We reverse the order of the equation (2) and rewrite

$$S_n = nC(n, 0) + (n-1)C(n, 1) + (n-2)C(n, 2) + \dots + C(n, n-1) \quad \dots (3)$$

Adding equations (1) and (3) we obtain

$$2S_n = n[C(n, 0) + C(n, 1) + C(n, 2) + \dots + C(n, n)]$$

$$= n \cdot 2^n$$

$$\therefore \quad S_n = \frac{n \cdot 2^n}{2} = n \cdot 2^{n-1}$$

Example 7: *Simplify the expression: C (3, 3) + C (4, 3) + C (5, 3) + ... + C (27, 3)*

Solution: Note that there are 25 terms in the series. We use the identity

$C(n, k) + C(n, k-1) = C(n+1, k)$

Also we can replace C (3, 3) by C (4, 4).

Hence the given sum is

$$[C(4, 4) + C(4, 3)] + C(5, 3) + C(6, 3) + \dots + C(27, 3)$$
$$= C(5, 4) + C(5, 3) + C(6, 3) + \dots + C(27, 3)$$
$$= C(6, 4) + C(6, 3) + \dots + C(27, 3)$$
$$= C(7, 4) + \dots + C(27, 3)$$

Continuing in this way, the last two terms

$$= C(27, 4) + C(27, 3) = C(28, 4)$$

EXERCISE – 5.2

1. Show that: $C(2n, n) + C(2n, n-1) = C(2n+2, n+1)$
2. If $C(n, 3) + C(n+2, 3) = P(n, 3)$, find n. **(Ans.:** n = 4)
3. Prove that: $C(2n, 2) = 2C(n, 2) + n^2$
4. Show that: $C(n, 1) + 6C(n, 2) + 6C(n, 3) = n^3$
5. Evaluate: $1^3 + 2^3 + 3^3 + \dots + n^3$
   (Ans.: $C(n+1, 2) + 6C(n+1, 3) + 6C(n+1, 4)$)
6. Evaluate the sum: $1 + 2C(n, 1) + \dots + (k+1)C(n, k) + \dots + (n+1)C(n, n)$ **(Ans.:** $2^n + n2^{n-1}$)
7. Find the sum: $C(n, 0) - 2C(n, 1) + 3C(n, 2) + \dots + (-1)^n (n+1)C(n, n)$ **(Ans.:** 0)
8. Evaluate: $C(n, 0) + 2C(n, 1) + C(n, 2) + 2C(n, 3) + \dots$ **(Ans.:** $3 \cdot 2^{n-1}$)

POINTS TO REMEMBER

- **Rule of Product:** If one experiment E_1 has n_1 possible outcomes and another experiment E_2 has n_2 possible outcomes, then there are $n_1 n_2$ possible outcomes when both the experiments ($E_1 \cap E_2$ or $E_1 E_2$) take place.
- This rule can be extended to a finite number of experiments $E_1, E_2, E_3, \dots, E_k$, with outcomes $n_1, n_2, n_3, \dots n_k$ respectively.
- **Rule of Sum:** If one experiment E_1 has n_1 possible outcomes and another experiment E_2 has n_2 possible outcomes, then there are $n_1 + n_2$ possible outcomes when exactly one of these experiments take place i.e. $(E_1 \cup E_2) - E_1 \cap E_2$.
- Let $0 \le r \le n$. The number of ways to have an ordered sequence of n distinct elements, taken r at a time is called as an r-permutation of n-elements and is denoted by P(n, r) or (^nP_r).

$$P(n, r) = n(n-1)(n-2)\dots(n-r+1)$$
$$= \frac{n!}{(n-r)!}, \quad \text{where } 0 \le r \le n$$

- The number of ways in which of the n elements can be arranged, where r_1 elements are of one kind, r_2 are of another kind and so on till r_k elements are of another kind, is given by the formula $\frac{P(n, r)}{r_1!\, r_2! \dots r_k!}$, where $r = r_1 + r_2 + \dots + r_k$.
- Let $0 \le r \le n$. A selection of a set of r elements from a set of n distinct elements is called a **Combination**.

◈◈◈

Unit IV : Graph Theory

CHAPTER 6
GRAPH THEORY

6.1 INTRODUCTION

- During the last four decades, graph theory has emerged as one of the important branch of Mathematics. Now-a-days, it is considered as a powerful tool to solve the large complex systems. The simplicity of the graph theory finds its applications in different fields like Computer Science, Chemistry, Operations research, economics, linguistics etc. In addition, graph theory has proved useful in the study of problems arising in other branches of Mathematics such as group theory and matrix theory.
- The theory of graph originated from the famous königsberg seven bridges problem which was solved by great Swiss Mathematician **Euler** in the year 1736. After hundred years, **Kirchhoff** developed the theory of trees (special type of graphs) for its application in the study of electrical networks. In recent times, the graph theory is widely used in various engineering applications such as network analysis, data structures, artificial intelligence, compiler writing, computer graphics etc.
- In this chapter, first we introduce some of the basic terminology of graph theory and then we study some important graphs and operations on the graphs.

6.2 BASIC TERMINOLOGY OF A GRAPH

- A graph is simply a collection of points, called **'Vertices'**, and a collection of lines, called **'Edges'**, each of which joins either a pair of points or a single point to itself.
- Mathematically, a graph G is an ordered pair (V, E) where V is the set of vertices and E is the set of edges. Each edge e_{ij} is associated with an unordered pair of vertices (v_i, v_j) The vertices v_i and v_j are called **end vertices** or **terminal vertices** of the edge e_{ij} or simply the edge e. For example, consider the graph G_1 shown in Fig. 6.1. It has four vertices namely v_1, v_2, v_3 and v_4 and five edges e_1, e_2, e_3, e_4 and e_5. The end vertices of the edge e_1 are v_1 and v_2 i.e. $e_1 = (v_1, v_2)$

Similarly $e_2 = (v_2, v_3)$, $e_3 = (v_3, v_4)$, $e_4 = (v_4, v_1)$ and $e_5 = (v_1, v_3)$ In short, we can represent G_1 as $G_1 = (V_1, E_1)$ where $V_1 = \{v_1, v_2, v_3, v_4\}$ and $E_1 = \{e_1, e_2, e_3, e_4, e_5\}$.

- A vertex is also referred to as a **node, a junction or a point**. Other terms used for an edge are **a line**, an **element** or **an arc**.

Following are various examples of the graphs given in Fig. 6.1.

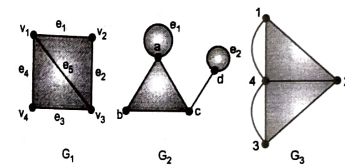


Fig. 6.1

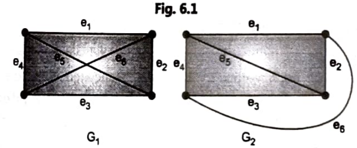
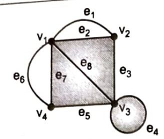


Fig. 6.2

The representation of the graph is not unique. It is immaterial whether the edges are drawn as straight lines or curves. What is important is the incidence between the vertices and edges. For example, the graphs G_1 and G_2 in Fig. 6.2 represent the same graph.

6.2.1 Self Loops and Parallel Edges

- If the end vertices v_i and v_j of any edge e_{ij} are same, then the edge e_{ij} is called as **Self Loop** or simply a **loop**. In Fig. 6.3, the edge $e_4 = (v_3, v_3)$ is a self loop.
- If there are more than one edges associated with a given pair of vertices then those edges are called **Parallel Edges** or **Multiple Edges**. In Fig. 6.3, $e_1 = (v_1, v_2)$ and $e_2 = (v_1, v_2)$ are parallel edges. Similarly, e_6 and e_7 are also parallel edges.
- A graph with one vertex and no edge is called **Trivial Graph**.



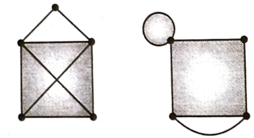
G_1

Fig. 6.3

6.2.2 Simple and Multiple Graphs

A graph that has neither self loops nor parallel edges is called a **Simple Graph**, otherwise, it is called a **Multiple Graph**.

In the following Fig. 6.4, G_1 is a simple graph but G_2 is not.

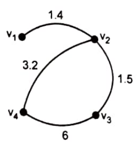


G_1 G_2

Fig. 6.4

6.2.3 Weighted Graph

Let G be a graph with vertex set V and edge set E. If each edge or each vertex or both are associated with some positive real numbers then the graph is called a **Weighted Graph**. For example, a graph representing a system of pipelines in which the weights assigned indicate the amount of some commodity transferred through the pipe is a weighted graph. Similarly, a graph of city streets may be assigned weights according to the length of each street or according to the traffic density on each street. Fig. 6.5 (a) shows a weighted graph where weights are assigned to each edge.



G

Fig. 6.5 (a)

6.2.4 Finite and Infinite Graphs

A graph with finite number of vertices as well as finite number of edges is called a **Finite Graph**, otherwise, it is an infinite graph.

Graph G in Fig. 6.5 (a) is a finite graph.

In most of the theory and applications, the graphs are taken as finite graphs. In the following sections and chapters, we are dealing with finite graphs only.

6.2.5 Labelled Graphs

A graph G = (V, E) is called a labeled graph if its edges are labeled with some name or data. For example, graph shown in Fig. 6.5 (b) is a labeled graph.

Here, $G = \{\{v_1, v_2, v_3, v_4\}, \{e_1, e_2, e_3, e_4, e_5\}\}$

where $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{e_1, e_2, e_3, e_4, e_5\}$

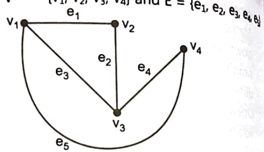


Fig. 6.5 (b)

6.3 ADJACENCY AND INCIDENCE

Let us define the relation between the vertices and edges in the graph.

Two vertices v_1 and v_2 of a graph G are said to be **Adjacent** to each other if they are the end vertices of the same edge. In other words, if two vertices are joined directly by at least one edge then there vertices are called **Adjacent Vertices**.

In Fig. 6.6, v_1 and v_2 are adjacent vertices but v_1 and v_4 are not.

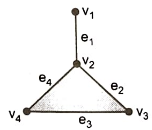


Fig. 6.6

For incidence, if the vertex v_i is the end vertex of an edge $e_{ij} = (v_i, v_j)$ then the edge e_{ij} is said to be incident on v_i. Similarly, e_{ij} is said to be incident on v_j. In Fig. 6.6, e_1 is incident on v_1 and v_2.

Two non-parallel edges are said to be **adjacent** if they are incident on a common vertex. For example, e_1 and e_2 are adjacent. Similarly, e_3 and e_4 are also adjacent edges.

6.3.1 Degree of a Vertex

The number of edges incident on a vertex v_i with self loop counted twice, is called the degree of the vertex v_i. It is denoted by $d(v_i)$ In the following Fig. 6.7, $d(v_1) = 3$, $d(v_2) = 3$, $d(v_3) = 1$, $d(v_4) = 3$, $d(v_5) = 4$, $d(v_6) = 0$.

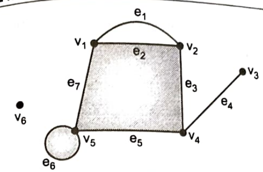


Fig. 6.7

6.3.2 Isolated Vertex and Pendant Vertex

A vertex with degree **zero** is called an **isolated vertex**. More formally, a vertex having no incident edge is an isolated vertex. A vertex of **degree** *1* is called a **pendant vertex**. In Fig. 6.7, v_6 is an isolated vertex and v_3 is a pendant vertex.

6.3.3 Handshaking Lemma

Consider a graph G with e number of edges and n number of vertices. Since each edge contributes two degrees, the sum of the degrees of all vertices in G is twice the number of edges in G. i.e.

$$\sum_{n=1}^{n} d(v_i) = 2e \qquad \text{... (6.1)}$$

This is called **Handshaking Lemma**. The result is so named because it implies that if several people shake hands, the total number of hands shaken must be even, precisely, because two hands are involved in one handshake.

From the equation (6.1), we shall derive the following interesting result.

Theorem : The number of vertices of odd degree in a graph is always even.

Proof:

Let G be a graph with e edges and n number of vertices. Then by handshaking lemma,

$$\sum_{i=1}^{n} = d(v_i) = 2e = \text{even number}$$

Now, the total degree of all the vertices can be expressed as the sum of degrees of even degree vertices and odd degree vertices.

$$\Rightarrow \sum d(v_j) + \sum d(v_k) = \sum_{i=1}^{n} d(v_i)$$

(even degree vertices) + (odd degree vertices)

$$\Rightarrow \sum d(v_j) + \sum d(v_k) = 2e = \text{an even number}$$

(even degree vertices)

But the sum of degrees of even degree vertices is always even i.e. $\sum d(v_j)$ is even and 2e is always an even number.

Therefore, the sum of degrees of odd degree vertices should be even.

$\Rightarrow \sum d(v_k)$ = an even number.

odd degree vertices

This shows that the sum of the degrees of odd vertices is an even number and which is possible only when the number of odd degree vertices is even.

6.4 SOME IMPORTANT AND USEFUL GRAPHS

After defining the basic terminology, now we are in position to define some important and useful graphs.

Directed Graphs or Digraphs: A **directed graph** or **digraph D** is an ordered pair (V, A) where V is a non empty set of elements called vertices and A is the set of ordered pair of elements called directed edges or arcs. In other words, we can say that if the each edge of the graph G has a direction then the graph is called **directed graph** or **digraph**.

In Fig. 6.8, we have a digraph D = (V, A) in which the vertex set $V = \{v_1, v_2, v_3, v_4\}$ and the directed edge set $A = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$. The edges are given by $e_1 = (v_1, v_3)$, $e_2 = (v_3, v_3)$, $e_3 = (v_2, v_3)$, $e_4 = (v_1, v_2)$, $e_5 = (v_1, v_2)$, $e_6 = (v_4, v_2)$, $e_7 = (v_2, v_4)$

In Fig. 6.8, the vertices v_1 and v_2 are joined by more than one arc with the same directions. Such arcs are called **multiple arcs**. The arcs e_6 and e_7 are not multiple arcs because their directions are different though their end vertices are same.

The self loop for the digraph is defined exactly in the same way. In Fig. 6.8, the arc e_2 is a self loop.

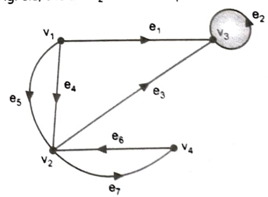


Fig. 6.8

Digraphs without loops and multiple arcs are known as **simple digraphs**.

Incidence

Let u be any vertex of the digraph D = (V, A) An arc 'a' in D is said to be incident into the vertex u if it is of the form (v, u) for some $v \in V$. Similarly, 'a' is said to be incident out of the vertex u if it is of the form (u, v) for some $v \in V$. For example, e_1 is incident into the vertex v_3 and incident out of the vertex v_1 in Fig. 6.8.

Indegree and Outdegree

The indegree of a vertex u of digraph D is defined as the **number of arcs which are incident into u** and is denoted by $\overrightarrow{d(u)}$. Similarly, the **outdegree of** a vertex u is defined as the **number of arcs which are incident out of u** and is denoted by $\overleftarrow{d(u)}$.

As shown in Fig. 6.8, indegree of different vertices are

$$\overrightarrow{d(v_1)} = 0,\quad \overrightarrow{d(v_2)} = 3,\quad \overrightarrow{d(v_3)} = 3,\quad \overrightarrow{d(v_4)} = 1,$$

The outdegree of vertices are

$$\overleftarrow{d(v_1)} = 3,\quad \overleftarrow{d(v_2)} = 2,\quad \overleftarrow{d(v_3)} = 1,\quad \overleftarrow{d(v_4)} = 1$$

Underlying Graph of a Digraph

The underlying graph of a digraph D is obtained by neglecting the directions of the arcs. For example, D_2 is a underlying graph of D_1 in Fig. 6.9.

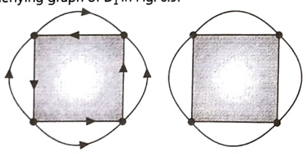


Fig. 6.9

Here D_1 is a simple digraph but its underlying graph D_2 is not a simple graph.

Null Graph: If the edge set of any graph with n vertices is an empty set, then the graph is known as **null graph or edgeless graph**. It is denoted by N_n. Following are the examples of null graph on 3 vertices and 4 vertices respectively.

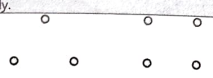


Fig. 6.10

Complete Graph: Let G be a simple graph on n vertices. If the degree of each vertex is (n – 1), then the graph G is called a **complete graph.** More generally, if every pair of vertices is adjacent in any simple graph, then the graph is said to be a **complete graph.**

Complete graph on n vertices is denoted by K_n.

Complete graph on two, three, four and five vertices are shown in the following Fig. 6.11.

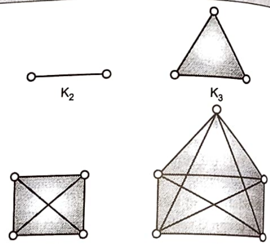


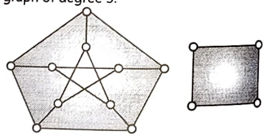
Fig. 6.11

In a complete graph K_n, the number of edges are $\frac{n(n-1)}{2}$.

Regular Graph: If the **degree of each vertex** is same say 'r' in any graph G, then the graph G is said to be a **regular graph of degree r.**

The regular graph of degree r is also called r-regular graph. For example, the regular graph of degree 2 is also called 2-regular graph.

For example, the Peterson graph shown in Fig. 6.12 (a) is a regular graph of degree 3.



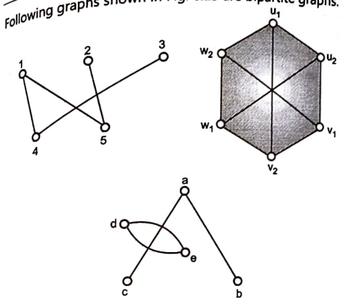
(a) Regular Graph of Degree 3 (b) Regular Graph of Degree 2

Fig. 6.12

Complete graph K_n is also a regular graph of degree (n – 1) But every regular graph need not be a complete graph. For example, Fig. 6.12 (b) is a regular graph of degree 2 but it is not a complete graph.

Bipartite Graph: Let G be a graph with vertex set V and edge set E, then G is called a **bipartite graph** if its vertex set V can be partitioned into two disjoint subsets say V_1 and V_2 such that $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \phi$ and also each edge of G joins a vertex of V_1 to a vertex of V_2.

Following graphs shown in Fig. 6.13 are bipartite graphs.



$V_1 = \{1, 2, 3\}$ $V_1 = \{u_1, v_1, w_1]$ $V_1 = \{a, d\}$

$V_2 = \{4, 5)$ $V_2 = \{u_2, v_2, w_2\}$ $V_2 = \{b, c, e\}$

Fig. 6.13

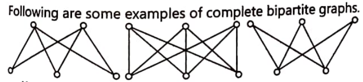
In bipartite graph, the vertices of V_1 should **not** be joined. Similarly, vertices of V_2 also should **not** be joined by edges. A bipartite graph cannot have a self loop.

Complete Bipartite Graph: A bipartite graph is called a **complete bipartite graph if each** vertex of V_1 is joined to every vertex of V_2 by a unique edge.

A complete bipartite graph is denoted by $\mathbf{K_{m,n}}$ if the number of vertices in V_1 is m and that in V_2 is n.

Total number of edges in any complete bipartite graph $K_{m,n}$ is given by (mn)

Following are some examples of complete bipartite graphs.



$K_{2,3}$	$K_{3,3}$	$K_{3,2}$
m = 2, n = 3	m = 3, n = 3	m = 3, n = 2
m < n	m = n	m > n

Fig. 6.14 (a)

The graph $K_{1,n}$ is known as **star**. A star $K_{1,5}$ shown in Fig. 6.14 (b)

Every complete bipartite graph is regular if m = n, $K_{3,3}$ is a regular graph of degree 3 shown in Fig. 6.14 (a)

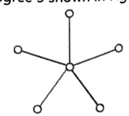


Fig. 6.14 (b)

6.5 MATRIX REPRESENTATION OF GRAPHS

A graph can also be represented by a matrix. Matrix representation of the graph is very convenient and useful for computer calculations. It is used in data structure to represent graphs. Two important ways are used for matrix representation of a graph; namely using

1. Adjacent matrix .
2. Incident matrix.

6.5.1 Adjacency Matrix

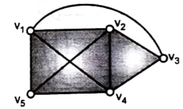
The adjacency matrix of a graph G with n vertices and no parallel edges is a symmetric binary matrix $A(G) = [a_{ij}]$ or order n × n where,

a_{ij} = 1, if there is an edge between vertices v_i and v_j (v_i and v_j are adjacent)

= 0, if v_i and v_j are not adjacent.

From the definition, it is clear that elements along principle diagonal of A(G) are all zeros if and only if the graph has no self loops. A self loop at vertex v_i corresponds to $a_{ij} = 1$.

Also if the graph has no self loops and no parallel edges, then the degree of a vertex is equal to the number of ones in the corresponding row or column vector of A(G)



G_1

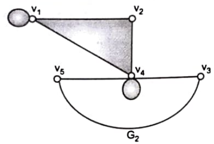
Fig. 6.15

Consider the graph G_1 shows in Fig. 6.15 which is without self loop.

Its adjacent matrix is

$$A(G_1) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Another graph G_2 and its adjacency matrix are shown in Fig. 6.16 below.



G_2

Fig. 6.16

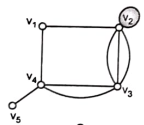
$$A(G_2) = \begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline v_1 & 1 & 1 & 0 & 1 & 0 \\ v_2 & 1 & 0 & 0 & 1 & 0 \\ v_3 & 0 & 0 & 0 & 1 & 1 \\ v_4 & 1 & 1 & 1 & 1 & 1 \\ v_5 & 0 & 0 & 1 & 1 & 0 \end{array}$$

The adjacency matrix for the multigraph G with n vertices is an n × n matrix. $A(G) = [a_{ij}]$, where the elements a_{ij} are defined as

a_{ij} = N, if one or more edges are there between vertices v_i and v_j and N is the number of edges between v_i and v_j

= 0, otherwise

The multigraph G and its adjacency A(G) are shown in Fig. 6.17.



G

Fig. 6.17

$$A(G_2) = \begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline v_1 & 0 & 1 & 0 & 1 & 0 \\ v_2 & 1 & 1 & 3 & 0 & 0 \\ v_3 & 0 & 3 & 0 & 2 & 0 \\ v_4 & 1 & 0 & 2 & 0 & 1 \\ v_5 & 0 & 0 & 0 & 1 & 0 \end{array}$$

From above, it is clear that though it is simple to implement the adjacency representation of the graphs on computers, it requires lot of memory space as these matrices are sparse matrices. For a graph with n vertices, then adjacency matrix representation requires space for n^2 elements.

6.5.2 Incidence Matrix

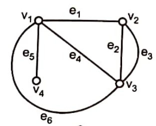
Given a graph G with n vertices, e edges and no self loops. The incidence matrix $X(G) = [x_{ij}]$ of the graph G is an n × e matrix where,

x_{ij} = 1; if j^{th} edge e_j is incident on i^{th} vertex v_i

= 0; otherwise

Here n rows correspond to n vertices and e columns correspond to e edges.

The graph G and its incidence matrix are given below in Fig. 6.18.



G

Fig. 6.18

$$X(G) = \begin{array}{c|cccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \hline v_1 & 1 & 0 & 0 & 1 & 1 & 1 \\ v_2 & 1 & 1 & 1 & 0 & 0 & 0 \\ v_3 & 0 & 1 & 1 & 1 & 0 & 1 \\ v_4 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}$$

It is observed that, since every edge is incident on two vertices, each column of X(G) has exactly two ones. Thus, the number of ones in each row represents the degree of the corresponding vertex if the graph has no self loops. The row with all elements zero, represent the isolated vertex. If the graph has self loop for the edge e_i, then the corresponding column of e_i in the incidence matrix will contain only single one. This is evident from the following example, consider the graph G with self loop defined in Fig. 6.19.

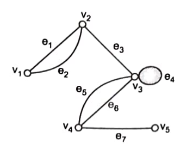


Fig. 6.19

Its incidence matrix X(G) is given by

$$X(G) = \begin{array}{c|ccccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ \hline v_1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ v_2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ v_3 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ v_4 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ v_5 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

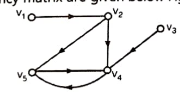
Adjacency Matrix of a Digraph:

Adjacency matrix of a directed graph is defined in a similar fashion. Let G be a directed graph with n vertices with no parallel edges. The adjacency matrix $A(D) = [a_{ij}]$ of the digraph is then defined as an n × n matrix containing zeros and ones with

a_{ij} = 1; if there is an edge directed from v_i to v_j

= 0; otherwise

Adjacency matrix is also called connection matrix (in network flow) or transition matrix. The directed graph D and its adjacency matrix are given below Fig. 6.20.



D

Fig. 6.20

$$A(D) = \begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline v_1 & 0 & 1 & 0 & 0 & 0 \\ v_2 & 0 & 0 & 0 & 1 & 1 \\ v_3 & 0 & 0 & 0 & 1 & 0 \\ v_4 & 0 & 0 & 0 & 0 & 1 \\ v_5 & 0 & 0 & 0 & 1 & 0 \end{array}$$

The adjacency matrix is used as a tool to investigate the various properties such as connectedness of a digraph by means of a digital computer.

Incidence Matrix of a Digraph:

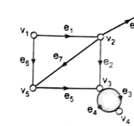
The incidence matrix of a diagraph with n vertices, e edges and no self loops is a matrix $X(G) = [x_{ij}]$ or order n × n, whose rows correspond to vertices and columns correspond to edges such that

x_{ij} = 1; if j^{th} edge e_j is incident out of i^{th} vertex v_i

= − 1; if j^{th} edge e_j is incident into i^{th} vertex v_i

= 0; if j^{th} edge e_j is not incident on i^{th} vertex v_i

A graph D and its incidence matrix are shown in Fig. 6.21.



D

Fig. 6.21

$$X(G) = \begin{array}{c|cccccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\ \hline v_1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ v_2 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ v_3 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 \\ v_4 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ v_5 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \\ v_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{array}$$

It is observed that, in the incident matrix for digraph the − number of ones in any row shows the out degree of the corresponding vertex. Similarly, the number of negative one's shows the in degree of the vertex.

6.5.3 Adjacency List Representation of Graph

- Graph with n number of vertices, the adjacency matrix representation requires space for n^2 elements. In many cases, the adjacency matrix is a sparse matrix, that is, it has a lot of zero elements, and thus considerable space is wasted in adjacency matrix representation of graph.
- To overcome with this problem, another representation of graph is used known as an **adjacency list representation of graph.** In this representation, each vertex in the graph has associated with it a list of all the vertices adjacent to it. Adjacency list of the graph specify the vertices that are adjacent to each vertex of the graph. For an undirected graph, if vertex v_i and vertex v_j are connected by an edge then adjacency list of vertex v_i will contain vertex v_j and adjacency list of vertex v_j must contain vertex v_i also. Consider the following undirected graph:

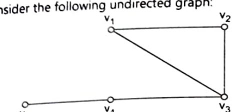


Fig. 6.22

The adjacency list of above undirected graph is

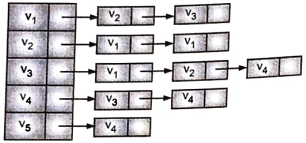
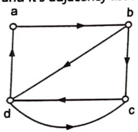


Fig. 6.23

For directed graph, if there is a directed edge from the vertex v_i to vertex v_j then adjacency list of vertex v_i will contain the vertex v_j.

A directed graph and it's adjacency list are given below:



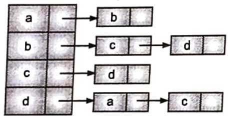


Fig. 6.24

It should be noted that for a directed graph, the space requirement for an adjacency list is n+e, where n is the number of vertices and e is the number of edges in the graph.

To represent a simple graph with relatively few edges (sparse graph), adjacency list is mostly preferable than adjacency matrix representation.

For a dense graph (graph with many edges), adjacency matrix is usually preferable over adjacency lists to represent the graph. This is because if we want to find whether there is an edge between the vertices v_i and v_j or not in a graph with n vertices, we have to search the list of vertices adjacent to either v_i and v_j to determine whether this edge is present or not. This require n comparisons when many edges are present in the graph. But in adjacency matrix representation of graph, we need to make only one comparison. We can check the edge between the vertices v_i and v_j by examining the (i, j)th entry in the adjacency matrix. If (i, j)th entry is 1, the graph contains the edge between the vertices v_i and v_j otherwise the entry is 0. Thus we make only one comparison, namely, comparing this entry with 0, to determine whether edge is present or not.

6.6 LINKED REPRESENTATION OF GRAPHS

- Many practical problems which involve large graphs are solved using digital computer. Various methods are available to store the graph in a computer. Sequential representation using adjacency matrix, edge listing, link representation is some of the methods used for this purpose. The link representation which is also called adjacency structure for storing the graph in computer is described in this section.
- In this method, the graph is represented by a linear array. After assigning the vertex, in any order, the number 1, 2, ... n, we represent each vertex k by a linear array, whose first element is a vertex k and whose remaining elements are the vertices that are immediate successor of k, that is, the vertices which have a directed path of length one from k (In an undirected graph these are the adjacent vertices to vertex k)
- For example, consider the digraph as shown in Fig. 6.25.

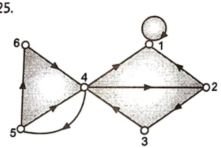


Fig. 6.25

- The vertices and their adjacent vertices are as follows:

1 : 1
2 : 1, 3
3 : 4
4 : 1, 2, 5
5 : 4, 6
6 : 4

The above adjacency can be represented as

G = [1: 1; 2: 1, 3; 3: 4; 4: 1, 2, 5; 5: 4, 6; 6: 4]

This is called link representation of the graph G.

- The link representation of a graph G stores the graph in the memory of computer by using its adjacency lists and will contain two files namely (i) vertex file and (ii) edge file. The vertex file will contain the list of vertices of G, usually maintained by a linked list and the edge file will contain all the edges of G. Each record of the edge file will correspond to a vertex in an adjacency list and hence, indirectly, to an edge of the graph G.

SOLVED EXAMPLES

Example 1: *Find the adjacency matrix of the following graphs:*

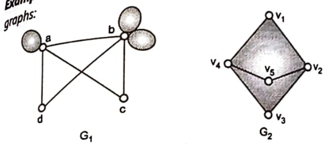


Fig. 6.26

Solution: The adjacency matrices for the graphs G_1 and G_2 are given below.

$$A(G_1) = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 1 & 1 & 1 \\ b & 1 & 2 & 1 & 0 \\ c & 1 & 1 & 0 & 0 \\ d & 1 & 1 & 0 & 0 \end{array}$$

$$A(G_2) = \begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline v_1 & 0 & 1 & 0 & 1 & 0 \\ v_2 & 1 & 0 & 1 & 0 & 1 \\ v_3 & 0 & 1 & 0 & 1 & 0 \\ v_4 & 1 & 0 & 1 & 0 & 1 \\ v_5 & 0 & 1 & 0 & 1 & 0 \end{array}$$

Example 2: *Draw the graph corresponding to each adjacency matrix.*

$$\text{(i)} \quad \begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline v_1 & 0 & 1 & 1 & 0 & 0 \\ v_2 & 1 & 0 & 1 & 0 & 0 \\ v_3 & 1 & 1 & 0 & 1 & 0 \\ v_4 & 0 & 0 & 1 & 0 & 1 \\ v_5 & 0 & 0 & 0 & 1 & 1 \end{array}$$

$$\text{(ii)} \quad \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 0 & 0 & 1 \\ b & 0 & 0 & 2 & 1 \\ c & 0 & 2 & 0 & 0 \\ d & 1 & 1 & 0 & 1 \end{array}$$

Solution: (i) The graph represented by adjacency matrix (i) is shown below.

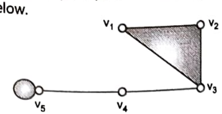


Fig. 6.27

(ii) The graph G with adjacency matrix (ii) is given by

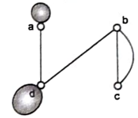


Fig. 6.28

Example 3: *Determine the incidence matrix of the following graphs.*

(i)

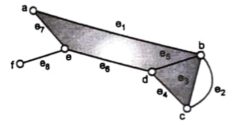


Fig. 6.29 (a)

(ii)

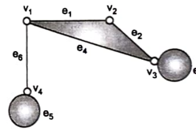


Fig. 6.29 (b)

Solution: (i) The incidence matrix for the graph shown in Fig. 6.29 (a) is as follows:

$$X(G) = \begin{array}{c|cccccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\ \hline a & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ b & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ c & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ e & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

(ii) The graph G in (b) has following incidence matrix.

$$X(G) = \begin{array}{c|cccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \hline v_1 & 1 & 0 & 0 & 1 & 0 & 1 \\ v_2 & 1 & 1 & 0 & 0 & 0 & 0 \\ v_3 & 0 & 1 & 1 & 1 & 0 & 0 \\ v_4 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}$$

Example 4: *Draw the graph for following incidence matrix.*

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Solution: The corresponding graph has the vertices and six edges shown below.

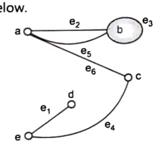


Fig. 6.30

Example 5: *Determine which of the graphs given in following Fig. 6.31 are Bipartite graphs.*

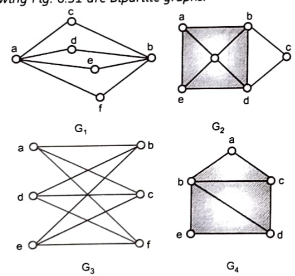


Fig. 6.31

Solution: Consider G_1, this is a bipartite graph because we can divide its vertex set V = {a, b, c, d, e, f} into two sets of vertices V = {a, b} and V_2 = {c, d, e, f} such that $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \emptyset$ also edges of G_1 join the vertices from V_1 to V_2.

For G_2, it is not a bipartite graph because we cannot partition its vertex set into two parts satisfying the conditions for bipartite graphs.

Consider G_3, it is a bipartite graph because its vertex set V can be partitioned into two subsets V_1 and V_1 where V_1 = {a, d, e}, V_2 = {b, c, f} such that $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$. Also edges of G_3, join the vertices of V_1 to vertices of V_2. In fact, G_3 is a complete bipartite graph because each vertex of V_1 is joined to each vertex of V_2. G_4 is not a bipartite graph.

Example 6: *Draw a complete bipartite graph which is not a regular graph.*

Solution: A complete bipartite graph $k_{m,\,n}$ is not a regular graph if m ≠ n. For, $k_{2,\,3}$ shown in Fig. 6.32 is not a regular graph.

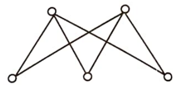


Fig. 6.32

Example 7: *How many edges has each of the following graphs: (i) k_{10} (ii) $k_{5,\,7}$.*

Solution: (i) k_{10} is a complete graph on 10 vertices and will have $\dfrac{10 \times 9}{2}$ = 45 edges.

(ii) $k_{5,\,7}$ is a complete bipartite graph and will have 5 × 7 = 35 edges.

Example 8: *Draw a graph which is regular of degree 3 (other than k_4 and $k_{3,3}$)*

Solution: The Peterson graph is a regular graph of degree 3 which is shown in Fig. 6.33.

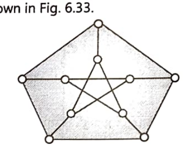


Fig. 6.33

Example 9: *Draw the directed graph whose adjacency matrix is given below.*

$$A(D) = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array}\begin{array}{c} \begin{array}{cccccc} a & B & c & d & e & f \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Solution: The directed graph with six vertices with adjacency matrix given above is as follows.

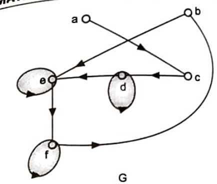


Fig. 6.34

Example 10: *Draw the directed graph with incidence matrix shown below.*

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\begin{array}{c} \begin{array}{ccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \end{array} \\ \begin{bmatrix} -1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 \end{bmatrix} \end{array}$$

Solution: The directed graph with five vertices and 9 edges with given incidence matrix is as follows:

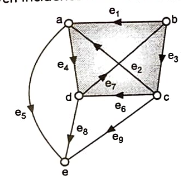


Fig. 6.35

Example 11: *Find the adjacency matrix, incidence matrix and linked representation for the graph shown in the following Fig. 6.36.*

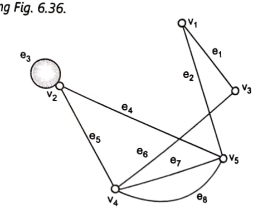


Fig. 6.36

Solution: Since the graph has five vertices, its adjacency matrix will contain five rows and five columns. It is given by

$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array}\begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2 \\ 1 & 1 & 0 & 2 & 0 \end{bmatrix} \end{array}$$

The given graph has five vertices and eight edges therefore its incidence matrix will be 5 × 8 matrix given by

$$X(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array}\begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

The adjacency structure of vertices is given by

v_1 : v_3, v_5

v_2 : v_2, v_4, v_5

v_3 : v_1, v_4

v_4 : v_2, v_3, v_5

v_5 : v_1, v_2, v_4

Hence the linked representation of given graph is

[v_1: v_3, v_5; v_2: v_2, v_4, v_5; v_3: v_1, v_4; v_4: v_2, v_3, v_5; v_5: v_1, v_2, v_4]

Example 12 : *Show that the maximum degree of any vertex in a simple graph with n vertices is (n – 1)*

Solution: Let G be a simple graph on n vertices. Consider any vertex v of G.

Since the graph is simple (i.e. without self loops and parallel edges), the vertex v can be adjacent to at most remaining (n – 1) vertices. Hence, the degree of the vertex v can be at the most (n – 1) Hence maximum degree of any vertex in a simple graph with n vertices is (n – 1)

Example 13: *Show that the maximum number of edges in a simple graph with n vertices is $\dfrac{n\,(n-1)}{2}$.*

Solution: Let G be a simple graph with n vertices. By handshaking lemma,

$$\sum_{i=1}^{n} d\,(v_i) = 2\,e$$ where, e is the number of edges in the graph G.

$\Rightarrow$ $d\,(v_1) + d\,(v_2) + \ldots + d\,(v_n) = 2e$

We know that the maximum degree of each vertex in the graph G can be at the most (n – 1)

Therefore, above equation reduces to

$$\underbrace{(n-1) + (n-1) + \ldots + (n-1)}_{n \text{ times}} = 2e$$

$$\Rightarrow \quad n(n-1) = 2e$$

$$\Rightarrow \quad e = \frac{n(n-1)}{2}$$

Hence, the maximum number of edges in any simple graph with n vertices is $\frac{n(n-1)}{2}$.

Example 14: *How many nodes are necessary to construct a graph with exactly 6 edges in which each node is of degree 2.*

Solution: Suppose there are n vertices (nodes) in the graph with 6 edges. Also, given the degree of each vertex is 2. Therefore, by handshaking lemma,

$$\sum_{i=1}^{n} d(v_i) = 2e = 2 \times 6$$

$$\Rightarrow d(v_1) + d(v_2) + \ldots + d(v_n) = 2 \times 6 = 12$$

$$\Rightarrow \underbrace{2 + 2 + \ldots + 2}_{n \text{ times}} = 12$$

$$\Rightarrow \quad 2n = 12$$

$$\Rightarrow \quad n = 6$$

Hence, 6 nodes are required to construct a graph with 6 edges in which each node is of degree 2.

Example 15: *Determine the number of edges in a graph with 6 nodes, 2 of degree 4 and 4 of degree 2. Draw two such graphs.*

Solution: Suppose the graph with 6 vertices has e number of edges. Therefore, by handshaking lemma,

$$\sum_{i=1}^{6} d(v_i) = 2e$$

$$\Rightarrow d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5) + d(v_6) = 2e$$

Now, given 2 vertices are of degree 4 and 4 vertices are of degree 2.

Hence, from the above equation

$$(4 + 4) + (2 + 2 + 2 + 2) = 2e$$

$$\Rightarrow \quad 16 = 2e$$

$$\Rightarrow \quad e = 8$$

Hence the number of edges in a graph with 6 vertices with given conditions is 8.

Two such graphs are shown below in Fig. 6.37.

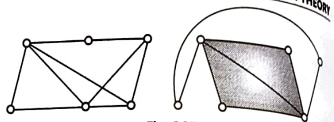


Fig. 6.37

Example 16: *Is it possible to construct a graph with 12 nodes such that 2 of the nodes have degree 3 and the remaining nodes have degree 4.*

Solution: Suppose it is possible to construct a graph with 12 nodes out of which 2 of them are having degree 3 and remaining vertices are having degree 4.

Hence, by handshaking lemma,

$$\sum_{i=1}^{12} d(v_i) = 2e$$

where e is the number of edges.

According to given conditions,

$$(2 \times 3) + (10 \times 4) = 2e$$

$$\Rightarrow \quad 6 + 40 = 2e$$

$$\Rightarrow \quad 46 = 2e$$

$$\Rightarrow \quad e = 23$$

It is possible to construct a graph with 23 edges and 12 nodes which satisfy given conditions.

Example 17: *A man is supposed to bring a dog, a sheep and a bag of cabbage across a river on a row boat. The boat is very small and he can carry only one of these items on the boat at a time. Furthermore, he cannot leave the dog alone with the sheep nor the sheep alone with the cabbage. Construct the graph and determine all possible ways for the man to transport all items across the river.*

Solution: Suppose 'M' denotes for man, 'S' for sheep, 'D' for dog and 'C' for cabbage.

In the beginning, suppose all the items are with man on one side of a river keeping in mind that sheep can eat the cabbage and the dog can eat the sheep, all the possible ways for the man to transport all items across the river are as follows:

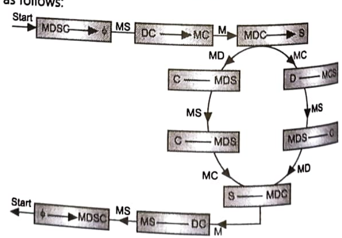


Fig. 6.38

Example 18: *An odd fellow wishes to have an odd party that is attended by an odd number of odd people, each of whom is acquainted with an odd number of other odd people at the party. Can this odd situation occur?*

Solution: Represent each person in a party (except host) by a vertex – and each pair of acquaintances by an edge. Then the odd party can be represented by a graph with odd number of vertices such that the degree of each vertex is odd. But which says that number of odd vertices in any graph is always even.

Hence the odd party is impossible.

Example 19: *Is it possible to draw a simple graph with 4 vertices and 7 edges? Justify.*

Solution: In the simple graph with n vertices, the maximum number of edges will be $\frac{n(n-1)}{2}$. Hence, a simple graph with 4 vertices will have at the most $\frac{4 \times 3}{2}$ = 6 edges. Therefore, the simple graph with 4 vertices can not have 7 edges. Hence such a graph does not exist.

Example 20 : *How many simple labeled graphs with n vertices are there?*

Solution: We know that a simple graph with n vertices has maximum possible number of edges $\frac{n(n-1)}{2}$.

To construct a simple graph with e number of edges with n vertices, can be done in $\binom{\frac{n(n-1)}{2}}{e}$ ways i.e. ${}^{n(n-1)/2}C_e$ ways.

The number of edges e can vary from zero to $\frac{n(n-1)}{2}$.

Hence the total number of ways to construct such type of graphs is given by

$$\binom{\frac{n(n-1)}{2}}{0} + \binom{\frac{n(n-1)}{2}}{1} + \binom{\frac{n(n-1)}{2}}{2} + \ldots\ldots + \binom{\frac{n(n-1)}{2}}{\frac{n(n-1)}{2}}$$

which is equal to $2^{\frac{n(n-1)}{2}}$ because

$$\binom{k}{0} + \binom{k}{1} + \binom{k}{2} + \ldots\ldots \binom{k}{k} = 2^k.$$

Example 21: *Consider a graph G=(V, E) where V={v_1 v_2 v_3} and degree(v_2) = 4.*

(i) *Does such simple graph exist? If not, why?*

(ii) *Does such multigraph exist? If yes, give example.*

Solution: (i) We know that in a simple graph with n vertices, the maximum number of edges are $\frac{n(n-1)}{2}$.

Hence, a simple graph with 3 vertices will have at the most 3 edges. These three edges will connect vertex v_1 to v_2, vertex v_2 to v_3 and vertex v_3 to v_1. Therefore the degree of the vertex v_2 can be at the most 2 in a simple graph with 3 vertices.

Thus a simple graph with 3 vertices v_1, v_2, v_3 where degree(v_2) =4 does not exist.

(ii) A multigraph with 3 vertices v_1, v_2, v_3 where degree(v_2) =4 exists. One such graph is shown below.

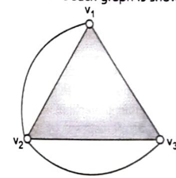


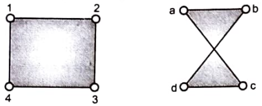
Fig. 6.39

6.7 ISOMORPHISM

- In geometry, two figures are thought of as equivalent (and called congruent) if they have identical behaviour in terms of geometric properties. Likewise, two graphs are thought of as equivalent (called **isomorphic**) if they have identical behaviour in terms of graph-theoretic properties. The problem of graph isomorphism arises in many field, such as chemistry, switching theory, information retrieval etc. Now we define isomorphism of graph as follows:
- Two graphs G_1 (V_1, E_1) and G' (V', E') are said to be isomorphic to each other if there is a one-one correspondence between their vertices and between their edges such that the incidence relationship is preserved. In other words, suppose that edge e is incident on vertices v_1 and v_2 of G, then the corresponding edge e' in G' must be incident on the vertices v_1' and v_2' that corresponds to v_1 and v_2.
- In particular, adjacency between vertices is preserved. More generally two graphs G_1 and G_2 are isomorphic if we can find the bijections f_V: $V_1 \to V_2$ and f_E: $E_1 \to E_2$ such that if $e \in E_1$, given by e = (a, b) where a and b are vertices in V_1 then $f_E(e) = (f_V(a), f_V(b))$, where $f_V(a)$ and $f_V(b)$ are vertices in V_2.

Isomorphic graphs are denoted by $G_1 \cong G_2$.

In following Fig. 6.40, G_1 is isomorphic to G_2.



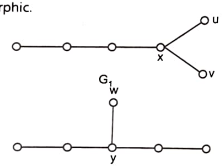
G_1 G_2

Fig. 6.40

The one-one correspondence between the vertices are

$1 \to a$
$2 \to b$
$3 \to d$
$4 \to c$

- It is immediately apparent by definition of isomorphism that two isomorphic graphs must have
  1. The same number of vertices.
  2. The same number of edges.
  3. An equal number of vertices with a given degree.
- However, these conditions are by no means sufficient. For instance, the two graphs shown in Fig. 6.41, satisfy all the three conditions given above, yet they are not isomorphic.



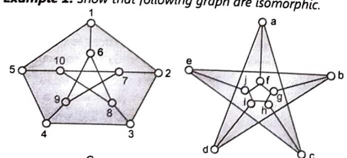
G_2

Fig. 6.41

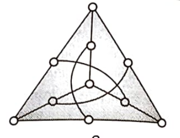
- In Fig. 6.41, the graph G_1 has a vertex x of degree 3 which is adjacent to two pendant vertices u and v and one vertex of degree 2. But in G_2, the vertex y of degree 3 is adjacent to only one pendant vertex w and two vertices of degree 2. Hence, adjacency is not preserved. Therefore graph G_1 is not isomorphic to G_2.

SOLVED EXAMPLES

***Example 1:** Show that following graph are isomorphic.*



G_1 G_2



G_3

Fig. 6.42

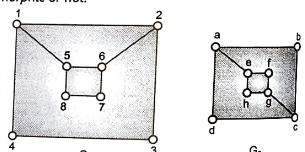
Solution: Consider the given graphs G_1 and G_2. Here both the graphs contain 10 vertices and 15 edges. The number of vertices of degree 3 is 10 in both the graphs. Also the adjacency is preserved. Hence, both the graphs are isomorphic.

Isomorphism is given by

$1 \to f$	$2 \to g$
$3 \to h$	$4 \to i$
$5 \to j$	$6 \to a$
$7 \to b$	$8 \to c$
$9 \to d$	$10 \to e$

In the similar way, we can show that G_2 and G_3 are also isomorphic.

***Example 2:** Determine whether following graphs are isomorphic or not.*



G_1 G_2

Fig. 6.43

Solution: Here both the graphs G_1 and G_2 contain 8 vertices and 10 edges. The number of vertices of degree 2 in both the graphs is four. Also the number of vertices of degree 3 in both the graphs is 4.

For adjacency, consider the vertex 1 of degree 3 in G_1. It is adjacent to two vertices of degree 3 and 1 vertex of degree 2. But in G_2 there does not exist any vertex of degree 3 which is adjacent to two vertices of degree 3 and 1 vertex of degree 2. Hence, adjacency is not preserved. Hence, given graphs are not isomorphic.

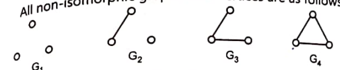
***Example 3:** Draw all non-isomorphic graphs on 2 and 3 vertices.*

Solution: All non-isomorphic graphs on 2 vertices are

G_1 G_2

Fig. 6.44

All non-isomorphic graphs on 3 vertices are as follows:

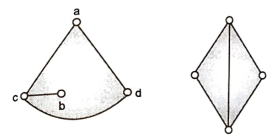


G_1 G_2 G_3 G_4

Fig. 6.45

***Example 4:** Find whether following pairs of graphs shown in Fig. 6.46, 6.47 and 6.48 and 6.49 are isomorphic or not.*

(i)

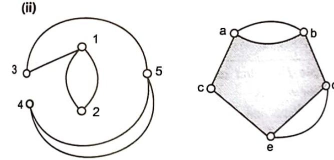


G_1 G_2

Fig. 6.46

Solution: Here G_1 and G_2 both have 4 vertices but G_1 has 4 edges and G_2 has 5 edges. Hence G_1 is not isomorphic to G_2.

(ii)

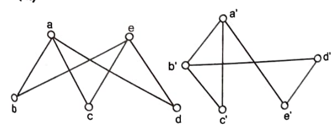


G_1 G_2

Fig. 6.47

Solution: Here again G_1 and G_2 both have 5 vertices but G_1 has 6 edges while G_2 has 7 edges. Hence $G_1 \equiv G_2$.

(iii)



G_1 G_2

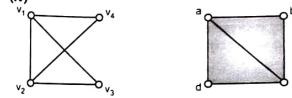
Fig. 6.48

Solution: G_1 and G_2 both have 5 vertices and 6 edges.

In G_1, the vertex 'a' of degree 3 is adjacent to 3 vertices of degree 2. But in G_2, both the vertices a' and b' of degree 3 are **not** adjacent to 3 vertices of degree 2. Hence, 'a' in G_1 can not be mapped to either a' or b' in G_2. Hence, bijection between vertices does not exist.

Hence G_1 is not isomorphic to G_2.

(iv)



G_1 G_2

Fig. 6.49

Solution: Here both the graphs have 4 vertices and 5 edges.

In G_1 and G_2 both, there are 2 vertices of degree 2 and 2 vertices of degree 3.

Also the adjacency is preserved. The one-one correspondence between vertices is given by

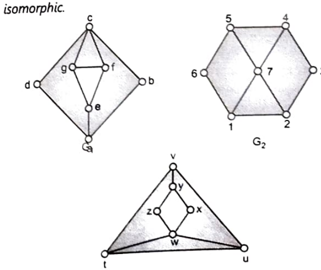
$v_1 \to a$
$v_2 \to c$
$v_3 \to d$
$v_4 \to b$

Hence $G_1 \equiv G_2$.

***Example 5:** Find whether k_6 and $k_{3,3}$ are isomorphic or not?*

Solution: k_6 and $k_{3,3}$ both contain 6 vertices. But k_6, the complete graph on 6 vertices has $\frac{6 \times 5}{2} = 15$ edges while $k_{3,3}$, the complete bipartite graph has $3 \times 3 = 9$ edges. Hence, k_6 is **not** isomorphic to $k_{3,3}$.

***Example 6:** Which of the graphs shown in Fig. 6.50 are isomorphic.*



G_3

Fig. 6.50

Solution: Note first that each of the graphs is simple, connected and has 7 vertices and 11 edges. Furthermore, each has one vertex of degree 4, four vertices of degree 3 and two vertices of degree 2. Now, consider the graph G_1. In this graph, the vertex C of degree 4 is adjacent to 2 vertices of degree 3, while in G_2, the vertex 7 of degree 4 is adjacent to four-vertices of degree 3. Hence, G_1 and G_2 are not isomorphic. Graphs G_1 and G_3 are isomorphic because adjacency is preserved. An isomorphism is defined by the following vertex bijection: $a \to y$, $b \to x$, $c \to w$, $d \to z$, $e \to v$, $f \to t$, $g \to u$. Hence $G_1 \cong G_3$ since, $G_1 \not\cong G_2$ and $G_1 \cong G_3$, therefore $G_2 \not\cong G_3$.

Hence, in the given Fig. 6.50 only G_1 and G_3 are isomorphic.

Example 7: *Are the graphs drawn below isomorphic? Why?*

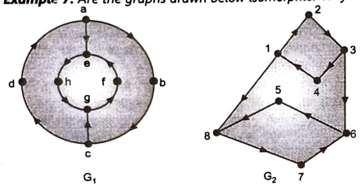


Fig. 6.51

Solution: Here both the graphs G_1 and G_2 have same number of vertices 8 and same number of edges 10. Furthermore, each has 4 vertices of degree 2 and 4 vertices of degree 3. Also the adjacency is preserved. The one-one correspondence between the vertices is given by

$a \to 1$ $b \to 2$
$c \to 3$ $d \to 4$
$e \to 8$ $f \to 5$
$g \to 6$
$h \to 7$

Hence G_1 and G_2 are isomorphic graphs.

Example 8: *State and justify whether the following graphs are isomorphic or not?*

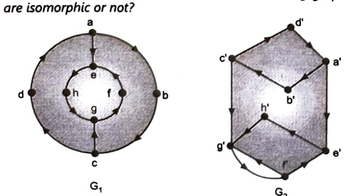


Fig. 6.52

Solution: Consider the graphs G_1 and G_2, both the graphs have same number of vertices 8, but the number of edges are not same in both the graphs. G_1 has 10 edges while G_2 has 11 edges. Therefore graphs are not isomorphic.

Example 9: *Show that pairs of graphs are isomorphic / not isomorphic.*

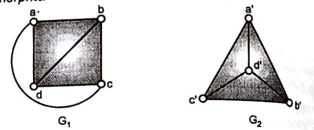


Fig. 6.53

Solution: Here both the graphs G_1 and G_2 have same number of vertices 4 and same number of edges 6. Also each vertex is of degree 3 and adjacency is preserved in both the graphs. The one-one correspondence between the vertices is given by

$a \to a'$

$b \to b'$

$c \to c'$

$d \to d'$

Hence, G_1 and G_2 are isomorphic graphs.

Here both the graphs are K_4 (the complete graph on 4 vertices)

Example 10: *Find all non-isomorphic connected graphs with four vertices.*

Solution: All non-isomorphic connected graphs with four vertices are

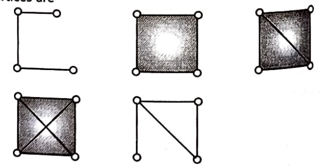


Fig. 6.54

Example 11: *Determine whether the following graphs G = (V, E) and G^* = (V^*, E^*) are isomorphic or not.*

G = ({a, b, c, d}, {(a, b), (a, d), (b, d), (c, d), (c, b), (d, c)})

G^* = ({1, 2, 3, 4}, {(1, 2), (2, 3), (3, 1), (3, 4), (4, 1), (4, 2)})

Solution: The graphs G and G^* are given by

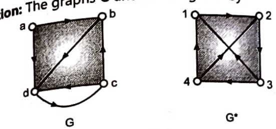


Fig. 6.55

Here G and G^* have same number of vertices and same number of edges but they are not isomorphic because indegree of vertex d in G is 3 and outdegree of d in G is 1 and there does not exist any vertex in G^* which has indegree 3 and outdegree 1. Therefore, G and G^* are not isomorphic.

Example 12: *State whether the given graphs are isomorphic or not.*

(i)

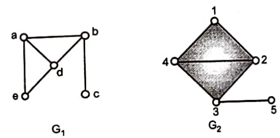


Fig. 6.56

(ii)

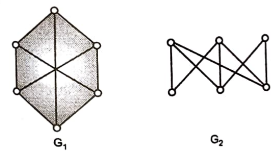


Fig. 6.57

Solution: (i) Here the graphs G_1 and G_2 have 5 vertices and six edges. Both the graphs have one vertex of degree 1, one vertex of degree 2 and three vertices of degree 3. Also the adjacency is preserved. Hence, both the graphs are isomorphic.

Isomorphism is given by

$c \to 5$

$b \to 3$

$d \to 2$

$e \to 1$

$a \to 4$

(ii) Here G_1 and G_2, both have six vertices but G_1 has 9 edges and G_2 has eight edges, therefore, G_1 and G_2 are non-isomorphic graphs.

Example 13: *Two graphs G and H with vertex sets V(G) and V(H) are drawn below. Determine whether or not G and H drawn below are isomorphic. If they are isomorphic, give a function g: V(G)→ V(H) that defines isomorphism. If they are not, explain why they are not.*

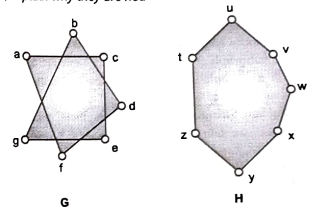


Fig. 6.58

Solution: Here the graphs G and H both have 7 vertices and seven edges. All seven vertices in both the graphs are of degree 2. Each vertex of degree 2 is adjacent to two vertices of degree two in both the graphs G and H. Hence adjacency is preserved in graphs G and H. Hence, both the graphs G and H are isomorphic.

Isomorphic function g: V(G)→ V(H) is given by

$g(a) \to t$

$g(c) \to u$

$g(f) \to z$

$g(e) \to v$

$g(g) \to w$

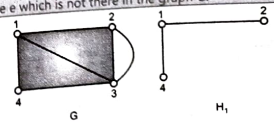
$g(b) \to x$

$g(d) \to y$

6.8 NEW GRAPHS FROM OLD ONES

In this section, we derive new graphs from old graphs.

Subgraph: Let G = (V, E) be any given graph. Then the graph G' = (V', E') is called a **subgraph** of G if V' ⊆ V and E' ⊆ E. In Fig. 6.59, H_1, H_2 and H_4 subgraphs of G but H_3 is not a **subgraph** of G because it contains the edge e which is not there in the graph G.



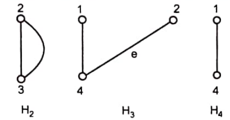


Fig. 6.59

Properties:

1. Each graph is a subgraph of itself.
2. A single vertex of a graph G is a subgraph of G.
3. A single edge together with its end vertices is also a subgraph of a graph G.
4. A subgraph of a subgraph of a graph G is a subgraph of G.

Edge Disjoint Subgraphs: Two subgraphs H_1 and H_2 of the graph G are said to be **edge disjoint subgraphs** of a graph G if there is **no edge** common between H_1 and H_2 (but may have vertex common)

In Fig. 6.59, H_1 and H_2 are edge disjoint subgraphs of graph G.

Vertex Disjoint Subgraphs: Two subgraphs H_1 and H_2 are said to be **vertex disjoint subgraphs** of a graph G if there is **no vertex** common between them (i.e. they do not have common edges also)

For example, in Fig. 6.59, H_2 and H_4 are vertex disjoint subgraphs of G but H_1 and H_2 are not.

Spanning Subgraph: Let G = (V, E) be any graph. Then the subgraph G' is said to be the **spanning subgraph** of the graph G if its vertex set V' is **equal to** the vertex set V of G.

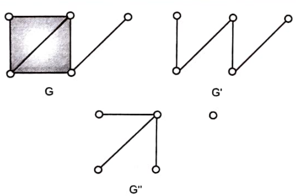


Fig. 6.60

In Fig. 6.60, G' and G'' are spanning subgraphs of the graph G.

Null Subgraph: A subgraph H of a graph G is called a null subgraph if its vertex set is same as the vertex set of G and its edge set is empty set. i.e. it does not contain any edges. A null subgraph is constructed from a graph G by deleting all the edges of G.

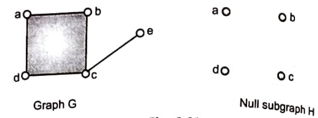


Fig. 6.61

Factors of a Graph: A **k-factor** of a graph is defined to be a spanning subgraph of the graph with the degree of each of its **vertex** being **k**.

For example, for the graph in Fig. 6.62 (a), its 1-factor graph is shown in Fig. 6.62 (b) and its 2-factor graph is shown in Fig. 6.62 (c)

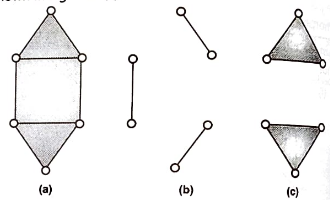


Fig. 6.62

A graph might have many different k-factors or might not have any k-factor at all for some k. Fig. 6.63 shows a graph which does not have any 1 factor graph.

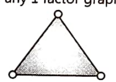


Fig. 6.63

Complement of a Graph: Let G be a simple graph. Then the **complement of G** denoted by $\bar{G}$ is the graph whose vertex set is the same as the vertex set of G and in which two vertices are adjacent if and only if they are **not** adjacent in G.

A graph and its complement are shown in Fig. 6.64.

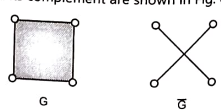


Fig. 6.64

Complement of a complete graph is a null graph and vice versa.

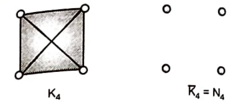
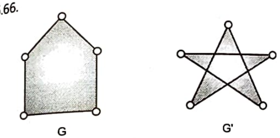


Fig. 6.65

A graph is said to be self complementary if it is isomorphic to its complement.

A graph G and its self complementary graph are shown in Fig. 6.66.



Graph G **Self Complementary Graph of G**

G ≅ G'

Fig. 6.66

SOLVED EXAMPLES

Example 1: *What is the complement of complete bipartite graph $K_{3,2}$. Is it a regular graph?*

Solution: The graph $K_{3,2}$ is given by

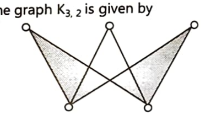


Fig. 6.67

It's complement is given by

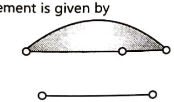


Fig. 6.68

It is not a regular graph.

Example 2: *Is the graph H shown in Fig. 6.69 is a subgraph of a graph G in Fig. 6.69. Is it a spanning subgraph of G?*

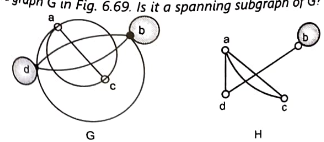


Fig. 6.69

Solution: Yes, H is a subgraph of G because its vertex set and edge set are subsets of vertex and edge sets of G. Also H contains all the vertices of G, hence, H is a spanning subgraph of G.

Example 3: *Which of the subgraphs of G are vertex disjoint and edge disjoint subgraphs.*

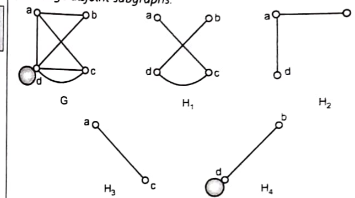


Fig. 6.70

Solution: Consider H_1 and H_2, both have common vertices. Hence they are **not** vertex disjoint subgraphs. In fact, they are edge disjoint subgraphs because edges are not common to them.

Consider (H_1, H_3) and (H_1, H_4), they are neither edge disjoint nor-vertex disjoint subgraphs. The vertex disjoint subgraphs are H_3 and H_4. For H_2 and H_3, they are edge disjoint subgraphs.

Example 4: *Find the complement of the following graph in Fig. 6.71. Is it self complementary?*

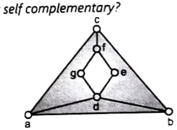


Fig. 6.71

Solution: The complement of the given graph is shown in Fig. 6.72.

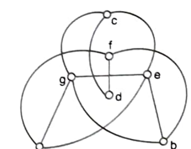


Fig. 6.72

The graph G is **not** self complementary because it is not isomorphic to its complement $\bar{G}$.

Example 5: *Find 1-factor graph of the following graph in Fig. 6.73. Find its 2-factor graph also, if possible.*

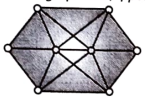


Fig. 6.73

Solution: 1-factor graph of the graph G in Fig. 6.74 is as follows:

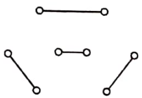


Fig. 6.74

2-factor graph of the graph G is

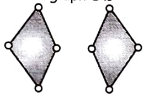


Fig. 6.75

Example 6: *For the following graphs, determine whether H = H(V', E') is a subgraph of G, where*

(i) V' = {A, B, F}, E' = [{A, B}, {A, F}]

(ii) V' = {B, C, D}, E' = [{B, C}, {B, D}]

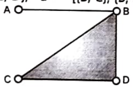


Fig. 6.76

Solution: (i) No, H is not a subgraph of G because vertex set V' of H contains the vertex F which does not belong to the graph G. Hence, H is not a subgraph of G.

(ii) Yes, all the vertices and edges of H belong to the graph G also. Hence, H is a subgraph of G.

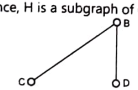


Fig. 6.77

Example 7: *Find all possible k-factors of the following graph ;* ***(Nov. 2014)***

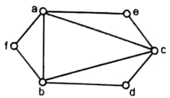


Fig. 6.78 (a)

Solution : 1-factor graph of the given graph is

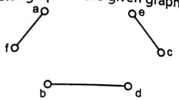


Fig. 6.78 (b)

2-factor graph of the given graph is

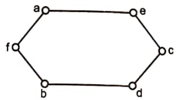
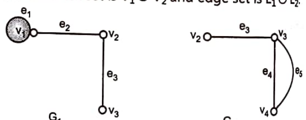


Fig. 6.78 (c)

6.9 OPERATIONS ON GRAPHS

In previous sections, we have defined graphs in terms of set of vertices and set of edges. Now we define some standard set theoretical operations like union, intersection etc. on the graphs.

1. **Union of Two Graphs:** Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be any two given graphs. The union of these two graphs is denoted by $G_1 \cup G_2$ and it is a graph whose vertex set is $V_1 \cup V_2$ and edge set is $E_1 \cup E_2$.

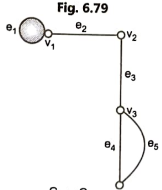


G_1 G_2

$V_1 = \{v_1, v_2, v_3\}$ $V_2 = \{v_2, v_3, v_4\}$

$E_1 = \{e_1, e_2, e_3\}$ $E_2 = \{e_3, e_4, e_5\}$

Fig. 6.79



$G_1 \cup G_2$

$V_1 \cup V_2 = \{v_1, v_2, v_3, v_4\}$

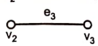
$E_1 \cup E_2 = \{e_1, e_2, e_3, e_4, e_5\}$

Fig. 6.80

G_1, G_2 are given in Fig. 6.79 and their union $G_1 \cup G_2$ is shown in Fig. 6.80. For any graph G, $G \cup G = G$.

2. **Intersection of Two Graphs:** The intersection of two graphs G_1 (V_1, E_1) and G_2 (V_2, E_2) is a graph whose vertex set is $V_1 \cap V_2$ and edge set is $E_1 \cap E_2$. It is denoted by $G_1 \cap G_2$.

Consider the graphs G_1 and G_2 shown in Fig. 6.79. Their intersection $G_1 \cap G_2$ is shown in the following Fig. 6.81.



$G_1 \cap G_2$

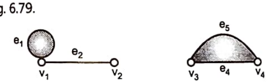
Fig. 6.81

If G_1 and G_2 are edge disjoint graphs then $G_1 \cap G_2$ is a null graph.

If G_1 and G_2 are vertex disjoint graphs then $G_1 \cap G_2$ is an empty set. For any graph G, $G \cap G = G$.

3. **Ring Sum of Two Graphs:** The ring sum of two graphs G_1 (V_1, E_1) and G_2 (V_2, E_2) is a graph consisting of the vertex set $V_1 \cup V_2$ and of edges that either in G_1 or G_2 but **not** in both. Ring sum is denoted by $G_1 \oplus G_2$.

In the following Fig. 6.82, the ring sum of two graphs is shown where G_1 and G_2 are the graphs given in Fig. 6.79.

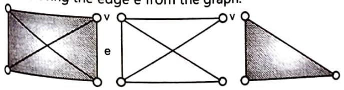


$G_1 \oplus G_2$

Fig. 6.82

For any graph G, $G \oplus G$ = a null graph.

4. **Removal of an Edge:** Let G (V, E) be any graph. Let $e \in E$. Then the graph (G – e) can be obtained by removing the edge e from the graph.



G G – e (a) G – v (b)

Fig. 6.83

In Fig. 6.83 for the graph G, the graph (G – e) is shown in Fig. 6.83 (a)

It is important to note that removal of any edge e from the graph G does **not** mean the removal of its **end vertices.**

5. **Removal of a Vertex:** Let G (V, E) by any graph. Let $v \in V$. The graph (G – v) can be obtained by removing the vertex v from the graph G. Removal of v means, removal of all these edges also which are incident on v.

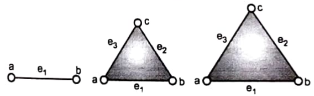
For the graph G in Fig. 6.83, the (G – v) is given in Fig. 6.83 (b)

SOLVED EXAMPLES

Example 1: *What is the union of (i) two null graphs N_3 and N_4 (ii) two complete graphs k_2 and k_3.*

Solution: (i) Union of two null graphs N_3 and N_4 is a null graph on seven vertices N_7.

(ii) Union of two complete graphs k_2 and k_3 is a complete graph k_3 as shown in Fig. 6.84.

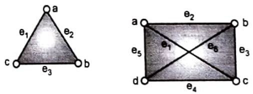


K_2 K_3 $K_2 \cup K_3 = K_3$

Fig. 6.84

Example 2: *What is the intersection of two complete graphs k_3 and k_4?*

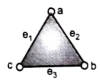
Solution: k_3 and k_4 are given by



K_3 K_4

Fig. 6.85

The intersection of k_3 and k_4 is k_3.



$K_2 \cap K_3 = K_3$

Fig. 6.86

Example 3: *Draw the graphs (i) G – v, (ii) G – e, where the graph G is shown in Fig. 6.87.*

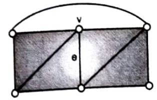


Fig. 6.87

Solution: The graph G – v, after deleting the vertex v from the graph G is shown in Fig. 6.88.

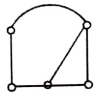


Fig. 6.88

The graph (G – e), after deleting the edge e from G is given by

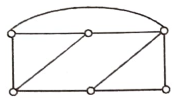


Fig. 6.89

EXERCISE - 6.1

1. Define the following graphs and give an example of each.
   (i) Bipartite graph **(April 2018)**
   (ii) Spanning subgraph **(April 2018)**
   (iii) Complement of a graph
   (iv) Subgraph **(April 2018, Oct. 2017)**
   (v) Complete graph
   (vi) Weighted graph.
   (vii) Multiple graphs.
   (viii) Complete Bipartite graph.
   (ix) Factors of a graph
   (x) Null subgraph
   (xi) Isomorphic graphs **(April 2018)**
   (xii) Directed and undirected graphs **(Oct. 2017)**
   (xiii) Adjacency & Incidence matrices of undirected graph **(April 2018)**
2. Define isomorphism of graphs. Are the graphs shown in the following Fig. 6.90 isomorphic? Justify your answer.

(i)

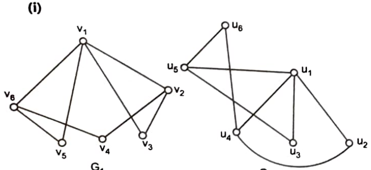


Fig. 6.90 (a)

(ii)

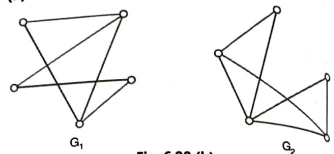


Fig. 6.90 (b)

(iii)

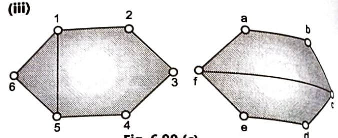


Fig. 6.90 (c)

3. Is there exist a regular graph of degree 5 on 9 vertices?
4. Find union and intersection of following graphs.

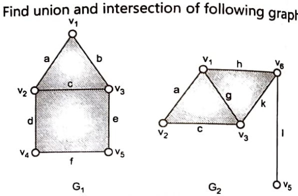


Fig. 6.91

5. Find (G – v) and (G – e) from the following graph.

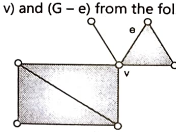


Fig. 6.92

6. Show that the number of edges in a complete graph K_n is $\frac{n(n-1)}{2}$.
7. Three married couples on a journey came to a river where they find a boat which cannot carry more than two persons at a time. The crossing of the river is complicated by the fact that the husbands are all very jealous and will not permit their wives to be left without them in a company where there are other men present. Construct a graph to show how it is possible.
8. Draw a graph with 4 nodes and 7 edges.

9. Draw all simple graphs on 4 nodes.
10. Draw a simple graph with 6 nodes all of degree 2 or greater and with at least 2 nodes of degree 3.
11. Check for isomorphism.

(i)

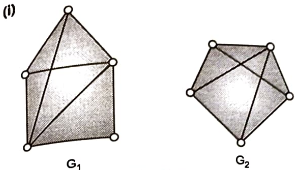


Fig. 6.93

(ii)

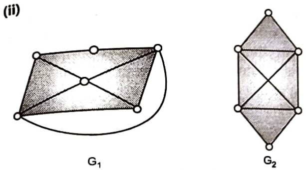


Fig. 6.94

12. State handshaking lemma. How many nodes are necessary to construct a graph with exactly 8 edges in which each node is of degree 2.

ANSWERS - 6.1

2. (i) Yes, $v_1 \to u_1$, $v_2 \to u_4$, $v_3 \to u_2$, $v_4 \to u_6$, $v_5 \to u_3$, $v_6 \to u_5$.
   (ii) No, graph G_1 has 6 edges and G_2 has 7 edges.
   (iii) No, vertex 3 of degree 2 in G_1 is adjacent to two vertices of degree 2. But in G_2, there does not exist any vertex of degree 2 fulfilling this condition. Hence, adjacency is not preserved.
3. No, the total degree of the graph = 5 × 9 = 45 which is not an even number.
4. $G_1 \cup G_2$ =

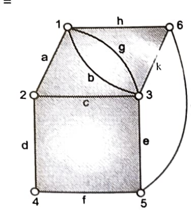


Fig. 6.95

$G_1 \cap G_2$

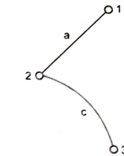


Fig. 6.96

5. G – v

G – e

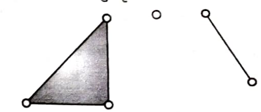


Fig. 6.97

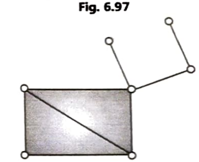
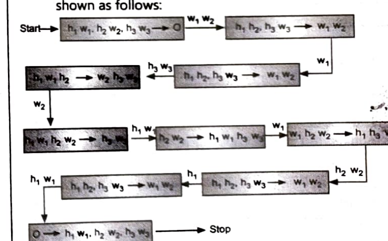


Fig. 6.98

7. Suppose 3 couple be denoted by (h_1, w_1), (h_1, w_2) and (h_3, w_3) The possible way to cross the river is shown as follows:



8.

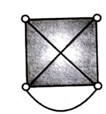


Fig. 6.99

9. Each simple graph with 4 nodes is isomorphic to one of the following:

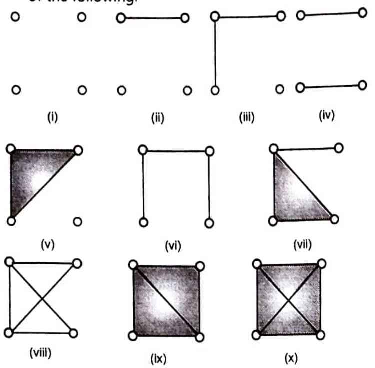


Fig. 6.100

10.

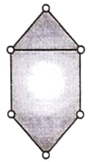


Fig. 6.101

6.10 PATHS AND CIRCUITS

- In this section, we will introduce the concept of paths, circuits, Euler graphs, Hamiltonian graphs which deal mainly with the nature of connectivity in graphs. Planarity of the graphs is also discussed in detail.
- The solutions of some practical problems like Konigsberg bridge problem, three utilities problem are also described here.
- Before defining Euler's and Hamiltonian graphs, first we will give a formal definition of a path and a circuit and discuss some related concepts.
- Let G = (V, E) be any graph and let v_o and v_n be any two vertices in V. A **path P of length n** from v_o to v_n is a sequence of vertices and edges of the form (v_o, e_1, v_1, e_2, ... e_n v_n) where each e_j is an edge between v_{j-1} and v_j. The vertices v_o and v_n are called the **end points** of the path and the other vertices v_1, v_2 .... v_{n-1} are called its **interior vertices**.

Following are same paths in the graph G shown in Fig. 6.102.

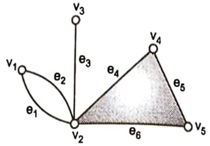


Fig. 6.102

Path I → v_1 e_2 v_2 e_4 v_4

Path II → v_4 e_4 v_2 e_2 v_1 e_1 v_2 e_3 v_3

Path III → v_3 e_3 v_2 e_2 v_1 e_1 v_2 e_3 v_3

In a simple graph with no loops and parallel edges, a path may be described by giving only the sequence of vertices traversed in the path. For example, in the following Fig. 6.103 the path (v_5 e_4 v_1 e_1 v_2 e_2 v_3) can be written as (v_5 v_1 v_2 v_3) also.

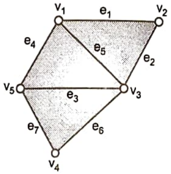


Fig. 6.103

6.10.1 Simple Path

A path in a graph G is called a **simple path** if the edges do **not** repeat in the path.

In Fig. 6.102, path I, and path II are simple paths but path III is not a simple path because the edge e_3 is repeated twice.

6.10.2 Elementary Path

A path is said to be an **elementary** path if vertices do **not** repeat in the path.

For example, the path I in Fig. 6.102 is an elementary path but path II is not.

6.10.3 Circuit

Suppose that C = (v_0, e_1, v_1 .... v_n) is a path in a graph G. If v_0 = v_n i.e. if the end vertices of the path are same then the path C is called a circuit.

In the following Fig. 6.104 C_1 = (v_1, e_1, v_2, e_2, v_1), C_2 = (v_3, e_4, v_1, e_1, v_2, e_3, v_3) are some examples of circuits.

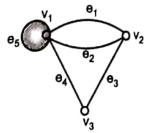


Fig. 6.104

6.10.4 Simple and Elementary Circuit

A circuit in the graph G is said to be a **simple circuit** if it does not include the same **edge** twice.

A circuit is said to be an **elementary circuit or cycle** if it does not meet the same **vertex** twice (except for first and last vertex)

Consider the graph G shown in Fig. 6.105. The simple circuit in G is (v_1 e_1 v_3 e_3 v_2 e_2 v_1) which is also an elementary circuit in G.

The circuit (v_1 e_2 v_2 e_6 v_4 e_7 v_5 e_8 v_3 e_4 v_4 e_5 v_1) is a simple circuit but it is not an elementary circuit because v_4 is repeated twice in the circuit.

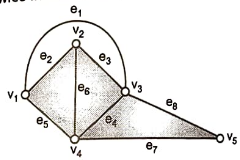


Fig. 6.105

Number of edges in any circuit or cycle is called the length of the circuit or cycle. If number of edges are even in the cycle, it is called even cycle and if number of edges are odd in the cycle, it is called odd cycle.

A graph is **bipartite** if and only if it does not contain an odd cycle.

Paths and circuits have been defined in a similar way for the directed graph also.

Now, we give a very important result which has a great significance in the vector space associated with the graph G.

Theorem : The ring sum of two circuits in a graph G is either a circuit or an edge disjoint union of circuits.

A graph which does not contain any circuit is called **acyclic graph.**

6.11 CONNECTED AND DISCONNECTED GRAPHS

Now, we define connected and disconnected graphs in terms of paths.

A graph is said to be a **connected graph** if there exists a **path** between every pair of vertices, otherwise the graph is **disconnected.**

It follows that disconnected graph consists of two or more parts called **components**, each of which is a connected graph and there is **no path** between two vertices if they belong to **different components**.

A connected graph has only one component.

Fig. 6.106

In Fig. 6.106, G_1 is a connected graph and G_2 is disconnected graph which has two components.

In a similar way, we can define a connected digraph also.

A directed graph or digraph is said to be **strongly connected** if for every pair of vertices a and b in the digraph, there is a path from a to b as well as a path from b to a.

A digraph is **weakly connected** if it is not strongly connected and its underlying graph is connected. Digraph which is neither strongly connected nor weakly connected is known as disconnected digraph.

As shown in Fig. 6.107, D_1 is strongly connected; D_2 is weakly connected diagraph because there does not exist any path from v_1 to v_5, also its underlying graph is connected as shown in Fig. 6.108.

Fig. 6.107

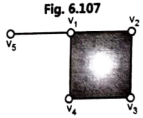


Fig. 6.108

(Underlying graph of D_2)

In a acyclic connected graph, there is exactly one path between any two vertices. These graphs are known as trees which we will study in next chapter. Thus a graph G is called as tree if it is connected acyclic graph.

Trees are connected graphs which do not contains any cycle. The number of edges in a tree with n vertices is (n – 1)

6.12 EDGE AND VERTEX CONNECTIVITY

In this section, we will see how the removal of edges and vertices disconnects the graph. Also we will define the terms edge connectivity and vertex connectivity which are useful in construction of communication network.

6.12.1 Edge Connectivity

In a **connected graph** G, a **cut-set** is a **minimal set** of edges whose removal disconnects the graph and increases the components of the graph by **one**.

In other words, a cutset in a connected graph G is a set of edges whose removal from G leaves G disconnected provided removal of **no** proper subset of these edges disconnects G.

For instance, in Fig. 6.109 $\{e_1, e_4, e_8\}$ is a cut set whereas $\{e_1, e_4, e_8, e_9\}$ is **not** a cut set because its subset $\{e_1, e_4, e_8\}$ is also a cut set.

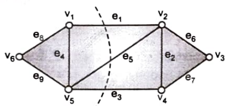


Fig. 6.109

Other cutsets in the graph are $\{e_6, e_7\}$, $\{e_8, e_9\}$, $\{e_1, e_3, e_5\}$ etc.

For the cutset $\{e_1, e_3, e_5\}$, the above graph G will have 2 components, namely G_1 and G_2 as given in Fig. 6.110.

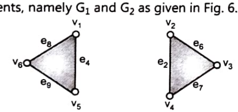


Fig. 6.110

Hence, if S is a cutset then (G – S) has exactly two components.

If the cutset of the connected graph contains only **one edge** then that edge is called an **isthmus** or **bridge** i.e. removal of an isthmus (edge) disconnects the graph. e is an isthmus in the graph given in Fig. 6.111.

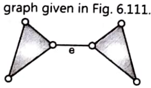


Fig. 6.111

The number of edges in a smallest cutset of a connected simple graph G is called an **edge connectivity** of the graph G and it is denoted by λ (G) In other words, λ(G) is the smallest number of edges in G whose removal disconnects G. In Fig. 6.110, λ (G) = 2. For Fig. 6.111 λ (G) = 1.

6.12.2 Vertex Connectivity

The vertex connectivity K (G) of a simple connected graph G is defined as the **smallest number of vertices whose removal disconnects the graph.**

For instance, k (G) = 2 in the following Fig. 6.112.

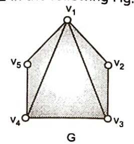


Fig. 6.112

A graph is said to be **k-connected** if its vertex connectivity is k.

A graph is said to be a **separable graph** if its vertex connectivity is **one.** In a separable graph, a vertex whose removal disconnects the graph is said to be a **cut vertex** or a **cut point**.

In Fig. 6.113, v is a cut point.

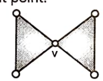


Fig. 6.113

The above defined terms; edge connectivity and vertex connectivity are related to the minimum degree of a vertex in the graph. This relation is given by

$$k\,(G) \le \lambda\,(G) \le \delta \qquad \text{... (6.2)}$$

where, k (G) is the vertex connectivity of G,

λ (G) is the edge connectivity of G, and

δ is the minimum degree of a vertex in the graph G.

The edge connectivity λ(G) is also related to the number of edges and vertices in the given graph and which is given by

$$\boxed{\lambda(G) \le \left[\frac{2e}{n}\right]} \qquad \text{... (6.3)}$$

where, e is the number of edges and n is the number of vertices in the graph G.

SOLVED EXAMPLES

Example 1: *Find the edge connectivity for the complete graph K_5.*

Solution: For complete graph K_5, the number of vertices is 5 and the degree of each vertex is 4 as shown in following Fig. 6.114. If we remove all four edges incident on any vertex, then the graph will become disconnected.

Fig. 6.114

Hence it edge connectivity λ (G) = 4.

Example 2: *Find k (G), λ (G) for $K_{4,3}$ the complete bipartite graph.*

Solution: The complete bipartite graph $K_{4,3}$ is given by,

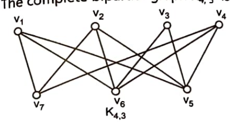


Fig. 6.115

To make the graph disconnected, the smallest cutset should contain 3 edges. Hence its edge connectivity λ (G) = 3.

If we remove v_5, v_6, v_7 vertices then the graph will be disconnected. Hence its vertex connectivity k (G) = 3.

Example 3: *Find the edge connectivity of the following graph.*

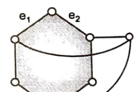


Fig. 6.116

Solution: In the given graph, total number of edges e = 9 and total number of vertices n = 7.

By the edge connectivity $\le \left[\frac{2\,e}{n}\right]$

$\Rightarrow$ The edge connectivity $\le \left[\frac{18}{7}\right] = 2.$

From the graph, it is clear that the edge connectivity = 2.

(Remove two edges e_1 and e_2, the graph becomes disconnected)

Example 4: *Suppose that we have 6 houses and enough wire to establish telephone links between 10 pair of houses. How should this be done so as to make it possible for any two houses to communicate and so as to minimize the danger of disrupted communications due to several lines?*

Solution: Represent the houses by nodes and wires by edges. Given that there are 6 houses i.e. 6 nodes and 10 edges in the graph.

According to the problem, we have to find the minimum number of edges where removal of disconnects the graph i.e. we have to find the edge connectivity of the graph.

From the edge connectivity is less than or equal to $\left[\frac{2 \times 10}{6}\right] = 3.$

i.e. the degree of each vertex should be at least 3 in the graph which has 6 vertices and 10 edges. The following graph satisfies all the above conditions and has edge connectivity 3.

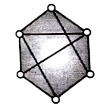


Fig. 6.117

Example 5: *For a given graph G, find*

(i) all simple paths from A to C,

(ii) all cycle,

(iii) subgraph H of G generated by H = {B, C, X, Y}

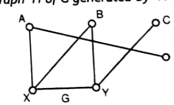


Fig. 6.118 (a)

Solution: (i) Simple paths from A to C are

(a) AXYC

(b) AXBYC

(ii) Cycle in G is XYBX.

(iii) Subgraphs H_1, H_2, H_3 generated by the vertices B, C, X, Y are given by

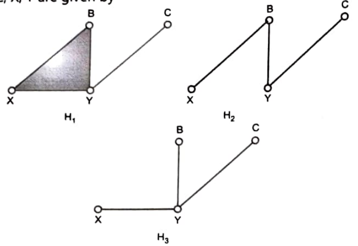


Fig. 6.118 (b)

6.13 SHORTEST PATH ALGORITHM

- A large number of optimization problems are mathematically equivalent to finding the shortest paths in a graph. For example, given a railway network connecting several cities, it is required to find shortest route between two cities.
- In this case, we represent the railway network by the graph in which vertices represent the cities and the edges represent the railway routes. The weight of the edge is the distance between two cities. Then the given problem of finding the shortest route between two cities reduces to finding the shortest distance between two vertices.
- We now give an algorithm for solving the shortest path problem. The algorithm was found by Dijkstra in 1959 and is known as Dijkstra's shortest path algorithm. This algorithm gives the shortest length of the path from the vertex 'a' to the vertex 'z' but it does not give the actual path for the shortest distance from the vertex a to the vertex z.

Dijkstra's Algorithm to find the shortest path from the vertex a to the vertex z.

Let G = (V, E) be a simple graph. Let a and z be any two vertices of the graph. Suppose L (x) denotes the label of the vertex z which represents the **length of the shortest path** from the vertex a to the vertex (z) w_{ij} denotes the weight of the edge $e_{ij} = (v_i, v_j)$

Step 1: Let P = ϕ where P is the set of those vertices which have permanent labels and T = {all vertices of the graph G}

Set L (a) = 0, L (x) = ∞ $\forall$ x $\in$ T and x $\neq$ a

Step 2: Select the vertex v in T which has the smallest label. This label is called the permanent label of v.

Also set P = P $\cup$ { v } and T = T – { v }. If v = z, then L (z) is the length of the shortest path from the vertex a to z and stop.

Step 3: If v $\neq$ z, then revise the labels of vertices of T i.e. the vertices which do not have permanent labels. The new lable of a vertex x in T is given by

L (x) = min {old L (x), L (v) + w (v, x)}

where w (v, x) is the weight of the edge joining the vertex v and x.

If there is no direct edge joining v and x then take w (v, x) = ∞.

Step 4: Repeat steps 2 and 3 until z gets the permanent label.

SOLVED EXAMPLES

Example 1: *Explain Dijkstra's shortest path algorithm to obtain a shortest path between two vertices in the graph.* **(Oct. 2017)**

Determine a shortest path between the vertices a and z as shown in the graph below. The numbers associated with the edges are the distances between vertices. **(April 2019, Dec. 14)**

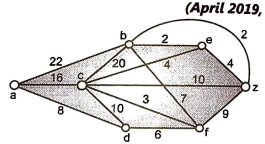


Fig. 6.119

Solution: Dijkstra's Algorithm to find the shortest path from a to z is as follows:

(i) P = ϕ, T = {a, b, c, d, e, f, z}

L (a) = 0, L (x) = ∞, $\forall$ x = T x $\neq$ a

(ii) V = a

P = { a } T = {b, c, d, e, f, z}

L (b) = min {old L (b), L (a) + w (a, b)}

L (b) = min {∞, 0 + 22}

L (b) = 22

Similarly, L (c) = 16

L (d) = 8

L (e) = ∞

L (f) = ∞

L (z) = ∞

(iii) V = d; the permanent label of d is 8

P = {a, d}, T = {b, c, e, f, z}

L (b) = min {old L (b), L (d) + w (b, d)}

= min {22, 8 + ∞} = 22

L (c) = min {16, 8 + 10} = 16

L (e) = min {∞, 8 + ∞} = ∞

L (f) = min {∞, 8 + 6} = 14

L (z) = min {∞, 8 + ∞} = ∞.

(iv) V = f, the permanent label of f is 14.

P = {a, d, f}, T = {b, c, e, z}

L (b) = min {22, 14 + 7} = 21

L (c) = min {16, 14 + 3} = 16

L (e) = min {∞, 14 + ∞} = ∞

L (z) = min {∞, 14 + 9} = 23

(v) V = c, the permanent label of c is 16.

P = {a, d, f, c}, T = {b, e, z}

L (b) = min {21, 16 + 20} = 21

L (e) = min {∞, 16 + 4} = 20

L (z) = min {23, 16 + 10} = 23

(vi) V = e, the permanent label of e is 20,

P = {a, d, f, c, e}, T = {b, z}

L (b) = min {21, 20 + 2} = 21

L (z) = min {23, 20 + 4} = 23

L (f) = min {∞, 0 + ∞} = ∞

L (g) = min {∞, 0 + ∞} = ∞

Similarly L (h) = ∞, L (i) = ∞, L (z) = ∞

(vii) V= b, the permanent label of b is 21.

P = {a, d, f, c, e, b}, T = { z }

L (z) = min {23, 21 + 2} = 23.

Now the permanent label of z is 23. Hence, the length of the shortest path from the vertex a to the vertex z is **23.**

The shortest path is adfz shown in the following Fig. 6.120

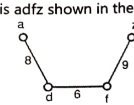


Fig. 6.120

Example 2: *Apply Dijkstra's shortest path algorithm to obtain the shortest path between vertices a and z in the Fig. 6.121 (a) shown below.*

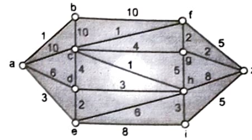


Fig. 6.121 (a)

Solution: The steps of Dijkstra's algorithm to find the shortest path from a to z are shown as follows.

(i) P = ϕ, T = {a, b, c, d, e, f, g, h, i, z}

L (a) = 0, L (x) = ∞ $\forall$ x $\in$ T, x $\neq$ a

(ii) V = a, the permanent label of a is 0

P = { a }, T = {b, c, d, e, f, g, h, i, z}

L (b) = min {∞, 0 + 1} = 1

L (c) = min {∞, 0 + 10} = 10

L (d) = min {∞, 0 + 6} = 6

L (e) = min {∞, 0 + 3} = 3

L (f) = min {∞, 0 + ∞) = ∞

L (g) = min {∞, 0 + ∞} = ∞

Similarly, L (h) = ∞, L (z) = ∞

(iii) V = b the permanent label of b is 1.

P = {a, b} T = {c, d, e, f, g, h, i, z}.

L (c) = min {10, 1 + 10} = 10

L (d) = min {6, 1 + ∞} = 6

L (e) = min {3, 1 + ∞} = 3

L (f) = min {∞, 1 + 10} = 11

L (g) = min {∞, 1 + ∞} = ∞

L (h) = ∞, L (i) = ∞, L (z) = ∞.

(iv) V = e, the permanent label of e is 3.

P = {a, b, e), T = {c, d, f, g, h, i, z}

L (c) = min {10, 3 + ∞} = 10

L (d) = min {6, 3 + 2} = 5

L (f) = min {11, 3 + ∞} = 11

L (g) = min {∞, 3 + ∞} = ∞

L (h) = min {∞, 3 + 6} = 9

L (i) = min {∞, 3 + 8} = 11

L (z) = min {∞, 3 + ∞} = ∞

(v) V = d, the permanent label of d is 5.

P = {a, b, e, d}, T = {c, f, g, h, i, z}

L (c) = min {10, 5 + 4} = 9

L (f) = min {11, 5 + ∞} = 11

L (g) = min {∞, 5 + ∞} = ∞

L (h) = min {9, 5 + 3} = 8

$L(i) = \min\{11, 5 + \infty\} = 11$

$L(z) = \min\{\infty, 5 + \infty\} = \infty$

(vi) V = h the permanent label of h is 8.

$P = \{a, b, e, d, h\}, T = \{c, f, g, i, z\}$

$L(c) = \min\{9, 8, + 1\} = 9$

$L(f) = \min\{11, 8 + \infty\} = 11$

$L(g) = \min\{\infty, 8 + 5\} = 13$

$L(i) = \min\{11, 8 + 3\} = 11$

$L(z) = \min\{\infty, 8 + 8\} = 16$

(vii) V = c, the permanent label of C is 9.

$P = \{a, b, e, d, h, c\}, T = \{f, g, i, z\}$

$L(f) = \min\{11, 9 + 1\} = 10$

$L(g) = \min\{13, 9 + 4\} = 13$

$L(i) = \min\{11, 9 + \infty\} = 11$

$L(z) = \min\{16, 9 + \infty\} = 16$

(viii) .V = f, the permanent label of f is 10.

$P = \{a, b, e, d, h, c, f\}, T = \{g, i, z\}$

$L(g) = \min\{13, 10 + 2\} = 12$

$L(i) = \min\{11, 10 + \infty\} = 11$

$L(z) = \min\{16, 10 + 5\} = 15$

(ix) V = i, the permanent level of i is 11.

$P = \{a, b, e, d, h, i, f, i\}, T = \{g, z\}$

$L(g) = \min\{12, 11 + \infty\} = 12$

$L(z) = \min\{15, 11 + 5\} = 15.$

v = g, the permanent label of g is 12.

$P = \{a, b, e, d, h, c, f, i, g\}, T = \{z\}$

$L(z) = \min\{15, 12 + 2\} = 14$

v = z, the permanent label of z is 14.

Hence, the length of shortest path from a to z is 14.

The shortest path is a e d c f g z which is shown in the following Fig. 6.121 (b)

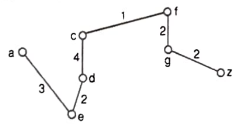


Fig. 6.121 (b)

Example 3: *The graph in Fig. 6.122 shows the communication channels and the communication time delays in the channels among eight communication centres. The centres are represented by vertices, the channels are represented by edges, and the communication time delay in minutes in each channel is represented by the weight of the edge. Suppose that at 3.00 p.m. communication centre 'a' broadcasts through all it channels the news that someone has found a way to build a better mouse trap. Other communication centres will then broadcast this news through their channels as soon as they receive it. For the communication centres b, c, d, e, f, g and h, determine the earliest time each receives the news.*

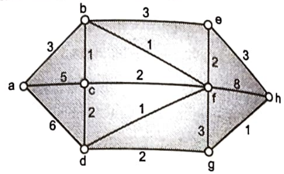


Fig. 6.122

Solution: In this problem, we have to find a shortest path from the communication centre 'a' to each centre b, c, d, e, f, g and h, which can be done by using Dijkstra's shortest path algorithm.

(i) $P = \phi, T = \{a, b, c, d, e, f, g, h\}$

$L(a) = 0, L(x) = \infty, \forall x \in T, x \neq a$

(ii) V = a, the permanent label of a is 0.

$P = \{a\}, T = \{b, c, d, e, f, g, h\}$

$L(b) = \min\{\infty, 0 + 3\} = 3$

$L(c) = \min\{\infty, 0 + 5\} = 5$

$L(d) = \min\{\infty, 0 + 6\} = 6$

$L(e) = \min\{\infty, 0 + \infty\} = \infty$

Similarly $L(f) = \infty$, $L(g) = \infty$, $L(h) = \infty$.

(iii) V = b, the permanent label of b is 3.

$P = \{a, b\}, T = \{c, d, e, f, g, h\}$

$L(c) = \min\{5, 3 + 1\} = 4$

$L(d) = \min\{6, 3 + \infty\} = 6$

$L(e) = \min\{\infty, 3 + 3\} = 6$

$L(f) = \min\{\infty, 3 + 1\} = 4$

$L(g) = \min\{\infty, 3 + \infty\} = \infty$

$L(h) = \min\{\infty, 3 + \infty\} = \infty$

(iv) V = c, the permanent label of c is 4.

$P = \{a, b, c\}, T = \{d, e, f, g, h\}$

$L(d) = \min\{6, 4 + 2\} = 6$

$L(e) = \min\{6, 4 + \infty\} = 6$

$L(f) = \min\{4, 4 + 2\} = 4$

$L(g) = \min\{\infty, 4 + \infty\} = \infty$

$L(h) = \min\{\infty, 4 + \infty\} = \infty$

(v) V = f, the permanent label of f is 4.

$P = \{a, b, c, f\}, T = \{d, e, g, h\}$

$L(d) = \min\{6, 4 + 1\} = 5$

$L(e) = \min\{6, 4 + 2\} = 6$

$L(g) = \min\{\infty, 4 + 4\} = 8$

$L(h) = \min\{\infty, 4 + 8\} = 12$

(vi) V = d, the permanent label of d is 5.

$P = \{a, b, c, f, d\}\ T = \{e, g, h\}$

$L(e) = \min\{6, 5 + \infty\} = 6$

$L(g) = \min\{8, 5 + 2\} = 7$

$L(h) = \min\{12, 5 + \infty\} = 12$

(vii) V = e, the permanent label of e is 6.

$P = \{a, b, c, f, d, e\}\ T = \{g, h\}$

$L(g) = \min\{7, 6 + \infty\} = 7$

$L(h) = \min\{12, 6 + 3\} = 9$

(viii) V = g, the permanent label of g = 7

$P = \{a, b, c, f, d, e, g,\}, T = \{h\}$

$L(h) = \min\{9, 7 + 1\} = 8$

(ix) V = h, the permanent label of g is 8.

$P = \{a, b, c, f, d, e, g, h\}$

$T = \phi.$

Hence according to Dijkstra's shortest path algorithm

b will receive the news after 3 minutes

c will receive the news after 4 minutes

f will receive the news after 4 minutes

d will receive the news after 5 minutes

e will receive the news after 6 minutes

g will receive the news after 7 minutes

h will receive the news after 8 minutes.

Example 4: *Apply Dijkstra's shortest path algorithm to find the shortest path between vertices a and z in following graphs shown in*

(i) Fig 6.123(a)

(ii) Fig 6.123(b) ***(Oct. 2018)***

(i)

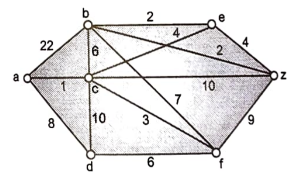


Fig. 6.123 (a)

(ii)

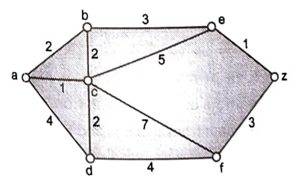


Fig. 6.123 (b)

Solution: (i) The steps involved in Dijkstra's Algorithm to find the shortest path from a to z are as follows:

(i) $P = \phi, T = \{a, b, c, d, e, f, z\}$

$L(a) = 0, L(x) = \infty \quad \forall x \in T, x \neq a$

(ii) V = a, the permanent label of a is 0.

$P = \{a\}, T = \{b, c, d, e, f, z\}$

$L(b) = \min(\infty, 0 + 22) = 22$

$L(c) = \min(\infty, 0 + 1) = 1$

$L(d) = \min(\infty, 0 + 8) = 8$

$L(e) = \min(\infty, 0 + \infty) = \infty$

$L(f) = \min(\infty, 0 + \infty) = \infty$

$L(z) = \min(\infty, 0 + \infty) = \infty$

(iii) V = c, the permanent label of c is 1.

$P = \{a, c\}, T = \{b, d, e, f, z\}$

$L(b) = \min(22, 1 + 6) = 7$

$L(d) = \min(8, 1 + 10) = 8$

$L(e) = \min(\infty, 1 + 4) = 5$

$L(f) = \min(\infty, 1 + 3) = 4$

$L(z) = \min(\infty, 1 + 10) = 11$

(iv) V = f, the permanent label of f = 4.

$P = \{a, c, f\}, T = \{b, d, e, z\}$

$L(b) = \min(22, 4 + 7) = 11$

$L(d) = \min(8, 4 + 6) = 8$

$L(e) = \min(5, 4 + \infty) = 5$

$L(z) = \min(11, 4 + 9) = 11$

(v) V = e, the permanent label of e = 5.

$P = \{a, c, f, e\}, T = \{b, d, z\}$

$L(b) = \min(11, 5 + 2) = 7$

$L(d) = \min(8, 5 + \infty) = 8$

$L(z) = \min(11, 5 + 4) = 9$

(vi) V = b, the permanent label of b = 7.

$P = \{a, c, f, e, b\}, T = \{d, z\}$

$L(d) = \min(8, 7 + \infty) = 8$

$L(z) = \min(9, 7 + 2) = 9$

(vii) V = d, the permanent label of d = 8.

$P = \{a, c, f, e, b, d\}, \quad T = \{z\}$

$L(z) = \min(9, 8 + \infty) = 9$

$\therefore$ The permanent label of z = 9.

Hence the length of shortest path from a to z is 9.

The shortest path is acez.

(2) According to Dijkstra's Algorithm, the shortest path from a to z can be calculated as follows:

(i) $P = \phi, T = \{a, b, c, d, e, f, z\}$

$L(a) = 0, \quad L(x) = \infty \quad \forall x \in T, \; x \neq a$

(ii) V = a, the permanent label of a = 0.

$P = \{a\}, \; T = \{b, c, d, e, f, z\}$

$L(b) = \min(\infty, 0 + 2) = 2$
$L(c) = \min(\infty, 0 + 1) = 1$
$L(d) = \min(\infty, 0 + 4) = 4$
$L(e) = \min(\infty, 0 + \infty) = \infty$
$L(f) = \min(\infty, 0 + \infty) = \infty$
$L(z) = \min(\infty, 0 + \infty) = \infty$

(iii) V = c, the permanent label of c = 1.

$P = \{a, c\}, T = \{b, d, e, f, z\}$

$L(b) = \min(2, 1 + 2) = 2$
$L(d) = \min(4, 1 + 2) = 3$
$L(e) = \min(\infty, 1 + 5) = 6$
$L(f) = \min(\infty, 1 + 7) = 8$
$L(z) = \min(\infty, 1 + \infty) = \infty$

(iv) V = b, the permanent label of b = 2.

$P = \{a, c, b\}, T = \{d, e, f, z\}$

$L(d) = \min(3, 2 + \infty) = 3$
$L(e) = \min(6, 2 + 3) = 5$
$L(f) = \min(8, 2 + \infty) = 8$
$L(z) = \min(\infty, 2 + \infty) = \infty$

(v) V = d, the permanent label of d = 3,

$P = \{a, c, b, d\}, T = \{e, f, z\}$

$L(e) = \min(5, 3 + \infty) = 5$
$L(f) = \min(8, 3 + 4) = 7$
$L(z) = \min(\infty, 3 + \infty) = \infty$

(vi) V = e, the permanent label of e = 5,

$P = \{a, c, b, d, e\}, T = \{f, z\}$

$L(f) = \min(7, 5 + \infty) = 7$
$L(z) = \min(\infty, 5 + 1) = 6$

(vii) V = z, the permanent label of z = 6.

Hence the length of the shortest path from a to z is 6. This shortest path is abez.

Example 5: *For the following graph in Fig. 6.124 (a), find the shortest path using Dijkstra's Algorithm.*

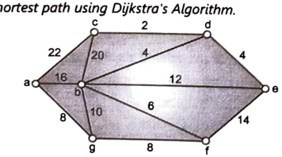


Fig. 6.124 (a)

Solution: According to the algorithm, the shortest path from a to e can be calculated as follows:

(i) $P = \phi \quad T = \{a, b, c, d, e, f, g\}$

$L(a) = 0, L(x) = \infty \quad \forall x \in T, \; x \neq a$

(ii) V = a, the permanent label of a = 0.

$P = \{a\}, \quad T = \{b, c, d, e, f, g\}$

$L(b) = \min(\infty, 0 + 16) = 16$
$L(c) = \min(\infty, 0 + 22) = 22$
$L(d) = \min(\infty, 0 + \infty) = \infty$
$L(e) = \min(\infty, 0 + \infty) = \infty$
$L(f) = \min(\infty, 0 + \infty) = \infty$
$L(g) = \min(\infty, 0 + 8) = 8$

(iii) V = g, the permanent label of g = 8.

$P = \{a, g\}, T = \{b, c, d, e, f\}$

$L(b) = \min(16, 8 + 10) = 16$
$L(c) = \min(22, 8 + \infty) = 22$
$L(d) = \min(\infty, 8 + \infty) = \infty$
$L(e) = \min(\infty, 8 + \infty) = \infty$
$L(f) = \min(\infty, 8 + 8) = 16$

(iv) V = b, the permanent label of b is 16.

$P = \{a, g, b\}, T = \{c, d, e, f\}$

$L(c) = \min(22, 16 + 20) = 22$
$L(d) = \min(\infty, 16 + 4) = 20$
$L(e) = \min(\infty, 16 + 12) = 28$
$L(f) = \min(16, 16 + 6) = 16$

(v) V = f, the permanent label of f is 16.

$P = \{a, g, b, f\}, \quad T = \{c, d, e\}$

$L(c) = \min(22, 16 + \infty) = 22$
$L(d) = \min(20, 16 + \infty) = 20$
$L(e) = \min(28, 16 + 14) = 28$

(vi) V = d, the permanent label of d is 20.

$P = \{a, g, b, f, d\}, T = \{c, e\}$

$L(c) = \min(22, 20 + 2) = 22$
$L(e) = \min(28, 20 + 4) = 24$

(vii) V = c, the permanent label of c is 22.

$P = \{a, g, b, f, d, c\}, \quad T = \{e\}$

$L(e) = \min(24, 22 + \infty)$
$= 24$

(viii) V = e, the permanent label of e is 24.

Hence, the length of shortest path from a to e is 24.

The shortest path from a to e is abde which is shown in Fig. 6.124 (b) below.

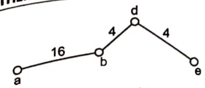


Fig. 6.124 (b)

6.14 EULERIAN PATH AND EULERIAN CIRCUIT

Now, we are in a position to define an eulerian path and an eulerian circuit which has many practical applications.

A path is called an **Eulerian Path** if every edge of the graph G appears exactly once in the path.

Similarly, the circuit which contains every edge of the graph G exactly once is called an **Eulerian Circuit**. A graph which has an Eulerian circuit is called an **Eulerian Graph**.

For example in Fig. 6.125, G_1 has an eulerian path $e_1\, e_2\, e_3\, e_4\, e_5$ but G_2 does not have an eulerian path. Similarly, G_3 contains an eulerian circuit 1 2 3 4 5 1 but G_4 does not.

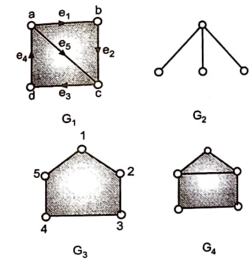


Fig. 6.125

A graph which has an eulerian path may not have an eulerian circuit. For instance, G_1 has an eulerian path but it does not have an eulerian circuit.

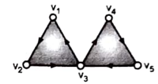
The existence of eulerian paths or circuits in a graph is related to the degree of vertices as given below:

Theorem : A graph possesses an eulerian path if and only if it is connected and has either zero or two vertices of odd degrees.

Theorem : A graph possesses an eulerian circuit if and only if it is connected and its vertices are all of even degrees.

Theorem : A directed graph possesses an eulerian circuit if it is connected and the incoming degree of every vertex is equal to its outgoing vertex.

For example, in the Fig. 6.126, the incoming degree of each vertex is equal to the outgoing degree of each vertex. Hence, the diagraph has an eulerian circuit $v_1\, v_2\, v_3\, v_4\, v_5\, v_3\, v_1$.



$\vec{d}(v_1) = \overleftarrow{d}(v_1) = 1$

$\vec{d}(v_2) = \overleftarrow{d}(v_2) = 1$

$\vec{d}(v_3) = \overleftarrow{d}(v_3) = 2$

Fig. 6.126

We shall now give the solution of famous Königsberg seven bridges problem which was solved by Swiss Mathematician Leonhard Euler in 1736, Euler presented the paper with the solution of seven bridges problem which is considered as the birth mark of graph theory. The problem is depicted in Fig. 6.127.

Two islands C and D formed by Pregel river in Königsberg (then the capital of East Prussia but now renamed Kaliningrad and in West Soviet Russia) were connected to each other and to the banks A and B with seven bridges as shown in Fig. 6.127. The problem was to start at any of the four land areas of the city A, B, C or D, walk over each of the seven bridges exactly once and return to the starting point.

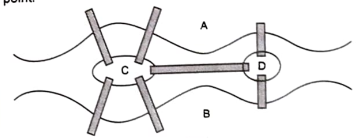


Fig. 6.127

Euler represented this situation by means of graph as shown in Fig. 6.128 and proved that the solution to this problem does not exist.

In Fig. 6.128, vertices present the land areas and the edges present the bridges of the problem. According to the problem, we have to walk through all the edges and come back to the original vertex. That is, it is required to find an eulerian circuit in a graph. As the degree of each vertex is not even, there does not exist any eulerian circuit in the graph.

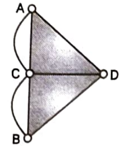


Fig. 6.128

Hence, it is not possible to walk through all seven bridges exactly once and come back to original position. Therefore, there is **no** solution to Konigsberg bridge problem.

SOLVED EXAMPLES

Example 1: *Find out the eulerian circuit in the following graph. Also find an euler path from the vertex 'a' to vertex 'b'.*

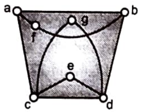


Fig. 6.129

Solution: In the above graph, the degree of the vertex a is 3 which is not an even number. Hence by the graph does not have an eulerian circuit.

Also the given graph has exactly two vertices a and b of odd degrees. The graph has an eulerian path. It is given by a c e d c g d b f a b.

Example 2: *Draw a graph which has an eulerian circuit and has a cut vertex also.*

Solution: A graph which has an eulerian circuit and cut vertex both is shown in the following Fig. 6.130.

Here, the eulerian circuit is a v c d v f a and the cut vertex is 'v'.

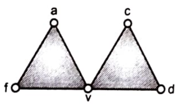


Fig. 6.130

Example 3: *Draw a graph which contains an eulerian path but does not contain an eulerian circuit.*

Solution: Consider a graph shown in Fig. 6.131.

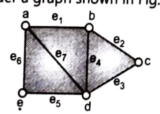


Fig. 6.131

The above graph has an eulerian path a e_1 e_2 e_3 e_5 e_6 e_7 e_4 b.

But it is does not contain an eulerian circuit because the degree of each vertex is not even.

Example 4: *For what values of n does K_n, the complete graph on n nodes, have an Euler circuit? For which it has an Euler path?*

Solution: In K_n, each vertex is joined to remaining $(n - 1)$ vertices i.e. the degree of each vertex is $(n - 1)$ If n is odd then the degree of each vertex will be even. Hence, by the complete graph K_n will contain an Euler circuit.

For an Euler path, the graph should have either zero or exactly two vertices of odd degree. Exactly two vertices of odd degree is possible only in K_2 given in Fig. 6.132.

K_2

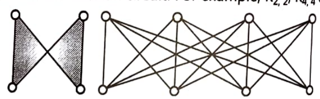
Fig. 6.132

Also, zero vertices of odd degree are possible in K_n when n is odd. Hence K_2 and all complete graphs K_n when n is odd have an Euler path.

Example 5: *Find under what conditions Km, n the complete bipartite graph will have an eulerian circuit.*

Solution: In complete bipartite graph $K_{m,\ n}$ consider the following cases:

(i) When m = n and both m and n are even, then degree of each vertex is even and hence the graph $K_{m,\ n}$ will contain an eulerian circuit. For example, $K_{2,\ 2}$, $K_{4,\ 4}$ etc.

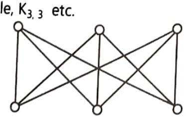


$K_{2,\ 2}$ $K_{4,\ 4}$

Fig. 6.133

(ii) If m = n and both are odd, then the degree of each vertex is odd. Hence, the graph will **not** contain an eulerian circuit.

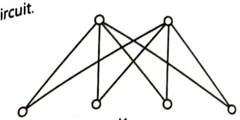
For example, $K_{3,\ 3}$ etc.



$K_{3,\ 3}$

Fig. 6.134

(iii) If m ≠ n and both m and n are even, then the graph has an eulerian circuit. For instance, $K_{2,\ 4}$ has an eulerian circuit.



$K_{2,\ 4}$

Fig. 6.135

(iv) If m ≠ n and either m is odd, n is odd or both are odd, then the graph will **not** possess an eulerian circuit. $K_{2,\ 3}$, $K_{3,\ 5}$ are the examples of graphs which do not possess an eulerian circuit.

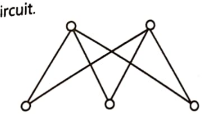
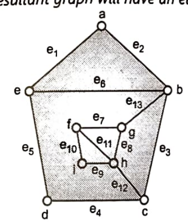


Fig. 6.136

Example 6: *Consider the graph G shown in Fig. 6.137. The edges in the graph can be partitioned into two edge disjoint paths. Show one such partition.*

Does the graph G possess an eulerian circuit? What is the minimum number of edges that can be added to the graph G so that the resultant graph will have an eulerian circuit?



G

Fig. 6.137

Solution: The two edge disjoint paths in the graph G are g e_8 e_{12} e_4 e_5 e_1 e_2 e_6 and c e_3 e_{13} e_7 e_{10} e_9 e_{11} f.

The above graph does not possess an eulerian circuit because the degree of vertices c, e, f and g are odd.

The graph will contain an eulerian circuit if we make odd degree vertices as even degree vertices by introducing the edges between them.

Join the vertices e and f by an edge and the vertices g and c by an edge so that the degree of each vertex becomes even and the graph will contain an eulerian circuit. The resultant graph is shown in Fig. 6.138.

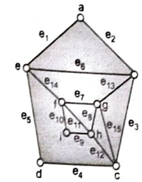


Fig. 6.138

An eulerian circuit in the resultant graph in Fig. 6.138 is f e_{14} e_1 e_2 e_6 e_5 e_4 e_3 e_{13} e_{15} e_{12} e_9 e_{10} e_{11} e_8 e_7 f.

Example 7: *Determine whether Eulerian Path and Eulerian circuit exist in the graphs G_1 and G_2 shown in Fig. 6.139.*

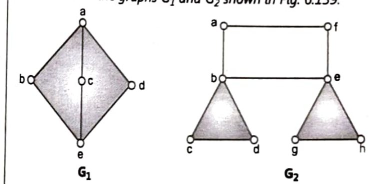


Fig. 6.139

Solution: If the path/circuit contains each edge of the graph exactly once then it is called an Eulerian path/circuit. In G_1, there are exactly two vertices of odd degree, hence by the graph G_1 contains an Eulerian path. It is given by a d e b a c e. Also, since the degree of each vertex is not even, the graph G_1 will not contain an Eulerian circuit.

In G_2, the degree of each vertex is even and hence by the graph will contain an Eulerian circuit. It is given by a f e h g e b d c b a. Since the graph has an Eulerian circuit it will contain an Eulerian path also which will be given by a f e h g e b d c b a.

Example 8: *Which of the following graphs possess Euler's path or circuit?*

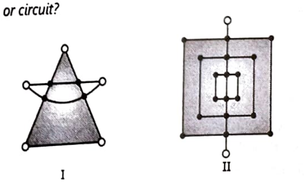


Fig. 6.140

Solution: In the graph (I), each vertex is of even degree. Hence it possesses an Euler's circuit.

Graph (II) is connected graph and it has exactly 2 vertices of odd degree. Hence this graph possess an Euler's path.

6.15 HAMILTONIAN PATH AND HAMILTONIAN CIRCUIT

In the last section, we have introduced the terms eulerian path and eulerian circuit in a connected graph. In a similar way, we will now define Hamiltonian path and Hamiltonian circuit in a connected graph which gives the solution to the famous game "all around the world" invented by Sir William Hamiltonian in 1859.

A circuit in a connected graph G is called a **Hamiltonian Circuit** if it contains every **vertex of G exactly once** (except the first and the last vertex)

Similarly a path in a connected graph G is a **Hamiltonian Path** if it contains every **vertex of G exactly once**. A graph which has a Hamiltonian circuit is called a **Hamiltonian graph**.

For example, for the graph G_1 in Fig. 6.141, the Hamiltonian circuit is given by (1 2 3 4 1)

A graph that contains the Hamiltonian path may not contain a Hamiltonian cycle. In Fig. 6.141, G_2 has a Hamiltonian path b a c but it does **not** contain any Hamiltonian circuit.

Now we present the famous "all around the world" problem.

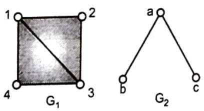


Fig. 6.141

All Around The World:

This problem was first posed by Irish Mathematician, Sir William Hamiltonian. He made a regular dodecahedron from wood with 20 vertices and 30 edges as shown in Fig. 6.142 in which each of 20 vertices were marked with the name of a city. The puzzle was to start from any city and find a route along the edge of the dodecahedron that passes through each city exactly once and return to the city of origin. That is, it was required to find a Hamiltonian circuit in the graph of dodecahedron.

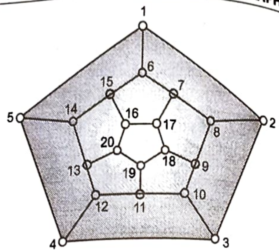


Fig. 6.142

One such route (Hamiltonian circuit) is 1 2 8 9 10 3 4 5 14 15 16 20 13 12 11 19 18 17 7 6 1.

Obviously not every connected graph has a Hamiltonian circuit. For example, the graph in the following Fig. 6.143 does not have any Hamiltonian circuit.

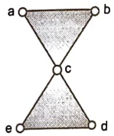


Fig. 6.143

A necessary and sufficient condition for a connected graph to have a Hamiltonian circuit is still unknown. However, there are some results which give sufficient conditions (not necessary) for a Hamiltonian circuit and Hamiltonian path.

We present now general conditions that are sufficient to guarantee for existence of a Hamiltonian path and a Hamiltonian circuit in a connected graph G.

Theorem :

Let G be a simple connected graph on n vertices. If the sum of the degree for each pair of vertices in G is $(n-1)$ or large, then there exists a Hamiltonian path in G.

For example, in Fig. 6.142, total number of vertices $n = 5$ and the degree sum of every pair of vertices is 4 or greater than 4. Hence, there exists a Hamiltonian path in G which is given by a b c e d.

It is easy to see that the condition in the Theorem is a **sufficient** condition but **not** a necessary condition for the existence of a Hamiltonian path in a graph. For instance, let G be a graph on 6 vertices given by Fig. 6.144.

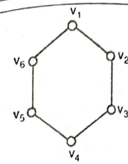


Fig. 6.144

Here the graph G has a Hamiltonian path $v_1\ v_2\ v_3\ v_4\ v_5\ v_6$ but the degree sum of any pair of vertices $= 4 \ngeq (6-1) = (n-1)$

The next theorem which was proved by Dirac in 1952 gives a **sufficient** condition for existence of a Hamiltonian circuit in a simple connected graph.

Theorem :

If $G = (V, E)$ is a simple connected graph on n vertices and if the degree of **each** vertex v is greater than or equal to $\frac{n}{2}$ i.e. $d(v) \geq n/2 \ \forall \ v \in V$ then G will contain a Hamiltonian circuit.

For example, consider the graph G with 6 vertices given in Fig. 6.145. In G, the degree of each vertex is greater than or equal to 3. Hence the graph has a Hamiltonian circuit which is given by $v_1\ v_2\ v_3\ v_4\ v_5\ v_6\ v_1$.

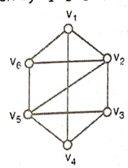


Fig. 6.145

The condition given in theorem is only a sufficient condition for a given graph to have a Hamiltonian circuit. For instance, the graph G with 8 vertices in Fig. 6.146 has a Hamiltonian circuit 1 2 3 4 5 6 7 8 1 but degree of each vertex $d(v) = 2 \geq 8/2 = 4$.

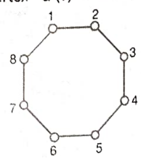


Fig. 6.146

Now, we give some necessary condition for a graph to have a Hamiltonian circuit.

Theorem :

Let G be a connected simple graph. If G has a Hamiltonian circuit then for **every** proper non-empty subset S of V (G), the components in the graph (G – S) is less than or equal to the number of vertices in S.

The above theorem can be used in another way also:

If for **every** proper non-empty subset S of V(G), the components in the graph (G – S) is **not** less than or equal to the number of vertices in S then G does **not** have a Hamiltonian circuit.

With the help of above necessary condition, we will show that the complete bipartite graph $K_{m,\,n}$ does not have a Hamiltonian circuit if $m \neq n$. In fact it has a Hamiltonian circuit when $m = n$.

Consider the complete bipartite graph $K_{m,\,n}$ when $m \neq n$.

Let (V_1, V_2) be a partition of the vertex set of $K_{m,\,n}$ where $|V_1| = m$, and $|V_2| = n$ and $m < n$. The graph $K_{m,\,n} - V_1$ is a null graph on n vertices and hence it is a disconnected graph with n components.

Therefore, the number of components in $K_{m,\,n} - V_1 = n \leq |V_1| = m$ and according to above theorem, if $K_{m,\,n}$ has Hamiltonian circuit then the number of components in $K_{m,\,n} - V_1 = n \leq m = |V_1|$ which is contradiction to $m < n$. $K_{m\,,\,n}$ does not contain a Hamiltonian circuit when $m \neq n$.

Consider the complete bipartite graph $K_{m,\,n}$ when $m = n = 2$ i.e. $K_{2,\,2}$. It is shown in Fig. 6.147.

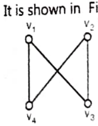


Fig. 6.147

Here the total number of vertices = 4 and the degree of each vertex v is $d(v) = 2$.

i.e. $d(v) = 2 \geq \frac{4}{2}$

$\Rightarrow$ $d(v) = 2 \geq 2$

Hence $K_{2,\,2}$ has a Hamiltonian circuit.

In general, we can say that $K_{m,\,n}$ has a Hamiltonian circuit (and also has a Hamiltonian path) if **m = n.**

6.16 TRAVELLING SALESMAN PROBLEM

- In the previous section, we have studied the theorems for hamiltonian graphs. One of the problems related to Hamiltonian circuit is "Travelling Salesman Problem" which is stated below.
- A salesman is required to visit a number of cities during a trip. Given the distances between the cities, in what order should he travel so as to visit every city precisely once and return home with the minimum distance travelled?
- The above problem can be represented by a weighted graph, in which vertices represent the cities and the roads between the cities represent the edges. The weight of each edge is the distance between the two cities. Thus the travelling salesman problem reduces to find the Hamiltonian circuit in the given graph with minimum weight.

For example, consider the following weighted graph.

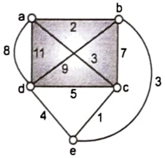


Fig. 6.148

There are several Hamiltonian circuits starting from a

1. abceda with weight 22
2. abecda with weight 21
3. acebda with weight 26 and so on.

The minimum Hamiltonian circuit is abecda.

If the graph is k_n(complete graph on n vertices) then the problem can be solved, theoretically by listing all possible $\frac{(n-1)!}{2}$ Hamiltonian circuits and picking the one which has the least weight. However, for a large value of n, this is highly inefficient algorithm.

In fact, no efficient algorithm is available to find the solution of travelling salesman problem. There are methods available that give a route very close to the shortest one but do not guarantee the shortest path. One such method is nearest-neighbour method which gives good results to the salesman problem.

6.16.1 Nearest-Neighbor Method

In this method, we start the Hamiltonian circuit with any arbitrary vertex and find the vertex which is nearest to it. Continuing this way and coming back to the starting vertex by travelling through all the vertices exactly once, we get the Hamiltonian circuit.

The nearest-neighbour method is described below:

1. Start with any arbitrary vertex (say v_1) and choose the vertex closest to the starting vertex to form an initial path of one edge. Construct this path by selecting different vertices as described in step (2)
2. Let v_n denote the latest vertex that was added to the path. Select the vertex v_{n+1} closest to v_n from all vertices that are not in the path and add this vertex to the path.
3. Repeat step (2) till all the vertices of the graph G are included in the path.
4. Lastly form a circuit by adding the edge connecting the starting vertex and the last vertex added.

The circuit obtained using the nearest-neighbour method will be the required Hamiltonian circuit.

SOLVED EXAMPLES

Example 1: *Use nearest-neighbour method to find out Hamiltonian circuit for the graph shown in Fig. 6.149*

(a) Starting at vertex a.

(b) Starting at vertex d.

(c) Determine the minimum Hamiltonian circuit.

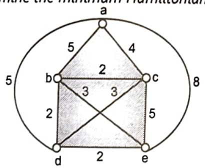
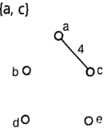


Fig. 6.149

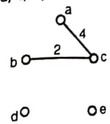
Solution: (a) Start with vertex a. There are 4 adjacent vertices to a namely b, c, d and e but nearest neighbour is c (weight of the edge ac is 4 which is minimum)

(i) Path = {a, c}

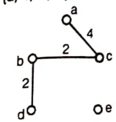


(ii) There are three vertices adjacent to c, namely b, d and e (except a) The nearest neighbour is b.

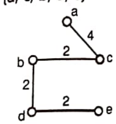
Path = {a, c, b}



(iii) Path = {a, c, b, d}



(iv) Path = {a, c, b, d, e}



(v) Since all the vertices are traversed, to complete the Hamiltonian circuit, we have to reach back to the vertex a from e.

Hamiltonian circuit = {a, c, b, d, e, a}

The weight of Hamiltonian circuit = 18.

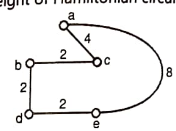
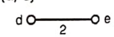


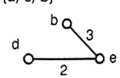
Fig. 6.150

(b) To find the Hamiltonian circuit starting from d, consider the starting vertex d. Since both the vertices b and e are nearest to d (at a distance 2), we can choose any one of them. Let the vertex e be chosen.

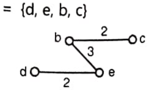
(i) Path = {d, e}



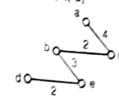
(ii) Path = {d, e, b}



(iii) Path = {d, e, b, c}



(iv) Path = {d, e, b, c, a}



(v) Hamiltonian circuit = {d, e, b, c, a, d}

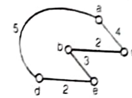


Fig. 6.151

Weight = 16.

(c) The minimum Hamiltonian circuit is debcad with weight 16.

Example 2: *Use nearest-neighbour method to find the Hamiltonian circuit starting from a in the following graph. Find its weight.*

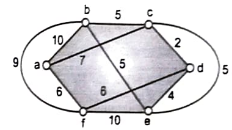
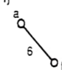
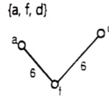


Fig. 6.152

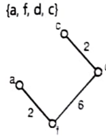
Solution: (i) Path= {a, f}



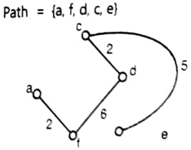
(ii) Path = {a, f, d}



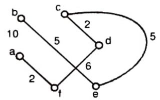
(iii) Path = {a, f, d, c}



(iv) Path = {a, f, d, c, e}



(v) Path = {a, f, d, c, e, b}



(vi) Hamiltonian circuit = {a, f, d, c, e, b, a}.

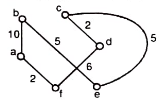


Fig. 6.153

Weight of the Hamiltonian circuit = 34.

Example 3: *Show that the complete graph K_n $(n \geq 3)$ has a Hamiltonian circuit. What is the length of that circuit? How many Hamiltonian circuits exist in K_n?*

Solution: The complete graph K_n $(n \geq 3)$ has n vertices and the degree of each vertex is $(n - 1)$

We know that when $n \geq 3$, $n - 1 \geq \frac{n}{2}$

$$\Rightarrow \quad d(v) \geq \frac{n}{2} \quad \forall\ v \in V(K_n)$$

Therefore the theorem is satisfied. Hence, the complete graph on n vertices has a Hamiltonian circuit.

For example, consider K_4 as shown in Fig. 6.154, the Hamiltonian circuit is a b c d a. The length of the Hamiltonian circuit is the number of edges in the circuit. For K_4, the Hamiltonian circuit will contain 4 edges.

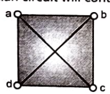


Fig. 6.154

Hence in K_n, the length of the Hamiltonian circuit is n.

There are $\frac{(n-1)!}{2}$ Hamiltonian circuits in the complete graph K_n.

Example 4: *Find a Hamiltonian path and a Hamiltonian circuit in $K_{4,3}$.*

Solution: The complete bipartite graph $K_{4,3}$ is given by

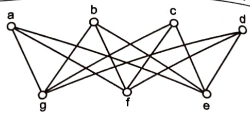


Fig. 6.155

Here the total number of vertices in the graph $K_{4,3}$ is 7.

Also the degree sum of any pair of vertices $\geq (7 - 1)$

Hence by theorem the graph $K_{4,3}$ has a Hamiltonian path.

It is given by a g b f c e d.

Also, the graph $K_{4,3}$ does not contain a Hamiltonian circuit because $m \neq n$.

Example 5: *Is the following graph contains a Hamiltonian circuit? If yes, find the Hamiltonian circuit.*

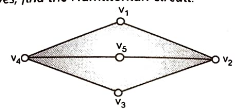


Fig. 6.156

Solution: Since there are five vertices, a Hamiltonian circuit must have five edges. Suppose that we could eliminate edges from the graph to produce a Hamiltonian circuit. We would have to eliminate one edge incident on v_2 and one edge incident on v_4, since each vertex in a Hamiltonian circuit has degree 2. But this leaves four edges - **not** enough for a Hamiltonian circuit. Therefore, the graph does not contain a Hamiltonian circuit.

Example 6 : *Consider the problem of scheduling seven examinations in seven days so that two examinations given by the same instructor are not scheduled on consecutive days. If no instructor gives more than four examinations, show that it is always possible to schedule the examination.*

Solution: Let G be a graph with seven vertices corresponding to the seven Examinations. There is an edge between two vertices if and only if two vertices correspond to two examinations given by **different** instructors.

First we will show that degree of each vertex is **at least** 3. Suppose not i.e. there exists a vertex v of degree less than 3. This means that at the most only two vertices will have an edge with the vertex v i.e. the remaining 4 vertices will have **no** edge with v. Now, these 4 vertices together with the vertex v correspond to 5 different examinations given by the same instructor which is a contradiction to the assumption that no instructor gives more than 4 examinations.

Hence, the degree of each vertex is at least 3 and therefore the degree sum of any pair of vertices is at least 6 which is greater than or equal to $(7 - 1) = 6$.

Hence, by the graph will have a Hamiltonian path. This shows that, it is possible to schedule the examination under given condition.

Example 7: *(i) Give an example of a graph that has both an Eulerian circuit and Hamiltonian circuit.*

(ii) Give an example of a graph that has an Eulerian circuit but no Hamiltonian circuit.

(iii) Give an example of a graph that has Hamiltonian circuit but no Eulerian circuit.

(iv) Give an example of a graph that has neither Hamiltonian circuit nor Eulerian circuit.

Solution: (i) Graph with Eulerian circuit and Hamiltonian circuit is as shown in Fig. 6.157.

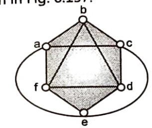


Fig. 6.157

Eulerian circuit b c e d c a f d b f e a b

Hamiltonian circuit b c d e f a b

(ii) Graph with Eulerian circuit but no Hamiltonian circuit.

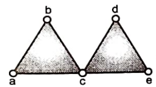


Fig. 6.158

Eulerian circuit a b c d e c a

No Hamiltonian circuit.

(iii) Graph with Hamiltonian circuit but not Eulerian circuit.

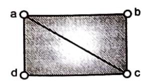


Fig. 6.159

Hamiltonian circuit a b c d a

No Eulerian circuit.

(iv) Graph with neither Hamiltonian nor Eulerian circuit.

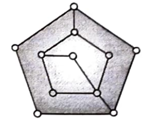


Fig. 6.160

Example 8: *Show that any Hamiltonian circuit in the graph shown in Fig. 6.161 that contains the edge x must also contain the edge y.*

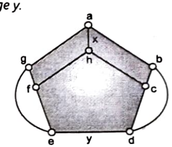


Fig. 6.161

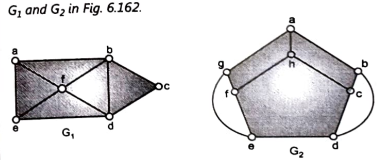
Solution: A Hamiltonian circuit contains each vertex of the graph exactly once. Consider the Hamiltonian circuit either starting from the vertex a and containing the edge x = (a, h) At h, if we take the path h c b d then to cover all the vertices in the circuit, we have to cross the edge y = (d, e) to traverse through the remaining vertices e, f and g.

Similarly, if the circuit starts from the vertex h or any other vertex from the graph and contains the edge x, then to traverse through all the vertices of the graph we have to travel through the edge y.

Hence, any Hamiltonian circuit in the graph that contains the edge x must contain the edge y.

Example 9: *(i) Is there a Hamiltonian path in a complete bipartite graph $K_{4,4}$ and $K_{4,5}$?*

(ii) Is there a Hamiltonian circuit in the graphs shown in G_1 and G_2 in Fig. 6.162.



G_1 G_2

Fig. 6.162

Solution: (i) In $K_{4,4}$ the number of vertices are 8 and the degree of each vertex is 4.

Hence the degree sum of any pair of vertices in $K_{4,4}$ is $4 + 4 = 8$.

Degree sum $(4 + 4) > (8 - 1) = (n - 1)$

$\Rightarrow 8 > 7$

Hence, $K_{4,4}$ contains a Hamiltonian path.

Similarly in $K_{4,5}$ there are 9 vertices and there are 4 vertices of degree 5 and 5 vertices of degree 4.

Hence, the degree sum of pair of vertices

$(4 + 4) = 8 > (9 - 1) = n - 1$

or $(4 + 5) = 9 > (9 - 1) = (n - 1)$

$(5 + 5) = 10 > (9 - 1)$

Hence the graph $K_{4,5}$ has a Hamiltonian path.

(ii) In G_1, the Hamiltonian path is a b c d f e.

A Hamiltonian circuit is a b c d f e a.

In G_2, the Hamiltonian path is a b d c h f e g

The Hamiltonian circuit is a b d c h f e g a.

***Example 10:** (i) Is there a Hamiltonian circuit in a complete bipartite graph $K_{4,4}$, $K_{4,5}$ and $K_{4,6}$?*

(ii) Is there a Hamiltonian circuit in the graph shown in Fig. 6.163? What about a Hamiltonian path?

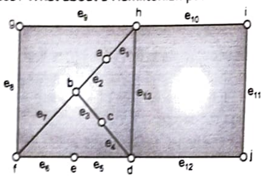


Fig. 6.163

Solution: (i) We know that in the complete bipartite graph $K_{m,n}$ if $m = n$, the Hamiltonian circuit exists, and if $m \neq n$, the Hamiltonian circuit does not exist.

Hence in $K_{4,4}$ the Hamiltonian circuit exists.

In $K_{4,5}$ and $K_{4,6}$ the Hamiltonian circuit does not exist.

(ii) We know that for any Hamiltonian circuit, all the vertices of the graph should be traversed exactly **once**. Each vertex in the **circuit** must have degree exactly 2 because if the vertex has degree more than 2 in the circuit, it means that the vertex is traversed more than once in the circuit. Hence we have to remove additional edges, if any, keeping in mind the connectedness of the graph.

Consider the vertex f of degree 3, if we remove e_6 or e_8 the vertices g and e will become of degree 1 which is not allowed for the circuit.

Hence remove the edge e_7 so that the vertex f becomes of degree 2. This implies b also becomes of degree 2. To make the vertex d of degree 2, we have to remove 2 edges incident on d. Remove the edge e_{13}. Now we cannot remove the edges e_4, e_5 or e_{12} otherwise the vertices c, e or j will become of degree 1. Hence it is not possible to make the vertex d of degree 2 by removing the edges. Therefore, the graph does not have a Hamiltonian circuit.

The Hamiltonian path is given by c b a h i j d e f g.

***Example 11:** Show that the graph shown in Fig. 6.164 has no Hamiltonian circuit.*

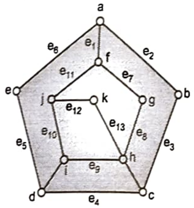


Fig. 6.164

Solution: For a Hamiltonian circuit, each vertex should have degree exactly 2 in the circuit. Hence we have to remove the additional edges.

Consider the vertex a of degree 3. Edges e_2 and e_6 incident on a can not be removed since they are incident on vertices b and e of degree 2 also. Hence we have to remove the edge e_1. Similarly, consider the vertex d of degree 3, with the same argument we cannot remove the edge e_5. Remove the edge e_{15}.

Consider the vertex c of degree 3, to maintain the connectivity of the graph, we have to retain the edges e_4 and e_3. This means we can remove the edge e_{14} only to make the vertex c of degree 2. But this is not possible because otherwise the graph will become disconnected. Hence the given graph has no Hamiltonian circuit.

***Example 12:** Does $K_{1,3}$ have an Eulerian circuit? A Hamiltonian circuit?*

Solution: The complete bipartite graph $K_{1,3}$ is shown below.

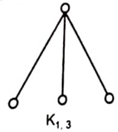

$K_{1,3}$

Fig. 6.165

The above graph does not have even degree vertices. Therefore it does not contain an Eulerian circuit. Also $K_{m,n}$ will have a Hamiltonian circuit, if $m = n$. Hence $K_{1,3}$ will not have a Hamiltonian circuit.

***Example 13:** Determine which of the graphs G_1 and G_2 represent Eulerian circuit, Eulerian path, Hamiltonian circuit, Hamiltonian path. Justify your answer.*

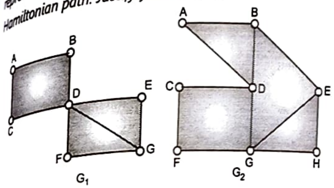


Fig. 6.166

Solution: For the graph G_1,

(i) There is no Eulerian circuit because degree of each vertex is not even.

(ii) There exists an Eulerian path in G_1 because G_1 contains exactly two vertices of odd degree. It is given by GEDBACDFGD.

(iii) No Hamiltonian circuit. To cover all the vertices exactly once, the vertex D has to traverse twice.

(iv) G_1 has Hamiltonian path EGFDBAC.

For the graph G_2,

(i) No Eulerian circuit because degree of each vertex is not even.

(ii) There is an Eulerian path BADCFGHEGDBE.

(iii) It has Hamiltonian path and circuit both. It is ABEHGFCDA.

***Example 14:** Draw the graphs formed by the vertices and edges of a tetrahedron, a cube and an octahedron. Find a Hamiltonian cycle in each graph.*

Solution: Tetrahedron has four vertices, six edges and four triangular faces. It is shown in Fig. 6.167.

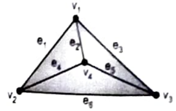


Fig. 1.167 (a)

The Hamiltonian circuit in tetrahedron given by Fig. 6.167 (a) is $v_1\ v_2\ v_3\ v_4\ v_1$.

The octahedron called a platonic solid has eight triangular faces with 6 vertices and 12 edges. It is shown in Fig. 6.167 (b)

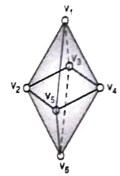


Fig. 6.167 (b)

The Hamiltonian circuit in octahedron shown in Fig. 6.167 (b) is given by $v_1\ v_2\ v_3\ v_4\ v_6\ v_5\ v_1$.

The cube is shown in Fig. 6.167 (c)

It has Hamiltonian circuit given by $v_1\ v_2\ v_3\ v_4\ v_5\ v_8\ v_7\ v_6\ v_1$.

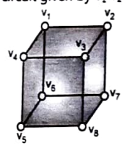


Fig. 6.167 (c)

6.17 PLANAR GRAPH

In this last section of the chapter, we study drawing of a graph in a plane without its edges crossing over, which is useful in many fields like technology of printed circuits, design of large scale integrated circuits etc.

A graph is said to be a **planar graph** if it can be drawn on the plane **with no intersecting edges** i.e. a graph is a planar graph if it is drawn on a plane such that no edges cross each other.

For instance, the graph shown in Fig. 6.168 is a planar graph.

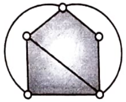


Fig. 6.168

A graph which is non planar in one representation may become a planar graph after redrawing it.

Consider a graph G_1 in Fig. 6.169 which appears a non planar graph because its edges e_5 and e_6 are seemed to be crossing each other.

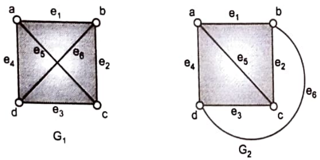


Fig. 6.169

The graph G_1 in the above Fig. 6.169 can be redrawn as G_2 given in Fig. 6.169. The graph G_2 is indeed a **planar graph or planar embedding of the graph in the plane.**

6.17.1 Regions

A planar representation of a graph divides the plane into regions (also called **windows, faces** and **meshes**) **A region is characterized by the set of edges forming its boundary.**

Fig. 6.170 shows different regions of the graph G which are marked by 1, 2, 3, 4, 5 and 6.

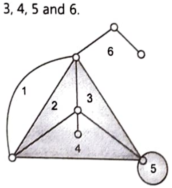


Fig. 6.170

A region is said to be finite if its **area** is finite and it is said to be infinite if its **area** is infinite. A planar graph has exactly **one infinite region**.

In Fig. 6.170, the regions 1, 2, 3 and 4 are finite but the region 6 is an infinite region.

6.17.2 Euler's Formula

Since a planar graph may have different planar representations, the number of regions resulting from each representation is the same. The number of regions in any planar representation depends upon the number of vertices and the number of edges in the graph. The relation between edges, vertices and faces of any graph is given by Euler's formula.

Theorem :

For any connected planar graph,

$$\boxed{v - e + r = 2}$$

where v, e, r are the total number of vertices, edges and regions in the graph respectively.

For instant, consider the graph G with 12 vertices and 15 edges as shown in Fig. 6.171.

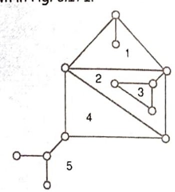


Fig. 6.171

According to Euler's formula, the graph G should have $r = 2 - v + e$ faces i.e. $r = 2 - 12 + 15 = 5$ faces. These 5 regions are shown in the given Fig. 6.171.

Proof of Euler's theorem:

Proof:

Let G be any connected planar graph with v vertices, e edges and r number of faces. Since G is connected, two cases arise: (i) G is a tree, (ii) G is not a tree. We will prove Euler's theorem for both the cases

Case I: If G is a tree then G contains $(v - 1)$ number of edges, does not contain any circuit and has only one unbounded face so that $r = 1$.

Therefore,

$$v - e + r = v - (v - 1) + 1$$
$$= 2$$
$$\Rightarrow v - e + r = 2$$

Thus, theorem holds if G is a tree.

Case II: Suppose G is not a tree. Then G contains cycles. We will prove the result by induction on the number of edges $e \geq 0$. If $e = 0$ then $v = 1$ and $r = 1$.

$\Rightarrow v - e + r = 1 - 0 + 1 = 2$

For induction step, suppose theorem is true for $(e - 1)$ number of edges. From the graph G, remove the edge e' which forms the cycle in G. The graph $(G - e')$ is connected and has $(e - 1)$ number of edges and $(r - 1)$ number of faces.

Now, by induction $v - (e - 1) + (r - 1) = 2$.

i.e. $v - e + r = 2$

Now, we give some results which are based on Euler's formula.

Corollary :

If G (V, E) is a simple connected planar graph, then

$$\boxed{e \leq 3v - 6}$$

where e is the total number of edges and v is the total number of vertices in the graph G.

Proof: Since the graph is a simple planar graph, therefore each region of the planar graph is bounded by at least three or more edges.

Hence the total number of edges $e \geq 3r$.

Also, each edge is included in at least 2 regions of the planar graph G, therefore

$$2e \geq 3r$$

or $$\frac{2e}{3} \geq r$$

Substitute this value of r in Euler's formula, we get

$$v - e + \frac{2e}{3} \geq 2$$

or $$3v - 6 \geq e.$$

The most important application to this corollary is to show that the complete graph K_5 on 5 vertices are **non planar**. K_5 is known as **Kuratowski's First Graph (named on Polish Mathematician Kazimierz Kuratowski)**

In K_5, the number of vertices $v = 5$

The number of edges $e = 10$

$\Rightarrow v - 6 \geq e$

$\Rightarrow 3(5) - 6 \geq 10$

$\Rightarrow 9 \geq 10$

which is impossible.

Hence, K_5 is non-planar.

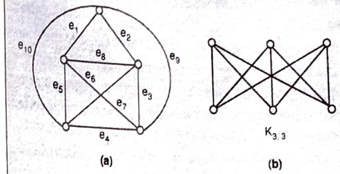


Fig. 6.172

As shown above in Fig. 6.172 (a) it is **not** possible to draw the edges e_6 and e_7 without crossing.

Hence, K_5 is non-planar.

Algebraically, we can show that in K_5, the crossing of edges cannot be avoided. K_5 is shown in Fig. 6.172 (a)

The inequality is only a **necessary** condition but it is **not** a sufficient condition for the planarity of a graph. In other words, every simple planar graph must satisfy the condition , but the converse need not be true. That is the satisfaction of the inequality does not guarantee for planarity of a graph.

For example, $K_{3,3}$ **Kuratowski's second** graph given in Fig. 6.172 (b) satisfies because $e = 9$, $v = 6$. Hence $3v - 6 \geq e$, yet the graph $K_{3,3}$ is a **non planar graph.**

To prove the non planarity of $K_{3,3}$ we use the additional fact that no regions in the graph K_{33} is bounded with fewer than 4 edges. Hence, if this graph were planar, we would have

$$e \geq 4r$$

or $$\frac{e}{2} \geq r$$

Hence, according to Euler's formula

$$v - e + \frac{e}{2} \geq 2$$

$$\Rightarrow 2v - 4 \geq e$$

In $K_{3,3}$ inequality becomes

$$(2 \times 6) - 4 \geq 9$$

$$\Rightarrow 8 \geq 9$$

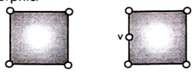
which is impossible. Hence $K_{3,3}$ is not a planar graph.

Kuratowski gave a necessary and sufficient condition for a graph to be planar in 1930.

Before we present a necessary and sufficient condition for planarity of a graph, we will define the new term

Isomorphism of two graphs within vertices of degree 2 (or homeomorphic graphs)

Two graphs G_1 and G_2 are said to be isomorphic within vertices of degree 2 (or homeomorphic) if they are isomorphic or if they can be transformed to isomorphic graphs by repeated insertion of vertices of degree 2 or by merging the edges which have exactly one common vertex of degree 2 as illustrated in Fig. 6.173. In Fig. 6.173, in G_1, the vertex v of degree 2 is inserted to get G_2. Hence G_1, G_2 are homeomorphic.



Insertion of a Vertex v

Fig. 6.173

In Fig. 6.174, in the graph G_3, two edges e_1 and e_2 are merged to get the graph G_4. Therefore, G_3 and G_4 are homeomorphic.

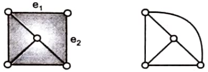


Fig. 6.174: Merging of Edges e_1 or e_2

Theorem (Kuratowski Theorem):

A graph is a planar graph if and only if it does not contain any subgraph that is isomorphic to within vertices of degree 2 (or homeomorphic) to either K_5 or $K_{3,3}$.

Consider the following graph G in Fig. 6.175.

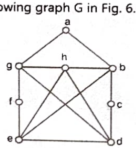


Fig. 6.175

Merge the edges which are incident on the vertices a, c and f. After merging the graph is shown in Fig. 6.176 (a) which is isomorphic to K_5 as shown in Fig. 6.176 (b)

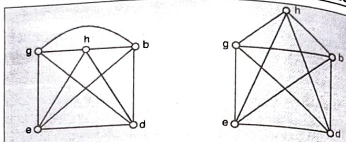


Fig. 6.176

Hence, the given graph G is non planar according to theorem .

6.18 GRAPH COLOURING

In this section, we describe the colouring of a graph and its chromatic number.

By means of colouring, we can find the partition of the graph which is useful to many practical problems, such as job scheduling, pattern matching, coding theory, state reduction of sequential machines etc.

A proper colouring of graph G is an assignment of colours to its vertices so that no two adjacent vertices have the same colour.

In other words, painting all the vertices of a graph with colours such that no two adjacent vertices have the same colour is called proper colouring (or simply colouring) of a graph. For colouring, we consider only simple connected graph.

In general, a given graph can be properly coloured in many different ways.

For example, the graph shown in Fig. 6.177 can be coloured in many ways.

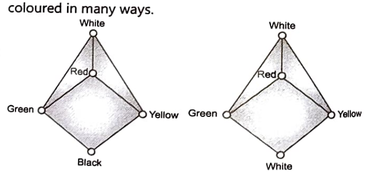


Fig. 6.177

As shown above in Fig. 6.177, the number of colours used in first graph is five and in second graph is four. Our main interest is in proper colouring of graph which uses minimum number of colours. This number is called chromatic number.

Hence, we define **chromatic number** as the minimum number of colours needed to produce a proper colouring of a graph G. The chromatic number of the graph G is denoted by $\chi(G)$ If graph G has chromatic number K then χ (G) = K and graph is called K-chromatic.

From the definition of colouring of graph, it is observed that

1. Null graph (graph without edges) is 1-chromatic.
2. A complete graph K_n with n vertices is n-chromatic because all its vertices are adjacent. Hence a graph containing a complete graph of p vertices is atleast p-chromatic.

For example, a graph having a triangle is at least 3-chromatic.

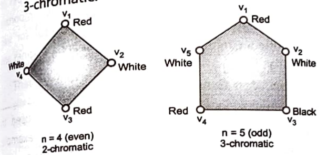


Fig. 6.178

3. A graph containing simply one circuit with number of vertices n (n ≥ 3) is 2-chromatics if n is even and 3-chromatic if n is odd.

   This can be explained from the following Fig. 6.178.

4. Every 2-chromatic graph (with at least one edge) is bipartite, since colouring partitions the vertex set V into two subsets V_1 and V_2 such that no two vertices in V_1 (or V_2) are adjacent which is the definition of bipartite graph. Thus a graph is **bipartite** if and only if it is 2-colorable. Also a graph is **bipartite** if and only if it does not contain an odd cycle, therefore, the chromatic number of an n vertex simple connected graph which does not contain any odd length cycle is 2.

As we have seen above, a proper colouring of a graph naturally induces a partitioning of the vertices into different subsets. For example, the colouring of graph in Fig. 6.178 with chromatic number 3 produces the partitioning.

$\{v_1, v_4\}, \{v_2, v_5\}, \{v_3\}$

such that no two vertices in any of these three subsets are adjacent. Such a subset of vertices is called an independent set (or an internally stable set).

A maximal independent set or (maximal internally stable set) is an independent set to which no other vertex can be added without destroying its independence property.

Consider the following graph shown in Fig. 6.179

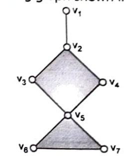


Fig. 6.179

Some of the maximal independent sets are $\{v_1, v_3, v_4, v_6\}$, $\{v_2, v_6\}$, $\{v_2, v_7\}$.

The number of vertices in the largest independent set is called the independence number (or coefficient of internal stability)

The concept of maximal independent set is used in coding theory for communication.

For example, consider the graph shown in Fig. 6.179 in which seven vertices of the graph represent some code word used in communication. If some code words are very close in sound then these code words can create confusion during communication and one word maybe mistaken for another close word. These types of code words (vertices) are joined by edges.

For reliable communication, it is required to find largest set of code words. In other words, it is required to find maximal independent set with largest number of vertices. For our example (Fig. 6.179), the set $\{v_1, v_3, v_4, v_7\}$ is the answer.

6.18.1 Chromatic Polynomial

As we have seen in the last section that a given graph G on V vertices can be properly coloured in many different ways using a sufficiently large number of colours. The polynomial which describes this property of the graph is called chromatic polynomial P_n (λ) of graph G and is defined as the number of ways of properly colouring of graph, using λ or fewer colours.

For example, with λ given colours, the complete graph k_1 can be coloured with λ different colouring.

Hence, $P_1(\lambda) = \lambda$

Theorem :

The chromatic polynomial of complete graph k_n on vertices is

$$P_n(\lambda) = \lambda(\lambda-1)(\lambda-2) \ldots (\lambda-n+1)$$

Proof : Consider any vertex v_1 of complete graph K_n. With λ given colours, this vertex v_1 can be coloured in λ different ways. Now, for each colouring of v_1, there remain (λ − 1) colours. Therefore the second vertex v_2 can be coloured in (λ − 1) ways. For each colouring of v_1 and v_2, the vertex v_3 can be coloured in (λ − 2) ways and so on and since each vertex of complete graph k_n is adjacent to remaining (n − 1) vertices, it shows that

$$P_n(\lambda) = \lambda(\lambda-1)(\lambda-2) \ldots (\lambda-n+1)$$

6.18.2 Colouring of Planar Graph

- We have seen the colouring of vertices in a graph. Colouring of edges in a graph can be defined on a similar line. A proper colouring of edges then requires that no two adjacent edges are of same colour (i.e. adjacent edges should be of different colours) Also the set of edges in which no two are adjacent is called a matching.
- In this section, we will describe the colouring of planar graph (graph without crossing of edges) for which the proper colouring of regions to be considered.
- The regions of a planar graph are said to be properly coloured if no two continuous or adjacent regions have the same colour. (If two regions have a common edge between them, they are called adjacent regions)
- For example, in atlas, adjacent countries (with common boundary) are coloured with different colours. For this reason, the proer colouring of regions is also called map colouring.
- The problem of finding minimum number of colours required for proper colouring of regions of planar graph has attracted many researchers, scientists and mathematicians over last hundred years. In 1852, Francis Guthrie hypothesized that any planar graph could be coloured with only four colours. This famous four colour conjecture was proved by Kenneth Appel and W. Haken in 1977 who used computer assisted proof methodology for the conjecture. However, 5-colourability of planar graphs can be proved without a computer search. Following Fig. 6.180 shows that a complete graph k_4 (planar graph also) uses four colours for its proper colouring.

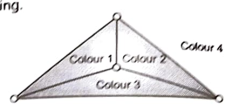


Fig. 6.180

6.19 APPLICATIONS OF GRAPH

In previous sections, we have described many types of graphs which can be used to model various types of problems. In this section we will study the important application of graph called web graph, useful for web crawling, enhancement of efficiency of search engines and in many other fields.

6.19.1 The Web Graph

- As we have seen that the graphs are used to represent many real life situations, for example, Konigsberg seven bridge problem, three utility problem etc. The world wide web (www) which was created by English Scientist Tim Berners-Lee (1989) can also be modeled using directed graph, which is called web graph. Web graph is a directed graph whose vertices represent the web pages and an edge between the vertices represented the link between the web pages.
- A vertex A which represents the web page is connected to vertex B by direct edge from A to B if there is a link from web page A to web page B. that is if there is a direct edge joining vertex A to vertex B, there exists a hyperlink on webpage A referring to page B. Hence web graph describes the direct link between the pages of world wide web. The number of vertices in the web graph is huge (billion of vertices) and changing continuously as every second, the new web pages are created and others are removed. Since some of the web pages (vertices) are not linked between, the underlying undirected graph of the web graph is not connected but it has a connected component with approximately 90% of the vertices of the graph. The sub graph of the original directed web graph corresponding to this connected component has one very large strongly connected component, called *Giant strongly connected component (GSCC)* and many small connected components starting from any web page in this component GSCC, following the series of links, we can reach to any other, web page in the component. We refer to the hyperlinks into the page as *in-links* and those out of a page as out-links. The number of in-links to a page is indegree of the vertex and number of out-links to a page is its out degree.
- There are three major categories of web pages (vertices) which are referred as *IN*, *OUT* and *GSCC*. From any page *IN*, a web surfer can go to any page in *GSCC* by following hyperlinks. Also, a surfer can go to any page from GSCC to any page in OUT. However, reverse is not possible in both the cases. That is, from GSCC we can not reach to In and similarly from OUT, we can not access to any page in GSCC (or subsequently IN)
- The sites of IN and OUT are roughly equal. Most of the web pages fall into one of these categories. The remaining pages form into *tubes* and *tendrils*. Tubes are small sets of pages outside GSCC that connect directly IN to OUT. Tendrils either lead from IN to nowhere or from nowhere to OUT. Following figure 6.181 shows this.

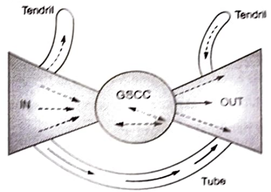


Fig. 6.181 : Structure of web graph

- The study of web Graph can be useful for web algorithms for crawling. Web Graph can also be used to attain the efficiency and comprehensiveness in web navigation as well as enhancing web tool, for example, better search engines and intelligent agents.

6.19.2 Google Map

- Google map is a web mapping service which provides the information about geographical regions and sites, view of streets and route planning for travelling by foot, car or public transport. (In addition to this google maps offers aerial and satellite view of place).
- To find the shortest route between two places, Google map is using a big giant graph of world whose vertices are locations (places) and edges represent the road or train/air connection between the locations. Weight of each edge represents distance or time taken to traverse, between locations which are connected by given edge. Thus Google map is a big graph with huge number of vertices ranging from large cities to small cities to villages to even street intersections. With the help of Google maps, one can find the shortest route from one location to another. Dijkestra's shortest path algorithm is used for this purpose. Since Google map is a big graph with large number of vertices, to reduce the computation time, the nodes in the graph are encompassed within other nodes. These giant nodes encompass the graph of related nodes (locations) This can be seen via 'Zoom' feature of Google maps.
- Further, Google has multiple high performance servers and databases which operate at extreme speeds to perform calculations in almost no time.

SOLVED EXAMPLES

Example 1: *By drawing the graph, show that following graphs are planar graphs.*

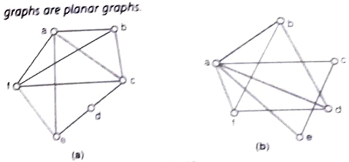


Fig. 6.182