. Then f and g are said to be equivalent or **identical** only if A = C, B = D and f(a) = g(a) for all a ∈ A.
- The function f: Z → Z given by f(x) = x and g: Z → Z' given by g(x) = x are not equivalent.
- Let f: A → B be a function.

  f is called a **surjective** (onto) function if f(A) = B, i.e. range of f is equal to the codomain of f.

  f is called an **injective** (one-to-one) function if for elements a, a' ∈ A, a ≠ a' implies (a) ≠ f(a'), or equivalently if f(a) = f(a'), then a = a'.

  f is called bijection (one-to-one and onto) function if f is both surjective and injective.
- Let f: A → B be a bijection from A to B. The inverse of f denoted by f^{-1} is the function f^{-1}: B → A such that

  $f^{-1} o f = 1_A$ and

  $f o f^{-1} = 1_B$
- A set A is infinite if there exists an injection f: A → A such that f(A) is a proper subset of A. If no such injection exists, the set is finite.
- An infinite set A is said to be **countable** if there exists a bijection f: N → A.
- A countably infinite set is also called a **denumerable** set.

◈◈◈

CHAPTER 5
COUNTING PRINCIPLES

5.1 INTRODUCTION

- This chapter is divided into two sections. In the first section, we shall study various counting techniques involving finite sets. Counting the objects in a finite set is a basic necessity in order to compare, evaluate and predict things. To compare the cost of applying two algorithms, one needs to estimate how many operations each of them executes to solve the same problem.
- Again to evaluate the cost efficiency of using a particular data structure for a file, one needs to determine the average and maximum lengths of searches for items stored in that particular data structure. Hence problems of this nature, ultimately involve counting the elements of a set.
- The methods developed in this section are applied to solve problems on probability, related to discrete sample spaces, which are discussed in the second section.

5.2 PERMUTATIONS AND COMBINATIONS

Problems involving counting are generally of the type where we have to find the number of ways to arrange some or all elements of a set; or to select some elements or their combinations from a given set. Problems such as these are called as **combinatorial** problems. Some of these problems can be quite tricky as is evident from the following examples.

1. How many non-negative integer solutions are there of the equation

   x + y + z + u + v = 10,000?
2. In how many ways can the four walls of a room be painted with three colours so that no two adjacent walls have the same colour?

Problems such as those stated above are interpreted as "experiments" or "tasks" and solutions of the problems as "outcomes" of the experiments.

Suppose we have to consider several experiments and their outcomes, simultaneously. We observe the following self-evident rules of operations:

1. **Rule of Product:** If one experiment E_1 has n_1 possible outcomes and another experiment E_2 has n_2 possible outcomes, then there are $n_1 n_2$ possible outcomes when both the experiments ($E_1 \cap E_2$ or $E_1 E_2$) take place.

   This rule can be extended to a finite number of experiments $E_1, E_2, E_3, ..., E_k$, with outcomes $n_1, n_2, n_3, ..., n_k$ respectively.
2. **Rule of Sum:** If one experiment E_1 has n_1 possible outcomes and another experiment E_2 has n_2 possible outcomes, then there are $n_1 + n_2$ possible outcomes when exactly one of these experiments take place i.e. ($E_1 \cup E_2$) – $E_1 \cap E_2$.

Example: Suppose there are 65 ways to select a class representative for FE-I and 75 ways to select a class representativefor SE-(COM), then to select a representative for the combined FE and SE classes, there will be 65 × 75 = 4875 ways, by the rule of product. On the other hand, there will be 65 + 75 = 140 ways to select a representative for either the FE class or the SE class.

The problems discussed in this section will deal with two basic ideas of counting, that of permutations and combinations.

5.2.1 Permutations

An arrangement in sequence of elements of a set is called a permutation of the elements. Essentially, there are three types of arrangement of elements to be considered.

1. Let 0 ≤ r ≤ n. The number of ways to have an ordered sequence of n distinct elements, taken r at a time is called as an r-permutation **of n-elements** and is denoted by P(n, r) or (^nP_r).

The first place in the sequence can be filled up in n ways, then the second in (n – 1) ways and proceeding in this manner the r^{th} place can be filled up in (n – r + 1) ways.

Hence the number of permutations of n-distinct elements taken r at a time is given by the formula

$$P(n, r) = n(n-1)(n-2)\ldots(n-r+1)$$
$$= \frac{n!}{(n-r)!}, \quad \text{where } 0 \le r \le n$$

SOLVED EXAMPLES

***Example 1:** (a) Find the permutations of the set A = {1, 2, 3, 4}, taking the elements two at a time.*

Solution: The permutations are the sequences

12, 13, 14, 21, 23, 24, 31, 32, 34, 41, 42, 43.

Note that the order in which the elements appear is important, 12 and 21 are **different** arrangements of the same elements.

(b) Find the permutations of the set A = {1, 2, 3}.

In this case, we permute all the three elements. Hence the permutations are the sequences: 123, 132, 231, 213, 312, 321.

***Example 2:** Four persons enter a bus in which there are six vacant seats. In how many ways can they take their places?*

Solution: The first person may seat himself in 6 ways, then the second person can seat himself in 5 ways. Proceeding in this manner, the total number of ways in which the 4 persons can seat themselves is $P(6, 4) = 6 \times 5 \times 4 \times 3 = 360$.

***Example 3:** A menu card in a restaurant displays four soups, five main courses, three desserts and 5 beverages. How many different menus can a customer select if*

(i) He selects one item from each group without omission?

(ii) He chooses to omit the beverages, but selects one each from the other groups?

(iii) He chooses to omit the desserts but decides to take a beverage and one item each from the remaining groups?

Solution: (i) The customer can select the soup in 4 ways, the main course in 5 ways, the dessert in 3 ways and beverage in 5 ways. Hence by the product rule, the number of ways in which he can select one item each, without omission, is $4 \times 5 \times 3 \times 5 = 300$.

(ii) The number of ways in which the selection is made is $4 \times 5 \times 3 = 60$.

(iii) The number of ways to make the required selection is $4 \times 5 \times 5 = 100$.

***Example 4:** In how many ways can one select a President, a General Secretary and a Treasurer from the members of a committee consisting of 9 men and 11 women, if the Treasurer must be a woman, and the General Secretary a man?*

Solution: If the President is a man, he can be selected in 9 ways, in which case the General Secretary will be selected in 8 ways. If the President is a woman, she can be selected in 11 ways and then the Treasurer in 10 ways. Hence by the rules of sum and product, the total number of ways to make the selection is $9 \times 8 \times 11 + 11 \times 10 \times 9 = 1782$.

***Example 5:** Find how many words (meaningful or otherwise) of length 3 can be formed from the word COMPUTER, the beginning letter being C, and allowing no repetition of the letters.*

Solution: The given word contains 8 distinct letters. Since the first letter of the word is already fixed, the remaining two letters can be selected in $7 \times 6 = 42$ ways.

Hence the number of words that can thus be formed is 42.

***Example 6:** Find how many symbol codes can be formed if the first two symbols are letters and the next 3 are digits, but no symbol is repeated.*

Solution: The first two symbols can be chosen in P (26, 2) ways and the next three symbols can be chosen in P (10, 3) ways. Hence by the product rule, the number of symbol codes that can be formed is $\frac{26!}{24!} \cdot \frac{10!}{7!} = 26 \times 25 \times 10 \times 9 \times 8 = 468000$.

***Example 7:** (i) Suppose that repetitions are not permitted, then how many 4 digit numbers can be formed from the six digits 1, 2, 3, 5, 7, 8?*

(ii) How many such numbers are less than 4000?

(iii) How many of the numbers in (i) are even?

(iv) How many of the numbers in (ii) are odd?

(v) How many of the numbers in (i) contain both the digits 3 and 5?

Solution: (i) The number of 4 digit numbers is given by $P(6, 4) = \frac{6!}{2!} = 360$.

(ii) The numbers beginning with 1, 2, 3, will be the numbers which are less than 4000. The number of integers beginning with 1 is $5 \times 4 \times 3 = 60$.

The number of integers beginning with 2 is again $5 \times 4 \times 3 = 60$.

Thus the total number of integers less than 4000 is $3 \times 60 = 180$.

(iii) Those numbers ending in 2 or 8 are even. Numbers ending in 2 are $5 \times 4 \times 3 = 60$. Similarly, the numbers ending in 8 are $5 \times 4 \times 3 = 60$. Hence the numbers ending in 2 or 8 are 120 numbers.

(iv) The numbers ending in 1, 3, 5, 7 are odd. The 4 digit numbers ending in 1 and less than 4000 should begin with either 2 or 3. Then there are $2 \times 4 \times 3 = 24$ such numbers. Similarly, the numbers ending in 3 are 24. However, the numbers ending in 5 or 7 are $3 \times 4 \times 3 = 36$ for each. Hence there are in all $2 \times 24 + 2 \times 36 = 120$ such numbers.

(v) The digit 3 can occupy any of the 4 positions and the remaining 3 will be occupied by the digit 5. Hence the number of ways in which the digits 3 and 5 can appear is $4 \times 3 = 12$. Hence the number of integers containing both the digits 3 and 5 is $12 \times 4 \times 3 = 144$.

***Example 8:** Suppose repetitions are not possible.*

1. How many three digit numbers can be formed from six digits 2, 3, 4, 5, 7 and 9?

2. How many of these numbers are less than 400?

3. How many are even?

4. How many are multiples of 5?

Solution:

1. Since repetition is not allowed, the first digit can be chosen in 6 ways, the second in 5 and the third in 4 ways.

   Hence the total number of such 3 digit numbers is $6 \times 5 \times 4 = 120$.

2. Since the number is less than 400, the first digit has to be 2 or 3 only. Hence the first digit can be chosen in 2 ways, then the second in 5 and the last in 4 ways only.

   Hence the total number of such numbers is $2 \times 5 \times 4 = 40$.

3. For the number to be even, the last digit has to be 2 or 4, then the middle digit can be chosen in 5 and the first in 4 ways.

   Hence the total number of such numbers is $2 \times 5 \times 4 = 40$.

4. Since the last digit is 5, it can be filled up in only one way. Hence, the total number of such numbers is $1 \times 5 \times 4 = 20$.

***Example 9:** Six different Mathematics books, four different Discrete Structures books and three different Computer Science books are to be arranged on a shelf. How many different arrangements are possible if*

(i) The books in each subject must all be together?

(ii) Only the Discrete Structures books must be together?

Solution: (i) The Mathematics books can be arranged among themselves in 6 ! ways, the Discrete Structures books in 4 ! ways and the Computer Science books in 3 ! ways.

Hence the total number of arrangements is 3 ! 3 ! 4 ! 6 ! = 103680 ways.

(ii) Consider the four Discrete Structures books as a single entity. Then we have 10 books which can be arranged in 10 ! ways. In each of these arrangements, the Discrete Structures books can be arranged among themselves in 4 ! ways. Hence, the number of arrangements is 4 ! 10 ! = 87091200 ways.

***Example 10:** In how many ways can 9 people be seated at a round table if*

(i) they can sit anywhere?

(ii) 2 particular persons must not sit next to each other?

Solution:

(i) Since the people are seated in a circle, the total number of ways in seating them is $(9-1)! = 8! = 40320$ ways.

(ii) Consider the two persons as a single entity. Then there are 8 people altogether and they can be seated in $(8-1)! = 7!$ ways. But the two particular persons can be arranged among themselves in 2! ways. Thus the number of ways of arranging 9 people at a round table with the two persons sitting together is $2!\ 7! = 10080$ ways.

Then from (i), the total number of ways in which 9 people can be seated at a round table so that 2 particular persons do not sit together $= 40320 - 10080 = 30240$ ways.

Example 11: *How many different ways are there to arrange the letters in the word "PROBLEM" if*

(i) the letter P must come first?

(ii) The letter P must come first and the letter M last?

Solution: (i) Since the letter P must come first, its position is fixed. Hence the remaining 6 letters can be arranged in 6 ! ways. Hence the required answer is 6 !.

(ii) Since the letter P must come first and the letter M last, the remaining 5 letters can be arranged in 5 ! ways.

Hence the required answer is 5 ! ways.

Example 12: *The number of injective functions from a set with r elements to a set with n elements is P (n, r), $r \leq n$.*

Solution: Let A and B be sets with cardinalities r, n respectively. Let $A = \{a_1, a_2, \dots a_r\}$ and $B = \{b_1, b_2, \dots, b_n\}$. Let f be an injective function from A to B. $f(a_1)$ has n choices from elements of B. Since f is injective, $f(a_1) \neq f(a_2)$. Hence $f(a_2)$ has (n − 1) choices, $f(a_3)$ has (n − 2) choices and so on till $f(a_r)$ which has (n − r + 1) choices. Hence by the product rule, the number of injective functions from A to B is n (n − 1) (n − 2) ... (n − r + 1) = P (n, r). We now deal with permutation problems of another type, which can be stated as follows:

> **2.** The number of ways in which of the n elements can be arranged, where r_1 elements are of one kind, r_2 are of another kind and so on till r_k elements are of another kind, is given by the formula $\frac{P(n, r)}{r_1!\, r_2! \dots r_k!}$, where $r = r_1 + r_2 + \dots + r_k$.

SOLVED EXAMPLES

Example 1: *In how many ways can the letters in the word MISSISSIPPI be arranged?*

Solution: The letter I occurs 4 times, hence there are 4 ! ways in which these I's can be re-arranged among themselves. But then as this does not change the word as such, we have to divide by 4 ! to count the word MISSISSIPPI once. Hence the I's are objects of one kind, similarly the 4 S's are objects of second kind and the two P's are objects of third kind. Hence the number of ways the letters of the word MISSISSIPPI can be rearranged is $\frac{11!}{4!\,4!\,2!}$.

Example 2: *How many numbers can be formed with the digits 1, 2, 3, 4, 3, 2, 1, so that the odd digits always occupy the odd places?*

Solution: The odd digits 1, 3, 3, 1 can be arranged in their four places in

$$\frac{4!}{2!\,2!} \text{ ways.}$$

The even digits 2, 4, 2 can be arranged in their three places in $\frac{3!}{2!}$ ways.

Hence the required number of ways is $\frac{4!}{2!\,2!} \cdot \frac{3!}{2!} = 6 \times 3 = 18$.

Example 3: *Fifteen new students are to be evenly distributed among three classes. Suppose there are three brilliant scholars among them. In how many ways can the distribution done, so that each class gets one? One class gets them all?*

Solution: Since each class must get one scholar, we shall first assign these 3 students which is done in 3 ! ways. The other 12 students can be equally distributed in the 3 classes. Hence in this case, the number of ways in which the distribution can be done is $3! \frac{12!}{4!\,4!\,4!}$.

Next, if one class gets them all, then there are 3 possibilities according to which class it is. Hence the number of ways in which distribution can be done is $3 \cdot \frac{12!}{5!\,5!\,2!}$.

Example 4: *In how many ways can the letters in the word "PIONEER" be arranged so that the two E's are always together?*

Solution: The other five letters can be rearranged in 5! ways and for each such arrangement, the two E's can occupy any of the six remaining places. Hence the number of ways in which the letters of the word can be arranged, so that the two E's are always together, is 6 · 5 = 6 !.

Example 5: *Find the number of ways in which letters in the word "MALYALAM" be arranged so that the two M's are always together.*

Solution: The remaining 6 letters in the word can be arranged in 6 ! ways and for each such arrangement the two M's can occupy any of the 7 remaining places; for instance one such arrangement is YMMALALA. For each such arrangement, since there are 3A's, 2L's (objects of the same kind), the total number of arrangements is $\frac{7.6!}{3!\,2!} = \frac{7!}{3!\,2!} = 420$.

Example 6: *In how many ways can 8 different books be divided among Sameer, Ajay and Leela if Sameer gets 4 books, Ajay and Leela get 2 each?*

Solution: One of the arrangements can be SALSSLSA. Each such ordering determines a distribution of books.

Hence the total number of ways in distributing the 8 books, in the required manner is $\frac{8!}{4!\,2!\,2!} = 420$.

The third type of problems on permutation can be described as follows:

> **3.** The number of permutations of n elements, r at a time, when each element may be repeated once, twice, ... upto r times in any arrangement.

In this case, the first place may be filled up in n ways and when it has filled up in any one way, the second place may also be filled up in n ways since we are not precluded from using the same element. Proceeding in this manner, the number of ways in which the r places can be filled up is $\mathbf{n^r}$.

SOLVED EXAMPLES

Example 1: *A die is rolled three times, find the number of faces that can appear on top.*

Solution: If the die is rolled once, the face appearing on the top can be any one of the six faces 1 to 6; when it is rolled the second time, then also there are 6 choices for the face appearing on top, and same is the situation when the die is rolled the third time. Hence the total number of ways of a face appearing on top is $6 \times 6 \times 6 = 6^3$.

Example 2: *Find the number of ways in which three examinations can be scheduled within a five day period, with no restriction on the number of examinations scheduled each day.*

Solution: The first examination can be held on any of the five days, hence it has 5 choices. There being no restriction, the second examination can also be held on any of the five days, and similar is the situation for the third examination. Hence the number of ways in which the examinations can be held is $5 \times 5 \times 5 = 5^3$.

Example 3: *(i) Find the number of binary sequences of length 5.*

(ii) Find the number of four digit decimal numbers that contain one or more repeated digits.

Solution: (i) Each position in the sequence has 2 choices viz. 0 or 1. Hence the number of binary sequences of length 5 is $2 \times 2 \times 2 \times 2 \times 2 = 2^5$.

(ii) There are 9×10^3 four digit decimal numbers, including the non-repeated digits. Since we are required to find only those 4 digit numbers that contain one or more repeated digits, we have to omit those n numbers with non-repeated digits. There are $9 \times 9 \times 8 \times 7$ such numbers.

Hence the number of 4 digit decimal numbers, with one or more repeated digits, is $10^4 - 9 \times 9 \times 8 \times 7 = 5464$.

Example 4: *How many auto license plates can be made if each is identified by 2 letters followed by 4 digits?*

Solution: The first two positions in the sequence can be occupied in $26 \times 26 = 26^2$ ways. The next four positions can be occupied in $10 \times 10 \times 10 \times 10 = 10^4$ ways.

Hence the number of auto license plates that can be made is $26^2 \times 10^4$.

Example 5: *In a Mathematics test, there are 10 multiple choice questions with 4 possible answers and 15 true-false questions. In how many ways can the 25 questions be answered?*

Solution: If the student attempts to answer all the questions, he can answer the 10 multiple choice questions in 4^{10} ways and the 15 true-false questions in 2^{15} ways. Hence the required answer is $4^{10} \times 2^{15} = 2^{35} = 3436 \times 10^{10}$ ways.

Sometimes it is necessary to find the total number of ways in which it is possible to make a selection by taking **some or all** of the n elements. In this case, each element may be dealt within 2 ways, either taken or rejected. Hence the number of ways of dealing with them is $2 \times 2 \times \dots \times 2$ (n times) i.e. 2^n. But since this includes the case when all the elements are rejected, omitting this case, the total number of ways is $2^n - 1$.

The following examples demonstrate this case.

Example 6: *A man has 10 friends. In how many ways can he go to dinner with 1 or more of them?*

Solution: Since he has to select some or all of his 10 friends, the number of ways is $2^{10} - 1$.

Example 7: *There are 15 true or false questions in an exam. In how many ways can a student answer the exam if he or she can also choose not to answer some of them?*

Solution: If the student attempts all the 15 questions, then he or she can do so in 2^{15} ways. But since he or she can choose not to answer some of them, the correct solution is $2^{15} - 1$.

Example 8: *A bit is either 0 or 1. A byte is a sequence of 8 bits. Find:*

(i) Number of bytes.

(ii) Number of bytes that begin with 11 and end with 11.

Solution: (i) Total number of bytes is $2 \times 2 \times 2$ (8 times) $= 2^8 = 256$.

(ii) Since the first two and last two bits are fixed, i.e. 11, the remaining bits in the sequence are either 0 or 1.

Hence, total number of bytes $= 2 \times 2 \times 2 \times 2 = 2^4 = 16$.

Example 9: *In how many ways the letters in the word 'ORGANISE' can be arranged in such a way that all vowels come together and all consonants always come together.*

Solution: The vowels are A, E, I, O, and consonants are R, G, N, S.

The vowels, being considered as a single block can be rearranged in 4 ! ways. Similarly, the consonants can be rearranged in 4 ! ways. The two blocks can be further arranged in 2 ! ways. Hence, required number of permutations is 2 ! 4 ! 4 ! = 1152.

5.2.2 Combinations

The counting methods that we have seen for permutations all apply to situations where order matters. In this section, we consider some problems where order does not matter.

Definition:

> Let $0 \le r \le n$. A selection of a set of r elements from a set of n distinct elements is called a **combination.**

It is clear from the definition that in order to find the total number of r combinations of n elements, all that we have to do is to find all subsets of cardinality r, of a set whose cardinality is n. This can be done as follows.

Let B be a subset of A containing r elements. Let C (n, r) denote the number of ways to choose B. Now since a permutation of n-elements taken r at a time involves a subset B of r elements and a particular permutation of these r elements, it follows that

$$C(n, r) \cdot r! = P(n, r)$$

$$\text{Hence } C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r!\,(n-r)!}$$

The notation ${}^nC_r$ is also commonly used to denote C (n, r).

SOLVED EXAMPLES

Example 1: *In how many ways can 25 late admitted students be assigned to three practical batches if the first batch can accommodate 10 students, the second 8 and third only 7?*

Solution: The first batch can be assigned 10 students in C (25, 10) ways, then the second batch can be assigned 8 students in C(15, 8) ways, and the third batch can be assigned C (7, 7) = 1 way. Therefore, by the product rule (similar to that of permutations), the total number of ways of assigning the students is C (25, 10). C (15, 8) = $\frac{25!}{15!\,10!} \cdot \frac{15!}{7!\,8!}$.

Example 2: *A and B are members of a club with a membership of 30. In how many ways can a committee of 10 be formed if*

(i) A must be included in the committee?

(ii) A or B should be included but not both?

Solution: (i) Since A's choice is fixed, the remaining 9 members can be chosen in C (29, 9) ways.

(ii) Since A's choice precludes the selection of B, the remaining 9 members can be chosen in C (28, 9) ways. Similarly if B is included, A should be excluded, again the number of ways to choose the remaining 9 members is C (28, 9). Hence by the rule of sum (similar to that of permutations), the number of ways to select a committee of 10 that includes A or B but not both is 2 C (28, 9).

Note that the rule of sum is based on the inclusion-exclusion principle of sets.

Example 3: *A die is rolled 6 times and the sequence of faces is noted. In how many sequences does the face "5" appear an even number of times? Also find the number of sequences in which "5" appears exactly twice or the face "3" appears exactly 4 times.*

Solution: There are 6 choices for each face and since the die is rolled 6 times, there are 6^6 possible sequences, each of length 6. For the face "5" to appear an even number of times, we have to find the number of times 5 does not appear, appears twice, four times or 6 times. The number of sequences in which 5 does not appear is C(6, 0) 5^6 since all the six positions in the sequence can be occupied by the remaining 5 numbers, in $5 \times 5 \times 5 \times 5 \times 5 \times 5 = 5^6$ ways. Similarly, if it has appear twice it can occupy any of the 6 positions in C(6, 2) ways and the remaining 4 slots can be filled by any of the remaining numbers in $5 \times 5 \times 5 \times 5 = 5^4$

ways. Thus the number of sequences in which the face "5" appears an even number of times is C(6, 0) 5^6 + C(6, 2) 5^4 + C(6, 4) 5^2 + C(6, 6) 5^0.

The number of sequences in which "5" appears exactly twice is C (6, 2) 5^4. The number of sequences in which the "3" appears exactly 4 times is C (6, 4) 5^2. The common case to both the experiments, when "5" appears twice and "3" appears 4 times is C(6, 2) ways. Hence by the mutual inclusion – exclusion principle for cardinalities of two sets, we have

C (6, 2) 5^4 + C (6, 4) 5^2 – C (6, 2) sequences in which "5" appears exactly twice or "3" appears exactly 4 times.

Example 4: *A fair coin is tossed 5 times. Find the number of sequences in which the head 'H' appears at the most 3 times.*

Solution: Since the coin is tossed 5 times, we have a sequence of 5 slots to be filled either by 'H' or 'T' (eg. HTTTT or HHHHH). The number of sequences in which no head appears is C (5, 0) = 1. The number of sequences in which 1 head appears is C (5, 1), 2 heads appear is C (5, 2) and 3 heads appear is C (5, 3). Since these are mutually exclusive cases, the total number of sequences in which H appears at the most 3 times is C (5, 0) + C (5, 1) + C (5, 2) + C (5, 3) = 26.

Example 5: *Mathematics students have to attempt six out of ten questions in an examination in any order. How many choices have they? How many choices do they have if they must answer at least three out of the first five?*

Solution: Since out of 10 distinct objects, we have to choose any six, the total number of selections is C (10, 6) = 210. For the second part we have three possibilities; one can answer three out of the first five and three out of the second five, or four out of the first five and two out of the second five, or five out of the first five and one out of the second five. Hence the total number of selections is: C (5, 3) C (5, 3) + C (5, 4) C (5, 2) + C (5, 5) C (5, 1) = 100 + 50 + 5 = 155.

Example 6: *There are 10 points in a plane of which 4 are collinear. Find the number of triangles that can be formed with vertices at these points.*

Solution: Since 4 points are collinear, it means that the remaining 6 points are non-collinear. These 6 points among themselves can form C (6, 3) = 20 triangles. Taking any two collinear points and one non-collinear point, we can form one triangle. Similarly, taking any two non-collinear points and any one of the collinear points, we can form another triangle. Hence taking the combination of collinear and non-collinear points, we can form C (4, 2) · C (6, 1) + C (4, 1) C (6, 2) = 36 + 60 = 96 triangles. Hence the total number of triangles thus formed is 20 + 96 = 116 triangles.

Example 7: *If no three diagonals of a convex decagon meet at the same point inside the decagon, into how many line segments are the diagonals divided by their intersections?*

Solution: A decagon is a polygon with 10 sides. Since any two vertices of a decagon are joined either by a side or a diagonal, the total number of diagonals = C(10, 2) – 10 = 35. Since for every four vertices, there corresponds exactly one intersection of the diagonals, there are a total of C(10, 4) = 210 intersections between the diagonals. These intersections will give rise to 2×210 line segments. A diagonal itself can be considered as a line segment divided by its own intersection. Hence the total number of required line segments is $35 + 2 \times 210 = 455$.

Quite often we deal with problems on counting, where one has to make r-selections from n-types of objects with **repetitions** freely allowed. This problem can also be described in the following way, using the analogy of identical objects and distinct boxes.

In how many ways is it possible to distribute r identical objects in n distinct boxes, with no restriction put on the number of objects, a box may contain?

Problems of this nature are quite common place. Consider a few examples:

1. In how many ways is it possible to distribute 10 apples among 4 children?
2. In how many ways can 5 balls be selected from 8 identical red balls and 8 identical white balls?

In such problems what we are interested in, is the number of items to be selected and not which ones of them are selected.

To derive a formula for this type of distribution, we adopt the following approach, although there are other ways to solve the problem. Let us represent the r items by r circles and the distribution of r items into n distinct boxes by a sequence of r circles and n – 1 slashes. The slashes indicate the distribution of the items. For example, for n = 4, r = 8, consider the distribution represented by

00 / 00 / 0 / 000

This figure indicates that the first box receives two items, the second also two, the third one item and the last three items.

Next, consider the figure

000 / / 0000 / 0

This indicates that the first box receives three items, the second none, the third four and the last one.

In this manner, we can obtain any distribution by moving the three slashes to occupy any of the $8 + 4 - 1 = 11$ positions.

Hence in the general case, the $n - 1$ slashes should occupy any of the $r + n - 1 = n + r - 1$ positions.

Hence the total number of ways to do this is $C(n + r - 1, n - 1)$.

Example 8: *In how many ways can one distribute 10 apples among 4 children?*

Solution: Consider the apples as identical objects and let the children correspond to distinct boxes. The problem can then be considered as finding the number of ways to distribute 10 identical objects in 4 distinct boxes. This can be done in $C(13, 3) = 286$.

Example 9: *In how many ways can 5 balls be selected from 8 identical red balls and 8 identical white balls?*

Solution: The problem is that of distributing 5 identical objects in two distinct boxes, corresponding to their colours. Since we are required to select 5 objects from two types of objects, one must assume that there are at least 5 objects of each type, which is the case in this problem. Taking $n = 2$, $r = 5$, the number of ways to make the required selection is $C(6, 1) = 6$.

Example 10: *Ten balls are picked from a pile of red, blue and white balls. Find how many such selections contain less than 5 red balls.*

Solution: The number of ways to select 10 balls from a pile of red, blue and white balls is equivalent to distributing 10 identical objects into 3 distinct boxes. Hence there are $C(10 + 3 - 1, 3 - 1) = C(12, 2) = 66$ ways of selection.

The number of ways to select 5 red balls is $C(5 + 3 - 1, 2) = 21$. Hence the number of ways to select 10 balls from a pile of red, blue and white balls, so that each selection contains less than 5 red balls is $66 - 21 = 45$.

Example 11: *How many non-negative integer solutions are there in the equation*

$$x + y + z + u + v = 10{,}000?$$

Solution: The problem is that of distributing 10,000 identical objects in 5 distinct boxes, where there is no restriction on the number of objects, the box may contain.

Hence putting $n = 5$, $r = 10{,}000$ in the formula, $C(n + r - 1, n - 1)$, we have $C(10004, 4) = 417.0834792 \times 10^{12}$ solutions.

Example 12: *Find the number of ways a person can distribute Rs. 601 as pocket money to his three sons, so that no son should receive more than the combined total of the other two. (Assume no fraction of a rupee is allowed).*

Solution: First without applying the restriction, the number of ways to distribute Rs. 601 among his three sons is $C(601 + 3 - 1, 2) = C(603, 2)$.

Let the sons receive x, y, z rupees respectively. Suppose the first son receives more than the combined total of the other two it will follow that $x + y + z = 601$ means $2(y + z) < 601$ which means that $y + z = 300$, in which case x must be 301 at least. Hence the total number of ways in which the first son will receive more money than the combined total of the other two is $C(300 + 3 - 1, 2) = C(302, 2)$.

The same result is applicable when the second son receives more than the combined total of the first and third sons or when the third son receives more than the combined total of the first and second.

Hence the number of ways so that no son receives more than the combined total of the other two is $C(603, 2) - 3\,C(302, 2)$

$$= \frac{603 \times 602}{2} - 3\left(\frac{302 \times 301}{2}\right)$$

$$= 181503 - 136353$$

$$= 45150$$

Example 13: *In how many ways can 15 different books be distributed among three students A, B, C so that A and B together receive twice as many books as C?*

Solution: Let x, y, z denote the number of books that A, B, C receive such that

$$x + y + z = 15 \quad \text{and} \quad x + y = 2z$$

Hence $z = 5$ which means that C receives 5 books.

Hence C can receive the number of books in one way only. Now since $z = 5$, we have $x + y = 10$. We have to find all non-negative integer solutions to this equation. The number of ways in which the solutions are possible is

$$C(n + r - 1, n - 1) = C(2 + 10 - 1, 2 - 1)$$

$$= C(11, 1) = 11.$$

Hence there are 11 ways in which the distribution can be done.

Example 14: *Determine the number of ways to place 2t + 1 indistinguishable balls in three distinct boxes so that any two boxes together will contain more balls than the other one.*

Solution: We shall first determine the number of ways in which the first box will contain more balls than the second and third considered together. For this, since $2t + 1 = t + 1 + t$, place $t + 1$ balls in the first box and then the remaining t balls in the three boxes arbitrarily.

The total number of ways in which this can be done is $C(3 + t - 1, t) = C(2 + t, t)$ (use the formula $C(n + r - 1, r)$, where $n = 3$, $r = t$).

The same applies to the case when the second box contains more balls than the first and third, or when the third box contains more balls than the first and second.

The total number of ways to place $2t + 1$ balls in 3 boxes without any constraint is

$$C(3 + 2t + 1 - 1, 2t + 1) = C(2t + 3, 2t + 1)$$

Hence the total number of ways to place $2t + 1$ balls in 3 boxes so that any two boxes together will contain more balls than the third is

$$C(2t + 3, 2t + 1) - 3\,C(t + 2, t) = \frac{(2t + 3)\,!}{2!\,(2t + 1)\,!} - 3\,\frac{(t + 2)\,!}{2!\,t\,!}$$

$$= \frac{(2t + 3)(2t + 2)}{2} - \frac{3(t + 2)(t + 1)}{2}$$

$$= \frac{(t + 1)}{2}\,4t + 6 - 3t - 6 = \frac{t(t + 1)}{2}$$

5.2.3 Generation of Permutations and Combinations

Procedure to Generate n ! Permutations: An important but interesting problem in combinatorics is to find a systematic procedure for generating all the n! permutations of a set of n distinct elements, with no omissions or repetitions. In order to ensure that all the n ! permutations are indeed generated or that there is no repetition, some kind of hierarchy or ordering needs to be introduced on the permutations. One natural way to do so is to adopt the lexicographic order, which we shall now discuss.

Let $\{a_1, a_2, \ldots, a_n\}$ be a set of n distinct positive integers to be permuted. For two permutations $<a_1, a_2, \ldots, a_n>$ and $<b_1, b_2, \ldots, b_n>$ we shall say $<a_1, a_2, \ldots a_n>$ comes before $<b_1, b_2, \ldots b_n>$ in the lexicographic order if

(i) for some m, $1 \le m < n$.

$a_1 = b_1, a_2 = b_2, \ldots\ldots, a_{m-1} = b_{m-1},$

(ii) $a_m < b_m$.

For example, $<1\ 3\ 4\ 2\ 5>$ comes before $<1\ 3\ 5\ 4\ 2>$, where $m = 3$.

On the other hand, the permutation $<2\ 3\ 6\ 4\ 5>$ comes after $<2\ 3\ 5\ 4\ 6>$. Under this ordering a permutation $<b_1, b_2, \ldots, b_n>$ will immediately succeed $< a_1, a_2, \ldots a_n >$ if

- $a_i = b_i$ for $1 \le i \le m - 1$ and $a_m < b_m$ for the **largest possible m**
- b_m is the smallest element from $\{a_{m+1}, a_{m+2}, \ldots, a_n\}$ that is larger than a_m.
- $b_{m+1} < b_{m+2} < \ldots < b_m$.

Consider for example, the permutation $<1\ 2\ 3\ 4\ 5>$. By conditions (i) and (ii), m is the largest possible value for which $a_m < a_{m+1}$ In this case, obviously, $m = 4$, so that the permutation immediately following $<1\ 2\ 3\ 4\ 5>$ is $<1\ 2\ 3\ 5\ 4>$. Next for the permutation $<1\ 2\ 3\ 5\ 4>$, $m = 3$. Hence by condition (ii), $b_3 = 4$ and by condition (iii), $b_4 = 3$ and $b_5 = 5$. Hence $<1\ 2\ 4\ 3\ 5>$ is the permutation, which comes immediately after $<1\ 2\ 3\ 5\ 4>$.

In this manner, all the 5 ! permutations of $< 1\ 2\ 3\ 4\ 5 >$ will be generated and order in which they appear, will be as follows:

$<1\ 2\ 4\ 3\ 5> \to <1\ 2\ 3\ 5\ 4> \to <1\ 2\ 4\ 3\ 5> \to <1\ 2\ 4\ 5\ 3> \to <1\ 2\ 5\ 3\ 4> \to <1\ 2\ 5\ 4\ 3> \to <1\ 3\ 2\ 4\ 5> \to <1\ 3\ 2\ 5\ 4> \to \ldots \to.$

The procedure followed above can be succinctly stated in the following steps:

1. Examine the permutation $<a_1\ a_2 \ldots a_n>$ element by element from right to left and let a_m be the right most element such that $a_m < a_{m+1}$ (this determines the value of m).
2. Next examine the permutation again, for the right most element such that $a_m < a_k$.
3. Interchange a_m and a_k.
4. Interchange a_{m+1} and a_n, a_{m+2} and a_{n-1}, a_{m+3} and a_{n-2}, and so on.

For example, consider $<1\ 2\ 5\ 4\ 3>$. Here $m = 2$ and $k = 5$. Hence interchange a_2 and a_5, i.e. 2 and 5. Hence $<1\ 2\ 5\ 4\ 3>$ becomes $<1\ 3\ 5\ 4\ 2>$. Next interchange a_3 and a_5. Hence we obtain $<1\ 3\ 2\ 4\ 5>$, which is the immediate successor to $<1\ 2\ 5\ 4\ 3>$.

Alternative Method:

In this method, we generate all the n! permutations of {1, 2, ..., n} by successively generating all the permutations of {1}, {1, 2}, {1, 2, 3}, ... and so on up to {1, 2, ... n – 1}. Let us see, how this is done. Beginning with <1>, we obtain the permutations <1 2> and <2 1> placing 2 on either side of 1. Next, in the permutation <1 2>, we place 3 at the extreme ends and between 1 and 2 to obtain the permutations <1 2 3>, <1 3 2> and <3 2 1>. Similarly from <2 1> we obtain <2 1 3>, <2 3 1>, <3 2 1>. Proceeding in this manner, we obtain all permutations of <1, 2, 3, ... n – 1> and then inserting n at both the ends and in then n – 2 gaps in between. We obtain all the n! permutations of {1, 2, 3, ... n}.

For example, suppose we wish to generate all the 24 permutations of {1, 2, 3, 4}. Then the sequential process, by which the permutations are generated is shown below:

<1>;

<1 2>, <2 1>;

<1 2 3>, <1 3 2>, <3 1 2>, <2 1 3>, <2 3 1>, <3 2 1>;

<1 2 3 4>, <1 2 4 3>, <1 4 2 3>, <4 1 2 3>;

<1 3 2 4>, <1 3 4 2>, <1 4 3 2>, <4 1 3 2>;

<3 1 2 4>, <3 1 4 2>, <3 4 1 2>, <4 3 1 2>;

<2 1 3 4>, <2 1 4 3>, <2 4 1 3>, <4 2 1 3>;

<2 3 1 4>, <2 3 4 1>, <2 4 3 1>, <4 2 3 1>;

<3 2 1 4>, <3 2 4 1>, <3 4 2 1>, <4 3 2 1>.

Procedure to Generate Subsets of {1, 2, 3, ... n}.

Let $\{a_1, a_2, ..., a_k\}$ be a subset of size k of {1, 2, ..., n}, with $a_1 < a_2 < ... < a_k$.

Then the maximum possible value of a_{k-1} is n – 1 and so on.

In general, the maximum possible value of a_i is n – k + i.

Consider the subset {1, 2, ..., k–1, k}. If k ≠ n, its maximum possible value, increase k by 1, so that the next subset {1, 2, ..., k – 1, k + 1} is generated. We continue this till the subset {1, 2, ..., k – 1, n} is reached.

Next repeat the procedure for k – 1. If k – 1 is not equal to its maximum value n – 1, increase it by 1 and let k take all large values. Continue this procedure with k – 1, till n – 1 is reached.

Then move to k – 2 and repeat the steps. In this manner, moving from right to left, we finally reach the a_n element a_j such that a_j can be increased to $a_j + 1$, but no a_i with i > j can be increased, which means that at some stage, a_i is equal to its maximum value n – k + i.

The procedure terminates when a_1 reaches its maximum value.

Let us apply this procedure to the following example.

Example 15: *Generate all the subsets of size 4 of {1, 2, 3, 4, 5, 6}.*

Solution: Begin with the subset {1, 2, 3, 4}. Now for any subset $\{a_1, a_2, a_3, a_4\}$ with $a_1 < a_2 < a_3 < a_4$, maximum possible value of a_4 is 6, of a_3 is 5, of a_2 is 4 and of a_1 is 3.

Hence increasing 4 by 1, we obtain the subset {1, 2, 3, 5}. Since a_4 has not still reached its maximum value, increasing 5 by 1, we obtain {1, 2, 3, 6}. We next move to the element 3 and repeat the procedure for 3 till 3 as a_3 reaches its maximum value 5, with the last element taking large values till it also reaches its maximum value. This gives us the subsets {1, 2, 4, 5}, {1, 2, 4, 6} and {1, 2, 5, 6}.

In this manner, we obtain the following 15 subsets:

{1, 2, 3, 4}, {1, 2, 3, 5}, {1, 2, 3, 6}, {1, 2, 4, 5}, {1, 2, 4, 6}, {1, 2, 5, 6}, {1, 3, 4, 5}, {1, 3, 4, 6}, {1, 3, 5, 6}, {1, 4, 5, 6}, {2, 3, 4, 5}, {2, 3, 4, 6}, {2, 3, 5, 6}, {2, 4, 5, 6}, {3, 4, 5, 6}.

SOLVED EXAMPLES

Example 1: *How may four-digit numbers are there formed from the digits 1, 2, 3, 4, 5 (with possible repetition) that are divisible by 4?*

Solution: A number is divisible by 4 if and only if the number formed by its two right most digits is divisible by 4. These numbers are 12, 32, 52, 24, 44. Numbers whose two right most digits are 12 are 5 × 5 numbers. Similarly in the other cases too, we have 5^2 numbers, whose last two digits are 32, 52, 24 or 44 respectively. Hence, in all there are $5^2 + 5^2 + 5^2 + 5^2 + 5^2 = 5^3$ numbers which are divisible by 4.

Example 2: *How many sequences of length 5 can be formed using the digits 0, 1, 2, ..., 9 with the property that exactly two of the digits appear, (e.g. 03330)?*

Solution: First we can select two digits from the ten digits in $^{10}C_2$ ways. Using these digits, we can form sequences of length 5 in 2 × 2 × 2 × 2 × 2 ways. However, since the condition is that both the digits should appear, we should omit the two sequences in which exactly one digit appears. For example, if we choose 0 and 1, we should omit the sequences 00000 and 11111. Hence, such sequences are $2^5 - 2$ for each pair of digits chosen.

Hence, in all there are

$$^{10}C_2 (2^5 - 2) = \frac{10 \times 9 \times 30}{2} = 1350 \text{ sequences.}$$

Example 3: *How many ways are there to pick 2 different cards from a standard 52-card deck such that*

(i) the first is an ace and the second card is not a queen?

(ii) the first card is a space and the second card is not a queen?

Assume that the cards are not replaced.

Solution: (i) The first card, being an ace, can be chosen in 4 ways. Since we are not replacing the card and no queen should be chosen, we have only 52 – 1 – 4 = 47 ways to choose the second card. Hence there are 3 × 47 = 188 ways to choose the pair.

(ii) The first card being a spade, there are 13 possibilities. If the first card happens to be a queen of spaces, for the second card there are 48 choices. If the first card is not a queen of spades, then for the second card there are 48 choices. If the first card is not a queen of spades, then for the second card there are 52 – 4 – 1 = 47 choices; in this case the first card already having been chosen in 12 ways. Hence the total number of choices = (1 × 48) + (12 × 47) = 612.

Example 4: *There are 10 different people at a party. How many ways are there to pair them of into a collection of 5 pairings?*

Solution: Let A be an arbitrary person in the party; then he can be paired with the remaining 9 persons in 9 ways. Let {A, B} be one such pair. Then the third person C has 7 choices. In this manner we have 9 × 7 × 5 × 3 × 1 = 945 ways of pairing.

Example 5: *How many arrangements of the word INSTRUCTOR are there in which there are exactly two consonants between successive pairs of vowels?*

Solution: There are 3 vowels I, O, U and 7 consonants in the word. The 3 vowels can rearrange themselves in 3! ways. Let I xx O xx U be one such arrangement in which 2 consonants are between the successive pairs of vowels. These pairs of consonants can be selected in $^7C_2 \times {}^5C_2$ ways, and then be arranged between I, O and O, U in 2 ways. The remaining 3 consonants can be regrouped in 3! ways and have 2 positions to occupy, either before I or after U. Hence, in all, there are

$3! \times {}^7C_2 \times {}^5C_2 \times 2 \times 3! \times 2$ = 30, 240 ways of arrangement.

Example 6: *A box contains 6 white balls and 5 black balls. Find the number of ways, 4 balls can be drawn from the box if*

(i) two must be white

(ii) all of them must have the same colour.

Solution: (i) The two white balls can be selected in C (6, 2) ways. The remaining 2 balls can be selected in C (9, 2) ways. Hence the number of ways to make the necessary selection is

C (6, 2) · C (9, 2) = 540 ways.

(ii) All of them must have the same colour implies that the balls must be all white or all black. The number of ways in which exactly one of these combinations can be done is C (6, 4) + C (5, 4) = 15 + 5 = 20.

Example 7: *Suppose a valid computer password consists of 4 characters, the first of which is a letter chosen from the set {A, B, C, ... Z} and remaining 3 are chosen from English alphabets or digits from 0 to 9. How many passwords are there?*

Solution: The first character has 26 choices. Each of the remaining characters have 26 + 10 = 36 choices. Hence the total number of passwords that are possible is 26×36^3.

Example 8: *Consider all positive integers with three different digits.*

(i) How many numbers are greater than 700?

(ii) How many numbers are even?

(iii) How many numbers are odd?

(iv) How many numbers are divisible by five?

Solution: (i) The first digit of a number greater than 700 is either 7, 8 or 9, hence can be chosen in three ways. The second digit of such a number will be any number from 0 to 9 but excluding the one which occurs as the first digit, hence can be chosen in 9 ways. Similarly the third digit will have 8 choices. Hence, the total number of numbers with three different digits and greater than 700 is 3 × 9 × 8 = 216.

(ii) The last digit of an even number is either 0, 2, 4, 6 or 8. If the last digit is 0, number of such even numbers is 9 × 8 = 72. If the last digit is 2, 4, 6, or 8, the number of even numbers is 8 × 8 × 4 = 256. Hence, there will be in all 72 + 256 = 328 even numbers.

(iii) The last digit on an odd number is 1, 3, 5 or 9. For each such choice, the first digit can be chosen in 8 ways, as 0 has to be excluded, the second digit also in 8 ways, as 0 can be included. Hence, there will be $8 \times 8 \times 5 = 320$ odd numbers.

(iv) For a number to be divisible by 5, the last digit is 0 or 5 (with non-repeated digits). Total number of numbers (with non-repeated digits) with last digit 0 is $8 \times 9 = 72$. If the last digit is 5, total number of such numbers is $8 \times 8 = 64$. Hence, total number of numbers, which are divisible by 5 is $72 + 64 = 136$.

Example 9: *Find the number of ways to paint 12 offices so that 3 of them will be green, 2 of them will be pink, 2 of them will be yellow and the remaining white. Also give the generalised formula.*

Solution: The 12 offices can be painted in 12 ! ways, in general. However, since 3 of them should be green, 2 of them pink, 2 of them yellow and 5 white, the number of ways is

$$\frac{12!}{3!\,2!\,2!\,5!}$$

Generalised formula: $\dfrac{P(n, r)}{r_1!\, r_2! \ldots r_k!}$

where, $r = r_1 + r_2 + \ldots + r_k$

Example 10: *From 12 mathematicians and 9 physicists, a committee of 8 is to be formed including two physicists. In how many ways can the committee be chosen so as to give majority of mathematicians?*

(i) 2 physicists and 6 mathematicians.

(ii) 3 physicists and 5 mathematicians.

Solution : (i) The two physicists can be chosen in ${}^9C_2$ ways and 6 mathematicians in ${}^{12}C_6$ ways.

Hence the number of ways of forming the committee is ${}^9C_2 \cdot {}^{12}C_6 = 33264$ ways.

(ii) A committee of 3 physicists and 5 mathematicians can be chosen in

${}^9C_3 \cdot {}^{12}C_5 = 66528$ ways.

Hence, the number of ways in which the committee can be chosen so as to give majority of mathematicians is ${}^9C_2 \cdot {}^{12}C_6 + {}^9C_3 \cdot {}^{12}C_5$ ways.

Example 11: *There are 50 students in each of the junior and the senior classes. Each class has 25 male and 25 female students. In how many ways can eight representatives be selected so that there are four females and three juniors?*

Solution: From the phrasing of the problem it is to be assumed that the four females belong to the senior class and so is the other 8th representative, obviously a male. The number of ways to select the four females from the senior class is C (25, 4). The number of ways to select the single male representative from the senior class is C (25, 1). Since the number of female students is specified in the group, it follows that the juniors are the male students from the junior class. Hence the number of ways in which the necessary selection can be made is C (25, 4) · C (25, 1) · C (25, 3).

Example 12: *In how many ways can the letters in the English alphabet be arranged so that there are exactly seven letters between the letters a and b?*

Solution: We have to permute the remaining 24 letters to obtain the desired result. Out of these 24 letters, 7 letters can be filled between a and b in P (24, 7) ways; also allowing for interchange between a and b, there are 2P (24, 7) strings of 9 letters, each beginning with a and ending with b or vice-versa. The remaining $26 - 9 = 17$ letters, together with the string form a group of 18 elements and hence be permuted in 18! ways. Hence, there are in all 2·P (24, 7) · 18 ! arrangements of the letters of the desired type.

Example 13: *There are ten political leaders gathered at a party and two are known to be staunch opponents. In how many ways can they be seated in a row so that these two persons do not sit next to each other?*

Solution: Let us name the two quarrelling leaders as A and B. The total number of ways to seat 10 persons is 10 !. Let us count the number of ways in which A and B can be seated together. We have to consider two types of arrangements where the order is AB (i.e. B is seated immediately after A) and BA (where B is seated before A). The number of ways for each such arrangement will however be the same. Treating AB as a single entity, the number of ways to seat the ten guests is $(10 - 1)! = 9!$. The other arrangement, involving BA, also has 9 ! ways. Hence A and B can be seated together in 2×9 ! ways. Hence the number of ways in which they are not seated next to each other is $10! - (2 \times 9!)$.

Example 14: *In how many ways can 5 girls and 7 boys are seated in a row so that no two girls are seated next to each other?*

Solution: First let the boys be seated, which can be done in 7 ! ways. Now, for any such arrangement there will be $(7 - 1) = 6$ gaps between adjacent pairs and two vacant spaces at the extreme ends. Hence, there are $6 + 2 = 8$ vacant slots, which can be filled by the 5 girls. Since no gap should be assigned to more than one girl, this can be done in P(8, 5) ways. Hence, the total number of ways in which no two girls are seated next to each other is $7! \times \dfrac{8!}{3!} = 338688 \times 10^2$.

Example 15: *Suppose we print all five digit numbers on slips of paper with one number on each slip. Find how many minimum distinct slips one has to make up for all the five digit numbers.*

Solution: Note that there are 10^5 distinct five digit numbers (including the five digit number beginning with 0). Also note that the digits 0, 1, 6, 8 and 9 become 0, 1, 9, 8 and 6 when they are read upside down. Hence for numbers involving these digits, we can have common slip. For example, 1689 and 1986 can share the same slip, if the slips are read right side up or upside down. There are in all $5 \times 5 \times 5 \times 5 \times 5 = 5^5$ such numbers. There are some numbers among these which read the same whether inverted or not. For example, 16891, 86198, 18081 are such numbers. In all these numbers, the centre digit is either 0, 1 or 8. Hence numbers of this type are $3\,(5^2)$ numbers. Consequently there are $5^5 - 3\,(5^2)$ numbers that can be read right side up or upside down, but read differently. Hence these numbers can be divided into pairs that can share the same slip. Hence the total number of slips required is $\left[10^5 - \dfrac{5^5 - 3\,(5^2)}{2}\right]$.

Example 16: *Five boys and five girls are to be seated in a row. In how many ways can they be seated if*

(i) all boys must be seated in the five left-most seats?

(ii) no two boys can be seated together?

(iii) John and Mary must be seated together?

Solution: (i) Since the boys must be seated in the five left most seats, the girls must be seated in the right most seats. The boys can be seated in 5 ! ways and so are the girls. Hence the total number of ways in which they can be seated in this manner is 5 ! 5 !.

(ii) First let the girls be seated. This arrangement can be done in 5 ! ways. Now for each such seating arrangement there will be $5 - 1 = 4$ gaps between any adjacent pair of girls, in addition to the 2 gaps at the extreme ends. Hence the seating arrangement of the boys should be such that these 6 gaps have to be filled in 5 ways so that no gap should be assigned to more than one boy. This will be done in P (6, 5) ways. Hence the number of ways of seating the boys, so that no two are seated adjacent is 5 ! P (6, 5) ways = 5 ! 6 ! ways.

(iii) The number of ways in which John and Mary are seated together is $2 \times 9 = 18$ ways. The remaining 8 people can be seated in 8 ! ways. Hence the number of ways in which the boys and girls are seated so that John and Mary are seated together is 8 ! · 18.

Example 17: *In a class of 100 students 40 are boys.*

(i) In how many ways can a 10 person committee be formed?

(ii) Repeat (i) if there must be an equal number of boys and girls in the committee.

(iii) Repeat (i) if the committee must consist of either 6 boys and 4 girls or 4 boys and 6 girls.

Solution: (i) A 10 person committee can be formed in C (100, 10) ways.

(ii) There should be 5 boys and 5 girls in the committee. This can be done in C (40, 5) · C (60, 5) ways.

(iii) The number of ways in which the committee consists of six boys and four girls is C (40, 6) · C (60, 4).

The number of ways in which the committee consists of four boys and six girls is C (40, 4) · C (60, 6).

Hence the number of ways in which the committee consists of either combination is C (40, 6) · C (60, 4) + C (40, 4) · C (60, 6).

Example 18: *A student has to answer 10 out of 13 questions in an examination.*

(i) How many choices he has?

(ii) How many choices he has if he has to answer the first two questions?

(iii) How many choices he has if he must answer the first or second but not both?

(iv) How many choices he has if he must answer exactly three out of first five?

(v) How many choices he has if he must answer at least three of the first five?

Solution: (i) C (13, 10) = 286

(ii) C (11, 8) = 165

(iii) [C (11, 9) + C (11, 9)] = $2 \times 55 = 110$

(iv) C (5, 3) · C (8, 7) = 80

(v) C (5, 3) · C (8, 7) + C (5, 4) · C (8, 6) + C (5, 5) · C (8, 5) = 80 + 140 + 56 = 276.

***Example 19:** Three thieves have stolen a cash of Rs. 10,000, all of which is in the notes of denomination 10. How many ways can they distribute the money amongst themselves?*

Solution: The number of notes of 10 are 1000, which can be treated as identical objects to be distributed amongst the three thieves. Hence the problem is equivalent to that of distributing 1000 identical objects in 3 distinct boxes, with no restriction on the number of objects a box may contain. Hence the number of ways is C (n + r – 1, n – 1) where n = 3, r = 1000, i.e. C (1002, 2).

(Problems On Derangement)

***Example 20:** Five gentlemen attend a party, where before joining the party they leave their overcoats in a checkroom. After the party, the overcoats get mixed up are returned to the gentlemen in a random manner. Find the number of ways in which none receives his own overcoat.*

Solution: Let us name the gentlemen as a, b, c, d, e. Let A denote the event a gets back his overcoat, B the event that b gets back his overcoat and similarly C, D, E are defined. Then $|A| = |B| = |C| = |D| = |E| = 4! = 24$. $A \cap B$ denotes the event that both A and B get their overcoats. Hence $|A \cap B| = 3! = 6$.

Similarly, $|A \cap C| = |A \cap D| = |A \cap E| = |B \cap C|$

$= |B \cap D| = |B \cap E| = |C \cap D| = |C \cap E|$

$= |D \cap E| = 3! = 6.$

Similarly taking any three events, say A, B, C $|A \cap B \cap C| = 2! = 2$ and there are 10 such sets.

Taking 4 sets at a time, there are 5 such sets whose cardinalities are 1 each.

Finally $|A \cap B \cap C \cap D \cap E| = 1$.

Hence by the principle of mutual inclusion-exclusion,

$|A \cup B \cup C \cup D \cup E| = \sum |A| - \sum |A \cap B| + \sum |A \cap B \cap C| - \sum |A \cap B \cap C \cap D| + |A \cap B \cap C \cap D \cap E|$

$$= (5 \times 24) - (10 \times 6) + (10 \times 2) - 5 + 1$$

$$= 120 - 60 + 20 - 4 = 76$$

Hence the number of ways in which any of the five gentlemen can get back his overcoat is 76. Therefore the number of ways in which no gentlemen gets back his overcoat is

$$5! - 76 = 120 - 76 = 44.$$

***Example 21:** In how many ways can the 4 walls of a room be painted with 3 colours so that no two adjacent walls have the same colour?*

Solution: The number of ways in which 4 walls can be painted with 3 colours is $3^4 = 81$ ways. Label the walls as a, b, c, d in the clockwise manner (or anticlockwise manner). Let A denote the event that walls a and b have the same colour, B the event that walls b and c have the same colour. Similarly C and D are defined. Then $|A| = |B| = |D| = 3^3$. This is explained as follows. For the event A since walls a and b have the same colour, together they have 3 choices. The remaining two walls will have $3 \times 3 = 3^2$ choices.

Similarly $|A \cap B| = |A \cap C| = |A \cap D| = |B \cap C| = |B \cap D| = |C \cap D| = 3^2$ and $|A \cap B \cap C \cap D| = 3$.

Hence the number of ways in which any two adjacent walls will have the same colour is

$|A \cup B \cup C \cup D| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$

$$= 4 \times 3^3 - 6 \times 3^2 + 4 \times 3 - 3$$

$$= 108 - 54 + 12 - 3 = 63$$

Hence the number of ways of painting 4 walls with 3 colours, so that no two adjacent walls have the same colour is $1 - 63 = 18$.

***Example 22:** A man has 7 relatives, 4 of them are ladies and 3 are gentlemen. His wife has 7 relatives, 3 of them are ladies and 4 are gentlemen. In how many ways can they invite a dinner party of 3 ladies and 3 gentlemen, so that there are 3 of man's relatives and 3 of wife's relatives?*

Solution: Let us prepare the box as follows:

M		W	
4L	3G	3L	4G

Since there must be 3 relatives of M and 3 relatives of W, we can form the following combinations.

M	W
3L	3G
2L, 1G	1L, 2G
1L, 2G	2L, 1G
3G	3L

Hence, the total number of ways to invite the guests is $4C_3 \cdot 4C_3 + 4C_2 \cdot 3C_1 \cdot 3C_1\, 4C_2 + 4C_1\, 3C_2 \cdot 3C_2 \cdot 4C_1 + 3C_3 \cdot 3C_3$

$$= 16 + 324 + 144 + 1 = 485$$

***Example 23:** (i) In how many ways can 6 men and 5 women be seated in a line so that no two women sit together.*

(ii) In how many ways can 6 men and 5 women sit in a line so that women occupy the even places.

Solution: (i) Let the men be first seated which can be done in 6! ways. For each such arrangement, there will be 6 – 1 = 5 gaps between adjacent pairs and two vacant spaces at the extreme ends. These 5 + 2 = 7 slots can be filled by the girls in P (7, 5) ways, as no gap can be filled by more than one girl. Hence the total number of ways is $\frac{6!\,7!}{2!}$.

(ii) Let the places be numbered 1 to 11, so that the women should occupy the places numbered as 2, 4, 6, 8, 10. This leaves the places numbered as 1, 3, 5, 7, 9 and 11 to be occupied by the men. Hence, the total number of ways to do so is $5! \times 6!$.

***Example 24:** A man, a woman, a boy, a girl, a dog and a cat are walking along a long and winding road, one after the other.*

(i) In how many ways can this happen?

(ii) In how many ways can this happen if the dog comes first.

(iii) In how many ways can this happen if the dog immediately follows the boy?

(iv) In how many ways can this happen if only dog is in between the man and boy.

Solution: Let M, W, B, G, D, C denote respectively man, woman, boy, girl, dog and cat.

(i) We have 6 ! permutations of the 6 letters. Hence the number of ways in which they walk in sequence is 6 ! ways.

(ii) ∴ D must come first, the remaining 5 letters can be permuted in 5 ! ways. Hence, if dog the dog leads the sequence, the total number of ways is 5 !.

(iii) We must have the combination BD in any arrangement of the 6 letters. Hence, this can be done in $5 \times 4! = 5!$ ways, since the block BD can occupy any of the 5 places as follows:

... M ... W ... G ... C ... for one arrangement of the remaining letters.

(iv) If only D is between the man and boy, we must have MDB or BDM in any arrangement. The remaining 3 letters have in all 3 ! permutations.

Hence, the required number of ways is $2 \times 4 \times 31 = 2 \times 4!$ ways.

***Example 25:** A family of 4 brothers and 3 sisters is to be arranged in a row of a photograph. In how many ways can they be seated if all sisters are together.*

Solution: Consider all the sisters as one single object and the brothers as 4 distinct objects. Then the 5 objects can be permuted in 5! ways. The three sisters among themselves, can be seated in 3! ways.

Hence, the total number of ways, all can be seated is 3! 5! ways.

***Example 26:** In how many ways can 10 examination papers be arranged so that the best and worst are never together.*

Solution: Total number of ways of arranging the 10 papers is 10!.

The number of ways in which the best worst papers are consecutive is 2! 9! ways.

Hence, total number of ways in which the ten papers can be arranged so that the best and worst papers are never together is 10! – 2! 9! = 8.9! ways.

***Example 27:** A menu card in a restaurant displays four soups, five main courses, three desserts and five beverages. How many different menus can a customer select if:*

(i) He selects one item from each group without omission.

(ii) He chooses to omit the beverages, but selects one each from the other groups.

(iii) He chooses to omit he desserts, but decides to take a beverage and one item each from the remaining groups.

Solution: (i) Selecting one item from each group without omission can be done in ${}^4C_1\, {}^5C_1\, {}^3C_1\, {}^5C_1$ ways.

(ii) Omitting beverages, selection can be done in ${}^4C_1\, {}^5C_1\, {}^3C_1$ ways.

(iii) Omitting the desserts, selection can be made in ${}^4C_1\, {}^5C_1\, {}^5C_1$ ways.

***Example 28:** How many automobile license plates can be made if each plate consists of different letters followed by three different digits. Solve the problem if first digit cannot be 0.*

Solution: It is not mentioned in the problem how many different letters, the license plate must contain, whether 2 letters followed by 3 digits or 3 letters followed by 3 digits. Suppose we assume that the plate contains 2 alphabets followed by 3 digits, then the number of ways $26 \times 25 \times 9 \times 9 \times 8$.

On the other hand if the plate contains 3 alphabets followed by 3 digits, the number of ways is $26 \times 25 \times 24 \times 9 \times 9 \times 8$.

***Example 29 :** In how many ways can three prizes be distributed among four winners so that no one gets more than one prize ?*

Solution : Since winners are more than the prizes available, a winner may received no prize or at the most one prize. This can be done in 4 + 4 = 8 ways.

***Example 30 :** In how many ways can three prizes be distributed among four winners so that*

(i) A winner may get any number of prizes.

(ii) No winner gets all the prizes.

Solution : (a) Since a winner may set any number of prizes, the 3 prizes may be distributed in $4 \times 4 \times 4 = 64$ ways.

(b) Since no winner gets all the prizes, the number of ways to distribute the three prizes is $4 \times 4 \times 4 - 4 = 60$ ways.

EXERCISE – 5.1

1. How many permutations are there of the 26 letters of the alphabet in which the 5 vowels are in consecutive places?

(**Ans.** $5! \times 22!$)

2. How many different necklaces can be designed from 6 different colours, using one bead of each colour?

(**Ans.** 60)

3. A car registration number is to consist of 2 letters followed by a 4 digit number. How many car numbers are possible?

(**Ans.** $26^2 \cdot (10^4 - 1)$)

How many numbers between 1000 and 3000 can be formed from the digits 1, 2, 3, 4, 5 if repetition of digits is (i) allowed (ii) not allowed?

(**Ans.** (i) 2×5^3, (ii) $2 \times 4!$)

In how many ways can a 5-letter word be formed from an alphabet of 26 letters if repetitions are (i) allowed, (ii) not allowed? (**Ans.** (i) 26^5, (ii) $\frac{26!}{21!}$)

How many different arrangements of the letters in the word MONDAY can be formed if the vowels must be kept next to each other? (**Ans.** 240)

In how many ways can the letters in the word COMMITTEE be rearranged? (**Ans.** 45360)

The names of the 12 months of the year are listed in random order. Given that May and June are not next to each other, how many possible lists are there?

(**Ans.** $10 \times 11!$)

9. An eight member committee is to be formed from a group of 10 men and 15 women. In how many ways can the committee be chosen if –

(i) the committee must contain 4 men and 4 women?

(ii) there must be more men than women?

(iii) there must be at least two men?

(**Ans.** (i) C (10, 4) · C (15, 4),

(ii) C (10, 5) · C (15, 3) + C (1, 6) · C (15, 2) + C (10, 7) · C (15, 1) + C (10, 8)

(iii) C (25, 8) – C (15, 8) – 10C (15, 7))

10. A team of 11 players is to be chosen from a pool of 15. How many team selections are possible? How many if one of the 15 has already been appointed captain and must play? (**Ans.** C (15, 11), C (14, 10))

11. A man plans to visit one friend on each evening of a given week. There are 12 friends whom he would like to visit. In how many ways can he plan his week if

(i) he can visit a friend more than once?

(ii) he will not visit a friend more than once?

(**Ans.** (i) 12^7, (ii) $\frac{12!}{5!}$)

12. 6 men are to be seated round a circular table. How many ways are there of achieving this? How many if A refuses to sit beside B? (**Ans.** (i) 120, (ii) 72)

13. 16 students, 4 each from FE, SE, TE and BE have to select 6 of their number to form a subcommittee. How many selections can be made if

(i) each class is represented?

(ii) no class can have more than two representatives?

(**Ans.** (i) $4^5 + 6^3 \cdot 4^2$ (ii) $4^2 6^3 + 4 \cdot 6^3$)

14. There are 10 different books and 2 copies of each. Find the number of ways in which a selection can be made from them. (**Ans.** $3^{10} - 1$)

15. A palindrome is a word that reads the same forward and backward. How many seven letter palindromes can be made out of the English alphabet? How many 6 letters palindromes? (**Ans.** (i) 26^4, (ii) 26^3)

16. 6 boys and 6 girls are to be seated in a row such that

(i) All boys sit together and girls sit together.

(ii) No two girls sit together.

(iii) Boys and girls sit alternately.

(iv) The extreme positions are occupied by boys.

Find the number of ways in each case.

(**Ans.** (i) $2 \times (6!)^2$, (ii) $7!$ (iii) $2\,(6!)^2$ (iv) $P(6, 2) \times 10!$)

17. From a class of 11 students, what is the number of ways to select a committee of 5 students? Also find the number of ways if

(i) Class representative should always be included.

(ii) Last ranker should always be excluded.

(**Ans.** ${}^{11}C_5$, (i) ${}^{10}C_4$, (ii) ${}^{10}C_5$)

18. A certain stationary shop has 6 types of ball pens available in 6 different colours. If a student wants to buy one, how many choices does he have? (**Ans.** 36)

19. In how many ways can 7 books be arranged on a shelf so that

(i) two particular books are together?

(ii) these two books are not together?

(**Ans.** (i) 1440, (ii) 4320)

20. In how many ways can 6 letters be placed in 6 envelopes, if 2 of the letters are too large for one of the envelopes?

(**Ans.** 480)

21. If 10 parallel straight lines are intersected by 8 other parallel straight lines, then find the number of different parallelograms so formed. (**Ans.** 1260)

22. 7 people enter the lift. The lift stops only at three floors. At each of the floor, none enters the lift, but at least 1 person leaves the lift. After the three floor stops, the lift is empty. In how many ways can this happen? (**Ans.** $3^7 - 3\,(2^7 - 1)$)

23. A student wishes to prepare for four subjects during a seven-day period; so that she may be able to devote at least one day for each subject. Find the number of ways in which she can plan her time table.

(**Ans.** $4^7 - 4.3^7 + 6.2^7 - 4$)

24. In how many ways can 21 white balls be distributed in three distinct boxes so that any two boxes together contain more balls than the other one?

(**Ans.** 55)

25. How many non-negative integer solutions are to the equation?

(i) $x + y + z = 8$ (ii) $x + y + z + t = 29$.

(**Ans.** (i) 45, (ii) 4960)

26. There are 10 copies of one book and 1 copy each of 10 other books. In how many ways can we select 10 books?

(**Ans.** 2^{10})

27. A fair coin is tossed 5 times. Determine how many heads occur exactly 3 times. Determine how many sequences have atmost 4 heads? (**Ans.** 10, 31)

28. Eight students are standing in line for an interview. Find the probability that there are exactly two freshman, two sophomores, two juniors and two seniors in the line.

29. A dice is rolled 6 times and sequence of faces is noted. In how many sequences does the face "5" appear an even number of times. Find the number of sequences in which "5" appears exactly twice or the face "3" appears exactly 4 times.

30. A box contains 6 white balls and 5 black balls. Find the number of ways in which 4 balls can be drawn from the box if:

(i) Two must be white.

(ii) All of them must have the same colour.

31. Five boys and five girls are to be seated in a row. In how many ways can they be seated if:

(i) No two boys can be seated together.

(ii) John and Mary must be seated together.

32. Find the number of ways that a party of seven persons can arrange themselves in:

(i) a row of seven chairs.

(ii) a round a circular table.

33. A women has 11 close friends:

(i) In how many ways, can she invite five of them to dinner?

(ii) In how many ways if two of the friends are married and will not attend separately?

(iii) In how many ways if two of them are not on speaking terms and will not attend together?

34. A pair of fair dice is thrown. Find the probability p that sum is 10 or greater if:

(i) 5 appears on the first die.

(ii) 5 appears on at least one die.

35. In how many ways, five boys and five girls are to be seated in a row if:

(i) All the boys must be seated in five leftmost seat?

(ii) No two boys can be seated together?

(iii) John and Mary may be seated together?

36. (i) Find the number m_1 of permutations that can be formed from all the letters of MISSISSIPPI.

(ii) Find the number m_2 for the above case if words are to begin with an I.

(iii) Find the number m_3 for the word in (i) if the two P's are to be next to each other.

(iv) Find the number m_4 for the word in (i) if the four S's are to be next to each other.

37. Suppose repetitions are not permitted, how many four digit numbers can be formed from six digits 1, 2, 3, 5, 7, 8?

(i) How many of such numbers are less than 4000?

(ii) How many in (1) are even?

(iii) How many in (1) are odd?

(iv) How many in (1) contain both 3 and 5?

(v) How many in (1) are divisible by 10?

38. A student has to answer 10 out of 13 questions in an examination.

(i) How many choices he has?

(ii) How many choices he has if he has to answer the first two questions?

(iii) How many choices he has if he must answer exactly three out of first five?

(iv) How many choices he has if he must answer at least three out of first five?

39. (i) Find the number m of permutations that can be formed from all the letters of ELEVEN.

(ii) Find the number of permutations if word in (1) begins with L.

(iii) Find the number of permutations if the words are to begin and end with E.

(iv) Find the number of permutations if the words are to begin with E and end with N.

40. (i) How many distinguishable permutations of the letters in the word "BANANA' are there?

(ii) Compute number of permutations of the set given: {1, 2, 3, 4, 5}.

(iii) Find the number of permutations of A taken r at a time:

A = {a, b, c, d, e, f}, r = 2.

(iv) In how many ways can six men and six women be seated in a row if any person may sit next to any other.

41. A bit is either 0 or 1: a byte is a sequence of 8 bits. Find:

(i) The number of bytes that can be formed from 8 bits.

(ii) The number of bytes than begin with 11 and end with 11.

(iii) The number of bytes that begin with 11 and do not end with 11.

(iv) The number of bytes that begin with 11 or end with 11.

42. Five fair coins are tossed and the results are recorded:

(i) How many different sequences of heads and tails are possible?

(ii) How many of the sequence in part (i) have exactly one head recorded?

(iii) How many of the sequences in part (i) have exactly three heads recorded?

43. Find the number of unordered samples of size five (repetition allowed) from the set {a, b, c, d, e, f}

(i) No further restriction.

(ii) 'a' occurs at least once.

(iii) 'a' occurs exactly twice.

44. A computer password consists of a letter of the alphabet followed by 3 or 4 digits. Find:

(i)The total number of passwords that can be formed.

(ii) Number of passwords in which no digits repeat.

45. Find the number of permutation of letters a, b, c, d, e, f, g so that neither the pattern 'beg' nor 'cad' appears.

46. Given 6 flags of different colours, how many different signals can be generated, if signal requires the use of two flags, one below the other?

47. In how many ways can three prizes be distributed among 4 boys when:

(i) No one gets more than one prize.

(ii) A body can get number of prizes.

48. Find the number of arrangements that can be made out of the letters.

(i) ASSASSINATION (ii) GANESHPURI

49. How many words with or without meanings can be formed using all the letters of the word 'EQUATION', using each letter exactly once.

50. Find the number of ways of arranging the letters of the word. TENNESSEE all at a time (i) if there is no restriction (ii) if the first two letters must be 'E'.

51. Suppose repetitions are permitted:

(i) How many ways three digit number can be formed from six digits 2, 3, 4, 5, 7 and 9?

(ii) How many of these numbers are less than 400?

(iii) How many are even?

(iv) How many are odd?

(v) How many are multiples of 5?

(vi) How many are multiples of 10?

52. Out of 4 officers and 10 clerks, a committee of 2 officers and 3 clerks is to be formed. In how many ways can committee be formed if:

(i) any officer and any clerk can be included.

(ii) a particular clerk must be in the committee,

(iii) a particular officer cannot be in the committee?

53. 12 persons are made to sit around a table. Find the number of ways they can sit such that 2 specific persons are not together.

54. A box contains 6 white and 6 black balls. Find the number of ways 4 balls can be drawn from the box if :

(i) Two must be white.

(ii) All of them must have the same colour.

55. Out of 6 makes and 6 females a committee of 5 is to be formed. Find the number of ways in which it can be formed so that among the person chosen in the committee there are

(i) Exactly 3 males and 2 females.

(ii) at least 2 males and one female.

56. Suppose license plate contains 3 English letters followed by 4 digits :

(i) How many different license plates can be manufactured if repetition of letetrs and digits are allowed ?

(ii) How many plates are possible if only the letters are repeated ?

(iii) How many plates are possible if only the digits are repeated.

57. In how many ways can 6 men and 5 women be seated in a line so that no two women sit together ?

58. What is the number of ways of choosing 4 cards from a pack of 52 playing cards ? In how many of these :

(i) Four cards are of the same suit.

(ii) Four cards belong to four different suits.

(iii) Cards are of the same colour.

59. An 8 member team is to be formed from a group of 10 men and 15 women. In how many ways can be team be chosen if :

(i) The team must contain 4 men and 4 women.

(ii) There must be more men than women.

(iii) There must be atleast two men.

5.3 BINOMIAL COEFFICIENTS AND COMBINATORIAL IDENTITIES

The numbers C (n, r) described in article 7.1.2, occur as coefficients in the binomial expansion $(a + b)^n$, and hence are called as binomial coefficients. The binomial coefficients surprisingly satisfy quite a large number of identities, which are well adapted to algebraic manipulation. Hence these identities find important applications in problems involving counting. (This branch of study is called as Combinatorics.)

In this section we will deal with some of the basic identities and their properties.

Let us first, take a review of the Binomial Theorem.

The Binomial Theorem gives the expansion of $(a + b)^n$ (n positive integer) as

$$(a+b)^n = C(n,0)\, a^n b^0 + C(n,1)\, a^{n-1} b^1 + C(n,2)\, a^{n-2} b^2 + \ldots + C(n, n-1)\, a^1 b^{n-1} + C(n,n)\, a^0 b^n$$

Although the Binomial Theorem is normally proved, using induction on n, a combinatorial approach will be more appropriate here.

For n = 3, consider

$$(a+b)^3 = (a+b)(a+b)(a+b)$$
$$= aaa + aab + aba + abb + baa + bab + bba + bbb$$

The first term aaa occurs, when we select a from each factor (a + b) and there is only one such term, i.e. a^3 in the final expansion. There are 3 terms viz. aab, aba, baa in which a appears twice and b only once. The same is the case with b twice and a only once. The last term bbb occurs only once, like aaa.

Putting the above observations differently, we note:

- There are two choices a or b for each term in the product and so there are $2 \times 2 \times 2 = 8$ formal products.
- In the 8 formal products, there are 3 in which a appears twice and same is the case for b.
- There is only a single product in which a (as well as b) has all the three choices.

Using the above combinational argument we conclude that

$$(a + b)^3 = a^3 + 3a^2 b + 3ab^2 + b^3$$

For the general case $(a + b)^n$, we can argue similarly. Each product in the expansion contains n factors. Suppose we want to find how many times the product $a^k b^{n-k}$ appears in the expansion. Since a must have k choices, these can be found in C (n, k) ways. The number of choices is exactly the same for b to have n – k choices. Hence it follows that

$$C(n, k) = C(n, n-k)$$

The binomial coefficient C (n, k) can also be interpreted differently using set theory, and this interpretation is found very often useful.

Let us suppose that we have a set of n people, from which we want to form a committee of k persons. This is equivalent to finding k-element subsets of a set containing n objects and this number is C (n, k).

Let us now consider some basic properties of the binomial coefficients, which are considered as **Combinational Identities.**

Theorem 1: $C(n + 1, k) = C(n, k-1) + C(n, k)$, for $1 \le k \le n$

Proof:

The theorem can easily be verified by algebraic method. However, a combinational proof will be more in place here.

Let A be a set containing n elements. Choose b / A, and form the set $B = A \cup \{b\}$.

Then C (n + 1, k) is the number of k-element subsets of B.

These can be divided into two disjoint classes.

Subsets of B not containing b.

Subsets of B containing b.

The subsets of class 1 are just k-element subsets of A, and those of class 2 consist of a k – 1 element subset of A together with b.

Number of subsets of the first type is C (n, k) and that of the second type is C (n, k – 1).

Hence $C(n + 1, k) = C(n, k-1) + C(n, k)$

Theorem 2: $C(n, k) \cdot C(k, m) = C(n, m) \cdot C(n-m, k-m)$; $1 \le m \le k \le n$

Proof:

L.H.S. gives the number of ways to select k-element subsets of a set of n elements and then to select m-element subsets from each of these k-subsets.

This can be equivalently substituted by selecting m-element subsets from the set of n elements and the remaining k – m element subsets from remaining n – m elements.

Theorem 3: $C(n, 0) + C(n, 1) + C(n, 2) + \ldots + C(n, n) = 2^n$

Proof:

Let A be a set containing n elements.

Then the number of subsets of A = 2^n = R.H.S. On the L.H.S., C (n, 0), C (n, 1) , ... , C (n, n) gives the count of all the subsets of A, containing 0 elements, 1 element, 2 elements, and so on.

Hence the total number of subsets by L.H.S. count is the same as the total number of subsets of A which is equal to 2^n.

Theorem 4: $C(k, k) + C(k + 1, k) + C(k + 2, k) + \ldots + C(n, k) = C(n + 1, k + 1)$.

Proof:

We know that

$$C(i + 1, k) = C(i, k-1) + C(i, k) \ ; \ \forall \ 1 \le k \le i \le n$$

Hence we can deduce from the above that

$$C(i, k) = C(i + 1, k + 1) - C(i, k + 1)$$

(replacing k by k + 1 in the above identity)

Now in L.H.S., C (k, k) = 1

$$C(k + 1, k) = C(k + 2, k + 1) - C(k + 1, k + 1)$$

(putting i = k + 1)

Similarly

$$C(k + 2, k) = C(k + 3, k + 1) - C(k + 2, k + 1)$$

and so on.

Hence, L.H.S. $= 1 + C(k + 2, k + 1) - C(k + 1, k + 1) + C(k + 3, k + 1) - C(k + 2, k + 1) + \ldots + C(n + 1, k + 1) - C(n, k + 1)$

$= C(n + 1, k + 1)$

as all other terms mutually cancel.

The following examples illustrate how the binomial identities can be applied to evaluate sum of a series, whose coefficients are closely related to binomial coefficients.

SOLVED EXAMPLES

Example 1: *Find the sum of 1 + 2 + ... + n*

Solution: $1 + 2 + \ldots + n = C(1, 1) + C(2, 1) + \ldots + C(n, 1)$

$= C(n + 1, 2)$ (by theorem)

$= \dfrac{(n+1)!}{2!\,(n-1)!} = \dfrac{n(n+1)}{2}$

Example 2: *Find the sum $1^2 + 2^2 + 3^2 + \ldots + n^2$*

Solution: The general term k^2, we can express as

$$k^2 = k(k-1) + k$$

By this strategy we can relate k (k – 1) to C (k, 2).

Hence the given sum can be rewritten as

$(1 \times 0 + 1) + [(2 \times 1) + 2] + [(3 \times 2) + 3] + \ldots + [n(n-1) + n]$

$= [2 \times 1 + 3 \times 2 + \ldots + n(n-1)] + (1 + 2 + 3 + \ldots + n)$

$= [2\,C(2, 2) + 2\,C(3, 2) + \ldots + 2\,C(n, 2)] + C(n + 1, 2)$

$= 2\,C(n + 1, 3) + C(n + 1, 2)$

$= 2 \cdot \dfrac{(n+1)!}{3!\,(n-2)!} + \dfrac{n(n+1)}{2}$

$= \dfrac{(n+1)\,n\,(n-1)}{3} + \dfrac{n(n+1)}{2}$

$= \dfrac{n(n+1)(2n+1)}{6}$

Example 3: *Evaluate the sum*

$C(n, 0) + 2\,C(n, 1) + 2^2\,C(n, 2) + \ldots + 2^k\,C(n, k) + \ldots + 2^n\,C(n, n)$

Solution: By binomial theorem,

$$(1 + x)^n = C(n, 0) + C(n, 1)\,x + C(n, 2)\,x^2 + \ldots + C(n, n)\,x^n$$

Hence putting x = 2,

R.H.S. $= C(n, 0) + 2\,C(n, 1) + \ldots + 2^n\,C(n, n)$

$= (1 + 2)^n = 3^n$

Example 4: *Evaluate the sum $1 \times 2 \times 3 + 2 \times 3 \times 4 + \ldots + (n-2)(n-1)\,n$*

Solution: The general term is $k(k-1)(k-2) = 3!\,C(k, 3)$

Hence the given sum can be rewritten as

$3!\,C(3, 3) + 3!\,C(4, 3) + \ldots + 3!\,C(n, 3)$

$= 3!\,[C(3, 3) + C(4, 3) + \ldots + C(n, 3)]$

$= 3!\,C(n + 1, 4) = \dfrac{3!\,(n+1)!}{4!\,(n-3)!}$

$= \dfrac{1}{4}(n + 1)\,n\,(n-1)(n-2)$

Example 5: *Prove that: $C(n, 0)^2 + C(n, 1)^2 + \ldots + C(n, n)^2 = C(2n, n)$*

Solution: We use the equality

$$(1 + x)^n + (1 + x)^n = (1 + x)^{2n} \qquad \ldots (1)$$

$$(1 + x)^n = \sum_{r=0}^{n} C(n, r)\,x^r$$

$$(1 + x)^{2n} = \sum_{k=0}^{2n} C(2n, k)\,x^k$$

Hence equation (1) can be expressed as

$$\left[\sum_{r=0}^{n} C(n, r)\,x^r\right]\left[\sum_{m=0}^{n} C(n, m)\,x^m\right] = \sum_{k=0}^{2n} C(2n, k)\,x^k$$

The coefficient of x^n should be equal on both the sides.

Hence,

$C(n, 0)\,C(n, n) + C(n, 1)\,C(n, n-1) + C(n, 2)\,C(n, n-2) + \ldots + C(n, n)\,C(n, 0) = C(2n, n)$

But C (n, 0) = C (n, n), C (n, 1) = C (n, n – 1) and so on.

Hence, $C(n, 0)^2 + C(n, 1)^2 + \ldots + C(n, n)^2 = C(2n, n)$

Example 6: *Show that $C(n, 1) + 2C(n, 2) + 3C(n, 3) + \ldots + nC(n, n) = n \cdot 2^{n-1}$*

Solution: We use the identity

$$C(n, n-k) = C(n, k)$$

Let, $S_n = C(n, 1) + 2C(n, 2) + 3C(n, 3) + \ldots + nC(n, n) \qquad \ldots (1)$

Then using the above identity, we see that

$S_n = C(n, n-1) + 2C(n, n-2) + 3C(n, n-3) + \ldots + nC(n, 0) \qquad \ldots (2)$