3A The linear dependence in $X^T X$ matrix indicates x_2, x_3, which represent province in Catholic or Protestant, are perfectly correlated. Even from given X matrix, we can observe when $x_2 = 0$ then $x_3 = 1$ and vice versa.

We know $X^T X \hat{\beta} = X^T y$ [from pg 60 Textbook]

Then $\hat{\beta} = (X^T X)^{-1} X^T y$

So to find $\hat{\beta}$ we need $(X^T X)^{-1}$.

The matrix $X^T X$ have linearly dependent columns, implies that matrix is singular and inverse of $X^T X$ is not possible.

Because the third and fourth columns of this matrix are related because $x_2 + x_3 = 1$. This creates redundancy making $X^T X$ non invertible.

This is a case of multicolinearity. It makes impossible to estimate individual effects of x_2 & x_3 only.

So to solve this issue we can drop either x_2 or x_3, which keeps ~~Keeping~~ $X^T X$ invertible, allowing to compute $\hat{\beta}$.

5A

Given $\hat{y} = 2 + 0.8x_1 + 0.5x_2 - 0.01(x_1 \times x_2)$

$\beta_0 = 2$, $\beta_1 = 0.8$, $\beta_2 = 0.5$, $\beta_3 = -0.01$

(a) Case 1 [Without Interaction term]

$y = 2 + 0.8x_1 + 0.5x_2$

keeping the humidity constant at 60%

The average change in plant growth for unit increase in temperature is given by coefficient of x_1.

$y = 2 + 0.8x_1 + 0.5(60)$

$y = 32 + 0.8x_1$

So $\beta_1 = 0.8 =$ change in plant growth for unit increase in temperature

(b) Case ② [With Interaction term]

$y = 2 + 0.8x_1 + 0.5x_2 - 0.01[x_1 \times x_2]$

The interaction term $x_1 \times x_2$ means that effect of temperature on plant growth depends on level of humidity

The Change in plant growth when temperature increases by 1 degree is:

$$\frac{dy}{dx_1} = \cancelto{0}{\frac{d\,x}{dx_1}} + \frac{d(0.8\,x_1)}{dx_1} + \cancelto{0}{\frac{d(0.5\,x_2)}{dx_1}} - \frac{d\,[0.01(x_1 \cdot x_2)]}{dx_1}$$

$$\boxed{\frac{dy}{dx_1} = 0.8 - 0.01\,x_2}$$

at $x_2 = 60\%$

$$\frac{dy}{dx_1} = 0.8 - 0.01 \times 60\% = 0.2$$

So with iteraction term, the change in plant growth rate is only 0.2 cm/day for 1° increase in temperature.