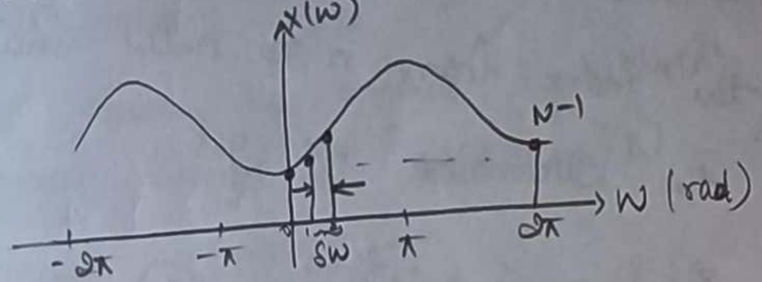
(10m)

Frequency domain Sampling and Reconstruction of DT signals

Consider DT signal with its fourier transform

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \quad \text{—— ①}$$

Sample $X(\omega)$ periodically in frequency at a spacing of $\delta\omega$ radians between successive samples.



where $\delta\omega = \frac{\text{total length}}{\text{No of samples}} = \frac{2\pi}{N}$

Let us consider magnitude of $X(\omega)$ for different values of ω

$\omega = 0.\delta\omega \qquad X(\omega) = X(0.\delta\omega)$

$\omega = 1\delta\omega \qquad X(\omega) = X(1.\delta\omega)$

$\omega = 2\delta\omega \qquad X(\omega) = X(2.\delta\omega)$

⋮

$\omega = (N-1)\delta\omega \qquad X(\omega) = X((N-1)\delta\omega)$

In general

$X(\omega) = X(k.\delta\omega)$ where $k = 0, 1, 2, \ldots, N-1$.

$X(\omega) = X\left(k.\frac{2\pi}{N}\right)$

$$\boxed{\omega = \frac{2\pi}{N}k}$$

Evaluate eqⁿ ① ~~and~~ at $\omega = \frac{2\pi}{N}k$ we get

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\frac{2\pi}{N}kn} \quad \text{—— ②}$$

Subdivide eq^n ② into infinite no's of summations and each summation contains N number of samples

$$X\left(\frac{2\pi}{N}k\right) = \cdots + \sum_{n=-N}^{-1} x(n)e^{-j\frac{2\pi}{N}kn} + \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}kn} + \sum_{n=N}^{2N-1} x(n)e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n)e^{-j\frac{2\pi}{N}kn}$$

Change in the index from n to $n-lN$ and interchanging the order of the summation we get

$$= \sum_{n=0}^{N-1} \sum_{l=-\infty}^{\infty} x(n-lN)e^{-j\frac{2\pi}{N}kn} \quad \text{——— ③}$$

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} x_p(n)e^{-j\frac{2\pi}{N}kn} \quad \text{——— ④}$$

always for DFT we consider only RHS that is only the value of N

where

$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$ is a periodic extension of $x(n)$ and it is expressed in a fourier form $x_p(n) = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi}{N}kn}$ ——— ⑤.

fourier coefficient → $$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n)e^{-j\frac{2\pi}{N}kn} \quad \text{——— ⑥}$$

compare eq^n ④ & ⑥

$$\boxed{c_k = \frac{1}{N}X\left(\frac{2\pi}{N}k\right)}$$

(or)

$$X\left(\frac{2\pi}{N}k\right) = \cancel{c_k}\ N.c_k$$

eq^n ⑤ becomes

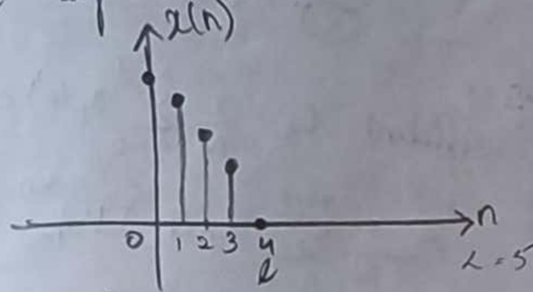
$$x_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right)e^{j\frac{2\pi}{N}kn} \qquad \text{—— (7)}$$

eq^n (7) provides reconstruction of periodic signal $x_p(n)$ from $X(\omega)$

However it does not imply that we can recover $X(\omega)$ of $x(n)$

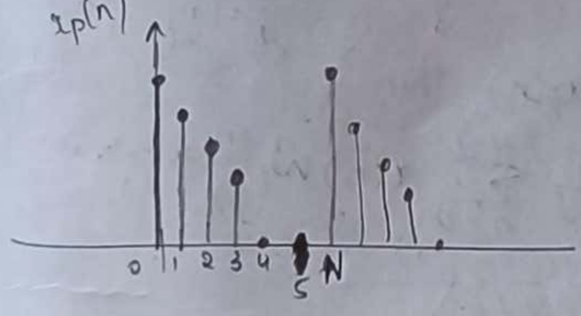
i.e; to recover the original signal $x(n)$ the length of the sequence should be less than N.

Consider signal $x(n)$



$L = 5$

eg 1

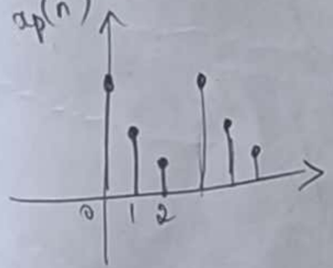


$N > L$ we can reconstruct

$L = 5$, $N = 7$ no overlapping so we can reconstruct

No aliasing.

eg 2



$N = 3$ we cannot reconstruct

$N < L$ → overlapping takes place

aliasing.

① if $N \geq L$; we can reconstruct the original signal $x(n)$. There is no loss of signals

② if $N < L$; it is not possible to recover $x(n)$. There is a loss of signal / information

Relationship between DFT and other transforms

① DFT with fourier series coefficient of a periodic Sequence

A periodic seq $x_p(n)$ with fundamental period N can be expressed in a fourier series form

$$x_p(n) = \sum_{k=0}^{N-1} c_k \, e^{j\frac{2\pi}{N}kn} \quad \text{— ①}$$

where fourier series coefficient $c_k = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n) \, e^{-j\frac{2\pi}{N}kn}$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x_p(n) \, W_N^{kn}$$

NOTE :- $\boxed{W_N = e^{-j\frac{2\pi}{N}}}$

if $x(n) = x_p(n)$ then

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n) \, W_N^{kn}$$

$$\boxed{\underbrace{c_k}_{\text{fourier series coefficient}} = \frac{1}{N} \underbrace{X(k)}_{DFT}}$$

(or)

$$\boxed{X(k) = N \cdot C_k}$$

② DFT with Z-transforms

Consider z-transform of a Sequence

$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) \, z^{-n} \quad \text{— ①}$$

ROC includes the unit $\bigcirc^{le}$ of $X(z)$ are sampled at a N- number of equally spaced points on a unit $\bigcirc^{le}$ with $z = e^{j\frac{2\pi}{N}k}$

then $X(k) = X(z) \Big|_{z = e^{j\frac{2\pi}{N}k}}$

$$X(k) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\frac{2\pi}{N}kn}$$

consider $X(z) = \sum_{n=0}^{N-1} x(n) z^{-n}$

$$= \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn} \right] z^{-n}$$

interchange order of summation we get

$$X(z) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \sum_{n=0}^{N-1} W_N^{-kn} z^{-n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \sum_{n=0}^{N-1} \left(W_N^{-k} z^{-1} \right)^n$$

$$\sum_{n=0}^{N-1} b^n = \frac{1-b^N}{1-b}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \frac{1 - \cancel{W_N^{-kN}}^{\,1} z^{-N}}{1 - W_N^{-k} z^{-1}}$$

$$X(z) = \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1 - W_N^{-k} z^{-1}}$$

1) Let $X(k)$ be the 14-point DFT of the seq $x(n)$ the first 8 samples of $X(k)$ are

$X(0) = 12$

$X(1) = -1 + j3$

$X(2) = 3 + j4$

$X(3) = 1 - j5$

$X(4) = -2 + j2$

$X(5) = 6 + j3$

$X(6) = -2 + j3$

$X(7) = 10$

find the remaining samples of $X(k)$

$N = 14$

$\rightarrow X(k) = X^*(N-k) = X^*(14-k)$

$X(8) = X^*(14-8) = X^*(6) = -2-j3$

$X(9) = X^*(14-9) = X^*(5) = 6-j3$

$X(10) = X^*(14-10) = X^*(4) = -2-j2$

$X(11) = X^*(14-11) = X^*(3) = 1+j5$

$X(12) = X^*(14-12) = X^*(2) = 3-j4$

$X(13) = X^*(14-13) = X^*(1) = -1-j3$

Assignment

1) Let $x_p(n)$ be a periodic sequence with fundamental period N. Let $X_1(k)$ denotes N-point DFT of ONE period of $x_p(n)$ & $X_3(k)$ denotes 3-N point DFT of 3-periods of $x_p(n)$. What is relation b/w $X_1(k)$ & $X_3(k)$

2) Let $x(n)$ be a real valued seq of length N & its N-point DFT is $X(k)$. Show that

i) $X(N-k) = X^*(k)$

ii) $X(0)$ is real

iii) if N is given then $X\left(\frac{N}{2}\right)$ is real

3) compute N-point DFT of the sequence

$x(n) = an, \; 0 \le n \le N-1$

MOD :- 02

Additional DFT properties

① Circular Shift of DFT input / Circular Shift in time

if DFT $[x(n)] \longrightarrow X(k)$ then

$$\text{DFT}\left[x((n-m))_N\right] \longrightarrow W_N^{mk}\, X(k)$$

Proof :- consider

$$\text{IDFT}\left[X(k)\right] = x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn} \quad \text{—— ①} \quad 0 \le n \le N-1$$

put $n = n-m$

$$x(n-m) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-k(n-m)}$$

$$x((n-m))_N = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn}\, W_N^{+km}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} \left\{X(k)\, W_N^{km}\right\} W_N^{-kn} \quad \text{—— ②}$$

$$x((n-m))_N = \text{IDFT}\left\{W_N^{km}\, X(k)\right\}$$

(OR)

$$\text{DFT}\left[x((n-m))_N\right] = W_N^{mk}\, X(k)$$

① Find ~~the~~ 4-point DFT of the seq $x(n) = \{1, -1, 1, -1\}$ and also compute DFT of the sequence $y(n) = x((n-2))_4$

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$N = 4$

$$X(k) = \sum_{n=0}^{3} x(n) W_N^{kn}$$

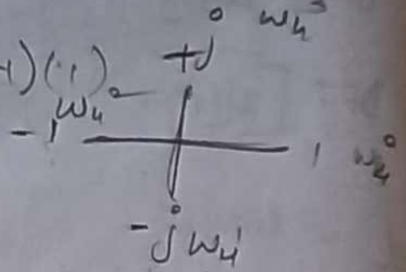
$$= x(0)W_4^{k(0)} + x(1)W_4^{k} + x(2)W_4^{2k} + x(3)W_4^{3k}$$

$$X(k) = 1 + (-1)W_4^{k} + 1W_4^{2k} + (-1)W_4^{3k}$$

$k=0$ $\quad X(0) = 1+(-1)+(1)(1)+(-1)(1)$

$= 1-1+1-1$

$= 0$



$k=1$ $\quad X(1) = 1+(-1)(1)+1(\ldots)+(-1)(\ldots)$

$= 1 + j - j - j$

$= \cancel{2-1-j} = 0$

$= 1-j$

$k=2$ $\quad X(2) = 1+(-1)(-1)+1(1)+(-1)(-1)$

$= 1+1+1+1$

$= 4$

$k=3$ $\quad X(3) = 1+(-1)(j)+1(-1)+(-1)(-j)$

$= 1-j-1+j$

$= 0$

$$X(k) = \{\underset{\uparrow}{0}, 0, 4, 0\}$$

given $y(n) = x((n-2))_4$

wkt

$$DFT\left[x((n-m))_N\right] = W_N^{mk} X(k)$$

$m = 2$

$$\boxed{Y(k) = W_4^{2k} X(k)}$$

$Y(0) = W_4^0 X(0) = 1 \times 0 = 0$

$Y(1) = W_4^2 X(1) = -1 \times 0 = 0$

$Y(2) = W_4^4 X(2) = 1 \times 4 = 4$

$Y(3) = W_4^2 X(3) = -1 \times 0 = 0$

$Y(k) = \{\underset{\uparrow}{0}, 0, 4, 0\}$

2) 4-point DFT of the sequence is given by

$X(k) = \{\underset{\uparrow}{8}, 2-j2, -2, 2+j2\}$

find the 4-pt DFT of the following seq.

a) $y_1(n) = x((n-1))_4$

b) $y_2(n) = x((n-2))_4$

→ given $y(n) = x((n-1))_4$

$m = 1, \; N = 4$

$Y(k) = W_N^{mk} X(k)$

$$\boxed{Y(k) = W_4^k X(k)}$$

$k=0, \; Y(0) = W_4^0 X(0) = 1 \times 8 = 8$

$Y(1) = W_4^1 X(1) = -j \times (2-j2) = -j2-2$

$$y(2) = W_4^2 X(2) = -1 \times -2 = 2$$

$$y(3) = W_4^3 X(3) = j(2+j2) = -2+j2.$$

$$y_1(k) = \{\underset{\uparrow}{2}, -2-j2, 2, -2+j2\}$$

5-6m

(2) Circular Shift in DFT output

Statement :- If DFT of $x(n)$ =

$$x(n) \xrightarrow[\text{Time}]{} \boxed{\text{DFT}} \xrightarrow[\text{Freq.}]{\text{o/p}} X(k)$$

If $DFT[x(n)] = X(k)$ then

$$DFT[W_N^{-ln} x(n)] = X((k-l))_N$$

Proof :- Consider DFT of the sequence $x(n)$

$$DFT[x(n)] = X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad \text{——— (1)}.$$

Let $k = k - l$ in eq (1).

$$X((k-l))_N = \sum_{n=0}^{N-1} x(n) W_N^{(k-l)n}$$

$$= \sum_{n=0}^{N-1} \{x(n) W_N^{-ln}\} W_N^{kn} \quad \text{——— (2)}.$$

compare (1) & (2)

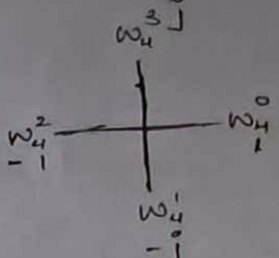
$$X((k-l))_N = DFT[W_N^{-ln} x(n)]$$

1) The 4-point ~~DFT of the~~ seq $x(n)$ is given by

~~$X(K) = \{8, 2-j2, -4, 2+j2\}$~~ $x(n) = \{\underset{\uparrow}{4}, 2, 0, 1\}$

find $y(n)$ if $Y(K) = X((K-2))_4$

→ wkt $DFT[W_N^{-ln} x(n)] = X((K-l))_N$



$l = 2 \quad N = 4$

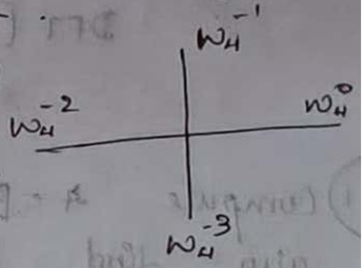
$y(n) = W_4^{-2n} x(n)$

$n = 0 \quad y(0) = W_4^0 x(0) = 1 \times 4 = 4$

$n = 1 \quad y(1) = W_4^{-2} x(1) = -1 \times 2 = -2$

$n = 2 \quad y(2) = W_4^{-4} x(2) = 1 \times 0 = 0$

$n = 3 \quad y(3) = W_4^{-6} x(3) = -1 \times 1 = -1$



$y(n) = \{\underset{\uparrow}{4}, -2, 0, -1\}$

3) <u>Time Reversal of a Sequence</u>

$x(n) = \{1, 2, 1, 0\}$

$x((-n)) = \{1, 0, 1, 2\}$

If DFT of $x(n) = X(K)$, then

$DFT[x((-n))_N] = X((-K))_N = X(N-K)$

<u>Proof</u> :- Consider $DFT[x(n)] = X(K) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$ → (1)

put $n = N-n$ in eq (1).

$$DFT[x(N-n)] = \sum_{n=0}^{N-1} x(N-n) W_N^{kn}$$

Let $m = N-n$ then $n = N-m$

$$= \sum_{m=0}^{N-1} x(m) W_N^{k(N-m)}$$

$$= \sum_{m=0}^{N-1} x(m) \cancel{W_N^{kN}}^{\,1} W_N^{-km}$$

$$= \sum_{m=0}^{N-1} x(m)\, W_N^{-km} = X((-k))_N \quad \text{— (2)}$$

$$DFT\left[x(N-n)\right] = \sum_{m=0}^{N-1} x(m)\, W_N^{-km}\, W_N^{Nm}$$

$$= \sum_{m=0}^{N-1} x(m)\, W_N^{(N-k)m}$$

$\boxed{\underline{\text{Note}} \quad W_N^{Nm} = 1}$

$$= X(N-k)$$

$$\therefore\ DFT\left[x((-n))\right] = x(N-n)] = X((-k))_N = X(N-k)$$

1) Compute 4-point DFT $x(n) = \{\underset{\uparrow}{1}, 2, 1, 0\}$ and also find $Y(K)$ if $y(n) = x((-n))_4$

$\rightarrow$ $$X(K) = \sum_{n=0}^{3} x(n)\, W_4^{kn}$$

$$= x(0)W_4^{k(0)} + x(1)W_4^{k} + x(2)W_4^{2k} + x(3)W_4^{3k}$$

$$X(K) = 1 + 2W_4^{k} + 1W_4^{2k} + 0$$

$j\ W_4^3$, $W_4^2\ -1$, $1\ W_4^0$, $-j\ W_4^1$

$K = 0$; $X(0) = 1 + 2(\cancel{-j}) + (1)(-1)$

$= \cancel{1+2j-1}\ \ 1+2+1$

$\cancel{-2j} = 4$

$K = 1$; $X(1) = 1 + 2 \times (-j) + 1(-1)$

$= 1 - 2j - 1$

$= -2j$

$k=2$, $X(2) = 1 + 2\times(-1) + 1(1)$
$= 1 - 2 + 1$
$= 0$

$k=3$; $X(3) = 1 + 2\times(j) + (1)(-1)$
$= 1 + 2j - 1$
$= 2j$

$$X(K) = \{\underset{\uparrow}{4}, -2j, 0, 2j\}$$

given :- $y(n) = x((-n))_4$

Taking DFT on b.s

$Y(K) = X((-K))_4$

$= \{\underset{\uparrow}{4}, 2j, 0, -2j\} = X^*(K)$

2) A 5-point DFT of the seq $x(n)$ is given by $X(K) = \{\underset{\uparrow}{j}, 1+j, 1+j^2, 2+j2, 4+j\}$. Compute $Y(K)$ if $y(n) = x^*(n)$.

→ given :- $y(n) = x^*(n)$
taking DFT on b.s
$Y(K) = X^*((-K))_5$

$X((-K))_5 = \{\underset{\uparrow}{j}, 4+j, 2+j2, 1+j2, 1+j\}$

$X^*((-K))_5 \cdot Y(K) = \{\underset{\uparrow}{-j}, 4-j, 2-j2, 1-j2, 1-j\}$

3) Consider the following seq of length = 8

(a) $x_1(n) = \{2, 2, 2, 0, 0, 0, 2, 2\}$

(b) $x_2(n) = \{2, 2, 0, 0, 0, 0, -2, -2\}$

(c) $x_3(n) = \{0, 2, 2, 0, 0, 0, -2, -2\}$

(d) $x_4(n) = \{0, 2, 2, 0, 0, 0, 2, 2\}$

Which seq have a real valued 8-point DFT?
Which seq have an imaginary valued 8-point DFT

→ $x(n) = x((-n)) \Rightarrow$ even valued & purely real

$x(n) = -x((-n)) \Rightarrow$ odd & imaginary

* $x_1(n) = \{2, 2, 2, 0, 0, 0, 2, 2\}$

$x_1((-n)) = \{2, 2, 2, 0, 0, 0, 2, 2\}$

even & purely real

* $x_2(n) = \{2, 2, 0, 0, 0, 0, -2, -2\}$

$x_2((-n)) = \{2, -2, -2, 0, 0, 0, 0, 2\}$

~~odde & im~~

$-x_2((-n)) = \{-2, 2, 2, 0, 0, 0, 0, -2\} \Rightarrow$ neither even nor odd

* $x_3(n) = \{0, 2, 2, 0, 0, 0, -2, -2\}$

$x_3((-n)) = \{0, -2, -2, 0, 0, 0, 2, 2\}$

odd & ~~imaginary real valued~~ purely imaginary

* $x_4(n) = \{0, 2, 2, 0, 0, 0, 2, 2\}$

$= \{0, 2, 2, 0, 0, 0, 2, 2\}$

even & purely real.

Complex Conjugate property

If $DFT[x(n)] = X(K)$ then

$$DFT[x^*(n)] = X^*((-k))_N = X^*(N-k)$$

Consider $DFT[x(n)] = X(K) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$

$$X^*(k) = \sum_{n=0}^{N-1} x^*(n) W_N^{-kn} \longrightarrow (1)$$

put $k = -k$

$$X^*(-k) = \sum_{n=0}^{N-1} x^*(n) W_N^{kn} = DFT[x^*(n)] - (2)$$

put $k = N-k$

$$X^*(N-k) = \sum_{n=0}^{N-1} x^*(n) W_N^{-(N-k)n}$$

$$= \sum_{n=0}^{N-1} x^*(n) \cancel{W_N^{-Nn}} \, W_N^{kn}$$

$$= \sum_{n=0}^{N-1} x^*(n) W_N^{kn} = DFT[x^*(n)] - (3)$$

$$DFT[x^*(n)] = X^*(k) = X^*(N-k)$$

Circular Convolution

If $x(n)$ and $y(n)$ are the complex valued Seq i.e;

$$x(n) \xrightarrow{DFT} X(K)$$

and $$y(n) \xrightarrow{DFT} Y(K) \text{ then}$$

$$\tilde{r}_{xy}(l) \xrightarrow{DFT} \tilde{R}_{xy}(K) = X(K) Y^*(K)$$

where

$$\tilde{r}_{xy}(l) = x(n) \circledast_N y(n) = \sum_{n=0}^{N-1} x(n) y((n-l))_N$$

Proof

Consider circular convolution of $x(l)$ & $y(-l)$

$$\tilde{r}_{xy}(l) = x(l)\, y^*(-l)$$

taking DFT

$$\tilde{R}_{xy}(k) = X(k)\, Y^*(k)$$

Multiplication of Two Time Domain Sequences

Let $x_1(n)$ and $x_2(n)$ are the two time Domain Sequences then

$$x_1(n) \xrightarrow[N]{DFT} X_1(k)$$

& $$x_2(n) \xrightarrow{DFT} X_2(k) \text{ then}$$

$$x_1(n)\, x_2(n) \xrightarrow[N]{DFT} \frac{1}{N}\left[X_1(k) \circledast_N X_2(k)\right]$$

Consider $$DFT\left[x(n)\right] = X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$$DFT\left[x_1(n)\,x_2(n)\right] = \sum_{n=0}^{N-1} \left[x_1(n)\, x_2(n)\right] W_N^{kn}$$

$$IDFT\left[X_2(l)\right] = x_2(n) = \frac{1}{N}\sum_{l=0}^{N-1} X_2(l)\, W_N^{-ln}$$

$$= \sum_{n=0}^{N-1} x_1(n)\, \frac{1}{N}\sum_{l=0}^{N-1} X_2(k)\, W_N^{-ln}\, W_N^{kn}$$

$$= \frac{1}{N}\sum_{l=0}^{N-1} X_2(l) \sum_{n=0}^{N-1} x_1(n)\, W_N^{(k-l)n}$$

$$= \frac{1}{N}\sum_{l=0}^{N-1} X_2(l)\, X_1(k-l)$$

$$DFT\left[x_1(n)\,x_2(n)\right] = \frac{1}{N}\left[X_1(k) \circledast_N X_2(k)\right]$$

Parsevals Theorem

It is used to find the energy of the given Signal

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

Consider

$$\sum_{n=0}^{N-1} |x(n)|^2 = \sum_{n=0}^{N-1} x(n)\, x^*(n)$$

$$= \sum_{n=0}^{N-1} x(n) \frac{1}{N} \sum_{K=0}^{N-1} X^*(K)\, W_N^{+kn}$$

$$= \frac{1}{N} \sum_{K=0}^{N-1} X^*(k) \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$$= \frac{1}{N} \sum_{K=0}^{N-1} X^*(k)\, X(k)$$

$$= \frac{1}{N} \sum_{K=0}^{N-1} |X(k)|^2$$

1) Find the energy of the sequence $X(K) = \{2, j2, -2, 0\}$

$2+j^2$
mag $= \sqrt{2^2+j2^2}$
$= \sqrt{4+4}$

$\Rightarrow$ $$E = \frac{1}{4} \sum_{K=0}^{3} |X(K)|^2$$

$$= \frac{1}{4} \left[|X(0) + X(1) + X(2) + X(3)|^2 \right]$$

$$= \frac{1}{4} [2 + 2 + 2 + 0]$$

$$= \frac{1}{\cancel{4}_2} [\cancel{6}^3] = \frac{3}{2}\ J$$

2) Let $g(n)$ & $h(n)$ are the Two sequence of length = 6. They have 6 point DFT'S $G(K)$ & $H(K)$ respectively. The Sequence $g(n)$ is given by $\{4.1, 3.5, 1.2, 5, 2, 3.3\}$ the relation b/w $G(K)$ & $H(K)$ are given by $H(K) = G((K-3))_6$; Determine $h(n)$ without computing DFT & IDFT

→ WKT $DFT[W_N^{-ln} x(n)] = X((K-l))_N$

$$l = 3 \qquad N = 6$$

$$DFT[W_N^{-ln} g(n)] = G((K-l))_N$$

$$h(n) = W_N^{-ln} g(n)$$

$$h(n) = W_6^{-3n} g(n)$$

$W_N = e^{-j\frac{2\pi}{N}}$

$W_6 = e^{-j\frac{2\pi}{6}} = e^{-j\frac{\pi}{3}}$

$W_6^0 = 1$

$W_6^{-3} = e^{j\frac{\pi}{3}\times 3}$

$= \cos\pi + j\sin\pi$

$= -1$

$n = 0;\ h(0) = W_6^0 g(0) = 1\times 4.1 = 4.1$

$h(1) = W_6^{-3} g(1) = -1\times 3.5 = -3.5$

$h(2) = W_6^0 g(2) = 1.2$

$h(3) = W_6^{-3} g(3) = -5$

$h(4) = W_6^0 g(4) = 2$

$h(5) = W_6^3 g(5) = -3.3$

3) Let $x(n)$ be a finite length seq with $X(K) = \{0, 1+j, 1, 1-j\}$
using properties of DFT. Find the DFT of the following seq

① $x_1(n) = e^{j\frac{\pi}{2}n}\, x(n)$

② $x_2(n) = \{\cos \frac{\pi}{2} n\}\, x(n)$

$X_1(K) = ?$

$X_2(K) = ?$

→① $x_1(n) = e^{j\frac{\pi}{2}n} x(n)$

$N = 4$

$W_N = e^{-j\frac{2\pi}{N}}$

$W_4 = e^{-j\frac{2\pi}{4}} = e^{-j\frac{\pi}{2}}$

$W_4^{-1} = e^{j\frac{\pi}{2}}$

$x_1(n) = W_N^{-n} x(n)$

using property of circular shift in freq

$e^{-j\frac{\pi}{3}}$ DFT $[W_N^{-ln} x(n)] = X((K-l))_N$

$l = 1 \qquad N = 4$

$X_1(K) = X((K-1))_4$

$n\pi$ $= \{1-j, 0, 1+j, 1\}$

Consider $x_2(n) = \{\cos \frac{\pi}{2} n\} x(n)$

$= \left\{\dfrac{e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}}{2}\right\} x(n)$

$= \frac{1}{2} e^{j\frac{\pi}{2}n} x(n) + \frac{1}{2} e^{-j\frac{\pi}{2}n} x(n)$

$W_N = e^{-j\frac{2\pi}{N}}$

$W_4 = e^{-j\frac{\pi}{2}}$

$W_4^{-1} = e^{j\frac{\pi}{2}}$

- right shift
+ left shift

$x_2(n) = \frac{1}{2} W_4^{-n} x(n) + \frac{1}{2} W_4^{n} x(n)$

$l = 1$

$DFT\left[W_N^{-ln} x(n)\right] = X((k-l))_N$

$X_2(k) = \frac{1}{2} X((k-1))_4 + \frac{1}{2} X((k+1))_4$

$= \frac{1}{2}\{1-j, 0, 1+j, 1\} + \frac{1}{2}\{1+j, 1, 1-j, 0\}$

$= \frac{1}{2}\{2, 1, 2, 1\}$

$X_2(k) = \{1, \frac{1}{2}, 1, \frac{1}{2}\}$

④ Let $x(n)$ be a finite length Sequence with $X(k) = \{\underset{\uparrow}{10}, -2-j2, -2, -2+j2\}$ using properties of DFT find the DFT of $x_1(n) = x(4-n)$

→ wkt $DFT\left[x((-n))_N = x(N-n)\right] = X((-k))_N = X(N-k)$

The given Seq is folded version of $x(n)$

$X_1(k) = X((-k))_4$

$\{10, -2+j2, -2, -2-j2\}$

⑤ Let $x(n)$ be a real valued seq and it is defined as $x(n) = \{1, 2, 3, 4\}$. without evaluating DFT find

① $\sum_{k=0}^{3} X(k)$

② $X(0)$

→ ① consider $x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-kn}$

with $n = 0$

$x(0) = \frac{1}{4}\sum_{k=0}^{3} X(k)$

$\sum_{k=0}^{3} X(k) = 4x(0) = 4 \times 1 = 4$

(a) Consider $X(K) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$

with $K = 0$

$$X(0) = \sum_{n=0}^{3} x(n)$$

$$X(0) = x(0) + x(1) + x(2) + x(3)$$
$$= 1+2+3+4$$
$$= 10.$$

(6) Consider a finite duration seq $x(n) = \{0, 1, 2, 3, 4, 5\}$ find the seq.

(1) $s(n)$ with $S(K) = W_2^{K} X(K)$

(2) Determine the seq $y(n)$ with 6-point DFT $Y(K) = R_e X(K)$

→ $S(K) = W_2^{K} X(K)$

wkt
$$W_N = e^{-j\frac{2\pi}{N}}$$
$$W_6 = e^{-j\frac{2\pi}{6}}$$
$$W_6^3 = e^{-j\frac{2\pi}{6}\times 3} = e^{-j\frac{2\pi}{2}} = e^{-j\pi}$$
$$W_2 = e^{-j\frac{2\pi}{2}} = e^{-j\pi}$$
$$W_6^3 = W_2$$
$$S(K) = W_6^{3k} X(K)$$

wkt $DFT[x((n-m))_N] = W_N^{mk} X(K)$

$$m = 3, \quad N = 6$$
$$s(n) = x((n-3))_6$$

$x((n-1))_6 = \{5, 0, 1, 2, 3, 4\}$

$x((n-2))_6 = \{4, 5, 0, 1, 2, 3\}$

$s(n) = x((n-3))_6 = \{3, 4, 5, 0, 1, 2\}$.

② $y(k) = Re\ X(k)$

$$Y(k) = \frac{1}{2}\{X(k) + X^*(k)\}$$

$$y(n) = \frac{1}{2}\{x(n) + x((-n))_N\}$$

$$= \frac{1}{2}\{(0,1,2,3,4,5) + (0,5,4,3,2,1)\}$$

$$= \{\underset{\uparrow}{0}, 3, 3, 3, 3, 3\}$$

Let $g(n)$ & $h(n)$ are the two sequence of length 6, they have 6-point DFT $G(K)$ & $H(K)$ respectively. The sequence $g(n) = \{\underset{\uparrow}{4.1}, 3.5, 1.2, 5, 2, 3.3\}$. The DFTs of $G(K)$ & $H(K)$ are related by circular freq shift has $H(K) = G((K-3))_6$. Determine $h(n)$ without computing DFT & IDFT consider finite length sequence $x(n) = \delta(n) + 2\delta(n-5)$

i) find the 10-point DFT of the sequence $x(n)$

ii) find the sequence $y(n)$ without computing DFT for $Y(k) = e^{j2k\frac{2\pi}{10}} X(k)$ where $X(k)$ is the 10-point DFT of the sequence $x(n)$

iii) find the 10-point sequence $y(n)$ that DFT $Y(k) = X(k)W(k)$ where $W(k)$ is a 10-point DFT of the sequence $w(n)$ & it is defined as $w(n) = \begin{cases} 1 & 0 \le n \le 2 \\ 0 & \text{otherwise} \end{cases}$

→ i) given $x(n) = \delta(n) + 2\delta(n-5)$

$N = 10$.

$x(n) = \{\underset{\uparrow}{1}, 0, 0, 0, 0, 2, 0, 0, 0, 0\}$

$X(k) = \sum_{n=0}^{N-1} x(n) W_n^{kn} \qquad 0 \le k \le N-1$

$X(k) = \sum_{n=0}^{9} x(n) W_{10}^{kn} \qquad 0 \le k \le 9.$

$= 1 W_{10}^{k\times 0} + 2 W_{10}^{5k} \qquad W_{10} = e^{-j\frac{2\pi}{10}} = e^{-j\frac{\pi}{5}}$

$X(k) = 1 + 2W_{10}^{5k}$

$W_{10}^{0} = 1$

$W_{10}^{5} = e^{-j\pi} = -1$

$k=0, \quad X(0) = 1+2 = 3.$

$k=1, \quad X(1) = 1 + 2W_{10}^{5} = 1-2 = -1$

$k=2, \quad X(2) = 1+2 = 1+2 = 3.$

$k=3, \quad X(3) = 1+2(-1) = -1$

$X(k) = \{\underset{\uparrow}{3}, -1, 3, -1, 3, -1, 3, -1, 3, -1\}$

$W_{10} = e^{-j\frac{2\pi}{10}}$

ii) $Y(k) = e^{j2k\frac{2\pi}{10}} = e^{j\frac{2\pi}{10} \cdot 2k} X(k) \qquad W_{10}^{-1} = e^{j\frac{2\pi}{10}}$

$Y(k) = W_{10}^{-2k} X(k)$ compare

wkt circular shift in time $x((n-m))_N = W_N^{mk} X(k)$

$N = 10, \quad m = -2$

$y(n) = x((n+2))_{10}$

$x(n) = \{1, 0, 0, 0, 0, 2, 0, 0, 0, 0\}$

$x((n+1))_{10} = \{0\ 0\ 0\ 0\ 2\ 0\ 0\ 0\ 0\ 1\}$

$x((n+2))_{10} = \{0\ 0\ 0\ 2\ 0\ 0\ 0\ 0\ 1\ 0\} = y(n)$

iii) $Y(k) = X(k)W(k)$

$w(n) : \begin{cases} 1 & 0 \le n \le 2 \\ 0 & \end{cases}$

$w(n) = \{1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\}$

$$W(k) = \sum_{n=0}^{N-1} w(n) W_N^{kn}$$

$$= \sum_{n=0}^{9} w(n) W_{10}^{kn}$$

$$= w(0)W_{10}^{0} + w(1)W_{10}^{k} + w(2)W_{10}^{2k} + 0 + 0 + 0$$

$$W(k) = 1 + W_{10}^{k} + W_{10}^{2k}$$

$$Y(k) = X(k)W(k)$$

$$= (1 + 2W_{10}^{5k})(1 + W_{10}^{k} + W_{10}^{2k})$$

$$Y(k) = 1 + W_{10}^{k} + W_{10}^{2k} + 2W_{10}^{5k} + 2W_{10}^{6k} + 2W_{10}^{7k}$$

Taking Inverse DFT

$$y(n) = \frac{1}{10} \sum_{k=0}^{9} Y(k) W_{10}^{-kn}$$

$$y(n) = \frac{1}{10} \sum_{k=0}^{9} \left(1 + W_{10}^{k} + W_{10}^{2k} + 2W_{10}^{5k} + 2W_{10}^{6k} + 2W_{10}^{7k}\right) W_{10}^{-kn}$$

$$= \frac{1}{10} \sum_{k=0}^{9} (W_{10}^{-n})^{k} + \frac{1}{10} \sum_{k=0}^{9} (W_{10}^{1-n})^{k} + \frac{1}{10} \sum_{k=0}^{9} (W_{10}^{2-n})^{k} +$$

$$\frac{2}{10} \sum (W_{10}^{5-n})^{k} + \frac{2}{10} \sum (W_{10}^{6-n})^{k} + \frac{2}{10} \sum (W_{10}^{7-n})$$

$$= 1\delta(n) + \delta(n-1) + \delta(n-2) + 2\delta(n-5) + 2\delta(n-6) + 2\delta(n-7)$$

$$y(n) = \{1, 1, 1, 0, 0, 2, 2, 2, 0, 0\}$$

(OR)

given $x(n) = \{1\ 0\ 0\ 0\ 0\ 2\ 0\ 0\ 0\ 0\}$
$w(n) = \{1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\}$

Sl No	$x(n)$	$w((n-m))_{10}$	$y(n)$
0	1 0 0 0 0 2 0 0 0 0	1 0 0 0 0 0 0 0 1 1	1
1	1 0 0 0 0 2 0 0 0 0	1 1 0 0 0 0 0 0 0 1	1
	1 0 0 0 0 2 0 0 0 0	1 1 1 0 0 0 0 0 0 0	1
2	1 0 0 0 0 2 0 0 0 0	0 1 1 1 0 0 0 0 0 0	0
3	1 0 0 0 0 2 0 0 0 0	0 0 1 1 1 0 0 0 0 0	0
4			
5	1 0 0 0 0 2 0 0 0 0	0 0 0 1 1 1 0 0 0 0	2
6	1 0 0 0 0 2 0 0 0 0	0 0 0 0 1 1 1 0 0 0	2
7	1 0 0 0 0 2 0 0 0 0	0 0 0 0 0 1 1 1 0 0	2
8	1 0 0 0 0 2 0 0 0 0	0 0 0 0 0 0 1 1 1 0	0
9	1 0 0 0 0 2 0 0 0 0	0 0 0 0 0 0 0 1 1 1	0

Compare Linear convolution & circular convolution

Linear Convolution	Circular Convolution
→ The output $y(n)$ contain $N = L+M-1$ no of samples.	→ $y(n)$ contains $N = \max\{L, M\}$
→ Shifting takes place linearly	→ Shifting takes place circularly.
→ Used to find the response of a linear filter	→ Cannot be used.
→ Zero's padding is not required to find the response of a linear filter	→ Zero's padding is required to find the response of linear filter.
→ Range of time index is $-\infty \le n \le \infty$	→ $0 \le n \le N-1$

W_{10}^{-kn}

$W_{10}^{2-n})^k +$

$\sum (W_{10}^{7-n})^k$

$\delta(n-7)$

7] given 8-point sequence $x(n) = \begin{cases} 1 & 0 \le n \le 3 \\ 0 & \text{otherwise} \end{cases}$

compute DFT of the sequence $x_1(n) = \begin{cases} 1 & n = 0 \\ 0 & 1 \le n \le 4 \\ 1 & 5 \le n \le 7 \end{cases}$

use properties of DFT.

$\rightarrow x(n) = \{ \underset{\uparrow}{1}\ 1\ 1\ 1\ 0\ 0\ 0\ 0 \}$

$x_1(n) : \{ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 1 \}$

$x_1((-n)) = \{ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0 \}$

$x(n) = x_1((-n))$ or $x((-n))_8 = x_1(n)$

Taking DFT

$X_1(k) = X((-k))_8$

$X(k) = \sum_{n=0}^{7} x(n) W_8^{kn}$

$X(k) = 1 + W_8^{k} + W_8^{2k} + W_8^{3k}$

$k = 0 \quad X(0) = 1+1+1+1 = 4$

$X(1) =$