0

Digital Signal Processing

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***Abstract*—This manual provides a simple introduction to digital signal processing.**

1 Software Installation

Run the following commands

```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1 python3
    −scipy python3−numpy python3−matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://raw.githubusercontent.
    com/gadepall/
EE1310/master/filter/codes/
    Sound_Noise.wav
```

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2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution:

2.4 $Theoutputofthepythonscriptin Problem 2.3 istheaudiofile$
Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{3.1}$$

Sketch $x(n)$.

3.2 Let

$$y(n) + \frac{1}{2} y(n-1) = x(n) + x(n-2),$$
$$y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.
Solution: The following code yields Fig. 3.2.

```
wget https://github.com/gadepall/
    EE1310/raw/master/filter/codes/
    xnyn.py
```

3.3 Repeat the above exercise using a C code. **solution:** The following code is the implementation in C.

```
https://github.com/dhanushpittala11/
    EE3900−2022/blob/main/filter/
    codes/xn_yn.ipynb
```

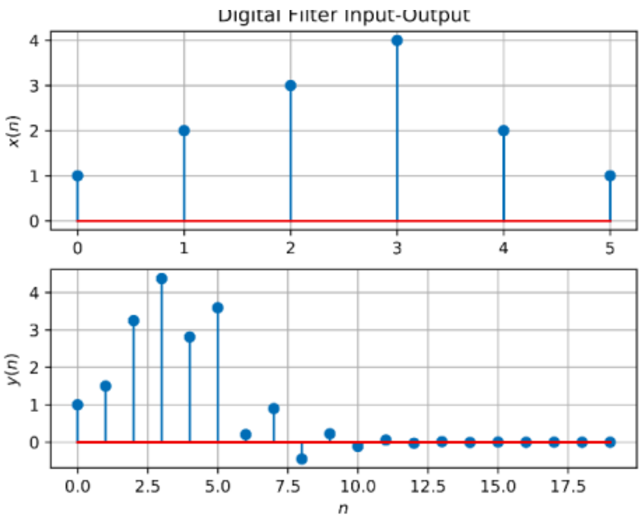


Fig. 3.2

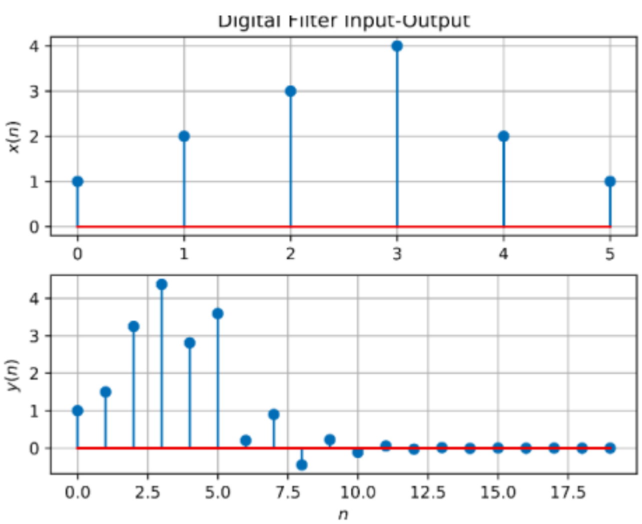


Fig. 3.3

4 Z-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: From (4.1),

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{4.4}$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1} \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.5}$$

resulting in (4.2). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \tag{4.6}$$

4.2 Obtain $X(z)$ for $x(n)$ defined in problem 3.1.
Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.7}$$

But

$$x(n) = \{1, 2, 3, 4, 2, 1\} \tag{4.8}$$

so,

$$X(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5} \tag{4.9}$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.10}$$

from (3.2) assuming that the Z-transform is a linear operation.
Solution: Applying (4.6) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.11}$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{4.12}$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.13}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.14}$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.15}$$

Solution: It is easy to show that

$$\delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} 1 \tag{4.16}$$

and from (4.14),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.17}$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.18}$$

using the formula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.19}$$

Solution: let

$$f(n) = a^n u(n) \tag{4.20}$$

$$f(n) = \left\{ \begin{array}{ll} a^n, & \text{if } n > 0 \\ 0, & \text{otherwise} \end{array} \right\} \tag{4.21}$$

Now the Z- Transform of f(n) is

$$F(z) = \mathcal{Z}\{f(n)\} = \sum_{n=-\infty}^{\infty} f(n)z^{-n} \tag{4.22}$$

$$F(z) = \sum_{n=0}^{\infty} a^n z^{-n} \tag{4.23}$$

This forms a infinite Geometric Progression.

$$F(z) = \frac{1}{1 - az^{-1}} \text{ for } z < a. \tag{4.24}$$

4.6 Let

$$H\left(e^{j\omega}\right) = H\left(z = e^{j\omega}\right). \tag{4.25}$$

Plot $\left|H\left(e^{j\omega}\right)\right|$.Is it periodic? If so, find the period. Comment. $H(e^{j\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $x(n)$.
Solution:

$$|H(e^{j\omega})| = \frac{|(1 + cos(2\omega) - isin(2\omega))|}{|(1 + \frac{1}{2}cos(\omega)| - i\frac{1}{2}sin(\omega))|} \tag{4.26}$$

$$|H(e^{j\omega})| = \frac{|2cos(\omega)|}{|\sqrt{\frac{5}{4} + cos(\omega)}|} \tag{4.27}$$

$$|H(e^{j\omega})| = \frac{|4cos(\omega)|}{|\sqrt{5 + 4cos(\omega)}|} \tag{4.28}$$

we can see that the period of $H(e^{j\omega})$ is same as of $cos(\omega)$ which is 2π.
Hence the period is 2π.
The following code plots Fig.

```
wget https://raw.githubusercontent.
    com/gadepall/EE1310/master/
    filter/codes/dtft.py
```

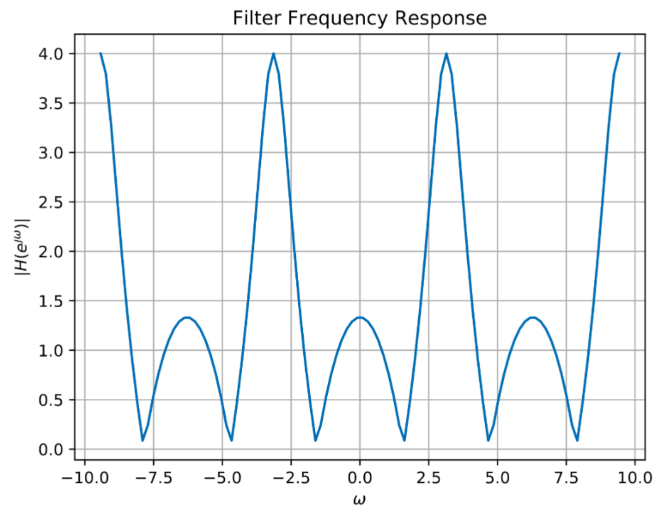


Fig. 4.6: $\left|H\left(e^{j\omega}\right)\right|$

4.7 Express $x(n)$ in terms of $H\left(e^{j\omega}\right)$.
Solution: We have,

$$H(e^{\jmath\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-\jmath\omega k} \tag{4.29}$$

However,

$$\int_{-\pi}^{\pi} e^{\jmath\omega(n-k)} d\omega = \begin{cases} 2\pi & n = k \\ 0 & \text{otherwise} \end{cases} \tag{4.30}$$

and so,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{j\omega n} d\omega \tag{4.31}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{-\pi}^{\pi} h(k)e^{\jmath\omega(n-k)} d\omega \tag{4.32}$$

$$= \frac{1}{2\pi} 2\pi h(n) = h(n) \tag{4.33}$$

which is known as the Inverse Discrete Fourier

Transform. Thus,

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega n}d\omega \tag{4.34}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1+e^{-2\jmath\omega}}{1+\frac{1}{2}e^{-\jmath\omega}}e^{\jmath\omega n}d\omega \tag{4.35}$$

5 Impulse Response

5.1 Using long divsion, find

$$h(n), n < 5 \tag{5.1}$$

for $H(z)$ in (4.12)
Solution: from (4.12)

$$H(z) = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.2}$$

$$\begin{array}{l} 1+\frac{1}{2}z^{-1}\overline{)1+z^{-2}}(2z^{-1} \\ \quad\ \underline{2z^{-1}+z^{-2}} \\ 1+\frac{1}{2}z^{-1})1-2z^{-1}(-4 \\ \quad\ \underline{-4-2z^{-1}} \\ \quad\ 5+0z^{-1} \end{array}$$

Hence by long division will be

$$H(z) = 2z^{-1} - 4 + \frac{5}{1+\frac{1}{2}z^{-1}} \tag{5.3}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.4}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).
Solution: From (4.12),

$$H(z) = \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.5}$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.6}$$

using (4.19) and (4.6).

5.3 Sketch $h(n)$. Is it bounded? Convergent?
Solution: The following code plots Fig. 5.3.

```
wget https://raw.
    githubusercontent.com/
    gadepall/EE1310/master/
    filter/codes/hn.py
```

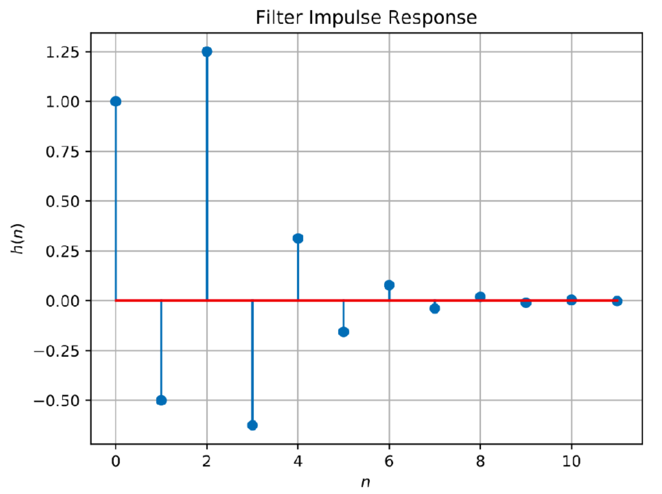


Fig. 5.3: $h(n)$ as the inverse of $H(z)$

5.4 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.7}$$

Is the system defined by (3.2) stable for the impulse response in (5.4)?
Solution: from 5.6

$$h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.8}$$

then

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n + \sum_{n=2}^{\infty}\left(-\frac{1}{2}\right)^{n-2} \tag{5.9}$$

$$\sum_{n=-\infty}^{\infty} h(n) = \frac{4}{3} \tag{5.10}$$

since

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.11}$$

$h(n)$ is stable.

5.5 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{5.12}$$

This is the definition of $h(n)$.
Solution: The following code plots Fig. 5.5. Note that this is the same as Fig. 5.3.

```
wget https://raw.
    githubusercontent.com/
    gadepall/EE1310/master/
```

filter/codes/hndef.py

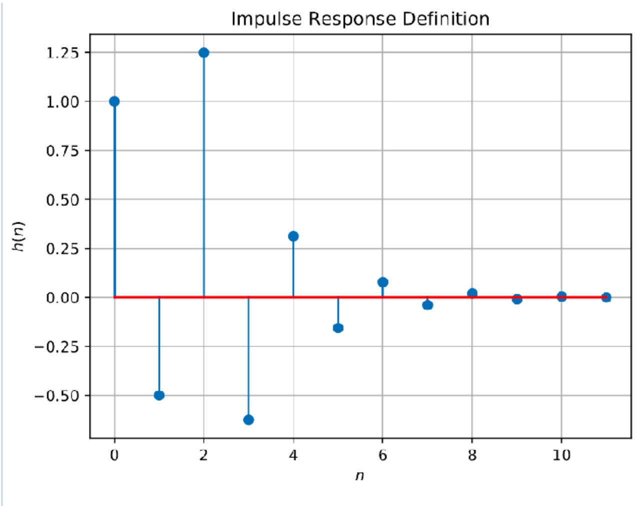


Fig. 5.5: $h(n)$ from the definition

5.6 Compute

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad (5.13)$$

Comment. The operation in (5.13) is known as *convolution*.

Solution: The following code plots Fig. 5.6. Note that this is the same as $y(n)$ in Fig. 3.2.

wget https://raw.githubusercontent.com/gadepall/EE1310/master/**filter**/codes/ynconv.py

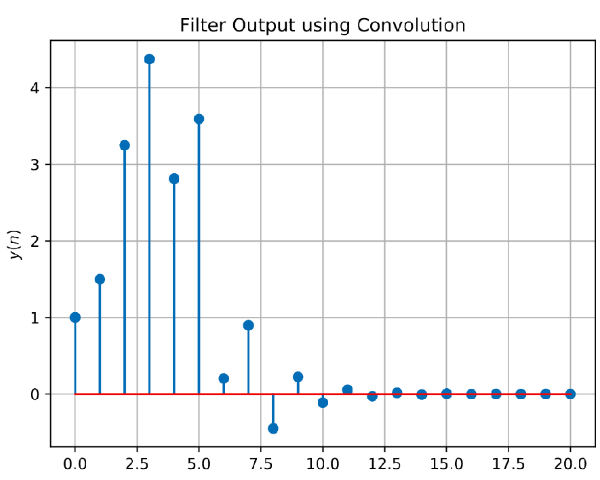


Fig. 5.6: $y(n)$ from the definition of convolution

5.7 Show that

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \quad (5.14)$$

Solution: from 5.13 ww know that

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad (5.15)$$

now consider

$$t = n - k \quad (5.16)$$

5.15 will transform into

$$y(n) = \sum_{n-t=-\infty}^{\infty} x(n-t)h(t) \quad (5.17)$$

since n is finite and $-\infty < \infty$, 5.17 is equivalent to

$$y(n) = \sum_{t=-\infty}^{\infty} x(n-t)h(t) \quad (5.18)$$

hence proved.

5.8 Express the above convolution using a Teoplitz matrix.

5.9

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad (5.19)$$

This can also be wrtten as a matrix-vector multiplication given by the expression,

$$\begin{bmatrix} y = T \\ h * x \end{bmatrix} \quad (5.20)$$

$\begin{bmatrix} T \\ h \end{bmatrix}$ is a Teoplitz matrix.

$$\begin{pmatrix} h_1 & 0 & . & . & . & 0 \\ h_2 & h_1 & . & . & . & 0 \\ h_3 & h_2 & h_1 & . & . & 0 \\ . & . & . & . & . & . \\ h_{n-1} & h_{n-2} & h_{n-3} & . & . & 0 \\ h_n & h_{n-1} & h_{n-2} & . & . & h_1 \\ 0 & h_n & h_{n-1} & h_{n-2} & . & h_2 \\ . & . & . & . & . & . \\ 0 & . & . & . & 0 & h_{n-1} \\ 0 & . & . & . & 0 & h_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix}$$

5.10

5.9 Show that

$$y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \tag{5.21}$$

$$y(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k) \tag{5.22}$$

Taking k = n-k

$$y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \tag{5.23}$$

6 DFT and FFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$.
Solution: The following code plots $X(k)$ and $H(k)$.

```
wget
https://github.com/
    dhanushpittala11/
    EE3900−2022/blob/
    main/filter/codes/
    XkHk_dft.ipynb
```

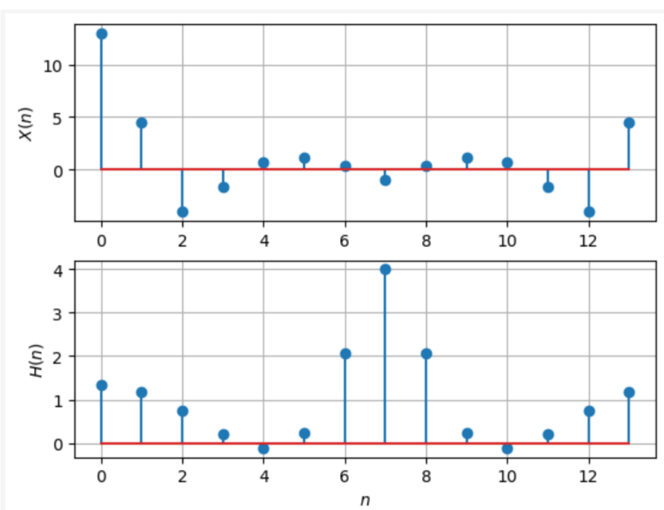


Fig. 6.1

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

Solution: The following code plots $Y(k)$.

```
wget
https://github.com/
    dhanushpittala11/
    EE3900−2022/blob/
    main/filter/codes/Y_k.
    ipynb
```

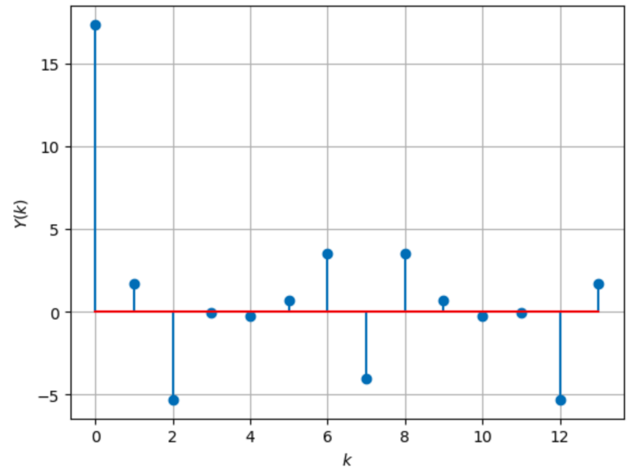


Fig. 6.2

6.3 Compute

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) \cdot e^{j2\pi kn/N}, \quad n = 0, 1, \dots, N-1 \tag{6.3}$$

Solution: The following code plots Fig. 5.6. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://github.com/
    dhanushpittala11/
    EE3900−2022/blob/
    main/filter/codes/yndft.
    ipynb
```

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.
Solution: The following code plots $X(n)$, $H(n)$ and $y(n)$ by fft.

```
wget
https://github.com/
    dhanushpittala11/
    EE3900−2022/blob/
    main/filter/codes/
    XkHk_fft.ipynb
```

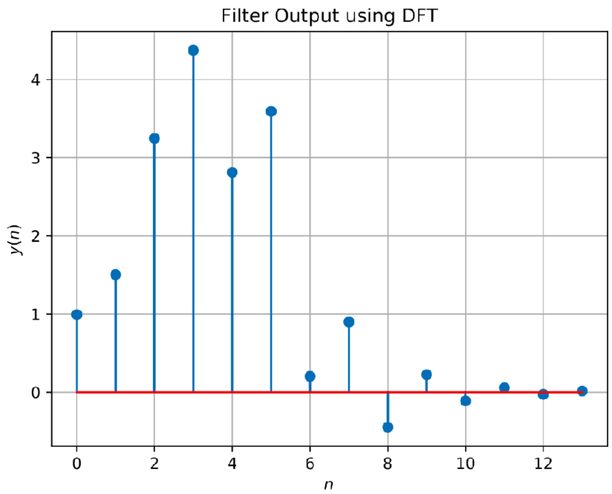


Fig. 6.3: $y(n)$ from the DFT

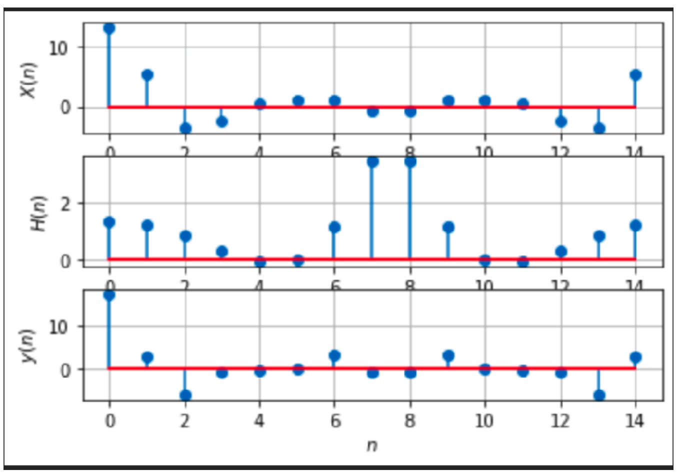


Fig. 6.4: $X(k)$, $H(k)$ and $y(n)$ from fft and IFFT

6.5 compare $y(n)$ obtained in 6.3 and 3.3 and IFFT.
Solution: The below code plots the plot of both the y(n)'s

```
wget
https://github.com/
    dhanushpittala11/
    EE3900−2022/blob/
    main/filter/codes/
    compare.ipynb
```

The plot will be 6.5
The following code compares $y(n)$'s obtained by 6.3, 3.3 and IFFT.

```
wget
https://github.com/
    dhanushpittala11/
    EE3900−2022/blob/
    main/filter/codes/
    compare2.ipynb
```

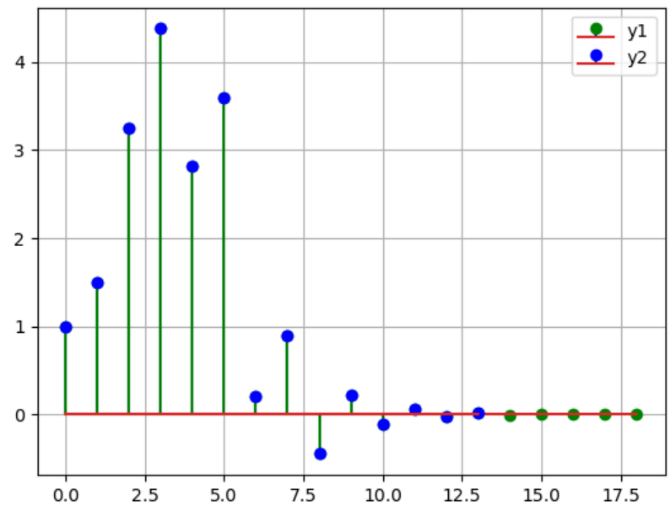


Fig. 6.5: y(n)'s from 6.3 and 3.3

The plot is 6.5

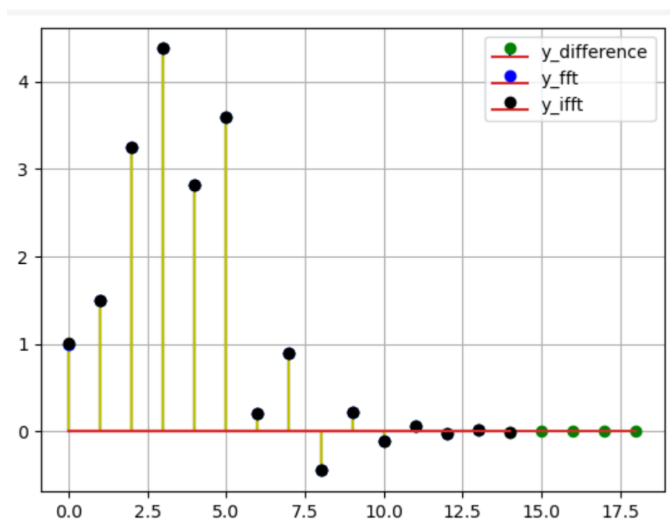


Fig. 6.5: y(n)'s from 6.3 and 3.3 and IFFT

6.6 Wherever possible, express all the above equations as matrix equations.
Solution:

$$x = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}$$

$$J = \begin{pmatrix} 0 \\ e^{\frac{-2\pi j(1)k}{N}} \\ e^{\frac{-2\pi j(2)k}{N}} \\ . \\ . \\ . \\ e^{\frac{-2\pi j(N-1)k}{N}} \end{pmatrix}$$

$$X(k) = x^T J \tag{6.4}$$

$$h = \begin{pmatrix} h[0] \\ h[1] \\ . \\ . \\ . \\ h[N-1] \end{pmatrix}$$

$$H(k) = h^T J \tag{6.5}$$

$$y = \begin{pmatrix} h[0]x[0] \\ h[1]x[1] \\ . \\ . \\ . \\ h[N-1]x[N-1] \end{pmatrix}$$

$$Y(k) = y^T J \tag{6.6}$$

7 FFT

7.1 Definitions

1. The DFT of $x(n)$ is given by

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{7.1}$$

2. Let

$$W_N = e^{-j2\pi/N} \tag{7.2}$$

Then the N-point *DFT matrix* is defined as

$$\vec{F}_N = [W_N^{mn}] \tag{7.3}$$

where W_N^{mn} are the elements of $\vec{F}_N$.

3. Let

$$\vec{I}_4 = \begin{pmatrix} \vec{e}_4^1 & \vec{e}_4^2 & \vec{e}_4^3 & \vec{e}_4^4 \end{pmatrix} \tag{7.4}$$

be the 4×4 identity matrix. Then the 4 point *DFT permutation matrix* is defined as

$$\vec{P}_4 = \begin{pmatrix} \vec{e}_4^1 & \vec{e}_4^3 & \vec{e}_4^2 & \vec{e}_4^4 \end{pmatrix} \tag{7.5}$$

4. The 4 point *DFT diagonal matrix* is defined as

$$\vec{D}_4 = diag \begin{pmatrix} W_N^0 & W_N^1 & W_N^2 & W_N^3 \end{pmatrix} \tag{7.6}$$

7.2 Problems

1. Show that

$$W_N^2 = W_{N/2} \tag{7.7}$$

Solution: We know that.

$$W_N = e^{-j2\pi/N} \tag{7.8}$$

Then

$$W_{N/2} = e^{-2*j2\pi/N} \tag{7.9}$$

$$W_{N/2} = {W_N}^2 \tag{7.10}$$

Hence Proved.

2. Show that

$$\vec{F}_4 = \begin{bmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & -\vec{D}_2 \end{bmatrix} \begin{bmatrix} \vec{F}_2 & 0 \\ 0 & \vec{F}_2 \end{bmatrix} \vec{P}_4 \tag{7.11}$$

Solution: Observe that for $n \in \mathbb{N}$, $W_4^{4n} = 1$ and $W_4^{4n+2} = -1$. Using (**??**),

$$\vec{D}_2\vec{F}_2 = \begin{pmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{pmatrix} \begin{pmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{pmatrix} \tag{7.12}$$

$$= \begin{pmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{pmatrix} \begin{pmatrix} W_4^0 & W_4^0 \\ W_4^0 & W_4^2 \end{pmatrix} \tag{7.13}$$

$$= \begin{pmatrix} W_4^0 & W_4^0 \\ W_4^1 & W_4^3 \end{pmatrix} \tag{7.14}$$

$$\implies -\vec{D}_2\vec{F}_2 = \begin{pmatrix} W_4^2 & W_4^6 \\ W_4^3 & W_4^9 \end{pmatrix} \tag{7.15}$$

and

$$\vec{F}_2 = \begin{pmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{pmatrix} \tag{7.16}$$

$$= \begin{pmatrix} W_4^0 & W_4^0 \\ W_4^0 & W_4^2 \end{pmatrix} \tag{7.17}$$

Hence,

$$\vec{W}_4 = \begin{pmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^2 & W_4^1 & W_4^3 \\ W_4^0 & W_4^4 & W_4^2 & W_4^6 \\ W_4^0 & W_4^6 & W_4^3 & W_4^9 \end{pmatrix} \tag{7.18}$$

$$= \begin{pmatrix} \vec{I}_2\vec{F}_2 & \vec{D}_2F_2 \\ \vec{I}_2\vec{F}_2 & -\vec{D}_2F_2 \end{pmatrix} \tag{7.19}$$

$$= \begin{pmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & \vec{D}_2 \end{pmatrix}\begin{pmatrix} \vec{F}_2 & 0 \\ 0 & \vec{F}_2 \end{pmatrix} \tag{7.20}$$

Multiplying (7.20) by $\vec{P}_4$ on both sides, and noting that $\vec{W}_4\vec{P}_4 = \vec{F}_4$ gives us.

3. Show that

$$\vec{F}_N = \begin{bmatrix} \vec{I}_{N/2} & \vec{D}_{N/2} \\ \vec{I}_{N/2} & -\vec{D}_{N/2} \end{bmatrix}\begin{bmatrix} \vec{F}_{N/2} & 0 \\ 0 & \vec{F}_{N/2} \end{bmatrix}\vec{P}_N \tag{7.21}$$

Solution: Observe that for even N and letting $\vec{f}_N^i$ denote the i^{th} column of $\vec{F}_N$, from (7.14) and (7.15),

$$\begin{pmatrix} \vec{D}_{N/2}\vec{F}_{N/2} \\ -\vec{D}_{N/2}\vec{F}_{N/2} \end{pmatrix} = \begin{pmatrix} \vec{f}_N^2 & \vec{f}_N^4 & \dots & \vec{f}_N^N \end{pmatrix} \tag{7.22}$$

and

$$\begin{pmatrix} \vec{I}_{N/2}\vec{F}_{N/2} \\ \vec{I}_{N/2}\vec{F}_{N/2} \end{pmatrix} = \begin{pmatrix} \vec{f}_N^1 & \vec{f}_N^3 & \dots & \vec{f}_N^{N-1} \end{pmatrix} \tag{7.23}$$

Thus,

$$\begin{pmatrix} \vec{I}_2\vec{F}_2 & \vec{D}_2\vec{F}_2 \\ \vec{I}_2\vec{F}_2 & -\vec{D}_2\vec{F}_2 \end{pmatrix} = \begin{pmatrix} \vec{I}_{N/2} & \vec{D}_{N/2} \\ \vec{I}_{N/2} & -\vec{D}_{N/2} \end{pmatrix}\begin{pmatrix} \vec{F}_{N/2} & 0 \\ 0 & \vec{F}_{N/2} \end{pmatrix}$$

$$= \begin{pmatrix} \vec{f}_N^1 & \dots & \vec{f}_N^{N-1} & \vec{f}_N^2 & \dots & \vec{f}_N^N \end{pmatrix} \tag{7.24}$$

and so,

$$\begin{pmatrix} \vec{I}_{N/2} & \vec{D}_{N/2} \\ \vec{I}_{N/2} & -\vec{D}_{N/2} \end{pmatrix}\begin{pmatrix} \vec{F}_{N/2} & 0 \\ 0 & \vec{F}_{N/2} \end{pmatrix}\vec{P}_N$$

$$= \begin{pmatrix} \vec{f}_N^1 & \vec{f}_N^2 & \dots & \vec{f}_N^N \end{pmatrix} = \vec{F}_N \tag{7.25}$$

4. Find

$$\vec{P}_6\vec{x} \tag{7.26}$$

Solution: We have,

$$\vec{P}_4\vec{x} = \begin{pmatrix} \vec{e}_4^1 & \vec{e}_4^3 & \vec{e}_4^2 & \vec{e}_4^4 \end{pmatrix}\begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{pmatrix} = \begin{pmatrix} x(0) \\ x(2) \\ x(1) \\ x(3) \end{pmatrix} \tag{7.27}$$

5. Show that

$$\vec{X} = \vec{F}_N\vec{x} \tag{7.28}$$

where $\vec{x}, \vec{X}$ are the vector representations of $x(n), X(k)$ respectively.

Solution: Writing the terms of X,

$$X(0) = x(0) + x(1) + \ldots + x(N-1) \tag{7.29}$$

$$X(1) = x(0) + x(1)e^{-\frac{j2\pi}{N}} + \ldots + \\ + x(N-1)e^{-\frac{j2(N-1)\pi}{N}} \tag{7.30}$$

$$\vdots$$

$$X(N-1) = x(0) + x(1)e^{-\frac{j2(N-1)\pi}{N}} + \ldots + \\ + x(N-1)e^{-\frac{j2(N-1)(N-1)\pi}{N}} \tag{7.31}$$

Clearly, the term in the m^{th} row and n^{th} column is given by ($0 \le m \le N-1$ and $0 \le n \le N-1$)

$$T_{mn} = x(n)e^{-\frac{j2mn\pi}{N}} \tag{7.32}$$

and so, we can represent each of these terms as a matrix product

$$\vec{X} = \vec{F}_N\vec{x} \tag{7.33}$$

where $\vec{F}_N = \left[e^{-\frac{-j2mn\pi}{N}}\right]_{mn}$ for $0 \le m \le N-1$ and $0 \le n \le N-1$.

6. Derive the following Step-by-step visualisation of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix}\begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.34}$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix}\begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.35}$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix}\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.36}$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.37}$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.38}$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.39}$$

$$P_8 \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \\ x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} \tag{7.40}$$

$$P_4 \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \end{bmatrix} \tag{7.41}$$

$$P_4 \begin{bmatrix} x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \tag{7.42}$$

Therefore,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.43}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.44}$$

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.45}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.46}$$

Solution: We write out the values of performing an 8-point FFT on $\vec{x}$ as follows.

$$X(k) = \sum_{n=0}^{7} x(n) e^{-\frac{\jmath 2kn\pi}{8}} \tag{7.47}$$

$$= \sum_{n=0}^{3} \left(x(2n) e^{-\frac{\jmath 2kn\pi}{4}} + e^{-\frac{\jmath 2k\pi}{8}} x(2n+1) e^{-\frac{\jmath 2kn\pi}{4}} \right) \tag{7.48}$$

$$= X_1(k) + e^{-\frac{\jmath 2k\pi}{4}} X_2(k) \tag{7.49}$$

where $\vec{X}_1$ is the 4-point FFT of the even-numbered terms and $\vec{X}_2$ is the 4-point FFT of the odd numbered terms. Noticing that for $k \geq 4$,

$$X_1(k) = X_1(k-4) \tag{7.50}$$

$$e^{-\frac{\jmath 2k\pi}{8}} = -e^{-\frac{\jmath 2(k-4)\pi}{8}} \tag{7.51}$$

we can now write out $X(k)$ in matrix form as in (**??**) and (**??**). We also need to solve the two 4-point FFT terms so formed.

$$X_1(k) = \sum_{n=0}^{3} x_1(n) e^{-\frac{\jmath 2kn\pi}{8}} \tag{7.52}$$

$$= \sum_{n=0}^{1} \left(x_1(2n) e^{-\frac{\jmath 2kn\pi}{4}} + e^{-\frac{\jmath 2k\pi}{8}} x_2(2n+1) e^{-\frac{\jmath 2kn\pi}{4}} \right) \tag{7.53}$$

$$= X_3(k) + e^{-\frac{\jmath 2k\pi}{4}} X_4(k) \tag{7.54}$$

using $x_1(n) = x(2n)$ and $x_2(n) = x(2n+1)$. Thus we can write the 2-point FFTs

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.55}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.56}$$

Using a similar idea for the terms X_2,

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.57}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.58}$$

But observe that from (7.27),

$$\vec{P}_8 \vec{x} = \begin{pmatrix} \vec{x}_1 \\ \vec{x}_2 \end{pmatrix} \tag{7.59}$$

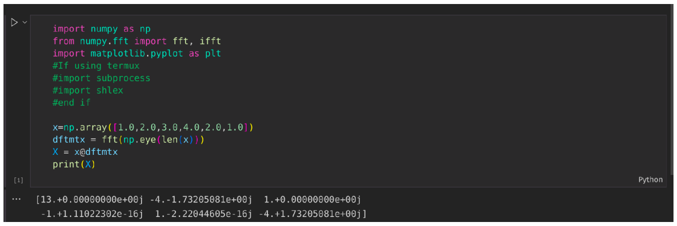
$$\vec{P}_4 \vec{x}_1 = \begin{pmatrix} \vec{x}_3 \\ \vec{x}_4 \end{pmatrix} \tag{7.60}$$

$$\vec{P}_4 \vec{x}_2 = \begin{pmatrix} \vec{x}_5 \\ \vec{x}_6 \end{pmatrix} \tag{7.61}$$

where we define $x_3(k) = x(4k)$, $x_4(k) = x(4k+2)$, $x_5(k) = x(4k+1)$, and $x_6(k) = x(4k+3)$ for $k = 0, 1$.

$$\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.62}$$

Solution:
The code below gives the answer



```
import numpy as np
from numpy.fft import fft, ifft
import matplotlib.pyplot as plt
#If using termux
#import subprocess
#import shlex
#end if

x=np.array([1.0,2.0,3.0,4.0,2.0,1.0])
dftmtx = fft(np.eye(len(x)))
X = x@dftmtx
print(X)
```

```
[13.+0.00000000e+00j -4.-1.73205081e+00j  1.+0.00000000e+00j
 -1.+1.11022302e-16j  1.-2.22044605e-16j -4.+1.73205081e+00j]
```

compte the DFT using (7.28)

7. Repeat the above exercise using (**??**)
8. Write a C program to compute the 8-point FFT.
Solution: The following code calculates the 8-point fft of $x(n)$ in 3.1

```
wget
https://github.com/
    dhanushpittala11/
    EE3900−2022/blob/
    main/filter/codes/8
    _point_FFT.c
```

The result is 7.8

8 Exercises

Answer the following questions by looking at the python code in Problem 2.3.

8.1 The command

```
output_signal = signal.
    lfilter(b, a, input_signal
    )
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^{M} a(m)\, y(n-m) = \sum_{k=0}^{N} b(k)\, x(n-k) \tag{8.1}$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.
Solution:

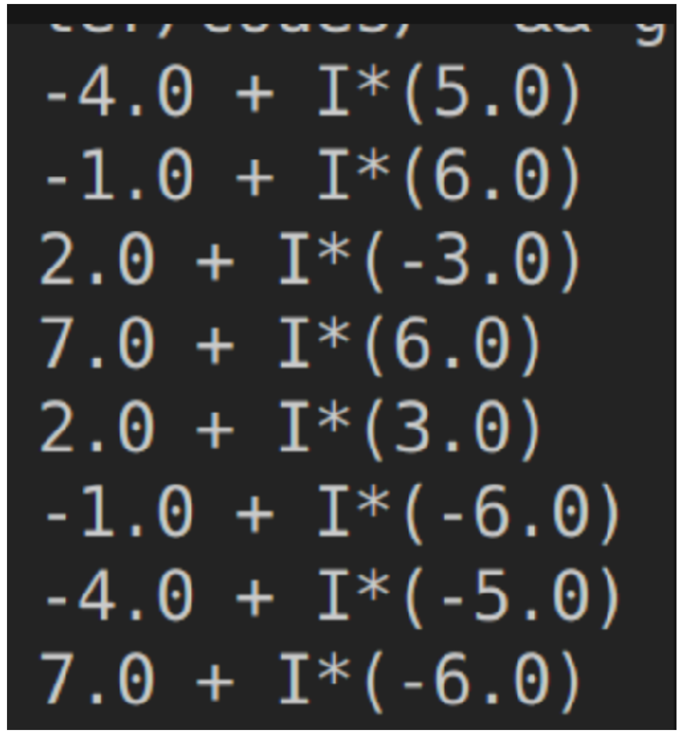


Fig. 7.8: 8-Point FFT

```
https://github.com/
    dhanushpittala11/
    EE3900−2022/blob/
    main/filter/codes/8_1.
    ipynb
```

8.2 Repeat all the exercises in the previous sections for the above a and b. **Solution:** For the given values, the difference equation is

$$\begin{aligned} &y(n) - (4.44)\, y(n-1) + (8.78)\, y(n-2) \\ &- (9.93)\, y(n-3) + (6.90)\, y(n-4) \\ &- (2.93)\, y(n-5) + (0.70)\, y(n-6) \\ &- (0.07)\, y(n-7) = \left(5.02 \times 10^{-5}\right) x(n) \\ &+ \left(3.52 \times 10^{-4}\right) x(n-1) + \left(1.05 \times 10^{-3}\right) x(n-2) \\ &+ \left(1.76 \times 10^{-3}\right) x(n-3) + \left(1.76 \times 10^{-3}\right) x(n-4) \\ &+ \left(1.05 \times 10^{-3}\right) x(n-5) + \left(3.52 \times 10^{-4}\right) x(n-6) \\ &+ \left(5.02 \times 10^{-5}\right) x(n-7) \end{aligned} \tag{8.2}$$

From (8.1), we see that the transfer function can be written as follows

$$H(z) = \frac{\sum_{k=0}^{N} b(k)z^{-k}}{\sum_{k=0}^{M} a(k)z^{-k}} \tag{8.3}$$

$$= \sum_i \frac{r(i)}{1 - p(i)z^{-1}} + \sum_j k(j)z^{-j} \tag{8.4}$$

where $r(i)$, $p(i)$, are called residues and poles respectively of the partial fraction expansion of $H(z)$. $k(i)$ are the coefficients of the direct polynomial terms that might be left over. We can now take the inverse z-transform of (8.4) and get using (4.19),

$$h(n) = \sum_i r(i)[p(i)]^n u(n) + \sum_j k(j)\delta(n - j) \tag{8.5}$$

Substituting the values,

$$\begin{aligned} h(n) = {} & [(2.76)\,(0.55)^n \\ & + (-1.05 - 1.84\jmath)\,(0.57 + 0.16\jmath)^n \\ & + (-1.05 + 1.84\jmath)\,(0.57 - 0.16\jmath)^n \\ & + (-0.53 + 0.08\jmath)\,(0.63 + 0.32\jmath)^n \\ & + (-0.53 - 0.08\jmath)\,(0.63 - 0.32\jmath)^n \\ & + (0.20 + 0.004\jmath)\,(0.75 + 0.47\jmath)^n \\ & + (0.20 - 0.004\jmath)\,(0.75 - 0.47\jmath)^n]u(n) \\ & + \left(-6.81 \times 10^{-4}\right)\delta(n) \end{aligned} \tag{8.6}$$

The values $r(i)$, $p(i)$, $k(i)$ and thus the impulse response function are computed and plotted at

https://github.com/dhanushpittala11/EE3900−2022/blob/main/**filter**/codes/8_2_1.ipynb

The filter frequency response is plotted at

https://github.com/dhanushpittala11/EE3900−2022/blob/main/**filter**/codes/8_2_2.ipynb

Observe that for a series $t_n = r^n$, $\frac{t_{n+1}}{t_n} = r$. By the ratio test, t_n converges if $|r| < 1$. We note that observe that $|p(i)| < 1$ and so, as $h(n)$ is the sum of convergent series, we see that $h(n)$ converges. From Fig. (8.2), it is clear that $h(n)$ is bounded. From (4.1),

$$\sum_{n=0}^{\infty} h(n) = H(1) = 1 < \infty \tag{8.7}$$

Therefore, the system is stable. From Fig. (8.2), $h(n)$ is negligible after $n \geq 64$, and we can apply a 64-bit FFT to get y(n). The following code uses the DFT matrix to generate $y(n)$ in Fig. (8.2).

https://github.com/dhanushpittala11/EE3900−2022/blob/main/**filter**/codes/8_2_3.ipynb

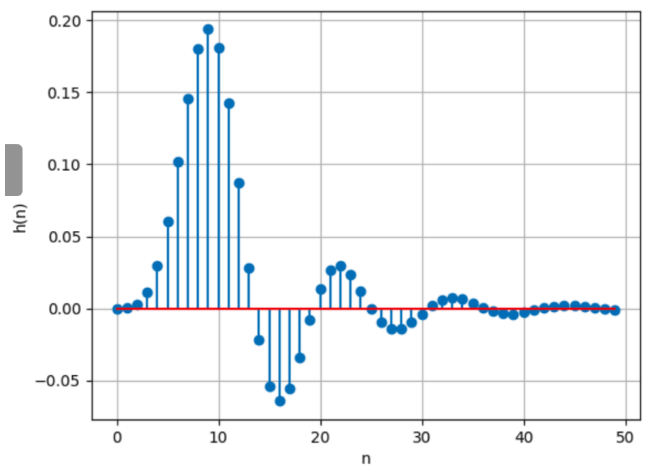


Fig. 8.2: Plot of $h(n)$

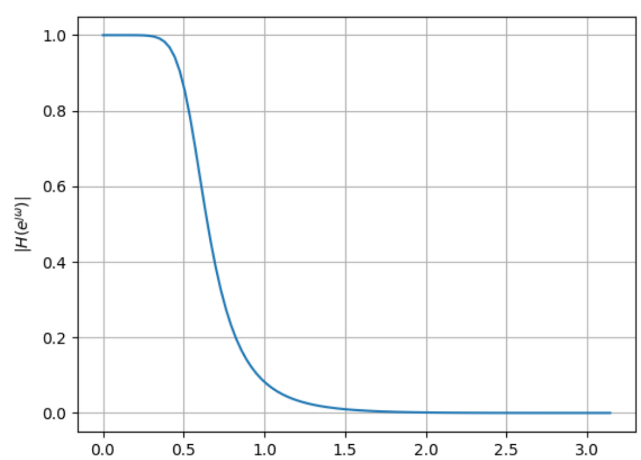


Fig. 8.2: Filter frequency response

8.3 What is the sampling frequency of the input signal?

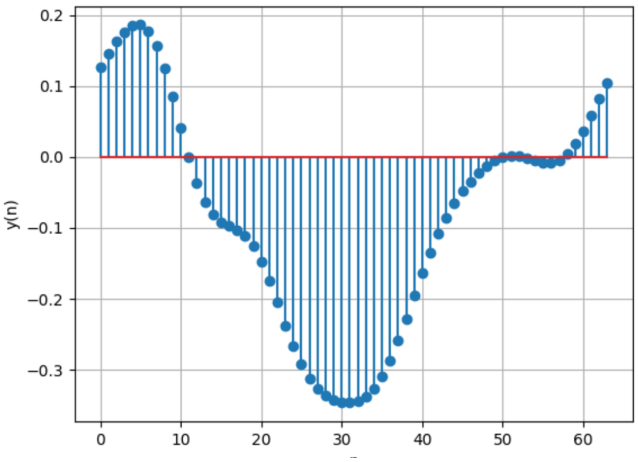


Fig. 8.2: Plot of $y(n)$

Solution: run the following code to Sampling frequency(fs)=44.1kHZ.

8.4 What is type, order and cutoff-frequency of the above butterworth filter
Solution: The given butterworth filter is low pass with order=2 and cutoff-frequency=4kHz.

8.5 Modifying the code with different input parameters and to get the best possible output. **Solution:** a better filtering was found on changing the order of filter to 7.