$|A \cup D| = |A| + |D| - |A \cap D|$ (by the previous theorem) ... (1)

$|D| = |B \cup C| = |B| + |C| - |B \cap C|$ ... (2)

$|A \cap D| = |A \cap (B \cup C)| = |(A \cap B) \cup (A \cap C)|$

$= |A \cap B| + |A \cap C| - |A \cap B \cap C|$ ... (3)

Substituting equations (2) and (3) in (1), we have

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

Thus, the principle is proved for three sets. We now have the general theorem for a finite collection of finite sets.

Theorem: Let $\{A_1, A_2, \ldots, A_n\}$ be a finite collection of sets.

Then $|A_1 \cup A_2 \cup \ldots \cup A_n|$

$$= \sum_{i=1}^{n} |A_i| - \sum_{1 \le i \le j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| + \ldots + (-1)^{n-1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

Proof:

Proof is by induction on n.

We have already proved the theorem for n = 2, 3.

Hence, let us assume the theorem for (n – 1) numbers of sets and prove it for n sets.

Regarding $A_1 \cup A_2 \cup \ldots \cup A_n$ as $(A_1 \cup A_2 \cup \ldots \cup A_{n-1}) \cup A_n$, we have

$|A_1 \cup A_2 \ldots \cup A_n| = |(A_1 \cup A_2 \cup \ldots \cup A_{n-1}) \cup A_n|$

$= |A_1 \cup A_2 \cup \ldots \cup A_{n-1}| + |A_n|$

$- |(A_1 \cup A_2 \cup \ldots \cup A_{n-1}) \cap A_n|$ ... (1)

By induction hypothesis,

$$|A_1 \cup A_2 \cup \ldots \cup A_{n-1}| = \sum_{i=1}^{n-1} |A_i| - \sum_{1 \le i < j \le n-1} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n-1} |A$$

Now $|(A_1 \cup A_2 \cup \ldots \cup A_{n-1}) \cap A_n|$

$= |(A_1 \cap A_n) \cup (A_2 \cap A_n) \cup \ldots \cup (A_{n-1} \cap A_n)|$

$$= \sum_{i=1}^{n-1} |A_i \cap A_n| - \sum_{1 \le i < j \le n-1} |A_i \cap A_j \cap A_n| + \sum_{1 \le i < j < k \le n-1} |A_i \cap A_j \cap A_k \cap A_n| - \ldots + (-1)^{n-2} |A_1 \cap A_2 \cap \ldots \cap A_{n-1} \cap A_n| \quad \ldots (3)$$

Substituting equations (2) and (3) in (1) we obtain the equation

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j < n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n-1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

SOLVED EXAMPLES

***Example 1:** In a computer laboratory out of 6 computers:*

(i) 2 have floating point arithmetic unit.

(ii) 5 have magnetic disk memory.

(iii) 3 have graphics display.

(iv) 2 have both floating point arithmetic unit and magnetic disk memory.

(v) 3 have both magnetic disk memory and graphic display.

(vi) 1 has both floating point arithmetic unit and graphics display.

(vii) 1 has floating point arithmetic, magnetic disk memory and graphics display.

How many have at least one specification?

Solution: Let A be the set of computers having floating point arithmetic unit, B having magnetic disk memory and C having graphics display.

Then $|A| = 2$, $|B| = 5$, $|C| = 3$,

$|A \cap B| = 2$, $|B \cap C| = 3$,

$|A \cap C| = 1$, $|A \cap B \cap C| = 1$

We have to determine $|A \cup B \cup C|$

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B|$

$- |B \cap C| - |A \cap C| + |A \cap B \cap C|$

$= 2 + 5 + 3 - 2 - 3 - 1 + 1 = 5.$

Hence, 5 computers out of 6, have at least one specification.

***Example 2:** Among the integers 1 to 1000:*

(i) How many of them are not divisible by 3, nor by 5, nor by 7?

(ii) How many are not divisible by 5 and 7 but divisible by 3?

Solution: (i) Let A, B, C denote respectively the set of integers from 1 to 1000 divisible by 3, by 5 and by 7. Then $\bar{A} \cap \bar{B} \cap \bar{C}$ denote the set of integers not divisible by 3, nor by 5, nor by 7.

By De Morgan's laws, $\bar{A} \cap \bar{B} \cap \bar{C} = \overline{(A \cup B \cup C)}$

Hence, $|\bar{A} \cap \bar{B} \cap \bar{C}| = 1000 - |A \cup B \cup C|$

$|A| = \left[\frac{1000}{3}\right] = 333$, $|B| = \left[\frac{1000}{5}\right] = 200$,

$|C| = \left[\frac{1000}{7}\right] = 142$,

$|A \cap B| = \left[\frac{1000}{15}\right] = 66$

$|B \cap C| = \left[\frac{1000}{35}\right] = 28$

$|A \cap C| = \left[\frac{1000}{21}\right] = 47$

$|A \cap B \cap C| = \left[\frac{1000}{105}\right] = 9.$

Hence, $|A \cup B \cup C| = |A| + |B| + |C|$

$= |A \cap B| - |B \cap C| - |A \cap C|$

$+ |A \cap B \cap C|$

$= 333 + 200 + 142 - 66 - 28 - 47 + 9$

$= 543.$

Hence $|\bar{A} \cap \bar{B} \cap \bar{C}| = 1000 - 543 = 457.$

(ii) Consider the Venn diagram

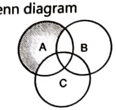


Fig. 2.41

The set of integers not divisible by 5 and 7 but divisible by 3 is the set $A \cap \bar{B} \cap \bar{C}$.

$A \cap \bar{B} \cap \bar{C} = A \cap \overline{(B \cup C)} = A - (B \cup C)$, the shaded portion shown in the Venn diagram.

It is clear from the diagram that

$|A - (B \cup C)| = |A| - |(A \cap B) \cup (A \cap C)|$

Now

$|(A \cap B) \cup (A \cap C)| = |A \cap B| + |A \cap C| - |A \cap B \cap C|$

$= 66 + 47 - 9 = 104$

$\therefore \ |A - (B \cup C)| = 333 - 104 = 229.$

Hence, 229 integers from 1 to 1000 are not divisible by 5 and 7 but divisible by 3.

***Example 3:** How many integers between 1 – 1000 are divisible by 2, 3, 5 or 7?*

Solution: Let A, B, C, D denote respectively the set of integers from 1 to 1000 divisible by 2, 3, 5 or 7.

$\therefore \ |A| = \left[\frac{1000}{2}\right] = 500$

$|B| = \left[\frac{1000}{3}\right] = 333$

$|C| = \left[\frac{1000}{5}\right] = 200$

$|D| = \left[\frac{1000}{7}\right] = 142$

$|A \cap B| = \left[\frac{1000}{6}\right] = 166$

$|A \cap C| = \left[\frac{1000}{10}\right] = 100$

$|A \cap D| = \left[\frac{1000}{14}\right] = 71$

$|B \cap C| = \left[\frac{1000}{15}\right] = 66$

$|B \cap D| = \left[\frac{1000}{21}\right] = 47$

$|C \cap D| = \left[\frac{1000}{35}\right] = 28$

$|A \cap B \cap C| = \left[\frac{1000}{30}\right] = 33$

$|B \cap C \cap D| = \left[\frac{1000}{105}\right] = 9$

$|A \cap C \cap D| = \left[\frac{1000}{70}\right] = 14$

$|A \cap B \cap D| = \left[\frac{1000}{42}\right] = 23$

$|A \cap B \cap C \cap D| = \left[\frac{1000}{210}\right] = 4$

$$\therefore |A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| - |A \cap B \cap C \cap D|$$

$$= 7723$$

Example 4: *An investigator interviewed 100 students to determine their preferences for the three drinks – Milk (M), Coffee (C) and Tea (T). He reported the following: 10 students had all the three drinks, 20 had 'M' and 'C', 30 had 'C' and 'T', 25 had 'M' and 'T', 12 had 'M' only, 5 had 'C' only and 8 had 'T' only.*

(i) How many did not take any of the three drinks?

(ii) How many take milk but not coffee?

(iii) How many take tea and coffee but not milk?

Solution: Consider the Venn diagram, incorporating the given data.

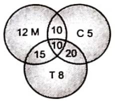


Fig. 2.42

(i) Taking the cardinalities of the disjoint sets into account,

$$|\bar{M} \cap \bar{C} \cap \bar{T}| = 100 - |M \cup C \cup T|$$

$$= 100 - [12 + 10 + 10 + 15 + 20 + 8 + 5]$$

$$= 100 - 80 = 20.$$

Hence, 20 students did not take any drink.

(ii) The set of students taking milk but not coffee is M – C.

$$|M - C| = 12 + 15 = 27$$

(iii) The set of students taking tea and coffee, but not milk is $(T \cap C) - M$.

$$|(T \cap C) - M| = |T \cap C| - |T \cap C \cap M|$$

$$= 30 - 10 = 20.$$

Example 5: *(i) Among 50 students in a class, 26 got an A in the first examination and 21 got an A in the second examination. If 17 students did not get an A in either examination, how many students got an A in both examinations?*

(ii) If the number of students who got an A in the first examination is equal to that in the second examination, if the total number of students who got an A in exactly one examination is 40 and if 4 students did not get an A in either examination, then determine the number of students who got an A in the first examination only, who got an A in the second examination only, and who got an A in both the examinations.

Solution: (i) Let F denote the set of students who got an A in the first examination, S that of students who got an A in the second examination.

The set of students who did not get an A in either is $\overline{F \cup S}$.

$$|\overline{F \cup S}| = 50 - |F \cup S| = 17$$

$$\therefore |F \cup S| = 50 - 17 = 33.$$

$$\therefore |F \cap S| = |F| + |S| - |F \cup S|$$

$$= 26 + 21 - 33 = 14.$$

(ii) $|F| = |S|$, by given condition.

The set of students who got an A in exactly one examination is $(F - S) \cup (S - F) = F \oplus S$.

$$|F \oplus S| = 40, \quad |\overline{F \cup S}| = 4 \text{ (given)}$$

$$\therefore |F \cup S| = 50 - 4 = 46$$

$$|F \oplus S| = |F \cup S| - |F \cap S|$$

$$\Rightarrow 40 = 46 - |F \cap S|$$

$$\therefore |F \cap S| = 46 - 40 = 6.$$

$$|F \cup S| = |F| + |S| - |F \cap S| = 2|F| - |F \cap S|$$

i.e. $46 = 2|F| - 6$

$$\therefore |F| = \frac{52}{2} = 26 = |S|.$$

The set of students who got an A in the first examination only is $F - S$.

$$\therefore |F - S| = |F| - |F \cap S| = 26 - 6 = 20.$$

Similarly, the number of students who got an A in the second examination only is

$$|S - F| = |S| - |S \cap F| = 20.$$

Example 6: *In a survey, it is reported that of 1000 programmers, 650 habitually flowchart their programs, 788 are skilled COBOL programmers, 675 are men, 278 of the women are skilled COBOL programmers, 440 programmers both habitually flowchart and are skilled in COBOL, 210 women habitually flowchart and 166 women are both skilled in COBOL and habitually flowchart. Would you accept these data as being accurately reported? Justify your answer.*

Solution: Let F denote the set of programs (both men and women) who habitually flow-chart their programs, and let C denote the set of all skilled COBOL programmers. Let M and W denote the set of men and women programmers respectively.

$$|M| = 675, \quad \therefore |W| = 1000 - 675 = 325.$$

$$|F| = 650, \quad |C| = 788$$

$$|W \cap C| = 278, \quad |W \cap F| = 210, \quad |F \cap C| = 440$$

$$|W \cap F \cap C| = 166.$$

$$\therefore |M \cap C| = |C| - |W \cap C| = 788 - 278 = 510.$$

$$|M \cap F| = |F| - |W \cap F| = 650 - 210 = 440.$$

$$|M \cap F \cap C| = |F \cap C| - |W \cap F \cap C|$$

$$= 440 - 166 = 274.$$

The set of $M \cap (F \cup C)$ is the set of male programmers who habitually flowchart their programs or are skilled COBOL programmers.

$$\therefore |M \cap (F \cup C)| = |M \cap F| + |M \cap C| - |M \cap F \cap C|$$

$$= 510 + 440 - 274$$

$$= 676.$$

Hence, there should be at least 676 men programmers. But this contradicts the given data that there are in all only 675 men programmers.

Hence, the data is inaccurately reported.

Example 7: *75 children went to an amusement park, where they can ride on the merry-go-round, roller coaster, and the Ferris wheel. It is known that 20 of them have taken all three rides, and 55 of them have taken at least 2. Each ride costs 5 rupees and the total collection of the park was 700 rupees. Determine the number of children who did not try any of the rides.*

Solution: The total number of rides $= \frac{700}{5} = 140.$

The number of children who have taken exactly 2 rides

$$= 55 - 20 = 35.$$

The number of children who have taken only one ride

$$= 140 - 2 \times 35 - 3 \times 20$$

$$= 140 - 70 - 60 = 10$$

Hence, the number of children who have not taken any ride

$$= 75 - (35 + 20 + 10) = 75 - 65 = 10.$$

Example 8: *It was found that in first year of computer science of 80 students 50 know Cobol, 55 know 'C', 46 know Pascal. It was also known that 37 know 'C' and Cobol, 28 know 'C' and Pascal, 25 know Pascal and Cobol. 7 students, however, know none of the languages.*

Find:

(i) How many know all the three languages?

(ii) How many know exactly two languages?

(iii) How many know exactly one language?

Solution: Let B, C and P denote the set of students who know Cobol, 'C' and Pascal respectively.

Then $|B \cup C \cup P| = 80 - 7 = 73$

is the number of students who know at least one of the languages.

(i) $|B \cup C \cup P| = |B| + |C| + |P| - |B \cap C| - |B \cap P| - |C \cap P| + |B \cap C \cap P|$

Hence, the number of students who know all the three languages is

$$|B \cap C \cap P| = 73 - 50 - 55 - 46 + 37 + 28 + 25$$

$$= 12$$

(ii) The number of students who know Cobol and 'C' but not Pascal is

$$|B \cap C \cap \bar{P}| = |B \cap C| - |B \cap C \cap P|$$

$$= 37 - 12 = 25$$

Similarly, the number of students who know Cobol and Pascal but not 'C' is

$$|B \cap P \cap \bar{C}| = 25 - 12 = 13$$

and the number of students who know Pascal and 'C' but not Cobol is

$$|\bar{B} \cap P \cap C| = 28 - 12 = 16$$

Hence, the number of students who known exactly two languages is

$$25 + 13 + 16 = 54$$

(iii) The number of students who know only Cobol (i.e. neither 'C' nor Pascal) is

$$|B| - |B \cap C| - |B \cap P| + |B \cap P \cap C| = 50 - 37 - 25 + 12 = 0$$

Similarly the number of students who know only 'C' is 55 – 37 – 28 + 12 = 2 and the number of students knowing only Pascal is 46 – 28 – 25 + 12 = 5.

Hence, the number of students who know exactly one language is 0 + 2 + 5 = 7

Example 9: *How many elements are in the union of five sets if the sets contain 10,000 elements each, each pair of sets has 1000 common elements, each triple of sets has 100 common elements, every four of the sets has 10 common elements, and there is 1 element common in all five sets?*

Solution: Let A_1, A_2, A_3, A_4, A_5 denote the five sets.

Then $|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5|$

$$= \sum_{1 \le i \le 5} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k|$$

$$- \sum_{i<j<k<l} |A_i \cap A_j \cap A_k \cap A_l|$$

$$+ |A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5|$$

Taking two sets at a time there are $5C_2 = 10$ such sets; taking three sets at a time, there are $5C_3 = 10$ such sets. Taking 4 sets at a time, there are $5C_4 = 5$ such sets.

$\therefore$ $|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5| = 5 \times 10{,}000 - 10 \times 1000 + 10 \times 100 - 5 \times 10 + 1 = 40{,}951$

Example 10: *Find the number of positive integers not exceeding 100 that are either odd or the square of an integer.*

Solution: Let A be the set of odd integers between 1 and 100, B the set of integers between 1 and 100, that are squares of an integer.

$$B = \{1, 4, 9, 16, 25, 36, 49, 64, 81, 100\}$$

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$= 50 + 10 - 5 = 55$$

Example 11: *A college record gives the following information: 119 students enrolled in Introductory Computer Science; of these 96 took Data Structures, 53 took Foundations, 39 took Assembly Language, 31 took both Foundations and Assembly Language, 32 took both Data Structures and Assembly Language, 38 took Data Structures and Foundations and 22 took all the three courses.*

Is the information correct? Why?

Solution: Let D, F and A denote the set of students who took Data Structure, Foundations and Assembly Language respectively.

Given: $|D| = 96$, $|F| = 53$, $|A| = 39$, $|F \cap A| = 31$, $|D \cap A| = 32$, $|D \cap F| = 38$ and $|F \cap D \cap A| = 22$.

$\therefore$ $|F \cup D \cup A| = |F| + |D| + |A| - |F \cap A| - |D \cap A| - |D \cap F| + |F \cap D \cap A|$

$$= 53 + 96 + 39 - 31 - 32 - 38 + 22$$

$$= 109 \text{ which is less than } 119.$$

Since, there were 119 students enrolled for the course assuming that all these students had taken at least one course, the given information is not correct.

Example 12: *A software company writes a new package which integrates a word processing program with a spread sheet program and they wish to run on a 64 K machine. The word processor requires 40 K for program and data and the spread sheet requires 32 K for the same. If 16 K must be reserved for the code integrator, what is the minimum amount of overlapping space that will be necessary?*

Solution: Let A denote the memory space reserved for word processor and B that for spread sheet.

$|A| = 40$, $|B| = 32$.

Available memory is $64 - 16 = 48$

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$\because$ $|A \cup B| \le 48$

$\therefore$ $|A| + |B| - |A \cap B| \le 48$

i.e. $|A \cap B| \ge |A| + |B| - 48$

$$= 40 + 32 - 48 = 24$$

Hence, the minimum amount of overlapping space that will be necessary is 24 K.

Example 13: *Among 130 students, 60 study Mathematics, 51 study Physics and 30 study both Mathematics and Physics. Out of 54 students studying Chemistry, 26 study Mathematics, 21 study Physics and 12 study both Mathematics and Physics. All the students studying neither Mathematics nor Physics are studying Biology.*

Find:

1. *How many are studying Biology?*
2. *How many not studying Chemistry are studying Mathematics but not Physics?*
3. *How many students are studying neither Mathematics nor Physics nor Chemistry?*

Solution:

1. $|M \cup P| = |M| + |P| - |M \cap P|$

$$= 60 + 51 - 30 = 81$$

$\therefore$ Number of students studying neither Mathematics nor Physics

$$= 130 - |M \cup P| = 130 - 81 = 49$$

Hence, the number of students studying Biology is 49.

2. The set of students studying Mathematics but neither Chemistry nor Physics is $M - M \cap (C \cup P)$.

$\therefore$ $|M - [M \cap (C \cup P)]| = |M| - |M \cap C| - |M \cap P| + |M \cap C \cap P|$

$$= 60 - 26 - 30 + 12 = 16$$

3. Set of students studying neither Mathematics nor Physics nor Chemistry is the complement of the set $M \cup P \cup C$, i.e. $\overline{M \cup P \cup C}$.

$\therefore$ $|\overline{M \cup P \cup C}| = 130 - |M \cup P \cup C|$

$$= 130 - |M| - |P| - |C| + |M \cap P| + |M \cap C| + |P \cap C| - |M \cap P \cap C|$$

$$= 130 - 60 - 51 - 54 + 30 + 26 + 21 - 12$$

$$= 30$$

Example 14: *During a survey of the ice cream preferences of students, it was found that 22 like mango, 25 like custard apple, 39 like grape, 9 like custard apple and mango, 17 like mango and grape, 20 like custard apple and grape, 6 like all flavours and 4 like none. Then how many students were surveyed ? How many students like exactly one flavour ? How many students like exactly two flavours ?*

Solution : Let M denote the set of students who like mango ice cream, C the students who like custard apple and G, who like grape ice cream. We have the following Venn diagram.

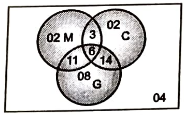


Fig. 2.43

$$|M \cup C \cup G| = |M| + |C| + |G| - |M \cap C| - |C \cap G| - |M \cap G| + |M \cap C \cap G|$$

$$= 22 + 25 + 39 - 9 - 20 - 17 + 6$$

$$= 46$$

Hence total number of students under went the survey is 46 + 04 = 50. Number of students who like exactly one flavour is 02 + 02 + 08 = 12.

The number of students who liked exactly two flavours is 11 + 3 + 14 = 28.

Example 15: *Consider a set of integers 1 to 500. Find how many of these are divisible by 3 or 5 or 11.* **(Nov./Dec. 14)**

Solution : Let A, B, C denote respectively the set of integers from 1 to 500, divisible by 3 or 5 or 11.

$$|A| = \left[\frac{500}{3}\right] = 166, \quad |B| = \left[\frac{500}{5}\right] = 100,$$

$$|C| = \left[\frac{500}{11}\right] = 45$$

$$|A \cap B| = \left[\frac{500}{15}\right] = 33$$

$$|A \cap C| = \left[\frac{500}{33}\right] = 15$$

$$|B \cap C| = 9$$

$$|A \cap B \cap C| = 3$$

The set of integers that are divisible by 3 or 5 or 11 is denoted as $A \cup B \cup C$.

Hence, $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

$$= 166 + 100 + 45 - 33 - 9 - 15 + 3$$

$$= 257$$

2.8.7 Infinite Sets

We have seen that if a set is finite; its elements can be counted or listed and this counting ceases in finite time. On the other hand if the counting is interminable or impossible, then such a set is said to be infinite. Familiar examples of infinite sets are:

(i) N = {1, 2, 3, ...} the set of natural numbers.

(ii) The set of prime positive integers {2, 3, 5, 7, ...}.

(iii) The set of all points in the first quadrant of the plane, whose x and y co-ordinates are integers.

(iv) The set of all binary strings of odd length.

The above examples are of infinite sets, whose elements are although infinitely many in number can be listed or 'counted', in other words these elements are put into one-to-one correspondence with the set of natural numbers. The cardinality of such a set is denoted by N_0 (pronounced as aleph nought).

If a set is not countable, then it is called "uncountable" set. The set of real numbers, denoted by IR, is uncountable. The open interval (0, 1), as a set is uncountable. Another important example of an uncountably infinite set is the power set of N, the set of natural numbers. The cardinality of this set is 2^{N_0} denoted by C and is called the 'continuum'.

We shall discuss countably infinite and uncountably infinite sets, more in detail, in the chapter on functions.

2.9 MULTISET

- Multiset is generalization of a set. A set, we know, is a collection of distinct objects. In a multiset however, an object can occur more than once. For example, the collection of books, in a library can contain multiple copies of the same book; such a collection is a multiset. Similarly names of persons, birth months of individuals, account numbers of transactions of a bank on a given day (the same account number may have more than one transaction), are practical examples of multisets. A multiset is also called as "bag", "heap", "bunch", "weighted set". In our discussion, we will use the term "multiset" (first coined by N.G. de Bruijn) or briefly "mset".
- To distinguish a set and a multiset, we denote the latter by enclosing the elements within square brackets. For example, [a, b, a, a] is an mset, whereas the underlying or "generic" set is {a, b}. A multiset containing no elements is denoted by [ ], corresponding to the empty set ϕ.
- The multiplicity of an element in an mset is defined as the number of times the element appears in the mset. Thus, in the mset [a, b, a, a] multiplicity of a is 3 whereas multiplicity of b is 1. If an element dsoes not belong to the multiset, its multiplicity is zero.
- Hence, it follows that sets are special cases of msets, in which multiplicity of the elements is either zero or one. In fact we can characterize a multiset as a pair (A, μ), where A is the generic set and μ is the multiplicity function defined as

  $$\mu \ : \ A \rightarrow \{1, 2, 3, \ldots\}$$

  so that $\mu(a) = k$ where k is the number of times the element a occurs in the mset.

  For example, if [a, b, c, c, a, c] is the mset, $\mu(a) = 2$, $\mu(b) = 1$, $\mu(c) = 3$.

2.9.1 Equality of Msets

If the number of occurrences of each element is the same in both the msets, then the msets are equal.

Example, [a, b, a, a] = [a, a, b, a]

However,[a, b, a] $\neq$ [a, b]

Multisubset (or msubset): A multiset A is said to be a multisubset of B if multiplicity of each element in A is less or equal to its multiplicity in B.

Example: [1, 2, 2, 3] $\subseteq$ [1, 1, 1, 2, 2, 3]

2.9.2 Union of Msets

If A and B are two msets, then $A \cup B$ is the mset such that for each element $x \in A \cup B$,

$$\mu(x) = \max. \{\mu_A(x), \ \mu_B(x)\}$$

Example: A = [a, b, b, c], B = [b, c, c, d]

Then $A \cup B$ = [a, b, b, c, c, d]

2.9.3 Intersection of Msets

If A and B are msets, then $A \cap B$ is defined as the mset such that for each element $x \in A \cap B$, $\mu(x) = \min(\mu_A(x), \mu_B(x))$.

Example: A = [1, 1, 1, 2, 2, 3]

B = [1, 2, 2, 2, 3, 3]

Then $A \cap B$ = [1, 2, 2, 3]

2.9.4 Difference of Msets

For multisets A and B, the difference A – B is an mset such that for each $x \in A - B$,

$\mu(x) = \mu_A(x) - \mu_B(x)$, if the difference is greater than zero.

$\mu(x) = 0$ if difference is zero or negative.

From the above definition it follows that

$A - A = \phi$ [ ]

Example: A = [a, b, c, c, c]

B = [b, c, d, d]

Then A – B = [a, c, c]

Sum of Msets:

This concept is not defined for ordinary sets. However, for multisets A and B, we define A + B as follows:

For each element $x \in A + B$,

$$\mu(x) = \mu_A(x) + \mu_B(x)$$

Example: A = [1, 1, 2, 3]

B = [2, 3, 3, 3]

A + A = [1, 1, 1, 1, 2, 2, 3, 3]

A + B = [1, 1, 2, 2, 3, 3, 3, 3]

An interesting observation = $A + A \neq A$.

(i.e. idempotent law is not true for sum).

The above definitions (excluding sum) are consistent with those defined for sets. Hence, one can easily see that the laws of associativity, commutativity, distributivity adsorption and idempotent are satisfied for union and intersections. It is also an easy exercise to verify that

$(A + B) \cup C = A \cup C + B \cup C$

$(A + B) \cap C = A \cap C + B \cap C$

$A \cup (B + C) = A \cup B + A \cup C$

$A \cap (B + C) = A \cap B + A \cap C$

The concept of symmetric difference, however, cannot be carried over to that of multisets. Recall that we define symmetric difference of two sets as:

$$A \oplus B = (A \cup B) - (A \cap B)$$

Symmetric difference satisfies the associative law for sets. For multisets, this law is not valid. Consider, for example,

A = [2, 2, 3, 3]

B = [1, 1, 2]

C = [3, 3, 2]

Then, if $A \oplus B = A \cup B - A \cap B$, this is equal to

[1, 1, 2, 2, 3, 3] – [2].

= [1, 1, 2, 3, 3]

$\therefore$ $(A \oplus B) \oplus C$ = [1, 1, 2, 3, 3] $\cup$ [3, 3, 2] – [1, 1, 2, 3, 3] $\cap$ [3, 3, 2]

= [1, 1, 2, 3, 3] – [2, 3, 3]

= [1, 1]

On the other hand consider,

$A \oplus (B \oplus C)$ $B \oplus C$ = [1, 1, 2] $\cup$ [3, 3, 2] – [1, 1, 2] $\cap$ [3, 3, 2]

= [1, 1, 2, 3, 3] – [2]

= [1, 1, 3, 3]

$\therefore$ $A \oplus (B \oplus C)$ = [2, 2, 3, 3] $\cup$ [1, 1, 3, 3] – [2, 2, 3, 3] $\cap$ [1, 1, 3, 3]

= [1, 1, 2, 2, 3, 3] – [3, 3]

= [1, 1, 2, 2]

Thus, we see that $(A \oplus B) \oplus C \neq A \oplus (B + C)$.

Cardinality of a multiset containing finitely many elements is defined as the total number of elements in the multiset, i.e. it is the sum of multiplicities of each element in the set. For example,

If A = [a, a, b, b, c], |A| = 2 + 2 + 1 = 5

Then extending the principle of mutual inclusion – exclusion to multisets, it follows that

$$|A \cup B| = |A| + |B| - |A \cap B|$$

For example, A = [1, 1, 1, 1], B = [1, 2, 2, 3]

then $|A \cup B|$ = 4 + 4 – 1 = 7

This is immediate because

$A \cup B$ = [1, 1, 1, 1, 2, 2, 3] and $A \cap B$ = [1]

2.9.5 Polynomial Representation of Multisets

A multiset can be represented by a monomial. The empty multiset [ ] corresponds to $x^0 = 1$, [x] corresponds to x^1, [x, x] corresponds to x^2 and [x, y] to xy. In fact the multiset of submultisets (equivalent to power set) can be represented by a suitable polynomial expression. For example, [[ ], [x], [x], [x, x]] correspond to the polynomial $1 + 2x + x^2$. Similarly $(1 + x)(1 + y) = 1 + x + y + xy$ correspond, s to [[ ], [x], [y], [x, y]]. In general the polynomial $(1 + x)^n = \sum_{0}^{n} {}^nC_k x^{n-k}$ represents an mset consisting of ${}^nC_k$ subsets of cardinality, k varying from 0 to n.

An interesting observation is that $(1 - x)^{-1} = 1 + x + x^2 \ldots$ corresponds to the infinite mset [[ ], [x], [x, x], [x, x, x], …]. Similarly $(1 - x)^{-2} = 1 + 2x + 3x^2 \ldots$ corresponds to the mset [[ ], [x], [x], [x, x], [x, x], [x, x] …].

Such polynomials are called as **Cumulant generating functions**.

Given a set of n (distinct) elements, one can determine how many multisets, consisting of k elements, can be formed, where $0 \leq k \leq n$. This is nothing but the combinatorial problem of distributing k identical objects in n distinct boxes, whose solution is given by $\binom{n+k-1}{k-1}$. (Refer to the chapter on combinatories). We denote this number by the symbol $\left(\!\binom{n}{k}\!\right)$, which is called as **n multichoose k**. For example, if A = {a, b, c} is the generic set, all multisets of cardinality 2 are given by [a, b], [a, c], [b, c], [a, a], [b, b], [c, c]. These total 6 msets are given by the number $\left(\!\binom{3}{2}\!\right) = \binom{3+2-1}{3-1} = \binom{4}{2} = 6$.

2.9.6 Application of Multisets

- In mathematics, the prime factorization of every non-negative integer is n > 0, corresponds to a multset. Hence, there is a one-to-one correspondence between the prime factors of the integer and the corresponding multiset. For example, if $n = 2^3 \cdot 3^2 \quad 5$, the corresponding multiset is [2, 2, 2, 3, 3, 5].
- The roots of an algebraic equation also form a multiset. For example the roots of the equation $x^3 - 4x^2 + 5x - 2 = 0$, given rise to the multiset [1, 1, 2]. Hence, multisets are useful in representing the zeros and poles of meromorphic (analytic) functions, invariants of a matrix (eigenvalues) in canonical forms.

- In Computer Science, multisets are applied in a variety of search and sort procedures, for fast retrieval of keys (of the same key value with multiple copies), which allows fast access to stored key values.

SOLVED EXAMPLES

Example 1: *Find the union and intersection of each of the following multisets:*

(a) [a, b] and [a, b, c]

(b) [a, b, b] and [a, b, a, b]

(c) [a, a, a, b] and [a, a, b, b, c]

(d) [1, 1, 3, 3, 3, 4] and [1, 2, 2, 4, 5, 5]

(e) [a, a, (b, b), (b, b)] and [a, a, b, b]

(f) [a, a, (b, b), [a, (b)]] and [a, a, (b), (b)]

Solution:

(a) [a, b] ∪ [a, b, c] = [a, b, c]

[a, b] ∩ [a, b, c] = [a, b]

(b) [a, b, b] ∪ [a, b, a, b] = [a, a, b, b]

[a, b, b] ∩ [a, b, a, b] = [a, b, b]

(c) [a, a, a, b] ∪ [a, a, b, b, c] = [a, a, a, b, b, c]

[a, a, a, b] ∩ [a, a, b, b, c] = [a, a, b]

(d) [1, 1, 3, 3, 3, 4] ∪ [1, 2, 2, 4, 5, 5] =

[1, 1, 2, 2, 3, 3, 3, 4, 5, 5]

[1, 1, 3, 3, 3, 4] ∩ [1, 2, 2, 4, 5, 5] = [1, 4]

(e) [a, a, (b, b), (b, b)] ∪ [a, a, b, b] =

[a, a, (b, b), (b, b), b, b]

[a, a, (b, b), (b, b)] ∩ [a, a, b, b] = [a, a]

(f) [a, a, (b, b), [a, (b)]] ∪ [a, a, (b), (b)] = [a, a, (b, b), [a, (b)], (b), (b)]

[a, a, (b, b), [a, (b)]] ∩ [a, a, (b), (b)] = [a, a]

Example 2: *Find a multiset that solves the equation*

A ∪ [a, b, b, c] = [a, a, b, b, c, c, d]

A ∩ [a, b, b, c, d] = [a, b, c, d]

Solution: Maximum multiplicity of each element is as follows

$\mu(a) = 2, \; \mu(b) = 2, \; \mu(c) = 2, \; \mu(d) = 1.$

Minimum multiplicity of each element is as follows:

$\mu(a) = 1, \; \mu(b) = 1, \; \mu(c) = 1, \; \mu(d) = 1.$

∴ A = [a, a, b, c, c, d]

2.10 MATHEMATICAL INDUCTION

- Mathematical Induction is a powerful technique in Mathematics; especially in Number Theory, where many properties of natural numbers are established b this method.
- In Mathematics, we are often required to generalise particular solution. In order to do this, we look for pattern in the particular solution. Mathematic induction generalises this pattern of solutions b proving that it is always possible to extend the solutio to a group that is one larger than the previous. Th generalisation is achieved by using a statemen involving a variable natural number.
- The logic underlying the principle of mathematica induction, makes it an extremely suitable method t solve problems related to real life. A few examples o this type will be discussed in this section.
- To the software engineer, mathematical induction is a important tool in algorithm verification, to chec whether a program statement is loop invariant, that i whether it is true before and after every pass through a programming loop.

2.10.1 Statement of the Principle of Mathematical Induction

Let P(n) be a statement involving a natural number n.

1. If P(n) is true for $n = n_0$, and
2. Assuming P(k) is true, $(k \geq n_0)$ we prove P(k + 1) is also true,then P(n) is true for all natural numbers n $\geq n_0$.

Step (1) is called as the **basis of induction**. Step (2) is called as the **induction step.** The assumption that P(n) is true for n = k is called as the **induction hypothesis.**

SOLVED EXAMPLES

Example 1 : *Prove that*

$$\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \text{.........} + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$$

Solution: Let P(n) be the statement:

$$\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \text{......} + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}.$$

For $n = 1$, $P(1) : \frac{1}{1.4} = \frac{1}{4}$

Hence, P(1) is true.

Assume P(k) is true, and prove P(k + 1) is also true.

$$P(k+1) : \frac{1}{1.4} + \frac{1}{4.7} + \ldots + \frac{1}{(3k-2)(3k+1)}$$

$$+ \frac{1}{(3(k+1)-2)(3(k+1)+1)}$$

$$= \frac{k}{3k+1} + \frac{1}{(3k+1)(3k+4)}$$

$$= \frac{k(3k+4)+1}{(3k+1)(3k+4)} = \frac{3k^2+4k+1}{(3k+1)(3k+4)}$$

$$= \frac{(3k+1)(k+1)}{(3k+1)(3k+4)}$$

$$= \frac{k+1}{3k+4} = \frac{k+1}{3(k+1)+1}$$

Hence, assuming P(k) is true, P(k + 1) is also true. Therefore, P(n) is true for all $n \geq 1$.

Example 2: *Prove that $5^n - 1$ is divisible by 4 for $n \geq 1$.*

Solution: (i) Basis of induction: For n = 1, $5^1 - 1 = 4$, divisible by 4.

(ii) Induction step: Assume that $5^k - 1$ is divisible by 4.

We have $5^{k+1} - 1 = (5^k \cdot 5 - 5) + 4$

$= 5(5^k - 1) + 4.$

By induction hypothesis, $5^k - 1$ is divisible by 4.

∴ Each term on the RHS is divisible by 4.

∴ $5^{k+1} - 1$ is divisible by 4.

Hence, $5^n - 1$ is divisible by 4 for $n \geq 1$.

Example 3: *Show that $n^3 + 2n$ is divisible by 3 for all $n \geq 1$.*

Solution: (i) Basis of induction:

$n^3 + 2n = 1 + 2 = 3$ divisible by 3.

(ii) Induction step: Assume that for n = k, $k^3 + 2k$ is divisible by 3.

For n = k + 1,

$(k+1)^3 + 2(k+1) = k^3 + 3k^2 + 3k + 1 + 2k + 2 = (k^3 + 2k) + 3(k^2 + k + 1)$

Since, each term in the above is divisible by 3, it follows that the result is true for n = k + 1. Hence, $n^3 + 2n$ is divisible by 3 for all $n \geq 1$.

Example 4: *Show that for any positive integer n > 1,*

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} > \sqrt{n}.$$

Solution: (i) Basis of induction:

For n = 2,

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = 1 + 0.7071 = 1.7071$$

$$\sqrt{2} = 1.4142$$

$$\therefore \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} > \sqrt{2}.$$

(ii) Induction step: Assume that for n = k,

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k}} > \sqrt{k}.$$

Now for n = k + 1

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k} + \frac{1}{\sqrt{k+1}}$$

To show that $\sqrt{k} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$

We have to show $\sqrt{k} > \sqrt{k+1} - \frac{1}{\sqrt{k+1}}$

i.e. $\sqrt{k}\,\sqrt{k+1} > k + 1 - 1 = k$

i.e. $k(k+1) > k^2$ which is true for $k \geq 1$.

Hence, the result is true for all $n \geq 1$.

Example 5: *Show that $2^n > n^3$ for $n \geq 10$.*

Solution: (i) Basis of induction: For n = 10, $2^{10} = 1024 > 10^3$

(ii) Induction step: Assume that $2^k > k^3$.

Then $2^{k+1} = 2^k \cdot 2 = 2^k(1+1)$

$$> 2^k\left(1 + \frac{1}{10}\right)^3 \geq 2^k\left(1 + \frac{1}{k}\right)^3 \qquad \text{(Note this step)}$$

$$> k^3\left(1 + \frac{1}{k}\right)^3 = k^3\,\frac{(k+1)3}{k3} = (k+1)^3.$$

Therefore, $2^n > n^3$ for $n \geq 10$.

Example 6: *Show that for any positive integer n,*

$(11)^{n+2} + (12)^{2n+1}$ is divisible by 133.

Solution: (i) Basis of induction: For n = 0, $(11)^{n+2} + (12)^{2n+1} = 121 + 12 = 133$, which is obviously divisible by 133.

For n = 1, $(11)^{n+2} + (12)^{2n+1} = 3059$ divisible by 133, since $133 \times 23 = 3059$.

(ii) Induction step: Assume the result for n = k.

Consider $(11)^{k+1+2} + (12)^{2(k+1)+1}$

$= 11 \cdot (11)^{k+2} + (12)^{2k+1} \cdot 144$

$= 11 \cdot (11)^{k+2} + (133 + 11) \cdot (12)^{2k+1}$

$= 11 \cdot (11)^{k+2} + (12)^{2k+1} + 133 \cdot (12)^{2k+1}$

Since, both these terms are divisible by 133, the result follows.

Example 7: *Formulate and prove by induction, a general formula stemming from the observations*

$$1^3 = 1$$
$$2^3 = 3 + 5$$
$$3^3 = 7 + 9 + 11$$
$$4^3 = 13 + 15 + 17 + 19.$$

Solution: Note that the first term on the R.H.S., starting with the second equality can be written as

$$3 = 2.1 + 1$$
$$7 = 3.2 + 1$$
$$13 = 4.3 + 1$$

Hence, the first term for the n^{th} equality can be written as

$$n(n-1) + 1.$$

Also note that in each equation, the number of terms of R.H.S. is equal to the value of n on L.H.S. Hence, the nth equation can be written as

$$n^3 = [n(n-1)+1] + [n(n-1)+3] + \ldots + [n(n-1)+(2n-1)]$$

Hence, $$n^3 = \sum_{i=1}^{n} [n(n-1) + 2i - 1]$$

Let us verify the formula.

For $n = 1$, $1^3 = 1(1-1) + 2 - 1 = 1$

For $n = 2$, $2^3 = [2(2-1)+1] + [2(2-1)+3]$
$= 3 + 5$

For $n = 3$, $3^3 = [3(3-1)+1] + [3(3-1)+3] + [3(3-1)+5]$
$= 7 + 9 + 11$

Now assume the result for $n = k$ and prove it for $n = k + 1$

$$\therefore \quad k^3 = \sum_{i=1}^{k} [k(k-1) + 2i - 1] \qquad \ldots (1)$$

Now $(k+1)^3 = k^3 + 3k^2 + 3k + 1$

Consider for $n = k + 1$, the term on R.H.S., i.e.

$$\sum_{i=1}^{k+1} [(k+1)k + 2i - 1] = \sum_{i=1}^{k} [(k+1)k + 2i - 1] + [(k+1)k + 2(k+1) - 1]$$

$$= \sum_{i=1}^{k} [k(k-1) + 2i - 1] + \sum_{i=1}^{k} 2k + [(k+1)k + 2k + 1]$$

(Note this step)

$$= k^3 + 2k^2 + k^2 + 3k + 1 \qquad \text{by (1)}$$
$$= k^3 + 3k^2 + 3k + 1 = (k+1)^3.$$

Example 8 : *Show that any integer composed of 3^n identical digits is divisible by 3^n.*

Solution: (i) Basis of induction: For $n = 1$, the result is true, since an integer is divisible by 3 if the sum of the digits is divisible by 3. For example 111, 444, 555 etc. are divisible by 3.

(ii) Induction step: Let x be an integer composed of 3^{k+1} identical digits. Then we may express x as $x = yz$, where y is an integer composed of 3^k identical digits and

$$z = 10^{2\cdot 3^k} + 10^{3^k} + 1$$

For example,

$777777777 = 777\,(10^6 + 10^3 + 1)$, putting $k = 1$.

In this case $y = 777$ and $z = 1001001$.

Now both y and z are divisible by 3 so that 777777777 is divisible by $3^2 = 9$.

In the general case y is divisible by 3^k, by induction hypothesis.

and $z = 10000 \ldots 01000 \ldots 01$

$3^k - 1$ 0's $\quad 3^k - 1$ 0's

Clearly z is divisible by 3. Hence, $x = yz$ is divisible by $3^k \cdot 3 = 3^{k+1}$.

Thus, the result is proved for any value of n.

Example 9: *Prove by induction that the sum of the cubes of three consecutive integers is divisible by 9.*

Solution: We have to show that $(n-1)^3 + n^3 + (n+1)^3$ is divisible by 9.

(i) Basis of induction: For $n = 1$, we have $0 + 1 + 2^3 = 9$, divisible by 9.

(ii) Induction step: Assume the result for $n = k$

$\therefore$ $(k-1)^3 + k^3 + (k+1)^3 = 3k^3 + 6k$ is divisible by 9.

For $n = k + 1$, we have

$$k^3 + (k+1)^3 + (k+2)^3 = 3k^3 + 9k^2 + 15k + 9$$
$$= 3k^3 + 6k + 9(k^2 + k + 1)$$

which is divisible by 9.

Hence, the result is proved.

Example 10: Show that $1 + 2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 1$.

Solution: (i) Basis of induction: For $n = 1$, we have

$$1 + 2 = 3 = 2^2 - 1$$

(ii) Induction step: Assume the result for $n = k$.

For $n = k + 1$, we have

$1 + 2 + 2^2 + \ldots + 2^k + 2^{k+1} = (2^{k+1} - 1) + 2^{k+1} = 2.2^{k+1} - 1$
$= 2^{k+2} - 1$

Hence, the result is proved.

Example 11: *Prove that $8^n - 3^n$ is a multiple of 5 by mathematical induction for $n \geq 1$.*

Solution: (i) Basis of induction: For $n = 1$, we have $8 - 3 = 5$, obviously a multiple of 5.

(ii) Induction step: Assume the result for $n = k$, i.e. $8^k - 3^k = 5$ m for some integer m.

For $n = k + 1$, we have

$$8^{k+1} - 3^{k+1} = 8^k \cdot 8 - 3^k \cdot 3$$
$$= 8^k(5 + 3) - 3^k \cdot 3 = 5.8^k + 3(8^k - 3^k)$$
$$= 5.8^k + 3.5\text{ m, clearly a multiple of 5.}$$

Hence, the result is proved.

Example 12: *Prove that $n^3 - n$ is divisible by 3, for a positive integer n.*

Solution: (i) Basis of induction: For $n = 1$, $1^3 - 1 = 0$ is divisible by 3.

(ii) Induction step: Assume for $n = k$, $n^3 - n$ is divisible by 3.

For $n = k + 1$, we have

$$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1$$
$$= (k^3 - k) + 3(k^2 + k)$$

Since, $k^3 - k$ is divisible by 3, by induction hypothesis, the result follows.

Example 13: *Prove that $1.1! + 2.2! + 3.3! + \ldots n.n! = (n+1)! - 1$, where n is a positive integer.*

Solution: (i) Basis of induction: For $n = 1$, $1.1! = 1$ and $(1+1)! - 1 = 2! - 1 = 1$.

(ii) Induction step: Assume for $n = k$, $1.1! + 2.2! + \ldots + k.k! = (k+1)! - 1$.

For $n = k + 1$, $1.1! + 2.2! + \ldots + k\,k! + (k+1)\cdot(k+1)!$

$$= (k+1)! - 1 + (k+1)\cdot(k+1)!$$

(by induction hypothesis)

$$= (k+1)! + (k+1)\cdot(k+1)! - 1$$
$$= (k+1)!\,(1 + k + 1) - 1$$
$$= (k+1)!\,(k+2) - 1$$
$$= (k+2)! - 1.$$

Hence, the result is proved.

Example 14: *For a positive integer $n > 1$, prove that $1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2} < 2 - \frac{1}{n}$.*

Solution: (i) Basis of induction:

For $n = 2$, $1 + \frac{1}{2^2} = 1 + \frac{1}{4} < 2 - \frac{1}{2}$ as $1 + \frac{1}{4} = \frac{5}{4} < \frac{3}{2}$.

(ii) Assume for n = k, i.e.

$$1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{k^2} < 2 - \frac{1}{k}.$$

For $n = k + 1$,

$$1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$$

(by induction hypothesis)

$$= 2 - \frac{1}{k} + \frac{1}{(k+1)^2} = 2 - \frac{(k+1)^2 - k}{k(k+1)^2}$$
$$= 2 - \frac{k^2 + k + 1}{k(k+1)^2} = 2 - \left(\frac{k(k+1) + 1}{k(k+1)^2}\right)$$
$$= 2 - \left(\frac{1}{k+1} + \frac{1}{k(k+1)^2}\right)$$
$$= \left(2 - \frac{1}{k+1}\right) - \frac{1}{k(k+1)^2}$$
$$< 2 - \frac{1}{k+1}$$

Hence, the result.

Example 15: *Show that $1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$.*

Solution: Proof: (i) Basis of induction: For $n = 1$, $1^3 = 1^2$.

(ii) Induction step: Assume for $n = k$; hence $1^3 + 2^3 + \ldots + k^3 = (1 + 2 + \ldots + k)^2$

For $n = k + 1$, $1^3 + 2^3 + \ldots + k^3 + (k+1)^3$

$$= (1 + 2 + \ldots + k)^2 + (k+1)^3$$
$$= \frac{k^2(k+1)^2}{4} + (k+1)^3$$
$$= (k+1)^2\left[\frac{k^2 + 4(k+1)}{4}\right]$$
$$= \frac{(k+1)^2(k+2)^2}{4}$$
$$= (1 + 2 + \ldots + k + 1)^2$$

Hence, the result is proved.

Following examples deal with real-life situations, where the principle of induction can be applied.

Example 16: *Using mathematical induction, prove that*

$$1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{n-1} n^2 = (-1)^{n-1} \frac{n(n+1)}{2}$$

Solution: For n = 1,

$$\text{L.H.S.} = 1^2 = 1$$

$$\text{R.H.S.} = \frac{(-1)\,1(2)}{2} = 1$$

$\therefore$ Result is true for n = 1.

Assume result is true for n = k,

i.e. $1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{k-1} k^2 = (-1)^{k-1} \frac{k(k+1)}{2}$

Consider $1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{k-1} k^2 + (-1)^k (k+1)^2$

$$= (-1)^{k-1} \frac{k(k+1)}{2} + (-1)^k (k+1)^2$$

$$= (-1)^{k-1} \left\{\frac{k(k+1) - 2(k+1)^2}{2}\right\} = (-1)^{k-1} \left\{\frac{-k^2 - 3k - 2}{2}\right\}$$

$$= \frac{(-1)^k (k+1)(k+2)}{2}$$

Hence, result is true for n = k + 1.

Hence, result is true for all n.

Example 17: *Prove by mathematical induction* $0 \cdot 2^0 + 1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + \ldots + n\,2^n = (n-1)\,2^{n+1} + 2;$ *for* $n \geq 0$.

Solution: Basis for induction : n = 0.

L.H.S. = 0, R.H.S. = $(0 - 1)\,2^1 + 2 = 0$

Induction hypothesis: Assume the result is true for n = k.

For n = k + 1 $(0 \cdot 2^0 + 1 \cdot 2^1 + 2 \cdot 2^2 + \ldots + k\,2^k) + (k+1)\,2^{k+1}$.

$$= (k-1)\,2^{k+1} + 2 + (k+1)\,2^{k+1}$$

$$= 2^{k+1}\,[k - 1 + k + 1] + 2$$

$$= 2^{k+1} \cdot 2k + 2 = k\,2^{k+2} + 2$$

Hence, the equation is true for n = k + 1; hence it is true $\forall$ n.

Example 18: *Prove that* $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \ldots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$, *for* $n \geq 1$

Solution: Basis of induction : n = 1.

$$\text{L.H.S.} = \frac{1}{2}$$

$$\text{R.H.S.} = 2 - \frac{1+2}{2} = 2 - \frac{3}{2} = \frac{1}{2}$$

Assume the equation for n = k. For n = k + 1

$$\left(\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \ldots + \frac{k}{2^k}\right) + \frac{k+1}{2^{k+1}}$$

$$= 2 - \frac{k+2}{2^k} + \frac{k+1}{2^{k+1}}$$

$$= 2 - \left[\frac{k+2}{2^k} - \frac{k+1}{2^{k+1}}\right]$$

$$= 2 - \left(\frac{2k + 4 - k - 1}{2^{k+1}}\right)$$

$$= 2 - \frac{(k+1) + 2}{2^{k+1}}$$

Thus, the result is true for $\forall$ n.

Example 19: *Let n be a positive integer. Show that any* $2^n \times 2^n$ *chessboard with one square removed can be covered using L-shaped pieces, where each piece covers three squares at a time.*

Solution: Let P(n) be the proposition that any $2^n \times 2^n$ chessboard with one square removed can be covered using L-shaped pieces.

Basis of induction: For n = 1, P(1) implies that any 2 × 2 chessboard with one square removed can be covered using L-shaped pieces. P(1) is true, as seen in the following Fig. 2.44 below.

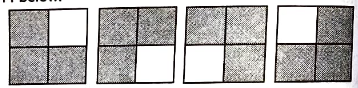


Fig. 2.44

Induction step: Assume P (n) is true and prove that P (n + 1) is true.

For this consider a $2^{n+1} \times 2^{n+1}$ chessboard with one square removed. Divide the chessboard into four equal halves of size $2^n \times 2^n$; as shown in the Fig. 2.45.

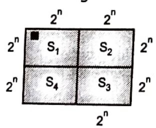


Fig. 2.45

Then the square which has been removed, would have been removed from one of the four chessboards, say S_1. Then by induction hypothesis, S_1 can be covered using L-shaped pieces. Now from each of the three remaining chessboards, remove that particular square lying at the centre of the larger chessboard. This is illustrated in the Fig. 2.46.

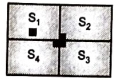


Fig. 2.46

Then by induction hypothesis, each of these $2^n \times 2^n$ chessboards with a square removed can be covered by the L-shaped pieces. Moreover, the three squares that have been temporarily removed can be covered by one L-shaped piece. Hence, the entire $2^{n+1} \times 2^{n+1}$ chessboard can be covered by L-shaped pieces. This completes the proof.

Example 20: Coin problem: *Suppose we have coins of two different denominations, 2 rupees and 5 rupees. It is possible to make up exactly any denomination of 7 rupees or more, using only these two denominations, assuming of course that we have an unlimited supply of these.*

Solution: For k = 7, we have one 5 rupee coin and one 2 rupee coin. For k = 8, we have four 2 rupee coins. Hence, let us assume that we can make up a denomination of k rupees (k ≥ 7). We discuss two cases. Suppose there is a 5 rupee coin in the k-denomination, we have made up. Replacing the 5 rupee coin by three 2 rupee coins, we can make up a denomination of k + 1 rupees. On the other hand, suppose that the k-denomination coins, we have made up, consist of only 2 rupee coins, then replacing two 2 rupee coins by one 5 rupee coin, we can still make up a denomination of k + 1 rupees. Thus, the process can be continued till the supply runs out.

Example 21: Solitaire game problem: *For every integer i, there is an unlimited supply of balls marked with the number i. Initially, a tray of balls is given and the balls are thrown away one at a time. If a ball marked i is thrown away, it is replaced by any finite number of balls marked 1, 2, ..., i – 1. There is no replacement for a ball marked 1. The game ends when the tray is empty. Show that the game always terminates for any tray of balls given initially.*

Solution:

(i) Basis of induction: For n = 1, there is a finite number of balls marked 1. Since, according to the rules of the game, there is no replacement if the balls are thrown away, the game terminates after a finite number of moves.

(ii) Induction step: Let us assume that the game terminates if the largest number that appears on the balls is k. Suppose k + 1 is the largest number appearing on the balls. If all these balls are thrown away, they are replaced by balls marked 1, 2, ..., k. Then the largest number appearing on the balls is k. Hence, by induction hypothesis, the game has to terminate after a finite number of steps.

2.10.2 Principle of Strong Mathematical Induction

We shall now state a more powerful form of the principle of mathematical induction.

Statement: Let P(n) be a statement involving a natural number n. If

(i) P(n) is true for n = n_0, and

(ii) P(n) is true for n = k + 1, **assuming that the statement is true for** $n_0 \leq n \leq k$, then the statement is true for all $n \geq n_0$.

Note that in the second principle of induction, we make a stronger assumption, that the statement is true for $n_0 \leq n \leq k$ (not merely for n = k).

SOLVED EXAMPLES

Example 1: *Show that any positive integer n greater than or equal to 2 is either a prime or a product of primes.*

Solution:

(i) Basis of induction: For n = 2, the statement is obviously true.

(ii) Induction step: Assume the result for 2 ≤ n ≤ k. Consider k + 1. If k + 1 is a prime, the statement is true. If k + 1 is not a prime, then k + 1 = pq, where p ≤ k, q ≤ k. Hence, by induction hypothesis, p is either a prime or product of primes. Similarly, q is either a prime or a product of primes. Hence, k + 1 = pq is a product of primes.

Example 2: Jigsaw Puzzle Problem: *Show that for a jigsaw puzzle with n pieces, it will always take n – 1 moves to solve the problem.*

Solution:

(i) Basis of induction: For n = 1, no moves are needed to solve the puzzle.

(ii) Induction step: Assume that for any jigsaw puzzle with m pieces, 1 ≤ m ≤ k, it takes m – 1 moves to solve the puzzle.

Consider a jigsaw puzzle with k + 1 pieces. For the last move that produces the solution of the puzzle two blocks with m_1 pieces and m_2 pieces respectively, where $m_1 + m_2$ = k + 1, are put together to form a single block.

Since, $1 \le m_1 \le k$, $1 \le m_2 \le k$, by induction hypothesis it takes $m_1 - 1$ moves to form one block and $m_2 - 1$ moves to form the other block.

Hence, the total number of moves to solve the puzzle

$$= (m_1 - 1) + (m_2 - 1) + 1 = m_1 + m_2 - 1$$
$$= k + 1 - 1 = k \text{ moves.}$$

2.11 POWER SET

2.11.1 Definition

Let A be any set. The power set of A, denoted by P(A) is the set of all subsets of A.

Examples:

(i) If $A = \{a\}$, then $P(A) = \{A, \phi\}$.

(ii) If $A = \{a, b\}$, then $P(A) = \{\phi, \{a\}, \{b\}, A\}$.

(iii) If $A = \{a, \{a\}\}$, then

$P(A) = \{\{a\}, \{\{a\}\}, A, \phi\}$.

The following theorem determines the size of the power set.

2.11.2 Theorem (Cardinality of a Power Set)

Let A be a finite set containing n elements. Then the power set of A has exactly 2^n elements.

Proof:

We prove the theorem by mathematical induction.

For $n = 1$, $A = \{a\}$, so that $P(A) = \{A, \phi\}$.

Hence, $|P(A)| = 2^1$ elements.

Assume that if $|A| = k$, $|P(A)| = 2^k$.

Let $|A| = k + 1$. For an element $a \in A$, consider the subset $B = A - \{a\}$. Since, $|B| = k$, by induction hypothesis $|P(B)| = 2^k$, i.e. there are exactly 2^k subsets of B.

Since, every subset of B is also a subset of A, it follows that A contains at least 2^k subsets.

In addition, for each subset of B, say C, we have another subset $C \cup \{a\}$ of A.

Hence, the total number of subsets of A is $2^k + 2^k = 2.2^k = 2^{k+1}$ subsets.

Hence, by induction, it follows that if $|A| = n$, $|P(A)| = 2^n$.

SOLVED EXAMPLES

Example 1: *If $A = \{\phi, a\}$, then construct the sets $A \cup P(A)$, $A \cap P(A)$.*

Solution : $P(A) = \{\phi, \{\phi\}, \{a\}, A\}$

$\therefore$ $A \cup P(A) = \{\phi, a, \{\phi\}, \{a\}, A\}$.

$A \cap P(A) = \{\phi\}$.

Example 2: *Let $A = \{\phi\}$. Let $B = P(P(A))$.*

(i) Is $\phi \in B$? $\phi \subseteq B$?

(ii) Is $\{\phi\} \in B$? $\{\phi\} \subseteq B$?

(iii) Is $\{\{\phi\}\} \in B$? $\{\{\phi\}\} \subseteq B$?

Solution: $P(A) = \{\phi, \{\phi\}\}$

$B = P(P(A)) = \{\phi, \{\phi\}, \{\{\phi\}\}, P(A)\}$.

(i) The element $\phi \in B$. The empty set ϕ is always a subset of B.

(ii) Both are true, one as element and the other as subset containing the single element ϕ.

(iii) Both are true, the first as element and the second as a single to n subset containing the element $\{\phi\}$.

Example 3: *If $A \subseteq B$, then $P(A) \subseteq P(B)$.*

Solution: Let $C \in P(A)$. Then $C \subseteq A$ which implies that $C \subseteq B$.

Hence, $C \in P(B)$. $\therefore$ $P(A) \subseteq P(B)$.

Example 4: *Let A and B be two arbitrary sets.*

(i) Show that $P(A \cap B) = P(A) \cap P(B)$ or give a counter example.

(ii) Show that $P(A \cup B) = P(A) \cup P(B)$ or give a counter example.

Solution: (i) Let $C \in P(A \cap B)$. Then $C \subseteq A \cap B$

$\Rightarrow$ $C \subseteq A$ and $C \subseteq B$ $\Rightarrow$ $C \in P(A)$ and $C \in P(B)$ $\Rightarrow$ $C \in P(A) \cap P(B)$

$\therefore$ $P(A \cap B) \subseteq P(A) \cap P(B)$.

Conversely, let $C \in P(A) \cap P(B)$.

This implies $C \in P(A)$ and $C \in P(B)$

$\Rightarrow$ $C \subseteq A$ and $C \subseteq B$

$\Rightarrow$ $C \subseteq A \cap B$, i.e. $C \in P(A \cap B)$.

Hence, $P(A) \cap P(B) \subseteq P(A \cap B)$

Hence, $P(A \cap B) = P(A) \cap P(B)$.

(ii) Equality is not true.

Consider $A = \{1\}$, $B = \{2\}$.

$A \cup B = \{1, 2\}$

$\therefore$ $P(A \cup B) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$

$P(A) = \{\phi, \{1\}\}$

$P(B) = \{\phi, \{2\}\}$

$\therefore$ $P(A) \cup P(B) = \{\phi, \{1\}, \{2\}\}$

$\neq P(A \cup B)$.

Example 5: *Let $A = \{\phi, b\}$; construct the following sets:*

(i) $A - \phi$

(ii) $\{\phi\} - A$

(iii) $A \cup P(A)$

(iv) $A \cap P(A)$

where P(A) is power set of A.

Solution: (i) $A - \phi = A$

(ii) $\{\phi\} - A = \phi$

(iii) $A \cup P(A) = \{\phi, b, \{\phi\}, \{b\}, A\}$

(iv) $A \cap P(A) = \{\phi\}$

EXERCISE - 2.1

1. If $A = \{\phi, \{\phi\}, \{\phi, \{\phi\}\}\}$, determine whether the following statements are true or false. Justify your answer.
   (i) $\phi \in A$
   (ii) $\{\phi\} \subseteq A$
   (iii) $\{\phi\} \in A$
   (iv) $\{\phi, \{\phi\}\} \subseteq A$
   (v) $\{\{\phi\}\} \in A$
2. If $u = \{n \in N \mid 1 \le n \le 9\}$,
   $A = \{1, 2, 4, 6, 8\}$, $B = \{2, 4, 5, 9\}$, $C = \{x \in Z^+ \mid x^2 \le 16\}$ and $D = \{7, 8\}$,
   find (i) $A \oplus B$, $B \oplus C$, $C \oplus D$
   (ii) $A - B$, $B - A$, $C - D$
   (iii) $\overline{A \cup B}$, $\overline{A \cap B}$
   (iv) $A \cap (\bar{C} \cup D)$
3. For $A = \{a, b, \{b, c\}, \phi\}$ determine the following sets:
   (i) $A - \{a\}$, (ii) $A - \{b, c\}$, (iii) $\{\{b, c\}\} - A$,
   (iv) $A - \{c, \phi\}$, (v) $\{a\} - \{A\}$.
4. Give an example of sets A, B, C such that $A \in B$, $B \in C$ and $A \notin C$.
5. Draw Venn diagrams for the following situations.
   (i) A, B, C are sets such that $A \subseteq B$, $A \subseteq C$, $(B \cap C) \subseteq A$ and $A \subseteq (B \cap C)$.
   (ii) $(A \cap B \cap C) = \phi$, $A \cap B \neq \phi$, $B \cap C \neq \phi$, $A \cap C \neq \phi$.
6. Using Venn diagrams, prove or disprove the following:
   (i) $(A - B) - C = (A - C) - B$
   (ii) $(A - B) - C = (A - C) - (B - C)$
   (iii) $(A - B) \cap (A - C) = A - (B \cup C)$
   (iv) $(A - C) \cup (B - C) = (A \cup B) - C$
   (v) $A - (B - C) = (A - B) \cup (A \cap C)$
   (vi) $A \cap (B - C) = (A \cap B) - (A \cap C)$
   (vii) $(A \cap B) - C = (A - C) \cap (B - C)$
   (viii) $(A \oplus B) \cap C = (A \cap C) \oplus (B \cap C)$
   (ix) $A \cup (\bar{B} \cap C) = (A \cup \bar{B}) \cap (A \cup C)$.
7. Using the rules of set operations, simplify the following:
   (i) $\overline{(A \cup B)} \cup (\bar{A} \cap B)$
   (ii) $[(A \cap B) \cup (A \cap \bar{B}) \cup (\bar{A} \cap B)] \cap B$
   (iii) $((A \cup B) \cap \bar{A}) \cup (\overline{B \cap A})$
   (iv) $\overline{[(A \cap B) \cup C]} \cap \bar{B}$.
8. What can you say about sets A and B, if
   (i) $A - B = B$?
   (ii) $A - B = B - A$?
   (iii) $A \oplus B = A$?
9. It is known that at the University, 60 percent of the professors play tennis, 50 percent of them play bridge, 70 percent jog, 20 percent play tennis and bridge, 30 percent play tennis and jog and 40 percent play bridge and jog. If someone claimed that 20 percent of the professors jog and play bridge and tennis, would you believe this claim? Why?
10. A survey was conducted among 1000 people. Of these 595 are graduates, 595 wear glasses and 550 like ice cream, 395 of them are graduates who wear glasses, 350 of them are graduates who like ice cream and 400 of them wear glasses and like ice cream; 250 of them are graduates who wear glasses and like ice cream. How many of them who are not graduates do not wear glasses and do not like ice cream? How many of them are graduates who do not wear glasses and do not like ice cream?
11. Consider a set of integers from 1 to 250. Find how many of these numbers are divisible by 3 or 5 or 7? Also indicate how many are divisible by 3 or 7 but not by 5.
12. How many integers between 1 and 2000 are divisible by 2, 3, 5 or 7?

13. A college record gives the following information: 119 students enrolled in Introductory Computer Science; of these, 96 took Data Structures, 53 took Foundations, 39 took Assembly Language. Also 38 took both Data Structures and Foundations, 31 took both Foundations and Assembly Language, 32 took both Data Structures and Assembly language and 22 took all the three courses. Is the information correct? Why?

14. A survey of 100 students of the Management Programme shows that 70 read India Today, 31 read Fortune and 54 read Business India. Also the people who read Business India do not read Fortune. Draw a Venn diagram to represent the situation.

15. A software company writes a new package which integrates a word processing program with a spreadsheet program, and they wish it to run on a 64 K machine. The word processor requires 40 K for program and data and the spreadsheet requires 32 K for the same. If 16 K must be reserved for the code integrator, what is the minimum amount of overlapping space that will be necessary?

16. Consider a set of integers 1 to 500. Find how many of these numbers are divisible by 3 or by 5 or by 11?
   (i) Also indicate how many are divisible by 3 or by 11 but not by all 3, 5 and 11.
   (ii) How many are divisible by 3 or 11 but not by 5?

17. It was found that in first year of computer engineering out of 80 students, 50 know 'C' language, 55 know 'basic' and 25 know 'C++', while 8 did not know any language.
   Find,
   (i) How many know all the three languages?
   (ii) How many know exactly two languages?

18. In the survey of 60 people, it was found that 25 read Newsweek Magazine, 26 read time, 26 read fortune. Also 9 read both Newsweek and Fortune, 11 read both Newsweek and Time, 8 read both Time and Fortune and 8 read no magazine at all.
   (i) Find out the number of people who read all the three magazines.
   (ii) Fill in the correct numbers in all the regions of the Venn diagram.
   (iii) Determine number of people who reads exactly one magazine.

19. Among 130 students, 60 study Mathematics, 5 Physics and 30 both Mathematics and Physics. Of th 54 students studying Chemistry, 26 stud Mathematics, 21 Physics and 12 both Mathemati and Physics. All the students studying neith Mathematics nor Physics are studying Biology.
   (i) How many students are studying Biology?
   (ii) How many students not studying Chemistry a studying Mathematics but not Physics?
   (iii) How many students are studying neith Mathematics nor Physics nor Chemistry.

20. It was found that in first year of computer science o 80 students, 50 know COBOL, 55 know C languag and 46 know Pascal. It was also known that 37 kno C and COBOL, 28 know C and Pascal, and 25 kno Pascal and COBOL. 7 students however know non of the language. Find:
   (i) How many know all the three languages?
   (ii) How many know exactly two languages?
   (iii) How many know exactly one language?

21. A survey has been taken on methods of compute travels. Each respondent was asked to check BUS TRAIN or AUTOMOBILE as a major method o traveling to work. More than one answer wa permitted. The results reported were as follows: BUS – 30 people, TRAIN – 35 people, AUTOMOBILE – 1 people, TRAIN and AUTOMOBILE – 20 people and a three methods-5 people. How many peopl completed the survey form?

22. A survey of 500 television watchers produced th following information. 285 watch football, 195 watc hockey, 115 watch basket ball. 45 watch football an basket ball, 70 watch football and hockey, 50 watc hockey and basketball and 50 do not watch any o the three games.
   (i) How many people in the survey watch all th three games?
   (ii) How many people watch exactly one game?

23. 100 of the 120 engineering students in a college tak part in at least one of the activities: group discussion debate and quiz. Also 65 participate in group discussion 45 participate in debate, 42 participate in quiz, 20 participate in group discussion and debate, 25 participate in group discussion and quiz, 15 participate in debate and quiz. Find the number of students:
   (i) Who participate in all the three activities
   (ii) Who participate in exactly one of the activities.

24. In a class of 55 students, the number of students studying different subjects are as follows: Maths 23, Physics – 24, Chemistry 19, Maths + Physics – 12, Maths + Chemistry – 9, Physics + Chemistry – 7, all three subjects - 4. Find the numbers of students who have taken: (i) At least one subject, (ii) Exactly one subject, (iii) Exactly two subjects.

25. In a survey of 100 new cars, it is found that 60 had Air Conditioner (AC), 48 had Power-Steering (PS), 44 had Power Windows (PW), 36 had AC + PW, 20 had AC + PS, 16 had PW + PS, 12 had all three. Find the number of cars that had: (i) Only PW, (ii) PS and PW but not AC, (iii) AC and PS but not PW.

Exercise on Mathematical Induction

26. Show that $n^3 + 2n$ is divisible by 3 for all $n \geq 1$.
27. Show that $n^4 - 4n^2$ is divisible by 3 for all $n \geq 2$.
28. Show that $2^n \times 2^n - 1$ is divisible by 3 for all $n \geq 1$.
29. Show that $5^n - 4n - 1$ is divisible by 16 for all $n \geq 1$.
30. Prove that $1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$
31. Prove mathematical induction for $n \geq 1$
$$1\cdot 2 + 2\cdot 3 + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$
32. Prove by mathematical induction, the given proposition
$$\frac{1}{1(3)} + \frac{1}{3(5)} + \frac{1}{5(7)} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}.$$
33. State the principle of mathematical induction and prove the following proposition
$$P(n) = 1 + 4 + 7 + \ldots + (3n-2) = \frac{n(3n-1)}{2}.$$
34. Prove that for any positive integer n, the number $n^5 - n$ is divisible by 5.
35. Use Mathematical induction show that $n(n^2 - 1)$ is divisible by 24, where n is a odd +ve integer.
(Nov./Dec. 15)
36. State the principle of Mathematical induction using Mathematical induction prove the following proposition :
$$P(n) = 1 + 4 + 7 + \ldots + (3n-1) = \frac{n(3n-1)}{2}.$$
(Nov./Dec. 14)
37. Prove De Morgan's laws for a finite collection of sets $A_1, A_2, \ldots, A_n$ using induction.
38. When n couples arrived at a party, they were greeted by the host and hostess at the door. After rounds of handshaking, the host asked the guests as well as his wife (the hostess) to indicate the number of hands each of them had shaken. He got 2n + 1 different answers. Given that no one shook hands with his or her spouse, how many hands had the hostess shaken? Prove your answer by induction.
39. We present a proof of the statement "Any n billiard balls are of the same colour", by induction.
   Basis of induction: For n = 1, the statement is trivially true.
   Induction step: Suppose we are given k + 1 billiard balls which we number 1, 2, ... (k + 1). According to the induction hypothesis, balls 1, 2, ..., k are of the same colour. Also balls 2, 3, ..., (k + 1) are of the same colour. Consequently, balls 1, 2, ..., k, (k + 1) are all of the same colour.
   What is wrong with the proof?

Problems on Power Sets:

40. Let A = {a, {a}}. Determine which of the following statements are true or false.
   (i) $\phi \in P(A)$
   (ii) $\phi \subseteq P(A)$
   (iii) $\{a\} \in P(A)$
   (iv) $\{a, \{a\}\} \in P(A)$
   (v) $\{\{\{a\}\}\} \subseteq P(A)$
41. Determine whether the following statements are true or false. Justify your answer.
   (i) $A \cup P(A) = P(A)$
   (ii) $\{A\} \cup P(A) = P(A)$
   (iii) $A - P(A) = A$
   (iv) $P(A) - \{A\} = P(A)$
   (v) $\{A\} \cap P(A) = A.$

42. For multisets, define in brief:
    (i) Multisets.
    (ii) Multiplicity of an element in a multiset.
    (iii) Cardinality of mulstiset.
    (iv) Union of multiset.
    (v) Intersection of multiset.
    (vi) Difference of multiset.

43. A survey has been taken on methods of computer travel. Each respondent was asked to check bus, train or automobile as a major method of travelling to work. More than one answer was permitted. The results reported were as follows:

    Bus - 30 people, train - 35 people, automobile - 100 people, bus and train - 15 people, bus and automobile - 15 people, train and automobile - 20 people and all three methods - 5 people. How many people completed a survey form?

44. In a survey of 260 college students, the following data were obtained: 64 had taken a Mathematical course, 94 has taken a Computer Science course, 58 had taken a Business course, 28 had taken both Mathematic and Business courses, 26 had taken both Mathematical and Computer Science course, 22 had taken both Computer Science and Business course and 14 had taken all 3 types of courses.
    (1) How many students were surveyed who had taken none of the three types of courses?
    (2) Of the students surveyed, how many had taken only Computer Science course?

45. A survey of 70 high school students revealed that 35 like folk music, 15 like classical music and 5 like both. How many of the students surveyed do not like either folk or classical music.

ANSWERS - 2.1

1. (i) True, (ii) True, (iii) True, (iv) True, (v) False.
2. (i) $A \oplus B = \{1, 5, 6, 8, 9\}$ (ii) $A - B = \{1, 6, 8\}$
   (iii) $\overline{A \cup B} = \{3, 7\}$.
3. (i) $A - \{a\} = \{b, \{b, c\}, \phi\}$ (v) $\{a\} - \{A\} = \{a\}$.
7. (i) $\bar{A}$, (iii) $\bar{A} \cup \bar{B}$.
9. Claim is false.
10. 155, 100
11. 86 numbers between 1 to 250 are divisible by 3 or but not by 5.
12. 1499
19. (i) 49, (ii) 16, (iii) 30.
20. (i) 12, (ii) 66, (iii) 7.
22. (i) 20

POINTS TO REMEMBER

- A set is a **collection of** objects.
- An object in the collection is called an **element** member of the set.
- The term **class** is also used to denote a set.
- A set may contain **finite** number of elements **infinite** number of elements.
- A set is called an **empty set** or a **null set** if it contain no element. An empty set is denoted by the letter ϕ.
- If every element of a set A is also an element of a set then we say A is a **subset** of B, or A is **contained** in This is denoted by '$A \subseteq B$'. This can be also denoted b '$B \supseteq A$'. If A is not a subset of B, this is indicated b '$A \not\subseteq B$'.
- If all sets, considered during a **specific discussion** ar subsets of a given set, then this set is called as th **Universal Set**, and is denoted by 'U'.
- A Venn diagram (named after the British logician Joh Venn) is a pictorial depiction of a set.
- Let A be a given set. **Complement** of A, denoted b $\bar{A}$ is defined as $\bar{A} = \{x \mid x \notin A\}$.
- The union of two sets A and B is the set consisting o all elements which are in A, or in B, or in both se A and B. It is denoted by $A \cup B$.
- The intersection of two sets A and B, denoted by $A \cap$ is the set consisting of elements which are in A **as we as** in B.
- If the counting of the elements of a set is interminabl or impossible, then such a set is said to be infinite.
- If a set **contains multiple occurrences of an objec then such set is called 'multiset'**
- **Multisubset (or msubset):** A multiset A is said to be multisubset of B if multiplicity of each element in A i less or equal to its multiplicity in B.

◈◈◈

Unit II : Relations and Functions

CHAPTER 3
RELATIONS

3.1 INTRODUCTION

- In the preceding chapter we dealt with sets, elements and general properties of sets. Now we progress further and study the various relationships that may exist between elements of a set. We study various properties of a relation, including its matrix and graphical representations.
- The concept of relation is of primary importance in computer science, especially in the study of data structure such as linked list, array, relational models etc. Relations are also important in the analysis of algorithms, information system etc.

3.2 RELATIONS

A common notion of relation is a type of association that exists between two or more objects. Consider the following examples:

- x is the father of y.
- x was born in the city y in the year z.
- The number x is greater than the number y.
- Prof. x teaches the subject y to the class z in classroom u.

In general, one can have relation among n objects, (where n is a positive integer). In describing a relation, it is necessary not only to specify the objects, but also the order in which they appear. In the example "x is the father of y", the respective positions of x and y matter; x should precede y, and not vice-versa. Hence in the following definition, we introduce the concept which gives the necessary ordering of the objects.

3.2.1 Definition

An **ordered n-tuple**, for $n > 0$, is a sequence of objects or elements, denoted by $(a_1, a_2, \ldots, a_n)$.

If $n = 2$, the ordered n-tuple is called an **ordered pair.**

If $n = 3$, the ordered n-tuple is called an ordered triple; and so on.

As pre-requisite to study relations, we consider sets whose elements are ordered n-tuples and study their properties.

3.3 PRODUCT SETS

Definition:

Let A and B be non-empty sets. The **product set** or the **Cartesian product** $A \times B$ is defined as

$$A \times B = \{ (a, b) \mid a \in A \text{ and } b \in B \}.$$

If $A = \phi$ or $B = \phi$, then $A \times B = \phi$.

Examples:

1. Let $A = \{a, b, c\}$, $B = \{1, 2\}$.
   Then $A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$
2. Let A = set of students = {Shilpa, Ramesh, Aparna}
   and B = set of marks in DSGT = {65, 56, 72},
   Then $A \times B$ = {(Shilpa, 65), (Shilpa, 56), (Shilpa, 72), (Ramesh, 65), (Ramesh, 56), (Ramesh, 72), (Aparna, 65), (Aparna, 56), (Aparna, 72)}.
3. If **R** denotes the set of all real numbers, then $\mathbf{R} \times \mathbf{R}$ denotes the set of all points in the co-ordinate plane.
4. We know that a complex number $x + iy$ can be considered as an ordered pair (x, y). Hence if **C** denotes the set of all complex numbers, then C is the Cartesian product $\mathbf{R} \times \mathbf{R}$.

From the above examples, it is clear that product of sets is non-commutative,

i.e. $A \times B \neq B \times A$.

The following theorem establishes certain important properties of the product operation.

3.3.1 Theorem (Product Operation)

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

Proof:

We shall show that every element (x, y) of $A \times (B \cup C)$ is an element of $(A \times B) \cup (A \times C)$ and vice-versa.

$(x, y) \in A \times (B \cup C) \; x \in A$ and $y \in (B \cup C) \; x \in A$ and ($y \in B$ or $y \in C$).

Since 'and' distributes over 'or', this implies ($x \in A$ and $y \in B$) or ($x \in A$ and $y \in C$)

$(x, y) \in (A \times B)$ or $(x, y) \in (A \times C)$

$x \in (A \times B) \cup (A \times C)$.

Thus (i) is proved.

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Proof:

$(x, y) \in A \times (B \cap C) \leftrightarrow x \in A$ and $y \in (B \cap C) \leftrightarrow x \in A$ and ($y \in B$ and $y \in C$)

$\leftrightarrow$ ($x \in A$ and $y \in B$) and ($x \in A$ and $y \in C$)

$\leftrightarrow (x, y) \in (A \times B)$ and $(x, y) \in (A \times C)$

$\leftrightarrow (x, y) \in (A \times B) \cap (A \times C)$

(iii) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

Proof:

$(x, y) \in A \times (B \cap C) \leftrightarrow x \in A$ and $y \in (B \cap C) \leftrightarrow x \in A$ and ($y \in B$ and $y \in C$)

$\leftrightarrow$ ($x \in A$ and $y \in B$) and ($x \in A$ and $y \in C$)

$\leftrightarrow (x, y) \in (A \times B)$ and $(x, y) \in (A \times C)$

$\leftrightarrow (x, y) \in (A \times B) \cap (A \times C)$

(iv) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

$(x, y) \in (A \cap B) \times C \leftrightarrow x \in (A \cap B)$ and $y \in C \leftrightarrow$ ($x \in A$ and $x \in B$) and $y \in C$

$\leftrightarrow$ ($x \in A$ and $y \in C$) and ($x \in B$ and $y \in C$)

$\leftrightarrow (x, y) \in (A \times C)$ and $(x, y) \in (B \times C)$

$\leftrightarrow (x, y) \in (A \times C) \cap (B \times C)$.

The next theorem gives an important result pertaining to the cardinality of the product set.

3.3.2 Theorem (Cardinality of the Product Set)

If A and B are finite sets with cardinalities m, n respectively then $|A \times B| = m \cdot n$.

Proof:

Since $|A| = m$,

Let $A = \{a_1, a_2, ..., a_m\}$.

Similarly, $B = \{b_1, b_2, ..., b_n\}$, as $|B| = n$.

Now $A \times B = \left\{ (a_i, b_j) \mid 1 \le i \le m,\ 1 \le j \le n \right\}$

Since for each element a_i in A, there exists a corresponding element $b_i \in B$ in the ordered pair (a_i, b_j), the set $A \times B$ consists of exactly $m \cdot n$ elements.

Hence $|A \times B| = m \cdot n$.

Example: If $A = \left\{ n \in N \mid 1 \le n \le 100 \right\}$

and $B = \left\{ n \in N \mid 1 \le n \le 50 \right\}$

then $|A \times B| = 100 \times 50 = 5000$

The definition of product of two sets is generalised fo finite collection of sets $\{A_1, A_2, ..., A_n\}$ by defining $A_1 \times A_2 \times$ ...... $\times A_n = \{(a_1, a_2, ..., a_n) \mid a_i \in A_i, 1 \le i \le n\}$.

If all $A_i = A$, then $A_1 \times A_2 \times ... \times A_n = A^n$.

If A, B, C are non-empty sets ;

then $A \times B \times C = \left\{ (a, b, c) \mid a \in A, b \in B, c \in C \right\}$

This set is clearly different from

$$A \times (B \times C) = \left\{ (a, (b, c)) \mid a \in A, (b, c) \in B \times C \right\}$$

and also different from the set

$$(A \times B) \times C = \left\{ ((a, b), c) \mid (a, b) \in A \times B, c \in C \right\}$$

Distinguishing the three types is quite a proble sometimes, though normally by product of sets A, B, C, w mean $A \times B \times C$.

SOLVED EXAMPLES

***Example 1:** If $A = \{1\}$, $B = \{a, b\}$, $C = \{2, 3\}$, find $A \times B \times C$, A^2, $B^2 \times A$, C^3.*

Solution:

$A \times B \times C = \{(1, a, 2), (1, b, 2), (1, a, 3), (1, b, 3)\}$

$A^2 = \{(1, 1)\}$

$B^2 = \{(a, a), (a, b), (b, a), (b, b)\}$

$B^2 \times A = \{((a, a), 1), ((a, b), 1), ((b, a), 1), ((b, b), 1)\}$

$C^3 = C \times C \times C = \{(2, 2, 2), (2, 2, 3), (2, 3, 3), (2, 3, 2), (3, 2, 2), (3, 3, 2), (3, 2, 3), (3, 3, 3)\}$

***Example 2:** If $A \subseteq C$ and $B \subseteq D$, prove that $A \times B \subseteq C \times D$.*

Solution: Let $(a, b) \in A \times B$.

This implies $a \in A$ and $b \in B$. Since $A \subseteq C$ and $B \subseteq D$, $a \in C$ and $b \in D$ so that $(a, b) \in C \times D$. Hence $A \times B \subseteq C \times D$.

***Example 3:** Show that $A \times B = B \times A \leftrightarrow A = \phi$ or $B = \phi$ or $A = B$.*

Solution: Let $A \times B = B \times A$. Then $(a, b) \in A \leftrightarrow B \times (a, b) \in B \times A \leftrightarrow a \in A$ iff $a \in B$ and $b \in B$ iff $b \in A \leftrightarrow A = B$ iff $A \times B \ne \phi$. If $A \times B = \phi$, then $A = \phi$ or $B = \phi$. Hence the result.

***Example 4:** If $A = \{1, 2\}$, $B = \{1, 2, 3\}$, find $(A \times B) \cap (B \times A)$*

Solution: $A \times B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}$

$B \times A = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2)\}$.

Hence $(A \times B) \cap (B \times A) = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$.

EXERCISE 3.1

1. Let $A = \{a, b\}$ and $B = \{4, 5, 6\}$. List the elements in
   (a) $A \times B$, (b) $B \times A$, (c) $A \times A$, (d) $B \times B$,
   (e) $(A \times B) \times A$, (f) $A \times A \times B$, (g) $A \times (B \times A)$.
2. If $A = \{a, b, c\}$, $B = \{1, 2\}$ and $C = \{\#, *\}$, list all the elements in $A \times B \times C$.
3. If $A = \{1, 2\}$ construct the set $P(A) \times A$.
4. If $A \times B \subseteq C \times D$, does it necessarily follow that $A \subseteq C$ and $B \subseteq D$?
5. Is it possible $A \subseteq A \times A$, for some set A?
6. Prove or disprove
   (i) $(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)$
   (ii) $(A - B) \times (C - D) = (A \times C) - (B \times D)$
   (iii) $(A \oplus B) \times (C \oplus D) = (A \times C) \oplus (B \times D)$
   (iv) $(A \oplus B) \times C = (A \times C) \oplus (B \times C)$
7. Prove that $A \times B = B \times A$ iff $A = B$.
8. Consider the ordered triple $\{1, 3, 5\}$.
   (a) Represent the ordered triple as an ordered pair.
   (b) Represent the ordered pair in (a) as a set.
9. Let $A = \left\{ a \mid 1 \le a \le 2 \right\}$ and $B = \left\{ b \mid 0 \le b \le 1 \right\}$.
   Find : (a) $A \times B$
   (b) $B \times A$
   (c) Represent $A \times B$ and $B \times A$ graphically.
10. Is $(A \times B) \times C = A \times (B \times C)$? Justify?

3.4 BASIC CONCEPTS OF RELATION

3.4.1 Definition

Let $\{A_1, A_2, ..., A_n\}$ be a finite collection of sets. A subset R of $A_1 \times A_2 \times ... \times A_n$ is called an **n-ary relation** on $A_1, A_2, ..., A_n$.

If $R = \phi$, then R is called **void** or empty relation.

If $R = A_1 \times A_2 \times ... \times A_n$, then R is called the **universal** relation.

If $A_i = A$ for all i, then R is called an n-ary relation on A.

If n = 1, 2 or 3, then R is called a **unary, binary** or **ternary** relation respectively.

Examples:

1. Let **Z** be the set of all integers. Then the property "x is an even integer", can be characterised as a relation which is unary. In this case, the relation
   $$R = \left\{ x \in Z \mid x \text{ is even} \right\}$$
2. Let $A = \{1, 2, 5, 6\}$ and let R be the relation characterised by the property "x is less than y". Then $R = \{(1, 2), (1,5), (1, 6), (2, 5), (2, 6), (5, 6)\}$, where R is binary.
3. Let $A = \{1, 2, 3\}$ and let R be the relation characterised by the property "x + y is less than z". Then $R = \{1, 1, 3\}$, which is a ternary.
4. Let $A = \{2, 3, 4\}$ and let R be the relation characterised by the property "x + y is divisible by z". Then $R = \{(2, 2, 2), (2, 2, 4), (2, 4, 2), (2, 4, 3), (3, 3, 2), (3, 3, 3), (4, 2, 2), (4, 2, 3), (4, 4, 2), (4, 4, 4)\}$, where R is a ternary relation.

Among the relations, binary relations are the most important, being widely used in various applications. Hence in what follows, we will discuss binary relations and their properties in detail.

3.4.2 Binary Relation

Let A and B be non-empty sets. Then a binary relation R from A to B is a subset of $A \times B$, i.e. $R \subseteq A \times B$. The **domain** of R, denoted by D(R), is the set of elements in A that are related to some element in B, i.e.

$$D(R) = \left\{ a \in A \mid \text{for some } b \in B,\ (a, b) \in R \right\}$$

The range of R denoted by Rn (R) is the set of elements in B, that are related to some element in A, i.e.

$$Rn\ (R) = \left\{ b \in B \mid \text{for some } a \in A,\ (a, b) \in R \right\}.$$

Clearly $D(R) \subseteq A$ and $Rn\ (R) \subseteq B$.

***Example 5:** Let $A = \{2, 3, 4, 5\}$ and let R be the relation on A defined as aRb iff a < b. Find D(R) and Rn(R).*

Solution: $R = \{(2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$

Then $D(R) = \{2, 3, 4\}$ and $Rn\ (R) = \{3, 4, 5\}$

As R is basically a set, all the rules of set operations are applicable to R. Hence if A, B are sets with binary relations R and S, then

$$R \cup S = \left\{ (a, b) \mid (a, b) \in R \vee (a, b) \in S \right\}$$

$$R \cap S = \left\{ (a, b) \mid (a, b) \in R \wedge (a, b) \in S \right\}$$

The set $A \times B$ is the **universal** relation and the empty set ϕ is the **void** relation.

3.4.3 Complement of a Relation

> The complement of a relation R, denoted by $\bar{R}$ is defined as the set
>
> $$\bar{R} = \left\{ (a, b) \,\middle|\, (a, b) \notin R \right\}, \text{ i.e. } a \bar{R} b \text{ iff } a \not R b.$$

Example 1:

Let $A = \{1, 2, 3, 4\}$ *and* $B = \{a, b, c\}$.

Let $R = \{(1, a), (1, b), (2, c), (3, a), (4, b)\}$

and $S = \{(1, b), (1, c), (2, a), (3, b), (4, b)\}$

Find (i) $\bar{R}$ *and* $\bar{S}$ *(ii) Verify De Morgan's laws for R and S.*

Solution:

(i) $A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c), (4, a), (4, b), (4, c)\}$.

$\therefore \quad \bar{R} = \{(1, c), (2, a), (2, b), (3, b), (3, c), (4, a), (4, c)\}$

$\bar{S} = \{(1, a), (2, b), (2, c), (3, a), (3, c), (4, a), (4, c)\}$.

(ii) De Morgan's laws states that

$$\overline{R \cup S} = \bar{R} \cap \bar{S} \quad \text{and} \quad \overline{R \cap S} = \bar{R} \cup \bar{S}$$

$R \cup S = \{(1, a), (1, b), (1, c), (2, a), (2, c), (3, a), (3, b), (4, b)\}$

$\overline{R \cup S} = \{(2, b), (3, c), (4, a), (4, c)\}$

$\bar{R} \cap \bar{S} = \{(2, b), (3, c), (4, a), (4, c)\}$

Hence $\overline{R \cup S} = \bar{R} \cap \bar{S}$

$R \cap S = \{(1, b), (4, b)\}$

$\therefore \quad \overline{R \cap S} = \{(1, a), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c), (4, a), (4, c)\}$

$\bar{R} \cup \bar{S} = \{(1, a), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c), (4, a), (4, c)\}$

Hence $\overline{R \cap S} = \bar{R} \cup \bar{S}$

Thus De Morgan's laws are verified for relations R and S.

3.4.4 Converse of a Relation

Given a relation from A to B, one may define a relation from B to A as follows.

> *Let R be a relation from* **A to B**. *Then the* **converse** *of R, denoted by* R^c ***is the relation from B to A, defined as***
>
> $$R^c = \left\{ (b, a) \,\middle|\, (a, b) \in R \right\}$$
>
> *Clearly* $R^c \subseteq B \times A$

For example, if A = N is the set of natural numbers and R is the relation <, then R^c is the relation >. The converse relation is also called as the **inverse** relation and is denoted by R^{-1}.

The following theorem gives important properties of the converse relation.

3.4.5 Theorem (Converse of a Relation)

Let R, S be the relations from A to B. Then

(i) $(R^c)^c = R$

Proof:

(i) is immediate, by the definition of the converse.

(ii) $(R \cup S)^c = R^c \cup S^c$

Proof:

$$(R \cup S)^c = \left\{ (b, a) \,\middle|\, (a, b) \in R \text{ or } (a, b) \in S \right\}$$

$$= \left\{ (b, a) \,\middle|\, (b, a) \in R^c \vee (b, a) \in S^c \right\}$$

$$= R^c \cup S^c$$

(iii) $(R \cap S)^c = R^c \cap S^c$

Proof:

$$(R \cap S)^c = \left\{ (b, a) \,\middle|\, (a, b) \in R \wedge (a, b) \in S \right\}$$

$$= \left\{ (b, a) \,\middle|\, (b, a) \in R^c \wedge (b, a) \in S^c \right\}$$

SOLVED EXAMPLES

Example 1: *Let* $A = \{1, 2, 3, 4\}$ *and* $B = \{a, b, c\}$

Let $R = \{(1, a), (3, a), (3, c)\}$.

Find (i) R^c, *(ii)* $D(R^c)$, *(ii)* $Rn(R^c)$.

Solution: (i) $R^c = \{(a, 1), (a, 3), (c, 3)\}$

(ii) $D(R^c) = \{a, c\} = Rn(R)$

(iii) $Rn(R^c) = \{1, 3\} = D(R)$

Example 2: *Let* $A = \{2, 3, 4, 6\}$. *Let R and S be relations on A such that*

$$R = \left\{ (a, b) \,\middle|\, a = b + 1 \text{ or } b = 2a \right\}$$

and $S = \left\{ (a, b) \,\middle|\, a \text{ divides } b \right\}$, *find* $(R \cap S)^c$.

Solution: $R = \{(3, 2), (4, 3), (2, 4), (3, 6)\}$

and $S = \{(2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (6, 6)\}$

$\therefore \quad R \cap S = \{(2, 4), (3, 6)\}$

$\therefore \quad (R \cap S)^c = \{(4, 2), (6, 3)\}$.

3.4.6 Composition of Binary Relations

We shall now discuss relations that are formed from an existing **sequence** of relations. These are called as composite relations. Real life abounds with such relations. Consider for example, the relationship of grandfather who is father's (or mother's) father.

The concept of composite relations plays an important role in the execution of programs, where a sequence of data conversions takes place from decimal to binary and from binary to floating point. Let us now formally define composite relation.

Definition:

> Let R_1 be a relation from A to B and R_2 a relation from B to C. The **composite relation** from A to C, denoted by $R_1 \cdot R_2$ (or $R_1 R_2$) is defined as
>
> $$R_1 \cdot R_2 = \left\{ (a, c) \,\middle|\, a \in A \wedge c \in C \wedge \exists b\, [b \in B \wedge (a, b) \in R_1 \wedge (b, c) \in R_2] \right\}$$

Note that if R_1 is a relation from A to B and R_2 from C to D, $R_1 \cdot R_2$ is not defined unless B = C. In general, if $\{A_1, A_2, \ldots, A_{n+1}\}$ is a finite collection of sets where R_1 is a relation from A_1 to A_2, R_2 from A_2 to A_3, ... R_n from A_n to A_{n+1}, then $R_1 \cdot R_2 \cdot \ldots \cdot R_n$ is a relation from A_1 to A_{n+1}. We shall denote this relation simply as $R_1 R_2 R_3 \ldots R_n$.

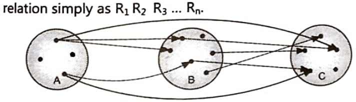


Fig. 3.1

In particular, if $A_1 = A_2 = \ldots = A_{n+1} = A$ and $R_1 = R_2 = \ldots = R_n = R$, then we denote $R_1 R_2 \ldots R_n$ by R^n which is a relation on A. Hence given R, one can compute R^2, R^3, and so on.

The operation of composition is clearly not commutative; i.e. $R_1 R_2 \neq R_2 R_1$. In fact $R_2 R_1$ may not be defined, even though $R_1 R_2$ is. However, the operation is associative, as established in the following theorem.

3.4.7 Theorem (Associatively Property of Relations)

> Let R_1, R_2 and R_3 be relations from A to B, B to C and C to D. Then $(R_1 R_2) R_3 = R_1 (R_2 R_3)$.

Proof:

Since we have to prove essentially the equality of two sets, we shall show that $(R_1 R_2) R_3 \subseteq R_1 (R_2 R_3)$ and conversely.

Let $(a, d) \in (R_1 R_2) R_3$, where $a \in A$, $d \in D$.

Note that $R_1 R_2$ is a relation from A to C.

Then this means that there exists an element $c \in C$ such that $(a, c) \in R_1 R_2$ and $(c, d) \in R_3$.

Now $(a, c) \in R_1 R_2$ implies there exists an element $b \in B$ such that $(a, b) \in R_1$ and $(b, c) \in R_2$.

Since $(b, c) \in R_2$ and $(c, d) \in R_3$, if follows that $(b, d) \in R_2 R_3$.

Again since $(a, b) \in R_1$ and $(b, d) \in R_2 R_3$, $(a, d) \in R_1 (R_2 R_3)$.

Similarly, we can prove $R_1 (R_2 R_3) \subseteq (R_1 R_2) R_3$. This proves the associatively property of relations. The next result deals with the converse of the composition.

3.4.8 Theorem

> Let R_1 be a relation from A to B and R_2 from B to C. Then $(R_1 R_2)^c = R_2^c R_1^c$.

Proof:

R_1^c is a relation from B to A and R_2^c from C to B.

Hence anyway both $(R_1 R_2)^c$ and $R_2^c R_1^c$ are relations from C to A.

We shall now prove that these relations are equal.

Let $(c, a) \in (R_1 R_2)^c$. This implies that $(a, c) \in R_1 R_2$.

Hence there exists an element $b \in B$ such that $(a, b) \in R_1$ and $(b, c) \in R_2$.

It follows that $(b, a) \in R_1^c$ and $(c, b) \in R_2^c$, so that $(c, a) \in R_2^c R_1^c$.

Hence $(R_1 R_2)^c \subseteq R_2^c R_1^c$. Similarly, we can prove $R_2^c R_1^c \subseteq (R_1 R_2)^c$. Hence the equality is proved.

SOLVED EXAMPLES

Example 1: *Let*

$A = \{a, b, c, d\}$ *where*

where $R_1 = \{(a, a), (a, b), (b, d)\}$

and $R_2 = \{(a, d), (b, c), (b, d), (c, b)\}$

Find $R_1 R_2$, $R_2 R_1$, R_1^2, R_2^3.

Solution: $R_1 R_2 = \{(a, d), (a, c)\}$

$R_2 R_1 = \{(c, d)\}$.

$R_1^2 = \{(a, a), (a, b), (a, d)\}$

$R_2^2 = \{(b, b), (c, c), (c, d)\}$

$R_2^3 = R_2 R_2^2 = \{(b, c), (c, b), (b, d)\}$

Example 2 : *Let* $A = \{2, 3, 4, 5, 6\}$ *and let* R_1, R_2 *be relations on A such that*

$$R_1 = \{(a, b) \mid a - b = 2\}$$

and $$R_2 = \{(a, b) \mid a + 1 = b \text{ or } a = 2b\}.$$

Find the composite relations

(i) $R_1 R_2$, *(ii)* $R_2 R_1$ *(iii)* $R_1 R_2 R_1$ *(iv)* R_1^2 *(v)* $R_1 R_2^2$.

Solution: $R_1 = \{(4, 2), (5, 3), (6, 4)\}$

$R_2 = \{(2, 3), (3, 4), (4, 5), (5, 6), (4, 2), (6, 3)\}$

(i) $R_1 R_2 = \{(4, 3), (5, 4), (6, 2), (6, 5)\}$

(ii) $R_2 R_1 = \{(3, 2), (5, 4), (4, 3)\}$

(iii) $R_1 R_2 R_1 = R_1 (R_2 R_1)$

$= \{(5, 2), (6, 3)\}$

Example 3 : *Le*

t$A = \{1, 2, 3, 4\}$,

Let R_1 *be the relation on A defined as*

$$R_1 = \{(x, y) \mid x + y = 5\}$$

and R_2 be the relation defined as

$$R_2 = \{(x, y) \mid y - x = 2\}.$$

Verify that $(R_1 R_2)^c = R_2^c R_1^c$.

Solution: $R_1 = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$

$R_2 = \{(1, 3), (2, 4)\}$

$R_1 R_2 = \{(3, 4), (4, 3)\}$

$\therefore$ $(R_1 R_2)^c = \{(4, 3), (3, 4)\}$

$R_1^C = \{(4, 1), (3, 2), (2, 3), (1, 4)\}$

$R_2^C = \{(3, 1) (4, 2)\}$

$R_2^C R_1^C = \{(3, 4), (4, 3)\} = (R_1 R_2)^c$

3.5 MATRIX REPRESENTATION OF A RELATION

Let $A = \{a_1, a_2, \ldots, a_n\}$

and $B = \{b_1, b_2, \ldots, b_n\}$

be finite sets containing respectively m and n elements.

Let R be a relation from A to B. By definition, $R \subseteq A \times B$. Hence we can represent R by a m × n matrix $M_R = [m_{ij}]$ which is defined as follows:

$$m_{ij} = 1 \quad ; \quad \text{if } (a_i, b_j) \in R$$
$$= 0 \quad ; \quad \text{if } (a_i, b_j) \notin R.$$

The matrix M_R is called as the matrix of R.

The matrix representation of R is useful in verifying certain properties of R.

SOLVED EXAMPLES

Example 1: *Let*

$A = \{a, b, c, d\}$ *and* $B = \{1, 2, 3\}$.

Let $R = \{(a, 1), (a, 2), (b, 1), (c, 2), (d, 1)\}$

Find the relation matrix.

Solution: M_R will have 4 rows and 3 columns.

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

Example 2: *Let* $A = \{1, 2, 3, 4, 8\}$, $B = \{1, 4, 6, 9\}$.

Let a R b iff a| b (a divides b).

Find the relation matrix.

Solution: $R = \{(1, 1), (1, 4), (1, 6), (1, 9), (2, 4), (2, 6), (3, 6), (3, 9), (4, 4)\}$

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 4 & 6 & 9 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Example 3: *Let* $A = \{1, 2, 3, 4, 8\} = B$; *a R b iff* $a + b \leq 9$

Find its relation matrix.

Solution: $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 8), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4), (8, 1)\}$

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 4: *Let* $A = \{a, b, c, d\}$ *and let*

$$\text{and let } M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Find R.

Solution: $R = \{(a, a), (a, b), (b, c), (b, d), (c, c), (c, d), (d, a), (d, c)\}$

3.5.1 Relation Matrix Operations

A relation matrix has entries which are either one or zero. Such a matrix is called a Boolean matrix. Let us see how to add or multiply two such matrices.

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be m × n Boolean matrix. We define

$A + B = [c_{ij}]$ where,

$$c_{ij} = 1 \quad ; \quad \text{if } a_{ij} = 1 \text{ or } b_{ij} = 1$$
$$= 0 \quad ; \quad \text{if } a_{ij} \text{ and } b_{ij} \text{ are both zero.}$$

Similarly, if $A = [a_{ij}]$ is an m × n Boolean matrix and $B = [b_{ij}]$ is an n × r Boolean matrix, then $A \cdot B = [d_{ij}]$ is an m × r matrix.

where, $d_{ij} = 1$; if $a_{ij} = b_{ij} = 1$

$= 0$; if $a_{ij} = 0$ or $b_{ij} = 0$

Example: Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

then $A + B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and

$$A \cdot B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

3.5.2 Properties of Relation Matrix

Let R_1 be a relation from A to B, R_2 from B to C. Then the relation matrices satisfy the following properties:

1. $M_{R_1 \cdot R_2} = M_{R_1} \cdot M_{R_2}$
2. M_{R^c} = transpose of M_R (for $R = R_1$ or $R = R_2$)
3. $M_{(R_1 \cdot R_2)^c} = M_{R_2^c R_1^c} = M_{R_2^c} \cdot M_{R_1^c}$

The proofs are left as exercises.

SOLVED EXAMPLES

Example 1: *Let*

$A = \{1, 2, 3, 4\}$,

and let $R_1 = \{(1, 1), (1, 2), (2, 3), (2, 4), (3, 4), (4, 1), (4, 2)\}$

and $R_2 = \{(3, 1), (4, 4), (2, 3), (2, 4), (1, 1), (1,4)\}$

Verify (i) $M_{R_1 R_2} = M_{R_1} \cdot M_{R_2}$

(ii) $M_{R_1^c}$ = Transpose of M_{R_1}

(iii) $M_{(R_1 R_2)^c} = M_{R_2^c} \cdot M_{R_1^c}$

Solution:

(i) $M_{R_1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$, $M_{R_2} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$R_1 R_2 = \{(1, 1), (1, 4), (1, 3), (2, 1), (2, 4), (3, 4), (4, 4), (4, 1), (4, 3)\}$

$$\therefore \quad M_{R_1 R_2} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$$M_{R_1} \cdot M_{R_2} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

(ii) $R_1^c = \{(1, 1), (2, 1), (3, 2), (4, 2), (4, 3), (1, 4), (2, 4)\}$

$$M_{R_1^c} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$= \text{Transpose of } M_{R_1}$

(iii) $(R_1 R_2)^c$ = {(1, 1), (4, 1), (3, 1), (1, 2), (4, 2), (4, 3), (4, 4), (1, 4), (3, 4)}

$$\therefore M_{(R_1 R_2)^c} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Now

$$M_{R_2^c} \cdot M_{R_1^c} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = M_{R_2^c} \cdot M_{R_1^c}$$

3.6 GRAPHICAL REPRESENTATION OF A RELATION

If A is a finite set and R is a relation on A, it is possible to represent R pictorially by means of a graph. The elements of A are represented by points or circles, called as **nodes** or **vertices.** If aRb, this is indicated by drawing an arc from a to b with an arrowhead pointing in the direction a → b. If aRa, this is shown by drawing a loop around a. These arcs (or loops) are called as **edges** of the graph. The resulting graph is called a **directed graph** or **digraph** of R.

The various types are illustrated in the following figures.

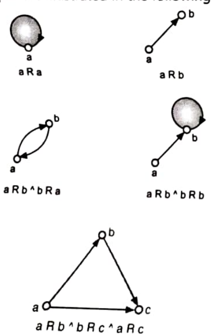


Fig. 3.2

SOLVED EXAMPLES

Example 1: *Let A = {2, 3, 4, 5} and let*

R = {(2, 3), (3, 2), (3, 4), (3, 5), (4, 3), (4, 4), (4, 5)}

Draw its digraph.

Solution:

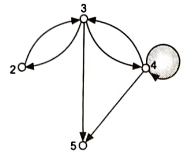


Fig. 3.3

Example 2: *Let A = {a, b, c, d}*

and $$M_R = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Draw the digraph of R.

Solution: R = {(a, a), (a, b), (a, d), (b, b), (b, c), (c, c), (c, d), (d, a)}

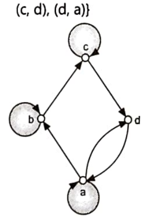


Fig. 3.4

Example 3: *Find the relation determined by the digraph and give its matrix.*

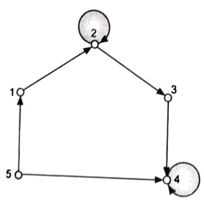


Fig. 3.5

Solution: A = {1, 2, 3, 4, 5}

R = {(1, 2), (2, 2), (2, 3), (3, 4), (4, 4), (5, 1), (5, 4)}

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

EXERCISE – 3.2

1. Let A be the product set {1, 2, 3} × {a, b}. How many relations are there on A?
2. A = {1, 2, 3, 4}, B = {1, 4, 6, 8, 9}; aRb if and only if $b = a^2$. Find the domain, range of R. Also find also its relation matrix and draw its digraph.
3. Let A = **R**, set of real numbers. Consider the following relation on A ; (a, b) ∈ R iff 2a + 3b = 6. Find domain of R and also its range.
4. Let A = {1, 2, 3, 4, 5} and let R = {(1, 1,), (1, 2), (2, 1), (1, 3), (1, 4), (4, 5), (5, 1), (1, 5), (4, 1)}. Draw the digraph of R.
5. For a set A = {1, 2, 3, 4, 5}, the relation matrix is
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$
Draw its digraph.
6. Let

   A = {1, 2, 3, 4}.

   If R = {(a, b) | (a – b) is an integral non-zero multiple of 2}

   and S = {(a, b) | (a – b) is an integral non-zero multiple of 3}

   Find R ∪ S and R ∩ S.
7. For a set

   A = {1, 2, 3, 4, 5} relations R_1 and R_2 are given by

   R_1 = {(1, 2), (3, 4), (2, 2)} and R_2 = {(4, 2), (2, 5), (3, 1), (1, 3)}

   Find (a) $R_1 R_2$, (b) $R_2 R_1$, (c) $R_1 (R_2 R_1)$, (d) $(R_1 R) R_1$, (e) R_1^3, (f) R_2^2.
8. If

   A = B = {1, 2, 3},

   R_1 = {(1, 1), (1, 2), (2, 3), (3, 1)}

   and

   R_2 = {(2, 1), (3, 1), (3, 2), (3, 3)}

   Compute

   (a) Complement of R_1,

   (b) Converse of R_2

   (c) $R_1 \oplus R_2$.
9. Let A = B = {1, 2, 3, 4}, R = {(1, 1), (1, 3), (2, 3), (3, 1), (4, 2), (4, 4)} and

   S = {(1, 2), (2, 3), (3, 1), (3, 2), (4, 3)}

   Compute: (a) $M_{R \cap S}$ (b) $M_{R \cup S}$

   (c) M_{R^c} (d) $M_{\bar{S}}$
10. Let A be set of workers and B be a set of jobs. Let R_1 be a binary relation from A to B such that (a, b) is in R_1 if worker a is assigned to job b. (We assume that a worker might be assigned to more than one job and more than one worker might be assigned to the same job.) Let R_2 be a binary relation on A such that (a_1, a_2) is in R_2 if a_1, a_2 can get along with each other if they are assigned to the same job. State a condition in terms of R_1, R_2 and (possibly) binary relations derived from R_1 and R_2 such that an assignment of the workers to the jobs according to R_1 will not put workers that cannot get along with one another on the same job.

3.7 SPECIAL PROPERTIES OF BINARY RELATIONS

In many applications to computer science, we deal with relations on a set A, rather then relations from A to B. These relations have certain properties which are useful in storing data, more efficiently, on the computer. Let R be a relation on a set A.

1. **Reflexive Relation:** R is reflexive if for **every** element a ∈ A, a $\not{R}$ a i.e. (a, a) ∈ R.

   R is not a reflexive relation if for **some** element a ∈ A, a R a, i.e. (a, a) ∉ R.

Examples:

1. Let A = {a, b} and let R = {(a, a), (a, b), (b, b)}.

   Then R is reflexive.
2. Let A = {1, 2} and let R = {(1, 1), (1, 2)}.

   R is not reflexive since (2, 2) ∉ R.

2. Irreflexive Relation R is said to be **irreflexive** if for every element $a \in A$, $a \not R a$, i.e. $(a, a) \notin R$.

Examples:

1. Let $A = \{1, 2\}$ and let $R = \{(1, 2), (2, 1)\}$. Then R is irreflexive since $(1, 1), (2, 2) \notin R$.
2. Let $A = \{1, 2\}$ and let $R = \{(1, 2), (2, 2)\}$. Then R is not irreflexive since $(2, 2) \in R$.

Note that R is not reflexive either; since $(1, 1) \notin R$.

If R is reflexive, the corresponding relation matrix M_R will have its diagonal entries as one. If R is irreflexive, the diagonal elements will be zeros.

Digraph of a reflexive relation.

Fig. 3.6

3. Symmetric Relation: R is said to be **Symmetric** if whenever a R b, then b R a. It then follows that, R is not symmetric if for some a and $b \in A$, a R b but $b \not R a$.

The relation matrix corresponding to a symmetric relation is a symmetric matrix. If its ij-th entry is 1, its ji^{th} entry is also 1. If its ji^{th} entry is 0, its ij-th entry is also 0 (for $i \neq j$).

Examples:

1. Let A be set of people. Let a R b if a is a friend of b. Then obviously b is related to a. Hence the relation of being "friend" is a symmetric relation.
2. Let A be set of lines in a plane. For lines $l_1, l_2 \in A$, let $l_1 R l_2$ if l_1 is parallel to l_2. Then $l_2 R l_1$ since the relation of being "parallel to" is a symmetric relation.
3. Let A be set of people and let a R b if a is brother of b. Then this is not a symmetric relation since b can be the sister of a. This relation will be symmetric only if A is the set of males.
4. Let $A = \{1, 2\}$ and let $R = \{(1, 1), (2, 2)\}$. This is an example of a symmetric relation which is also reflexive.

Digraph of a Symmetric Relation:

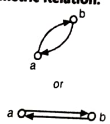


Fig. 3.7

4. Asymmetric relation R is said to be **asymmetric** if whenever a R b, then $b \not R a$. Hence R is not asymmetric if for some a and $b \in A$, we have both a R b and b R a.

Examples:

1. Let A = **R** the set of real numbers and let R be the relation '<'. Then $a < b \rightarrow b < a$. Hence < is asymmetric.
2. Let $A = \{2, 4, 5\}$ and let R be the relation "is a divisor of".

   Then $R = \{(2, 2), (2, 4), (4, 4), (5, 5)\}$.

   R is not asymmetric since (2, 2) (also (4, 4), (5, 5)) $\in R$.

5. Antisymmetric Relation R is antisymmetric if whenever a R b and b R a then a = b. It follows that R is not antisymmetric if we have elements $a, b \in A$ such that $a \neq b$ but both a R b and b R a.

An equivalent definition of antisymmetric relation R is: If $a \neq b$, then either $a \not R b$ or $b \not R a$.

This definition is sometimes useful to verify whether a given relation is antisymmetric.

Examples:

1. Let A = **R** and let R be the relation ' ≤ '. Then $a \le b$ and $b \le a \rightarrow a = b$. Hence ' ≤ ' is an antisymmetric relation.
2. Let $A = \{1, 2, 3\}$ and let $R = \{(1, 2), (2, 1), (2, 3)\}$.

   R is not antisymmetric since (1, 2) and (2, 1) $\in R$.

   R is not symmetric either since $(2, 3) \in R$ but $(3, 2) \notin R$.

   R is also not asymmetric since both (1, 2) and (2, 1) $\in R$.

6. Transitive Relation R is said to be **transitive** if whenever a R b and b R c, then a R c. It follows that a relation R is not transitive if there exist elements $a, b, c \in A$ such that a R b and b R c, but $a \not R c$. If such elements a, b, c do not exist, then R is transitive.

Examples:

1. Let A = **R** and let R be the relation '≤'. Then R is clearly transitive.
2. Let A = set of triangles and let R be the relation of being congruent. Then for triangles $a, b, c \in A$, a R b and b R c → a R c. Hence R is transitive.
3. Let A be set of people and let R be the relation of being "brother of". Then a is brother of b and b is brother of c implies a is brother of c. Hence R is transitive.

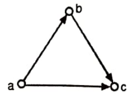
4. Let A = N the set of natural numbers, and let

$$R = \left\{ (a, b)\ a, b \in N \,\middle|\, a + b \text{ is an odd number} \right\}$$

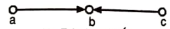
Then R is not transitive since (1, 2) and $(2, 1) \in R$, but $(1, 1) \notin R$.

Digraphs of Transitive Relation.

1.



2.



(a R b but $b \not R c$)

Fig. 3.8

3.8 EQUIVALENCE RELATION

A binary relation R on a set A is called as equivalence relation if it is reflexive, symmetric and transitive.

The following are some of the common but important examples of equivalence relations.

Examples:

1. Let A = **R** and R be 'equality' of numbers.
2. Consider all subsets of a universal set and R be the relation, "equality" of sets.
3. A is the set of triangles and R is 'similarity' of triangles.
4. A is a set of students and R is the relation of being in "the same class or division."
5. Let A be set of statement forms and R be the relation of "logical equivalence".
6. A is set of lines in a plane and R is the relation of lines being "parallel."

The digraph of an equivalence relation will have the following characteristics. Every vertex will have a loop; if there is an arc from a to b, there should be an arc from b to a; if there is an arc from a to b and one from b to c, there should be an arc from a to c. In short, the following is a typical digraph of an equivalence relation.

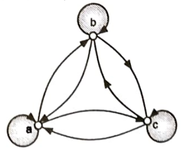


Fig. 3.9

SOLVED EXAMPLES

Example 1 : *Let $A = \{a, b, c, d\}$, $R = \{(a, a), (b, a), (b, b), (c, c), (d, d), (d, c)\}$.*

Determine whether R is an equivalence relation.

Solution:

R is reflexive since (a, a), (b, b), (c, c) and $(d, d) \in R$.

But R is not symmetric since $(b, a) \in R$ but $(a, b) \notin R$. Hence R is not an equivalence relation.

Example 2 : *Let $A = \{a, b, c\}$ and let*

$$M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$. Determine whether R is an equivalence relation.

Solution: $R = \{(a, a), (b, b), (b, c), (c, b), (c, c)\}$.

R is reflexive since $(a, a), (b, b), (c, c) \in R$.

R is symmetric since $(b, c) \in R \rightarrow (c, b) \in R$,

R is transitive since

(b, b) and $(b, c) \in R$ implies $(b, c) \in R$,

(b, c) and $(c, b) \in R$ implies $(b, b) \in R$,

(c, c) and $(c, b) \in R$ implies $(c, b) \in R$,

(c, b) and $(b, b) \in R$ implies $(c, b) \in R$

(c, b) and $(b, c) \in R$ implies $(c, c) \in R$

(b, c) and $(c, c) \in R$ implies $(b, c) \in R$.

Hence R is an equivalence relation.

Example 3. *Determine whether the relation R whose digraph is given below is an equivalence relation.*

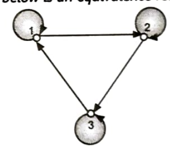


Fig. 3.10

Solution: R is reflexive since there is a loop around each vertex. But R is not symmetric, since $(1, 2) \in R$ but $(2, 1) \notin R$. Hence R is not an equivalence relation.

Example 4 : *Determine whether the relation R whose digraph is given below is an equivalence relation.*

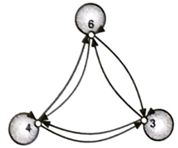


Fig. 3.11

Solution: R is reflexive since each vertex has a loop. R is also symmetric since every pair of distinct vertices is connected by a pair of double arcs, each pointing in the opposite direction.

In fact R = {(3, 3), (3, 4), (3, 6), (4, 3), (4, 4), (4, 6), (6, 3), (6, 4), (6, 6)}

Hence it is clear that R is also transitive. Therefore R is an equivalence relation.

3.8.1 Some Important Properties of Equivalence Relations

1. *If R_1 and R_2 are equivalence relations on a set A, then* $R_1 \cap R_2$ is an equivalence relation.

Proof:

For each $a \in A$, $(a, a) \in R_1$ and $(a, a) \in R_2$; hence $(a, a) \in R_1 \cap R_2$.

Therefore $R_1 \cap R_2$ is reflexive.

Let $(a, b) \in R_1 \cap R_2$; then $(a, b) \in R_1$ and $(a, b) \in R_2$.

Since R_1 and R_2 are both symmetric this implies $(b, a) \in R_1$ and $(b, a) \in R_2$, so that $(b, a) \in R_1 \cap R_2$.

Hence $R_1 \cap R_2$ is symmetric. Let (a, b), and $(b, c) \in R_1 \cap R_2$.

Then $(a, b), (b, c) \in R_1$ and $(a, b), (b, c) \in R_2$. But R_1 and R_2 are transitive; therefore $(a, c) \in R_1$ and $(a, c) \in R_2$, so that $(a, c) \in R_1 \cap R_2$.

Hence $R_1 \cap R_2$ is transitive. This proves that $R_1 \cap R_2$ is an equivalence relation.

2. If R_1 and R_2 are equivalence relations, it is not necessary that $R_1 \cup R_2$ is also an equivalence relation.

Proof:

Counter-example:

Let A = {a, b, c}

R_1 = {(a, a), (b, b), (c, c), (a, b), (b, a)}

and R_2 = {(a, a), (b, b), (c, c), (a, c), (c, a)}

Both R_1 and R_2 are equivalence relations.

$R_1 \cup R_2$ = {(a, a), (b, b), (c, c), (a, b), (b, a), (a, c), (c, a)}

is not an equivalence relation since it is not transitive (b, a) and (a, c) $\in R_1 \cup R_2$ but $(b, c) \notin R_1 \cup R_2$.

In general, if a relation R is not transitive, we can find a relation containing R which is transitive and is the smallest set with this property.

This set is called as the **transitive closure** of R.

3.8.2 Equivalence Classes

Let R be an equivalence relation on a set A. For every a ∈ A, let $[a]_R$ denote the set $\{x \in A \mid x R a\}$. Then $[a]_R$ is called as the **equivalence class of a with respect to R.**

$[a]_R \neq \phi$ since $a \in [a]_R$.

The **rank** of R is the number of distinct equivalence classes of R if the number of classes is finite; otherwise the rank is said to be infinite.

In what follows, we will drop the suffix R, and denote the equivalence class of a simply as [a].

The following theorem gives an important characterisation of the equivalence classes.

(i) For all a, b ∈ A, either [a] = [b] or $[a] \cap [b] = \phi$.

Proof:

If $A = \phi$, there is nothing to prove.

Hence assume $A \neq \phi$. If A = {a}, singleton set, the result is trivially true.

Therefore consider elements a, b ∈ A.

Suppose $[a] \neq [b]$. Then we have to show that $[a] \cap [b] = \phi$.

Suppose this is not true.

Let $c \in [a] \cap [b]$ then c R a and c R b.

Since R is symmetric it follows that a R c and c R b. But R is transitive as well.

Hence we have a R b, i.e. b ∈ [a] and a ∈ [b], which means that [a] = [b], a contradiction. Hence $[a] \cap [b] = \phi$.

(ii) $A = \bigcup_{a \in A} [a]$

a ∈ A

Proof:

If $A = \phi$, there is nothing to prove.

Hence assume $A \neq \phi$. If A = {a}, singleton set, the result is trivially true.

Therefore consider elements a, b ∈ A.

Suppose $[a] \neq [b]$. Then we have to show that $[a] \cap [b] = \phi$.

Suppose this is not true.

Let $c \in [a] \cap [b]$ then c R a and c R b.

Since R is symmetric it follows that a R c and c R b.

But R is transitive as well.

Hence we have a R b, i.e. b ∈ [a] and a ∈ [b], which means that [a] = [b], a contradiction. Hence $[a] \cap [b] = \phi$.

SOLVED EXAMPLES

Example 1 : *Let*

A = {a, b, c} and let

R = {(a, a), (b, b), (c, c), (a, b), (b, a)}

where, R is clearly an equivalence relation.

The equivalence classes of the elements of A are:

[a] = {a, b}

[b] = {b, a} = [a]

[c] = {c}.

The rank of R is 2.

Example 2 : *Let A = {1, 2, 3, 4} and let R = {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1), (2, 3), (3, 2), (3, 3), (4, 4)}. Show that R is an equivalence relation and determine the equivalence classes, and hence find the rank of R.*

Solution: R is reflexive since (1, 1), (2, 2), (3, 3), (4, 4) ∈ R.

R is symmetric since both (1, 2), (2, 1) ∈ R.

Similarly (2, 3), (3, 2), (1, 3), (3, 1) ∈ R.

R is transitive since (1, 2) and (2, 1) ∈ R implies (1, 1) ∈ R.

Similarly

(1, 3), (3, 1) ∈ R → (1, 1) ∈ R,

(2, 3), (3, 2) ∈ R → (2, 2) ∈ R

(3, 1), (1, 3) ∈ R → (3, 3) ∈ R

(3, 2), (2, 1) ∈ R → (3, 1) ∈ R.

Hence R is an equivalence relation. The equivalence classes of A are:

[1] = {1, 2, 3}

[2] = {1, 2, 3} = [1]

[3] = {3, 1, 2} = [1]

[4] = {4}.

Hence there two distinct equivalence classes. Hence rank of R is 2.

The following is an important example of an equivalence relation and equivalence classes.

Example 3 : *(Residue classes modulo a positive integer)*

Let Z denote the set of integers. Let n be a positive integer and define a relation R on Z by setting a R b iff $n \mid (a - b)$. *Show that R is an equivalence relation and determine its equivalence classes.*

Solution: R is reflexive since $n \mid (a - a)$ i.e. n divides zero.

R is symmetric, since $a R b \to n \mid a - b$ which implies $n \mid (b - a)$, i.e. b R a. Let a R b and b R c. Then $n \mid (a - b)$ and $n \mid (b - c)$ which implies $n \mid [(a - b) + (b - c)]$ $\to n \mid (a - c)$, i.e. a R c. Hence R is transitive. R is therefore an equivalence relation.

The equivalence classes are [0], [1], [2], ..., [n − 1]. This is because [n] = [0], [n + 1] = [1] and so on. Also note that for any integer m, [−m] = [m].

We denote the set of these equivalence classes by Z_n.

Z_1 = {[0]} = {... − 1, 0, 1, 2 ... } = Z

Z_2 = {[0], [1]}

Z_3 = {[0], [1], [2]}

Z_4 = {[0], [1], [2], [3]} and so on.

The relation R is known as the congruence relation.

3.8.3 Partitions

We shall now discuss the concept of partition which is closely related to that of equivalence relation.

Definition:

A partition of a non-empty set A is a collection of sets {A_1, A_2, ..., A_n} such that

(i) $A = \bigcup_{i=1}^{n} A_i$

(ii) $A_i \cap A_j = \phi$, for $i \neq j$ (i.e. the sets A_i are mutually disjoint).

We denote a partition of A by the symbol π. An element of a partition is called a **block.** The **rank** of π is the number of blocks of π.

For a given non-empty set, its partition is not unique; we can have different partitions of the same set.

Examples :

1. Let A = {1, 2, 3}

Then π_1 = {{1, 2}, {3}} is a partition of A.

Similarly, π_2 = {{1, 3}, {2}} is another partition of A. A third partition is

π_3 = {{1}, {2}, {3}} and so on.

2. Let Z = set of all integers,
   E = set of all even integers,
   O = set of all odd integers.

   Then {E, O} is a partition of Z.
3. The rooms (flats) in a building block form a partition.
4. The main memory of a multi-programmed computer system is partitioned and a separate program is stored in each block of the partition.

The following diagram represents a partition of a set.

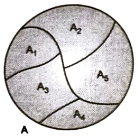


Fig. 3.12

The following theorem establishes the fact that equivalence relations and partitions are different descriptions of the same concept.

> **Theorem 1: Let A be a non-empty set and R an equivalence relation on A, then the set of equivalence classes $\left\{ [a]_R \mid a \in A \right\}$ constitutes a partition of A.**
>
> **Proof:**
>
> This theorem is actually a corollary to Theorem 1 of Art 3.9.2 in which we have shown that
>
> (i) $A = \bigcup_{a \in A} [a]$
>
> (ii) $[a] \cap [b] = \phi$ if $[a] \neq [b]$.
>
> The above conditions are the same as that at required for a partition of A. Thus the theorem is proved.

In the above theorem, we have shown that an equivalence relation induces a partition on A. The converse is also true, as proved in the following theorem.

> **Theorem 2:** Let A be a non-empty set, and let π be a partition of A. Then π induces an equivalence relation on A.
>
> **Proof:**
>
> Let π be a partition of the set A, and define the binary relation R on A as a R b iff there exists a set A_i in π such that a, b $\in A_i$. We shall show that R is an equivalence relation.
>
> (i) R is reflexive since a R a. This is because π being a partition of A, a $\in A_i$ for some i.
>
> (ii) R is symmetric by definition of R.
>
> (iii) R is transitive. Let a R b and b R c.
>
> This implies that for some i, a, b $\in A_i$, and for some j, b, c $\in A_j$.
>
> Therefore b $\in A_i \cap A_j$.
>
> But since π is a partition, this is possible only if $A_i = A_j$.
>
> Hence a, c $\in A_i$ which means that a R c. This proves that R is transitive.

The above theorems, thus, establish a natural correspondence between partition of a set and equivalence relation on the set.

SOLVED EXAMPLES

Example 1 : *Let A = {a, b, c, d}, π = {{a, b}, {c}, {d}}. Find the equivalence relation induced by π and construct is digraph.*

Solution: R = {(a, a), (a, b), (b, a), (b, b), (c, c), (d, d)}.

The digraph of R is:

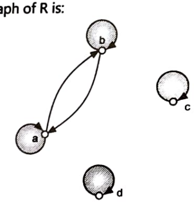


Fig. 3.13

Example 2 : *Let A = {1, 2, 3, 5} and π = {{1, 2}, {3}, {4, 5}}. Find the equivalence relation determined by π and draw its digraph.*

Solution: R = {(1, 1), (1 2), (2, 1), (2, 2), (3, 3), (4, 4), (4, 5), (5, 4), (5, 5)}.

Digraph of R

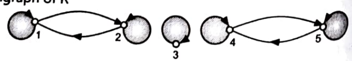


Fig. 3.14

Let R be an equivalence relation on A. We denote by A/R the partition induced by R. Hence a partition of A is called as a **quotient** set of A.

Example 3: *Let A = {1, 2, 3} and let R = {(1, 1), (2, 2), (1, 3), (3, 1), (3, 3)}. Find A/R.*

Solution: A/R is the partition of A induced by R.

Hence A/R = {{1, 3}, {2}}.

Example 4 : *Let Z be the set of integers. Define a relation R on Z as a R b iff $6 \mid (a - b)$, show that R is an equivalence relation and find Z/R.*

Solution: Since $6 \mid (a - a)$, a R a. Hence R is reflexive.

If $6 \mid (a - b)$, then $6 \mid (b - a)$, which shows that R is symmetric.

If $6 \mid (a - b)$ and $6 \mid (b - c)$ then obviously $6 \mid ((a - b) + (b - c))$, i.e. $6 \mid (a - c)$. Hence R is also transitive. R is therefore an equivalence relation.

Z/R = {[0], [1], [2], [3], [4], [5]}

where [0] = {... – 12, – 6, 0, 6, 12, 18 ... }

[1] = {, – 11, – 5, 1, 7, 13, ...}

[2] = {, ... – 10, – 4, 2, 8, 14, ...}

[3] = {... , ... , – 3, 3, 9, 15, ...}

[4] = {..., ... , – 2, 4, 10, 16, ...}

[5] = {..., – 7, –1, 5, 11, 17, ...}

The quotient set Z/R is denoted by Z_6 and is called as the set of congruence classes modulo 6. R is also called as a congruence relation.

Example 5 : *Let X = {a, b, c, d, e} and C = {{a, b}, {c}, {d, e}}. Show that the partition C defines an equivalence relation on X.*

Solution: C induces a relation R on X, as follows: For elements x, y$\in$X, xRy if and only if x and y are members of the same subset element of C.

For example, aRb since both a, b$\in${a, b} in C. R is reflexive since aRa, bRb, cRc, dRd, eRe. Also R is symmetric since aRb and bRa. R is also transitive since aRb, bRa, aRa, bRb, dRe, eRd, dRd, eRe. Hence R is an equivalence relation.

3.8.4 Refinement of Partition

Definition:

> Let π and π' be partitions of a non-empty set A. Then π' is called a refinement of π if every block (element) of π' is contained in a block of π.

Example: Let A = {a, b, c}.

Let π = {{a}, {b, c}} and π' = {{a}, {b}, {c}}.

Then π' is a refinement of π.

Let R and R' be the equivalence relations induced by π and π' respectively. Then the following theorem relates R' and R.

> **Theorem 1:** Let π and π' be partitions of a non-empty set A and let R, R' be the equivalence relations induced by π and π' respectively. Then π' refines π if and only if R' $\subseteq$ R.
>
> **Proof:**
>
> Let π' be a refinement of π.
>
> We have to prove that R' $\subseteq$ R. Let a R' b.
>
> Then there is some block $A_i' \in \pi'$ such that a, b $\in A_i'$.
>
> Since π' refines π, $A_i' \subseteq A_i$ for some block $A_i \in \pi$. Hence a, b $\in A_i$ which implies a R b.
>
> Hence R' $\subseteq$ R.
>
> Next, let R' $\subseteq$ R. We have to prove π' is a refinement of π.
>
> Let $A_i' \in \pi'$ and let a $\in A_i'$.
>
> Then $A_i' = [a]_{R'}$. Let x $\in A_i'$. This implies x R' a and hence x R a since R' $\subseteq$ R.
>
> This means that $[a]_{R'} \subseteq [a]_R$. Denote by A_i the block $[a]_R$.
>
> Then $A_i' \subseteq A_i$ which means that π' is a refinement of π

Remark on Product and Sum of Partitions:

The product $\pi_1 \cdot \pi_2$ consists of the set of intersections of every element of π_1 with every element of π_2, omitting the empty intersections.

For example,

if S = {a, b, c, d, e, f}

π_1 = {{a, b}, {c, d, e}, {f}},

π_2 = {{a, b, c}, {d}, {e, f}}

then $\pi_1 \cdot \pi_2$ = {{a, b} {c}, {d}, {e}, {f}}

Let $\pi(S)$ denote the set of all partitions of S. Then a subset T of S is in the sum $\pi_1 + \pi_2$ if

(i) T is the union of one or more elements of π_1.

(ii) T is the union of one or more elements of π_2.

(iii) No subset of T satisfies (i) and (ii) except T itself.

For example, if $\pi_1 = \{\{a, b\}, \{c\}, \{d, e, f\}\}$

$\pi_2 = \{\{a, b, c\}, \{d\}, \{e, f\}\}$

then $\pi_1 + \pi_2 = \{\{a, b, c\}, \{d, e, f\}\}$

3.8.5 Sums and Products of Partitions

Definition:

Let π_1 and π_2 be partitions of a non-empty set A. The product of π_1 and π_2, denoted $\pi_1 \cdot \pi_2$ is a partition π of A such that

(i) π refines both π_1 and π_2.

(ii) If π' refines both π_1 and π_2, then π' refines π.

The following theorem tells us how to determine the product $\pi_1 \cdot \pi_2$.

Theorem 1: Let R_1, R_2 be the equivalence relations induced by partitions π_1 and π_2 respectively. Then the relation $R = R_1 \cap R_2$ induces the product partition $\pi_1 \cdot \pi_2$.

Proof:

By property 1 (article 3.9.1) we know that $R_1 \cap R_2$ is an equivalence relation.

Let π be the partition induced by $R_1 \cap R_2$.

Then by theorem 1 (3.9.4) π refines both π_1 and π_2.

Next suppose π' refines both, π_1 and π_2.

If π' induces R', then by theorem 1 (3.9.4) $R' \subseteq R_1$ and $R' \subseteq R_2$ which implies

$R' \subseteq R_1 \cap R_2$.

Hence again by theorem 1 (3.9.4) π' refines π.

We now deal with sum of two partitions.

Definition:

Let π_1 and π_2 be partitions of a non-empty set. The sum of π_1 and π_2, denoted by $\pi_1 + \pi_2$, is a partition π such that

(i) Both π_1 and π_2 refine π.

(ii) If π' is a partition of A such that both π_1 and π_2 refine π', then π refines π'.

We have the following theorem.

Theorem 2: Let R_1 and R_2 be equivalence relations on a non-empty set A induced by the partitions π_1 and π_2. Let $R = (R_1 \cup R_2)^*$ the transitive closure of $R_1 \cup R_2$. Then R is an equivalence relation on A and induces the partition $\pi_1 + \pi_2$.

Proof:

$R_1 \cup R_2$ is reflexive and symmetric.

Hence $(R_1 \cup R_2)^*$ be also transitive is the smallest equivalence relation containing both R_1 and R_2.

Since $R_1 \subseteq R$ and $R_2 \subseteq R$ both π_1 and π_2 induce an equivalence relation which contains both R_1 and R_2.

Since $(R_1 \cup R_2)^*$ is the smallest such equivalence relation, it follows that π refines all such partitions.

Hence π is a sum of π_1 and π_2.

Note that the product and sum of two partitions always exists and is unique.

Example :

1. Let $A = \{a, b, c, d, e, f, g, h, i, j, k\}$

$\pi_1 = \{ \{a, b, c, d\}, \{e, f, g\}, \{h, i\}, \{j, k\} \}$

and $\pi_2 = \{ \{a, b, c, h\}, \{d, i\}, \{e, f, j, k\}, \{g\} \}$

be two partitions of A. Find $\pi_1 \cdot \pi_2$ and $\pi_1 + \pi_2$.

Solution: $\pi_1 \cdot \pi_2 = \{ \{a, b, c\}, \{d\}, \{e, f\}, \{g\}, \{h\}, \{i\}, \{j, k\} \}$

$\pi_1 + \pi_2 = \{ \{a, b, c, d, h, i\}, \{e, f, g, j, k\} \}$

3.9 COMPATIBLE RELATION

Definition:

A relation R on a set A is said to be compatible if it is reflexive and symmetric.

Examples:

1. All equivalence relations are compatible relations.
2. The relation of 'being friend of' is a compatible relation.
3. Let, $A = \{a, b, c\}$

$B = \{b, c, d\}$

$C = \{d, e, f\}$.

Define a relation R on these sets as: A set X is related to a set Y, i.e. X R Y iff $X \cap Y \neq \phi$.

In this example A R B, B R C, but A R̸ C. R is therefore a compatible relation which is not an equivalence relation. The graph of a compatible relation is drawn by omitting the loop at each vertex and using a single edge with no arrow between them. If R is a compatible relation, its relation matrix is symmetric with the diagonal elements being 1.

Definition:

A **covering** of a set A is a collection of subsets $\{A_1, A_2, \ldots, A_k\}$ of A, such that $\bigcup_{i=1}^{k} A_i = A$.

We have seen that an equivalence relation induces a partition on A. We shall now show that a compatible relation induces a covering on A.

Definition:

Let A be a non-empty set and let R be a compatible relation on A. A subset M of A is called a **maximal compatibility block** if every element of M is compatible with every other element of M and no element of A – M is compatible with all elements of M.

For example, consider the diagram given below for the compatible relation R on the set A = {1, 2, 3, 4, 5}.

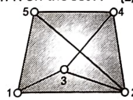


Fig. 3.15

The maximal compatibility blocks are

$M_1 = \{1, 2, 3\}$, $M_2 = \{2, 3, 4\}$, $M_3 = \{1, 2, 5\}$, $M_4 = \{2, 4, 5\}$.

$A = M_1 \cup M_2 \cup M_3$. Hence $\{M_1, M_2, M_3\}$ forms a covering for A.

Similarly, $\{M_1, M_4\}$ forms a covering for A.

3.10 TRANSITIVE CLOSURE

Definition:

The transitive closure of a relation R is the smallest transitive relation containing R. We denote transitive closure of R by R^*.

The following theorem gives a method to find the transitive closure.

Theorem: Let A be a set with |A| = n, and let R be a relation on A. Then

$$R^* = R \cup R^2 \cup R^3 \cup \ldots \cup R^n.$$

Proof:

Let $(a, b) \in R^*$. Then since R^* is the transitive closure of R, there exists a sequence of elements $x_1, x_2, \ldots, x_R$ in A such that $a = x_1$, $b = x_R$ and $(x_i, x_{i+1}) \in R$ for $1 \le i \le R - 1$.

This means that $(a, b) \in R^{k-1}$. Hence, $R^* \subseteq R \cup R^2 \cup R^3 \cup \ldots \cup R^n$.

Conversely $R^i \subseteq R^*$ for $1 \le i \le n$. Hence $R^* = R \cup R^2 \cup R^3 \cup \ldots \cup R^n$.

Now we have only to show that R^* is the smallest transitive relation, containing R.

Let S be a transitive relation containing R. Let $(a, c) \in R^2$.

This implies that there exists an element $b \in A$ such that $(a, b) \in R$ and $(b, c) \in R$.

Since $R \subseteq S$ and S is transitive it follows that $(a, c) \in S$.

Hence $R^2 \subseteq S$. Proceeding thus, we can show that $R^3, R^4, \ldots, R^n$ are all subsets of S.

Hence $R \cup R^2 \cup \ldots \cup R^n \subseteq S$, which proves the claim.

SOLVED EXAMPLES

Example 1 : *Let, A = {1, 2, 3, 4} and R = {(1, 2), (2, 3), (3, 4)} be a relation on A. Find R^* and draw its digraph.*

Solution: $R = \{(1, 2), (2, 3), (3, 4)\}$

$R^2 = \{(1, 3), (2, 4)\}$

$R^3 = \{(1, 4)\}$

$R^4 = \phi$

Hence $R^* = R \cup R^2 \cup R^3$

$= \{(1, 2), (2, 3), (3, 4), (1, 3), (2, 4), (1, 4)\}$

Digraph of R^*

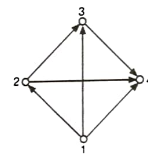


Fig. 3.16

Example 2 : *Let,*

A = {a, b, c, d},

R_1 = {(a, a), (b, b), (c, c), (a, b)} and

R_2 = {(a, a), (b, d), (d, c)}

Find $(R_1 \cup R_2)^*$ and draw its digraph.

Solution: $R = R_1 \cup R_2 = \{(a, a), (b, b), (c, c), (a, b), (b, d), (d, c)\}$

$R^2 = \{(a, b), (b, d), (a, d), (b, c), (a, a), (b, b), (c, c) (d, c)\}$

$R^3 = \{(a, a), (a, b), (a, c), (a, d), (b, b), (b, c), (b, d), (d, c), (c, c)\}$

$R^4 = \{(a, a), (a, b), (a, c), (a, d), (b, b), (b, c), (b, d), (c, c), (d, c)\}$

$\therefore \quad R^* = R \cup R^2 \cup R^3$

$= \{(a, a), (b, b), (c, c), (a, b), (b, d), (d, c), (a, d), (b, c), (a, c)\}$

Digraph of R^*

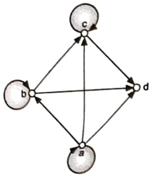


Fig. 3.17

3.10.1 Warshall's Algorithm

Finding the transitive closure of a relation, by computing various powers of R or products of the relation matrix M_R, is quite impractical for large sets and relations. Warshall's Algorithm offers an alternative but efficient method for computing the transitive closure.

Let R be a relation on a set $A = \{a_1, a_2, \ldots a_n\}$ and let R^* denote the transitive closure of R. A **path** of length m in R from a to b is a finite sequence a, x_1, x_2, ..., x_{m-1}, b, beginning with a and ending with b, such that a R x_1, x_1 R x_2, ... , x_{m-1} R b. Note that a path of length m involves m + 1 elements of A, not necessarily distinct. All vertices in the path, except a and b are called as **interior vertices** of the path. For $1 \le k \le n$, define a Boolean matrix $\mathbf{W_k}$ as $\mathbf{W_R}$ has 1 in position i, j if and only if there is a path from a_i to a_j in R whose interior vertices, if any, come from the set $\{a_1, a_2, \ldots, a_k\}$.

Suppose we have a path as shown in the diagram below,

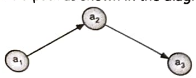


Fig. 3.18

then W_2 will have a 1 in the first row and third column.

Since any vertex must come from the set $\{a_1, a_2, \ldots a_n\}$, it follows that the matrix W_n has 1 in position i, j if and only if some path in R connects a_i and a_j. Hence $W_n = M_{R^*}$. Define $W_o = M_R$. Then we will have a sequence $\mathbf{W_o}$, $\mathbf{W_1}$, ..., $\mathbf{W_n}$ whose first term is M_R and last term is M_{R^*}. Suppose we have a path as shown in the diagram below,

Warshall's algorithm gives a procedure to compute each matrix $\mathbf{W_R}$ from the previous matrix W_{R-1}. Beginning with the matrix of R, we proceed one step at a time, until we reach the matrix of R^*, in n steps. The matrices W_R, being different from powers of the matrix M_R, results in a considerable saving of steps in the computation of the transitive closure of R.

Suppose $W_{k-1} = [u_{ij}]$ and $W_k = [v_{ij}]$. If $v_{ij} = 1$, there is a path from a_i to a_j whose interior vertices come from the set $\{a_1, a_2, \ldots a_k\}$. If a_k is not an interior vertex of this path, then all the interior vertices must come actually from $\{a_1, a_2, \ldots, a_{k-1}\}$, hence $u_{ij} = 1$. If a_k is an interior vertex of the path, then we have the situation as shown below.

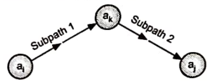


Fig. 3.19

Since there is a sub path from a_i to a_k whose interior vertices come from $\{a_1, a_2, \ldots, a_{k-1}\}$, we must have $u_{ik} = 1$. Similarly $u_{kj} = 1$.

Hence, $v_{ij} = 1$ if and only if

(1) $u_{ij} = 1$

(2) $u_{ik} = 1$ and $u_{kj} = 1$.

This is the basis for Warshall's algorithm. If $\mathbf{W_{k-1}}$ has 1 in position i, j then by (1) so will $\mathbf{W_k}$. A new 1 can be added in position i, j of $\mathbf{W_k}$ if and only if column k of W_{k-1} has a 1 in position i, and row k of W_{k-1} has a 1 in position j.

Thus we have the following procedure for computing $\mathbf{W_k}$ from W_{k-1}.

Step 1: Transfer to W_k, all the 1's in W_{k-1}.

Step 2: List the locations p_1, p_2, ... in column k of W_{k-1} where the entry is 1, and the locations q_1, q_2, ... in row k of W_{k-1}, where the entry is 1.

Step 3: Put 1's in all the positions p_i, q_j of $\mathbf{W_R}$ (if they are not already there).

The above procedure is illustrated in the following problems.

SOLVED EXAMPLES

Examples 1: *Let, A = {1, 2, 3, 4} and R = {(1, 2), (2, 4), (1, 3), (3, 2)}.*

Find the transitive closure of R by Warshall's algorithm.

Solution:
$$W_0 = M_R = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For W_1, k = 1, $a_k = 1$ is not an interior vertex for any path in R. Hence $W_1 = W_0$, without any addition to the entries.

For W_2, k = 2; $a_k = 2$ is an interior vertex for the path from 3 to 4. It is also an interior vertex for the path from 1 to 4. Hence W_2 has 1 in the position (3, 4) and also 1 in the position (1, 4).

Hence
$$W_2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For W_3, k = 3, a_k =3. Although 3 is an interior vertex for the path from 1 to 2, since the entry 1 is already in the position (1, 2), there is no new addition to the entries in W_2. Hence $W_3 = W_2$.

For W_4, k = 4, $a_k = 4$ which is not an interior vertex of any path in R. Hence $W_4 = W_3 = W_2$.

But $M_{R^*} = W_4$

Hence
$$M_{R^*} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, we obtain the transitive closure R^* as {(1, 2), (1, 3), (1, 4), (2, 4), (3, 2), (3, 4)}.

Example 2: *Let, A = $\{a_1, a_2, a_3, a_4, a_5\}$ and let R be a relation on A whose matrix is*

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Find M_{R^*} by Warshall's algorithm.

Solution: $R = \{(a_1, a_1), (a_1, a_4), (a_2, a_2), (a_3, a_4), (a_3, a_5), (a_4, a_1), (a_5, a_2), (a_5, a_5)\}$

$W_0 = M_R$

For k = 1, a_1 is an interior vertex for the path a_4 to a_4. Hence W_1 has 1 in the position (4, 4).

$$W_1 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$W_2 = W_1$ as there is already 1 in the position (5, 2). a_3 is not an interior vertex for any path in R. Hence $W_3 = W_2 = W_1$.

a_4 is an interior vertex for the path a_3 to a_1. Hence W_4 has 1 in the position (3, 1).

$$\therefore \quad W_4 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

a_5 is an interior vertex in the path from a_3 to a_2. Hence

$$W_5 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$W_5 = M_{R^*}$

Hence
$$M_{R^*} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

The algorithm for Warshall's method is given below:

Example 3: *Find the transitive closure of R by Warshall's algorithm,*

where A = {1, 2, 3, 4, 5, 6} and R = {(x, y) / |x − y| = 2}.

Solution: R = {(1, 3), (3, 1), (2, 4), (4, 2), (3, 5), (5, 3), (4, 6), (6, 4)}

$$M_R = W_0 = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

For k = 1, 1 is in the 3^{rd} row and 3^{rd} and 5^{th} columns.

Hence, put 1 in the positions (3, 3) and (3, 5).

$$\therefore \quad W_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

For k = 2, 1 is in 4^{th} row and 4^{th} column.

Hence, put 1 in the position (4, 4).

$$\text{Thus,} \quad W_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Proceeding in this manner.

$$W_3 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$W_4 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$W_5 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$M_R = W_6 = W_5$

Hence, $R^* = \{(1, 1), (1, 3), (1, 5), (2, 2), (2, 4), (2, 6), (3, 1), (3, 3), (3, 5), (4, 2), (4, 4) (4, 6), (5, 1), (5, 3), (5, 5), (6, 2), (6, 4), (6, 6)\}$

Warshall Algorithm

```
for i: = 1 to n do
    for j: = 1 to n do
        if a [j, i] = 1 then
            for k: = 1 to n do
                if a [i, k] = 1 then
                    a [j, k]: = 1;
                end;
                        end;
                end;
        end;
end;
```

Example 4 : *Let, A = {1, 2, 3, 4} and let R = {(1, 1), (1, 2), (1, 4), (2, 4), (3, 1), (3, 2), (4, 2), (4, 3), (4, 4)}. Find Transitive closure of R, using Warshall Algorithm.* ***(Nov./Dec. 14)***

$$\textbf{Solution :}\ M_R = W_0 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Fro k = 1, 1 is in the 1^{st} and 3^{rd} row, 1^{st}, 2^{nd} and 4^{th} columns. Hence, put 1 in the additional position (3, 4) to obtain

$$W_1 = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\text{Similarly, } W_2 = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$$W_3 = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Transitive closure is {(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)}

3.11 N-ARY RELATION AND ITS APPLICATIONS

Recall the definition of n-ary relation.

Definition : Let $A_1, A_2, \ldots, A_n$ be sets. An n-ary relation R on $A_1 \times A_2 \times \ldots \times A_n$ is a subset of $A_1 \times A_2 \times \ldots \times A_n$. The sets A_i $(1 \le i \le n)$ are called as the **domains** of the relation.

As an example, let us consider the following :

At a certain college, let

A_1 = The set of course numbers for courses offered in Computer Engineering

A_2 = The set of course titles offered in Computer Engineering

A_3 = The set of Computer Engineering faculty

A_4 = The set of division named alphabetically

Then any subset $R \subseteq A_1 \times A_2 \times A_3 \times A_4$ will be a quaternary relation.

A tuple (111, Discrete Mathematics, A. B. Rao, A) will be an element of R.

This representation of R as a set of n-tuples, each element in the n-tuple, belonging to different domains has an important application in developing **Relation Databases**, for various types of data. A data base is essentially a collection of interrelated data, belonging to several domains. Hence it can conveniently be arranged in a tabular form.

For example a students record : (NAME, ROLL NO, CLASS, BRANCH, PERCENTAGE OF MARKS) can be put in tabular form as

Table 3.1

NAME	ROLL NO.	CLASS	BRANCH	MARKS (%)
ASHWIN	101	A	MECH.	85
CLARA	212	B	I.T.	82
LOKESH	542	D	MECH.	74
PRIYA	410	C	ETC.	78

- The entries in each column, are all of the same type. Hence each column is called as an **attribute**, and the entries are called as **fields**.
- Each column has unique name.
- Order of the rows is immaterial.
- A single attribute is called a **primary key** if the values of the attributes **uniquely** identifies a given tuple. For example ROLL NO. is a primary key.

Relational Operators :

Relational Databases possess well defined operators that act on relations. There are three main operations : Selection, Projection and Join.

Selection : Given a relational database, users may be interested in only some of the values of an attribute. For example while considering the performance report of a group of students, we wish to focus only on these students, whose performance is **below** a certain level (say 50%).

Example :

Table 3.2

NAME	ROLL NO.	CLASS	PERCENTAGE
ALKA	105	A	67
BHARAT	125	A	4
TARUN	232	B	72
BHAVANA	312	C	48
KALPESH	453	D	40
ROHIT	526	E	74

Then using the SELECT operator, we will extract the subtable :

Table 3.3

NAME	ROLL NO.	CLASS	PERCENTAGE
BHARAT	125	A	44
BHAVANA	312	C	48
KALPESH	453	D	40

Note that the select operators only selects the **required rows** of the table.

Projection : Using the operation of projection we can extract the necessary attributes (column) from the database table.

For example consider the database table 3.4.

Table 3.4

NAME	ROLL NO.	CLASS	BRANCH	MARKS (%)
ASHWIN	101	A	MECH.	85
CLARA	212	B	I.T.	82
LOKESH	542	D	MECH.	74
PRIYA	410	C	ETC.	78

Suppose we are interested only in the information relating to the student and marks then we will get to the projection table 3.5.

Table 3.5

NAME	PERCENTAGE
BHARAT	44
BHAVANA	48
KALPESH	40

Join: If R and S represent two relational databases, then R × S represents the join of R and S.

Example : Define the relations R and S as shown table 3.6.

Table 3.6

R

NAME	ROLL NO.
ALKA	101
DAVID	237
SWATI	312
ASHOK	423

S

NAME	BRANCH
ALKA	COMP.
DAVID	MECH.
SWATI	MECH.
ASHOK	CIVIL

The join of R and S on Name is :

R × S		
NAME	**ROLL NO.**	**BRANCH**
ALKA	101	COMP.
DAVID	237	MECH.
SWATI	312	MECH.
ASHOK	423	CIVIL

3.12 PARTIAL ORDERING RELATIONS

An **order relation** is a transitive relation on a set by means of which we can compare elements of set.

Definition:

> A binary relation R on a non-empty set A is a **partial order** if R is reflexive, antisymmetric and transitive.
>
> The ordered pair (A, R) is called a **partially ordered set** or **poset**.

Examples:

1. *The relation ' ≤ ' is a partial order relation on the set of real numbers.*
2. *The relation of 'being a subset' is a partial order on any collection of subsets of a set A; i.e. the ordered pair (P(A), ⊆) is a poset.*
3. The lexicographic ordering on the set of alphabets is a partial order.

We will use the symbol ≤ to denote an arbitrary partial order. This notation should not be confused with the '≤' of number systems. Thus a ≤ b will mean a R b for an arbitrary partial order relation R, and (A, ≤) will be the corresponding poset.

3.12.1 Hasse Diagrams

The posets can be depicted by digraphs. However a more economical way to describe a poset is by **Hasse diagram**. A Hasse diagram is a simpler version of a digraph incorporating the following rules:

- All arrow heads that appear on the edges are omitted.
- Loops are omitted as reflexivity is implied, by definition of a partial order.
- Similarly an arc (or edge) is not present in the diagram if it is implied by transitivity. There is an arc from a to b only if there is no element c such that a ≤ c and c ≤ b.
- An arc pointing upward is drawn from a to b if a ≠ b and a ≤ b. Arrow heads are not used.

SOLVED EXAMPLES

***Examples 1:** Let A = {2, 3, 4, 6} and let a R b if a divides b. Show that R is a partial order and draw its Hasse diagram.*

Solution: R = {(2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (6, 6)}

R is reflexive since (2, 2), (3, 3), (4, 4), (6, 6) ∈ R. R is antisymmetric since if a | b b | a unless a = b. R is also transitive since a | b and b | c implies a | c. Hence R is a partial order.

Hasse diagram for R

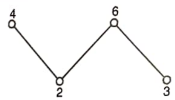


Fig. 3.20

***Example 2:** If A = {1, 2, 3, 4},*

R = {(1, 1), (1, 2), (2, 2), (2, 4), (1, 3), (3, 3), (3, 4), (1, 4), (4, 4)}

then show that R is a partial order and draw its Hasse diagram.

Solution: R is reflexive since (1, 1), (2, 2), (3, 3), (4, 4) ∈ R. R is antisymmetric since (1, 2) ∈ R but (2, 1) ∉ R, (2, 4) ∈ R but (4, 2) ∉ R. Similarly (1, 3), (1, 4), (3, 4) ∈ R but (3, 1), (4, 1), (4, 3) ∉ R. One can also similarly check that R is transitive.

Hasse Diagram:

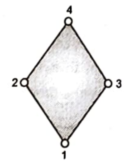


Fig. 3.21

***Example 3:** Let A = {a, b, c}. Show that (P(A), ⊆) is a poset and draw its Hasse diagram.*

Solution: P(A) = {A, ϕ, {a, b}, {b, c}, {a, c}, {a}, {b}, {c}}

Set containment ⊆ is always a partial order since for any subset B of A, B ⊆ B, i.e. ⊆ is reflexive. If B ⊆ C and C ⊆ B, B = C antisymmetry. If B ⊆ C and C ⊆ D then B ⊆ D (transitivity).

Hasse diagram:

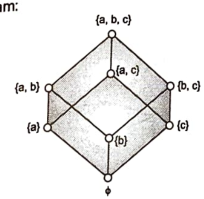


Fig. 3.22

3.13 CHAINS AND ANTICHAINS

Definition:

> Let (A, ≤) be a poset. A subset of A is called a **chain** if every pair of elements in the subset is related.

In any chain with a finite number of elements $\{a_1, a_2, ..., a_k\}$ there is an element a_{i1} that is less than (i.e. related to) every element in the chain, and there is an element a_{i2} that is less than every other element except a_{i1}. Continuing in this manner, we shall have a sequence $a_{i1} \le a_{i2} \le a_{i3} \le ... \le a_{ik}$. The number of elements in the chain is called as the length of the chain.

If A itself is a chain, the poset (A, ≤) is called a totally ordered set or linearly ordered set.

Definition:

> A subset of A is called an antichain if no two distinct elements in the subset are related.

Examples:

1. Let, A = {1, 2, 3} and let the partial order ≤ mean " less than or equal to ". Then (A, ≤) is a chain and its Hasse diagram is

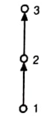


Fig. 3.23

2. Let, A = {a, b} and consider its poset {P(A), ⊆}

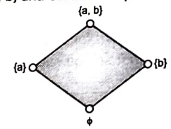


Fig. 3.24

Then the following subsets are chains

{ϕ, {a}, {a, b}}, {ϕ, {b}, {a, b}}, {ϕ, {a}}, {ϕ, {b}}, {{a}, {a, b}}, {{b}, {a, b}}.

The following subset is an antichains is {{a}, {b}}.

In the above example the length of the longest chain is 3 and the number of elements in the antichain is 2.

We state the following theorem (without proof) that shows a close relation between chains and antichains.

Theorem 1: Set (A, ≤) be a poset. Suppose the length of the longest chain in A is n. Then the elements is A can be partitioned into n disjoint antichains.

SOLVED EXAMPLES

***Example 1:** Let R_1 be a binary relation on A such that (a, b) is in R_1 if book a costs more and contains fewer pages than book b. Is R_1 reflexive? Symmetric? Antisymmetric? Transitive?*

Solution: R_1 is obviously not reflexive and not symmetric. If a R b, then b R a. Similarly if b R a, then a R b. Hence both the conditions a R b and b R a cannot be fulfilled simultaneously. Hence by the law of contrapositive, R_1 is antisymmetric. Let a R b and b R c. This implies a costs more than b, b costs more than c, a contains fewer pages than b, b contains fewer pages than c. Hence combining all these statements a costs more than c and contains few pages than c. Hence R is a transitive relation.

Example 2: *Let R be a binary relation on the set of all positive integers such that*

$$R = \{ (a, b) \mid a - b \text{ is an odd positive integer} \}.$$

Is R reflexive, symmetric, antisymmetric, transitive?

Is R an equivalence relation? a partial ordering relation.

Solution: R is not reflexive since $(a, a) \notin R$ as $a - a = 0$.

R is not symmetric since $a - b$ is odd implies $b - a$ is odd, but it is not positive. $(3, 2), \in R$ but $(2, 3) \in R$. R is also not transitive, since although $a - b$ is odd and $b - c$ is odd, $a - c = (a - b) + (b - c)$ which is even. Hence $(a, c) \notin R$ whenever $(a, b) \in R$ and $(b, c) \in R$. R is antisymmetric since $a R b \rightarrow b R a$. None of the conditions for an equivalence relation are satisfied. Hence R is not an equivalence relation. R is also not a partial order.

Example 3: *Let R be a binary relation on the set of all strings of 0's and 1's such that*

$$R = \{ (a, b) \mid a \text{ and } b \text{ are strings that have the same number of 0's} \}$$

Is R reflexive? Symmetric? Antisymmetric? Transitive? An equivalence relation? A partial order?

Solution: R is reflexive since $(a, a) \in R$. R is symmetric since if a and b have the same number of 0's, then b and a will have the same number of 0's. R is transitive since if a and b have the same number of 0's, b and c have the same number of 0's, then obviously a and c have the same number of 0's. R is obviously not antisymmetric. R is an equivalence relation but not a partial order.

Example 4: *Let S be the set of all points in a plane. Let R be the relation such that for any two points a and b, $(a, b) \in R$ if b is within one inch from a. Examine if R will be an equivalence relation.*

Solution: R is reflexive since a is **within** one inch (i.e. 0 inch) from itself.

R is symmetric since if b is within 1 inch from a, a is also within 1 inch from b.

But R is not transitive since b is within 1 inch from a, c is within 1 inch from b need not imply that c is within 1 inch from a. Hence R is not an equivalence relation.

Example 5: *Let T be a set of triangles in a plane and define R as the set*

$$R = \{ (a, b) \mid a, b \in T, \ a \text{ is congruent to } b \}.$$

Show that R is an equivalence relation.

Solution: a triangle is congruent to itself. Hence R is reflexive. If a is congruent to b, then b is congruent to a. Hence R is symmetric. If a is congruent to b, b is congruent to c, then a is congruent to c. Hence R is transitive. The relation satisfies all the three properties of an equivalence relation.

Example 6: *Consider subset as a relation on a given set. Check whether it is reflexive, symmetric, antisymmetric, equivalence or partial ordering relation.*

Solution: Let A be the given set. Consider subsets B, C of A. Any set is its own subset. Hence the relation is reflexive. If $B \subseteq C$, then $C \subseteq B$ only if $B = C$. Hence the relation is not symmetric but antisymmetric. If $B \subseteq C$ and $C \subseteq D$ then $B \subseteq D$. Hence the subset relation is transitive. The relation is therefore not an equivalence relation but a partial ordering relation.

Example 7: *From the following digraphs, write the relation a set of ordered pairs. Are the relations equivalence relations?*

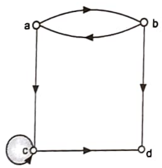


Fig. 3.25

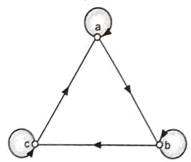


Fig. 3.26

Solution: R_1 = {(a, b), (b, a), (a, c), (c, d), (b, d), (c, c)}

and

R_2 = {(a, a), (b, b), (c, c), (c, a), (c, c), (a, b)}

R_1 is not an equivalence relation since R_1 is not reflexive. R_2 is also not an equivalence relation since R_2 is not symmetric, as $(a, b) \in R$ but $(b, a) \notin R$.

Example 8: *Consider the following relation on {1, 2, 3, 4, 5, 6}, $R = \{ (i, j) \mid |i - j| = 2 \}$. Is R reflexive, symmetric, transitive? Draw a graph of R.* **(Nov./Dec. 14)**

Solution: R is not reflexive. R is symmetric but not transitive.

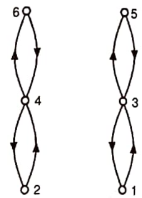


Fig. 3.27

Example 9: *Consider the relations defined by the digraphs. Determine whether the given relations are reflexive, symmetric, antisymmetric or transitive. Which graphs are equivalence relations and which are partial orders?*

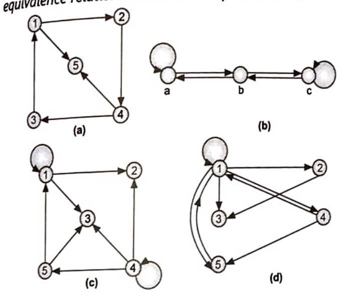


Fig. 3.28

Solution:

R_1 = {(1, 2), (1, 5), (2, 4), (3, 1), (4, 3), (4, 5)}

R_2 = {(a, a), (a, b), (b, a), (b, c), (c, b), (c, c)}

R_3 = {(1, 1), (1, 2), (1, 3), (2, 4), (4, 3), (4, 4), (4, 5), (5, 1)}

R_4 = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (4, 1), (4, 5), (5, 1)}

R_1 is not reflexive, not symmetric and not transitive also.

R_1 is also antisymmetric since $a \neq b \rightarrow a R b \vee b R a$.

R_2 is not reflexive, not transitive but symmetric. R_2 is not antisymmetric.

R_3 is not reflexive, not symmetric, but is transitive and antisymmetric.

R_4 is not reflexive, not symmetric and not transitive since $(4, 1) \in R$ and $(1, 3) \in R$ but $(4, 3) \notin R$. R_4 is not antisymmetric.

None of the relations are equivalence relations or partial orders.

Example 10: *The following relations R_1 and R_2 are defined over the set A = {1, 2, 3, 4, 5}. Show that they are partial order relations and draw their Hasse diagram.*

R_1	1	2	3	4	5
1	✓				
2	✓	✓			
3	✓	✓	✓		
4	✓	✓		✓	✓
5	✓				✓

R_2	1	2	3	4	5
1	✓	✓	✓	✓	
2		✓	✓	✓	
3			✓	✓	
4				✓	
5			✓	✓	✓

Solution: R_1 is reflexive since (1, 1), (2, 2), (3, 3), (4, 4), (5, 5) $\in R_1$. R_1 is also antisymmetric since $(1, 2) \in R_1$, but $(2, 1) \in R_1$ $(1, 3) \notin R_1$ but $(3, 1) \in R_1$. Likewise we can check for other elements. R_1 is transitive $(4, 5) \in R_1$, $(5, 1) \in R_1 \rightarrow (4, 1) \in R_1$. Hence R_1 is a partial order relation.

R_2 is reflexive since ✓ appears along the diagonal of the square. R_2 is antisymmetric since whenever $(a, b) \in R_2$, $(b, a) \notin R_2$, unless $a = b$. R_2 is also transitive.

$(1, 3)$ and $(3, 4) \in R_2 \rightarrow$ $(1, 4) \in R_2$. $(5, 3)$ and $(3, 4) \in R_2 \rightarrow$ $(5, 4) \in R_2$. Likewise we can check transitivity for all possible combinations. Hence R_2 is also a partial order.

Hasse Diagrams:

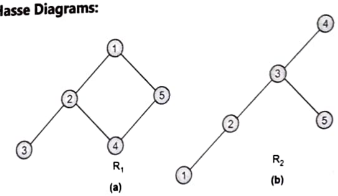


Fig. 3.29

Example 11: *Let R be a symmetric and transitive relation on a set A. Show that if for every a in A there exists b in A such that (a, b) is in R, then R is an equivalence relation.*

Solution: We have only to show that R is reflexive. Let a ∈ A, then there exists b ∈ A, such that (a, b) ∈ R. Since R is symmetric, this implies (b, a) ∈ R. Now (a, b) and (b, a) ∈ R. Hence by transitivity (a, a) ∈ R, i.e. R is reflexive.

Example 12: *Show that the transitive closure of a symmetric relation is symmetric.*

Solution: Let R^* denote the transitive closure of R. $R^* = R \cup R^2 \cup$. Let $(a, b) \in R^*$. Then $(a, b) \in R^k$ for some positive integer k. Then there exists a sequence of elements a, x_1, x_2, ... x_{k-1}, b, such that $(a, x_1) \in R$, $(x_1, x_2) \in R$, ..., $(x_{k-1}) \in R$.

Since R is symmetric this implies (b, x_{k-1}), (x_{k-1}, x_{k-2}), ... , (x_2, x_1), $(x_1, a) \in R$. Since R^* is transitive this implies that $(b, a) \in R^*$. Hence R^* is transitive.

Example 13: *Let X = {1, 2, 3, 4} and R = {(x, y) | x > y}. Draw the graph of R and also give its matrix.*

$$R = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$$

Graph of R:

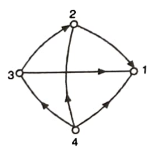


Fig. 3.30

$$M_R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Example 14: *Let X = {1, 2, ... 7} and R = {(x, y) | x − y is divisible by 3}. Show that R is an equivalence relation. Draw the graph of R.*

Solution: R is reflexive since ∀ x ∈ X, x − x = 0 is divisible by 3. Hence for ∀ x ∈ X, (x, x) ∈ R. R is symmetric since for every (x, y) ∈ R, (y, x) ∈ R, as y − x = − (x − y) is divisible by 3. Let (x, y) and (y, z) ∈ R. Then ± x − z = (x − y) + (y − z) is clearly divisible by 3. Hence R is an equivalence relation.

Graph of R: R = {(1, 1), (1, 4), (1, 7), (2, 2), (2, 5), (3, 3), (3, 6), (4, 1), (4, 4), (4, 7), (5, 2), (5, 5), (6, 3), (6, 6), (7, 1), (7, 4), (7, 7)}

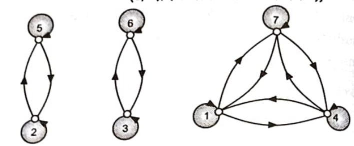


Fig. 3.31

Example 15: *Draw the Hasse diagram.*

Let A be the set of factors of a particular positive integer m and let ≤ be a relation divides, i.e.

$$\leq \; = \{(x, y) \mid x \in A \wedge y \in A \wedge (x \text{ divides } y)\}.$$

(i) m = 2, (ii) m = 6, (iii) m = 12, (iv) m = 45.

Solution: (i) m = 2,

A = {1, 2}, R = {(1, 1), (1, 2), (2, 2)}

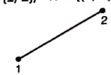


Fig. 3.32

(ii) m = 6, A = {1, 2, 3, 6}

R = {(1, 1), (1, 2), (1, 3), (1, 6), (2, 2), (2, 6), (3, 3), (3, 6), (6, 6)}

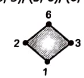


Fig. 3.33

(iii) m = 12, A = {1, 2, 3, 4, 5, 6, 12}

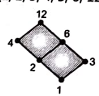


Fig. 3.34

(iv) m = 45, A = {1, 3, 9, 15, 45}

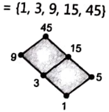


Fig. 3.35

Example 16: *Let R be the relation on the set A = {5, 6, 8, 10, 28, 36, 48}. Let R = {(a, b) | a is a divisor of b}. Draw the Hasse diagram and compare it with digraph. Determine whether R is reflexive, transitive and symmetric.*

Solution: R = {(5, 5), (5, 10), (6, 6), (6, 36), (6, 48), (8, 8), (8, 48), (10, 10), (28, 28), (36, 36), (48, 48)}.

Hasse Diagram:

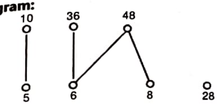


Fig. 3.36

Digraph:

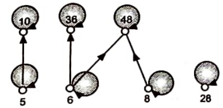


Fig. 3.37

R is reflexive, but not symmetric or transitive.

Example 17: *Find the transitive closure of R by Warshall Algorithm where A = {1, 2, 3, 4, 5, 6} and R = {(x, y) | |x − y| = 2}.*

Solution: R = {(1, 3), (3, 1), (2, 4), (4, 2), (3, 5), (5, 3), (4, 6), (6, 4)}

$$W_R = M_R = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

By notation, let $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 4$, $a_5 = 5$, $a_6 = 6$. For k = 1, $W_k = W_1$, $a_k = 1$ is an interior vertex for the path from 3 to 1 and 1 to 3. Hence W_1 has 1 in the position (3, 3).

$$\therefore \quad W_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Similarly for k = 2, $a_2 = 2$ is an interior vertex for the path (4, 2) and (2, 4). k = 3, $a_3 = 3$ is an interior vertex for the path (1, 3) and (3, 1). k = 4, $a_4 = 4$ is an interior vertex for the path (2, 4) and (4, 2). k = 5, $a_5 = 5$ is an interior vertex for the path (3, 5) and (5, 3). k = 6, $a_k = 6$ is an interior vertex for the path (4, 6) and (6, 4).

Hence the final matrix becomes

$$W_6 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Hence transitive closure of R is

R^* = {(1, 1), (1, 3), (1, 5), (2, 2), (2, 4), (2, 6), (3, 1), (3, 3), (3, 5), (4, 2), (4, 4), (4, 6), (5, 1), (5, 3), (5, 5), (6, 2), (6, 4), (6, 6)}

Example 18: *Use Warshall's Algorithm to find the transitive closure of R, where*

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and } A = \{1, 2, 3\}.$$

Solution: R = {(1, 1), (1, 3), (2, 2), (3, 1), (3, 2)}

$$W_0 = W_R$$

For k = 1, $a_k = 1$ is an interior vertex for the path (3, 1) and (1, 3). Hence W_1 will have 1 as its (3, 3) entry.

For k = 2, $a_k = 2$ is an interior vertex for the path (3, 2) and (2, 2), but already there is 1 at (3, 2) position.

For k = 3, $a_k = 3$ is an interior vertex for the paths (1, 3), (3, 2) and (1, 3), (3, 1). Hence we have to include 1 at the (1, 2) position, whereas there is already 1 at the (1, 1) position.

$$W_3 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Hence transitive closure of R

R^* = {(1, 1), (1, 3), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3)}

Example 19: *Let $R = \{(1, 4), (2, 1), (2, 5), (2, 4), (4, 3), (5, 3), (3, 2)\}$ Use Warshall's algorithm to find the matrix of transitive closure.*

Solution: $W_0 = M_R = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

For k = 1, place 1 is already in the position (2, 4). Hence, $W_0 = W_1$.

For k = 2, place 1 in the positions (3, 1), (3, 4) and (3, 5).

$$\therefore \quad W_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\text{Similarly,} \quad W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\text{Finally,} \quad W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Example 20: *Find the transitive closure of the relation R on $A = \{1, 2, 3, 4\}$ defined by $R = \{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (3, 4), (3, 2), (4, 2), (4, 3)\}$.* ***(May 15)***

Solution: $W_0 = M_R = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

For k = 1, place 1 in the positions (2, 2), (2, 3) and (2, 4).

$$\therefore \quad W_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

For k = 2, we have 1 in (1, 2), (2, 2), (3, 2) and (3, 4) (in 2nd column) and (2, 1), (2, 2), (2, 3) and (2, 4) (in the 2nd row).

Hence, we place 1 in the new locations (1, 1), (3, 1), (3, 3), keeping infact the 1's, which are already there in other locations.

$$\text{Hence,} \quad W_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

For k = 3, as before we place 1 in the position (4, 4), considering the combination of (4, 3) and (3, 4) positions, where 1 is located.

$$\therefore \quad W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = M_R^*$$

$\therefore$ Transitive closure

$R^* = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)\}$

Example 21: *Determine the properties of the relations given by the graphs below:*

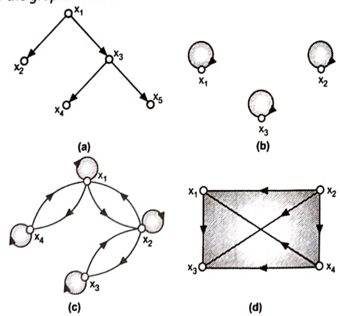


Fig. 3.38

Solution: (a) $M_{R_1} = \begin{array}{c|c} & x_1 \;\; x_2 \;\; x_3 \;\; x_4 \;\; x_5 \\ \hline \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$

$\therefore$ R_1 is not reflexive, not symmetric and not transitive. R_1 is reflexive and antisymmetric.

(b) $M_{R_2} = \begin{array}{c|c} & x_1 \;\; x_2 \;\; x_3 \\ \hline \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$

R_2 is reflexive.

(c) $M_{R_3} = \begin{array}{c|c} & x_1 \;\; x_2 \;\; x_3 \;\; x_4 \\ \hline \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$

R_3 is reflexive, symmetric R_3 is not transitive and not antisymmetric.

(d) $M_{R_4} = \begin{array}{c|c} & x_1 \;\; x_2 \;\; x_3 \;\; x_4 \\ \hline \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$

R_4 is not reflexive, not symmetric and not antisymmetric. R_4 is reflexive and transitive.

Example 22: *Let R, S and T be the relations on*

$A = \{1, 2, 3\}$ defined by

$R = \{(1, 1), (2, 2), (3, 3)\}$

$S = \{(1, 2), (2, 1), (3, 3)\}$

$T = \{(1, 2), (2, 3), (1, 3)\}$

Determine which of the relations are reflexive, symmetric, transitive, antisymmetric.

Solution: R is reflexive, symmetric and transitive.

S is not reflexive and not transitive. But R is symmetric. T is irreflexive, not symmetric.

T is transitive and antisymmetric.

Example 23: *Consider the set of words*

W = {Sheet, Last, Sky, Wash, Wind, Sit}

Let R be the equivalence relation on W defined by

(i) "has the same number of letter as"

(ii) "begins with the same letter as".

Find the quotient W/R.

Solution: (i) Quotient set W/R is the set of equivalence clauses of R

$\therefore$ W/R = {{Sheet}, {Last, Wash, Wind}, {Sky, Sit}}

(ii) W/R = {{Last}, {Sheet, Sky, Sit} {Wash, Wind}}

Example 24: *For $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Consider the poset (A, R) whose Hasse diagram is shown below.*

Find:

(i) GLB {2, 3}, (ii) GLB = {2, 7}, (iii) GLB {5, 8}, (iv) LUB {3, 2}, (v) LUB {4, 8}, (vi) LUB {3, 5}.

Solution:

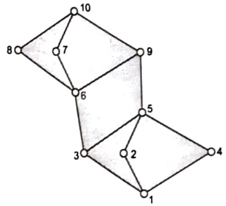


Fig. 3.39

(i) GLB {2, 3} = 1

(ii) GLB {2, 7} = 1

(iii) GLB {5, 8} = 3

(iv) LUB {3, 2} = 5

(v) LUB {4, 8} = 10

(vi) LUB {3, 5} = 5.

EXERCISE - 3.3

1. Let A = {1, 2, 3, 4}. Consider the following relations on A.

   $R_1 = \{(1, 3), (2, 3), (4, 1)\}$

   $R_2 = \{(1, 1), (2, 1)\}$

   $R_3 = \{(3, 4)\}$

   $R_4 = \{(1, 1), (2, 2), (3, 3)\}$

   $R_5 = \{(1, 3), (2, 4)\}$.

   Determine which relations are (i) reflexive, (ii) symmetric, (iii) transitive,

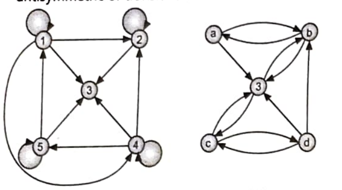
   (iv) equivalence, (v) partial ordering relation?

2. Let A = {a, b, c}. Determine whether the relation R whose matrix M_R is given is an equivalence relation.

   (a) $M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

   (b) $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3. Determine whether the relation R whose digraph is given is reflexive, irreflexive, symmetric, antisymmetric or transitive?



(a) (b)

Fig. 3.40

4. Let Z be the set of integers and let a R b iff b is a multiple of a. Determine which of the five properties are satisfied by R?

5. Let A be a set of lines in a plane. Define the following relation on A: l_1 A l_2 if and only if l_1 is perpendicular to l_2. Determine what properties of a relation are satisfied by R?

6. Let P be the set of all people. Let R be a binary relation on P such that a R b iff a is a brother of b. What properties of a relation are satisfied by R?

7. Let R_1 and R_2 be relations on a set A. Prove the following:

(a) If R_1 is symmetric, then so are R_1^c and $\bar{R}_1$.

(b) If R_1 and R_2 are symmetric, then so are $R_1 \cap R_2$ and $R_1 \cup R_2$.

(c) If R_1 and R_2 are transitive, then so is $R_1 \cap R_2$. Is $R_1 \cup R_2$ transitive?

8. If R_1 and R_2 are relations on any set A, prove or disprove the following:

(a) If R_1 and R_2 are reflexive then so is $R_1 R_2$.

(b) If R_1 and R_2 are irreflexive, then so is $R_1 R_2$.

(c) If R_1 and R_2 are symmetric, then so is $R_1 R_2$.

(d) If R_1 and R_2 are antisymmetric, then so is $R_1 R_2$.

(e) If R_1 and R_2 are transitive, then so is $R_1 R_2$.

9. If A = {1, 2, 3, 4, 5} and R = {(1, 2), (3, 4), (4, 5), (4, 1), (1, 1)}, find its transitive closure.

10. Let R be a transitive and reflexive relation on A. Let T be a relation on A such that (a, b) ∈ T iff both (a, b) and (b, a) are in R. Show that T is an equivalence relation.

11. Let R be a reflexive relation on a set A. Show that R is an equivalence relation iff (a, b) and (a, c) ∈ R implies that (b, c) ∈ R.

12. Let R be the relation defined on the integers by a R b iff a − b is even. Show that R is an equivalence relation and determine the equivalence classes.

13. Partition the set A = {1, 2, 3, 4, 5} by collection of sets {{1, 2}, {3}, {4, 5}}.

Determine the equivalence relation induced by the partition.

14. Let,

A = {1, 2, 3, 4} and

R = {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1), (2, 3), (3, 2), (3, 3), (4, 4)}

Show that R is an equivalence relation and determine the equivalence classes.

15. Let,

A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

and let, A_1 = {1, 2, 3, 4}, A_2 = {5, 6, 7}

A_3 = {8, 9, 10}

From a partition

π_1 = {A_1, A_2, A_3} of A.

Let, A_4 = {4, 8, 10}, A_5 = {3, 7, 9}, A_6 = {1, 2, 5, 6} form another partition

π_2 = {A_4, A_5, A_6} of A.

Find the sum and product of the two partitions.

16. Let A be a set of people and R be a binary relation on A such that (a, b) is in R if a is a friend of b. Show that R is a compatible relation.

17. Let R_1 and R_2 be two compatible relations on A. Is $R_1 \cap R_2$ a compatible relation?

18. Let A be a set of English words and R be a binary relation on A such that two words in A are related if they have one or more letters in common. Show that R is a compatible relation.

19. Determine whether the relation R is a partial order on the set A.

(i) A = Z, and a R b iff a = 2b

(ii) A = Z, and a R b iff b^2 | a.

20. Let A = {1, 2, 3, 4} and R = {(1, 1), (1, 2), (2, 2), (2, 4), (1, 3), (3, 3), (3, 4), (1, 4), 4)}. Show that R is a partial order and draw its Hasse diagram. Determine the chains and antichains.

21. Consider the poset whose Hasse diagram is given below. Find the length of the longest chain. Find also the antichains.

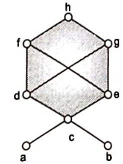


Fig. 3.41

22. Given S = {1, 2, 3, 4, 5} and relation R on S, where R = {(x, y) | x + y = 5}. What are the properties of R?

23. Let R be a relation on a set A,

A = {2, 3, 4, 6, 8, 12, 38, 48} defined by

R = {(a, b) | a is divisor of b|

Draw the diagraph and Hasse diagram.

24. Draw the Hasse diagram of the following sets under the partial ordering relation 'divides' and indicate those which are chains:

(i) {2, 4, 12, 24}

(ii) {1, 3, 5, 15, 30}

25. Draw the diagraph for the following relation and determine whether the relation is reflexive, symmetric, transitive and antisymmetric.

A = {1, 2, 3, 4, 5, 6, 7, 8} and let xRy whenever y is divisible by x.

26. Let A = {1, 2, 3} and consider two reflexive relations R = {(1, 1), (1, 2), (1, 3), (2, 2), (3, 3)} and S = {(1, 1), (1, 2), (2, 2), (3, 2), (3, 3)}. Determine whether the following relations are reflexive or irreflexive.

(i) R^{-1}, (ii) $\bar{R}$, (iii) $R \cap S$, (iv) $R \cup S$.

27. For the relation R whose matrix is given, find the matrix of transitive closure, using Warshall's Algorithm:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

28. Let R be the relation on the set A = {a, b, c, d, e, f} and let R = {(a, c), (b, d), (c, a), (c, e), (d, b), (d, f), (e, c), (f, d)}. Find the transitive closure of R, using Warshall's alorithm. **(Nov./Dec. 14)**

29. Find the transitive closure by using Warshall's algorithm for the given relation as : **(May 15)**

R = {(1, 1), (1, 4), (2, 1), (2, 2), (3, 3), (4, 4)}

3.14 JOB-SCHEDULING PROBLEM

Concept of a partial order relation, especially that of chain, finds an important application in Job-Scheduling Problems in Process Management. For example, in a multiprocessor computing system, the objective is to increase CPU utilisation and work output in a given interval of time. Hence in this case scheduling decides the optimum sequence and allocation of tasks to various processors, so that no processor is left intentionally idle.

In scheduling a given set of tasks, one must take into account the fact that certain tasks have to be completed before some other tasks can be executed; however in some situations, several different tasks can also be performed simultaneously. For example in the construction of a building doors and windows can be installed only after the walls have been raised. However, the plumbing and electrical work can be done simultaneously. Hence it is necessary to consider all these aspects for drawing up an optimum schedule for the various tasks.

There are many different types of scheduling problems which are solved by Directed Network (finding the shortest path) or Graph Colouring Methods. In this section, we shall deal only with a particular scheduling problem, encountered in a computing system.

Consider a multiprocessor system with n identical processors $P_1, P_2, \ldots, P_n$. Let $T = \{T_1, T_2, \ldots, T_m\}$ be a set of distinct tasks to be executed on the system. Let us assume that the execution of a task occupies one and only one processor, and since all the processors are identical, any task can be executed on any one of the processors.

Define a relation '≤' on T as follows:

(i) $T_i \le T_i$, $\forall\, 1 \le i \le m$.

(ii) If $i \ne j$, $T_i \le T_j$ if and only if the execution of T_i is completed before that of T_j can begin.

It is clear that '≤' defines a partial order on T. A sequence of tasks $T_{i1} \le T_{i2} \le \ldots \le T_{ir}$ will form a chain, $T_i \le T_j$ will imply that both these tasks can be performed independently.