Pimpri Chinchwad Education Trust's

Pimpri Chinchwad College of Engineering

Department of Mechanical Engineering

ENGINEERING MATHEMATICS III

UNIT V – VECTOR CALCULUS

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Vector Algebra

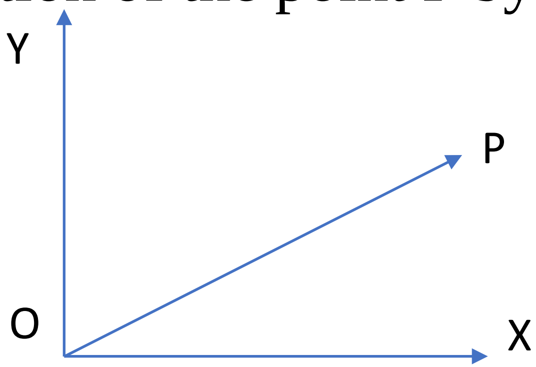
Scalar Quantity: A quantity which has magnitude only is called a scalar quantity. eg. Temp., Time, Speed, etc.

Vector Quantity: A quantity which has both magnitude as well as direction is called vector quantity.
eg. Force, Velocity, Acceleration, etc.

Unit Vector: A vector which has magnitude 1 is called a unit vector.

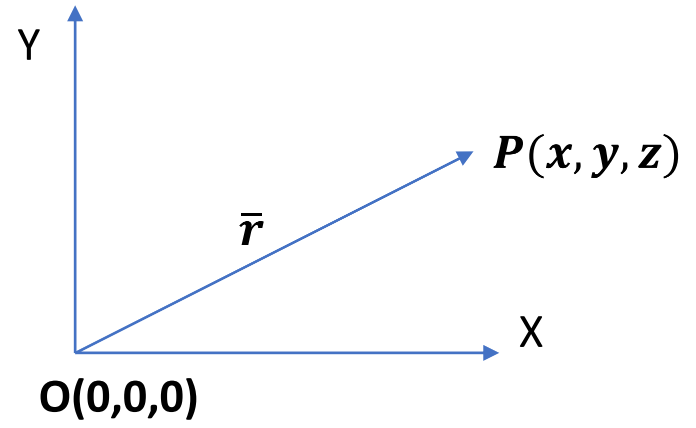
Unit vector in the direction of $\overline{AB}$ is given by $\dfrac{\overrightarrow{AB}}{|\overrightarrow{AB}|}$.

Position Vector: If P(x, y, z) is any point in space and O is the origin then vector $\overrightarrow{OP}$ which indicates the position of the point P by a vector, known as a position vector of P.



Vector Algebra

The position vector of P(x, y, z) is $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ where $\hat{\imath}, \hat{\jmath}, \hat{k}$ are the unit vectors in the direction of X, Y, Z axes resp.

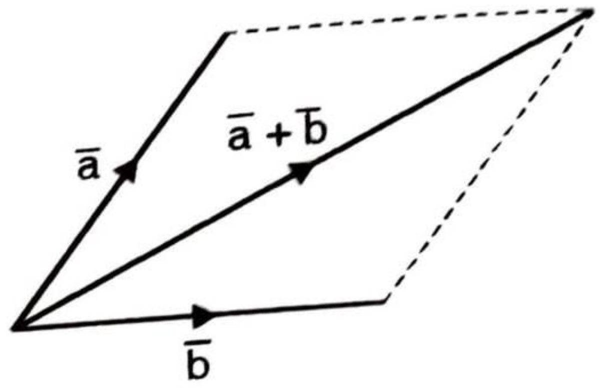


Algebra of Vectors

Let , $\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ be any two vectors

Addition :

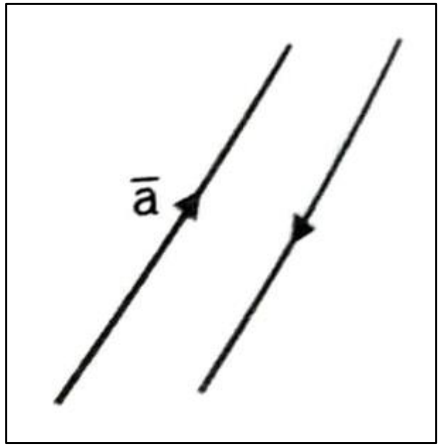
$$\bar{a} + \bar{b} = (a_1+b_1)\hat{\imath} + (a_2+b_2)\hat{\jmath} + (a_3+b_3)\hat{k}$$



Vector Algebra

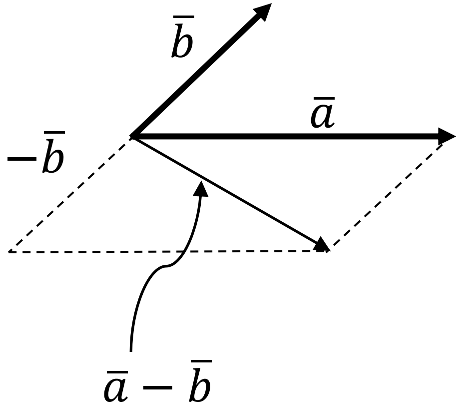
Negative Vector :

$$-\bar{a} = -a_1\hat{\imath} - a_2\hat{\jmath} - a_3\hat{k}$$



Subtraction of Vector:

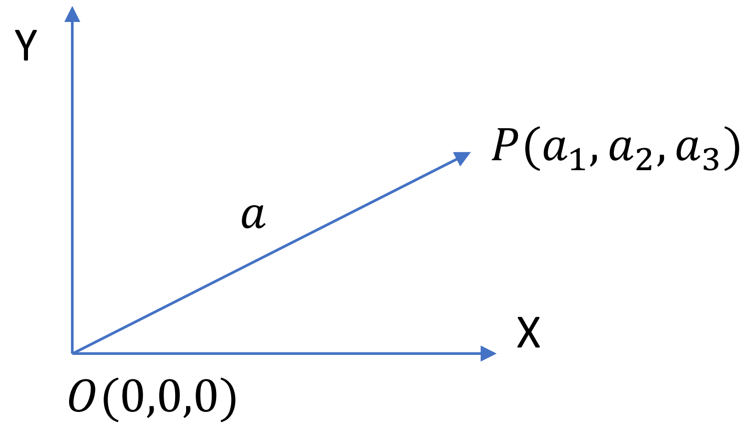
$$\bar{a} - \bar{b} = \bar{a} + \left(-\bar{b}\right)$$
$$= (a_1 - b_1)\hat{\imath} + (a_2 - b_2)\hat{\jmath} + (a_3 - b_3)\hat{k}$$



Vector Algebra

Magnitude of vector $\bar{a}$

$$|\bar{a}| = a = \sqrt{a_1^2 + a_2^2 + a_3^2}$$



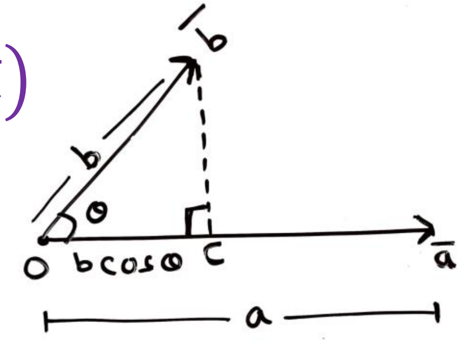
Product of Vector

Dot product: The dot product of two vectors $\bar{a}$ and $\bar{b}$ is defined as

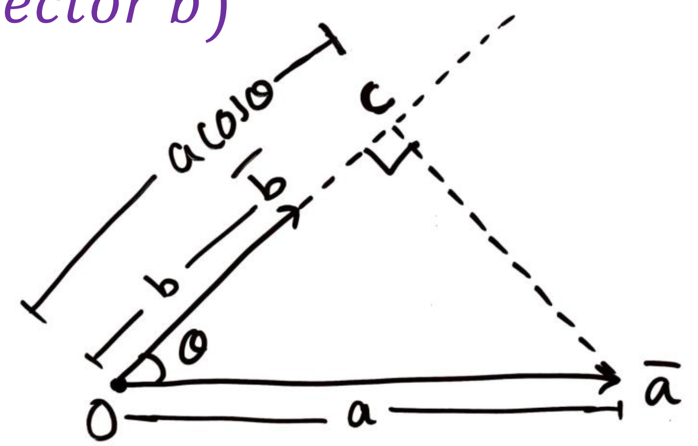
$$\bar{a} \cdot \bar{b} = |\bar{a}||\bar{b}|\cos\theta = ab\cos\theta$$

Where θ is the angle between the two vectors $\bar{a}$ and $\bar{b}$ such that $0 \le \theta \le \pi$.

$$\bar{a} \cdot \bar{b} = (b\cos\theta)a = (\text{Projection of vector } \bar{b} \text{ on vector } \bar{a}) \times (\text{Magnitude of vector } \bar{a})$$



$$\bar{a}\ .\ \bar{b} = (a\cos\theta)b = (Projection\ of\ vector\ \bar{a}\ on\ vector\ \bar{b}) \times (Magnitude\ of\ vector\ \bar{b})$$



Note:

1. $\bar{a}\cdot\bar{b}$ is a scalar quantity hence, it is also known as a scalar product or inner product.
2. The angle between the two vectors $\bar{a}$ and $\bar{b}$ can be calculated by

$$cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|}$$

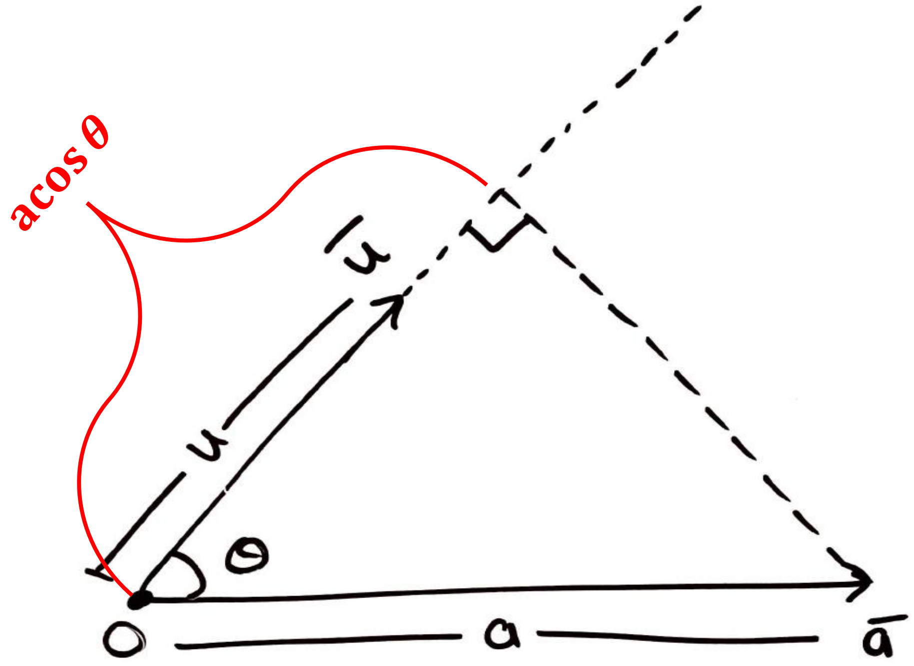
Vector Algebra

3. If $\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ then
$$\bar{a}\cdot\bar{b} = a_1b_1 + a_2b_2 + a_3b_3.$$
4. Dot product is commutative. i.e. $\bar{a}\cdot\bar{b} = \bar{b}\cdot\bar{a}$
5. As $\cos 0 = 1$ and $\cos 90 = 0$,
$$\hat{\imath}\cdot\hat{\imath} = \hat{\jmath}\cdot\hat{\jmath} = \hat{k}\cdot\hat{k} = 1 \text{ and } \hat{\imath}\cdot\hat{\jmath} = \hat{\jmath}\cdot\hat{k} = \hat{k}\cdot\hat{\imath} = 0.$$
6. The component of $\bar{a}$ in the direction of another vector $\bar{u}$ is given by $\bar{a}\cdot\hat{u}$.

The component of $\bar{a}$ in the direction of another vector $\bar{u}$ $=$ **$\mathbf{a\cos\theta}$**

$$\bar{a}\ .\ \bar{u} = au\cos\theta$$

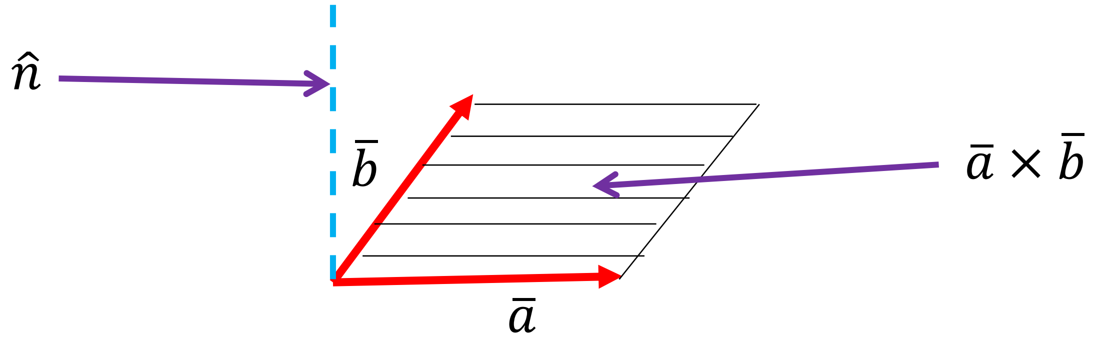
$$a\cos\theta = \frac{\bar{a}\ .\ \bar{u}}{u} = \bar{a}.\frac{\bar{u}}{u} = \bar{a}\ .\ u$$



Cross product: The cross product of two vectors $\bar{a}$ and $\bar{b}$ is defined as

$$\bar{a} \times \bar{b} = |\bar{a}||\bar{b}| sin\theta\, \hat{n}$$

Where θ is the angle between the two vectors $\bar{a}$ and $\bar{b}$ such that $0 \leq \theta \leq \pi$ and $\hat{n}$ is the unit vector perpendicular to the plane of $\bar{a}$ and $\bar{b}$.



Note:

1. $\bar{a} \times \bar{b}$ is a vector quantity hence, it is also known as a vector product.
2. $\bar{a} \times \bar{b} = -(\bar{b} \times \bar{a})$.
3. $\bar{a} \times \bar{a} = 0$.

3. If $\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\bar{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ then

$$\bar{a}\times\bar{b} = \begin{vmatrix} \hat{i} & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

4. As $sin\,0 = 0$ and $\sin 90 = 1$,

$$\hat{\imath}\times\hat{\imath} = \hat{\jmath}\times\hat{\jmath} = \hat{k}\times\hat{k} = 0 \text{ and } \hat{\imath}\times\hat{\jmath} = \hat{k},\ \hat{\jmath}\cdot\hat{k} = \hat{\imath},\ \hat{k}\cdot\hat{\imath} = \hat{\jmath}.$$

5. $\bar{a}$ and $\bar{b}$ are parallel iff $\bar{a}\times\bar{b} = 0$.

& $\bar{a}$ and $\bar{b}$ are perpendicular iff $\bar{a}.\bar{b} = 0$.

Scalar Triple product or Box product: The scalar triple product of three vectors $\bar{a}, \bar{b}$ and $\bar{c}$ is defined as

$$[\bar{a}\ \bar{b}\ \bar{c}] = (\bar{a}\times\bar{b})\cdot\bar{c}$$

Thus,

$$\left[\bar{a}\ \bar{b}\ \bar{c}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Note:

1. Box product is a scalar quantity.
2. $[\bar{a}\ \bar{b}\ \bar{c}] = [\bar{b}\ \bar{c}\ \bar{a}] = [\bar{c}\ \bar{a}\ \bar{b}]$. i.e. Cyclic changes are allowed.
3. $(\bar{a} \times \bar{b}) \cdot \bar{c} = \bar{a} \cdot (\bar{b} \times \bar{c})$, i.e. Dot and cross products are interchangeable.
4. $[\bar{a}\ \bar{a}\ \bar{b}] = 0$, i.e. if two vectors are same in S. T. P. then it is 0.

Vector Triple product: The vector triple product of three vectors $\bar{a}, \bar{b}$ and $\bar{c}$ is defined as

$$\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c} = (I.III)II - (I.II)III$$

Note: $\bar{a} \times (\bar{b} \times \bar{c}) \neq (\bar{a} \times \bar{b}) \times \bar{c}$.

$\bar{a} = a_1\hat{i} + a_2\bar{j} + a_3\bar{k}$, $\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$

$\bar{c} = c_1\bar{i} + c_2\bar{j} + c_3\bar{k}$

$$I \times (II \times III) = (I \cdot III)II - (I \cdot II)III$$

④ vector triple product

$$\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a}\cdot\bar{c})\bar{b} - (\bar{a}\cdot\bar{b})\bar{c}$$

① $\bar{a} \cdot \bar{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

$2 \cdot 3 = 6$

② $$\bar{b} \times \bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = i(b_2c_3 - c_2b_3) - j(b_1c_3 - c_1b_3) + k(b_1c_2 - c_1b_2)$$

③ Angle betn two vectors $\bar{a}$ & $\bar{c}$

$$\cos\alpha = \frac{\bar{a}\cdot\bar{c}}{|\bar{a}|\cdot|\bar{c}|}$$

$$|\bar{a}| = a = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$|\bar{c}| = c = \sqrt{c_1^2 + c_2^2 + c_3^2}$$

④ scalar triple product : $\bar{a} \cdot (\underbrace{\bar{b} \times \bar{c}}) = (\bar{a} \times \bar{b}) \cdot \bar{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

3 three vectors

Examples

Q.1 If, $\bar{a} = 2\hat{\imath} + 2\hat{\jmath} + \hat{k}$, $\bar{b} = \hat{\imath} - \hat{\jmath} + 2\hat{k}$, $\bar{c} = \hat{\imath} + \hat{\jmath} - \hat{k}$ then find

a) $\bar{a}.\bar{b}$

b) $\bar{b} \times \bar{c}$

c) $\hat{b}$

d) Angle between $\bar{a}$ & $\bar{c}$

e) $\bar{a}.(\bar{b} \times \bar{c})$

Solution :

$$(a)\ \bar{a}.\bar{b} = \left(2\hat{i} + 2j + k\right).\left(\hat{i} - j + 2k\right)$$

$$= (2)(1) + (2)(-1) + (1)(2)$$

$$= 2$$

Vector Algebra

$\bar{a} = 2\hat{\imath} + 2\hat{\jmath} + \hat{k}$,
$\bar{b} = \hat{\imath} - \hat{\jmath} + 2\hat{k}$,
$\bar{c} = \hat{\imath} + \hat{\jmath} - \hat{k}$

$$(b)\ \bar{b}\times\bar{c} = \left(2\hat{i}+2j+k\right)\times\left(\hat{i}-j+2k\right)$$

$$= \begin{vmatrix} \hat{i} & j & k \\ 1 & -1 & 2 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \hat{i}(1-2) - j(-1-2) + k(1+1)$$

$$= -\hat{i} + 3j + 2k$$

$$(c)\ \mathrm{b} = \frac{b}{|\bar{b}|} = \frac{\hat{i}-j+k}{\sqrt{(1)^2+(-1)^2+(2)^2}} = \frac{\hat{i}-j+k}{\sqrt{6}}$$

$\bar{a} = 2\hat{\imath} + 2\hat{\jmath} + \hat{k},$
$\bar{b} = \hat{\imath} - \hat{\jmath} + 2\hat{k},$
$\bar{c} = \hat{\imath} + \hat{\jmath} - \hat{k}$

(d) Angle between $\bar{a}$ and $\bar{c}$ =

$$\cos\theta = \frac{\bar{a}.\bar{c}}{|\bar{a}||\bar{c}|} = \frac{\left(2\hat{i}+2j+k\right).\left(\hat{i}+j-k\right)}{\sqrt{(2)^2+(2)^2+(1)^2}\sqrt{(1)^2+(1)^2+(-1)^2}}$$

$$= \frac{2+2-1}{3\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$(e)\ \bar{a}.\left(\bar{b}\times\bar{c}\right) = \begin{vmatrix} 2 & 2 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & -1 \end{vmatrix} = 2(1-2)-2(-1-2)+1(1+1)$$

$$= 6$$

Vector Function

Vector Function of a scalar variable: Let t be a scalar variable. If $\bar{F}$ is defined for every value of t then it can be expressed as

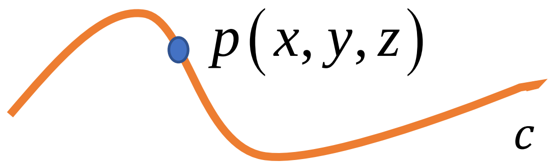
$$\bar{F} = \bar{F}(t) = f_1(t)\bar{\imath} + f_2(t)\bar{\jmath} + f_3(t)\bar{k}.$$

$\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$

As $\bar{F}$ gives a vector for every value of scalar t, hence it is known as a vector function of a scalar variable t.

Differentiation of Vectors: Let $x = x(t), y = y(t), z = z(t)$ be any point on given curve c, then vector equation of curve c is given by

$$\bar{r} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$$



Then,

$$\frac{d\bar{r}}{dt} = \frac{dx}{dt}\bar{i} + \frac{dy}{dt}\bar{j} + \frac{dz}{dt}\bar{k}$$

Vector Function

Note:

a) $\frac{d\bar{r}}{dt}$ is vector along the direction of tangent to curve $\bar{r}$.

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$\frac{d\bar{r}}{dt}$ = tangent vector

b) In particular if $\bar{r}$ is displacement vector of time t then $\frac{d\bar{r}}{dt}$ is velocity vector, denoted by $\bar{v}$.

$\frac{d^2\bar{r}}{dt^2}$ is acceleration vector, denoted by $\bar{a}$.

c) If θ is the angle between the tangent vectors $\bar{v}_1$ & $\bar{v}_2$, then

$$\cos\theta = \frac{\bar{v}_1 \cdot \bar{v}_2}{v_1 v_2}$$

$v_1 = |\bar{v}_1|$

Vector Function

Standard Results:

1. $\frac{d}{dt}\left(\bar{u}\pm\bar{v}\right)=\frac{d\bar{u}}{dt}\pm\frac{d\bar{v}}{dt}$

2. $\frac{d}{dt}\left(\phi\bar{u}\right)=\frac{d\phi}{dt}\bar{u}+\phi\frac{d\bar{u}}{dt}$

3. $\frac{d}{dt}\left(\bar{u}\cdot\bar{v}\right)=\bar{u}\cdot\frac{d\bar{v}}{dt}+\frac{d\bar{u}}{dt}\cdot\bar{v}$

4. $\frac{d}{dt}\left(\bar{u}\times\bar{v}\right)=\bar{u}\times\frac{d\bar{v}}{dt}+\frac{d\bar{u}}{dt}\times\bar{v}$

5. $\frac{d}{dt}\bar{u}\times\left(\bar{v}\times\bar{w}\right)=\frac{d\bar{u}}{dt}\times\left(\bar{v}\times\bar{w}\right)+\bar{u}\times\left(\frac{d\bar{v}}{dt}\times\bar{w}\right)+\bar{u}\times\left(\bar{v}\times\frac{d\bar{w}}{dt}\right)$

6. $\frac{d}{dt}\bar{u}\cdot\left(\bar{v}\times\bar{w}\right)=\frac{d\bar{u}}{dt}\cdot\left(\bar{v}\times\bar{w}\right)+\bar{u}.\left(\frac{d\bar{v}}{dt}\times\bar{w}\right)+\bar{u}\cdot\left(\bar{v}\times\frac{d\bar{w}}{dt}\right)$

$\bar{u}\cdot\bar{v}$

$= u_1v_1+u_2v_2$

$+u_3v_3$

$2\cdot 3 = 6$

MCQs on Vector Differentiation

1. If $\bar{r}(t)$ is position vector of a point on the curve C where t is a scalar variable then $\frac{d\bar{r}}{dt}$ represents ......

(A) Tangent vector (B) Normal vector (C) Radius vector (D) Orthogonal vector

2. If $\bar{r}(t) = x(t)\,\bar{i} + y(t)\,\bar{j} + z(t)\,\bar{k}$ be the position vector of a particle moving along the curve at time t then $\frac{d\bar{r}}{dt}$ represents ......

(A) Acceleration vector (B) Velocity vector (C) Radius vector (D) Normal vector

3. If $\bar{r}(t) = x(t)\,\bar{i} + y(t)\,\bar{j} + z(t)\,\bar{k}$ be the position vector of a particle moving along the curve at time t then $\frac{d^2\bar{r}}{dt^2}$ represents ......

(A) Radius vector (B) Velocity vector (C) Acceleration vector (D) Orthogonal vector

9. If $\bar{r} = r\cos\theta\bar{i} + r\sin\theta\bar{j}$, then $\hat{r}$ is given by ......

(A) $\cos\theta\bar{i} + \sin\theta\bar{j}$ (B) $\sin\theta\bar{i} + \sec\theta\bar{j}$ (C) $\cos\theta\bar{i} + \operatorname{cosec}\theta\bar{j}$ (D) $\tan\theta\bar{i} + \cos\theta\bar{j}$

$$\hat{r} = \frac{\bar{r}}{|\bar{r}|} = \frac{r\cos\theta\,\bar{i} + r\sin\theta\,\bar{j}}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}} = \cos\theta\,\bar{i} + \sin\theta\,\bar{j}$$

MCQs on Vector Differentiation

10. A curve is given by $x = t^2 + 1$, $y = 4t - 3$, $z = 2t^2 - 6t$. Tangent vectors to the curve at $t = 1$ and $t = 2$ are ……

(A) $2\bar{i} + 4\bar{j} + 2\bar{k}$, $2\bar{i} + 4\bar{j} + \bar{k}$

(B) $2\bar{i} + 4\bar{j} - 2\bar{k}$, $4\bar{i} + 4\bar{j} + 2\bar{k}$ ✓

(C) $2\bar{i} + 4\bar{j} - 2\bar{k}$, $2\bar{i} + 4\bar{j} - 2\bar{k}$

(D) $3\bar{i} + 4\bar{j} + 2\bar{k}$, $5\bar{i} + 4\bar{j} - 2\bar{k}$

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} = (t^2+1)\bar{i} + (4t-3)\bar{j} + (2t^2 - 6t)\bar{k}$

Tangent vector $= \dfrac{d\bar{r}}{dt} = 2t\,\bar{i} + 4\bar{j} + (4t-6)\bar{k}$

$\left(\dfrac{d\bar{r}}{dt}\right)_{t=1} =$

$\left(\dfrac{d\bar{r}}{dt}\right)_{t=2} =$

11. A curve is given by $\bar{r} = (t^3 + 2)\,\bar{i} + (4t - 5)\,\bar{j} + (2t^2 - 6t)\,\bar{k}$. Tangent vectors to the curve at $t = 0$ and $t = 2$ are ……

(A) $3\bar{i} + 4\bar{j} - 6\bar{k}$, $6\bar{i} + 4\bar{j} + 2\bar{k}$

(B) $3\bar{i} - 6\bar{k}$, $12\bar{i} + 4\bar{j} + 2\bar{k}$

(C) $4\bar{j} - 6\bar{k}$, $12\bar{i} + 4\bar{j} + 2\bar{k}$

(D) $4\bar{j} - 6\bar{k}$, $12\bar{i} + 2\bar{k}$

$\left(\dfrac{d\bar{r}}{dt}\right)_{t=0} =$

$\left(\dfrac{d\bar{r}}{dt}\right)_{t=2} =$

MCQs on Vector Differentiation

13. The tangent vector to the curve x = a cos t, y = a sin t, z = at tan α at $t = \frac{\pi}{4}$, where a and α are constants is ......

✓ (A) $-\frac{a}{\sqrt{2}}\bar{i} + \frac{a}{\sqrt{2}}\bar{j} + a\tan\alpha\,\bar{k}$

(B) $\frac{a}{\sqrt{2}}\bar{i} - \frac{a}{\sqrt{2}}\bar{j} + a\tan\alpha\,\bar{k}$

(C) $-\frac{a}{2}\bar{i} + \frac{a}{2}\bar{j} + a\tan\alpha\,\bar{k}$

(D) $-\frac{a}{\sqrt{2}}\bar{i} + \frac{a}{\sqrt{2}}\bar{j} + \alpha\,\bar{k}$

$\bar{r} = a\cos t\,\bar{i} + a\sin t\,\bar{j} + at\tan\alpha\,\bar{k}$

$\frac{d\bar{r}}{dt} = -a\sin t\,\bar{i} + a\cos t\,\bar{j} + a\tan\alpha\,\bar{k}$

$\left(\frac{d\bar{r}}{dt}\right)_{t=\frac{\pi}{4}} = -a\frac{1}{\sqrt{2}}\bar{i} + a\frac{1}{\sqrt{2}}\bar{j} + a\tan\alpha\,\bar{k}$

14. A curve is given by $\bar{r} = (e^t\cos t)\,\bar{i} + (e^t\sin t)\,\bar{j} + (e^t)\,\bar{k}$. Tangent vector to the curve at t = 0 is ......

(A) $-\bar{i} - \bar{j} - \bar{k}$

(B) $\bar{j} + \bar{k}$

(C) $2\bar{i} + 2\bar{j} + \bar{k}$

(D) $\bar{i} + \bar{j} + \bar{k}$

$\left(\frac{d\bar{r}}{dt}\right)_{t=0} =$

MCQs on Vector Differentiation

16. For the curve $x = t^3 + 1, y = t^2, z = t$, velocity and acceleration vectors at $t = 1$ are ......

(A) $4\bar{i} + 2\bar{j}, 6\bar{i} + 2\bar{j}$ (B) $3\bar{i} + 2\bar{j} + \bar{k}, 6\bar{i} + 2\bar{j}$ (C) $2\bar{i} + 2\bar{j} + \bar{k}, 3\bar{i} + 2\bar{j}$ (D) $3\bar{i} + 2\bar{j}, 6\bar{i} + \bar{j}$

$\bar{r} = (t^3+1)\bar{i} + (t^2)\bar{j} + t\bar{k}$

velo $= \frac{d\bar{r}}{dt} = 3t^2\,\bar{i} + 2t\,\bar{j} + \bar{k}$ at $t = 1 =$

acc^n $= \frac{d^2\bar{r}}{dt^2} = 6t\,\bar{i} + 2\bar{j}$ at $t = 1 =$

17. For the curve $x = t, y = t^2, z = t^3$, angle between tangents at $t = 0$ and $t = 1$ is given by ......

(A) $\frac{\pi}{2}$ (B) $\cos^{-1}\frac{1}{\sqrt{5}}$ (C) $\cos^{-1}\frac{1}{3}$ (D) $\cos^{-1}\left(\frac{1}{\sqrt{14}}\right)$

$\bar{r} = t\,\bar{i} + t^2\,\bar{j} + t^3\,\bar{k}$

tangent $= \frac{d\bar{r}}{dt} = \bar{i} + 2t\,\bar{j} + 3t^2\,\bar{k}$

$T_{t=0} = \bar{i}$

$T_{t=1} = \bar{i} + 2\bar{j} + 3\bar{k}$

$\cos\theta = \frac{T_{t=0} \cdot T_{t=1}}{|T_{t=0}| \cdot |T_{t=1}|} = \frac{1}{\sqrt{14}}$

MCQs on Vector Differentiation

20. Angle between tangent to the curve $\bar{r} = (e^t \cos$ [illegible] $(e^t \sin t)\,\bar{j} + (e^t)\,\bar{k}$ at $t = 0$ and z axis is given by ......

(A) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (B) $\cos^{-1}\left(\frac{2}{\sqrt{3}}\right)$ (C) $\cos^{-1}(\sqrt{3})$ (D) $\frac{\pi}{2}$

$$T_1 = \left(\frac{d\bar{r}}{dt}\right)_{t=0} = \qquad T_2 = \bar{k}$$

$$\cos\theta = \frac{T_1 \cdot T_2}{|T_1| \cdot |T_2|} =$$

21. If $\bar{r} = \bar{a}\,e^{5t} + \bar{b}\,e^{-5t}$ where $\bar{a}$ and $\bar{b}$ are constant vectors then $\frac{d^2\bar{r}}{dt^2} - 25\bar{r}$ is equal to ......

(A) 1 (B) 2 (C) zero ✓ (D) 5

$$\frac{d\bar{r}}{dt} = 5e^{5t}\,\bar{a} - 5e^{-5t}\,\bar{b}$$

$$\frac{d^2\bar{r}}{dt^2} = 25e^{5t}\,\bar{a} + 25e^{-5t}\,\bar{b} = 25\left[e^{5t}\,\bar{a} + e^{-5t}\,\bar{b}\right] = 25\bar{r}$$

MCQs on Vector Differentiation

23. If $\bar{r} = at\cos t\,\bar{i} + bt\sin t\,\bar{j}$ where a and b are constants then $\frac{d^2\bar{r}}{dt^2}$ at t = 0 is equal to ......

(A) $2b\bar{j}$ (B) $-2a\bar{i}$ (C) $a\bar{i} + b\bar{j}$ (D) $\bar{0}$

25. If acceleration vector $\frac{d^2\bar{r}}{dt^2} = -\bar{i} + 6m\bar{k}$, m is constant, is normal to the position vector $\bar{r} = \bar{i} + m\bar{k}$ then value of m is ......

(A) $\pm\sqrt{6}$ (B) $\pm\frac{1}{\sqrt{6}}$ (C) 0 (D) ± 1

$\bar{a} = -\bar{i} + 6m\bar{k}$

$\bar{b} = \bar{i} + m\bar{k}$

If two vectors are ⊥lar

$\Rightarrow \bar{a}\cdot\bar{b} = 0$

$-1 + 6m^2 = 0$

$\Rightarrow 6m^2 = 1 \Rightarrow m = \pm\frac{1}{\sqrt{6}}$

MCQs on Vector Differentiation

26. If $\bar{r} = \cos(t-1)\,\bar{i} + \sinh(t-1)\,\bar{j} + t^3\bar{k}$ then $\bar{r} \cdot \frac{d^2\bar{r}}{dt^2}$ at $t = 1$ is given by ......

(A) 4 (B) 5 (C) 2 (D) 1

$= (\bar{i} + \bar{k}) \cdot (-\bar{i} + 6\bar{k}) = -1 + 6 = 5$

$3t^2$

$6t$

$\bar{r} = \bar{i} + \bar{k}$ at $t = 1$

$\frac{d^2\bar{r}}{dt^2} = -\cos(t-1)\,\bar{i} + \sinh(t-1)\,\bar{j} + 6t\bar{k}$ at $t = 1$ $= -\bar{i} + 6\bar{k}$

27. If $\bar{r}(t) = t^2\,\bar{i} + t\,\bar{j} - 2t^3\,\bar{k}$ then the value of $\bar{r} \times \frac{d^2\bar{r}}{dt^2}$ is ......

(A) $12t^2\,\bar{i} + 8t^3\,\bar{j} + 2t\,\bar{k}$

(B) $-12t^2\,\bar{i} + 8t^3\,\bar{j}$

(C) $-12t^2\,\bar{i} + 16t^3\,\bar{j} + (t^2 - 2t)\,\bar{k}$

(D) $-12t^2\,\bar{i} + 8t^3\,\bar{j} - 2t\,\bar{k}$

$2t$

$-6t^2$

$-12t$

$\bar{r}(t) = t^2\bar{i} + t\bar{j} - 2t^3\bar{k}$

$\frac{d^2\bar{r}}{dt^2} = 2\bar{i} - 12t\,\bar{k}$

$\bar{r} \times \frac{d^2\bar{r}}{dt^2} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ t^2 & t & -2t^3 \\ 2 & 0 & -12t \end{vmatrix}$

$=$

MCQs on Vector Differentiation

28. If $\bar{r} = \bar{a}\cosh t + \bar{b}\sinh t$ where $\bar{a}$ and $\bar{b}$ are constant vectors then $\frac{d\bar{r}}{dt} \times \frac{d^2\bar{r}}{dt^2}$ is equal to ......

(A) $\bar{b} \times \bar{a}$ (B) $\bar{a} \times \bar{b}$ (C) $\bar{r}$ (D) zero

$\bar{a}k \times \bar{b}l = kl(\bar{a} \times \bar{b})$

$\bar{a} \times \bar{b} = -(\bar{b} \times \bar{a})$

$\sinh^2\alpha + \cos\alpha^2 = 1$

and

$\cosh^2\alpha - \sinh^2\alpha = 1$

$\frac{d\bar{r}}{dt} = \bar{a}\sinh t + \bar{b}\cosh t$

$\frac{d^2\bar{r}}{dt^2} = \bar{a}\cosh t + \bar{b}\sinh t$

$= (\bar{a}\sinh t + \bar{b}\cosh t) \times (\bar{a}\cosh t + \bar{b}\sinh t)$

$=$

$= 0 + \sinh^2 t\,(\bar{a} \times \bar{b}) + (\bar{b} \times \bar{a})\cosh^2 t + 0$

$= -\sinh^2 t\,(\bar{b} \times \bar{a}) + (\bar{b} \times \bar{a})\cosh^2 t = \bar{b} \times \bar{a}$

29. If $\bar{r} = t\,\bar{i} + 2t\,\bar{j} + t^2\,\bar{k}$ then $\bar{r} \cdot \left(\frac{d\bar{r}}{dt} \times \frac{d^2\bar{r}}{dt^2}\right)$ is equal to ......

(A) 1 (B) -1 (C) 0 (D) $\bar{k}$

$\bar{r} = t\bar{i} + 2t\bar{j} + t^2\bar{k}$

$\frac{d\bar{r}}{dt} = \bar{i} + 2\bar{j} + 2t\bar{k}$

$\frac{d^2\bar{r}}{dt^2} = 2\bar{k}$

$$= \begin{vmatrix} t & 2t & t^2 \\ 1 & 2 & 2t \\ 0 & 0 & 2 \end{vmatrix}$$

$= t(4) - 2t(2)$

$= 0$

MCQs on Vector Differentiation

32. $\frac{d}{dt}\left[\bar{r}\cdot\left(\frac{d\bar{r}}{dt}\times\frac{d^2\bar{r}}{dt^2}\right)\right]=$ ......

(A) $\left(\frac{d\bar{r}}{dt}\times\frac{d^2\bar{r}}{dt^3}\right)$ (B) $\bar{r}\cdot\left(\frac{d^2\bar{r}}{dt^2}\times\frac{d^3\bar{r}}{dt^3}\right)$ (C) $\bar{r}\cdot\left(\frac{d\bar{r}}{dt}\times\frac{d^3\bar{r}}{dt^3}\right)$ ✓ (D) 0

$$\frac{d}{dt}\left[\bar{a}\cdot(\bar{b}\times\bar{c})\right]=\frac{d\bar{a}}{dt}\cdot(\bar{b}\times\bar{c})+\bar{a}\cdot\left(\frac{d\bar{b}}{dt}\times\bar{c}\right)+\bar{a}\cdot\left(\bar{b}\times\frac{d\bar{c}}{dt}\right)$$

$$=\frac{d\bar{r}}{dt}\cdot\left(\frac{d\bar{r}}{dt}\times\frac{d^2\bar{r}}{dt^2}\right)+\bar{r}\cdot\left(\frac{d^2\bar{r}}{dt^2}\times\frac{d^2\bar{r}}{dt^2}\right)+\bar{r}\cdot\left(\frac{d\bar{r}}{dt}\times\frac{d^3\bar{r}}{dt^3}\right)$$

$$=0+0+\bar{r}\cdot\left(\frac{d\bar{r}}{dt}\times\frac{d^3\bar{r}}{dt^3}\right)$$

33. If $\frac{d\bar{u}}{dt}=\bar{w}\times\bar{u}$ and $\frac{d\bar{v}}{dt}=\bar{w}\times\bar{v}$ then $\frac{d}{dt}(\bar{u}\times\bar{v})=$ ......

(A) $(\bar{v}\cdot\bar{w})\,\bar{u}-(\bar{u}\cdot\bar{w})\,\bar{v}$ ✓ (B) $(\bar{v}\cdot\bar{w})\,\bar{u}+(\bar{v}\cdot\bar{w})\,\bar{u}$

(C) $(\bar{u}\cdot\bar{w})\,\bar{v}-(\bar{u}\cdot\bar{v})\,\bar{w}$ (D) $(\bar{v}\cdot\bar{w})\,\bar{u}+(\bar{u}\cdot\bar{v})\,\bar{w}$

$$\frac{d}{dt}(\bar{u}\times\bar{v})=\bar{u}\times\frac{d\bar{v}}{dt}+\frac{d\bar{u}}{dt}\times\bar{v}$$

$$=\underline{\bar{u}\times(\bar{w}\times\bar{v})}+\underline{(\bar{w}\times\bar{u})\times\bar{v}}$$

$$=\bar{u}\times(\bar{w}\times\bar{v})-\bar{v}\times(\bar{w}\times\bar{u})$$

$\bar{A}\times\bar{B}$
$=-\bar{B}\times\bar{A}$

vector triple product:

$\mathrm{I}\times(\mathrm{II}\times\mathrm{III})$
$=(\mathrm{I}\cdot\mathrm{III})\,\mathrm{II}-(\mathrm{I}\cdot\mathrm{II})\,\mathrm{III}$

$=(\bar{u}\cdot\cancel{\bar{v}})\,\bar{w}-(\bar{u}\cdot\bar{w})\,\bar{v}$
$-(\bar{v}\cdot\cancel{\bar{u}})\,\bar{w}+(\bar{v}\cdot\bar{w})\,\bar{u}$

$=$

MCQs on Vector Differentiation

34. If $\bar{a}$ is a constant vector then $\frac{d}{dt}\left[r^3\bar{r} + \bar{a} \times \frac{d^2\bar{r}}{dt^2}\right] = \ldots\ldots$

(A) $r^3\frac{d\bar{r}}{dt} + \bar{a} \times \frac{d^2\bar{r}}{dt^2}$

(B) $3r^2\frac{dr}{dt}\bar{r} + r^3\frac{d\bar{r}}{dt} + \bar{a} \times \frac{d^3\bar{r}}{dt^3}$ ✓

(C) $3r^2\bar{r} + r^3\frac{d\bar{r}}{dt}$

(D) $r^2\bar{r} + r^2\frac{d\bar{r}}{dt} + \bar{a} \times \frac{d^2\bar{r}}{dt^2}$

$$\frac{d}{dt}(\phi\bar{u}) = \phi\frac{d\bar{u}}{dt} + \frac{d\phi}{dt}\bar{u}$$

$$= \frac{d}{dt}(\underbrace{r^3}_{\phi}\ \underbrace{\bar{r}}_{u}) + \frac{d}{dt}\left(\bar{a} \times \frac{d^2\bar{r}}{dt^2}\right)$$

$$= r^3\frac{d\bar{r}}{dt} + \frac{d}{dt}(r^3)\bar{r} + \bar{a} \times \frac{d^3\bar{r}}{dt^3} + \underbrace{\frac{d\bar{a}}{dt}}_{=0} \times \frac{d^2\bar{r}}{dt^2}$$

$$= r^3\frac{d\bar{r}}{dt} + \left(3r^2 \times \frac{dr}{dt}\right)\bar{r} + \bar{a} \times \frac{d^3\bar{r}}{dt^3}$$

MCQs on Vector Differentiation

35. If $\bar{v} = t^2\,\bar{i} + 2t\,\bar{j} + (4t-5)\,\bar{k}$ then the value of $\bar{v}\cdot\left(\frac{d\bar{v}}{dt}\times\frac{d^2\bar{v}}{dt^2}\right)$ is ......

(A) $t^2 - 4t + 5$ (B) 10 (C) $16t + 10$ (D) 20

$$\bar{a}\cdot(\bar{b}\times\bar{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} t^2 & 2t & 4t-5 \\ 2t & 2 & 4 \\ 2 & 0 & 0 \end{vmatrix} =$$

36. If $\bar{r} = t^2\,\bar{i} + t\,\bar{j}$, value of $\int_0^1 \left(\bar{r}\times\frac{d\bar{r}}{dt}\right) dt$ is given by ......

$$= \int_0^1 -t^2\,\bar{k}\,dt = \left[-\frac{t^3}{3}\right]_0^1 \bar{k}$$

(A) $\bar{i} + \bar{j}$ (B) $-\frac{1}{3}\bar{k}$ (C) $\frac{2}{3}(\bar{i} + \bar{k})$ (D) $(\bar{i} - \bar{k})$

$$\bar{r}\times\frac{d\bar{r}}{dt} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ t^2 & t & 0 \\ 2t & 1 & 0 \end{vmatrix} = \bar{k}(t^2 - 2t^2)$$

$$= -t^2\,\bar{k}$$

MCQs on Vector Differentiation

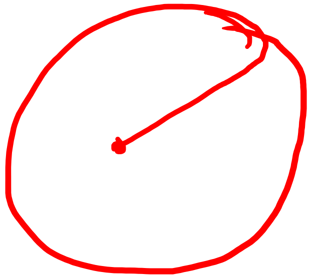
30. If $\bar{r} \cdot \frac{d\bar{r}}{dt} = 0$ then $\bar{r}$ has ......

$\bar{r} \times \frac{d\bar{r}}{dt} = 0$

(A) Constant direction
(B) Constant magnitude
(C) Both constant magnitude and direction
(D) None of these

31. An electron moves such that its velocity is always perpendicular to its radius vector then its path is......

(A) Ellipse
(B) Hyperbola
(C) Straight line
(D) Circle



Examples:

Ex. 1. If $\bar{r} = t^2\hat{\imath} + t\hat{\jmath} - 2t^3\hat{k}$ evaluate $\int_1^2 \bar{r} \times \frac{d^2\bar{r}}{dt^2}\,dt$.

Solution: Differentiate $\bar{r}$ w. r. to t, $\frac{d\bar{r}}{dt} = 2t\hat{i} + j - 6t^2 k$

Again, $\frac{d^2\bar{r}}{dt^2} = 2\hat{i} + 0j - 12tk$ and $\bar{r} = t^2\hat{\imath} + t\hat{\jmath} - 2t^3\hat{k}$

Now, $\bar{r} \times \frac{d^2\bar{r}}{dt^2} = \begin{vmatrix} \hat{i} & j & k \\ t^2 & t & -2t^3 \\ 2 & 0 & -12t \end{vmatrix} = \hat{i}\left(-12t^2\right) - j\left(-12t^3 + 4t^3\right) + k\left(-2t\right) = -12t^2\hat{i} + 8t^3 j - 2tk$

Integrating w. r. to t within the limits 1 to 2,

$$\int_1^2 \bar{r} \times \frac{d^2\bar{r}}{dt^2}\,dt = \int_1^2 \left(-12t^2\hat{i} + 8t^3 j - 2tk\right)dt = \left[-12\frac{t^3}{3}\hat{i} + 8\frac{t^4}{4} j - t^2 k\right]_1^2$$

$$= -28\hat{i} + 30j - 3k$$

Examples:

Ex. 2. Find the angle between the tangents to the curve $x = t, y = t^2, z = t^3$ at $t = 1$ and $\mathrm{t} = -1$.

Solution: Here, $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} = t\hat{\imath} + t^2\hat{\jmath} + t^3\hat{k}$

Differentiate $\bar{r}$ w. r. to t, $\dfrac{d\bar{r}}{dt} = \bar{v} = \hat{i} + 2t\,j + 3t^2 k$

Now, $\bar{v}_1 = \left(\dfrac{d\bar{r}}{dt}\right)_{t=1} = \hat{i} + 2j + 3k$ $\quad \bar{v}_2 = \left(\dfrac{d\bar{r}}{dt}\right)_{t=-1} = \hat{i} - 2j + 3k$

Let θ be the angle between the tangent vectors $\overline{v_1}$ and $\overline{v_2}$, then

$$\cos\theta = \frac{\bar{v}_1 \cdot \bar{v}_2}{v_1 v_2} = \frac{\left(\hat{i} + 2j + 3k\right) \cdot \left(\hat{i} - 2j + 3k\right)}{\sqrt{1+4+9}\sqrt{1+4+9}} = \frac{1-4+9}{\sqrt{14}\sqrt{14}}$$

$$\therefore \cos\theta = \frac{6}{14} \qquad \therefore \theta = \cos^{-1}\frac{3}{7}$$

Examples:

Ex. 3. Show that the necessary and sufficient condition for $\bar{F}$ to have constant magnitude is $\bar{F} \cdot \frac{d\bar{F}}{dt} = 0$.

Solution: To prove the necessary part, Let $\bar{F}$ be a vector with constant magnitude F.

Consider, $\bar{F} \cdot \bar{F} = FFcos0 = F^2$ (constant)

Differentiating w. r. to t $\frac{d}{dt}\left(\bar{F} \cdot \bar{F}\right) = \frac{d\bar{F}}{dt} \cdot \bar{F} + \bar{F} \cdot \frac{d\bar{F}}{dt} = 0$

Since, dot product is commutative,

$$2\frac{d\bar{F}}{dt} \cdot \bar{F} = 0 \qquad i.e. \ \frac{d\bar{F}}{dt} \cdot \bar{F} = 0$$ which proves the necessary part.

Conversely, let $\bar{F} \cdot \frac{d\bar{F}}{dt} = 0$

Multiply by 2 on both side $\Rightarrow 2\frac{d\bar{F}}{dt} \cdot \bar{F} = 0$

Vector Function

$$\Rightarrow \overline{F}.\frac{d\overline{F}}{dt}+\frac{d\overline{F}}{dt}\cdot\overline{F}=0$$

$$\Rightarrow \frac{d}{dt}\left(\overline{F}.\overline{F}\right)=0$$

$$\Rightarrow \frac{d}{dt}\left(F^{2}\right)=0$$

$$\Rightarrow F^{2}=C \qquad \Rightarrow F=\text{constant}$$

$\therefore \ \overline{F}$ to have constant magnitude

Examples:

Ex. 4. The position vector of particle at time t is $\bar{r} = cos(t-1)\bar{i} + sinht(t-1)\bar{j} + mt^3\bar{k}$. Find the condition imposed on m by requiring that at time $t = 1$, acceleration is normal to position vector.

Solution :

We have

$$\bar{r} = \cos(t-1)\bar{i} + \sinh(t-1)\bar{j} + mt^3\bar{k}$$

To find acceleration

$$\frac{d\bar{r}}{dt} = \frac{d}{dt}\left[\cos(t-1)\right]\bar{i} + \frac{d}{dt}\left[\sinh(t-1)\right]\bar{j} + \frac{d}{dt}\left[mt^3\right]\bar{k}$$

$$= -\sin(t-1)\bar{i} + \cosh(t-1)\bar{j} + 3mt^2\bar{k}$$

$$\therefore \bar{a} = \frac{d^2\bar{r}}{dt^2} = \frac{d}{dt}\left[-\sin(t-1)\right]\bar{i} + \frac{d}{dt}\left[\cosh(t-1)\right]\bar{j} + \frac{d}{dt}\left[3mt^2\right]\bar{k}$$

Vector Function

$$\therefore \bar{a} = \frac{d^2\bar{r}}{dt^2} = -\cos(t-1)\bar{i} + \sinh(t-1)\bar{j} + 6mt\bar{k}$$

For $t = 1$

Position vector $\bar{r}$:

$$\bar{r} = \cos 0\bar{i} + \sinh 0\bar{j} + m\bar{k} \qquad \boxed{\because \bar{r} = \cos(t-1)\bar{i} + \sinh(t-1)\bar{j} + mt^3\bar{k}}$$

$$\bar{r} = \bar{i} + m\bar{k}$$

Acceleration $\bar{a}$:

$$\bar{a} = -\cos(0)\bar{i} + \sinh(0)\bar{j} + 6m\bar{k}$$

$$\bar{a} = -\bar{i} + 6m\bar{k}$$

Since, acceleration $\bar{a}$ is normal to position vector $\bar{r} \Rightarrow \bar{a}.\bar{r} = 0$

$$\bar{a}\ .\ \bar{r} = 0 \Rightarrow \left(-\bar{i} + 6m\bar{k}\right).\left(\bar{i} + m\bar{k}\right) = 0$$

$$\Rightarrow (-1)(1) + (6m)(m) = 0$$

$$\Rightarrow m^2 = \frac{1}{6}$$

$$\Rightarrow m = \pm\frac{1}{\sqrt{6}}$$

An Operator Del or Nabla

Point Function:

A variable quantity whose value at any point in a region of space depends upon the position of the point, is called point function.

Example : Temperature, pressure, density, mass, volume, entropy, internal energy.

Scalar Point Function:

If a scalar quantity $\emptyset$ depends for its values on its position say (x, y, z) in space, then $\emptyset(x, y, z)$ is called a scalar point function.

Example : density of rigid body, temperature distribution in a medium

Vector Point Function:

If a vector quantity $\bar{F}$ depends for its values on its position say (x, y, z) in space, then $\bar{F}(x, y, z)$ is called a vector point function.

Example : velocity of moving fluid, gravitational force

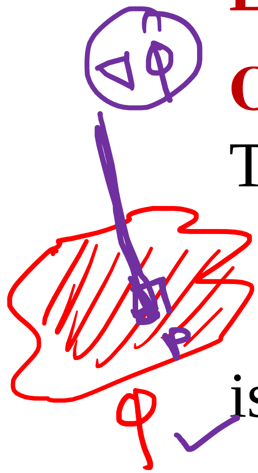
Gradient of scalar

Level surface:
A surface S in R^3 is called level surface of $\emptyset(x, y, z)$ if the value of $\emptyset$ on every point of $\emptyset$ is some fixed constant c.
Example : density of uniform body

$\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ — (2)

$\nabla\cdot\bar{F}$ = Divergence of vector

$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \rightarrow$ scalar value



Operator 'Del' or 'Nabla':
The vector differential operator

$$\nabla = \hat{i}\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \quad — (1)$$

is denoted by the symbol ∇ called 'Del' or 'Nabla'.

$\nabla\times\bar{F}$ = curl of vector

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Gradient of a Scalar:
If the del operator ∇ operates on a scalar point function $\emptyset(x, y, z)$, we get a vector function $\nabla\emptyset$ and is called as Gradient of a scalar point function $\emptyset$ or simply Grad $\emptyset$.

$$\boxed{\therefore Grad\,\phi = \nabla\phi_P = \hat{i}\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}}$$

✓ → vector value

Gradient of scalar

Geometrical meaning of Grad $\emptyset$:

Let P(x, y, z) be any pt. on the level surface $\emptyset(x, y, z) = c$.

$\therefore \bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and d$\bar{r} = dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}$

Now,

$$\nabla\phi \cdot d\bar{r} = \left(\hat{i}\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}\right)\cdot\left(dx\hat{i} + dy\,j + dz\,k\right)$$

$$= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= d\phi = 0 \qquad \left\{\because \phi(x, y, z) = c \quad \therefore d\phi = 0\right\}$$

$\therefore \nabla\emptyset$ is perpendicular to $d\bar{r}$.

As $d\bar{r}$ acts along the tangent at P(x, y, z). Hence, $\nabla\emptyset$ acts along the normal to the surface $\emptyset(x, y, z) = c$ at P(x, y, z).

Thus, $(\nabla\emptyset)_P = N = \boldsymbol{Vector\ normal\ to\ the\ surface\ at\ point\ P}$.

And $\hat{N} = \frac{\nabla\emptyset}{|\nabla\emptyset|} = Unit\ vector\ normal$.

φ

Gradient of scalar

Standard Results:

For any scalars u and v

1. $\nabla(u \pm v) = \nabla u \pm \nabla v$
2. $\nabla(uv) = u\nabla v + v\nabla u$
3. $\nabla\left(\frac{u}{v}\right) = \frac{v\nabla u - u\nabla v}{v^2}$
4. $\nabla(au) = a\nabla u$ where a is constant.
5. $\nabla f(u) = f'(u)\nabla u$

Examples:

Ex. 1. Prove that $\nabla f(r) = f'(r)\frac{\bar{r}}{r}$, where $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$.

Solution: From above results, $\nabla f(r) = f'(r)\nabla r$

$$\therefore \nabla f(r) = f'(r)\left(\hat{i}\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)r \quad = f'(r)\left(\hat{i}\frac{\partial r}{\partial x} + j\frac{\partial r}{\partial y} + k\frac{\partial r}{\partial z}\right)$$

$$\because r = \sqrt{x^2 + y^2 + z^2} \qquad \frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r}$$ similarly, $$\frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\therefore \nabla f(r) = f'(r)\left(\hat{i}\frac{x}{r} + j\frac{y}{r} + k\frac{z}{r}\right)$$

$$\boxed{\therefore \nabla f(r) = f'(r)\frac{\bar{r}}{r}}$$

Ex. 2. Find the angle between the normals to the surface $xy = z^2$ at $(1,4,2)$ and $(-3,-3,3)$.

Solution: Let $\emptyset(x,y,z) = xy - z^2$. Then

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} = y\hat{i} + xj - 2zk$$

As $\nabla\emptyset$ represents the vector normal.

Let,

$$\overline{N_1} = (\nabla\phi)_{(1,4,2)} = 4\hat{i} + j - 4k \qquad \text{and} \qquad \overline{N_2} = (\nabla\phi)_{(-3,-3,3)} = -3\hat{i} - 3j - 6k$$

Then,

$$\cos\theta = \frac{\overline{N_1}\cdot\overline{N_2}}{N_1 N_2} = \frac{\left(4\hat{i} + j - 4k\right)\cdot\left(-3\hat{i} - 3j - 6k\right)}{\sqrt{16+1+16}\sqrt{9+9+36}} = \frac{9}{\sqrt{33}\sqrt{54}}$$

$$\boxed{\therefore \theta = \cos^{-1}\frac{9}{\sqrt{33}\sqrt{54}}}$$

Orthogonal surfaces

Orthogonal surfaces :

Two surfaces are said to be intersect orthogonally if normal's to surfaces at the point of contact intersect at right angle.

Perpendicular vectors:

If two vectors $\bar{a}$ & $\bar{b}$ are perpendicular then their , dot product is zero.
$\bar{a}.\bar{b} = 0$

Ex. 4. Find the angle between the tangents to the curve $\bar{F} = t^2\bar{\imath} + 2t\bar{\jmath} - t^3\bar{k}$ at the points $t = -1$ & $t = 1$.

Solution:

We have

$$\overline{F} = t^2\overline{i} + 2t\overline{j} - t^3\overline{k}$$

Let T_1 & T_2 be the tangent vectors to $\bar{F}$ at $t = -1$ & $t = 1$.

$$\therefore T_1 = \left[\frac{d\overline{F}}{dt}\right]_{t=-1} = \left[2t\overline{i} + 2\overline{j} - 3t^2\overline{k}\right]_{t=-1} = -2\overline{i} + 2\overline{j} - 3\overline{k}$$

$$\therefore T_2 = \left[\frac{d\overline{F}}{dt}\right]_{t=1} = \left[2t\overline{i} + 2\overline{j} - 3t^2\overline{k}\right]_{t=1} = 2\overline{i} + 2\overline{j} - 3\overline{k}$$

Let θ be the angles between the vectors T_1 & T_2

Orthogonal surfaces

$$\cos\theta = \frac{\overline{T_1}.\overline{T_2}}{\left|\overline{T_1}\right|\left|\overline{T_2}\right|}$$

$$= \frac{\left(-2\bar{i}+2\bar{j}-3\bar{k}\right).\left(2\bar{i}+2\bar{j}-3\bar{k}\right)}{\sqrt{(-2)^2+(2)^2+(-3)^2}\sqrt{(2)^2+(2)^2+(-3)^2}}$$

$$= \frac{(-2)(2)+(2)(2)+(-3)(-3)}{\sqrt{17}\sqrt{17}}$$

$$= \frac{9}{17}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{9}{17}\right)$$

MCQs on Gradient, Divergence & Curl

1. Vector differential oprator ∇ is defined by ......

(A) $\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$

(B) $\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$

(C) $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

(D) $\bar{i}\frac{\partial^2}{\partial x^2} + \bar{j}\frac{\partial^2}{\partial y^2} + \bar{k} + \frac{\partial^2}{\partial z^2}$

$= \nabla\phi$

2. Gradient of scalar point function $\phi(x, y, z)$ is ......

(A) $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$

(B) $\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y} + \frac{\partial\phi}{\partial z}$

(C) $\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial z}$

(D) $\frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$

$\nabla\phi$

3. For the level surface $\phi(x, y, z) = c$, gradient of ϕ represents ......

(A) unive vector (B) tangent vector (C) normal vector (D) radius vector

4. For the scalar point functions ϕ and ψ, $\nabla(\phi\psi) =$ ......

(A) $\phi\nabla\psi - \psi\nabla\phi$ (B) $\phi\nabla\psi + \psi\nabla\phi$ (C) $\phi(\nabla^2\psi) + \psi(\nabla^2\phi)$ (D) $\frac{\phi\nabla\psi - \psi\nabla\phi}{\psi^2}$

5. For the scalar point function ϕ and ψ, $\nabla\left(\frac{\phi}{\psi}\right) =$ ......

(A) $\phi\nabla\psi + \psi\nabla\phi$ (B) $\frac{\phi\nabla\psi - \psi\nabla\phi}{\psi^2}$ (C) $\frac{\psi\nabla\phi + \phi\nabla\psi}{\psi^2}$ (D) $\frac{\psi\nabla\phi - \phi\nabla\psi}{\psi^2}$

MCQs on Gradient, Divergence & Curl

$\nabla \cdot \bar{F} = \text{scalar}$

6. If $\bar{F} = F_1(x, y, z)\,\bar{i} + F_2(x, y, z)\,\bar{j} + F_3(x, y, z)\,\bar{k}$ is a vector field then divergence of $\bar{F}$ is ......

(A) $\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$ ✓
(B) $\frac{\partial F_1}{\partial x}\bar{i} + \frac{\partial F_2}{\partial y}\bar{j} + \frac{\partial F_3}{\partial z}\bar{k}$
(C) $\frac{\partial F_1}{\partial x}\,\frac{\partial F_2}{\partial y}\,\frac{\partial F_3}{\partial z}$
(D) $\left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \times (F_1\bar{i} + F_2\bar{j} + F_3\bar{k})$

9. Vector field $\bar{F}$ is solenoidal if ......

(A) $\nabla \times \bar{F} = 0$
(B) $\nabla \cdot \bar{F} = 0$ ✓
(C) $\nabla^2 \bar{F} = 0$
(D) $\bar{F} \cdot \nabla = 0$

10. Vector field $\bar{F}$ is irrotational if ......

(A) $\nabla \cdot \bar{F} = 0$
(B) $\bar{F} \times \nabla = 0$
(C) $\nabla^2 \bar{F} = 0$
(D) $\nabla \times \bar{F} = \bar{0}$ ✓

15. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = \sqrt{x^2 + y^2 + z^2}$ then ∇r is given by ......

(A) $\frac{\bar{r}}{r}$ ✓
(B) $\bar{r}$
(C) $\frac{\bar{r}}{r^2}$
(D) $\frac{1}{r^3}$

$\nabla r = 1 \cdot \frac{\bar{r}}{r}$

$\nabla f(r) = f'(r) \cdot \frac{\bar{r}}{r}$

MCQs on Gradient, Divergence & Curl

17. If $\phi = mx^2 + y + z$, $\bar{b} = 2\bar{i} + 3\bar{j} - \bar{k}$ and $\nabla\phi$ at the point (1, 0, 1) is perpendicular to $\bar{b}$ then m is equal to ……

(A) 0 (B) $\frac{3}{2}$ (C) $-\frac{1}{2}$ (D) $-\frac{5}{2}$

$$\nabla\phi = \bar{i}\,\phi_x + \bar{j}\,\phi_y + \bar{k}\,\phi_z = \bar{i}(2mx) + \bar{j}(1) + \bar{k}(1)$$

$$(\nabla\phi)_{(1,0,1)} = 2m\bar{i} + \bar{j} + \bar{k} \quad \perp \text{lar} \quad \bar{b} = 2\bar{i} + 3\bar{j} - \bar{k}$$

$$\Rightarrow (\nabla\phi)_{(1,0,1)} \cdot \bar{b} = 0 \Rightarrow 2m(2) + (1)(3) + (1)(-1) = 0 \Rightarrow \boxed{m = -\frac{1}{2}}$$

18. The divergence of vector field $\bar{F} = 3xz\,\bar{i} + 2xy\,\bar{j} - yz^2\,\bar{k}$ at a point (1, 1, 1) is ……

(A) 3 (B) 4 (C) 7 (D) 0

$$\nabla\cdot\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 3z + 2x - 2yz$$

$$(\nabla\cdot\bar{F})_{(1,1,1)} = 3 + 2 - 2 = 3$$

MCQs on Gradient, Divergence & Curl

20. If vector field $\bar{v} = (x + 3y)\,\bar{i} + (y - 2z)\,\bar{j} + (x + az)\,\bar{k}$ is solenoidal then value of a is ......

(A) 0 (B) 3 (C) 2 (D) -2

$\bar{V}$ is solenoidal V.F. $\Rightarrow \nabla \cdot \bar{V} = 0$

$$\Rightarrow \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) \cdot \left((x+3y)i + (y-2z)j + (x+az)k\right)$$

$$\Rightarrow \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+az) = 0 \Rightarrow 1 + 1 + a = 0$$

$$\Rightarrow \boxed{a = -2}$$

22. The curl of vector field $\bar{F} = x^2y\,\bar{i} + xyz\,\bar{j} + z^2y\,\bar{k}$ at the point (0, 1, 2) is ......

(A) $4\bar{i} - 2\bar{j} + 2\bar{k}$ (B) $4\bar{i} + 2\bar{j} + 2\bar{k}$ (C) $4\bar{i} + 2\bar{k}$ (D) $2\bar{i} + 4\bar{k}$

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & xyz & z^2y \end{vmatrix} = i(z^2 - xy) - j(0 - 0) + \bar{k}(yz - x^2)$$

$$(\nabla \times \bar{F})_{(0,1,2)} = 4\bar{i} + 2\bar{k}$$

MCQs on Gradient, Divergence & Curl

23. If the vector field $\bar{F} = (x + 2y + az)\,\bar{i} + (2x - 3y - z)\,\bar{j} + (4x - y + 2z)\,\bar{k}$ is irrotational then the value of a is ......

(A) -4 (B) 3 (C) -3 (D) 4 ✓

$F_1 \quad F_2 \quad F_3$

irrotation $\Rightarrow \nabla \times \bar{F} = 0 \Rightarrow \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix} = i(-1-(-1)) - j(4-a) + \bar{k}(2-2) = 0$

$-j(4-a) = 0$

$\Rightarrow a = 4$

$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$

24. If $\bar{u} = x^2y\,\bar{i} + y^2x^3\,\bar{j} - 3x^2z^2\,\bar{k}$ and $\phi = x^2yz$, then $(\bar{u}\cdot\nabla)\,\phi$ at the point (1, 2, 1) is ......

(A) 6 ✓ (B) 9 (C) 18 (D) 5

$(\bar{u}\cdot\nabla)\phi$

$\bar{u}\cdot\nabla = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(y^2x^3) + \frac{\partial}{\partial z}(-3x^2z^2) = \nabla\cdot\bar{u}$

$\bar{u}\cdot\nabla \neq \nabla\cdot\bar{u}$

$= \underbrace{\left(x^2y\frac{\partial}{\partial x} + y^2x^3\frac{\partial}{\partial y} + (-3z^2x^2)\frac{\partial}{\partial z}\right)}_{\bar{u}\cdot\nabla} x^2yz = x^2y(2xyz) + y^2x^3(x^2z) - 3z^2x^2(x^2y) =$

at (1,2,1) $\rightarrow 6$

MCQs on Gradient, Divergence & Curl

25. If $u = x + y + z$, $v = x + y$, $w = -2xz - 2yz - z^2$ then $\nabla u \cdot (\nabla v \times \nabla w)$ is ......

(A) $-2y - 2z$ (B) 0 (C) $-4x - 4y - 4z$ (D) $-2x - 2y - 2z$

$\bar{u} \cdot (\bar{v} \times \bar{w}) \rightarrow$ scalar triple product

$\nabla \phi$

$\nabla u = u_x \bar{i} + u_y \bar{j} + u_z \bar{k}$

$\nabla u = \bar{i} + \bar{j} + \bar{z}$

$\nabla v = \bar{i} + \bar{j} + 0$

$\nabla w = (-2z)\bar{i} + (-2z)\bar{j} + (-2x - 2y - 2z)\bar{k}$

$$\nabla u \cdot (\nabla v \times \nabla w) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -2z & -2z & -2x-2y-2z \end{vmatrix}$$

$= -2x - 2y - 2z - (-2x - 2y - 2z) + (-2z + 2z)$

26. Unit vector in the direction normal to the surface $x^2 + y^2 + z^2 = 9$ at $(1, 2, 2)$ is ......

(A) $\frac{1}{3}(\bar{i} + 2\bar{j} + 2\bar{k})$ (B) $\frac{1}{3}(\bar{i} - 2\bar{j} - 2\bar{k})$ (C) $\frac{1}{3}(\bar{i} + \bar{j} + \bar{k})$ (D) $\frac{1}{9}(\bar{i} + 2\bar{j} + 2\bar{k})$

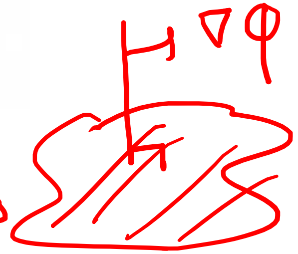
$x =$

$y =$

$z =$

$\bar{r} =$

$\phi = x^2 + y^2 + z^2 - 9 = 0$



Normal vector to surface $\phi = \nabla \phi$

$= (2x)\bar{i} + (2y)\bar{j} + (2z)\bar{k}$

$(\nabla \phi)(1,2,2) = 2\bar{i} + 4\bar{j} + 4\bar{k}$

$$\hat{\nabla \phi}_{(1,2,2)} = \frac{2\bar{i} + 4\bar{j} + 4\bar{k}}{\sqrt{2^2 + 4^2 + 4^2}}$$

MCQs on Gradient, Divergence & Curl

27. Unit vector in the direction normal to the surface $xy = z^2$ at (1, 1, 1) is ......

(A) $\frac{1}{\sqrt{6}}(2\bar{i} + \bar{j} + 2\bar{k})$ (B) $\frac{1}{\sqrt{6}}(\bar{i} - \bar{j} + 2\bar{k})$ (C) $\frac{1}{6}(\bar{i} - \bar{j} - 2\bar{k})$ (D) $\frac{1}{\sqrt{6}}(\bar{i} + \bar{j} - 2\bar{k})$

$\phi = xy - z^2$ Normal vector $= (\nabla\phi)_{(1,1,1)}$

29. Unit vector in the direction of tangent to the curve $x = \sin t$, $y = \cos t$, $z = t$ at $t = \frac{\pi}{4}$ is ......

(A) $\frac{1}{2}(\bar{i} - \bar{j} + \bar{k})$ (B) $-\frac{1}{2}\bar{i} + \frac{1}{2}\bar{j} + \frac{1}{\sqrt{2}}\bar{k}$ (C) $\frac{1}{2}\bar{i} - \frac{1}{2}\bar{j} + \frac{1}{\sqrt{2}}\bar{k}$ (D) $\frac{1}{4}\bar{i} - \frac{1}{4}\bar{j} + \frac{1}{\sqrt{2}}\bar{k}$

$\sqrt{\frac{1}{2} + \frac{1}{2} + 1}$

$= \sqrt{2}$

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$\bar{r} = \sin t\,\bar{i} + \cos t\,\bar{j} + t\,\bar{k}$

$\left|\frac{d\bar{r}}{dt}\right| = \sqrt{2}$

Tangent vector at $t = \pi/4$ $= [\bar{T}]_{t=\frac{\pi}{4}} = \left(\frac{d\bar{r}}{dt}\right)_{t=\frac{\pi}{4}} = \cos\frac{\pi}{4}\bar{i} - \sin\frac{\pi}{4}\bar{j} + \bar{k}$

$= \frac{1}{\sqrt{2}}\bar{i} - \frac{1}{\sqrt{2}}\bar{j} + \bar{k}$

32. Unit vector along the line equally inclined with co-ordinate axes is ......

(A) $\frac{1}{\sqrt{3}}(\bar{i}+\bar{j}+\bar{k})$ ✓ (B) $\frac{1}{\sqrt{3}}(\bar{i}-\bar{j}-\bar{k})$ (C) $\frac{1}{3}(\bar{i}+\bar{j}+\bar{k})$ (D) $\frac{1}{\sqrt{3}}(-\bar{i}+\bar{j}-\bar{k})$

vector equally inclined with co-ordinate axes $= \bar{i}+\bar{j}+\bar{k} = \bar{a}$

$\rightarrow (1,1,1)$

$\hat{a} = \frac{1}{\sqrt{3}}(\bar{i}+\bar{j}+\bar{k}$

34. Unit vector along the direction of line $\frac{x-1}{2}=\frac{y+2}{1}=\frac{z-3}{5}$ is ......

$\frac{x-\alpha}{a}=\frac{y-\beta}{b}=\frac{z-\gamma}{c}$ a, b, c are direction ratios

(A) $\frac{1}{\sqrt{14}}(\bar{i}-2\bar{j}-3\bar{k})$ (B) $\frac{1}{\sqrt{30}}(\bar{i}+2\bar{j}+5\bar{k})$ (C) $\frac{1}{30}(2\bar{i}+\bar{j}-5\bar{k})$ (D) $\frac{1}{\sqrt{30}}(2\bar{i}+\bar{j}+5\bar{k})$ ✓

drs of given line = 2, 1, 5

$\therefore$ vector $\bar{a} = 2\bar{i}+1\bar{j}+5\bar{k}$

$$\hat{a} = \frac{1}{\sqrt{4+1+25}}(2\bar{i}+\bar{j}+5\bar{k})$$

51. The angle between the surfaces $\phi = x \log z - y^2 - 1 = 0$ and $\psi = x^2 y + z + 2 = 0$ at (1, 1, 1) is ......

[Given : $\nabla\phi = \log z\,\bar{i} + (-2y)\,\bar{j} + \frac{x}{z}\bar{k}$ and $\nabla\psi = 2xy\,\bar{i} + x^2\,\bar{j} + \bar{k}$]

(A) $\cos^{-1}\left(-\frac{3}{\sqrt{10}}\right)$ (B) $\cos^{-1}\left(-\frac{1}{\sqrt{30}}\right)$ (C) $\cos^{-1}\left(-\frac{1}{2\sqrt{3}}\right)$ (D) $\cos^{-1}-\left(\frac{2}{\sqrt{30}}\right)$

≡ Angle betn normals at pt. of intersection

$\phi = x\log z - y^2 - 1 = 0$

$(\nabla\phi)_{(1,1,1)} = -2\bar{j} + \bar{k}$

$\psi = x^2 y + z + 2 = 0$

$(\nabla\psi)_{(1,1,1)} = 2\bar{i} + \bar{j} + \bar{k}$

$$\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}|\,|\bar{b}|} = \frac{(0)(2)+(-2)(1)+(1)(1)}{\sqrt{0^2+2^2+1^2}\times\sqrt{2^2+1^2+1^2}} = \frac{-1}{\sqrt{30}}$$

53. If the surfaces $\phi_1 = xyz - 1 = 0$ and $\phi_2 = x^2 + ay^2 + z^2 = 0$ are orthogonal at (1, 1, 1) then a is equal to ......

(A) −1 (B) 2 (C) 1 (D) −2

orthogonal ≡ Angle betn the normals at pt. of intersection is 90°

$\nabla\phi_1 = yz\,\bar{i} + xz\,\bar{j} + xy\,\bar{k} \Rightarrow (\nabla\phi_1)_{(1,1,1)} = \bar{i} + \bar{j} + \bar{k} = \bar{a}$

$\nabla\phi_2 = 2x\,\bar{i} + 2ay\,\bar{j} + 2z\,\bar{k} = (\nabla\phi_2)_{(1,1,1)} = 2\bar{i} + 2a\bar{j} + 2\bar{k} = \bar{b}$

we have $\bar{a}\cdot\bar{b} = 0 \Rightarrow 2 + 2a + 2 = 0 \Rightarrow a = -2$

Orthogonal surfaces

Ex. 3. Find constant a and b so that the surface a$x^2 - 2byz = (a+4)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at $(1,-1,2)$.

Solution: Let $\emptyset_1 = \text{a}x^2 - 2byz - (a+4)x = 0$ and

$$\emptyset_2 = 4x^2y + z^3 - 4 = 0.$$

Then

$$(\nabla\phi_1) = (2ax - a - 4)\hat{i} - 2bz\,j - 2byk = 0 \qquad (\nabla\phi_2) = 8xy\hat{i} + 4x^2\,j + 3z^2k = 0$$

At point $(1,-1,2)$,

$$\overline{N_1} = (\nabla\phi_1)_{(1,-1,2)} = (a-4)\hat{i} - 4b\,j + 2bk = 0 \qquad \overline{N_2} = (\nabla\phi_2)_{(1,-1,2)} = -8\hat{i} + 4\,j + 12k = 0$$

As two surfaces are orthogonal, we have $\overline{N_1} \cdot \overline{N_2} = 0$

$$\therefore \overline{N_1} \cdot \overline{N_2} = -8(a-4) - 16b + 24b = 0$$

$$\therefore a = 4 + b \qquad \ldots\ldots(1)$$

Since, the point $(1,-1,2)$ lies on the surface a$x^2 - 2byz = (a+4)x$, hence it will satisfy the equation of surface.

$$\therefore a + 4b = a + 4 \qquad \boxed{\therefore b = 1}$$

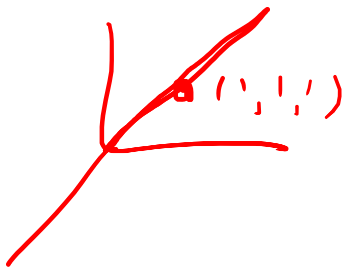
Using in eq. (1) we get $\boxed{\therefore a = 5}$

Important Remarks

Remarks

1) Vector in the direction of the line $\frac{x-\alpha}{a} = \frac{y-\beta}{b} = \frac{z-\gamma}{c}$ is $\boldsymbol{a\bar{\imath} + b\bar{\jmath} + c\bar{k}}$.

2) Tangent vector to the surface $x = x(t), y = y(t), z = z(t)$ is $\boldsymbol{\frac{d\bar{r}}{dt}}$.



where $\bar{r} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$ & $\frac{d\bar{r}}{dt} = \frac{dx}{dt}\bar{\imath} + \frac{dy}{dt}\bar{\jmath} + \frac{dz}{dt}\bar{k}$

3) Normal vector to the surface $\emptyset$ at point P is $(\nabla\emptyset)_{\boldsymbol{P}}$.

4) Vector which is equally inclined with the coordinate axes is $\boldsymbol{\bar{\imath} + \bar{\jmath} + \bar{k}}$.

Directional Derivative (D.D.)

Directional Derivative: The directional derivative of a scalar point function $\emptyset(x, y, z)$ in the direction of a vector $\bar{u}$, **is a component of $\nabla\emptyset$ in the direction of $\bar{u}$.**

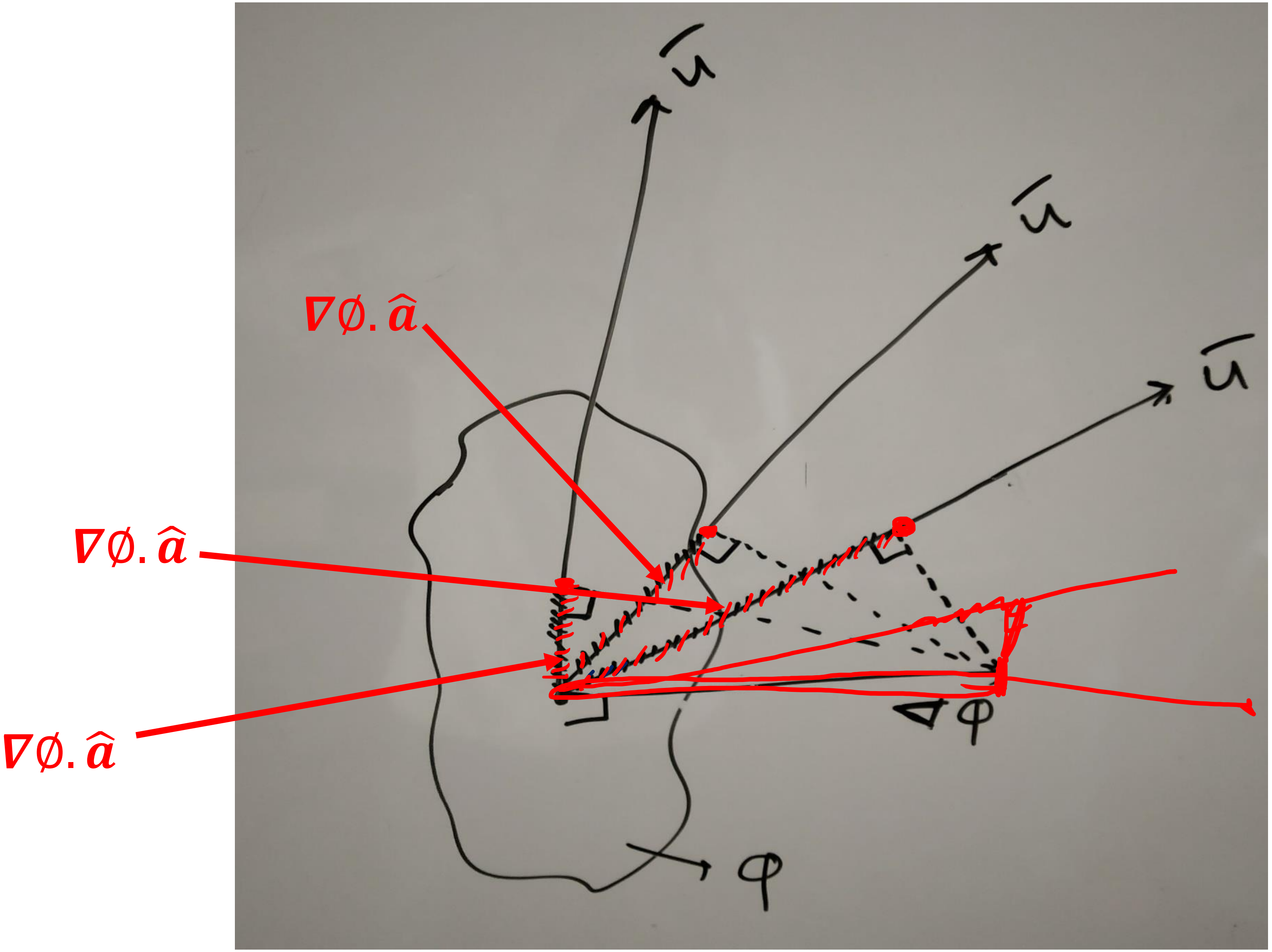
Thus, the directional derivative of $\emptyset$ at a point P is a dot product of $(\nabla\emptyset)_P$ and $\hat{u}$ a unit vector in the direction of $\bar{u}$.

i.e $\quad D.D. = (\nabla\emptyset)_P \cdot \hat{u}$

Note: As a component of a vector is maximum in its own direction. Therefore D. D. is maximum in the direction of $\nabla\emptyset$ only and its maximum magnitude is given by $|\nabla\emptyset|$.

$$(D.D)_{max} = (\nabla\phi)_p \cdot \widehat{\nabla\phi}$$

Directional Derivative (D.D.)



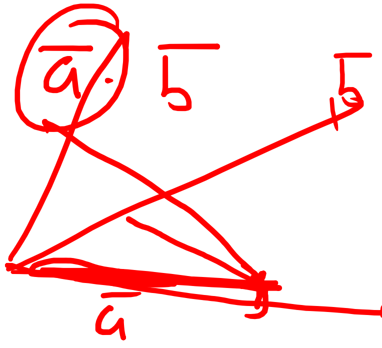
MCQs on Directional Derivative (D.D.)

11. Directional derivative of scalar point function $\phi(x, y, z)$ at a point $P(x_1, x_2, x_3)$ in the direction of vector $\bar{u}$ is ......

$(\nabla\phi)_P \cdot \hat{u}$

(A) $\nabla \cdot (\phi\,\hat{u})(x_1, x_2, x_3)$ (B) $(\nabla\phi)(x_1, x_2, x_3) \times \hat{u}$

(C) $(\nabla\phi)(x_1, x_2, x_3) \cdot \hat{u}$ (D) $(\nabla^2\phi)(x_1, x_2, x_3) \cdot \hat{u}$



12. Magnitude of maximum directional derivative of scalar point function $\phi(x, y, z)$ in the given direction is ......

$|\nabla\phi|$

(A) $|\nabla\phi|$ (B) $|\nabla^2\phi|$ (C) $|\phi\nabla\phi|$ (D) zero

13. Maximum directional derivative of scalar point function $\phi(x, y, z)$ is in the direction of ...... $\nabla\phi$

(A) tangent vector (B) $\bar{i} + \bar{j} + \bar{k}$ (C) radius vector (D) normal vector

14. If $\phi = xy^2 + yz^2$ and $(\nabla\phi)_{(1,-1,1)} = \bar{i} - \bar{j} - 3\bar{k}$ then the value of maximum directional derivative is ......

(A) $\dfrac{\bar{i} - \bar{j} - 3\bar{k}}{\sqrt{11}}$ (B) $\dfrac{1}{\sqrt{11}}$ (C) $\sqrt{4}$ (D) $\sqrt{11}$

$$- \ (D.D)_{max} = |\nabla\phi| = \sqrt{1^2 + 1^2 + 3^2}$$

$$= \sqrt{11}$$

MCQs on Directional Derivative (D.D.)

15. If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = \sqrt{x^2 + y^2 + z^2}$ then ∇r is given by ......

(A) $\frac{\bar{r}}{r}$ (B) $\bar{r}$ (C) $\frac{\bar{r}}{r^2}$ (D) $\frac{1}{r^3}$

16. If $\phi = x + y + z$, $\bar{a} = \bar{i} + \bar{j} + \bar{k}$ then $\nabla\phi \cdot \hat{a}$ is equal to ......

(A) $\frac{3}{2}$ (B) $\sqrt{3}$ (C) 0 (D) $-\frac{5}{2}$

$\nabla\phi = \bar{i} + \bar{j} + \bar{k}$

$\hat{a} = \frac{\bar{a}}{|\bar{a}|} = \frac{\bar{i} + \bar{j} + \bar{k}}{\sqrt{3}}$

$\nabla\phi \cdot \hat{a} = (1)\left(\frac{1}{\sqrt{3}}\right) + (1)\left(\frac{1}{\sqrt{3}}\right) + (1)\frac{1}{\sqrt{3}}$

$= \frac{3}{\sqrt{3}} = \sqrt{3}$

MCQs on Directional Derivative (D.D.)

35. The directional derivative of $\phi = 2x^2 + 3y^2 + z^2$ at the point (2, 1, 3) in the direction of vector $\bar{u} = \bar{i} - 2\bar{j} + 2\bar{k}$ is ......

(A) $\frac{8}{3}$ ✓ (B) 8 (C) $\frac{4}{3}$ (D) $\frac{16}{3}$

$$(\nabla\phi)_{(2,1,3)} \cdot \hat{u} = (8)\left(\frac{1}{3}\right) + (6)\left(-\frac{2}{3}\right) + (6)\left(\frac{2}{3}\right) = 8/3 \checkmark$$

$$\nabla\phi = 4x\bar{i} + 6y\bar{j} + 2z\bar{k} \Rightarrow (\nabla\phi)_{(2,1,3)} = 8\bar{i} + 6\bar{j} + 6\bar{k}$$

$$\hat{u} = \frac{\bar{u}}{|\bar{u}|} = \frac{\bar{i} - 2\bar{j} + 2\bar{k}}{\sqrt{9}} = \frac{1}{3}\bar{i} - \frac{2}{3}\bar{j} + \frac{2}{3}\bar{k}$$

39. The directional derivative of $\phi = e^{2x}\cos(yz)$ at origin in the direction of vector $\bar{u} = \bar{i} + \bar{j} + \bar{k}$ is ......

(A) $\frac{4}{\sqrt{3}}$ (B) $\frac{2}{\sqrt{3}}$ ✓ (C) 0 (D) $\frac{5}{\sqrt{3}}$

$$\phi = e^{2x}\cos(yz)$$

$$(\nabla\phi)_{(0,0,0)} \cdot \hat{u} = \frac{2}{\sqrt{3}}$$

$$(\nabla\phi)_{(0,0,0)} =$$

$$\hat{u} = \frac{\bar{u}}{|\bar{u}|}$$

MCQs on Directional Derivative (D.D.)

42. For what values of a, b, c the directional derivative of $\phi = axy + byz + czx$ at (1, 1, 1) has maximum magnitude 4 in a direction paralllel to x-axis. (2)

[Given : $(\nabla\phi)_{(1,\,1,\,1)} = (a + c)\,\bar{i} + (a + b)\,\bar{j} + (b + c)\,\bar{k}$]

(A) $a = -2, b = 2, c = -2$

(B) $a = 1, b = -1, c = 1$

(C) $a = 2, b = -2, c = 2$

(D) $a = 2, b = 2, c = 2$

MCQs on Directional Derivative (D.D.)

44. The directional derivative of $\phi = x^2yz^3$ at (2, 1, –1) has maximum value in the direction of vector is ……

(A) $-4\bar{i} - 4\bar{j} - 2\bar{k}$ (B) $-4\bar{i} - 4\bar{j} + 12\bar{k}$ (C) $-\bar{i} + 4\bar{j} + 12\bar{k}$ (D) $4\bar{i} - 4\bar{j} - 12\bar{k}$

47. If the directional derivatived of $\phi = ax + by$ has maximum magnitude 2 along x-axis, then a, b are respectively given by …

(A) 1, 0 (B) 0, 1 (C) 2, 0 (D) 1, 1

MCQs on Directional Derivative (D.D.)

49. Maximum value of directional derivative of $\phi = xyz^2$ at (1, 0, 3) is ......

(A) 12 (B) 9 (C) 3 (D) 17

51. The angle between the surfaces $\phi = x \log z - y^2 - 1 = 0$ and $\psi = x^2y + z + 2 = 0$ at (1, 1, 1) is ......

[Given : $\nabla\phi = \log z\,\bar{i} + (-2y)\,\bar{j} + \frac{x}{z}\bar{k}$ and $\nabla\psi = 2xy\,\bar{i} + x^2\,\bar{j} + \bar{k}$]

(A) $\cos^{-1}\left(-\frac{3}{\sqrt{10}}\right)$ (B) $\cos^{-1}\left(-\frac{1}{\sqrt{30}}\right)$ (C) $\cos^{-1}\left(-\frac{1}{2\sqrt{3}}\right)$ (D) $\cos^{-1}-\left(\frac{2}{\sqrt{30}}\right)$

MCQs on Directional Derivative (D.D.)

53. If the surfaces $\phi_1 = xyz - 1 = 0$ and $\phi_2 = x^2 + ay^2 + z^2 = 0$ are orthogonal at (1, 1, 1) then a is equal to ......

(A) -1 (B) 2 (C) 1 (D) -2

Directional Derivative (D.D.)

Ex. 1. Find the D. D. of $\emptyset = xy^2 + yz^2$ at $(2,-1,1)$

i) in the direction $2\hat{\imath} + \hat{\jmath} + 3\hat{k}$

ii) Along the line $2(x-2) = (y+1) = z-1$

iii) Along the direction normal to the surface $x^2 + y^2 + z^2 = 9$ at $(1,2,2)$.

Solution: From given, $\nabla\emptyset = y^2\hat{\imath} + (2xy + z^2)\hat{\jmath} + 2yz\hat{k}$

At pt. $(2,-1,1)$, $(\nabla\emptyset)_{(2,-1,1)} = \hat{\imath} - 3\hat{\jmath} - 2\hat{k}$

i) Let $\bar{u} = 2\hat{\imath} + \hat{\jmath} + 3\hat{k}$ then

$$u = \frac{u}{|\bar{u}|} = \frac{2\hat{i} + j + 3k}{\sqrt{4+1+9}} = \frac{2\hat{i} + j + 3k}{\sqrt{14}}$$

Therefore, $D.D. = (\nabla\emptyset)_P \cdot \hat{u}$

$$D.D. = \left(\hat{i} - 3j - 2k\right) \cdot \frac{2\hat{i} + j + 3k}{\sqrt{14}}$$

$$\therefore D.D. = \frac{2-3-6}{\sqrt{14}} = \frac{-7}{\sqrt{14}}$$

Directional Derivative (D.D.)

ii) To find $\bar{u}$ consider the equation of line.

$$\frac{x-x_1}{\alpha}=\frac{y-y_1}{\beta}=\frac{z-z_1}{\gamma}$$

Where α,β,γ are d.rs. of the line and its direction is given by $\alpha\hat{\imath}+\beta\hat{\jmath}+\gamma\hat{k}$.

Thus, for given line $\alpha=1,\beta=2,\gamma=2$ and

hence, $\bar{u}=\hat{\imath}+2\hat{\jmath}+2\hat{k}$.

$i)\quad 2(x-2)=(y+1)=z-1$

Then,

$$u=\frac{\bar{u}}{|\bar{u}|}=\frac{\hat{i}+2j+2k}{\sqrt{1+4+4}}=\frac{\hat{i}+2j+2k}{3}$$

Therefore, $D.D.=(\nabla\emptyset)_P\cdot\hat{u}$

$$D.D.=\left(\hat{i}-3j-2k\right)\cdot\frac{\hat{i}+2j+2k}{3}$$

$$\therefore D.D.=\frac{1-6-4}{3}=\frac{-11}{3}$$

Directional Derivative (D.D.)

i) Along the direction normal to the surface $x^2 + y^2 + z^2 = 9$ at (1,2,2).

iii) Let $\emptyset_1 = x^2 + y^2 + z^2 - 9$

Then, $\nabla\emptyset = 2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}$

At pt. (1,2,2), let $\bar{u} = (\nabla\emptyset)_{(1,2,2)} = \widehat{2i} + 4\hat{\jmath} + 4\hat{k}$

Then, $$u = \frac{\bar{u}}{|\bar{u}|} = \frac{2\hat{i} + 4j + 4k}{\sqrt{4+16+16}} = \frac{2\hat{i} + 4j + 4k}{6}$$

Therefore, $D.D. = (\nabla\emptyset)_P \cdot \hat{u}$

$$D.D. = \left(\hat{i} - 3j - 2k\right) \cdot \frac{2\hat{i} + 4j + 4k}{6}$$

$$\therefore D.D. = \frac{2-12-8}{6} = \frac{-18}{6} = -3$$

Directional Derivative (D.D.)

Ex. 2. Find the D. D. of $\emptyset = e^{2x} cosyz$ at origin in the direction tangent to the curve $x = asint, y = acost, z = at,$ at $t = \pi/4$.

Solution: From given,

$$\nabla\emptyset = (2e^{2x}cosyz)\hat{\imath} + (e^{2x}(-z)sinyz)\hat{\jmath} + (e^{2x}(-y)sinyz)\hat{k}$$

At origin, $(\nabla\emptyset)_{(0,0,0)} = 2\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$

Now, Let $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ then $\bar{r} = asint\hat{\imath} + acost\hat{\jmath} + at\hat{k}$

Then, $\bar{u}$ will be a tangent to the curve at $t = \pi/4$.

Let $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ then $\frac{d\bar{r}}{dt} = a\cos t\hat{i} - a\sin t\, j + ak$

$$\bar{u} = \left(\frac{d\bar{r}}{dt}\right)_{t=\pi/4} = \frac{a}{\sqrt{2}}\hat{i} - \frac{a}{\sqrt{2}} j + ak \qquad \therefore u = \frac{\bar{u}}{|\bar{u}|} = \frac{\frac{a}{\sqrt{2}}\hat{i} - \frac{a}{\sqrt{2}} j + ak}{\sqrt{\frac{a^2}{2} + \frac{a^2}{2} + a^2}}$$

Therefore, $D.D. = (\nabla\emptyset)_P \cdot \hat{u}$

$$D.D. = \left(2\hat{i}\right)\cdot\frac{\frac{a}{\sqrt{2}}\hat{i} - \frac{a}{\sqrt{2}}j + ak}{\sqrt{\frac{a^2}{2}+\frac{a^2}{2}+a^2}}$$

$$\therefore D.D. = \frac{\sqrt{2}a}{\sqrt{2}a} = 1$$

Ex. 3. Find the D. D. of $\emptyset = 4xz^3 - 3x^2y^2z$ at $(2, -1, 2)$ along a line equally inclined with coordinate axes.

Solution: From given,

$$\nabla\emptyset = (4z^3 - 6xy^2z)\hat{\imath} + (-6x^2yz)\hat{\jmath} + (12xz^2 - 3x^2y^2)\hat{k}$$

At $(2, -1, 2)$, $\quad (\nabla\emptyset)_{(2,-1,2)} = 8\hat{\imath} + 48\hat{\jmath} + 84\hat{k}$

Now, let $\bar{u} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, where x, y, z are the d.cs of a line equally inclined with coordinate axes i.e $\bar{u} = \hat{\imath} + \hat{\jmath} + \hat{k}$.

Directional Derivative (D.D.)

$$\therefore u = \frac{\bar{u}}{|\bar{u}|} = \frac{(\hat{i} + j + k)}{\sqrt{1+1+1}} = \frac{1}{\sqrt{3}}(\hat{i} + j + k)$$

Therefore, $D.D. = (\nabla\emptyset)_P \cdot \hat{u}$

$$D.D. = (8\hat{i} + 48j + 84k) \cdot \frac{1}{\sqrt{3}}(\hat{i} + j + k)$$

$$\therefore D.D. = \frac{140}{\sqrt{3}}$$

Ex. 4. If the directional derivative of $\emptyset = axy + byz + czx$ at $(1, 1, 1)$ has maximum magnitude 4 in a direction parallel to X- axis, find a, b, c.

Solution: From given,

$$\nabla\emptyset = (ay + cz)\hat{\imath} + (ax + bz)\hat{\jmath} + (by + cx)\hat{k}$$

At$(1, 1, 1)$, $\quad (\nabla\emptyset)_{(1,1,1)} = (a + c)\hat{\imath} + (a + b)\hat{\jmath} + (b + c)\hat{k}$

Directional Derivative (D.D.)

We know that the D. D. is maximum in the direction of $\nabla\emptyset$, but from given it is maximum in the direction $\bar{u}$ parallel to X- axis.
Therefore, the direction of $\nabla\emptyset$ and $\bar{u}$ must be same.

$$\text{i.e, } \frac{\nabla\emptyset}{|\nabla\emptyset|} = \hat{u} \qquad\qquad \therefore \nabla\emptyset = |\nabla\emptyset|\hat{u}$$

$$\therefore (a+c)\hat{\imath} + (a+b)\hat{\jmath} + (b+c)\hat{k} = 4\hat{\imath}$$

Equating coefficients, we get $(a+c) = 4,\ (a+b) = 0,\ (b+c) = 0$

which gives, a $= 2,\ b = -2,\ c = 2.$

Ex. 5. If the directional derivative of $\emptyset = ax^2y + by^2z + cz^2x$ at $(1,1,1)$ has maximum magnitude 15 in a direction parallel to the line

$$\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$$

find a, b, c.

Directional Derivative

Solution: From given,

$$\nabla\emptyset = (2axy + cz^2)\hat{\imath} + (ax^2 + 2byz)\hat{\jmath} + (by^2 + 2czx)\hat{k}$$

At $(1, 1, 1)$, $\quad (\nabla\emptyset)_{(1,1,1)} = (2a + c)\hat{\imath} + (a + 2b)\hat{\jmath} + (b + 2c)\hat{k}$

Now, the direction of the given line is $\bar{u} = 2\hat{\imath} - 2\hat{\jmath} + \hat{k} \quad \therefore \hat{u} = \frac{2\hat{\imath}-2\hat{\jmath}+\hat{k}}{3}$

We know that the D. D. is maximum in the direction of $\nabla\emptyset$, but from given it is maximum in the direction of $\bar{u}$.
Therefore, the direction of $\nabla\emptyset$ and $\bar{u}$ must be same.

i.e, $\frac{\nabla\emptyset}{|\nabla\emptyset|} = \hat{u} \qquad \therefore \nabla\emptyset = |\nabla\emptyset|\hat{u}$

$$\therefore (2a + c)\hat{\imath} + (a + 2b)\hat{\jmath} + (b + 2c)\hat{k} = 15\left(\frac{2\hat{\imath} - 2\hat{\jmath} + \hat{k}}{3}\right)$$

Equating coefficients, $\quad (2a + c) = 10,\ (a + 2b) = -10,\ (b + 2c) = 5$

By solving these equations, we get $a = \frac{20}{9},\ b = -\frac{55}{9},\ c = \frac{50}{9}$.

Ex. 5. If the directional derivative of $\emptyset = ax^2y + by^2z + cz^2x$ at $(1, 1, 1)$ has maximum magnitude 15 in a direction parallel to the line

$$\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$$

find a, b, c.

Directional Derivative

Example 6 :

Find the directional derivative of $\varphi = x^2 - y^2 - 2z^2$ at the point $P(2, -1, 3)$ in direction of PQ where Q is $(5, 6, 4)$.

Ans : $\frac{-14}{\sqrt{59}}$

Example 7 :

Find the directional derivative of $\varphi = xy + yz^2$ at the point $(1, -1, 1)$ towards the point $(2, 1, 2)$.

Ans : $\frac{1}{\sqrt{6}}$

Divergence of Vector

Let $\overline{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be any vector point function

Divergence of a Vector point function: The expression

$$\therefore \nabla \cdot \overline{F} = \left(\hat{i}\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \cdot \left(F_1\hat{i} + F_2 j + F_3 k \right)$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

which is a scalar quantity, called as Divergence of a vector point function $\overline{F}$ or simply Div $\overline{F}$.

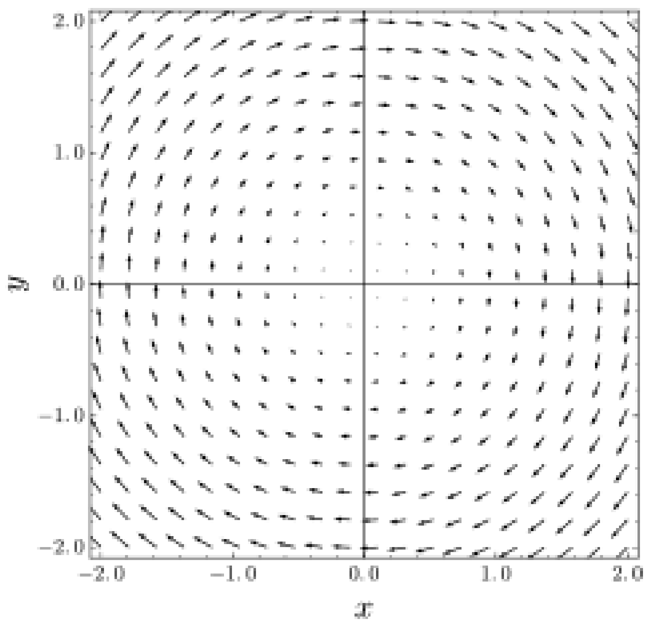
Thus,

$$Div\,\overline{F} = \nabla \cdot \overline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Divergence of Vector

Note:

1) If $\nabla \cdot \bar{F} = 0$, then $\bar{F}$ is said to be **Solenoidal.**
2) In vector calculus a solenoidal vector field also known as an incompressible vector field, a divergence-free vector field, or a transverse vector field is a vector field v with divergence zero at all points in the field
3) A common way of expressing this property is to say that the field has no sources or sinks. The field lines of a solenoidal field are either closed loops or end at infinity.
4) Examples
   a) The magnetic field B (Maxwell's equations)
   b) The velocity field of an incompressible fluid



Curl of Vector

Let $\bar{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$ be any vector point function

Curl of a Vector point function: The expression

$$\therefore \nabla \times \overline{F} = \left(\hat{i}\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times \left(F_1\hat{i} + F_2 j + F_3 k\right)$$

$$= \begin{vmatrix} \hat{i} & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\therefore Curl\,\overline{F} = \nabla \times \overline{F} = \hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - j\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + k\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

which is a vector quantity, called as Curl of a vector point function $\bar{F}$ or simply Curl $\bar{F}$.

Curl of Vector

Note:

1) If $\nabla \times \bar{F} = 0$, then $\bar{F}$ is said to be **Irrotational or Conservative.**
2) In this case there exists a scalar point function $\emptyset$ such that $\bar{F} = \nabla\emptyset$, where $\emptyset$ is called as scalar potential of $\bar{F}$.

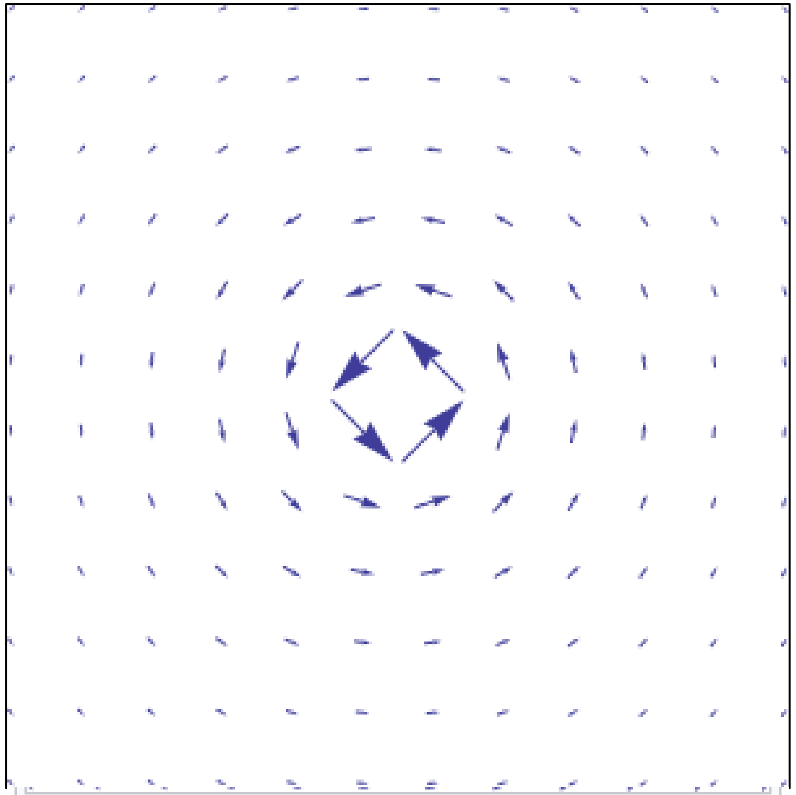
The formula to obtain such $\emptyset$ for an irrotational field $\bar{F}$ is

$$d\emptyset = F_1 dx + F_2 dy + F_3 dz$$

$$\therefore \phi = \int_{\substack{y,z-\text{constant}}} F_1 dx + \int_{\substack{,z-\text{constant} \\ \text{free } x}} F_2 dy + \int_{\text{free } x,y} F_3 dz$$

3) Conservative vector fields have the property that the line integral is path independent; the choice of any path between two points does not change the value of the line integral. Path independence of the line integral is equivalent to the vector field being conservative. A conservative vector field is also irrotational.

Curl of Vector



The above vector field

$$\mathbf{v} = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$$ defined on

$U = \mathbb{R}^3 \setminus \{(0,0,z) \mid z \in \mathbb{R}\}$ has zero curl almost everywhere and is thus irrotational. However, it is neither conservative nor does it have path independence.

Irrotational(or Conservative) Vector Field

Ex. 1. Show that $\bar{F} = (6xy + z^3)\hat{\imath} + (3x^2 - z)\hat{\jmath} + (3xz^2 - y)\hat{k}$ is irrotational. Find scalar $\emptyset$ such that $\bar{F} = \nabla\emptyset$.

Solution:

$$\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix}$$

$$\therefore \nabla \times \overline{F} = \hat{i}(-1+1) - j\left(3z^2 - 3z^2\right) + k(6x - 6x) = 0$$

$\therefore$ $\bar{F}$is **Irrotational.**

Now, to find corresponding scalar $\emptyset$

$$\boxed{\therefore \phi = \int\limits_{y,z-\text{constant}} F_1 dx + \int\limits_{\substack{,z-\text{constant}\\ \text{free x}}} F_2 dy + \int\limits_{free\,x,y} F_3 dz}$$

$$\therefore \phi = \int\limits_{y,z-\text{constant}} \left(6xy + z^3\right) dx + \int\limits_{\substack{z-\text{constant}\\ \text{free x}}} (-z)\, dy + \int\limits_{free\,x,y} 0 dz$$

$$\boxed{\therefore \phi = 3x^2 y + xz^3 - zy + c}$$

Irrotational(or Conservative) Vector Field

Ex. 2. Show that $\bar{F} = (2xz^3 + 6y)\hat{\imath} + (6x - 2yz)\hat{\jmath} + (3x^2z^2 - y^2)\hat{k}$ is irrotational. Find scalar $\emptyset$ such that $\bar{F} = \nabla\emptyset$.

Solution:

$$\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xz^3 + 6y & 6x - 2yz & 3x^2z^2 - y^2 \end{vmatrix}$$

$$\therefore \nabla \times \overline{F} = \hat{i}(-2y + 2y) - j\left(6xz^2 - 6xz^2\right) + k(6-6) = 0$$

$\therefore$ $\bar{F}$ is **Irrotational.** Now, to find corresponding scalar $\emptyset$

$$\boxed{\therefore \phi = \int\limits_{y,z-\text{cons tant}} F_1 dx + \int\limits_{\substack{,z-\text{cons tant} \\ \text{free x}}} F_2 dy + \int\limits_{free\, x,y} F_3 dz}$$

$$\therefore \phi = \int\limits_{y,z-\text{cons tant}} \left(2xz^3 + 6y\right) dx + \int\limits_{\substack{z-\text{cons tant} \\ \text{free x}}} (-2yz)\, dy + \int\limits_{free\, x,y} 0 dz$$

$$\boxed{\therefore \phi = x^2z^3 + 6xy - y^2z + c}$$

Irrotational(or Conservative) Vector Field

Ex. 3. Show that $\bar{F} = (y^2 cosx + z^2)\hat{\imath} + (2ysinx)\hat{\jmath} + (2xz)\hat{k}$ is conservative. Find scalar $\emptyset$ such that $\bar{F} = \nabla\emptyset$.

Solution:

$$\nabla \times \overline{F} = \begin{vmatrix} \hat{i} & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 \cos x + z^2 & 2y \sin x & 2xz \end{vmatrix}$$

$$\therefore \nabla \times \overline{F} = \hat{i}(0-0) - j(2z-2z) + k(2y\cos x - 2y\cos x) = 0$$

$\therefore$ $\bar{F}$ is **Irrotational.** Now, to find corresponding scalar $\emptyset$

$$\boxed{\therefore \phi = \int\limits_{y,z-\text{constant}} F_1 dx + \int\limits_{\substack{,z-\text{constant} \\ \text{free } x}} F_2 dy + \int\limits_{free\, x,y} F_3 dz}$$

$$\therefore \phi = \int\limits_{y,z-\text{constant}} \left(y^2 \cos x + z^2\right) dx + \int\limits_{\substack{z-\text{constant} \\ \text{free } x}} 0 dy + \int\limits_{free\, x,y} 0 dz$$

$$\boxed{\therefore \phi = y^2 \sin x + z^2 x + c}$$

Some Useful Results

If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ & $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$

$$\boxed{1)\ \nabla . \bar{r} = 3}$$

$$LHS = \nabla . \bar{r} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z} \right)\left(x\bar{i} + y\bar{j} + z\bar{k} \right)$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$$

$$= 1 + 1 + 1 \quad = 3$$

$$\boxed{2)\ \nabla . \bar{a} = 0}$$

$$LHS = \nabla . \bar{a} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z} \right)\left(a_1\bar{i} + a_2\bar{j} + a_3\bar{k} \right)$$

Some Useful Results

$$= \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}$$

$$= 0 + 0 + 0 = 0$$

$\boxed{3)\ \nabla \times \bar{a} = 0}$

$$LHS = \nabla \times \bar{a} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right] - j\left[\frac{\partial a_3}{\partial x} - \frac{\partial a_1}{\partial z}\right] + k\left[\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y}\right] = 0$$

Some Useful Results

$$\boxed{4)\ \nabla\times\bar{r}=0}$$

$$LHS=\nabla\times\bar{r}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x & y & z\end{vmatrix}$$

$$=\bar{i}\left[\frac{\partial z}{\partial y}-\frac{\partial y}{\partial z}\right]-j\left[\frac{\partial z}{\partial x}-\frac{\partial x}{\partial z}\right]+k\left[\frac{\partial y}{\partial x}-\frac{\partial x}{\partial y}\right]$$

$$=0$$

$$\boxed{5)\ \nabla f(r)=\frac{f'(r)}{r}\bar{r}}$$

Vector Identities

Vector Identities:

1. Laplacian of ∅:

$$\boxed{div\left(grad\phi\right)=\nabla^2\phi}$$

$$\boxed{Grad\ \ \phi=\nabla\phi}$$

$$\boxed{Div\ \ \overline{F}=\nabla.\overline{F}}$$

$$\boxed{Curl\ \ \overline{F}=\nabla\times\overline{F}}$$

$$LHS=\nabla\cdot\left(\nabla\phi\right)=\left(\hat{i}\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}\frac{\partial\phi}{\partial x}+j\frac{\partial\phi}{\partial y}+k\frac{\partial\phi}{\partial z}\right)$$

$$=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}$$

$$=\nabla^2\phi\quad=RHS$$

is known as Laplacian of ∅. Here, the operator

$$\boxed{\nabla^2=\nabla\cdot\nabla=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}}$$

is known as Laplacian operator.

2. Curl (grad ∅):

$$\boxed{Curl\left(grad\,\phi\right)=0}$$

$$\boxed{Grad\ \ \phi=\nabla\phi}$$

$$\boxed{Div\ \ \overline{F}=\nabla.\overline{F}}$$

$$\boxed{Curl\ \ \overline{F}=\nabla\times\overline{F}}$$

$$LHS=Curl\left(grad\,\phi\right)=\nabla\times\left(\nabla\phi\right)$$

$$=\begin{vmatrix} \hat{i} & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z \end{vmatrix}$$

$$=\bar{i}\left(\frac{\partial^2\phi}{\partial y\partial z}-\frac{\partial^2\phi}{\partial z\partial y}\right)-\bar{j}\left(\frac{\partial^2\phi}{\partial x\partial z}-\frac{\partial^2\phi}{\partial z\partial x}\right)+\bar{k}\left(\frac{\partial^2\phi}{\partial x\partial y}-\frac{\partial^2\phi}{\partial y\partial x}\right)$$

$$=0\quad=RHS$$

Vector Identities

3. Curl (Curl $\overline{F}$):

$$\boxed{Curl\left(Curl\,\overline{F}\right) = Grad\left(Div\ \overline{F}\right) - \nabla^2\overline{F}}$$

$$LHS = Curl\left(Curl\,\overline{F}\right) \quad = \nabla\times\left(\nabla\times\overline{F}\right)$$

$$= \nabla\left(\nabla\cdot\overline{F}\right) - \left(\nabla\cdot\nabla\right)\overline{F}$$

$$\because\ \bar{a}\times\left(\bar{b}\times\bar{c}\right) = (\bar{a}\cdot\bar{c})\bar{b} - \left(\bar{b}\cdot\bar{a}\right)\bar{c}$$
$$I\times(II\times III) = (I.III)II - (II.I)III$$

Since, $\nabla\cdot\nabla = \nabla^2$

$$= \nabla\left(\nabla\cdot\overline{F}\right) - \nabla^2\overline{F}$$

$$= Grad\left(Div\ \overline{F}\right) - \nabla^2\overline{F} \ = RHS$$

Standard Formulae for Vector Identities

1) $\nabla . \bar{r} = 3$

2) $\nabla . \bar{a} = 0$

3) $\nabla \times \bar{a} = 0$

4) $\nabla \times \bar{r} = 0$

5) $\nabla f(r) = \frac{f'(r)}{r} \bar{r}$

6) $Curl(grad\phi) = \nabla \times (\nabla \phi) = 0$

7) $div(grad\phi) = \nabla \cdot (\nabla \phi) = \nabla^2 \phi$

8) $I \times (II \times III) = (I.III)II - (I.II)III$

9) $c(\bar{u} \times \bar{v}) = (c\bar{u}) \times \bar{v} = \bar{u} \times (c\bar{v})$

10) $c(\bar{u}.\bar{v}) = (c\bar{u}).\bar{v} = \bar{u}.(c\bar{v})$

11) $Curl(Curl\bar{F}) = Grad(Div\ \bar{F}) - \nabla^2 \bar{F}$
$= \nabla(\nabla \cdot \bar{F}) - (\nabla \cdot \nabla)\bar{F}$

12) $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$

Standard Formulae for Vector Identities

For any scalars u and v

1. $\nabla(u \pm v) = \nabla u \pm \nabla v$
2. $\nabla(uv) = u\nabla v + v\nabla u$
3. $\nabla\left(\frac{u}{v}\right) = \frac{v\nabla u - u\nabla v}{v^2}$
4. $\nabla(au) = a\nabla u$ where a is constant.
5. $\nabla f(u) = f'(u)\nabla u$

Standard Formulae for Vector Identities

Let $\emptyset$ be a scalar function and $\bar{u}, \bar{v}$ be the vector functions.

1. $\nabla \times (\emptyset \bar{u}) = \nabla\emptyset \times \bar{u} + \emptyset(\nabla \times \bar{u})$

2. $\nabla . (\emptyset \bar{u}) = \nabla\emptyset . \bar{u} + \emptyset(\nabla . \bar{u})$

3. $\nabla \cdot (\bar{u} \times \bar{v}) = \bar{v} \cdot (\nabla \times \bar{u}) - \bar{u} \cdot (\nabla \times \bar{v})$

4. $\nabla \times (\bar{u} \times \bar{v}) = \bar{u}(\nabla \cdot \bar{v}) - (\bar{u} \cdot \nabla)\bar{v} + (\bar{v} \cdot \nabla)\bar{u} - \bar{v}(\nabla \cdot \bar{u})$

$$\boxed{8)\ I \times (II \times III) = (I.III)\,II - (I.II)\,III}$$

5. $\nabla(\bar{u} \cdot \bar{v}) = \bar{u} \times (\nabla \times \bar{v}) + (\bar{u} \cdot \nabla)\bar{v} + \bar{v} \times (\nabla \times \bar{u}) + (\bar{v} \cdot \nabla)\bar{u}$

MCQs on Vector Identities

1. $\nabla f(r)$ is equal to ......

(A) $\frac{f(r)}{r}\bar{r}$ (B) $\frac{f'(r)}{r}\bar{r}$ ✓ (C) $\frac{r}{f'(r)}\bar{r}$ (D) $f'(r)\,\bar{r}$

2. For a constant vector $\bar{a}$, $\nabla(\bar{a}\cdot\bar{r})$ is equal to ......

(A) $\bar{a}$ ✓ (B) $3\bar{a}$ (C) $\bar{r}$ (D) 0

$\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$, $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ $\therefore$ $\bar{a}\cdot\bar{r} = a_1x + a_2y + a_3z$

$\therefore \nabla(\bar{a}\cdot\bar{r}) = \nabla(a_1x + a_2y + a_3z) = a_1\bar{i} + a_2\bar{j} + a_3\bar{k} = \bar{a}$

3. For constant vectors $\bar{a}$ and $\bar{b}$, $\nabla(\bar{a}\cdot\bar{b})$ is equal to ......

(A) $\bar{a}\cdot\bar{b}$ (B) $\bar{a}$ (C) $\bar{b}$ (D) 0 ✓

$\bar{a}\cdot\bar{b} = a_1b_1 + a_2b_2 + a_3b_3$ $\therefore \nabla(\bar{a}\cdot\bar{b}) = \bar{i}(0) + \bar{j}(0) + \bar{k}(0) = 0$

MCQs on Vector Identities

6. For a constant vector $\bar{a}$, $(\bar{a} \cdot \nabla)\,\bar{r}$ is equal to ......

(A) $\bar{a}$ ✓ (B) $\bar{a} \cdot \bar{r}$ (C) $\bar{a} \cdot \frac{1}{r}\bar{r}$ (D) 3

$$\bar{a} \cdot \nabla = a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} + a_3 \frac{\partial}{\partial z} \quad \therefore (\bar{a} \cdot \nabla)\bar{r} = \left(a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} + a_3 \frac{\partial}{\partial z}\right) x\bar{i}$$

$$+ \left(a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} + a_3 \frac{\partial}{\partial z}\right) y\bar{j} + \left(a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} + a_3 \frac{\partial}{\partial z}\right) z\bar{k} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k} = \bar{a}$$

10. For the scalar function ϕ, div (grad ϕ) is equal to ......

(A) 1 (B) $\frac{\partial \phi}{\partial x}\bar{i} + \frac{\partial \phi}{\partial y}\bar{j} + \frac{\partial \phi}{\partial z}\bar{k}$

(C) $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$ ✓ (D) 0

$$\operatorname{div}(\operatorname{grad} \phi) = \nabla \cdot (\nabla \phi) = \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

MCQs on Vector Identities

11. For the scalar function ϕ, curl (grad ϕ) is equal to ......

(A) $\frac{\partial^2\phi}{\partial x^2}\bar{i} + \frac{\partial^2\phi}{\partial y^2}\bar{j} + \frac{\partial^2\phi}{\partial z^2}\bar{k}$

(B) $\frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$

(C) $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z}$

(D) $\bar{0}$ ✓

$$\text{curl}(\text{grad}\,\phi) = \nabla \times (\nabla\phi) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z \end{vmatrix} = \bar{i}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right)$$

$$-\bar{j}\left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right) + \bar{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right) = \bar{0}$$

12. For vector function $\bar{u}$, div (curl $\bar{u}$) is equal to ......

(A) $(\nabla\cdot\bar{u}) - \nabla^2\bar{u}$

(B) 0 ✓

(C) $\nabla(\nabla\cdot\bar{u}) - \nabla^2\bar{u}$

(D) $\nabla(\nabla\cdot\bar{u}) + \nabla^2\bar{u}$

$$\text{div}(\text{curl}\,\bar{u}) = \nabla\cdot(\nabla\times\bar{u}) = \frac{\partial}{\partial x}\left[\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right] - \frac{\partial}{\partial y}\left[\frac{\partial u_3}{\partial x} - \frac{\partial u_1}{\partial z}\right] + \frac{\partial}{\partial z}\left[\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right] = 0$$

$$\nabla\times\bar{u} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u_1 & u_2 & u_3 \end{vmatrix} = \bar{i}\left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right) - \bar{j}\left(\frac{\partial u_3}{\partial x} - \frac{\partial u_1}{\partial z}\right) + \bar{k}\left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right)$$

MCQs on Vector Identities

13. For vector function $\bar{u}$, curl (curl $\bar{u}$) is equal to ......

(A) $\nabla(\nabla\cdot\bar{u}) - \nabla^2\bar{u}$ ✓

(B) $\nabla(\nabla\cdot\bar{u}) + \nabla^2\bar{u}$

(C) $\nabla(\nabla\times\bar{u}) - \nabla\cdot\bar{u}$

(D) $\nabla\cdot(\nabla\times\bar{u}) + \nabla^2\bar{u}$

$$\text{curl}(\text{curl } \bar{u}) = \nabla(\nabla\cdot\bar{u}) - (\nabla\cdot\nabla)\bar{u}$$
$$= \nabla(\nabla\cdot\bar{u}) - \nabla^2\bar{u}$$

$$11)\ Curl\left(Curl\,\overline{F}\right) = Grad\left(Div\ \overline{F}\right) - \nabla^2\overline{F}$$
$$= \nabla\left(\nabla\cdot\overline{F}\right) - \left(\nabla\cdot\nabla\right)\overline{F}$$

14. $\nabla^2 f(r)$ is equal to ......

(A) $\frac{f'(r)}{r} - \frac{}{r}$

(B) $\frac{d^2f}{dr^2} + \frac{df}{dr}$

(C) $\frac{d^2f}{dr^2} - \frac{2}{r}\frac{df}{dr}$

(D) $\frac{d^2f}{dr^2} + \frac{2}{r}\frac{df}{dr}$ ✓

Standard formula

$$12)\ \nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$$

15. If $\bar{F}$ is irrotational vector field then there exists scalar potential ϕ such that ......

(A) $\bar{F} = \nabla^2\phi$

(B) $\bar{F} = \nabla\phi$ ✓

(C) $\phi = \nabla\cdot\bar{F}$

(D) $\nabla\times\bar{F} = \nabla\phi$

Standard result

MCQs on Vector Identities

16. ∇e^r is equal to ......

(A) $e^r \bar{r}$ (B) $\frac{e^r}{r}$ (C) $\frac{e^r}{r}\bar{r}$ ✓ (D) $\frac{r}{e^r}\bar{r}$

We know that, $\nabla f(r) = f'(r)\frac{\bar{r}}{r} \Rightarrow \nabla e^r = e^r \frac{\bar{r}}{r}$

17. $\nabla \log r$ is equal to ......

(A) $\frac{\log r}{r}\bar{r}$ (B) $\frac{1}{r^2}\bar{r}$ ✓ (C) $\bar{r}$ (D) $\frac{1}{r}\bar{r}$

$\nabla(\log r) = \frac{1}{r}\cdot\frac{\bar{r}}{r} = \frac{\bar{r}}{r^2}$

18. ∇r^n is equal to ......

(A) $n r^{n-1}$ (B) $\frac{r^{n+1}}{n+1}\bar{r}$ (C) $\frac{3r^{n-2}}{r}$ (D) $n r^{n-2}\bar{r}$ ✓

$\nabla(r^n) = n r^{n-1}\cdot\frac{\bar{r}}{r} = n r^{n-2}\cdot\bar{r}$

MCQs on Vector Identities

19. $\nabla (r^2 e^{-r})$ is given by ……

(A) $(2 - r)\,\bar{r}\,e^{-r}$ ✓

(B) $(2 + r^2)\,\bar{r}\,e^{-r}$

(C) $(2 - r)\,e^{-r}$

(D) $\bar{r}\,e^{-r}$

$$\nabla (r^2 e^{-r}) = \frac{d}{dr}(r^2 e^{-r}) \cdot \frac{\bar{r}}{r}$$

$$= \left[r^2(-e^{-r}) + e^{-r}(2r)\right] \frac{\bar{r}}{r}$$

$$= \left(-r e^{-r} + 2e^{-r}\right) \bar{r}$$

$$= (2 - r)\, e^{-r} \cdot \bar{r}$$

MCQs on Vector Identities

20. $\nabla(r^2 \log r)$ is equal to ......

(A) $(2 \log r + 1)\, r\, \bar{r}$ (B) $(2r + 1) \log r\, \bar{r}$ (C) $(2 \log r + 1)\, \bar{r}$ (D) $(2 \log r + 1)$

$$\nabla(r^2 \log r) = \frac{d}{dr}(r^2 \log r) \cdot \frac{\bar{r}}{r}$$

$$= \left[r^2 \cdot \frac{1}{r} + \log r\,(2r) \right] \frac{\bar{r}}{r}$$

$$= (1 + 2 \log r)\, \bar{r}$$

MCQs on Vector Identities

21. For constant vector $\bar{a}$, $\nabla\left(\frac{\bar{a}\cdot\bar{r}}{r^n}\right)$ is equal to ......

(A) $\frac{\bar{a}\cdot\bar{r}}{r^n} - \frac{1}{r^{n+2}}\bar{r}$ (B) $\frac{\bar{a}}{r^n} - \frac{n(\bar{a}\cdot\bar{r})}{r^{n+2}}\bar{r}$ (C) $\frac{\bar{a}}{r^n} + \frac{(\bar{a}\cdot\bar{r})}{r^{n+2}}\bar{r}$ (D) $\frac{\bar{a}}{r^n} - \frac{n(\bar{a}\cdot\bar{r})}{r^{n+1}}$

$$\nabla\left(\frac{1}{r^n}(\bar{a}\cdot\bar{r})\right) = \frac{1}{r^n}\nabla(\bar{a}\cdot\bar{r}) + (\bar{a}\cdot\bar{r})\nabla\left(\frac{1}{r^n}\right)$$

$$\because \nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$$

$$= \frac{1}{r^n}\nabla(a_1x + a_2y + a_3z) + (\bar{a}\cdot\bar{r})\left[-n r^{-n-1}\frac{\bar{r}}{r}\right]$$

$$= \frac{1}{r^n}\left[a_1\bar{i} + a_2\bar{j} + a_3\bar{k}\right] - \frac{n(\bar{a}\cdot\bar{r})}{r^{n+2}}\bar{r}$$

$$= \frac{\bar{a}}{r^n} - \frac{n(\bar{a}\cdot\bar{r})}{r^{n+2}}\bar{r}$$

MCQs on Vector Identities

22. $\nabla \cdot (r^n \bar{r})$ is equal to ......

(A) $(n+3)\, r^n$ ✓ (B) $3r^n + \frac{n}{r^{n-2}}$ (C) $(n-3)\, r^n$ (D) $(n+3)\, r^{-n}$

$$\nabla \cdot (\phi \bar{u}) = (\nabla \phi) \cdot \bar{u} + \phi (\nabla \cdot \bar{u})$$

$$\nabla (\underset{\phi}{\underset{\downarrow}{r^n}}\ \underset{\bar{u}}{\underset{\downarrow}{\bar{r}}}) = (\nabla r^n) \cdot \bar{r} + r^n (\nabla \cdot \bar{r})$$

$$= n r^{n-1} \frac{\bar{r}}{r} \cdot \bar{r} + 3 r^n$$

$$= n r^{n-1} \frac{r^2}{r} + 3 r^n \qquad (\because \bar{r} \cdot \bar{r} = r^2)$$

$$= (n+3)\, r^n$$

MCQs on Vector Identities

23. For constant vector $\bar{a}$, $\nabla \cdot [(\bar{a} \cdot \bar{r})\, \bar{a}]$ is equal to ......

(A) $\bar{a} \cdot \bar{r}$ (B) 0 ✓(C) $\bar{a} \cdot \bar{a}$ (D) $|\bar{a}|$

$$\nabla \cdot (\underbrace{(\bar{a} \cdot \bar{r})}_{\phi}\, \underbrace{\bar{a}}_{\bar{u}}) = \nabla(\bar{a} \cdot \bar{r}) \cdot \bar{a} + (\bar{a} \cdot \bar{r})(\nabla \cdot \bar{a})$$

$$= \bar{a} \cdot \bar{a} + 0$$

$$= \bar{a} \cdot \bar{a}$$

$$\nabla(\phi \bar{u}) = \nabla\phi \cdot \bar{u} + \phi(\nabla \cdot \bar{u})$$

24. $\nabla \cdot [(\log r)\, \bar{r}]$ is equal to ......

(A) $3 \log r + \frac{1}{r}$ (B) $3 \log r + \frac{1}{r^2}\bar{r}$ (C) $5 + 6 \log r$ ✓(D) $1 + 3 \log r$

$$\nabla \cdot (\underbrace{(\log r)}_{\phi}\, \underbrace{\bar{r}}_{\bar{u}}) = \nabla(\log r) \cdot \bar{r} + (\log r)(\nabla \cdot \bar{r})$$

$$= \frac{1}{r} \cdot \frac{\bar{r}}{r} \cdot \bar{r} + 3 \log r$$

$$= 1 + 3 \log r \qquad (\because \bar{r} \cdot \bar{r} = r^2)$$

MCQs on Vector Identities

25. $\nabla \cdot \left[r\nabla\left(\frac{1}{r^3}\right)\right]$ is equal to ......

(A) $\frac{3}{r^4}$ ✓
(B) $\frac{3}{r^2}$
(C) $\frac{1}{r^4}$
(D) $3r^4$

$$\nabla \cdot \left(r\, \nabla (r^{-3})\right) = \nabla \cdot \left(r\,(-3r^{-4})\,\frac{\bar{r}}{r}\right)$$

$$= \nabla \cdot \left(\underbrace{\left(\frac{-3}{r^4}\right)}_{\phi}\ \underbrace{\bar{r}}_{\bar{u}}\right)$$

$$= \nabla(-3r^{-4}) \cdot \bar{r} + (-3r^{-4})(\nabla \cdot \bar{r})$$

$$= (12r^{-5})\,\frac{\bar{r}}{r} \cdot \bar{r} - 9r^{-4}$$

$$= 3r^{-4}$$

MCQs on Vector Identities

26. If $\nabla^2\phi = 0$ and $\nabla^2\psi = 0$ then $\nabla \cdot [\phi\nabla\psi - \psi\nabla\phi]$ is equal to ......

(A) 0 (B) $2\nabla\phi \cdot \nabla\psi$ (C) $\nabla\phi + \nabla\psi$ (D) $[\phi\nabla\psi - \psi\nabla\phi]$

$$\nabla\cdot(\phi\nabla\psi - \psi\nabla\phi) = \nabla\cdot(\phi\nabla\psi) - \nabla\cdot(\psi\nabla\phi)$$

$$= \phi\nabla^2\psi - \psi\nabla^2\phi$$

$$= 0$$

30. $\nabla \times \left(\frac{\bar{a} \times \bar{r}}{r^n}\right) = \ldots\ldots$

(A) $\frac{2+n}{r^n}\bar{a} + \frac{1}{r^{n+2}}(\bar{a}\cdot\bar{r})\bar{r}$

(B) $\frac{2-n}{r^n} + \frac{n}{r^n}(\bar{a}\cdot\bar{r})\bar{r}$

(C) $\frac{2-n}{r^n}\bar{a} + \frac{n}{r^{n+2}}(\bar{a}\cdot\bar{r})\bar{r}$

(D) $\frac{2-n}{r^n}\bar{a} + \frac{n}{r^{-n-2}}(\bar{a}\cdot\bar{r})$

Lengthy question

MCQs on Vector Identities

27. $\nabla\left[\bar{b}\cdot\nabla\left(\frac{1}{r}\right)\right] = \ldots\ldots$

(A) $\frac{\bar{b}}{r^3}-\frac{3}{r^4}(\bar{b}\cdot\bar{r})\,\bar{r}$

(B) $-\frac{\bar{b}}{r^3}+\frac{3}{r^5}\bar{r}$

(C) $\frac{\bar{b}}{r^3}-\frac{3}{r^5}(\bar{b}\cdot\bar{r})$

✓(D) $-\frac{\bar{b}}{r^3}\cancel{+}-\frac{3}{r^5}(\bar{b}\cdot\bar{r})\,\bar{r}$

$$\nabla\cdot\left[\bar{b}\cdot\nabla\left(\frac{1}{r}\right)\right] = \nabla\cdot\left[\bar{b}\cdot\left(-\frac{1}{r^2}\right)\frac{\bar{r}}{r}\right]$$

$$= \nabla\left[\underbrace{\left(-\frac{1}{r^3}\right)}_{\phi}\underbrace{(\bar{b}\cdot\bar{r})}_{\psi}\right]$$

$$= \left(-\frac{1}{r^3}\right)\nabla(\bar{b}\cdot\bar{r}) + (\bar{b}\cdot\bar{r})\,\nabla\left(-\frac{1}{r^3}\right)$$

$$= -\frac{\bar{b}}{r^3} + (\bar{b}\cdot\bar{r})\left(-3r^{-4}\frac{\bar{r}}{r}\right)$$

$$= -\frac{\bar{b}}{r^3} - \frac{3}{r^5}(\bar{b}\cdot\bar{r})\,\bar{r}$$

MCQs on Vector Identities

28. $\nabla\,[\bar{a}\cdot\nabla \log r] = \ldots\ldots$

(A) $\frac{\bar{a}}{r^2}+\frac{2}{r^4}\bar{r}$ (B) $\frac{\bar{a}}{r}+\frac{1}{r^3}(\bar{a}\cdot\bar{r})\,\bar{r}$ ✓(C) $\frac{\bar{a}}{r^2}-\frac{2}{r^4}(\bar{a}\cdot\bar{r})\,\bar{r}$ (D) $\frac{\bar{a}}{r^2}-\frac{2}{r^3}(\bar{a}\cdot\bar{r})$

$$\nabla[\bar{a}\cdot\nabla \log r] = \nabla\left[\bar{a}\cdot\left(\frac{1}{r}\cdot\frac{\bar{r}}{r}\right)\right]$$

$$= \nabla\left[\underbrace{\left(\frac{1}{r^2}\right)}_{\phi}\underbrace{(\bar{a}\cdot\bar{r})}_{\psi}\right]$$

$$= \frac{1}{r^2}\nabla(\bar{a}\cdot\bar{r}) + (\bar{a}\cdot\bar{r})\,\nabla\left(\frac{1}{r^2}\right)$$

$$= \frac{\bar{a}}{r^2} + (\bar{a}\cdot\bar{r})\left(-2r^{-3}\frac{\bar{r}}{r}\right)$$

$$= \frac{\bar{a}}{r^2} - \frac{2}{r^4}(\bar{a}\cdot\bar{r})\,\bar{r}$$

MCQs on Vector Identities

29. $\nabla \times \left(\frac{\bar{r}}{r^3}\right)$ is equal to ......

(A) $\frac{3}{r^2}$

(B) $\bar{0}$

(C) $-\frac{2}{r^2}$

(D) $\frac{1}{r^2}\bar{r}$

$$\nabla \times \left(\frac{\bar{r}}{r^3}\right) = \frac{1}{r^3}\left(\nabla \times \bar{r}\right)$$

$$= \bar{0}$$

MCQs on Vector Identities

31. $\nabla \times \left((\bar{a} \cdot \bar{r}) \dfrac{\bar{r}}{r} \right) = \ldots\ldots$

(A) $\bar{a} \times \dfrac{\bar{r}}{r}$ (B) $\dfrac{\bar{r}}{r} \times \bar{a}$ (C) $\bar{a} \times \bar{r}$ (D) $\dfrac{\bar{r}}{r} + \dfrac{1}{r^2}(\bar{a} \cdot \bar{r})$

$$\nabla \times \left(\underbrace{(\bar{a} \cdot \bar{r})}_{\phi} \underbrace{\frac{\bar{r}}{r}}_{\bar{u}} \right) = \nabla(\bar{a} \cdot \bar{r}) \times \frac{\bar{r}}{r} + (\bar{a} \cdot \bar{r}) \left(\nabla \times \frac{\bar{r}}{r} \right)$$

$$= \bar{a} \times \frac{\bar{r}}{r} + \frac{(\bar{a} \cdot \bar{r})}{r} (\nabla \times \bar{r})$$

$$= \bar{a} \times \frac{\bar{r}}{r} + 0$$

MCQs on Vector Identities

32. Given $\bar{v} = 2y^2z\,\bar{i} + (3xy - yz^4)\,\bar{j} + 2x^3z\,\bar{k}$, the value of $\nabla(\nabla \cdot \bar{v})$ at (1, 1, 2) is ......

(A) $7\bar{i} + 8\bar{j} - 32\bar{k}$ (B) $2\bar{i} + 3\bar{j} + 2\bar{k}$ (C) $9\bar{i} + 32\bar{k}$ ✓(D) $9\bar{i} - 32\bar{k}$

$$\nabla \cdot \bar{v} = \frac{\partial}{\partial x}(2y^2 z) + \frac{\partial}{\partial y}(3xy - yz^4) + \frac{\partial}{\partial z}(2x^3 z)$$

$$= 0 + (3x - z^4) + 2x^3$$

$$\nabla(\nabla \cdot \bar{v}) = \nabla(3x - z^4 + 2x^3)$$

$$= \bar{i}(3 + 6x^2) + \bar{j}(0) + \bar{k}(-3z^3)$$

At (1, 1, 2)

$$= 9\bar{i} - 32\bar{k}$$

MCQs on Vector Identities

33. $\nabla^2\left(\frac{1}{r^2}\right)$ is equal to ……

(A) $\frac{1}{r^3}$

(B) $\frac{2}{r^4}$

(C) $-\frac{2}{r^4}\bar{r}$

(D) $\frac{6}{r^4}$

34. $\nabla^2 e^r$ is equal to ……

(A) $e^r + \frac{2}{r}e^r$

(B) $e^r + \frac{1}{r}e^r$

(C) $\frac{e^r}{r}\bar{r}$

(D) $e^r - \frac{2}{r}e^r$

MCQs on Vector Identities

35. $\nabla^2 (r^2 \log r)$ is equal to ......

(A) $\frac{(1+\log r)}{r} - \frac{}{r}$ (B) $(3 + 2\log r)$ (C) $(5 + 6\log r)$ (D) $(5 + 6\log r)\, r$

$$\nabla^2 (r^2 \log r) = \frac{d^2}{dr^2}(r^2 \log r) + \frac{2}{r}\frac{d}{dr}(r^2 \log r) = (3 + 2\log r) + \frac{2}{r}(r + 2r\log r)$$

$$= 5 + 6\log r$$

where $\frac{d}{dr}(r^2 \log r) = \frac{r^2}{r} + \log r\,(2r) = r + 2r\log r$

$$\frac{d^2}{dr^2}(r^2 \log r) = 1 + 2r \cdot \frac{1}{r} + \log r\,(2)$$

$$= 1 + 2 + 2\log r = 3 + 2\log r$$

MCQs on Vector Identities

36. $\nabla^2\left(\frac{\bar{a}\cdot\bar{b}}{r}\right)$ is equal to ......

(A) $-(\bar{a}\cdot\bar{b})\frac{1}{r^2}\bar{r}$ (B) $\frac{4}{r^3}(\bar{a}\cdot\bar{b})$ (C) $(\bar{a}\cdot\bar{b})\left(\frac{2}{r^3}-\frac{1}{r^2}\right)$ (D) 0 ✓

$$\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$$

$$f(r) = \frac{\bar{a}\cdot\bar{b}}{r} \Rightarrow f'(r) = -\frac{\bar{a}\cdot\bar{b}}{r^2} \Rightarrow f''(r) = \frac{2(\bar{a}\cdot\bar{b})}{r^3}$$

$$\nabla^2\left(\frac{\bar{a}\cdot\bar{b}}{r}\right) = \frac{2(\bar{a}\cdot\bar{b})}{r^3} + \frac{2}{r}\left(\frac{-\bar{a}\cdot\bar{b}}{r^2}\right)$$

$$= \frac{2(\bar{a}\cdot\bar{b})}{r^3} - \frac{2(\bar{a}\cdot\bar{b})}{r^3}$$

$$= 0$$

MCQs on Vector Identities

37. If $\nabla^2 (r^2 \log r) = 5 + 6 \log r$ then $\nabla^4 (r^2 \log r) = \ldots\ldots$

(A) $\frac{18}{r^2}$ (B) $\frac{6}{r^2}$ ✓ (C) $-\frac{6}{r^2}$ (D) $-\frac{6}{r^2} + \frac{6}{r}$

$$\nabla^4 (r^2 \log r) = \nabla^2 \left(\nabla^2 (r^2 \log r) \right)$$

$$= \nabla^2 (\underbrace{5 + 6 \log r}_{f(r)})$$

$$= f''(r) + \frac{2}{r} f'(r)$$

$$= -\frac{6}{r^2} + \frac{2}{r}\left(\frac{6}{r}\right)$$

$$= -\frac{6}{r^2} + \frac{12}{r^2}$$

$$= \frac{6}{r^2}$$

$$f(r) = 5 + 6 \log r$$

$$f'(r) = \frac{6}{r}$$

$$f''(r) = -\frac{6}{r^2}$$

MCQs on Vector Identities

38. If $\phi = 2xz + 2yz + z^2$ then $\nabla^2\phi$ is ......

(A) $2(x + y + z)$ (B) 2 (C) 0 (D) $6z$

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0 + 0 + 2 = 2$$

$\varphi_x = 2z \quad \varphi_y = 2z \quad \varphi_z = 2x + 2y + 2z$

$\varphi_{xx} = 0 \quad \varphi_{yy} = 0 \quad \varphi_{zz} = 2$

39. For constant vector $\bar{a}$, $\nabla \times (\bar{a} \times \bar{r}) =$

(A) $3\bar{a}$ (B) $\bar{a}$ (C) 0 (D) $2\bar{a}$

$$\bar{a} \times \bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= \bar{i}(a_2 z - a_3 y) - \bar{j}(a_1 z - a_3 x) + \bar{k}(a_1 y - a_2 x)$$

$$\nabla \times (\bar{a} \times \bar{r}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_2 z - a_3 y & a_3 x - a_1 z & a_1 y - a_2 x \end{vmatrix}$$

$$= \bar{i}(2a_1) + \bar{j}(2a_2) + \bar{k}(2a_3) = 2\bar{a}$$

MCQs on Vector Identities

40. div (grad r^3) = $\nabla \cdot (\nabla r^3)$ = ...

(A) 12r (B) 8r (C) 2r (D) 4r

$$\nabla \cdot (\nabla r^3) = \nabla \cdot \left(3r^2 \frac{\bar{r}}{r}\right) = \nabla \cdot (\underset{\phi}{3r} \cdot \underset{\bar{u}}{\bar{r}})$$

$$= \nabla(3r) \cdot \bar{r} + (3r)(\nabla \cdot \bar{r})$$

$$= 3\frac{\bar{r}}{r} \cdot \bar{r} + 9r = \frac{3r^2}{r} + 9r$$

$$= 12r$$

41. If $\phi = 2x^2 - 3y^2 + 4z^2$ then curl (grad ϕ) is ......

(A) 3 (B) $4x\,\bar{i} - 6y\,\bar{j} + 8z\,\bar{k}$ (C) 0 (D) $4x - 6y + 2z$

$$\boxed{6)\ Curl\left(grad\,\phi\right) = \nabla \times \left(\nabla \phi\right) = 0}$$

MCQs on Vector Identities

42. If $\bar{F}$ is a solenoidal vector field then curl curl $\bar{F}$ is ......

(A) $\nabla^2\bar{F}$ (B) $-\nabla^2\bar{F}$ ✓ (C) $\nabla^4\bar{F}$ (D) $\nabla(\nabla\cdot\bar{F})$

$$\text{curl curl } \bar{F} = \nabla\times(\nabla\times\bar{F}) = \nabla(\nabla\cdot\bar{F}) - \nabla^2\bar{F}$$
$$= -\nabla^2\bar{F}$$

$$11)\, Curl\left(Curl\,\overline{F}\right) = Grad\left(Div\ \overline{F}\right) - \nabla^2\overline{F}$$
$$= \nabla\left(\nabla\cdot\overline{F}\right) - \left(\nabla\cdot\nabla\right)\overline{F}$$

43. If $\bar{F}$ is a solenoidal vector field and curl curl $\bar{F} = -\nabla^2\bar{F}$ then curl curl curl curl $\bar{F}$ is ......

(A) $\nabla^2\bar{F}$ (B) $\nabla^4\bar{F}$ ✓ (C) $-\nabla^4\bar{F}$ (D) $\bar{0}$

$$\text{curl curl curl curl } \bar{F} = \text{curl curl}(-\nabla^2\bar{F})$$
$$= \nabla\times(\nabla\times(\underbrace{-\nabla^2\bar{F}}_{\bar{w}}))$$
$$= \nabla\times(\nabla\times\bar{w}) = -\nabla^2\bar{w} = -\nabla^2(-\nabla^2\bar{F})$$
$$= \nabla^4\bar{F}$$

MCQs on Vector Identities

44. For the vector field $\bar{F} = (6xy + z^3)\,\bar{i} + (3x^2 - z)\,\bar{j} + (3xz^2 - y)\,\bar{k}$, $\nabla \times \bar{F}$ is ......

(A) $6y\,\bar{i} + 6xz\,\bar{k}$ (B) $-2\bar{i} + 6z^2\,\bar{j} + 12x\,\bar{k}$ (C) $\bar{0}$ (D) $6y + 6xz$

45. For the vector field $\bar{F} = (2xz^3 + 6y)\,\bar{i} + (6x - 2yz)\,\bar{j} + (3x^2z^2 - y^2)\,\bar{k}$, $\nabla \times \bar{F}$ is ......

(A) $2z^3\bar{i} - 2z\bar{j} + 6xz^2\,\bar{k}$ (b) $4y\,\bar{i} - 12xz^2\,\bar{j} + 12\bar{k}$ (C) $2z^3 - 2z + 6xz^2$ (D) $\bar{0}$

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xz^3+6y & 6x-2yz & 3x^2z^2-y^2 \end{vmatrix} = \bar{i}\left(-2y-(-2y)\right) - \bar{j}\left(6xz^2 - 6xz^2\right) + \bar{k}\,(6-6) = \bar{0}$$

MCQs on Vector Identities

46. If for the vector field $\bar{u}$ and $\bar{v}$ are irrotational vectors then the value of $\nabla \cdot (\bar{u} \times \bar{v})$ is ......

(A) 2 (B) 1 (C) 3 (D) 0 ✓

$$\nabla \cdot (\bar{u} \times \bar{v}) = \bar{v} \cdot (\nabla \times \bar{u}) - \bar{u} \cdot (\nabla \times \bar{v}) = 0$$

$$\nabla \times \bar{u} = 0$$

$$\nabla \times \bar{v} = 0$$

47. The vector field $\bar{F} = (6xy + z^3)\,\bar{i} + (3x^2 - z)\,\bar{j} + (3xz^2 - y)\,\bar{k}$ is irrotational. Corresponding scalar function ϕ satisfying $\bar{F} = \nabla\phi$ is ......

(A) $3x^2y + z^3x - yz + c$

(B) $3x^2y + z^2x + c$

(C) $6x^2y + x^3 + xy - yz + c$

(D) $x^2y + z^3x - y^3 + c$

MCQs on Vector Identities

48. For irrotational vector field $\bar{F} = (x + 2y + 4z)\,\bar{i} + (2x - 3y - z)\,\bar{j} + (4x - y + 2z)\,\bar{k}$, scalar function ϕ such that $\bar{F} = \nabla\phi$ is ......

(A) $\frac{x^2}{2} + 2xy + 4xz - \frac{3}{2}y^2 - yz + z^2 + c$ ✓

(B) $x^2 + xy + xz - y^2 - yz + z^2 + c$

(C) $\frac{x^2}{2} + 2xy + 4xz - \frac{1}{2}y^2 - yz + c$

(D) $\frac{x^2}{2} + y^2 + 4xz - yz + 2z^2 + c$

$$\phi = \int (x + 2y + 4z)\,dx + \int (-3y - z)\,dy + \int 2z\,dz + c$$

$$= \frac{x^2}{2} + 2xy + 4xz - \frac{3}{2}y^2 - zy + z^2 + c$$

49. For irrotational vector field $\bar{F} = (2xz^3 + 6y)\,\bar{i} + (6x - 2yz)\,\bar{j} + (3x^2z^2 - y^2)\,\bar{k}$, scalar function ϕ such that $\bar{F} = \nabla\phi$ is ......

(A) $x^2z^3 + 3y^2 + 3x^2 - \frac{y^3}{3} + c$

(B) $x^2z^3 + 6xy + 3x^2 - 2y^2z + x^2z^3 + c$

(C) $xz^3 + 6xy + y^2z + \frac{y^3}{3} + c$

(D) $x^2z^3 + 6xy - y^2z + c$ ✓

$$\phi = \int (2xz^3 + 6y)\,dx + \int -2yz\,dy + \int 0\,dz + c$$

$$= x^2z^3 + 6xy - y^2z + c$$

MCQs on Vector Identities

50. For irrotational vector field $\bar{F} = (y^2 \cos x + z^2)\,\bar{i} + (2y \sin x - 4)\,\bar{j} + (2xz + 2)\,\bar{k}$, scalar function ϕ such that $\bar{F} = \nabla\phi$ is ......

(A) $-y^2 \sin x + z^2x + y^2 \sin x + xz^2 + c$

(B) $y^2 \sin x + z^2x - 4y + 2z + c$ ✓

(C) $y^2 \cos x + z^2x + y^2 \sin x - 4y + xz^3 + c$

(D) $\frac{y^2}{3} \sin x + z^3y + 2y \cos x - 4x + c$

$$\phi = \int (y^2 \cos x + z^2)\,dx + \int -4\,dy + \int 2\,dz + c$$

$$= y^2 \sin x + xz^2 - 4y + 2z + c$$

51. If $\bar{F} = yz\,\bar{i} + zx\,\bar{j} + xy\,\bar{k}$ and $\bar{F} = \nabla\phi$, then ϕ is given by ......

(A) $x + y + z + c$

(B) $x^2 + y^2 + z^2 + c$

(C) $xyz + c$ ✓

(D) $x^2 + y + z + c$

$$\phi = \int yz\,dx + \int 0\,dy + \int 0\,dz + c$$

$$= xyz + c$$

MCQs on Vector Identities

52. If $\nabla\phi = (y^2 + 2y + z)\,\bar{i} + (2xy + 2x)\,\bar{j} + x\,\bar{k}$ and $\phi(1, 1, 0) = 5$ then ϕ is ......

(A) $xy^2 + 4xy + 2zx + xy^2 - 5$
(B) $xy^2 + 2xy + zx + 2$ ✓
(C) $xy^2 + xy + zx + 2$
(D) $xy^2 + 2xy + 2zx + y^2 - 2$

$$\phi = \int (y^2 + 2y + z)\,dx + \int 0\,dy + \int 0\,dz + c$$

$$\phi(x, y, z) = xy^2 + 2xy + xz + c \Rightarrow \phi = xy^2 + 2xy + xz + 2$$

$$5 = \phi(1, 1, 0) = 1 + 2 + c \Rightarrow \boxed{c = 2}$$

53. If $\bar{F} = r^2\,\bar{r}$ is conservative, then scalar ϕ associated with it is given by ......

(A) $\frac{r^4}{4} + c$
(B) $\frac{r^2}{2} + c$
(C) $\frac{r^3}{3} + c$
(D) $r + c$

$$\bar{F} = \nabla\phi$$

$$r^2\,\bar{r} = \nabla\left(\frac{r^4}{4} + c\right) = \frac{\cancel{4}\,r^3}{\cancel{4}}\,\frac{\bar{r}}{r} = r^2\,\bar{r}$$

MCQs on Vector Identities

54. If $\nabla \cdot \{f(r)\,\bar{r}\} = 0$, then f(r) is given by (c is constant) ......

(A) $\frac{c}{r^2}$ (B) $\frac{c}{r}$ (C) $\frac{c}{r^4}$ (D) $\frac{c}{r^3}$

Examples on Vector Identities:

Ex. 1. Show that $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$.

Solution:

$$\nabla^2 f(r) = \nabla \cdot \nabla f(r) = \nabla \cdot \left\{\frac{f'(r)}{r}\bar{r}\right\}$$

$$= \frac{f'(r)}{r}(\nabla \cdot \bar{r}) + \left(\nabla \frac{f'(r)}{r}\right) \cdot \bar{r}$$

$\boldsymbol{\nabla}.(\boldsymbol{\emptyset\bar{u}}) = \nabla\emptyset.\bar{u} + \emptyset(\nabla.\bar{u})$

$$= \frac{f'(r)}{r}(3) + \left(\frac{rf''(r) - f'(r)}{r^2}\right)\frac{\bar{r}}{r} \cdot \bar{r}$$

$\nabla\left(\frac{u}{v}\right) = \frac{v\nabla u - u\nabla v}{v^2}$

$$= \frac{3f'(r)}{r} + \left(\frac{rf''(r) - f'(r)}{r^3}\right)\bar{r} \cdot \bar{r}$$

$\bar{r} \cdot \bar{r} = r^2$

$$= \frac{3f'(r)}{r} + \left(\frac{rf''(r) - f'(r)}{r^3}\right)r^2$$

$$= \frac{3f'(r)}{r} + f''(r) - \frac{f'(r)}{r}$$

$$\boxed{\therefore \nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)}$$

Vector Identities

Ex. 2. Show that $\nabla^2\left[\nabla\cdot\frac{\bar r}{r^2}\right]=\frac{2}{r^4}$.

Solution:

$$\nabla.\frac{\bar r}{r^2}=\nabla.\left(r^{-2}\bar r\right)=\left(\nabla r^{-2}\right).\,\bar r+r^{-2}\left(\nabla.\,\bar r\right)$$

$$\boxed{\because \nabla.\left(\phi\bar u\right)=\left(\nabla\phi\right).\bar u+\phi\left(\nabla.\bar u\right)}$$

$$=\frac{-2r^{-3}}{r}\bar r.\,\bar r+3r^{-2}$$

$$\boxed{\because \nabla f\left(r\right)=\frac{f'\left(r\right)}{r}\bar r}$$

$$=\frac{-2r^{-3}}{r}r^2+3r^{-2}$$

$$\boxed{\because \bar r.\bar r=r^2}$$

$$=-r^{-2}=-\frac{1}{r^2}$$

$$LHS=\nabla^2\left[\nabla.\frac{\bar r}{r^2}\right]=\nabla^2\left[\frac{-1}{r^2}\right]=-6r^{-4}+\frac{2}{r}\left(2r^{-3}\right)$$

$$\boxed{\because \nabla^2 f\left(r\right)=f''\left(r\right)+\frac{2}{r}f'\left(r\right)}$$

Vector Identities

$$= -6r^{-4} + \frac{2}{r}\left(2r^{-3}\right)$$

$$= -6r^{-4} + 4r^{-4}$$

$$= -2r^{-4}$$

$$LHS = \nabla^2\left[\nabla.\frac{\bar{r}}{r^2}\right] = \frac{2}{r^4} = RHS$$

Vector Identities

Ex. 3. Show that $\nabla^4 e^r = \left(1+\frac{4}{r}\right)e^r$.

Solution:

$$LHS = \nabla^4 e^r \quad = \nabla^2\left(\nabla^2 e^r\right)$$

$$where, \nabla^2 e^r = e^r + \frac{2}{r}e^r \qquad \boxed{\because \nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r)}$$

$$LHS = \nabla^4 e^r \quad = \nabla^2\left(e^r + \frac{2}{r}e^r\right)$$

$$= \left(e^r + \frac{2}{r}e^r - e^r\frac{2}{r^2} - \frac{2}{r^2}e^r + e^r\frac{4}{r^3}\right) + \frac{2}{r}\left(e^r + \frac{2}{r}e^r - e^r\frac{2}{r^2}\right) \qquad \boxed{\because \nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r)}$$

$$= e^r + \frac{2}{r}e^r - e^r\frac{2}{r^2} - \frac{2}{r^2}e^r + e^r\frac{4}{r^3} + \frac{2}{r}e^r + \frac{4}{r^2}e^r - e^r\frac{4}{r^3}$$

Vector Identities

$$= e^r + \frac{2}{r}e^r + \frac{2}{r}e^r$$

$$LHS = \nabla^4 e^r = \left(1 + \frac{4}{r}\right)e^r = RHS$$

Ex. 4. Prove that $\nabla^4\left(r^2 \log r\right) = \frac{6}{r^2}$

Solution :

$$LHS = \nabla^4\left(r^2\log r\right) \quad = \nabla^2\left[\nabla^2\left(r^2\log r\right)\right]$$

$$= \nabla^2\left[\left(1+2+2\log r\right)+\frac{2}{r}\left(r+2r\log r\right)\right]$$

$$\because \nabla^2 f(r) = f''(r)+\frac{2}{r}f'(r)$$

$$= \nabla^2\left[6\log r+5\right]$$

$$\because \nabla^2 f(r) = f''(r)+\frac{2}{r}f'(r)$$

$$= -\frac{6}{r^2}+\frac{2}{r}\left(\frac{6}{r}\right)$$

$$= \frac{6}{r^2} \quad = RHS$$

Vector Identities

Ex. 5. Show that $\bar{F} = \left(\frac{\bar{a}\times\bar{r}}{r^n}\right)$ is solenoidal.

Or Show that $\nabla\cdot\left(\frac{\bar{a}\times\bar{r}}{r^n}\right) = 0.$

Solution: Consider

$$\nabla\cdot\left(\frac{\bar{a}\times\bar{r}}{r^n}\right) = \nabla\cdot\left(\bar{a}\times\frac{\bar{r}}{r^n}\right) = \frac{\bar{r}}{r^n}\cdot(\nabla\times\bar{a}) - \bar{a}\cdot\nabla\times\left(\frac{\bar{r}}{r^n}\right)$$

$$\boxed{\ldots\left(\because \nabla\cdot(\bar{u}\times\bar{v}) = \bar{v}\cdot(\nabla\times\bar{u}) - \bar{u}\cdot(\nabla\times\bar{v})\right)}$$

But, $(\nabla\times\bar{a}) = 0$ and $\nabla\times\left(\frac{\bar{r}}{r^n}\right) = r^{-n}(\nabla\times\bar{r}) = 0$

$$\therefore \nabla\cdot\left(\frac{\bar{a}\times\bar{r}}{r^n}\right) = 0 - 0 = 0$$

$\therefore \bar{F}$ is solenoidal.

Ex. 6. Prove that $\nabla \cdot \left[r \nabla \left(\frac{1}{r^n} \right) \right] = \frac{n(n-2)}{r^{n+1}}$.

Solution: Consider

$$\nabla \cdot \left[r \nabla \left(\frac{1}{r^n} \right) \right] = \nabla \cdot \left[r \frac{(-n)}{r^{n+1}} \frac{\bar{r}}{r} \right] \qquad \boxed{\ldots \left(\because \nabla f(r) = f'(r) \frac{\bar{r}}{r} \right)}$$

$$= \nabla \cdot \left[\frac{(-n)\bar{r}}{r^{n+1}} \right]$$

$$= (-n) \nabla \cdot \left[\frac{1}{r^{n+1}} \bar{r} \right] \qquad \boxed{\because \nabla . \left(\phi \bar{u} \right) = \left(\nabla \phi \right) . \bar{u} + \phi \left(\nabla . \bar{u} \right)}$$

$$= (-n) \left\{ \nabla \frac{1}{r^{n+1}} \cdot \bar{r} + \frac{1}{r^{n+1}} \nabla \cdot \bar{r} \right\}$$

$$= (-n) \left\{ \frac{-(n+1)}{r^{n+2}} \frac{\bar{r}}{r} \cdot \bar{r} + \frac{3}{r^{n+1}} \right\}$$

$$= (-n) \left\{ \frac{-(n+1)}{r^{n+1}} + \frac{3}{r^{n+1}} \right\}$$

$$= (-n) \left\{ \frac{-n+2}{r^{n+1}} \right\} = \frac{n(n-2)}{r^{n+1}}.$$

Vector Identities

Ex. 7. Show that

i. $\nabla\left(\frac{\bar{a}\cdot\bar{r}}{r^n}\right)=\frac{\bar{a}}{r^n}-\frac{n(\bar{a}\cdot\bar{r})}{r^{n+2}}\bar{r}$

ii. $\nabla\times\left(\frac{\bar{a}\times\bar{r}}{r^n}\right)=\frac{(2-n)\bar{a}}{r^n}+\frac{n(\bar{a}\cdot\bar{r})}{r^{n+2}}\bar{r}$

Solution:

$$(i) LHS=\nabla\left(\frac{\bar{a}.\bar{r}}{r^n}\right)\quad=\nabla\left(\frac{1}{r^n}\left(\bar{a}.\bar{r}\right)\right)$$

$$=\frac{1}{r^n}\nabla\left(\bar{a}.\bar{r}\right)+\left(\bar{a}.\bar{r}\right)\nabla\left(\frac{1}{r^n}\right)\qquad \boxed{\because\nabla(uv)=u\nabla v+v\nabla u}$$

$$=\frac{1}{r^n}\nabla\left(a_1x+a_2y+a_3z\right)+\left(\bar{a}.\bar{r}\right)\nabla\left(\frac{1}{r^n}\right)$$

$$=\frac{1}{r^n}\bar{a}+\left(\bar{a}.\bar{r}\right)\frac{-nr^{-n-1}}{r}\bar{r}\qquad \boxed{\because\nabla f(r)=\frac{f'(r)}{r}\bar{r}}$$

Vector Identities

$$= \frac{1}{r^n}\bar{a} + \left(\bar{a}.\bar{r}\right)\frac{-nr^{-n-1}}{r}\bar{r}$$

ii. $\nabla \times \left(\frac{\bar{a}\times\bar{r}}{r^n}\right) = \frac{(2-n)\bar{a}}{r^n} + \frac{n(\bar{a}\cdot\bar{r})}{r^{n+2}}\bar{r}$

$$(i) LHS = \nabla\left(\frac{\bar{a}.\bar{r}}{r^n}\right) = \frac{1}{r^n}\bar{a} - \left(\bar{a}.\bar{r}\right)\frac{n}{r^{n+2}}\bar{r} = RHS$$

Vector Identities

$$\nabla\times\left(\frac{\bar{a}\times\bar{r}}{r^n}\right)=\frac{(2-n)\bar{a}}{r^n}+\frac{n(\bar{a}\cdot\bar{r})}{r^{n+2}}\bar{r}$$

$$(ii)LHS=\nabla\times\left(\frac{\bar{a}\times\bar{r}}{r^n}\right)$$

$$=\nabla\times\left(\bar{a}\times r^{-n}\bar{r}\right)$$

$$\boldsymbol{\nabla\times(\bar{u}\times\bar{v})}=\bar{u}(\nabla\cdot\bar{v})-(\bar{u}\cdot\nabla)\bar{v}+(\bar{v}\cdot\nabla)\bar{u}-\bar{v}(\nabla\cdot\bar{u})$$

$$=\bar{a}\left(\nabla.r^{-n}\bar{r}\right)-\left(\bar{a}.\nabla\right)r^{-n}\bar{r}+\left(r^{-n}\bar{r}.\nabla\right)\bar{a}-r^{-n}\bar{r}\left(\nabla.\bar{a}\right)$$

$$(1)\ \bar{a}\left(\nabla.r^{-n}\bar{r}\right)=\bar{a}\left(\nabla.\left(r^{-n}\bar{r}\right)\right)$$

$$=\bar{a}\left[\left(\nabla r^{-n}\right)\bar{r}+r^{-n}\left(\nabla.\bar{r}\right)\right]$$

$$\because\nabla.\left(\phi\bar{u}\right)=\left(\nabla\phi\right).\bar{u}+\phi\left(\nabla.\bar{u}\right)$$

$$=\bar{a}\left[\left(\frac{-nr^{-n-1}}{r}\bar{r}\right)\bar{r}+r^{-n}(3)\right]$$

Vector Identities

$$\nabla \times \left(\frac{\bar{a} \times \bar{r}}{r^n}\right) = \frac{(2-n)\bar{a}}{r^n} + \frac{n(\bar{a}\cdot\bar{r})}{r^{n+2}}\bar{r}$$

$$= \bar{a}\left(\nabla.r^{-n}\bar{r}\right) - \left(\bar{a}.\nabla\right)r^{-n}\bar{r} + \left(r^{-n}\bar{r}.\nabla\right)\bar{a} - r^{-n}\bar{r}\left(\nabla.\bar{a}\right)$$

(1) $\bar{a}\left(\nabla.r^{-n}\bar{r}\right) = \bar{a}\left[\left(\frac{-n}{r^{n+2}}\right)\bar{r}.\bar{r} + r^{-n}(3)\right]$

$$= \bar{a}\left[\left(\frac{-n}{r^{n+2}}\right)r^2 + 3r^{-n}\right]$$

$$= \bar{a}\left[-nr^{-n} + 3r^{-n}\right]$$

$$\boxed{= \bar{a}(3-n)r^{-n}}$$

(2) $\left(\bar{a}.\nabla\right)r^{-n}\bar{r} = \left(\bar{a}.\nabla r^{-n}\right)\bar{r}$

$$= \left(\bar{a}.\frac{-nr^{-n-1}}{r}\bar{r}\right)\bar{r}$$

$$\boxed{\because \nabla f(r) = \frac{f'(r)}{r}\bar{r}}$$

Vector Identities

$$\nabla \times \left(\frac{\bar{a} \times \bar{r}}{r^n}\right) = \frac{(2-n)\bar{a}}{r^n} + \frac{n(\bar{a}\cdot\bar{r})}{r^{n+2}}\bar{r}$$

$$= \bar{a}\left(\nabla . r^{-n}\bar{r}\right) - \left(\bar{a}.\nabla\right) r^{-n}\bar{r} + \left(r^{-n}\bar{r}.\nabla\right)\bar{a} - r^{-n}\bar{r}\left(\nabla . \bar{a}\right)$$

(2) $\left(\bar{a}.\nabla\right) r^{-n}\bar{r} = \left(\bar{a}.\frac{-nr^{-n-1}}{r}\bar{r}\right)\bar{r}$

$$\boxed{= \frac{-n}{r^{n+2}}\left(\bar{a}.\bar{r}\right)\bar{r}}$$

(3) $\left(r^{-n}\bar{r}.\nabla\right)\bar{a} = \left(\bar{r}.\nabla r^{-n}\right)\bar{a}$

$$= \left(\bar{r}.\frac{-nr^{-n-1}}{r}\bar{r}\right)\bar{a} \qquad \boxed{\because \nabla f(r) = \frac{f'(r)}{r}\bar{r}}$$

$$= \frac{-n}{r^{n+2}}\left(\bar{r}.\bar{r}\right)\bar{a}$$

$$= \frac{-n}{r^{n+2}} r^2 \bar{a} \qquad \boxed{= -nr^{-n}\bar{a}}$$

$$\nabla\times\left(\frac{\bar{a}\times\bar{r}}{r^n}\right)=\frac{(2-n)\bar{a}}{r^n}+\frac{n(\bar{a}\cdot\bar{r})}{r^{n+2}}\bar{r}$$

(4) $r^{-n}\bar{r}\left(\nabla.\bar{a}\right)\boxed{=0}$ $\quad\boxed{\because\left(\nabla.\bar{a}\right)=0}$

$$(ii)LHS=\nabla\times\left(\frac{\bar{a}\times\bar{r}}{r^n}\right)=\bar{a}\left(\nabla.r^{-n}\bar{r}\right)-\left(\bar{a}.\nabla\right)r^{-n}\bar{r}+\left(r^{-n}\bar{r}.\nabla\right)\bar{a}-r^{-n}\bar{r}\left(\nabla.\bar{a}\right)$$

$$=\bar{a}\left(3-n\right)r^{-n}-\frac{-n}{r^{n+2}}\left(\bar{a}.\bar{r}\right)\bar{r}\quad-nr^{-n}\bar{a}\quad-0$$

$$\boxed{=\left(2-n\right)r^{-n}\bar{a}+\frac{n}{r^{n+2}}\left(\bar{a}.\bar{r}\right)\bar{r}}$$

Ex. 8. Show that

i. $\nabla\left(\frac{\bar{a}\cdot\bar{r}}{r^3}\right)=\frac{\bar{a}}{r^3}-\frac{3(\bar{a}\cdot\bar{r})}{r^5}\bar{r}$

ii. $\nabla\times\left(\frac{\bar{a}\times\bar{r}}{r^3}\right)=-\frac{\bar{a}}{r^3}+\frac{3(\bar{a}\cdot\bar{r})}{r^5}\bar{r}$

Vector Identities

Ex. 6. For a solenoidal vector field $\bar{F}$, show that curlcurlcurlcurl$\bar{F} = \nabla^4 \bar{F}$.

Solution: From given, $\bar{F}$ being solenoidal, i.e. $\nabla \cdot \bar{F} = 0$

$$\begin{aligned} curl\ curl\ \bar{F} &= \nabla \times (\nabla \times \bar{F}) \\ &= \nabla(\nabla \cdot \bar{F}) - (\nabla \cdot \nabla)\bar{F} \\ &= -\nabla^2 \bar{F} \end{aligned}$$

$$\boxed{8)\ I \times (II \times III) = (I.III)II - (I.II)III}$$

Let $\bar{E} = -\nabla^2 \bar{F}$

$$\begin{aligned} curl\ curl\ \bar{E} &= \nabla \times (\nabla \times \bar{E}) \\ &= \nabla(\nabla \cdot \bar{E}) - (\nabla \cdot \nabla)\bar{E} \\ &= \nabla\left(\nabla \cdot (-\nabla^2 \bar{F})\right) - (\nabla^2)(-\nabla^2 \bar{F}) \\ &= \nabla\left(-\nabla^2(\nabla \cdot \bar{F})\right) + (\nabla^4 \bar{F}) \\ &= (\nabla^4 \bar{F}) \end{aligned}$$