CONTENTS

Unit I : Set Theory and Logic

Unit II : Relations and Functions

Unit IV : Graph Theory

Unit V : Trees

Unit VI : Algebraic Structures and Coding Theory

◈◈◈

CHAPTER 1
LOGIC AND PROOFS

1.1 INTRODUCTION

- Logic is central to the theme of Discrete Structures. A discrete structure is defined by a **set of axioms**. Properties of the structure are then derived from the axioms as theorems. These theorems are proved using valid rules of reasoning. The science of logic provides rules by which one can determine whether any particular proof (argument) is valid or not. These rules should be framed in a language which is precise and unambiguous. An ordinary language is unsuitable for this purpose, since languages are prone to be misinterpreted or misconstrued. For example consider the statement "He tries to study hard" and "He tries hard to study".
- A mere shuffling of the words, changes the entire meaning. To avoid such mistakes, a formal language is developed using symbols. A set of well defined rules is also built to perform. This is Symbolic Logic or Propositional Calculus. A 'calculus' is a set of rules for calculating with symbols. In Differential Calculus we do calculations on functions. In propositional calculus, we do calculations on propositions, with well-defined symbols, to determine the truth or falsehood of compound propositions.
- Propositional logic is an essential tool for the computer engineer in system development; the system specifications (requirements), made in the natural language (English), are translated into logical expressions, which are precise and unambiguous.
- The "formalism" in Symbolic Logic is well suited to be programmed on the computer. Computer programs are themselves "formal proofs", so that their construction and verification closely resemble logical manipulation. Logic plays a major role in formal hardware and software, verification of proofs and in the theory of programming languages.

1.2 PROPOSITIONS (STATEMENTS)

1.2.1 Definition

A **proposition** or **statement** is a declarative sentence which is either true or false, but not both.

Examples:

1. There are seven days in a week.
2. $2 + 2 = 5$.
3. The earth is flat.
4. The equation $x^2 + x + 1 = 0$ has no real root.
5. It will rain tomorrow.

Examples 1 and 4 are true statements, whereas statements 2 and 3 are obviously false. In the case of example 5, although its truth value cannot be predicted at this point of time, it can be definitely determined in the future (i.e. tomorrow). Hence it is also a statement.

The following are, however, examples of sentences which are not propositions.

1. $x + 3 = 5$.
2. Bring that book!
3. When is your interview?
4. What a beautiful painting!
5. This statement is false.

Example 1 is not a statement, since its truth value depends on the value of x. If $x = 2$, the sentence is true; if $x \neq 2$, the sentence is false. It is in fact a propositional function.

Examples 2, 3, 4 are not declarative sentences; one is a command, the other is a question while the third is an exclamation. Hence these are not statements.

Example 5, although a declarative sentence cannot be assigned a truth value, it is neither true nor false. Hence it is also not a proposition.

1.2.2 Notation

- The statements in Examples 1.2.1 are sentences, which cannot be further split or broken down into simpler sentences. Such statements are called **Primary, Primitive** or **Atomic** statements.

- We denote primary statements by the lower case letters p, q, r ......
- In Mathematics, we have the concept of real or complex variable. Similarly in logic, we have the concept of **Propositional Variable** or **Statement Variable**. We denote statement variables also by the same symbols p, q, r. This dual use of the same symbols to denote either a definite statement, called a **constant**, or an arbitrary statement called a **variable**, should not cause any confusion, as its use will be clear from the context.

  (Note that when 'p' is used as a statement variable, it has **no** truth value and hence is not a statement. It should be replaced only by a statement and then its truth value can be determined --). For example, we may replace p by a true statement "2 + 3 = 5" or by a false statement "3 + 3 = 5".

1.3 LOGICAL CONNECTIVES

- It is possible to form new and rather complicated statements from the primary statements by using certain connecting words, with whose usage; we are already familiar in the English language. These statements are called **Compound Statements.**
- In our everyday use of the English language, we use the words "not", "and", "or", "but", "while" etc., to connect two or more statements. However, these connectives, being quite flexible in their usage, lead to inexact and ambiguous interpretations. Hence we shall borrow only some of these connectives, redefine and symbolize them, to suit our purpose.

1.3.1 Negation

> The negation of a statement is formed either by introducing the word "not" at a proper place or by prefixing the statement with the phrase "It is not the case that".

If "p" denotes a statement, then negation of p is denoted by "~ p" (or). If the truth value of p is true, then the truth value of ~ p is false. If the truth value of p is false, then the truth value of ~ p is true. Consider the following examples:

1. If p is the statement "I am going for a walk", ~ p is the statement "I am not going for a walk" or "It is not the case that I am going for a walk".
2. If q is the statement "3 is not a prime number", ~ q is the statement "3 is a prime number".

1.3.2 Conjunction ("And")

> If p and q are the statements, the compound statement "p and q" is called as the conjunction of p and q; and is denoted by
>
> **$p \wedge q$.**

Examples:

1. Let us consider the statements

   p : The sun is shining.

   q : The birds are singing.

Then $p \wedge q$ is the statement" The sun is shining and the birds are singing."

2. Let, p : 2 is a prime number.

   q : Ram is an intelligent boy.

Then $p \wedge q$ is the statement "2 is a prime number and Ram is an intelligent boy."

This statement is perfectly acceptable in logic, although it makes no sense in our everyday language, as we cannot see the connection between the two statements.

The words "but" and "while" are treated as equivalent words to "and". Consider the following examples:

3. Translate into symbolic form the statement

   Amar is poor but happy.

Solution: Let, p : Amar is poor.

q : Amar is happy.

Then the given statement in the symbolic form is $p \wedge q$.

4. Translate into symbolic form the statement

We watch television while we have dinner.

Solution: Let, p : We watch television.

q : We have dinner.

Then the given statement in the symbolic form is $p \wedge q$.

1.3.3 Disjunction ("or")

> If p and q are statements, then the compound statement "p or q" is called as the disjunction of p and q, and is denoted by **$p \vee q$.**

Examples:

1. There is an error in the program or the data is wrong.

   Let, p : There is an error in the program.

   q : The data is wrong.

   Then $p \vee q$: There is an error in the program or the data is wrong.

In the above example, the connective "or" is used in the **inclusive** sense, i.e. at least one possibility exists or even both the possibilities exist.

Consider the next example.

2. Either I will read a book or go to sleep.

   Let, p : I will read a book.

   q : I will go to sleep.

   Then $p \vee q$: I will read a book or go to sleep.

In everyday language as this example demonstrates the connective "or" is used in the **exclusive sense,** i.e. either one or the other activity can happen, but not both.

In logic, the symbol ' • ' is used in the inclusive sense only; where we wish to specify that "exclusive or" is to be used, we use a new symbol "".

Hence in the above example, the more correct notation is p q.

Normally in our everyday language, the relation "or" is used between two statements, which have some kind of relationship between them. In logic, this is not necessary, as the following example demonstrates.

3. Let, p : It is raining today.

   q : Aarti is an intelligent girl.

Then $p \vee q$: It is raining today and Aarti is an intelligent girl.

This Statement makes perfect sense in logic, though not in English

1.3.4 Conditional ("If ...... then")

If p and q are statements, the compound statement "If p then q", denoted by **$p \to q$** is called a conditional statement or implication.

p is called the **antecedent** or hypothesis, while q is called the **consequent**.

The **converse** of $p \to q$ is the conditional $q \to p$, and the **contrapositive** of $p \to q$ is the conditional $\sim q \to \sim p$.

Examples:

1. Let, p : Hari works hard.

   q : Hari will pass the exam.

   Then $p \to q$: If Hari works hard, then he will pass the exam.

2. Give the converse and contrapositive of the conditional

   If it rains, then I carry an umbrella.

Solution: Let, p : It rains.

q : I carry an umbrella.

Converse of $p \to q$ is

$q \to p$: If I carry an umbrella, then it rains.

Contrapositive of $p \to q$ is

$\sim q \longrightarrow \sim p$: If I do not carry an umbrella, then it does not rain.

In ordinary usage, the conditional statement may not always have the form "If ... then". While putting the statement into symbolic form, we have to interpret the statement correctly. Consider the following examples.

3. Write in symbolic form the statement:

   Farmers will face hardship if the dry spell continues.

Solution: Let, p : Farmers will face hardship.

q : The dry spell continues.

Then $q \to p$ is the correct symbolic representation of the given statement.

4. Write in symbolic form the statement:

Rahul's father will buy him a computer only if he passes the exam, with distinction.

Solution: Let, p : Rahul's father will buy him a computer.

q : Rahul passes the exam with distinction.

Then $p \to q$ is the correct symbolic version of the statement.

5. Write in symbolic form the statement:

A sufficient condition for a function to be continuous is that the function is differentiable.

Solution: Let, p : The function is differentiable.

q : The function is continuous.

The statement has to be interpreted as p is sufficient for q or in other words p implies q.

Hence the correct solution is $p \to q$.

6. Put into symbolic form the statement:

A necessary condition for a candidate to get admission is to pass the entrance exam.

Solution: Let, p : A candidate gets admission.

q : The candidate has passed the entrance exam.

In this example, the statement q is necessary for p. Hence $p \to q$ is the required symbolic form.

7. Put into symbolic form the statement:

Unless I reach the station on time, I will miss the train.

Solution: Let, p : I reach the station on time.

q : I will miss the train.

Then $\sim p \rightarrow q$ is the required symbolic form.

> Hence it is **IMPORTANT** to REMEMBER that the statement forms
>
> If p, then q,
>
> q if p,
>
> p only if q,
>
> p is sufficient for q,
>
> q is necessary for p.
>
> All are equivalent to the statement form $p \rightarrow q$.

1.3.5 Biconditional ("If and only if")

If p and q are statements, the compound statement "p if and only if q", denoted by **p q**, is called a bi-conditional statement.

Often "if and only if" is shortened as "iff".

p q is also read as

"If p then q, and conversely".

Many of the theorems in Mathematics are of the type "if and only if".

Examples:

1. An integer is even if and only if it is divisible by 2.
2. A right angled triangle is isosceles if and only if the other two angles are equal to forty five degrees.
3. Two lines are parallel if and only if they have the same slope.

1.4 PROPOSITIONAL OR STATEMENT FORM

We know what a statement variable is (refer to article 1.2.2). Using the logical connectives defined above, we can construct or form an expression, involving the statement variables.

The following are examples of statement forms:

1. $\sim (p \vee q) \rightarrow p$

2. $(p \rightarrow q) \leftrightarrow (p \wedge \sim q)$

3. $((p \wedge q) \vee (p \wedge \sim r)) \rightarrow ((p \wedge r) \vee q)$

One can thus construct any number of complicated statement forms, from the statement variables by using the logical connectives.

A **statement form has no fixed truth value.** It is only when the statement variables in a form, are **assigned** definite truth values, that we obtain the truth value of the statement form.

Hence the truth value of a statement **assumes** the truth value "true" or the truth value "false", depending on the truth values assigned to the statement variables, appearing in the statement form.

We denote by 'T' the truth value true, and 'F' by 'F' the truth value false.

In circuit logic, 'T' is denoted by '1' and 'F' by '0'.

SOLVED EXAMPLES

The following examples illustrate the use of connectives in translating statements in English into logical forms.

Example 1: *Using the following statements:*

p : Mohan is rich

q : Mohan is happy

Write the following statements in symbolic form.

(i) Mohan is rich but unhappy.

(ii) Mohan is poor but happy.

(iii) Mohan is neither rich nor happy.

(iv) Mohan is poor or he is both rich and unhappy.

Solution:

(i) $p \wedge \sim q$

(ii) $\sim p \wedge q$

(iii) $\sim (p \vee q)$ or $\sim p \wedge \sim q$

(iv) $\sim p \vee (p \wedge \sim q)$.

Example 2: *Using the following statements:*

p : Rajini is tall

q : Rajini is beautiful

Write the following statements in symbolic form.

(i) Rajini is tall and beautiful.

(ii) Rajini is tall but not beautiful.

(iii) It is false that Rajini is short or beautiful.

(iv) Rajini is tall or Rajini is short and beautiful.

Solution:

(i) $p \wedge q$

(ii) $p \wedge \sim q$

(iii) $\sim (\sim p \vee q)$

(iv) $p \vee (\sim p \wedge q)$.

Example 3: *Using the following statements:*

p : I will study discrete structures

q : I will go to a movie

r : I am in a good mood

Write the following statements in symbolic form.

(i) If I am not in a good mood, then I will go to a movie.

(ii) I will not go to a movie and I will study discrete structures.

(iii) I will go to a movie only if I will not study discrete structures.

(iv) If I will not study discrete structures, then I am not in a good mood.

Solution:

(i) $\sim r \rightarrow q$

(ii) $\sim q \wedge p$

(iii) $q \rightarrow \sim p$

(iv) $\sim p \rightarrow \sim r$

Example 4: *Write the following statements in symbolic form:*

(i) Indians will win the world-cup if their fielding improves.

(ii) If I am not in a good mood or I am not busy, then I will go for a movie.

(iii) If you know Object Oriented Programming and Oracle, then you will get a job.

(iv) I will score good marks in the exam if and only if I study hard.

Solution:

(i) Let, p : Indians will win the world-cup.

q : Their fielding improves.

Then $q \rightarrow p$ is the required form.

(ii) Let, p : I am in a good mood.

q : I am busy.

r : I will go for a movie.

Then $(\sim p \vee \sim q) \rightarrow r$.

(iii) Let, p : You know Object Oriented Programming.

q : You know Oracle.

r : You will get a job.

Then $p \wedge q \rightarrow r$ is the required symbolic form.

(iv) Let, p : I will score good marks in the exam.

q : I study hard.

Then p q is the required symbolic form.

Example 5: *Put the following statements into symbolic form:*

(i) Whenever weather is nice, then only we will have a picnic.

(ii) If either Anil takes Mathematics or Aparna takes Biology, then Deepa will take Chemistry.

(iii) Program is readable only if it is well structured.

(iv) Unless he studies, he will fail in the examination.

Solution:

(i) Let, p : Weather is nice.

q : We will have a picnic.

The statement is equivalent to "we will have a picnic only if the weather is nice".

Hence $q \rightarrow p$ is the required form.

(ii) Let, p : Anil takes Mathematics.

q : Aparna takes Biology.

r : Deepa will take Chemistry.

Then $(p \vee q) \rightarrow r$ is the required form.

(iii) Let, p : Program is readable.

q : Program is well structured.

Then $p \rightarrow q$ is the required form.

(iv) Let, p : He studies.

q : He will fail in the examination.

The statement is equivalent to the statement "If he does not study, then he will fail in the examination".

Hence $\sim p \rightarrow q$ is the required form.

Example 6: *To describe the various restaurants in the city, let p denote the statement "the food is good", q the statement "the service is good" and r the statement "the rating is three-star". Write the following statements in symbolic form:*

(i) Either the food is good or service is good, or both.

(ii) Either the food is good or service is good, but not both.

(iii) The food is good while the service is poor.

(iv) It is not the case that both the food is good and the rating is three-star.

(v) If both the food and service are good, then the rating is three-star.

(vi) It is not true that a three-star rating always means good food and good service.

Solution:

(i) As "$\vee$" means "inclusive or", the statement required is $p \vee q$.

(ii) We have to use "exclusive or" i.e." $\bar{\vee}$". Hence the statement is $p \bar{\vee} q$.

Equivalently the statement is also $p \wedge \bar{q} \vee \bar{p} \wedge q$.

(iii) "While" is interpreted as "and". Hence the statement is $p \wedge \sim q$.

(iv) "It is not the case" implies negation. Hence the statement is $\sim (p \wedge r)$.

(v) The statement is $(p \wedge q) \rightarrow r$.

(vi) Here we have negation of implication. Hence the statement is $\sim (r \rightarrow (p \wedge q))$.

Example 7: *Using the following propositions:*

p : I am bored

q : I am waiting for one hour

r : There is no bus

translate the following into English.

(i) $(q \vee r) \rightarrow p$

(ii) $\sim q \rightarrow \sim p$

(iii) $(q \rightarrow p) \vee (r \rightarrow p)$

Solution:

(i) If I am waiting for one hour or there is no bus, then I get bored.

(ii) If I am not waiting for one hour, then I am not bored.

(iii) If I am waiting for one hour then I am bored, or if there is no bus, then I am bored.

Example 8: *Write the logical negation of the following statements in the symbolic form:*

(i) Gopal is intelligent and rich.

(ii) Gopal is intelligent but not rich.

(iii) Gopal is either intelligent or rich.

Solution: Let, p : Gopal is intelligent.

q : Gopal is rich.

(i) $\sim (p \wedge q)$ which is equivalent to $\sim p \vee \sim q$.

(ii) $\sim (p \wedge \sim q)$ which is equivalent to $\sim p \vee q$.

(iii) $\sim (p \vee q)$ which is equivalent to $\sim p \wedge \sim q$.

Example 9: *Translate the following statements into symbolic form:*

If the utility cost goes up or the request for additional funding is desired, then a new computer will be purchased if and only if we can show that the current computing facilities are indeed not adequate.

Solution: Let, p : The utility cost goes up.

q : The request for additional funding is desired.

r : A new computer will be purchased.

s : we can show that the current computing facilities are indeed adequate.

Then $(p \vee q) \rightarrow (r \times \sim s)$.

Example 10: *Consider the following advertisement for a game:*

(i) There are three statements in this advertisement.

(ii) Two of them are not true.

(iii) The average increase in IQ scores of people who learn this game is more than 20 points.

Is the statement (iii) true? Justify your answer.

Solution:

Let us suppose statement (iii) is false. Statement (i) is true since there are actually three statements in the advertisement. This leaves us with statement (ii). Statement (ii) cannot be true, since if it so, we shall have only one statement which is false, which contradicts statement (ii) itself. Hence statement (ii) is false, which means that there are actually two statements which are true. Hence statement (iii) cannot be false. It has to be a true statement.)

Example 11: *Write the following statements in symbolic form:*

(i) The sun is bright and humidity is not high.

(ii) It is already 9.00 a.m., I should start my job.

(iii) If the requirement of Computer Engineers is increased, then more seats will be offered by University and more computers will be purchased by the University Computer Department if the rates are competitive.

Solution: (i) Let,

p : The sun is bright,

q : Humidity of high.

Then the given statement in symbolic form is: $p \wedge \sim q$.

(ii) Let, r : It is already 9.00 a.m.,

s : I should start my job.

Then the statement in symbolic form is: $r \wedge s$.

(iii) Let, p : The requirement of Computer Engineers is increased,

q : More seats will be offered by University,

r : More computers will be purchased by the University Computer, Department

s : The rates are competitive.

Then the given statement in symbolic form is: $p \rightarrow (q \wedge (s \rightarrow r))$.

EXERCISE - 1.1

1. Let p denote the statement, "The material is interesting", and q denote the statement, "The exercises are challenging", and r denote the statement, "The course is enjoyable".

   Write the following statements in symbolic form:

   (i) The material is interesting and the exercises are challenging.

   (ii) The material is uninteresting, the exercises are not challenging and the course is not enjoyable.

   (iii) If the material is not interesting and the exercises are not challenging, then the course is not enjoyable.

   (iv) The material is interesting means the exercises are challenging and conversely.

   (v) Either the material is interesting, or the exercises are challenging, but not both.

2. Let the propositions p : Gopal is tall.

   q : Gopal is handsome.

   Write the following sentences in symbolic form, using p, q and appropriate connectives.

   (i) Gopal is tall and handsome.

   (ii) Gopal is tall but not handsome.

   (iii) It is false that Gopal is short or handsome.

   (iv) Gopal is neither tall nor handsome.

   (v) Gopal is tall means he is also handsome.

   (vi) It is not true that Gopal is short or not handsome.

3. Write the following statements in symbolic form:

   (i) The sun is bright and humidity is not high.

   (ii) If I finish my homework before dinner and it does not rain, then I will go to the ball game.

   (iii) If you do not see me tomorrow, it means that I am not in town.

4. Let 'p' be the proposition " in high speed driving is dangerous" and 'q' be the proposition "Rajesh was a wise man."

   Write down the meaning of the following propositions:

   (i) $p \wedge q$

   (ii) $\sim p \wedge q$

   (iii) $\sim (p \wedge q)$

   (iv) $(p \wedge q) \vee (\sim p \wedge \sim q)$

   (v) $(p \vee q) \wedge \sim (p \wedge q)$.

5. Write the following compound statements in symbolic form:

   (i) It is humid and cloudy, or it is raining, but at the same time it is false that it is both humid and raining.

   (ii) Being able to type is sufficient to learn word processing.

   (iii) If Manisha is not sick, then if she goes to the picnic, she will have a good time.

   (iv) I can study only if I am not tired or hungry.

6. State the converse and contrapositive of each of the following statements:

   (i) If it rains, I am not going to the city.

   (ii) I can't complete the task if I don't get help.

   (iii) I will come only if I am not too busy.

   (iv) If you complete this job, you can take a holiday.

7. Express the following statements in propositional form:

   (i) There are many clouds in the sky but it did not rain.

   (ii) I will get first class if and only if I study well and score above 80 in Mathematics.

   (iii) Computers are cheap but softwares are costly.

   (iv) It is very hot and humid or Ramesh is having heart problem.

   (v) In small restaurants the food in good and service is poor.

   (vi) If I finish my submission before 5.00 in the evening and it is not very hot, I will go and play a game of hockey.

8. Write the following statements in symbolic form:
   (i) The sun is bright and humidity is not high.
   (ii) It is already 11.00 a.m., I should start my job.
   (iii) If the requirement of computer engineers increased, then more seats will be offered by University and more computers will be purchased by University Computer Department if the rates are competitive.
9. Negate each of the statement:
   (i) If there is a riot, then someone is killed.
   (ii) It is day light and all the people are arisen.
10. Use p : The user enters a valid password.
    q : Access is granted.
    r : The user has paid the subscription fee.
    to write an English sentence that corresponds to each of the following:
    (1) $\sim r \wedge q$
    (2) $\sim q \vee r$
    (3) $\sim (p \vee q)$
    (4) $p \vee \sim r$
11. The following statements are given. Write their converse and contrapositive statements.
    (i) If he is considerate of others, then a man is a gentleman.
    (ii) If a steel root is stretched, then it has been heated.

1.5 TRUTH TABLES

A table giving all possible truth values of a statement form, corresponding to the truth values assigned to its variables, is called a truth table.

If a statement form consists of two distinct variables, the table will contain $2^2 = 4$ values. If it consists of three distinct variables, it will contain 2^3 values. In general, if a statement form contains n distinct variables, then the table will contain 2^n values.

Let us consider first, the truth tables of the basic statement forms $\sim p$, $p \wedge q$, $p \vee q$, $p \rightarrow q$ and p q.

Table 1.1

p	$\sim p$
T	F
F	T

Table 1.2

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Table 1.3

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Table 1.4

p	q	$p \rightarrow q$
T	T	T
F	T	T
T	F	F
F	F	T

Table 1.5

p	q	p q
T	T	T
F	T	F
T	F	F
F	F	T

Note the following important points:

1. As shown in Table 1.2 $p \wedge q$ has truth value T if and only if both p and q have truth values T.
2. As shown in Table 1.3 $p \vee q$ has truth value F only when both p and q have truth values F. Otherwise, it has truth value T.
3. As shown in Table 1.4 $p \rightarrow q$ has truth value F only when p has truth value T and q has truth value F. If p has truth value F, $p \rightarrow q$ has truth value T, whatever may be the truth value of q.
4. $p \leftrightarrow q$ has truth value T only when both have the same truth values.

Truth table for "Exclusive or".

Recall that p q implies either p or q is true but not **both**. Hence we have:

Table 1.6

p	q	p q
T	T	F
T	F	T
F	T	T
F	F	F

SOLVED EXAMPLES

Example 1: *Construct the truth tables for the following statement forms:*

(i) $(\sim p \vee q) \rightarrow q$
(ii) $\sim (p \wedge q) \vee (p \times q)$
(iii) $(\sim p \rightarrow r) \wedge (p \times q)$
(iv) $((\sim p \wedge q) \vee (q \wedge r)) \rightarrow r$

Solution:

(i)

p	q	$\sim p$	$\sim p \vee q$	$(\sim p \vee q) \rightarrow q$
T	T	F	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	F

(ii)

p	q	$p \wedge q$	$\sim (p \wedge q)$	$p \times q$	$\sim (p \wedge q) \vee (p \times q)$
T	T	T	F	T	T
T	F	F	T	F	T
F	T	F	T	F	T
F	F	F	T	T	T

(iii)

p	q	r	$\sim p$	$\sim p \rightarrow r$	$p \times q$	$(\sim p \rightarrow r) \wedge (p \times q)$
T	T	T	F	T	T	T
T	T	F	F	T	T	T
T	F	T	F	T	F	F
T	F	F	F	T	F	F
F	T	T	T	T	F	F
F	T	F	T	F	F	F
F	F	T	T	T	T	T
F	F	F	T	F	T	F

(iv)

p	q	r	$\sim p$	$\sim p \wedge q$	$q \wedge r$
T	T	T	F	F	T
T	T	F	F	F	F
T	F	T	F	F	F
T	F	F	F	F	F
F	T	T	T	T	T
F	T	F	T	T	F
F	F	T	T	F	F
F	F	F	T	F	F

$(\sim p \wedge q) \vee (q \wedge r)$	$((\sim p \wedge q) \vee (q \wedge r)) \rightarrow r)$
T	T
F	T
F	T
F	T
T	T
F	T
F	T
F	T

Example 2: *If $p \rightarrow q$ is false, determine the truth value of $(\sim (p \wedge q)) \rightarrow q$.*

Solution: $p \rightarrow q$ has truth value F only when p has truth value T and q has truth value F. Now, we have

p	q	$p \wedge q$	$\sim (p \wedge q)$	$(\sim (p \wedge q)) \rightarrow q$
T	F	F	T	F

Hence the truth value of $(\sim (p \wedge q)) \rightarrow q$ is false.

Example 3: *If p and q are false propositions, find the truth value of $(p \vee q) \wedge (\sim p \vee \sim q)$.*

Solution:

p	q	$p \vee q$	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$(p \vee q) \wedge (\sim p \vee \sim q)$
F	F	F	T	T	T	F

Example 4: *If $p \rightarrow q$ is true, can we determine the truth value of $\sim p \vee (p \rightarrow q)$? Explain your answer.*

Solution: Since $p \rightarrow q$ is true, we have to consider the following possible truth values of p and q.

p	q	$\sim p$	$p \rightarrow q$	$\sim p \vee (p \rightarrow q)$
T	T	F	T	T
F	T	T	T	T
F	F	T	T	T

Yes, it is possible to determine the truth value of $\sim p \vee (p \rightarrow q)$ and it is T.

Example 5: *Given the truth values of p and q as T and that of r and s as F, find the truth values of the following:*

(i) $p \vee (q \wedge r)$ (ii) $p \rightarrow (r \wedge s)$
(iii) $(p \wedge (q \wedge r)) \vee \sim ((p \vee q) \wedge (r \vee s))$.

Solution: (i)

p	q	r	$(q \wedge r)$	$p \vee (q \wedge r)$
T	T	F	F	T

(ii)

p	r	s	$r \wedge s$	$p \rightarrow (r \wedge s)$
T	F	F	F	F

(iii)

p	q	r	$q \wedge r$	$p \wedge (q \wedge r)$
T	T	F	F	F

$p \vee q$	$r \vee s$	$(p \vee q) \wedge (r \vee s)$	$\sim((p \vee q) \wedge (r \vee s))$	$(p \wedge (q \wedge r)) \vee \sim((p \vee q) \wedge (r \vee s))$
T	F	F	T	T

The following are Puzzle type problems based on the biconditional $p \quad q$.

***Example 6:** An island has two tribes of natives. Any native from the first tribe always tells the truth, while any native from the other tribe always lies. We arrive at the island and ask a native if there is gold on the island. He answers, "There is gold on the island if and only if I always tell the truth". Which tribe is he from? Is there gold on the island?*

Solution: We cannot determine the tribe from which the native is. However, we can determine if there is gold on the island.

Let, p : He always tells the truth.

q : There is gold on the island.

Then the native's answer is $q \quad p$. or equivalently $p \quad q$. We shall consider the following two cases.

Case I: The native always tells the truth. Then in this case, the truth table for $p \leftrightarrow q$ is :

p	q	p q
T	T	T

Since $p \quad q$ is T and p is T, then q has to be T. Hence in this case, there is gold on the island. Now consider, case II.

Case II: The native always lies. In this case, the truth table for

$p \quad q$ is :

p	q	p q
F	T	F

Since $p \quad q$ is F and p is also F, then q must be T.

Hence in this case also, we find that there is gold on the island.

***Example 7:** A certain country is inhabited only by people who either always tells the truth or always tells lies, and who will respond to questions only with a "yes" or a "no". A tourist comes to a fork in the road, where one branch leads to the capital and the other does not. There is no sign indicating which branch to take, but there is an inhabitant, Mr. Z, standing at the fork. What single question should the tourist ask him to determine which branch to take?*

Solution: Let, p : "Mr. Z always tells the truth".

q : "The left-hand branch leads to the capital".

Let, $A : p \quad q$.

Then the single question which the tourist should ask Mr. Z is, Is "A" true?

We shall consider two cases.

Case I: Mr. Z always speaks the truth. Consider the following table:

p	q	p q	Answer of Mr. Z
T	T	T	Yes
T	F	F	No

Hence when Mr. Z says "Yes", q is T, i.e. the left-hand branch leads to the capital.

Now let us consider, case II.

Case II: Mr. Z always tells lies.

Consider the truth table:

p	q	p q	Answer of Mr. Z
F	T	F	Yes
F	F	T	No

If Mr. Z answers "Yes", in this case also q being T, the tourist will take the left-hand branch leading to the capital.

***Example 8:** Given that the value of $p \quad q$ is false, determine the truth value of $(\sim p \quad \sim q) \quad q$.*

Solution: Consider the truth table.

p	Q	p q	~p	~q	~p ~q	(~p ~q) q
T	F	F	F	T	T	F

***Example 9:** Given that the value of $p \quad q$ is true, can you determine the value of $\sim p \quad (p \quad q)$?*

Solution: Consider the truth table

p	Q	p q	~p	p q	~p (p q)
T	T	T	F	T	T
F	T	T	T	F	T
F	F	T	T	T	T

1.6 TAUTOLOGY

We have seen how to construct the truth table of various statement forms. The last column in the truth table gives the truth values of the statement form for all possible assignment of truth values to its variables.

1.6.1 Definitions

A statement form is called a Tautology, if it always assumes the truth value 'T' irrespective of the truth values assigned to its variables.

A statement form is called a **contradiction** if it always assumes the truth value 'F' irrespective of the truth values assigned to its variables.

A statement form which is neither a tautology nor a contradiction is called a **contingency.**

Examples:

1. $p \vee \sim p$ is a tautology ; $p \wedge \sim p$ is a contradiction.

Solution: Consider the truth table:

p	~p	p∨~p	p∧~p
T	F	T	F
F	T	T	F

2. $p \rightarrow p$ is a tautology.

Solution:

p	p	p→p
T	T	T
F	F	T

SOLVED EXAMPLES

***Example 1:** Construct truth tables to determine whether each of the following is a tautology, a contingency or a contradiction.*

(i) $p \rightarrow (q \rightarrow p)$

(ii) $(p \wedge q) \wedge \sim (p \vee q)$

(iii) $(p \wedge q) \rightarrow p$

(iv) $(p \rightarrow q) \leftrightarrow (q \vee \sim p)$

(v) $(p \wedge (\sim p \vee q)) \wedge \sim q$

Solution:

(i) Consider the truth table:

p	q	q→p	p→(q→p)
T	T	T	T
T	F	T	T
F	T	F	T
F	F	T	T

Hence $p \rightarrow (q \rightarrow p)$ is a tautology.

(ii) Consider the truth table:

p	q	p∧q	p∨q	~(p∨q)	(p∧q)∧~(p∨q)
T	T	T	T	F	F
F	T	F	T	F	F
T	F	F	T	F	F
F	F	F	F	T	F

Hence $(p \wedge q) \wedge \sim (p \vee q)$ is a contradiction.

(iii)

p	q	p∧q	(p∧q)→p
T	T	T	T
F	T	F	T
T	F	F	T
F	F	F	T

Hence $(p \wedge q) \rightarrow p$ is a tautology.

(iv) Consider the truth table:

p	q	~p	(p→q)	(q∨~p)	(p→q) (q∨~p)
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Hence $(p \rightarrow q) \quad (q \vee \sim p)$ is a tautology.

(v) Consider the truth table:

p	Q	~p	(~p∨q)	p∧(~p∨q)	~q	(p∧(~p∨q))∧~q
T	T	F	T	T	F	F
T	F	F	F	F	T	F
F	T	T	T	F	F	F
F	F	T	T	F	T	F

Hence $(p \wedge (\sim p \vee q)) \wedge \sim q$ is a contradiction.

***Example 2:** Show that $(p \wedge (p \rightarrow q)) \rightarrow q$ is a tautology, without using truth table.*

Solution: We have only to show that wherever $p \wedge (p \rightarrow q)$ is true, q is also true, since in the other cases $(p \wedge (p \rightarrow q)) \rightarrow q$ is anyway true.

Now $p \wedge (p \rightarrow q)$ is T implies p is T and $p \rightarrow q$ is T. These together means that q is T.

Hence the required form is a tautology.

Example 3: *Show that* $(p \to q) \wedge \sim q \to \sim p$ *is a tautology.*

Solution: We need only to show that $p \to q$ and $\sim q$ both true imply $\sim$ **p** is true.

Since truth value of $\sim q$ is T, truth value of q is F. Since $p \to q$ is true, this means that p is false, i.e. truth value of $\sim p$ is T. Hence the proof.

1.7 EQUIVALENCE OF STATEMENT FORMS

The concept of equivalence of statement forms is similar to the concept of equality of algebraic expressions in mathematics.

Definition:

Two statement forms are **logically equivalent** if both have the **same** truth values, whatever may be the truth values assigned to the statement variables, occurring in both the forms.

In other words, if we consider the truth tables for both the statement forms, the final columns in both the tables should be identical.

Examples:

1. p and $p \wedge p$ are logically equivalent **(Idempotence)**

Solution: Consider the truth table:

p	$p \wedge p$
T	T
F	F

Since the truth tables for p and $p \to p$ are identical, it follows that p and $p \to p$ are logically equivalent.

2. The following are some basic logically equivalent forms. Proofs are left as exercises for the students.
   - (i) $p \wedge q$ and $q \wedge p$ are logically equivalent.
   - (ii) $p \vee q$ and $q \vee p$ are logically equivalent.
   - (iii) p and $\sim(\sim p)$ are logically equivalent.

3. $p \to q$ and $\sim q \to \sim p$ are logically equivalent. **(Contrapositive)**

Solution:

p	q	$p \to q$	$\sim q$	$\sim p$	$\sim q \to \sim p$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Note that the columns for $p \to q$ and $\sim q \to \sim p$ are identical.

Hence $p \to q$ and $\sim q \to \sim p$ are logically equivalent forms.

The logical equivalence proved above is very often practiced in everyday language.

Consider the two statements "He will pass if he works hard" and "If he does not work hard, he will not pass the exam". Both these statements obviously convey the same meaning.

4. $p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$ are logically equivalent. **(Distributivity)**

Solution: Consider the following truth tables:

(a)

p	q	r	$(q \vee r)$	$p \wedge (q \vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

(b)

p	q	r	$(p \wedge q)$	$(p \wedge r)$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	F	F
F	T	T	F	F	F
F	T	F	F	F	F
F	F	T	F	F	F
F	F	F	F	F	F

The last columns in both (a) and (b) are identical. Hence the two forms are logically equivalent.

5. $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent. Proof is similar to (4) and is left as an exercise.

6. $\sim (p \wedge q)$ and $\sim p \vee \sim q$ are logically equivalent **(De Morgan's Laws)**

Solution: Consider the following truth tables:

p	q	$p \wedge q$	$\sim(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

p	q	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

The last columns in both the tables are identical. Hence $\sim (p \wedge q)$ and $\sim p \vee \sim q$ are logically equivalent.

7. $\sim (p \vee q)$ and $\sim p \wedge \sim q$ are logically equivalent. Proof is left as an exercise.

Examples:

The following examples on logical equivalence show that the conditional can be replaced by the connectives $\sim$ and $\vee$.

8. $p \to q$ and $\sim p \vee q$ are logically equivalent **(Elimination of conditional).**

Solution:

p	q	$p \to q$	$\sim p$	$\sim p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Similarly, we have the next example, in which the biconditional can also be eliminated.

9. p q is logically equivalent to $(p \to q) \wedge (q \to p)$ which in turn is logically equivalent to $(\sim p \vee q) \wedge (\sim q \vee p)$.

Proof is left as an exercise to the students.

Notation: Hereafter we shall denote logical equivalence by the symbol '≡' modifying the equal sign '=' in Mathematics.

The following is a list of some important equivalences (many of which have been already proved), called as **identities.**

1.8 LOGICAL IDENTITIES

1.	$p \equiv p \vee p$	Idempotence of ∨
2.	$p \equiv p \wedge p$	Idempotence of ∧
3.	$p \vee q \equiv q \vee p$	Commutativity of ∨
4.	$p \wedge q \equiv q \wedge p$	Commutativity of ∧
5.	$p \vee (q \vee r) \equiv (p \vee q) \vee r$	Associativity of ∨
6.	$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$	Associativity of ∧
7.	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributivity of ∧ over ∨
8.	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	Distributivity of ∨ over ∧
9.	$p \equiv \sim(\sim p)$	Double negation
10.	$\sim (p \vee q) \equiv \sim p \wedge \sim q$	De Morgan's laws
11.	$\sim (p \wedge q) \equiv \sim p \wedge \sim q$	De Morgan's laws
12.	$p \vee \sim p \equiv$ Tautology	
13.	$p \wedge \sim p \equiv$ Contradiction	
14.	$p \vee (p \wedge q) \equiv p$	Absorption laws
15.	$p \wedge (p \vee q) \equiv p$	Absorption laws

Students are advised to work out the proofs of the Absorption laws, as an exercise.

The following theorem relates logical equivalence and a tautology.

1.9 THEOREM

Let **A** and **B** be two statement forms. Then **A** is logically equivalent to **B** if and only if **A** **B** is a tautology.

Proof:

Let **A** and **B** be logically equivalent.

Then **A** and **B** assume the **same** truth values for any assignment of truth values to the statement variables in **A** and **B**.

Hence either both **A** and **B** are true or both are false.

Therefore **A** **B** is a tautology.

Conversely if **A** **B** is a tautology, it means that wherever **A** is true, **B** is **true** (and conversely), and wherever **A** is false, **B** is also false (and conversely).

Therefore **A** and **B** are logically equivalent.

1.10 NORMAL FORMS

One of the main problems in logic is to determine whether a given statement form is a tautology or a contradiction. Constructing truth tables for this purpose may not always be practical (even with the help of a computer), especially where the statement form may contain a large number of variables or has a complicated structure. Hence it is necessary to consider alternate methods, such as reducing the statement form to so called **normal forms.**

1. **Disjunctive Normal Form (DNF):**

A conjunction of statement variables and (or) their negations is called as a **fundamental conjunction.** (It is also called as a **min term).**

For example, p, $\sim p$, $\sim p \wedge q$, $p \wedge q$, $p \wedge \sim p \wedge q$ are fundamental conjunctions.

We know that $p \wedge \sim p$ is always false. Hence if a fundamental conjunction contains at least one pair of factors, in which one is the negation of the other, it will be false.

A statement form which consists of a **disjunction** of **fundamental conjunctions**, is called a **disjunctive normal** form (abbreviated as dnf).

Examples of DNF:

(i) $(p \wedge q) \vee \sim q$

(ii) $(\sim p \wedge q) \vee (p \wedge q) \vee q$

(iii) $(p \wedge q \wedge r) \vee (p \wedge \sim r) \vee (q \wedge r)$

(iv) $(p \wedge \sim q) \vee (p \wedge r)$

(v) $(p \wedge q \wedge r) \vee \sim r$

In the following illustrative examples, we will reduce the given statement form to dnf, by using logical equivalence.

Examples:

1. Obtain the dnf of the form $(p \to q) \wedge (\sim p \wedge q)$.

Solution: $p \to q \equiv \sim p \vee q$ (Elimination of biconditional)

Hence $(p \to q) \wedge (\sim p \wedge q)$

$\equiv (\sim p \vee q) \wedge (\sim p \wedge q)$

$\equiv (\sim p \wedge \sim p \wedge q) \vee (q \wedge \sim p \wedge q)$

$\equiv (\sim p \wedge q) \vee (q \wedge \sim p)$

(by using the distributive laws, idempotence laws and commutative laws).

2. Obtain the dnf of $(p \wedge (p \to q)) \to q$

Solution: $(p \wedge (p \to q)) \to q$

$\equiv \sim (p \wedge (\sim p \vee q)) \vee q$

$\equiv \sim p \vee \sim (\sim p \vee q) \vee q$

$\equiv \sim p \vee (p \wedge \sim q) \vee q$

3. Obtain the dnf of the form $\sim (p \to (q \wedge r))$.

Solution: $\sim (p \to (q \wedge r))$

$\equiv \sim (\sim p \vee (q \wedge r))$

$\equiv \sim (\sim p \vee (q \wedge r))$

$\equiv \sim (\sim p) \wedge \sim (q \wedge r)$ (De Morgan's Laws)

$\equiv p \wedge (\sim q \vee \sim r)$(Idempotent laws and De Morgan's laws)

$\equiv (p \wedge \sim q) \vee (p \wedge \sim r)$.

2. Conjunctive Normal Form (CNF):

A disjunction of statement variables and (or) their negations is called a **fundamental disjunction** or **maxterm**.

For example, p, $\sim p$, $\sim p \vee q$, $p \vee q$, $p \vee \sim p \vee q$ are fundamental disjunctions.

We know that $p \vee \sim p$ is always true. Hence if a fundamental disjunction contains at least one pair of factors, in which one is the negation of the other, it will be true ($p \vee \sim p \vee q$ is logically equivalent to a tautology).

A statement form which consists of a **conjunction** of **fundamental disjunctions**, is called a **conjunctive normal form** (abbreviated as cnf).

Note that a cnf is a tautology if and only if every fundamental disjunction contained in it is a tautology.

Examples of CNF:

(i) $p \wedge q$

(ii) $\sim p \wedge (p \vee q)$

(iii) $(p \vee q \vee r) \wedge (\sim p \vee r)$

The following examples illustrate the procedure to obtain cnf of a given statement form, without using truth tables.

Examples: (i) *Obtain the cnf of the form $(\sim p \to r) \wedge (p \;\; q)$.*

Solution: $(\sim p \to r) \wedge (p \;\; q)$

$\equiv (\sim p \to r) \wedge ((p \to q) \wedge (q \to p))$

$\equiv (\sim (\sim p) \vee r) \wedge ((\sim p \vee q) \wedge (\sim q \vee p))$

$\equiv (p \vee r) \wedge (\sim p \vee q) \wedge (\sim q \vee p)$

(ii) *Obtain the cnf of the form $(p \wedge q) \vee (\sim p \wedge q \wedge r)$.*

Solution: $(p \wedge q) \vee (\sim p \wedge q \wedge r)$

$\equiv (p \vee (\sim p \wedge q \wedge r)) \wedge (q \vee (\sim p \wedge q \wedge r))$ (Distributive law)

$\equiv ((p \vee \sim p) \wedge (p \vee q) \wedge (p \vee r)) \wedge ((q \vee \sim p) \wedge (q \vee q) \wedge (q \vee r))$

$\equiv (p \vee q) \wedge (p \vee r) \wedge (q \vee \sim p) \wedge q \wedge (q \vee r)$.

3. Truth Table Method (to find dnf):

Let **p** be a statement form containing n variables $p_1, p_2, \ldots, p_n$. We obtain its dnf from the truth table as follows. For each row in which P assumes value T, form the conjunction $p_1 \wedge p_2 \wedge \& \wedge \sim p_k \wedge \& \wedge p_n$ where we take p_k if there is T in the k-th position in the row and $\sim p_k$ if there is F in that position. Such a term is called a minterm. The disjunction of the minterms is the dnf of the given form.

Examples:

1. *Find the dnf of the form $(\sim p \to r) \wedge (p \times q)$.*

Solution: Consider the truth table.

p	q	r	~ p	~ p → r	p q	(~ p → r) ∧ (p q)
T	T	T	F	T	**T**	**T**
T	T	F	F	T	**T**	**T**
T	F	T	F	T	**F**	**F**
T	F	F	F	T	**F**	**F**
F	T	T	T	T	**F**	**F**
F	T	F	T	F	**F**	**F**
F	F	T	T	T	**T**	**T**
F	F	F	T	F	**T**	**F**

Consider the rows of p, q, r in which T appears in the **last column.**

Then the required dnf is $(p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge r)$.

2. *The statement form f (p, q, r) is given by the following table. Find its dnf.*

p	q	r	f (p, q, r)
F	F	F	T
F	F	T	F
F	T	F	T
F	T	T	F
T	F	F	T
T	F	T	F
T	T	F	F
T	T	T	T

Then the required dnf is $(\sim p \wedge \sim q \wedge \sim r) \vee (\sim p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge \sim r) \vee (p \wedge q \wedge r)$

SOLVED EXAMPLES

Example 1: *Prove that the following formulae are logically equivalent and show that the operator "Exclusive or" is symmetric.*

(i) $p \;\bar{\vee}\; q$ and $q \;\bar{\vee}\; p$.

(ii) $(p \;\bar{\vee}\; q) \;\bar{\vee}\; r$ and $p \;\bar{\vee}\; (q \;\bar{\vee}\; r)$

Solution: (i) Consider the truth tables.

p	q	$p \bar{\vee} q$	$q \bar{\vee} p$
T	T	**F**	**F**
T	F	**T**	**T**
F	T	**T**	**T**
F	F	**F**	**F**

Hence "Exclusive or" is a symmetric operator and also the two formulae are logically equivalent.

(ii) Consider the truth tables.

p	q	r	(p q)	(p q) r	(q r)	q
T	T	T	F	**T**	F	**T**
T	T	F	F	**F**	T	**F**
T	F	T	T	**F**	T	**F**
T	F	F	T	**T**	F	**T**
F	T	T	T	**F**	F	**F**
F	T	F	T	**T**	T	**T**
F	F	T	F	**T**	T	**T**
F	F	F	F	**F**	F	**F**

The columns for $(p \;\bar{\vee}\; q) \;\bar{\vee}\; r$ and $p \;\bar{\vee}\; (q \;\bar{\vee}\; r)$ are identical. Hence the formulae are logically identical.

Example 2: *Eliminating conditional and biconditional, find the logical equivalent forms of*

(i) $(\bar{q} \to \bar{p}) \to (p \to q)$

(ii) $p \leftrightarrow (\bar{p} \vee \bar{q})$

(Note that $\bar{q}$ is $\sim q$)

Solution: (i) $\bar{q} \to \bar{p} \equiv (\bar{\bar{q}} \vee \bar{p}) = q \vee \bar{p}$

$p \to q \equiv (\bar{p} \vee q)$

$\therefore \quad (\bar{q} \to \bar{p}) \to (p \to q)$

$\equiv (q \vee \bar{p}) \to (\bar{p} \vee q)$.

Let, $q \vee \bar{p} \equiv s$ and $\bar{p} \vee q \equiv t$

then we have $s \to t \equiv (\bar{s} \vee t)$

$\bar{s} \equiv \overline{(q \vee \bar{p})} \equiv \bar{q} \wedge \bar{\bar{p}}$ (by De Morgan's law)

$\equiv \bar{q} \wedge p$.

$\therefore \quad \bar{s} \vee t \equiv (\bar{q} \wedge p) \vee (\bar{p} \vee q)$

$\equiv (\bar{q} \vee \bar{p} \vee q) \wedge (p \vee \bar{p} \vee q)$

(by Distributive law)

$\equiv ((q \vee \bar{q}) \vee \bar{p}) \wedge ((p \vee \bar{p}) \vee q)$

$\equiv (T \vee \bar{p}) \wedge (T \vee q)$

$\equiv T \wedge T \equiv T.$

$\therefore$ The given form is logically equivalent to a tautology.

(ii) $p \leftrightarrow (\bar{p} \vee \bar{q})$

$\equiv (p \to (\bar{p} \vee \bar{q})) \wedge ((\bar{p} \vee \bar{q}) \to p)$

$\equiv (\bar{p} \vee (\bar{p} \vee \bar{q})) \wedge ((\overline{\bar{p} \vee \bar{q}}) \vee p)$

$\equiv ((\bar{p} \vee \bar{p}) \vee \bar{q}) \wedge ((\bar{\bar{p}} \wedge \bar{\bar{q}}) \vee p)$

$\equiv (\bar{p} \vee \bar{q}) \wedge ((p \wedge q) \vee p)$

$\equiv (\bar{p} \wedge ((p \wedge q) \vee p)) \vee (\bar{q} \wedge ((p \wedge q) \vee p))$

$\equiv (\bar{p} \wedge p \wedge q) \vee (\bar{p} \wedge p) \vee (\bar{q} \wedge p \wedge q) \vee (\bar{q} \wedge p)$

$\equiv (c \wedge q) \vee c \vee (c \wedge p) \vee (\bar{q} \wedge p)$

$\equiv c \vee c \vee c \vee \bar{q} \wedge p$

$\equiv p \wedge \bar{q}$

Example 3: *Eliminating conditional and biconditional, find logical equivalent forms of*

(i) $(p \leftrightarrow (q \vee r)) \to \bar{p}$

(ii) $((p \to q) \to q) \to p$

Solution: (i) Here we shall solve the problems, by using the truth table method, i.e. finding a form in dnf which is logically equivalent to the given form.

p	q	r	$\bar{p}$	$(q \vee r)$	$p \leftrightarrow (q \vee r)$	$(p \times (q \vee r)) \to \bar{p}$
T	T	T	F	T	T	F
T	T	F	F	T	T	F
T	F	T	F	T	T	F
T	F	F	F	F	F	T
F	T	T	T	T	F	T
F	T	F	T	T	F	T
F	F	T	T	T	F	T
F	F	F	T	F	T	T

Consider only the values of p, q, r corresponding to T in the last column of the truth table.

Then the logically equivalent form is

$(p \wedge \bar{q} \wedge \bar{r}) \vee (\bar{p} \wedge q \wedge r) \vee (\bar{p} \wedge q \wedge \bar{r}) \vee (\bar{p} \wedge \bar{q} \wedge r) \vee (\bar{p} \wedge \bar{q} \wedge \bar{r})$

(ii) $((p \to q) \to q) \to p$

Solution :

p	q	$p \to q$	$(p \to q) \to q$	$((p \to q) \to q) \to p$
T	T	T	T	T
T	F	F	T	T
F	T	T	T	F
F	F	T	F	T

The logically equivalent form is $(p \wedge q) \vee (p \wedge\ ) \vee (\wedge\ )$.

Example 4: *Prove that* $p \to (q \to r)$ *and* $(p \wedge \bar{r}) \to \bar{q}$ *are logically equivalent.*

Solution:

p	q	r	$\bar{q}$	$\bar{r}$	$(q \to r)\ p \to (q \to r)$	$(p \wedge \bar{r})$	$(p \wedge \bar{r}) \to \bar{q}$
T	T	T	F	F	T	F	T
T	T	F	F	T	F	T	F
T	F	T	T	F	T	F	T
T	F	F	T	T	T	T	T
F	T	T	F	F	T	F	T
F	T	F	F	T	T	F	T
F	F	T	T	F	T	F	T
F	F	F	T	T	T	F	T

The marked columns are identical. Hence the two forms are logically equivalent.

Example 5: *There are two restaurants next to each other. One has a sign that says "Good food is not cheap" and the other has a sign that says "Cheap food is not good". Are the signs saying the same thing?*

Solution: We shall show that the two statements are logically equivalent.

Let, p : Food is good.

q : Food is cheap.

Then the symbolic form of "Good food is not cheap" is $p \to \sim q$ and symbolic form of "Cheap food is not good" is $q \to \sim p$.

Now $p \to \sim q$ and $q \to \sim p$ are logically equivalent. Consider the truth table.

p	q	~p	~q	$p \to \sim q$	$q \to \sim p$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

Both the marked columns are identical. Hence the two forms are logically equivalent.

Example 6: *Show that* $p \,\bar{\vee}\, q$ *and* $(p \vee q) \wedge \sim (p \wedge q)$ *are logically equivalent.*

Solution: Consider the truth tables.

p	q	$p \,\bar{\vee}\, q$	$p \vee q$	$p \wedge q$	$\sim(p \wedge q)$	$(p \vee q) \wedge \sim(p \wedge q)$
T	T	F	T	T	F	F
T	F	T	T	F	T	T
F	T	T	T	F	T	T
F	F	F	F	F	T	F

Since the marked columns are identical $(p \,\bar{\vee}\, q)$ and $(p \vee q) \wedge \sim (p \wedge q)$ are logically equivalent.

Example 7: *Obtain the disjunctive normal form of*

(i) $(p \to q) \wedge (\sim p \wedge q)$,

(ii) $(p \wedge (p \to q)) \to q$.

Solution: (i) $p \to q$ is logically equivalent to $\sim p \vee q$

$\therefore\ (p \to q) \wedge (\sim p \wedge q) \equiv (\sim p \vee q) \wedge (\sim p \wedge q)$

$\equiv (\sim p \wedge \sim p \wedge q) \vee (q \wedge \sim p \wedge q)$

$\equiv (\sim p \wedge q) \vee (q \wedge \sim p).$

(ii) $(p \wedge (p \to q)) \to q \equiv \sim (p \wedge (\sim p \vee q)) \vee q$

$\equiv \sim p \vee \sim (\sim p \vee q) \vee q$

$\equiv \sim p \vee (p \wedge \sim q) \vee q$

Example 8: *Show that the following statement is tautological:*

$(p \wedge (p \to q)) \to q$

Solution: $(p \wedge (p \to q)) \to q \equiv \sim (p \wedge (\sim p \vee q)) \vee q$

$\equiv (\sim p \vee \sim (\sim p \vee q)) \vee q$

$\equiv (\sim p \vee (p \wedge \sim q)) \vee q$

$\equiv (\sim p \vee \sim q) \vee q$

$\equiv \sim p \vee T \equiv T.$

Example 9: *Obtain the conjunctive normal form and disjunctive normal form of the following formulae given below:*

(i) $p \wedge (p \to q)$

(ii) $\sim (p \vee q) \rightleftarrows (p \wedge q)$

Solution:

(i) $p \wedge (p \to q) \equiv p \wedge (\sim p \vee q)$ – cnf

$p \wedge (\sim p \vee q) \equiv (p \wedge \sim p) \vee (p \wedge q)$

$\equiv F \vee (p \wedge q)$

$\equiv (p \wedge q)$– dnf. (a single conjunct)

(ii) $\sim (p \vee q) \rightleftarrows (p \wedge q)$

$\equiv (\sim \sim (p \vee q) \vee (p \wedge q)) \wedge (\sim (p \wedge q) \vee \sim (p \vee q))$

$\equiv ((p \vee q) \vee (p \wedge q)) \wedge ((\sim p \vee \sim q) \vee (\sim p \wedge \sim q))$

$\equiv (p \vee q) \wedge ((\sim p \vee \sim q \vee \sim p) \wedge (\sim p \vee \sim q \vee \sim q))$

$\equiv (p \vee q) \wedge (\sim p \vee \sim q) \wedge (\sim p \vee \sim q)$

$\equiv (p \vee q) \wedge (\sim p \vee \sim q)$ – cnf

Further,

$(p \vee q) \wedge (\sim p \vee \sim q) \equiv ((p \vee q) \wedge \sim p) \vee ((p \vee q) \wedge \sim q)$

$\equiv (p \wedge \sim p) \vee (q \wedge \sim p) \vee ((p \wedge \sim q) \vee (q \wedge \sim q))$

$\equiv F \vee (q \wedge \sim p) \vee (p \wedge \sim q) \vee F$

$\equiv (q \wedge \sim p) \vee (p \wedge \sim q)$ – dnf.

Example 10: *Find the conjunctive normal form and disjunctive normal form for the following:*

(i) $(p \vee \bar{q}) \to q$ *(ii)* $p \leftrightarrow (\bar{p} \vee \bar{q})$

Solution:

(i) $(p \vee \bar{q}) \to q \equiv \overline{(p \vee \bar{q})} \vee q$

$\equiv (\bar{p} \wedge \bar{\bar{q}}) \vee q$

$\equiv (\bar{p} \wedge q) \vee q$ – dnf

$(\bar{p} \wedge q) \vee q \equiv (\bar{p} \vee q) \wedge (q \vee q)$

$\equiv (\sim p \vee q) \wedge q$ – cnf.

(ii) $p \times (\bar{p} \vee \bar{q}) \equiv (\bar{p} \vee (\bar{p} \vee \bar{q})) \wedge ((\overline{\bar{p} \vee \bar{q}}) \vee p)$

$\equiv (\bar{p} \vee \bar{p} \vee \bar{q}) \wedge ((\bar{\bar{p}} \wedge \bar{\bar{q}}) \vee p)$

$\equiv (\bar{p} \vee \bar{q}) \wedge (p \vee p) \wedge (q \vee p)$

$\equiv (\bar{p} \vee \bar{q}) \wedge p \wedge (q \vee p)$ – cnf

$\equiv ((\bar{p} \wedge p) \vee (\bar{q} \wedge p)) \wedge (q \vee p)$

$\equiv (F \vee (\bar{q} \wedge p)) \wedge (q \vee p)$

$= (\bar{q} \wedge p) \wedge (q \vee p)$

$\equiv (\bar{q} \wedge p \wedge q) \vee (\bar{q} \wedge p \wedge p)$

$\equiv (F \wedge p) \vee (\bar{q} \wedge p)$

$\equiv F \vee (\bar{q} \wedge p)$

$\equiv (\bar{q} \wedge p)$ – dnf

(Single conjunct)

Example 11: *Find the conjunctive and disjunctive normal forms for the following without using truth table.*

(i) $(p \rightarrow q) \wedge (q \rightarrow p)$

(ii) $((p \wedge (p \rightarrow q)) \rightarrow q)$

Solution: (i) $(p \rightarrow q) \wedge (q \rightarrow p) \equiv (\sim p \vee q) \wedge (\sim q \vee p)$ ... (cnf)

Further, using the distributive law on the above cnf, we have,

$((\sim p \vee q) \wedge \sim q) \vee ((\sim p \vee q) \wedge p) = (\sim p \wedge \sim q) \vee (q \wedge \sim q) \vee (\sim p \wedge p) \vee (q \wedge p)$

$= (\sim p \wedge \sim q) \vee (q \wedge p) \rightarrow$ dnf

$(\because p \wedge \sim p \equiv q \wedge \sim q)$

(ii) Solution in 7 (ii).

Example 12: *If p and q are false propositions, state whether $(p \vee q) \wedge (\sim p \vee \sim q)$ is true or false. Verify.*

Solution: The statement is false. Consider the truth table.

p	q	~p	~q	p ∨ q	~p ∨ ~q	(p ∨ q) ∧ (~p ∨ ~q)
F	F	T	T	F	T	F

Example 13: *Show that (i) $(p \wedge (\sim p \vee q)) \vee (q \wedge \sim (p \wedge q))$ is equivalent to q.*

(ii) $((p \vee \sim q) \wedge (\sim p \vee \sim q)) \vee q$ is a tautology.

Solution: (i) $p \wedge (\sim p \vee q) \equiv (p \wedge \sim p) \vee (p \wedge q)$

$\equiv c \vee (p \wedge q) \equiv p \wedge q$

Similarly, $q \wedge \sim (p \wedge q) \equiv q \wedge (\sim p \vee \sim q)$

$\equiv (q \wedge \sim p) \vee (q \wedge \sim q)$

$\equiv (q \wedge \sim p) \vee c$

$\equiv q \wedge \sim p$

Here, $(p \wedge (\sim p \vee q)) \vee (q \wedge \sim (p \wedge q))$

$\equiv (p \wedge q) \vee (q \wedge \sim p)$

$\equiv (p \vee \sim p) \wedge q$

$\equiv T \wedge q \equiv q$

(iii) $((p \vee \sim q) \wedge (\sim p \vee \sim q)) \vee q$

$\equiv (p \vee \sim q \vee q) \wedge (\sim p \vee \sim q \vee q)$

$\equiv (p \vee T) \wedge (\sim p \vee T)$

$\equiv T \wedge T \equiv T$

Example 14: *Obtain cnf of each of the following:*

(i) $p \wedge (p \rightarrow q)$

(ii) $\sim (p \vee q) \leftrightarrow (p \wedge q)$

(iii) $q \vee (p \wedge \sim q) \vee (\sim p \wedge \sim q)$

Solution: (i) $p \wedge (p \rightarrow q) \equiv (p \wedge (\sim p \vee q)$

$\equiv (p \wedge \sim p) \vee (p \wedge q) \equiv c \vee (p \wedge q)$

$\equiv (p \wedge q)$ – cnf

(ii) Solved Ex. 9

(iii) $q \vee (p \wedge \sim q) \vee (\sim p \wedge \sim q)$

$\equiv ((q \vee p) \wedge (q \vee \sim q) \vee (\sim p \wedge \sim q)$

$\equiv (q \vee p) \wedge T \vee (\sim p \wedge \sim q)$

$\equiv (q \vee p) \vee (\sim p \wedge \sim q)$

$\equiv (q \vee p \vee \sim p) \wedge (q \wedge p \vee \sim q)$

$\equiv (q \vee T) \wedge (p \vee q \vee \sim q) \equiv T \wedge (p \vee T) \equiv T \wedge T$

$\equiv T \equiv (p \vee \sim p)$ – cnf (single disjunct)

1.11 LOGICAL IMPLICATION

Let **A** and **B** be two statement forms. Then **A** logically implies **B**, denoted by **A** ⇒ **B** if **A** → **B** is a tautology.

In other words, whenever **A** is true, **B** should be true.

The following are some basic examples of logical implication.

Examples 1: *Find DNF of $((p \rightarrow q) \wedge (q \rightarrow p) \vee p$. Find CNF of $p \leftrightarrow (\sim p \vee \sim q)$.* **(Nov./Dec. 14)**

Solution : $((p \rightarrow q) \wedge (q \rightarrow p)) \vee p \equiv ((\sim p \vee q) \wedge (\sim q \vee p)) \vee p$

$\equiv (\sim p \wedge (\sim q \vee p)) \vee (q \wedge (\sim q \vee p)) \vee p$

$\equiv (\sim p \wedge \sim q) \vee (\sim p \wedge p) \vee (q \wedge \sim q) \vee (q \wedge p) \vee p$

$\equiv (\sim p \wedge \sim q) \vee F \vee F \vee (q \wedge p) \vee p \equiv (\sim p \wedge \sim q) \vee (q \wedge p) \vee p)$ (DNF)

$p \leftrightarrow (\sim p \vee \sim q) \equiv (p \rightarrow (\sim p \vee \sim q)) \wedge ((\sim p \vee \sim q) \rightarrow p)$

$\equiv (\sim p \vee (\sim p \vee \sim q)) \wedge (\sim (\sim (p \vee \sim q) \vee p)$

$\equiv (\sim p \vee \sim q) \wedge ((p \wedge q) \vee p)$

$\equiv (\sim p \vee \sim q) \wedge (p \vee p) \wedge (q \vee p)$

$\equiv (\sim p \vee \sim q) \wedge p \wedge (p \vee q)$ (CNF)

SOLVED EXAMPLES

Example 1 : *Show that $(p \wedge q) \Rightarrow (p \rightarrow q)$*

Solution: Consider the truth table

p	q	p ∧ q	p → q	(p ∧ q) → (p → q)
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

Hence $(p \wedge q) \Rightarrow (p \rightarrow q)$.

Example 2: *Show that $(p \rightarrow q) \Rightarrow p \rightarrow (p \wedge q)$ without constructing truth table.*

Solution:

We have to show that if $p \rightarrow q$ is true, then so is $p \rightarrow (p \wedge q)$.

Suppose this is not the case.

Then this means that $p \rightarrow (p \wedge q)$ has truth value F, i.e. p is T and $p \wedge q$ is F.

This can happen only when q is F.

But if q is F, then so is p, since we assume $p \rightarrow q$ is T.

This contradicts the truth value of p which is T.

Hence $p \rightarrow (p \wedge q)$ is true. Hence, this is the logical implication.)

Example 3 : *Prove that*

$(p \rightarrow (q \rightarrow r)) \Rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$.

Solution: Consider the truth table:

p	q	r	q → r	p → (q → r)	p → q	p → r	((p → q) → (p → r))
T	T	T	T	T	T	T	T
T	F	T	T	T	F	T	T
T	T	F	F	F	T	F	F
T	F	F	T	T	F	F	T
F	T	T	T	T	T	T	T
F	F	T	T	T	T	T	T
F	T	F	F	T	T	T	T
F	F	F	T	T	T	T	T

Since the given statement forms have the same truth values, each logically implies the other, i.e. they are in fact equivalent forms.

EXERCISE - 1.2

1. Show that the truth values of the following forms are independent of their components:
   (i) $(p \wedge (p \rightarrow q)) \rightarrow q$
   (ii) $(p \leftrightarrow q) \leftrightarrow ((p \wedge q) \vee (\sim p \wedge \sim q))$
2. Determine which of the forms given below are tautologies, contradictions or neither.
   (i) $(p \rightarrow \sim p) \rightarrow \sim p$
   (ii) $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$
   (iii) $(p \vee q) \wedge \sim p \rightarrow q$
   (iv) $(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$
   (v) $(p \wedge q) \leftrightarrow p$.
3. Show that $p \rightarrow (q \rightarrow r)$ and $p \rightarrow (\sim q \vee r)$ are logically equivalent.
4. Show that $p \leftrightarrow q$ and $(p \wedge q) \vee (\sim p \wedge \sim q)$ are logically equivalent.
5. Find the dnf of $(q \rightarrow p) \wedge (\sim p \wedge q)$.
6. Find the dnf of $(p \rightarrow (q \wedge r)) \wedge (\sim p \rightarrow (\sim p \wedge \sim r))$ by truth table method.
7. Obtain cnf of (i) $(\sim p \rightarrow r) \wedge (p \rightarrow q)$, (ii) $(p \wedge q) \vee (\sim p \wedge q \wedge r)$
8. Obtain dnf of: (i) $(p \rightarrow q) \wedge (\sim p \wedge q)$, (ii) $(p \wedge (p \rightarrow q)) \rightarrow q$.
9. Show that $(p \rightarrow (q \rightarrow r))$ logically imply $(p \rightarrow q) \rightarrow (p \rightarrow r)$.
10. Without constructing truth tables, show that
    (i) $(p \rightarrow q) \rightarrow q \Rightarrow p \vee q$
    (ii) $(p \wedge q) \Rightarrow (p \rightarrow q)$
11. (i) Show that: $(p \wedge (p \rightarrow q)) \rightarrow q$.
    (ii) $(p \rightarrow q) \wedge \sim q \rightarrow \sim p$ are tautologies, without using truth table.
12. Obtain the conjunctive and disjunctive normal forms.
    (i) $p \wedge (p \rightarrow q)$
    (ii) $(p \vee \sim q) \rightarrow q$
13. Show that the following statements are tautological.
    (i) $(p \wedge (p \rightarrow q)) \rightarrow q$
    (ii) $(p \rightarrow q) \leftrightarrow (q \vee \sim p)$
14. Find the dnf of:
    (i) $(p \rightarrow q) \wedge (\sim p \wedge q)$
    (ii) $(p \rightarrow (q \wedge r)) \wedge (\sim p \rightarrow (\sim p \wedge \sim r))$ by truth table method.

15. A computer has been built to answer any yes or no question but it has been programmed either to answer all questions truthfully or to give incorrect answers to all questions. If we wish to find out whether Fermat's last Theorem is true, what question should we put to the computer, so that it will give a correct answer.

1.12 METHODS OF PROOF

- Whenever an assertion is made, which is claimed to be true, one has to state an argument, which establishes the truth of the assertion.
- In Mathematics, we prove theorems; a theorem being a mathematical assertion which is shown to be true. A proof consists of a sequence of statements. Some of these statements may be axioms (universal truths), some may be previously proved theorems and other statements may be hypothesis (assumed to be true). To construct a proof, we need to derive new assertions from existing ones. This is done using Rules of **Inference.** Before we discuss some of these rules of inference, let us formally define a **valid argument**.

Definition:

A valid argument is a finite sequence of statements $p_1, p_2, \ldots, p_n$, called **as premises** together with a statement C called the **conclusion** such that $p_1 \wedge p_2 \wedge \ldots \wedge p_n \rightarrow C$ is a tautology.

This concept is used in the following methods.

1. Modus Ponens (Law of Detachment):

This rule is presented in the following form

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

The assertions above the horizontal line are called premises (or hypothesis).

The assertion below the line is called the conclusion.

This rule constitutes a valid argument since $(p \wedge (p \rightarrow q)) \rightarrow q$ is a tautology.

Example: If Suresh gets a first class, he will get a job easily. Suresh gets a first class. Therefore he will get a job easily.

Let, p : Suresh gets a first class.

q : Suresh will get a job easily.

Then the premises are $p \rightarrow q$ and p; the conclusion is q.

The inferential form is thus

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$$

Hence this form of argument is valid.

2. Modus Tollens (Law of Contraposition):

This rule is presented in the following form

$$\begin{array}{l} p \rightarrow q \\ \sim q \\ \hline \therefore \sim p \end{array}$$

This argument is valid since $(p \rightarrow q) \wedge \sim q \rightarrow \sim p$ is a tautology.

Example: If Suresh gets a first class, he will get a job. Suresh does not get a job. Therefore Suresh does not get a first class.

Let, p : Suresh gets a first class.

q : Suresh gets a job.

Then the inferential form is

$$\begin{array}{l} p \rightarrow q \\ \sim q \\ \hline \therefore \sim p \end{array}$$

Hence the above argument is valid.

3. Disjunctive Syllogism: The rule of inference is presented in the form

$$\begin{array}{l} p \vee q \\ \sim p \\ \hline \therefore q \end{array}$$

Note that $(p \vee q) \wedge \sim p \rightarrow q$ is a tautology ($\sim p$ is T, $p \vee q$ is T implies q is T)

Example: Either it is raining or it is windy. It is not raining. Therefore it is windy.

Let, p : It is raining.

q : It is windy.

Then the inferential form is

$$\begin{array}{l} p \vee q \\ \sim p \\ \hline \therefore q \end{array}$$

Hence the argument is valid.

4. Hypothetical Syllogism: Rule is in the form

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

Note that $(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$ is a tautology.

Hence the above argument is a valid argument.

Example: If Suresh studies hard, he will obtain a first class. If he obtains a first class, he will get a good job. Therefore if Suresh studies hard, he will get a good job.

Let, p : Suresh studies hard.

q : He obtains a first class.

r : He will get a good job.

Hence the inferential form is

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

Hence the argument is valid.

SOLVED EXAMPLES

Example 1: *Determine whether the following is a valid argument:*

If Geeta goes to class, she is on time.

But Geeta is late.

She will therefore miss class.

Solution: Let, p : Geeta goes to class.

q : Geeta is on time.

The rule of inference is

$$\begin{array}{l} p \rightarrow q \\ \sim q \\ \hline \therefore \sim p \end{array}$$

This is the law of contraposition or Modus tollens. Hence the argument is valid.

Example 2: *I am happy if my program runs. A necessary condition for the program to run is it should be error free. I am not happy. Therefore the program is not error free.*

Solution: Let, p : I am happy.

q : My program runs.

r : It should be error free.

Then the argument is

$q \rightarrow p$

$q \rightarrow r$

$$\begin{array}{l} \sim p \\ \hline \therefore \sim r \end{array}$$

Consider the following assignment of truth values to p, q, r. Let the truth values of p, q, r be F, F, T respectively. Then $q \rightarrow p$ is T, $q \rightarrow r$ is T and $\sim p$ is T. But the conclusion $\sim r$ is F. Hence the above argument is invalid.

Example 3: *If today is Tuesday, then there is a test in Computer Science or in Discrete Mathematics. If the Discrete Mathematics professor is sick, there will be no test in Discrete Mathematics. Today is Tuesday and the professor of Discrete Mathematics is sick. Hence there will be a test in Computer Science.*

Solution: Let, p : Today is Tuesday.

q : There is a test in Computer Science.

r : There is a test in Discrete Mathematics.

s : Discrete Mathematics professor is sick.

Argument is

$$\begin{array}{l} p \rightarrow (q \vee r) \\ s \rightarrow \sim r \\ p \wedge s \\ \hline \therefore q \end{array}$$

Wherever $p \rightarrow (q \vee r)$, $s \rightarrow \sim r$, $p \wedge s$ are true, q is also true. This is because

$p \wedge s$ is T implies p is T, s is T. s is T implies $\sim r$ is T, $\therefore$ r is F.

$\therefore$ $p \rightarrow (q \vee r)$ is T implies q is T.

Hence the argument is valid.

Example 4: *Ramesh is studying ORACLE or he is not studying JAVA. If Ramesh is studying JAVA, then he is not studying ORACLE. Therefore he is studying ORACLE.*

Write the above statement in symbolic form and test the validity of the argument using laws of logic.

Solution: Let, p : Ramesh is studying ORACLE.

q : Ramesh is studying JAVA.

Argument is

$$\begin{array}{l} p \vee \sim q \\ \underline{q \rightarrow \sim p} \\ \therefore \quad p \end{array}$$

Argument is invalid, since consider the following assignment of truth values to p and q. Let the truth values of p and q be both F respectively. Then $p \vee \sim q$ has truth value T and $q \rightarrow \sim p$ has truth value T. Hence for validity of the argument p should have truth value T, which is not the case.

Example 5: *Show that the following premises are inconsistent.*

(i) *If Jack misses many classes through illness, then he fails high school.*

(ii) *If Jack fails high school, then he is uneducated.*

(iii) *If Jack reads a lot of books, then he is not uneducated.*

(iv) *If Jack misses many classes through illness, then he reads a lot of books.*

Solution: Let, p : Jack misses many classes through illness

q : Jack fails high school

r : Jack is uneducated

s : Jack reads a lot of books.

Then the premises are:

$$p \to q,$$
$$q \to r,$$
$$s \to \sim r,$$
$$p \to s.$$

Consider the following assignment of truth values to p, q, r and s.

p	q	r	s
T	T	T	F

Then $p \to q$ is T,

$q \to r$ is T,

$s \to \sim r$ is T.

ut $p \to s$ is F.

Hence, the premises are inconsistent.

Example 6: *Determine the validity of the argument given:*

S_1 : If I like Mathematics then I will study

S_2 : Either I will study or I will fail

∴ S : If I fail then I do not like Mathematics

Solution: Let, p : I like Mathematics

q : I will study

r : I will fail.

S_1 : $p \to q$

S_2 : $q \vee r$

S : $r \to \sim p$

For validity, $S_1 \wedge S_2$ should logically imply S.

Assign the truth values T, T, T to p, q, r respectively. Then S_1 is T, S_2 is T but S is false. Hence, the argument is invalid.

Example 7: *Test the validity of the argument: If a person is poor, he is unhappy. If a person is unhappy, he dies young, therefore poor person dies young.*

Solution: Let, p : Person is poor

q : Person is unhappy

r : Person dies young

In symbolic form the argument is:

S_1 : $p \to q$,

S_2 : $q \to r$,

S : $p \to r$

The above argument is the value of **Hypothetical Syllogism**. Hence it is valid.

Alternatively, let $p \to r$ be F, then p is T and r is F. If $p \to q$ is T, then q is T. But $q \to r$ is F. Hence, conclusion is false implies one of the premises is false. Therefore the argument is valid.

Example 8: *Determine whether the argument given is valid or not.*

If I try hard and I have talent, then I will become a musician.

If I become a musician, then I will be happy.

Therefore, if I will not be happy, then I did not try hard or I do not have talent.

Solution: Let, p : I try hard

q : I have talent

r : I will become a musician

s : I will be happy

The argument is then put in symbolic form as:

S_1 : $(p \wedge q) \to r$

S_2 : $r \to s$

∴ S : $\sim s \to \sim p \vee \sim q$

Suppose argument is invalid. This means that for some assignment of truth values, S_1 is T, S_2 is T, but S is F. S will have truth value F is $\sim s$ is T, and $\sim p \vee \sim q$ is F, i.e. s is F, p is T and q is T. Since we have assumed S_2 to be true, the truth values of r and s are both F. since S_1 is also, by assumption, true, r is F implies either p or q is F. This is a contradiction, since by assumption both p and q are T.

The given argument is valid.

1.13 PREDICATES

Consider the following sentences:

1. "x is tall and handsome".
2. "x + 3 = 5"
3. "x + y ≥ 10".

These sentences are not propositions, since they do not have any truth value. However, if values are assigned to the variables, each of them becomes a proposition, which is either true or false. For example, the above sentences can be converted into

1. "He is tall and handsome".
2. "2 + 3 = 5" (true statement)
3. "2 + 5 ≥ 10" (false statement)

Hence we have the following definition.

Definition:

An assertion that contains one or more variables is called a **predicate;** its truth value is predicated after assigning truth values to its variables.

A predicate P containing n variables $x_1, x_2, \ldots, x_n$ is called an **n-place** predicate.

Examples (i) and (ii) are one-place predicates while Example (iii) is a 2-place predicate.

If we want to specify the variables in a predicate, we denote the predicate by $P(x_1, x_2, \ldots x_n)$. Each variable x_i is also called as an **argument.**

For example,

(i) "x is a city in India" is denoted by P (x).

(ii) "x is the father of y" is denoted by P (x, y).

(iii) "x + y ≥ z" is denoted by P (x, y, z).

The values which the variables may assume constitute a collection or (a) set called as the **universe of discourse.**

When we specify a value for a variable appearing in a predicate, we **bind** that variable.

A predicate becomes a proposition only when all its variables are bound.

Consider the following examples:

1. P (x): x + 3 = 5.

Let the universe of discourse be the set of all integers. Binding x by putting x = – 1, we get a false proposition. Binding x by putting x = 2, we get a true proposition.

2. P (x, y): x + y = 10. Let the universe of discourse be the set of natural numbers.

Putting x = 1, we get the one-place predicate P (1, y): 1 + y = 10. Further setting y = 10, we obtain the proposition P (1, 9) which is true. However, if we set y = 10, P (1, 10) is a false proposition. In each case, we have bound both the variables (x by 1, y by 9 and y by 10).

A second method of binding individual variables in a predicate is by **quantification** of the variable.

1.13.1 Universal Quantifier

If P(x) is a predicate with the individual variable x as an argument, then the assertion "For all x, P(x)" which is interpreted as "For all values of x, the assertion P(x) is true", is a statement in which the variable x is said to be **universally quantified.**

We denote the phrase "For all" by ∀, called the **universal quantifier.** The meaning of ∀ is "for all" or "For every" or "For each".

If P(x) is true for every possible value of x, then **∀ x P(x)** is true; otherwise ∀ x P(x) is false.

Example: Let P(x) be the predicate "x ≥ 0"; where x is any positive integer. Then the proposition ∀ x P(x) is true. However, if x is any real number, then ∀ x P(x) is a false proposition.

1.13.2 Existential Quantifier

Suppose for the predicate P(x), ∀ x P(x) is false, but there exists at least one value of x for which P(x) is true, then we say that in this proposition, x is bound by **existential quantification.**

We denote the words "there exists" by the symbol ∃.

Then the notation **∃ x P(x)** means "there exists a value of x (in the universe of discourse) for which P(x) is true."

Example: Let P(x) be the predicate "x + 3 = 5"and let the universe of discourse be the set of all integers. Then the proposition ∃ x P(x) is true (by setting x = 2) but ∀ x P (x) is false.

Let P (x, y) be a two-place predicate, then

∃ x ∀ y P(x, y) is the proposition "There exists a value of x such that for all values of y, P(x, y) is true".

∀ y ∃ x P(x, y) is the proposition "For each value of y, there exists an x such that P(x, y) is true."

∃ x ∃ y P (x, y) is the proposition "There exist a value of x and a value of y such that P (x, y) is true."

∀ x ∀ y P(x, y) is the proposition "For all values of x and y, P(x, y) is true".

Example: Consider the universe as the set of all integers. Let P(x, y) denote the predicate x + y = 10.

1. Then the symbolic statement ∀x ∃y P(x, y) is interpreted as "For every integer x, there exists an integer y such that x + y = 10 (i.e. y = 10 – x)".

2. Now consider the statement $\exists y\ \forall x\ P(x, y)$. This statement is read as "there exists an integer y so that for all integers x, x + y = 10". This statement is of course false.
3. The statement $\exists x\ \exists y\ P(x, y)$ is read as "there exist integers x and y such that x + y = 10". This is a true statement.

1.13.3 Negation of a Quantified Statement

Consider the statement $\forall x\ P(x)$. Its negation is "it is not the case that for all x, P (x) is true". This means that for some x = a, P(a) is not true, or in other words there exists an x such that $\sim P(x)$ is true. Hence $\forall x\ P(x)$ is logically equivalent to $\exists x\ [\sim P(x)]$.

Example: Consider the example" All the invited guests were present for the dinner.

The negation is: "All the invited guests were not present for the dinner, equivalently".

Some guests were not present for the dinner, i.e. $\exists x\ (\sim P(x))$; where,

x : x is a guest

P(x) : x was present for the dinner.

Example: Consider the statement "There is a student in this class, who is not familiar with C programming". The negation of the above statement is "All students in this class are familiar with C programming".

Hence symbolically, $\exists x\ (\sim P(x))$ and $\forall x\ (P(x))$ are logically equivalent.

We summarize these results in the following table.

Statement	Negation
$\forall x\ (P(x))$	$\exists x\ (\sim P(x))$
$\exists x\ (\sim P(x))$	$\forall x\ (P(x))$
$\forall x\ (\sim P(x))$	$\exists x\ (P(x))$
$\exists x\ P(x)$	$\forall x\ (\sim P(x))$

1.14 RULES OF INFERENCE FOR PREDICATES

In this section, we consider rules of inference, to test the validity of arguments, involving predicates, similar to those we considered for arguments involving propositions.

We have the famous lines:

"All men are mortal,

Socrates is a man.

Therefore Socrates is mortal".

This is a universally quantified statement, about an attribute or property that is of mortality, over the universe of men. The argument is that Socrates, being a member of the set, satisfies the attribute.

This form of argument is called as **universal instantiation**, as we are taking a particular instance of a general (universal statement).

The above type of statement can be formulated in symbolic language as: $\dfrac{\forall x\ P(x)}{\therefore\ P(a)}$

where x is an element of the universe, P(x) the attribute of x and a, a particular member of the universe.

Next, we have a second type of statement, as illustrated by the following example:

2 is a square-free integer. Therefore, there are integers, that are square free.

In symbolic form, we can express the above statement as:

$$\frac{P(a)}{\therefore\ \exists x\ P(x)}$$

Such a type of statement is called as **Existential generalization** that is making a general statement, from a particular instance.

The rule of universal instantiation leads to the following rules:

1. Universal Modus Ponens:	$\forall x\ (P(x) \rightarrow Q(x))$ $\underline{P(a)}$ $\therefore\ Q(a)$
2. Universal Modus Tollens:	$\forall x, (P(x) \rightarrow Q(x))$ $\underline{\sim Q(a)}$ $\therefore\ \sim P(a)$

SOLVED EXAMPLES

***Example 1:** All law-abiding citizens obey the traffic rules.*

Solution: Mr. Joshi is a law-abiding citizen. Therefore he obeys the traffic rules.

Translate into symbolic language, we have,

$$\forall x\ (L(x) \rightarrow T(x))$$
$$\frac{L(a)}{\therefore\ T(a)}$$

where the universe of discourse is citizens, $L(x) \equiv$ x is a law-abiding citizen,

$T(x) \equiv$ x obeys the traffic rules.

***Example 2:** All the first year students know C-programming Manisha is a first year student. Therefore, Manisha knows C-programming.*

Solution: In symbolic form, the statements:

$$\forall x\ (C(x) \rightarrow P(x))$$
$$\frac{C(a)}{\therefore\ P(a)}$$

***Example 3:** All cats like milk. Timmy does not like milk. Therefore, Timmy is not a cat.*

Solution: Symbolically,

$$\forall x\ (C(x) \rightarrow M(x))$$
$$\frac{\sim M(a)}{\therefore\ \sim C(a)}$$

Apart from these standard forms, we test the validity of quantified forms of argument, along similar lines as we did in the case of propositional arguments.

***Example 4:** Determine the validity of the following argument:*

S_1 : All my friends are musicians

S_2 : John is my friend.

S_3 : None of my neighbours are musicians.

S : John is not my neighbour.

Solution: Let the universe of discourse be the set of people.

Let, F(x) : x is my friend

M(x) : x is a musician

N(x) : x is my neighbour

Then argument in symbolic form is

S_1 : $\forall x\ (F(x) \rightarrow M(x))$

S_2 : F(a) (a = John)

$\underline{S_3 : \forall x\ (N(x) \rightarrow \sim M(x))}$

$\therefore$ S : $\sim N(a)$

Suppose $\sim N(a)$ has value F, i.e. N(a) is T. Since S_3 is T, we must have $\sim M(a)$ is T or M(a) is F. But S_1 is T, hence we must have F(a) to be false, but this is a contradiction. Hence, if S is false, either of S_1 or S_3 should be false. Hence argument is valid.

***Example 5:** Babies are illogical. Nobody is despised who can manage a crocodile. Illogical persona is despised. Therefore, babies cannot manage crocodiles.*

Solution: Let a be the domain of people.

Let, B(x) : x is a baby.

L(x) : x is logical

D(x) : x is despised.

C(x) : x can manage a crocodile.

Statement is as follows:

S_1 : $\forall x\ [B(x) \rightarrow \sim L(x)]$

S_2 : $\forall x\ [C(x) \rightarrow \sim D(x)]$

S_3 : $\forall x\ [\sim L(x) \rightarrow D(x)]$

S $\therefore$ $\forall x\ [B(x) \rightarrow \sim C(x)]$

Suppose $\forall x\ [B(x) \rightarrow C(x)]$ is F.

This means $\exists x$ = a, such that $\therefore$ B(a) is T and $\sim C(a)$ is F, i.e. C(a) is T. For S_2 to be T, this means that $\sim D(a)$ is T or D(a) is F. For S_3 to be T, this mean that $\sim L(a)$ is F or L(a) is T. But then B(a) is F, a contradiction. Hence argument is valid.

***Example 6:** Suppose the universe of discourse is the set of integers. Let P (x, y) be the predicate x − y = 0. Indicate which of the propositions are true and which are false.*

(i) P (2, 3)

(ii) P (3, 3)

(iii) $\forall x\ \exists y\ P(x, y)$

(iv) $\exists x\ \forall y\ P(y, x)$

Solution:

(i) P (2, 3) is false.

(ii) P (3, 3) is true.

(iii) For each value of x, set y = x. i.e. if x = 0, y = 0, if x = 1, y = 1 etc.

Then P (x, y) is true.

Hence $\forall x\ \exists y\ P(x, y)$ is true.

(iv) In this case, there should be a specific value $x = x_0$ such that P (y, x_0) is true, no matter what the value of y is. This means $x_0 - y = 0$ for any value of y, obviously false.

***Example 7:** Transcribe the following into logical notation. Let the universe of discourse be the real numbers.*

(i) For any value of x, x^2 is non-negative.

(ii) For every value of x, there is some value of y such that $x \cdot y = 1$.

(iii) There are positive values of x and y such that $x \cdot y > 0$.

(iv) There is a value of x such that if y is positive, then x + y is negative.

(v) For every value of x, there is some value of y such that $x - y = 1$.

Solution:

(i) $\forall x\, [x^2 \geq 0]$

(ii) $\forall x\, \exists y\, [x \cdot y = 1]$

(iii) $\exists x\, \exists y\, [(x > 0) \wedge (y > 0) \wedge (x \cdot y > 0)]$

(iv) $\exists x\, \forall y\, [(y > 0) \rightarrow (x + y < 0)]$

(v) $\forall x\, \exists y\, [x - y = 1]$.

Example 8: *Negate the following in such a way that the symbol ~ does not appear outside the square brackets.*

(i) $\forall x\, [x^2 \geq 0]$

(ii) $\exists x\, [x \cdot 2 = 1]$

(iii) $\forall x\, \exists y\, [x + y = 1]$

(iv) $\forall x\, \forall y\, [(x > y) \rightarrow (x^2 > y^2)]$.

Solution:

(i) Negation of the statement is "There is a value of x such that $x^2 < 0$".

Hence Negation of the statement in logical notation is $\exists x\, [x^2 < 0]$.

(ii) Negation is "For all values of x, $x \cdot 2 \neq 1$".

Hence the required form is $\forall x\, [x \cdot 2 \neq 1]$.

(iii) Negation is "There is a value of x such that for all values of y, $x + y \neq 1$". Hence the required form is $\exists x\, \forall y\, [x + y \neq 1]$.

Example 9: *Write the following statements in symbolic form, using quantifiers.*

(i) All students have taken a course in communication skills.

(ii) There is a girl student in the class who is also a sports person.

(iii) Some students are intelligent, but not hardworking.

Solution:

(i) Let P(x): Student x has taken a course in communication skills.

Then the statement can be written as $\forall x\, P(x)$.

(ii) Let, P(x) : x is a student

Q(x) : x is a girl

R(x) : x is a sports person.

The statement is interpreted as x is a student, x is a girl and x is a sports person. Hence this statement can be written as $\exists x\, [P(x) \wedge Q(x) \wedge R(x)]$.

(iii) There exists an x such that x is intelligent, but x is not hardworking. Here in symbolic form the statement is $\exists x\, [P(x) \wedge \sim Q(x)]$.

Example 10: *For the universe of all integers, let P(x), Q(x), R(x), S(x) and T(x) be the following statements:*

P(x) : x > 0

Q(x) : x is even

R(x) : x is a perfect square

S(x) : x is divisible by 4

T(x) : x is divisible by 5.

Write the following statements in symbolic form:

(i) At least one integer is even.

(ii) There exists a positive integer that is even.

(iii) If x is even, then x is not divisible by 5.

(iv) No even integer is divisible by 5.

(v) There exists an even integer divisible by 5.

(vi) If x is even and x is a perfect square, then x is divisible by 4.

Solution:

(i) $\exists\, Q(x)$

(ii) $\exists x\, [P(x) \wedge Q(x)]$

(iii) $\forall x\, [Q(x) \rightarrow \sim T(x)]$

(iv) $\forall x\, [Q(x) \rightarrow \sim T(x)]$

(v) $\exists x\, [Q(x) \wedge T(x)]$

(vi) $\forall x\, [Q(x) \wedge R(x) \rightarrow S(x)]$.

In the above example, determine the truth table of each statement.

Solution: (i), (ii), (iv) and (vi) are true statements. Statements (iii) and (v) are false.

Example 11: *Rewrite the following statements using quantifier variables and predicate symbols.*

(i) All birds can fly.

(ii) Not all birds can fly.

(iii) Some men are genius.

(iv) Some numbers are not rational.

(v) There is a student who likes Mathematics but not Geography.

(vi) Each integer is either even or odd.

Solution: (i) Let, B(x) : x is a bird.

F(x) : x can fly.

Then the statement can be written as: $\forall x\, [B(x) \rightarrow F(x)]$.

(ii) $\exists x\, [B(x) \wedge \sim F(x)]$ or equivalently

$\sim [\forall x\, (B(x) \rightarrow F(x))]$.

(iii) M(x) : x is a man

G(x) : x is a genius.

Statement is interpreted as $\exists x\, [M(x) \wedge G(x)]$.

(iv) N(x) : x is a number

R(x) : x is rational

Statement is in symbolic form, $\exists x\, [N(x) \wedge \sim R(x)]$ or equivalently $\sim [\forall x\, (N(x) \rightarrow R(x))]$

(v) S(x) : x is a student

M(x) : x likes Mathematics

G(x) : x likes Geography

Statement is: $\exists x\, [S(x) \wedge M(x) \wedge \sim G(x)]$.

(vi) I(x) : x is an integer.

E(x) : x is even

O(x) : x is odd.

Statement in symbolic form: $\forall x\, [I(x) \rightarrow E(x) \vee O(x)]$.

Example 12: *Negate each of the following statements:*

(i) $\forall x, |x| = x$

(ii) $\exists x,\ x^2 = x$. **(Dec. 15; May 15)**

Solution:

(i) $\exists x,\ |x| \neq x$,

(ii) $\forall x,\ x^2 \neq x$.

Example 13: *Determine the truth value of each of the statement and negate every statement.*

(i) $\exists x,\ x + 2 = x$.

(ii) $\forall x,\ x + 1 > x$.

Solution:

(i) F, $\forall x,\ x + 2 \neq x$,

(ii) T, $\exists x,\ x + 1 \leq x$.

1.15 TECHNIQUES OF THEOREM PROVING

The properties of quantifiers that we have studied in the previous section, help us to evolve techniques for theorem proving. The word theorem at once reminds us of some of the most famous theorems in mathematics such as Pythagoras Theorem, Binomial Theorem etc.

A theorem is nothing but an exercise in logical reasoning or argument. It consists of two distinct parts - hypothesis and conclusion. Hypothesis is a set of statements, which if true, implies the conclusion. Hence a theorem is usually in the form of a conditional or biconditional statement form.

Example 1: If a function is differentiable, then it is continuous.

Example 2: For real numbers x, y; $x^2 = y^2 \leftrightarrow x = \pm y$.

However, there are also many theorems, which are not strictly in the form of conditional or biconditional.

Example 3: $\sqrt{2}$ is an irrational number.

Example 4: Every positive integer is a product of prime numbers.

In what follows, we shall consider some typical examples, which illustrate some of the techniques of theorem proving.

An important principle applied in theorem proving is the **Rule of universal specification**, which states that if P(x) is a statement for a given universe, and if $\forall x\, P(x)$ is true, then P(a) is true for each a in the universe.

The following example illustrates this idea.

All men are mortal.

Socrates is a man.

Therefore Socrates is mortal.

In the above argument, if

P(x) : x is a man

Q(x) : x is mortal,

then let s represent the particular person Socrates, then the argument in symbolic form is

$$\begin{array}{l} \forall x\, [P(x) \rightarrow Q(x)] \\ \underline{P(s)} \\ \therefore\ Q(s) \end{array}$$

Observe that in the above example we have used the rule of universal specification in conjunction with Modus Ponens (or the rule of detachment).

Another example in similar vein. All integers are rational numbers.

π is not a rational number. Therefore π is not an integer.

In symbolic form, we present the argument as:

Let, $P(x)$: x is an integer

$Q(x)$: x is a rational number.

Let a represent π.

Then the argument is $\forall x, (P(x) \rightarrow Q(x))$

$$\begin{array}{l} \underline{\sim Q(a)} \\ \therefore \quad \sim P(a) \end{array}$$

In the above argument we have combined the rule of universal specification together with Modus Tollens, and hence the argument is valid.

Let us now turn our attention to some theorems in mathematics.

Theorem 1:	If n is divisible by 6, then n^2 is divisible by 4.
Proof:	Since n is divisible by 6, it is divisible by 2, 2 being a factor of 6. **Hence n^2 is divisible by 4.**
Theorem 2:	Let n be an integer. Prove that $7n^2 + 5n - 4$ is even.
Proof:	We consider the following cases.
Case I:	n is even. Since the product of an even integer and any integer is even, $7n^2$ and $5n$ are even. Hence, $7n^2 + 5n - 4$ is even.
Case II:	n is odd. The product of odd integers is odd and sum of two odd integers is even. Hence $7n^2 + 5n$ is even and since sum of even integers being even, $7n^2 + 5n - 4$ is even.
Theorem 3:	If $n^2 + 1 = 2m$, for integers m, n; then m is the sum of squares of two integers.
Proof:	Since $n^2 + 1 = 2m$, this implies that $n^2 + 1$ is even. Hence n^2 must be odd, which in turn implies that n is odd. Let $n = 2k + 1$ for an integer k. Then $n^2 + 1 = (2k+1)^2 + 1 = 2m$, i.e. $4k^2 + 4k + 1 = 2m$ Hence, $m = 2k^2 + 2k + 1 = (k+1)^2 + k^2$. Thus the theorem is proved.

The above examples are illustrations of **Direct proofs**. However, in many cases, a direct proof is rather difficult; hence an indirect approach of proving the **contrapositive** yields the result immediately. Recall that by contrapositive method we mean that instead of proving "$P \rightarrow Q$" is true, its logical equivalence "$\sim Q \rightarrow \sim P$" is proved to be true.

Theorem 4:	If I is a non-zero rational number, and m is an irrational number, then Im is irrational.
Proof:	Let, P: I is a non-zero rational number Q:Im is irrational We prove the contrapositive, i.e. "$\sim Q \rightarrow \sim P$" is true. Suppose Im is not irrational; then Im is rational. Hence, we can express Im as $Im = \frac{a}{b}$, where $a \neq 0$, $b \neq 0$. Further I being non-zero rational, $I = \frac{c}{d}$, $c \neq 0$, $d \neq 0$. Hence, $Im = \frac{a}{b}$ gives $m = \frac{a}{Ib} = \frac{ad}{bc}$, where $ad \neq 0$, $bc \neq 0$. But this contradicts the fact that m is irrational. Hence "$\sim Q \rightarrow \sim P$" is true, i.e. "$P \rightarrow Q$" is true.
Theorem 5:	For all positive real numbers x, y, if the product xy is greater than 49, then $x > 7$ or $y > 7$.
Proof:	Consider the negation of the conclusion, $x > 7$ or $y > 7$, that is, suppose $0 < x \leq 7$ and $0 < y \leq 7$. This would imply that $0 < x \cdot y \leq 7.7 = 49$. Hence the product does not exceed 49.

Similar to the method of indirect proof is the method of contradiction. This is illustrated in the following examples.

Theorem 6:	There are infinitely many prime numbers.
Proof:	Let us assume the contrary. Hence let $p_1, p_2, \ldots, p_n$ be exhaustive list of all prime numbers, n in number. Let $p_1 < p_2 < \ldots < p_n$. Consider $P = p_1 p_2 \ldots p_n + 1$ P is not a prime number since $P > p_i \ \forall\ 1 \leq i \leq n$. Hence P must have prime factors. Let p_k be a prime factor of P. Then p_k divides $(P - p_1 p_2 \ldots p_n)$, i.e. p_k divides 1, which is absurd. Hence p_k is not any one of the p_i's listed. This means that there are infinitely many prime numbers.

Another beautiful theorem is the following.

Theorem 7:	$\sqrt{2}$ is an irrational number.
Proof:	Suppose $\sqrt{2}$ is not an irrational number. Then $\sqrt{2}$ is a rational number. Hence we can express $\sqrt{2}$ as $\sqrt{2} = p/q$, where p and q are positive integers, having no common factor except 1. Squaring we obtain $2 = p^2/q^2$. Hence $p^2 = 2q^2$. This implies that p^2 is even and hence p is also even. Hence p^2 contains a factor of 4. Since $q^2 = p^2/2$, q^2 is even and therefore q should also be even. Since p and q are both even, they have a common factor 2, which contradicts our assumption that p and q have a common factor other than 1. Hence $\sqrt{2}$ cannot be rational number, it is irrational.

Proof of another important theorem (real numbers are uncountable) is based on the same principle – proof by contradiction, and will be discussed in the chapter on sets.

Often we have to deal with statements in mathematics, which have to be proved or disproved. To disprove a given statement, all that we have to do is to produce a counter example. Here are few illustrations.

SOLVED EXAMPLES

Example 1: *Prove or disprove $n^2 + 41n + 41$ is a prime number for every integer n.*

Solution: If $n = 1$, expression $= 43$ and for $n = 2$, it is 47, both being prime numbers. However, this by no means imply that the number is prime for all n. Consider $n = 41$. Then $41^2 + 41 + 41 = 41 \times 43$. Hence, $n = 41$ is a counter example.

Example 2: *Prove or disprove: If a and b are irrational, then a^b is also irrational.*

Solution: Counter example:

Consider,

$$e^{\pm \log 2} = 2 \text{ or } \frac{1}{2}$$

Example 3: *Let A and B be two matrices, with product AB valid, then $AB = 0 \rightarrow A = 0$ or $B = 0$.*

Solution: Counter example:

Let, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

then, $AB = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

EXERCISE - 1.3

Determine the validity of the following arguments:

1. If I like computer science, then I will study.
   Either I don't study or I pass computer science.
   If I don't graduate, it means that I did not pass computer science.
   Therefore, if I like computer science, then I will graduate.
2. If I stay up late, watching T.V., then I will be late next morning.
   I did not stay up late.
   Therefore, I am not late the next morning.

3. If I study, then I will pass.

   If I do not go to a movie, then I will study.

   I passed.

   Therefore, I did not go to a movie.

4. If Ram's computer program is correct, then he will be able to complete his assignment in almost 2 hours.

   It takes Ram over two hours to complete his assignment.

   Therefore Ram's computer program is not correct.

5. If I drive to work, then I will arrive tired. I do not drive to work. Therefore I will not arrive tried.

6. I will become famous or I will not become a writer. I will become a writer. Therefore, I will become famous.

7. If I work, I cannot study either I work, or I pass Mathematics. I passed Mathematics. Therefore, I studied.

8. If 3 is a prime number, then 2 does not divide 5. Either 4 is not even or 2 divides 5. But 4 is even. Therefore, 3 is not a prime number.

9. All integers are rational numbers. The real numbers π is not a rational number. Therefore, π is not an integer.

10. No human beings are quadrupeds. All men are human beings. Therefore, no man is a quadruped.

11. In a heterogeneous group of students, there are some who are freshers, some seniors and some are already graduates. There are students who are specialising in Mathematics or Physics or Computer science (a student can specialise in only one). Considering the above description, write the following statements using quantifiers.

    (i) There is a senior student, who is not specialising in Mathematics.

    (ii) No graduate student specialises in Physics.

    (iii) Some graduate students are neither taking Mathematics nor Physics.

    (iv) Every senior student in the group is specialising in Mathematics or Computer Science.

12. Write each of the following in terms of quantifiers:

    (i) Every integer is either odd or even.

    (ii) There are no even prime numbers.

    (iii) The sum of two odd integers is even.

    (iv) All prime numbers are not odd.

13. Let P(x, y) denote the sentence "x divides y", where the universe for both x and y comprises all integers. Determine the truth value of

    (i) $\forall y \exists x\, P(x, y)$.

    (ii) $\exists y \forall x\, P(x, y)$.

    (iii) $\forall x \forall y\, [(P(x, y) \wedge P(y, x)) \rightarrow (x = y)]$

14. Let P(x, y, z) be the predicate $x \cdot y = z$. Transcribe each of the following into English.

    (i) P (z, 3, x)

    (ii) $\forall x \exists y\, P(x, y, 1)$

    (iii) $\forall x\, P(x, 1, x)$

    (iv) $\exists y \exists x\, P(x, y, x)$

15. Let the universe of discourse be the set of real numbers. Transcribe the following into an English sentence and indicate which are true and which are false.

    (i) $\exists x\, [x^2 = 2]$

    (ii) $\exists x \forall y\, [x + y = y]$

    (iii) $\exists y \exists z \forall x\, [x \cdot y = z]$

    (iv) $\exists z \forall x \forall y\, [x \cdot y = z]$

16. Negate each of the following in such a way so that the symbol ~ does not appear before a quantifier.

    (i) $\exists x \forall y\, [x > y]$ (ii) $\forall y \exists x\, [x^2 = y]$

    (iii) $\forall x \forall y\, [(y > 0) \rightarrow (x \cdot y > 0)]$

17. Give a direct proof of the following:

    (i) For all integers m and n, if both m and n are even, then m + n is even.

    (ii) For all integers if both m and n are even, then mn is even.

18. Give an indirect proof of the following:

    (i) If $x + y \geq 100$, then $x \geq 50$ or $y \geq 50$.

    (ii) If n^2 is odd, then n is odd.

19. Prove by contradiction the following:

    If the average of four different integers is 9, then at least one number should be greater than 10.

20. Prove that if x is an irrational number, then 1 − x is also irrational.

21. Prove that if two lines are each perpendicular to a third line, in the plane, then the two lines are parallel.

POINTS TO REMEMBER

- A proposition or statement is a declarative sentence which is either true or false, but not both.
- Combining primary statements using certain connecting words like "not", "and", "or", "but", "while" etc is called Compound Statements.
- The negation of a statement is formed either by introducing the word "not" at a proper place or by prefixing the statement with the phrase "It is not the case that".
- If p and q are the statements, the compound statement "p and q" is called as the conjunction of p and q; and is denoted by $p \wedge q$.
- If p and q are statements, then the compound statement "p or q" is called as the disjunction of p and q, and is denoted by $p \vee q$.
- If p and q are statements, the compound statement "If p then q", (is) denoted by $p \rightarrow q$ (which) is called a conditional statement or implication.
- If p and q are statements, the compound statement "p if and only if q", denoted by $p \leftrightarrow q$, is called a bi-conditional statement.
- A table giving all possible truth values of a statement form, corresponding to the truth values assigned to its variables, is called a truth table.
- A statement form is called a Tautology if it always assumes the truth value 'T' irrespective of the truth values assigned to its variables.
- A statement form is called a contradiction if it always assumes the truth value 'F' irrespective of the truth values assigned to its variables.
- A statement form which is neither a tautology nor a contradiction is called a contingency.
- Two statement forms are logically equivalent if both have the same truth values, whatever may be the truth values assigned to the statement variables, occurring in both the forms.
- A conjunction of statement variables and (or) their negations is called as a fundamental conjunction. (It is also called as a min term).
- A statement form which consists of a disjunction of fundamental conjunctions, is called a disjunctive normal form (abbreviated as dnf).
- A disjunction of statement variables and (or) their negations is called a fundamental disjunction or maxterm.
- A statement form which consists of a conjunction of fundamental disjunctions, is called a conjunctive normal form (abbreviated as cnf).
- An assertion that contains one or more variables is called a predicate; its truth value is predicated after assigning truth values to its variables. If we want to specify the variables in a predicate, we denote the predicate by $P(x_1, x_2, \ldots x_n)$. Each variable x_i is also called as an argument.
- The values which the variables may assume constitute a collection or (a) set called as the universe of discourse.
- If P(x) is a predicate with the individual variable x as an argument, then the assertion "For all x, P(x)" which is interpreted as "For all values of x, the assertion P(x) is true", is a statement in which the variable x is said to be universally quantified.
- A theorem is nothing but an exercise in logical reasoning or argument. It consists of two distinct parts - hypothesis and conclusion. Hypothesis is a set of statements, which if true, implies the conclusion. Hence a theorem is usually in the form of a conditional or biconditional statement form.

- **Historical Footnotel:** The study of Logic as a formal system was first undertaken by the Greek Philosopher Aristotle(384-322 BC). He formulated the Principles of Deductive Reasoning . These Principles are universally adopted as standard methods to prove theorems and also to test the validity of an argument.Among his notable contributions to Logic are the Syllogisms . A Syllogism consists of a set of statements called as Premises together with another statement called as Conclusion. Conclusion is deduced from the Premises.

The famous quotation:

" All men are mortal,

Socrates is a man,

Therefore Socrates is mortal" is an example of a Syllogism.

◈◈◈

CHAPTER 2
THEORY OF SETS

2.1 INTRODUCTION

Set is a fundamental concept in the theory of Discrete Structures. Any algebraic structure, be it a 'group' or 'graph', has its 'underlying structure'. Hence, one ought to have a clear understanding of the term **set.**

Naive set Theory (Cantorian Set Theory) :

The concept of a set was first introduced by the German mathematician Cantor, who felt the need of such a concept to compare the magnitudes of infinite sets of numbers. Cantor's 'naive' or 'intuitive' approach was to define a set in the following way :

Definition : A set is a collection of definite, distinguishable objects of our intuition to be conceived as a whole. The objects are called as elements or members of the set.

The Implications of the Definition are as Follow :

- The main point of interest is in the collection of objects as a whole, not in its individual members.

  In everyday life, we are quite familiar with this aspect; For example we refer to 'class of students', 'bunch of grapes.' 'flock of sheep', etc.

- The phrase 'objects of our intuition' allows considerable freedom in choosing the members arbitrarily. For example, as per the definition {pen, apple, 5} is a set. However, in actual practice, we always consider a set of elements, possessing some common characteristic or property, such as 'set of prime numbers', 'students of the first division' etc.

  In notation form, we can describe this as follows :

  Let us denote an element of a set by the letter x, and let p(x) denote a condition (or property) which in satisfied by x. P(x) is also called as formula (or predicate) in x. Then a set is described as $\{x \mid P(x)\}$, to be read as that an element a is in the set if and only if the statement P(a) is true.

- The word 'distinguishable' is used to determine whether a pair of objects are same or different. The word 'definite' is interpreted to mean whether a given object is or is not a member of the set.

Cantor's theory of sets, although acceptable to mathematicians, for practical purpose, was found to have inherent contradictions or paradoxes. One famous paradox is due to Bertrand Russell.

Russell asked the following question.

Suppose the universe of discourse is the *set of all sets*, and let S be a set whose objects are sets, which are not members of themselves, then is S a member of itself?

Attempt to answer this question leads to a paradox. If S is not a member of itself, then by the condition imposed on S, S should belong to itself. On the other hand, if S does not belong to itself, then S should be a member of itself, as per the definition. In Cantor's definition of set, it is implied that given any property there exists a set whose members are just those entities having that property. The root cause of the problem, was the unrestricted use of this principle. Russell's Paradox in symbolic form can be stated as $\{x \mid x \notin x\}$, where $p(x) : x \notin x$ is the property that x should satisfy.

2.2 AXIOMATIC SET THEORY

An alternative or modified version of Cantor's set theory, free from contradictions was developed by Zermelo and Fraenkel. This version is called as Axiomatic Set Theory, which is as follows.

- **ZF1** (Axiom of extension) If A and B are sets and if, for all x, $x \in A$ if and only if $x \in B$, then $A = B$.
- **ZF2** (Axiom Schema of subsets). For any set A, there exists a set B such that for all x, $x \in B$ iff $x \in A$ and p(x), where p(x) is any condition on x which contains no free occurrence of B.

The above statement means for each choice of p(x), there is a scheme for producing axioms; hence the word 'schema', plural form of scheme.

The second axiom is the required substitute to remove the fallacy in Cantor's definition. As per Cantor's definition, given any property or condition there exists a set. In the second version (ZF2), the axiom only provides the existence of a set corresponding to a condition, and *which is a subset of an existing set*. Thus the ZF_2 version is restrictive. We shall now show how Russell's paradox does not exist in ZF version.

According to ZF_2 let p(x) denote the condition x | x. Since we are considering a set as a member of itself, we shall use small letters x, y, a, b for sets as well.

Hence if $b = \{x \in a \mid x \mid x\}$, then for all y.

$$y \in b \text{ iff } y \in a \text{ and } y \mid y. \qquad \ldots(1)$$

It follows that b | a. The proof is by contradiction. Assume that $b \in a$. Now either $b \in b$ or b | b. If $b \in b$, then in view of our assumption and (1) b does not belong to b; hence a contradiction. If b | b, then this and our assumption yield in view of (1), $b \in b$, again a contradiction. Hence we conclude that b | a, i.e. such a set does not exist. Since the set a is any unspecified set, we conclude that there is no set which contains every set.

Hence Russell's phrase in the opening line of his paradox 'set of all sets', has no meaning in the axiomatic version of set theory.

In the ensuing discussion, we shall adopt the modified version of Cantor's theory. For each separate situation, we shall define a universal set and within its framework, the set under discussion will be a collection of objects, the objects being also members of the universal set.

2.3 SETS

2.3.1 Definition

A set is a **collection of** objects.

An object in the collection is called an **element** or member of the set.

The term **class** is also used to denote a set.

A set may contain **finite** number of elements or **infinite** number of elements.

A set is called an **empty set** or a **null set** if it contains no element. An empty set is denoted by the letter ϕ.

Examples:

- The set of letters forming the word 'PASCAL', is a finite set, whose elements are the five **distinct** letters of the word.
- The set of all telephone numbers in the directory. This is also a finite, though a large, set.
- The set of persons in a moving queue. This is also a finite set, but difficult to list, due to the constant flux (At any instant of time, people are entering as well as leaving the queue).
- The set of whole (natural) numbers greater than 10. This is an infinite set, but the elements in the set can be listed, i.e. 11, 12, 13, ...
- The set of all points in the plane. This is also an infinite set, but the elements cannot be listed, as the points are 'dense' not 'discrete.'
- The set consisting of a circle, the number 5, a tree and Bill Gates.

  This example shows that the elements in a set can be totally different in character, they need not have a common characteristic.
- The set of real roots of the equation $x^2 + 1 = 0$. This is obviously an empty set.

2.3.2 Notations

A set is generally denoted by capital letters A, B, C, ..., X, Y, Z.

Elements of the set are denoted by small letters a, b, c, ..., x, y, z.

If x is an element of the set A, we express this fact by writing

'$x \in A$' ($\in$ means 'belongs to')

If x is not an element of A, we write

'$x \notin A$'

There are various ways of describing a set.

1. **Listing Method:** In this method, the elements are listed within braces.

   e.g. (i) $A = \{\text{pencil, byte, } 5\}$

   (ii) $B = \{2, 4, 6, 8, \ldots\ldots\}$

2. **Statement Form:** A statement describing the set, especially where the elements share a common characteristic. e.g.

   (i) The set of all equilateral triangles.

   (ii) The set of all Prime Ministers of India.

3. **Set-Builder Notation:** It is not always possible or convenient to describe a set by the Listing method or the Statement form. A more concise or compact way of describing the set is to specify the property shared by all the elements of the set. This property is denoted by P(x), where P is a statement concerning an element x of the set. The set is then simply written as

$\{x \mid P(x)\}$ where the braces { } denote the clause " the set of", and the slash or stroke | denotes "such that" (read as ". A is the set of all x such that x is greater than 10").

Examples: (i) $A = \{x \mid x > 10\}$

(ii) $B = \{x \mid x \text{ is real and } x^8 - 5x^4 + 4 = 0\}$.

2.3.3 Some Special Sets (Number Sets)

The following sets occur frequently in our discussion. We give below Table 2.1 the standard notations used to denote these sets.

Table 2.1

N	the set of all natural numbers {1, 2, 3, ... }.
Z	the set of all integers {...... −2, −1, 0, 1, 2, ... }.
Z^+	the set of all positive integers {0, 1, 2, ... }.
Q	the set of rational numbers.
Q^+	the set of non-negative rational numbers.
$\mathbb{R}$	the set of real numbers.
$\mathbb{C}$	the set of complex numbers.

2.4 SUBSETS

2.4.1 Definition

If every element of a set A is also an element of a set B, then we say A is a **subset** of B, or A is **contained** in B. This is denoted by writing '$A \subseteq B$'. This can be also denoted by '$B \supseteq A$'

If A is not a subset of B, this is indicated by writing '$A \subseteq B$'.

Examples:

1. $N \subseteq Z^+ \subseteq Z \subseteq Q \subseteq \mathbb{R} \subseteq \mathbb{C}$

2. $A = \{1, 3, 6\}$, $B = \{-1, 1, 2, 3, 4, 6\}$

   $C = \{1, 2, 3\}$

   Then $A \subseteq B$,

   But $A \subseteq C$

It is clear from the definition that **Every set is a subset of itself. The empty set is a subset of any set.**

2.4.2 Universal Set

If all sets, considered during a **specific discussion** are subsets of a given set, then this set is called as the **Universal Set**, and is denoted by 'U'.

Hence, the universal set is a relative concept dependent on the specific discussion. Therefore, it is also referred to a s the **universe of discourse**.

2.4.3 Equality of Sets

Two sets A and B are equal if A is a subset of B and B is also a subset of A, i.e. $A \subseteq B$ and $B \subseteq A$ implies $A = B$.

Examples: (i) If

$A = \{\text{BASIC, COBOL, FORTRAN}\}$

and $B = \{\text{FORTRAN, COBOL, BASIC}\}$

then $A = B$.

(ii) If $A = \{x \mid x^2 + 1 = 0\}$

and $B = \{i, -i\}$ $\left(i = \sqrt{-1}\right)$

then $A = B$

2.4.4 Important Remark

A set itself can be an element of some other set. Hence, one should be able to clearly distinguish between an element of a set and subset of a set.

Examples:

1. Let $A = \{a, b, \{a, b\}, \{\{a, b\}\}\}$.

   Identify each of the following statements as true or false. Justify your answers.

   (a) $a \in A$, (b) $\{a\} \in A$, (c) $\{a, b\} \in A$ (d) $\{\{a, b\}\} \subseteq A$, (e) $\{a, b\} \subseteq A$ (f) $\{a, \{b\}\} \subseteq A$.

   Solution:

   (a) True, as a is an element of A.

   (b) False, as {a} is not an element but a subset of A.

   (c) True, as {a, b} is an element of A, listed third in the set.

   (d) True, as a subset containing the single element {a, b} of A.

   (e) True, as the subset containing the elements a, b of A.

   (f) False, as {b} is not an element of A.

2. Determine whether each of the following statements is true for arbitrary sets A, B, C. Justify your answers.

   (a) If $A \in B$ and $B \subseteq C$, then $A \in C$

   (b) If $A \in B$ and $B \subseteq C$, then $A \subseteq C$.

   (c) If $A \subseteq B$ and $B \in C$, then $A \in C$.

   (d) If $A \subseteq B$ and $B \in C$, then $A \subseteq C$.

Solution: (a) True, as A being an element of B, it should also belong to C as B is a subset of C.

(b) False, as A is not a subset but an element of B.

(c) False. Consider $A = \{a\}$, $B = \{a, b\}$, $C = \{\{a, b\}\}$.

(d) False. Consider the same example as in (c).

2.5 VENN DIAGRAMS

A Venn diagram (named after the British logician John Venn) is a pictorial depiction of a set. A rectangle represents the universal set. The interior of the rectangle represents the elements in the set. A circle drawn within the rectangle depicts an arbitrary set. It is not compulsory to show an arbitrary set, always by a circle. An oval shaped or elliptical curve could also be drawn to represent a set. In fact any closed curve of any shape can be used to depict a set.

Venn Diagram

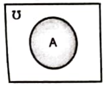
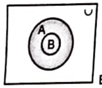


Fig. 2.1



$B \subset A$

Fig. 2.2

2.6 SET OPERATIONS

We shall now define various set operations, which will combine the given sets to yield new sets. These operations are analogous to the algebraic operations of addition, multiplication of numbers.

2.6.1 Complement of a Set

Let A be a given set. **Complement** of A, denoted by $\bar{A}$ is defined as

$$\bar{A} = \{x \mid x \notin A\}$$

Examples:

1. If $A = \{x \mid x \text{ is a real number and } x \le 7\}$, then

$\bar{A} = \{x \mid x \text{ is a real number and } x > 7\}$

where the universal set $\bar{U} = R$.

2. If $U = N = \{1, 2, 3, 4, 5, \ldots\ldots\}$

and $E = \{2, 4, 6, \ldots\ldots\}$

then $\bar{E} = \{1, 3, 5, \ldots\}$

Note that $\bar{\phi} = U$

and $\bar{U} = \phi$

2.6.2 Union of Sets

The union of two sets A and B is the set consisting of all elements which are in A, or in B, or in both sets A and B. It is denoted by $A \cup B$.

In the set – builder notation,

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

Examples:

1. If $A = \{2, 4, 6, 8, 10\}$

$B = \{1, 2, 6, 8, 12, 15\}$

then $A \cup B = \{1, 2, 4, 6, 8, 10, 12, 15\}$.

2. If $A = \{n \mid n \in N, 4 < n < 12\}$

$B = \{n \mid n \in N, 8 < n < 15\}$

then $A \cup B = \{5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$.

3. If $A = \{\phi\}$

$B = \{a, \phi, \{\phi\}\}$

then $A \cup B = \{\phi, a, \{\phi\}\}$

$= B$.

This is because $A \subseteq B$

Note that for any set

$A\ A \cup \phi = A$

$A \cup U = U$

$A \cup \bar{A} = U$.

2.6.3 Intersection of Sets

The intersection of two sets A and B, denoted by $A \cap B$ is the set consisting of elements which are in A **as well** as in B.

Thus, $A \cap B = \{x \mid x \in A \textbf{ and } x \in B\}$

→ If $A \cap B = \phi$, the sets are said to be **disjoint**.

Examples:

1. If $A = \{a, b, c, g\}$

$B = \{d, e, f, g\}$

then $A \cap B = \{g\}$.

2. If $A = \{n \mid n \in N, 4 < n < 12\}$

$B = \{n \mid n \in N, 5 < n < 10\}$

then $A \cap B = \{6, 7, 8, 9\} = B$.

3. If $A = \{\phi\}$

$B = \{a, \phi, \{\phi\}\}$

then $A \cap B = \{\phi\} = A$.

Note that for any set

$A \cap \phi = \phi$

$A \cap U = A$

$A \cap \bar{A} = \phi$.

2.6.4 Difference of Sets (Relative Complement)

Let A and B be any two sets.

The difference $A - B$ is the set defined as

$A - B = \{x \mid x \in A \text{ and } x \notin B\}$ is the (relative) complement of B in A.

Similarly,

$B - A = \{x \mid x \in B \text{ and } x \notin A\}$ is the complem

Examples:

1. If $A = \{1, 2, 3, \ldots\ldots, 10\}$

$B = \{1, 3, 5, \ldots 9\}$

then $A - B = \{2, 4, 6, 8, 10\}$ ✓

$B - A = \phi$.

2. If $A = \{a, b, \{a, c\}, \phi\}$ ✓

$A - \{a, b\} = \{\{a, c\}, \phi\}$

$\{a, c\} - A = \{c\}$.

2.6.5 Properties of Difference

Let A and B be any two sets.

Then (i) $\bar{A} = U - A$

(ii) $A - A = \phi$

(iii) $A - \bar{A} = A$, $\bar{A} - A = \bar{A}$

(iv) $A - \phi = A$

(v) $A - B = A \cap \bar{B}$

(vi) $A - B = B - A$ if and only if $A = B$

(vii) $A - B = A$ if and only if $A \cap B = \phi$

(viii) $A - B = \phi$ if and only if $A \subseteq B$.

Proofs of the properties (i) to (v) are immediate consequences of the definition. We shall prove the remaining properties.

(vi) If $A = B$, then $A - B = \phi = B - A$ by (ii). Conversely let $A - B = B - A$. Let $x \in A$. Assume $x \notin B$, then x should be in $A - B$. But since $A - B = B - A$, it follows that $x \in B - A$ which means $x \in B$, a contradiction. Hence, x should be an element of B. Therefore, $A \subseteq B$. Similarly we can prove $B \subseteq A$. Hence, $A = B$.

(vii) $A - B = A$ implies $A \cap \bar{B} = A$, i.e. $A \subseteq \bar{B}$. Hence, $A \cap B = \phi$. Conversely $A \cap B = \phi$ implies $A \subseteq \bar{B}$, which in turn means that $A \cap \bar{B} = A$, i.e. $A - B = A$.

(viii) If $A - B = \phi$, it implies that $A \cap \bar{B} = \phi$, i.e. $A \subseteq B$. Converse is proved by reversing the steps.

2.6.6 Symmetric Difference

The symmetric difference of two sets A and B, denoted by $A \oplus B$, is defined as

$A \oplus B = \{x \mid x \in A - B \text{ or } x \in B - A\}$

In other words,

$A \oplus B = (A - B) \cup (B - A)$.

Examples:

1. If $A = \{a, b, e, g\}$

$B = \{d, e, f, g\}$

then $A \oplus B = \{a, b, d, f\}$.

2. If $A = \{2, 4, 5, 9\}$

$B = \{x \in Z^+ \mid x^2 \le 16\}$

then $A \oplus B = \{0, 1, 3, 5, 9\}$

3. If $A = \{\phi\}$,

$B = \{a, \phi, \{\phi\}\}$, ✓

then $A \oplus B = \{a, \{\phi\}\}$.

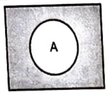
2.6.7 Properties of Symmetric Difference

1. $A \oplus A = \phi$
2. $A \oplus \phi = A$
3. $A \oplus U = \bar{A}$ ✓
4. $A \oplus \bar{A} = U$
5. $A \oplus B = A \cup B - A \cap B$.

The properties (i) to (iv) are immediate consequences of the definition.

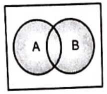
We shall prove the last property. Let $x \in A \cup B - A \cap B$. Then, $x \in A \cup B$ but $x \notin A \cap B$. This means that if $x \in A$, $x \notin B$. Similarly, if $x \in B$, then $x \notin A$. Hence, $x \in A - B$ or $x \in B - A$, which means that $x \in (A - B) \cup (B - A) = A \oplus B$. Conversely, let $x \in A \oplus B$. Then, $x \in A - B$ or $x \in B - A$. This means that $x \in A \cup B$ but $x \notin A \cap B$, i.e. $x \in (A \cup B) - (A \cap B)$. Hence, the two sets are equal.

2.6.8 Representation of Set Operations on Venn Diagrams



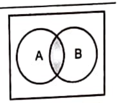
$\bar{A}$ =

Fig. 2.3



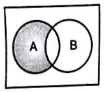
$A \cup B$ =

Fig. 2.4



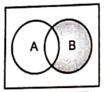
$A \cap B$ =

Fig. 2.5



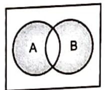
$A - B$ =

Fig. 2.6



$B - A$ =

Fig. 2.7



$A \oplus B$ =

Fig. 2.8

2.7 ALGEBRA OF SET OPERATIONS

The set operations obey the same rules as those of numbers, such as associativity, commutativity and distributivity. However, the cancellation rule which is true for numbers, is not true for sets in general. In addition, there are rules such as Idempotent laws, Absorption laws, De Morgan's laws, which are true only for sets.

Theorem: The set operations satisfy the following properties, for any sets A, B, C.

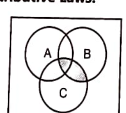
1. **Commutativity:**
   (i) $A \cup B = B \cup A$
   (ii) $A \cap B = B \cap A$
2. **Associativity:**
   (i) $A \cup (B \cup C) = (A \cup B) \cup C$, hence written as $A \cup B \cup C$
   (ii) $A \cap (B \cap C) = (A \cap B) \cap C$, hence written as $A \cap B \cap C$
3. **Distributivity:**
   (i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
4. **Idempotent Laws:**
   (i) $A \cup A = A$
   (ii) $A \cap A = A$
5. **Absorption Laws:**
   (i) $A \cup (A \cap B) = A$
   (ii) $A \cap (A \cup B) = A$
6. **De Morgan's Laws:**
   (i) $\overline{A \cup B} = \bar{A} \cap \bar{B}$
   (ii) $\overline{A \cap B} = \bar{A} \cup \bar{B}$
7. **Double Complement:**
   $$\bar{\bar{A}} = A$$

Proof: We shall prove properties 3, 6 and 7. The remaining are easy exercises for the reader.

8. **Distributive Laws:**
   (i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
   Let $x \in A \cup (B \cap C)$. Then, $x \in A$ or $x \in B \cap C$. This further implies that $x \in A$ or ($x \in B$ and $x \in C$).
   i.e. ($x \in A$ or $x \in B$) and ($x \in A$ or $x \in C$)
   i.e. $x \in A \cup B$ and $x \in A \cup C$
   i.e. $x \in (A \cup B) \cap (A \cup C)$
   Hence, $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$
   Similarly one can prove $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.
   (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ can be proved on similar lines.
   Refer to the figures below:
9. **De Morgan's Laws:**
   (i) $\overline{A \cup B} = \bar{A} \cap \bar{B}$
   $$\overline{A \cup B} = \{x \mid x \notin A \cup B\}$$
   $$= \{x \mid x \notin A \text{ and } x \notin B\}$$
   $$= \{x \mid x \in \bar{A} \text{ and } x \in \bar{B}\} = \bar{A} \cap \bar{B}$$
   (ii) $\overline{A \cap B} = \bar{A} \cup \bar{B}$ can be proved in the same way, as above.
10. **Double Complement:**
   $$\bar{\bar{A}} = A$$
   $$\bar{\bar{A}} = \{x \mid x \notin \bar{A}\} = \{x \mid x \in A\} = A.$$

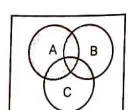
The above properties can also be demonstrated by drawing suitable Venn diagrams, as shown below.

For Distributive Laws:



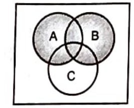
$B \cap C$

Fig. 2.9



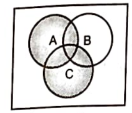
$A \cap (B \cap C)$

Fig. 2.10



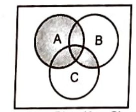
$A \cap B$

Fig. 2.11



$A \cap C$

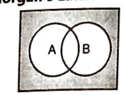
Fig. 2.12



$(A \cap B) \cap (A \cap C)$

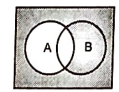
Fig. 2.13

De Morgan's Laws:



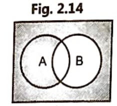
$\overline{A \cap B}$

Fig. 2.14



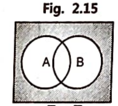
$\bar{A}$

Fig. 2.15



$\bar{B}$

Fig. 2.16



$\bar{A} \cap \bar{B}$

Fig. 2.17

SOLVED EXAMPLES

Example 1: *Determine whether the following statements are true or false. Justify your answers.*

(i) $\{a, \phi\} \in \{a, \{a, \phi\}\}$.
(ii) $\{a, b\} \subseteq \{a, b, \{a, b\}\}$.
(iii) $\{a, b\} \in \{a, b, \{a, b\}\}$.
(iv) $\{a, c\} \in \{a, b, c, \{a, b, c\}\}$.

Solution: (i) True, since $\{a, \phi\}$ is an element of $\{a, \{a, \phi\}\}$.

(ii) True, since $\{a, b\}$ is a subset of $\{a, b, \{a, b\}\}$ containing the elements a, b.

(iii) True, since $\{a, b\}$ is an element in $\{a, b, \{a, b\}\}$.

(iv) False, since $\{a, c\}$ is not an element but a subset of $\{a, b, c, \{a, b, c\}\}$ containing the elements a, c.

Example 2 : *If $A = \{a, b, \{a, c\}, \phi\}$, determine the following sets*

(i) $A - \{a, c\}$
(ii) $\{\{a, c\}\} - A$
(iii) $A - \{\{a, b\}\}$
(iv) $\{a, c\} - A$.

Solution : (i) $A - \{a, c\} = \{b, \{a, c\}, \phi\}$
(ii) $\{\{a, c\}\} - A = \phi$
(iii) $A - \{\{a, b\}\} = A$
(iv) $\{a, c\} - A = \{c\}$.

Example 3: *If*
$$U = \{n \mid n \in N, n \le 15\},$$
$$A = \{n \mid n \in N, 4 < n < 12\},$$
$$B = \{n \mid n \in N, 8 < n < 15\},$$
$$C = \{n \mid n \in N, 5 < n < 10\},$$
find $\bar{A} - \bar{B}$ and $\bar{C} - \bar{A}$.

Solution:
$$\bar{A} = \{1, 2, 3, 4, 12, 13, 14, 15\}$$
$$\bar{B} = \{1, 2, 3, 4, 5, 6, 7, 8, 15\}$$
$$\bar{C} = \{1, 2, 3, 4, 5, 10, 11, 12, 13, 14, 15\}$$
$$\therefore \quad \bar{A} - \bar{B} = \{12, 13, 14\}$$
$$\bar{C} - \bar{A} = \{5, 10, 11\}.$$

Example 4: *Let A, B, C be subsets of the universal set U. Given that $A \cap B = A \cap C$ and $\bar{A} \cap B = \bar{A} \cap C$, is it necessary that B = C? Justify your answer.*

Solution: Yes, B = C.

We can express B as

$$B = B \cap U = B \cap (A \cup \bar{A})$$
$$= (B \cap A) \cup (B \cap \bar{A}) \quad \text{(Distributive law)}$$
$$= (A \cap B) \cup (\bar{A} \cap B) \quad \text{(Commutative law)}$$
$$= (A \cap C) \cup (\bar{A} \cap C) \quad \text{(Given condition)}$$
$$= (A \cup \bar{A}) \cap C \quad \text{(Distributive law)}$$
$$= U \cap C$$
$$= C.$$

Example 5 : *(i) Given that $A \cup B = A \cup C$, is it necessary that B = C?*

(ii) Given that $A \cap B = A \cap C$, is it necessary that B = C?

Solution: (i) No. Let
$$A = \{1, 2, 3\}$$
$$B = \{1\}$$
$$C = \{3\}$$
$$A \cup B = \{1, 2, 3\} = A \cup C.$$
but $B \ne C$.

(ii) No. Let $A = \{1, 2\}$

$B = \{2, 3, 4, 5\}$

$C = \{2, 6, 7\}$

then $A \cap B = \{2\} = A \cap C$.

but $B \neq C$

Example 6: *If $A \oplus B = A \oplus C$, is $B = C$?*

Solution: Yes. Consider any element $x \in B$. This element is then in A or not in A. Suppose $x \in A$. Then, $x \in A \cap B$ which implies that $x \notin A \oplus B$ and hence $\notin A \oplus C$. Now $A \oplus C = A \cup C - A \cap C$. Therefore, it follows that $x \in A \cap C$ which means that $x \in C$. Hence, if $x \in A$, then $B \subseteq C$.

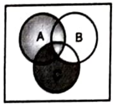
Suppose we have the other possibility that $x \notin A$. Then $x \notin A \cap B$ so that $x \in A \oplus B$ which in turn implies that $x \in A \oplus C$. This means that $x \in A \cup C$. Therefore, $x \in C$. Hence, in this case also $B \subseteq C$.

Similarly we can show that $C \subseteq B$. Hence, $B = C$.

Problems Involving Venn Diagrams:

Example 7: *Show that $A \cup (\bar{B} \cap C) = (A \cup \bar{B}) \cap (A \cup C)$, using Venn diagram.*

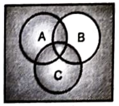
Solution:



$\bar{B} \cap C$ = ■

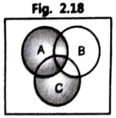
$A \cap (\bar{B} \cap C)$ = ▢ & ■

Fig. 2.18



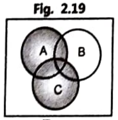
$A \cap \bar{B}$

Fig. 2.19



$A \cap C$

Fig. 2.20

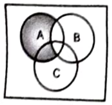


$(A \cap \bar{B}) \cap (A \cap C)$

Fig. 2.21

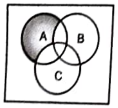
Example 8: *Show that $(A - B) - C = A - (B \cup C)$ using Venn diagram.*

Solution:



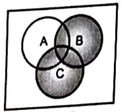
$A - B$

Fig. 2.22



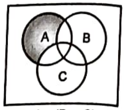
$(A - B) - C$

Fig. 2.23



$B \cap C$

Fig. 2.24



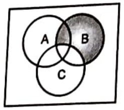
$A - (B \cap C)$

Fig. 2.25

Example 9: *Drawing Venn diagram, prove that*

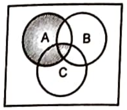
$A - (B - C) = (A - B) \cup (A \cap B \cap C)$.

Solution:



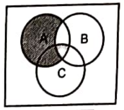
$B - C$

Fig. 2.26



$A - (B - C)$

Fig. 2.27



$A - B$ = ■

$A \cap B \cap C$ = ▢

Fig. 2.28

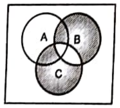
Example 10: *Using Venn diagram, prove or disprove*

(i) *$A \oplus (B \oplus C) = (A \oplus B) \oplus C$*

(ii) *$A \cap B \cap C = A - [(A - B) \cup (A - C)]$*

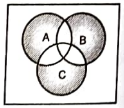
Solution:

(i)



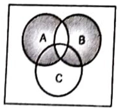
$B \oplus C$

Fig. 2.29



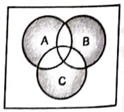
$A \oplus (B \oplus C)$

Fig. 2.30



$A \oplus B$

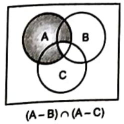
Fig. 2.31



$(A \oplus B) \oplus C)$

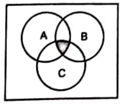
Fig. 2.32

(ii)



$(A - B) \cap (A - C)$

Fig. 2.33



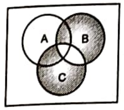
$A - [(A - B) \cap (A - C)]$
$= A \cap B \cap C$

Fig. 2.34

Example 11: *Using Venn diagram, prove or disprove*

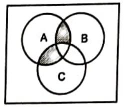
$A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$

Solution:



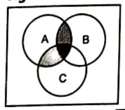
$B \oplus C$

Fig. 2.35



$A \cap (B \oplus C)$

Fig. 2.36



$(A \cap B) \oplus (A \cap C)$ = ■ + ▢

Fig. 2.37

Hence, the equation is true.

Example 12: *Let A denote the set of students who study Data Structures, B denote the set of students who study Discrete Mathematics, C denote the set of students who study Assembly Language Programming, D denote the set of students studying Theory of Computer Science. Let E denote the set of students who are staying in the Hostel and F denote the set of students who went to watch a cricket match last Monday. Express the following statements in set theoretic notation.*

(i) *All hostelites who study neither Data Structures nor Discrete Mathematics went to watch cricket match last Monday.*

(ii) *The students who went to see cricket match are only those who study Assembly Language Programming or Data Structures.*

(iii) *No student who is studying Data Structures went to see cricket match.*

(iv) *Those and only those students who are studying Theory of Computer Science and Discrete Mathematics went for a cricket match.*

(v) *All went to see cricket match.*

Solution:

(i) $E \cap \bar{A} \cap \bar{B} \subseteq F$ or $(E - A) - B \subseteq F$

(ii) $F \subseteq C \cup A$

(iii) $A \cap F = \phi$

(iv) $F \subseteq D \cap B$

(v) If the universal set is $A \cup B \cup C \cup D \cup E$, then $A \cup B \cup C \cup D \cup E = F$.

Otherwise $A \cup B \cup C \cup D \cup E \subset F$.

Example 13: *Let A denote the set of all automobiles that are manufactured domestically. Let B denote the set of all imported automobiles. Let C denote the set of all automobiles manufactured before 1977. Let D denote the set of all automobiles with a current market value of less than 2000 \$. Let E denote the set of all automobiles owned by students at the University.*

Express the following in set theoretic notation.

(i) *The automobiles owned by the students at the University are either domestically manufactured or imported.*

(ii) *All domestic automobiles manufactured before 1977 have a market value of less than 2000 \$.*

(iii) *All imported automobiles manufactured after 1977 have a market value of more than 200 \$.*

Solution:

(i) $E \subseteq A \cup B$.

(ii) $A \cap C \subseteq D$.

(iii) $B \cap \bar{C} \subseteq \bar{D}$.

where the universal set U = set of all automobiles $= A \cup B$.

i.e. $B \cap ((A \cup B) - C) \subseteq (A \cup B) - D$.

Example 14: *Tony, Mike and John belong to the Alpine Club. Every Club member is either a skier or mountain climber or both. No mountain climber likes rain and all skiers like snow. Mike dislikes whatever Tony likes and likes whatever Tony dislikes. Tony likes rain and snow. Is there a member of the Alpine Club who is a mountain climber but not skier?*

Solution: Let A denote the set of all members of the Alpine Club. Let S denote the set of skiers and M the set of all mountain climbers. Then $A \subset M \cup S$. If $x \in M$, x does not like rain and if $y \in S$, y likes snow. Since, Tony likes both rain and snow, Tony $\in$ S-M. Since, Mike dislikes whatever Tony likes and likes what Tony dislikes, it follows that Mike $\in$ M-S. Hence, there is a member (that is Mike), of the Alpine Club, who is a mountain climber but not skier.

Example 15: *Consider the following assumptions.*

S_1 : Poets are happy people.

S_2 : Every doctor is wealthy.

S_3 : No one who is happy is also wealthy.

Determine the validity of the following arguments, using Venn diagram.

(1) No poet is wealthy.

(2) Doctors are happy people.

(3) No one can be both a poet and a doctor.

Solution: Let H be the set of happy people, P set of poets, W set of wealthy people and D, set of doctors. By S_1, $P \subseteq H$, S_2 implies $D \subseteq W$, S_3 implies $H \cap W = \phi$.

We have the Venn diagram.

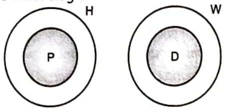


Fig. 2.38

From the Venn diagram, we observe that $P \cap W = \phi$, i.e. no poet is wealthy. Hence argument (1) is valid $D \cap H = \phi$, here doctors are not happy people so that (2) is invalid. $P \cap D = \phi$, hence no-one can be both a poet and doctor. Hence, (3) is valid.

2.8 CARDINALITY OF FINITE SET

A very important problem in Discrete Structures is that of determining the number of objects in a finite set. In the analysis of computer algorithms, one is often required to count the number of operations executed by various algorithms. This is necessary to estimate the cost effectiveness of a particular algorithm. In the study of data structures of files, determining the average and maximum lengths of searches for items stored in a data structure, also involve counting. Hence, in this section, we shall introduce the concept of **cardinality** of a finite set, and study its properties.

2.8.1 Definition

Let A be a finite set. The cardinality of A, denoted by $|A|$ is the number of elements in the set.

If $A = \phi$, then $|A| = 0$.

If $A \subseteq B$, where B is a finite set, then $|A| \le |B|$.

The following theorem enables us to find the cardinality of disjoint union of two sets.

2.8.2 Theorem (The Addition Principle)

Theorem (The Addition Principle): Let A and B be finite sets which are disjoint. Then $|A \cup B| = |A| + |B|$.

Proof:

If A or B is the empty set, the proof is trivial.

Hence, let us assume that $A \ne \phi$, $B \ne \phi$.

Since, A and B are finite disjoint sets, let $A = \{a_1, a_2, ..., a_m\}$ and $B = \{b_1, b_2, ..., b_n\}$, where $a_i \ne b_j$ for $1 \le i \le m$, $1 \le j \le n$. $|A| = m$ and $|B| = n$.

Then $A \cup B = \{a_1, a_2, ... a_m, b_1, b_2, ... b_n\}$, i.e. $A \cup B$ contains exactly m + n elements.

Hence, $|A \cup B| = m + n = |A| + |B|$. Thus, the theorem is proved.

The above theorem can be extended to a finite collection of finite mutually disjoint sets.

2.8.3 Corollary

Let $A_1, A_2, ... A_n$ be a finite collection of mutually disjoint finite sets.

Then $|A_1 \cup A_2 \cup ... \cup A_n| = |A_1| + |A_2| + ... + |A_n|$.

Proof is left as an exercise.

2.8.4 Theorem (Finite Set)

Let A be a finite set and let B be any set (not necessarily finite).

Then $|A - B| = |A| - |A \cap B|$.

Proof:

Consider the Venn diagram.

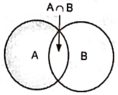


Fig. 2.39

From the Venn diagram, it is clear that $A = (A - B) \cup (A \cap B)$ (Disjoint union of two sets)

Hence, by the addition principle,

$$|A| = |A - B| + |A \cap B|,$$

so that $|A - B| = |A| - |A \cap B|$.

2.8.5 Theorem (Principle of Inclusion–Exclusion)

Theorem: Let A and B be finite sets.

Then $|A \cup B| = |A| + |B| - |A \cap B|$

Proof:

Consider the Venn diagram.

A – B is the shaded portion.

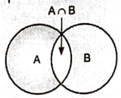


Fig. 2.40

We may express $A \cup B$ as disjoint union of two sets, by writing

$$A \cup B = (A - B) \cup B.$$

Hence, by the addition principle,

$$|A \cup B| = |A - B| + |B|$$

$$= |A| - |A \cap B| + |B|$$

(by the previous theorem)

Hence, $|A \cup B| = |A| + |B| - |A \cap B|$.

SOLVED EXAMPLES

Example 1: *In a survey, 2000 people were asked whether they read India Today or Business Times. It was found that 1200 read India Today, 900 read Business Times and 400 read both. Find how many read at least one magazine and how many read none.*

Solution: Let A denote the set of people who read India Today, B denote the set of people who read Business Times.

Now $|A| = 1200$, $|B| = 900$

and $|A \cap B| = 400$.

By the mutual inclusion-exclusion principle,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$= 1200 + 900 - 400 = 1700$$

and $|U - (A \cup B)| = |U| - |A \cup B|$

$$= 2000 - 1700 = 300.$$

Hence, 1700 read at least one magazine and 300 read neither.

Example 2: *Among the integers 1 to 300, find how many are not divisible by 3, nor by 5. Find also, how many are divisible by 3, but not by 7.*

Solution: Let A denote the set of integers 1 – 300, divisible by 3; B, the set of integers divisible by 5; C, the set of integers divisible by 7. We have to find $|\bar{A} \cap \bar{B}|$ and $|A - C|$.

By De Morgan's laws, $\bar{A} \cap \bar{B} = \overline{A \cup B}$.

Hence, $|\overline{A \cup B}| = |U| - |A \cup B|$

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A| = \left[\frac{300}{3}\right] = 100.$$

$$|B| = \left[\frac{300}{5}\right] = 60.$$

$$\therefore \quad |A \cap B| = \left[\frac{300}{15}\right] = 20.$$

$$\therefore \quad |A \cup B| = 100 + 60 - 20 = 140.$$

$$\therefore \quad |\overline{A \cup B}| = 300 - 140 = 160.$$

120

Hence, 160 integers between 1 to 300 are not divisible by 3, nor by 5.

Now $|A - C| = |A| - |A \cap C|$

$$|A \cap C| = \left[\frac{300}{21}\right] = 14.$$

Hence, $|A - C| = 100 - 14 = 86$.

Hence, 86 integers between 1 – 300 are divisible by 3, but not by 7.

2.8.6 Theorem (Mutual Inclusion–Exclusion Principle for Three Sets)

Theorem: Let A, B, C be finite sets.

Then $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

Proof:

Let D denote the union set $B \cup C$. Then $A \cup B \cup C = A \cup D$.