1. Let's say we have two frames: Body & Inertial. There is a point referred by $\bar{x}$ in inertial frame and $\bar{x}_b$ in body frame.

Initially, $\bar{x}_b^0 = \bar{x}$

Now rotate the moveable body frame yaw axis (z) through an angle ϕ.

$$\bar{x}_b^1 = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \bar{x}_b^0$$

$$= R(\phi)\,\bar{x}_b^0$$

Second rotation, pitch about new y-axis by the angle θ:

$$\bar{x}_b^2 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \bar{x}_b^1$$

$$= R(\theta)\,\bar{x}_b^1$$

Third, rotate an angle ψ about its newest x-axis:

$$\bar{x}_b^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix} \bar{x}_b^2 = R(\psi)\,\bar{x}_b^2$$

$$\bar{x}_b = R(\psi)\,R(\theta)\,R(\phi)\,\bar{x}$$

$$= \begin{bmatrix} c\theta c\phi & c\theta s\phi & -s\theta \\ -c\psi s\phi + s\psi s\theta c\phi & c\psi c\phi + s\psi s\theta s\phi & s\psi c\theta \\ s\psi s\phi + c\psi s\theta c\phi & -s\psi c\phi + c\psi s\theta s\phi & c\psi c\theta \end{bmatrix} \bar{x}$$

$= R(\phi, \theta, \psi)\, \bar{x}$

$\bar{x} = R^T \bar{x}_b$

In case body frame has a different frame.

$\bar{x} = \bar{x}_0 + R^T \bar{x}_b$

Let's consider small rotation:

$$R \simeq \begin{bmatrix} 1 & \delta\phi & -\delta\theta \\ -\delta\phi & 1 & \delta\psi \\ \delta\theta & -\delta\psi & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \delta\phi & -\delta\theta \\ -\delta\phi & 0 & \delta\psi \\ \delta\theta & -\delta\psi & 0 \end{bmatrix} + I$$

$$= -\delta\bar{E} \times + I$$

where $\delta\bar{E} = [\delta\psi, \delta\theta, \delta\phi]$

For smaller rotations are completely decoupled their order doesn't matter.

$R^{-1} = I + \delta\bar{E}\times$

$\bar{x}_b = \bar{x} - \delta\bar{E} \times \bar{x}$

$\bar{x} = \bar{x}_b + \delta\bar{E} \times \bar{x}_b$

Now, we fix a point on body frame instead of inertial frame by $\bar{r}$

$\bar{x}(t) = \bar{r}$

$\bar{x}(t + \delta t) = R^T \bar{r} = \bar{r} + \delta\bar{E} \times \bar{r}$

$$\frac{\delta \bar{x}}{\delta t} = \frac{\delta \bar{\theta}}{\delta t} \times \bar{r}$$

$$= \bar{\omega} \times \bar{r}$$

$$\bar{\omega} \simeq d\bar{\theta}/dt.$$

Similarly, $\bar{x}_b(t) = R\bar{r}$

$$= \bar{r} - \delta\bar{\theta} \times \bar{r}$$

$$\bar{x}_b(t+\delta t) = \bar{r}$$

$$\therefore \quad \frac{\delta \bar{x}_b}{\delta t} = \frac{\delta \bar{\theta}}{\delta t} \times \bar{r}$$

$$= \bar{\omega} \times \bar{r}$$

When radius vector changes with respect to body frame,

$$\frac{d\bar{x}_b}{dt} = \bar{\omega} \times \bar{r} + \frac{\partial \bar{r}}{\partial t}$$

Finally allowing the origin to move as well:

$$\frac{d\bar{x}_b}{dt} = \bar{\omega} \times \bar{r} + \frac{\partial \bar{r}}{\partial t} + \frac{d\bar{x}_0}{dt}$$

$$\bar{v} = \bar{\omega} \times \bar{r} + \frac{\partial \bar{r}}{\partial t} + \bar{v}_0$$

Now, in the previous derivation we used small euler angle approx. to get the exp. But proper treatment is given below

$$\bar{\omega} = R(\psi)\,R(\theta)\begin{bmatrix} 0 \\ 0 \\ d\phi/dt \end{bmatrix} + R(\psi)\begin{bmatrix} 0 \\ d\theta/dt \\ 0 \end{bmatrix} + \begin{bmatrix} d\psi/dt \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\psi & \sin\psi\cos\theta \\ 0 & -\sin\psi & \cos\psi\cos\theta \end{bmatrix}\begin{bmatrix} d\psi/dt \\ d\theta/dt \\ d\phi/dt \end{bmatrix}$$

$$\frac{d\bar{E}}{dt} = \begin{bmatrix} 1 & \sin\psi\tan\theta & \cos\psi\tan\theta \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi/\cos\theta & \cos\psi/\cos\theta \end{bmatrix}\bar{\omega}$$

2. Let's say R is a rotation matrix and it can be represented as

$$R = e^{A}$$

where A is skew-symmetric matrix.

So, according to matrix exponential,

$$e^{A} = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

Since, R is a rotation matrix then it should follow all its properties:

(i) R is an orthogonal matrix, i.e.

$$R R^T = I = R^T R$$

(ii) $$\det(R) = 1$$

Let's prove both of them:

(i) given $R = e^{A}$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

$$\therefore R^T = (e^{A})^T$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} (A^T)^k$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} (-1)^k A^k$$

following from the property of skew-symmetric matrix $(A^T = -A)$

$$\therefore R^T = \sum_{k=0}^{\infty} \frac{1}{k!}(-A)^k$$

$$= e^{-A}$$

$$\therefore RR^T = e^{A} e^{-A}$$

$$= e^{0}$$

$$= I$$

Similarly $R^T R = e^{-A} e^{A} = I$

(ii) from (i) we have $R^T R = I$

$$\therefore |R^T R| = 1$$

$$\Rightarrow |R^T||R| = 1$$

$$\Rightarrow |R|^2 = 1$$

$$\Rightarrow |R| = \pm 1$$

Since R is ~~not~~ a rotation matrix therefore $|R| = -1$ is not possible :

For this we ~~dete~~ go into the properties of $SO(3)$.

$\because \quad R \in SO(3)$

$\therefore \quad R^T R = I = R R^T, \ \det(R) = 1$

The tangent space at an element in the Lie group $SO(3)$ consists of vectors tangent to all differentiable curves $t \to R(t) \in SO(3)$. This vector space is isomorphic to the identity element $R = I$, which is defined as Lie algebra $so(3)$ of Lie group.

$$R(t)^T R(t) = I = R(t) R(t)^T$$

$R(t) \in SO(3)$ such that $R(0) = I$ and $\dot{R}(0) = S$, where $S \in so(3)$

Differentiating at $t = 0$, we see that

$$S^T + S = 0$$

S is a skew matrix

$\therefore \ so(3)$ is the matrix Lie algebra of SO_3

$\therefore$ element of $so(3)$ can be represented as

$$\exp(S) = I + S + \frac{1}{2!} S^2 + \dots$$

3. In a general euclidean geometry we can use the operation of additn and subtractn

$$X' = X + \Delta X \quad , \quad X'' = X_0 - \Delta X$$

X', X'' belongs to the same vector space as X.

We can't say that in the case of rotatn matrix :

$$R' = R + \Delta R$$

$$R' \notin SO(3)$$

So, inorder to find the error between R and R_d we have to take a difference

Let's say that initially a frame is at R_0 and It made a rotation of R. Therefore total rotation will be described by

$$R' = R R_0$$

$$\Rightarrow R = R' R_0^T$$

So, we see R denote a difference in the rotation angles betwn R' and R_0:
we can use analogy for calculating error betwn R_d and R.

$$e_R = R_d^T R \quad \text{or} \quad e_R = R^T R_d$$

Both describe an expression to define error for attitude control.

Also, if we observe:

$$e_R = \left(\exp(\omega_d^{\times} t) R_0\right)^T \exp(\omega^{\times} t) R_0$$

$$= \exp(-\omega_d^{\times} t)\, R_0^T \exp(\omega^{\times} t) R_0$$

$$= \exp\left((\omega^{\times} - \omega_d^{\times}) t\right) R_0^T R_0$$

$$= \exp(\Delta\theta^{\times})$$

$\exp(\Delta\theta^{\times})$ can be rotation matrix for error angle.

But, the expression has a problem. Let's say we change the sign of $\Delta\theta^{\times}$ to $-\Delta\theta^{\times}$, then the error should ~~be~~ change sign too, for the controller to work properly.

$$e_R = \exp(\Delta\theta^{\times}) > 0$$

$$\therefore e'_R = \exp(-\Delta\theta^{\times}) > 0$$

So, we need to find an expression which will change signs.

$$e_R = \frac{R_d^T R - R^T R_d}{2}$$

So, if we $e_R^T = -\frac{(R_d^T R - R^T R_d)}{2}$

$$= -e_R$$

So, $e_R = \frac{R_d^T R - R^T R_d}{2}$

4.

J = [[1, 0, 0],
[0, 1, 0],
[0, 0, 2]]

Kp = 10, Kd = 10

Pseudocode:

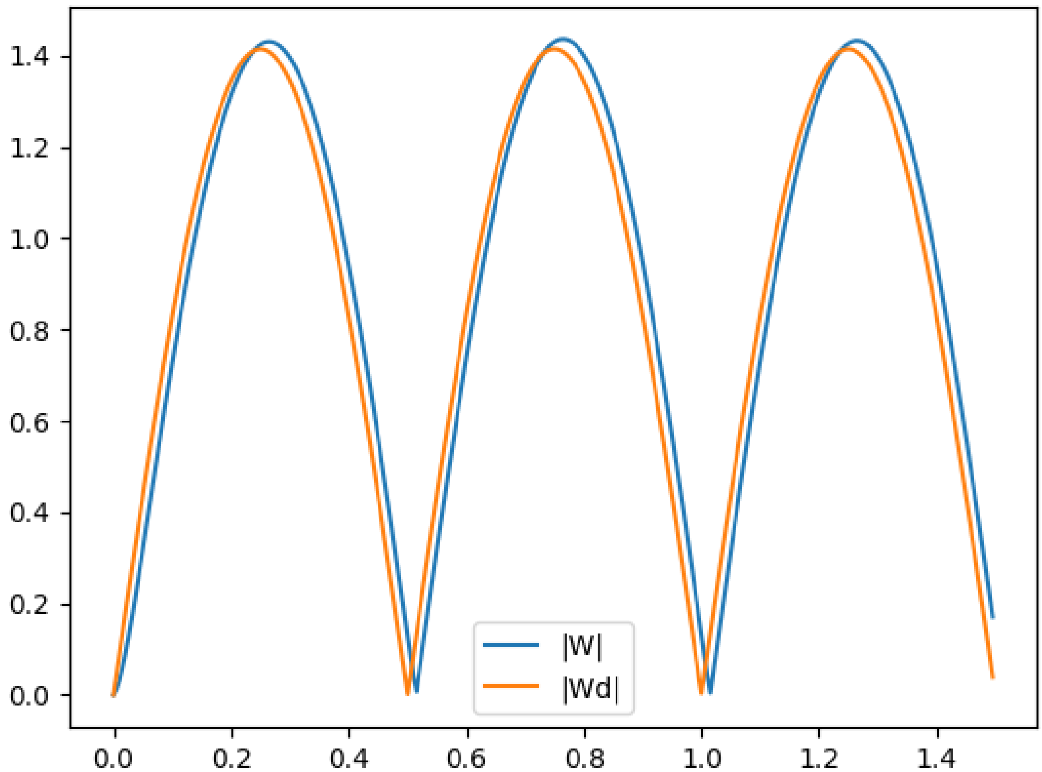
Initialize start time and end time.
Initialize step time as well.
Initialize R_t and W_t.

1. Define function for **Wd**. ([sin (2pi * f * t), sin (2pi * f * t), 0] taken here)
2. **Rd** is calculated using the formula $\mathbf{Rd_{t+1}} = \exp(\mathbf{Wd_t}^{x}\Delta t)\,\mathbf{Rd_t}$.
3. Since we have already had $\mathbf{R_t}$ and $\mathbf{W_t}$, we can calculate the respective errors. The **ew** is calculate using $\mathbf{ew} = \mathbf{W_t - (R^TRd)Wd_t}$. The $\mathbf{er} = \mathbf{(Rd^TR - R^TRd)/2}$.
4. When we have **ew** and **er**, we want to have **u** (control input that we are going to feed into the system). **er** is a 3 x 3 skew-symmetric matrix so we first need to convert it to 3 x 1 vector so that we can get a control input out of it. The control input **u** is calculated using **u = -Kp * VMap (er) -Kd * ew.**
5. The next step is to use the above control input and calculate $\mathbf{W_{t+1}}$ using **Jw'(t) = Jw(t) x w(t) + u.** And using $\mathbf{W_{t+1}}$ we can calculate $\mathbf{R_{t+1}}$ using $\mathbf{R_{t+1}} = \exp(\mathbf{w_t}^{x}\Delta t)\,\mathbf{R_t}$.

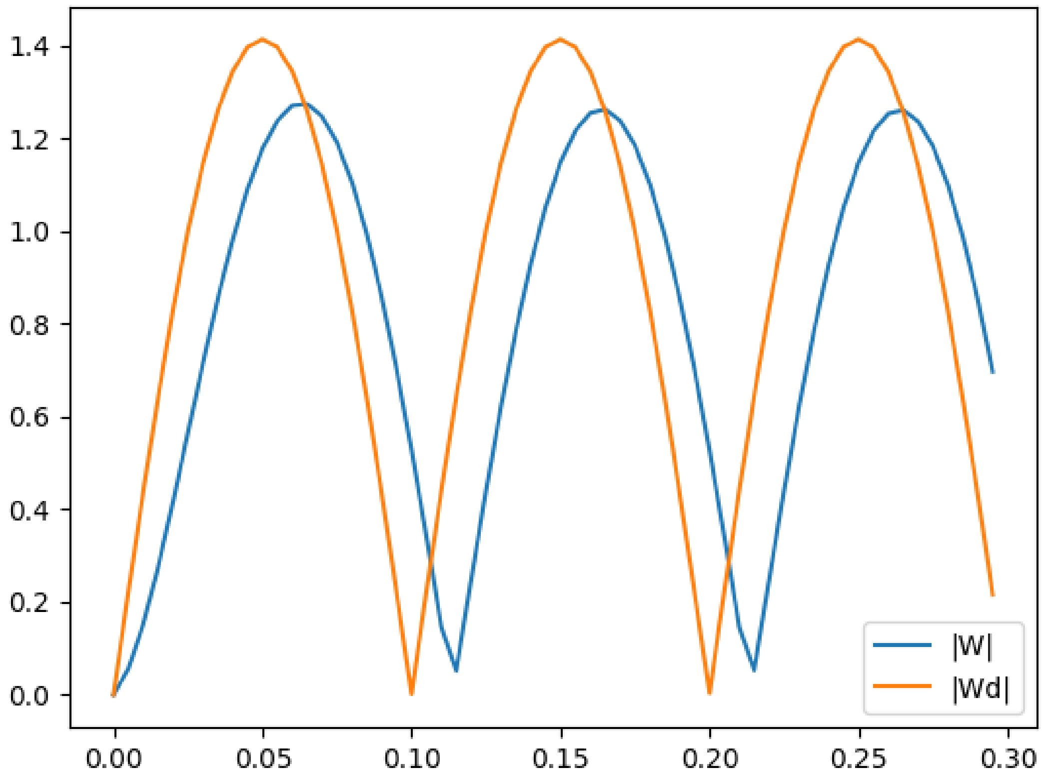
Below are the images showing the tracing of **|Wd|** with **|W|**. It can be inferred from the plot that:

1. It is harder to trace a high-frequency angular velocity than a lower one.
2. Higher PD gains work better for any frequency.

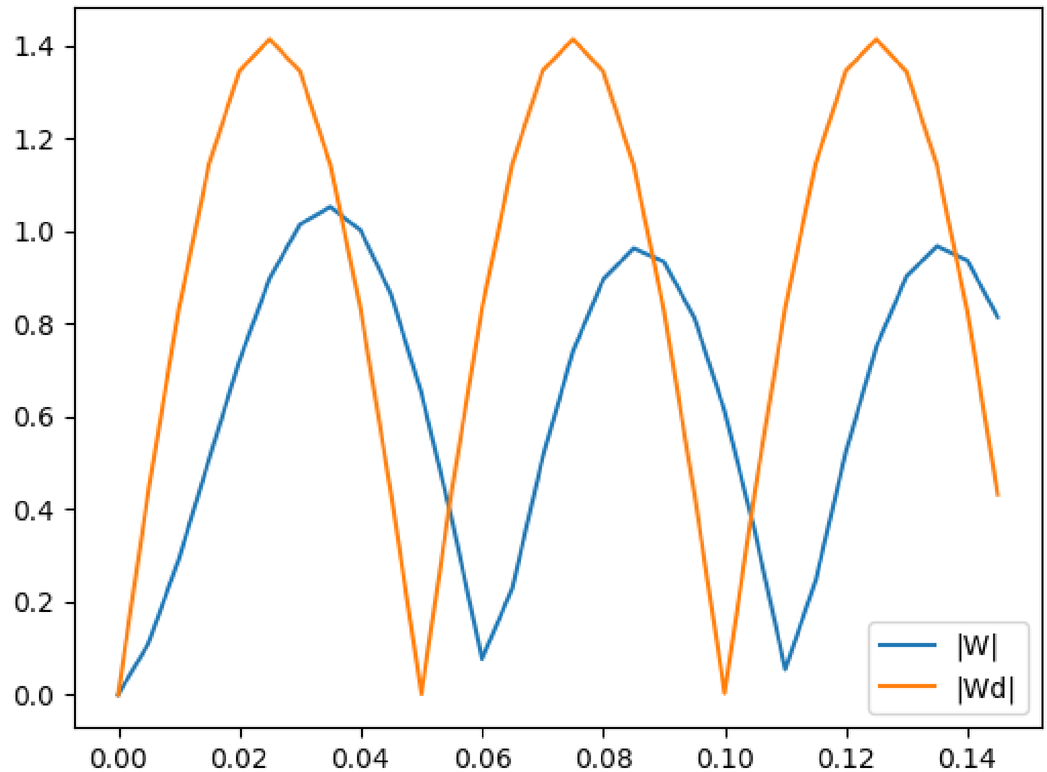
Case 1) Kp = Kd = 50



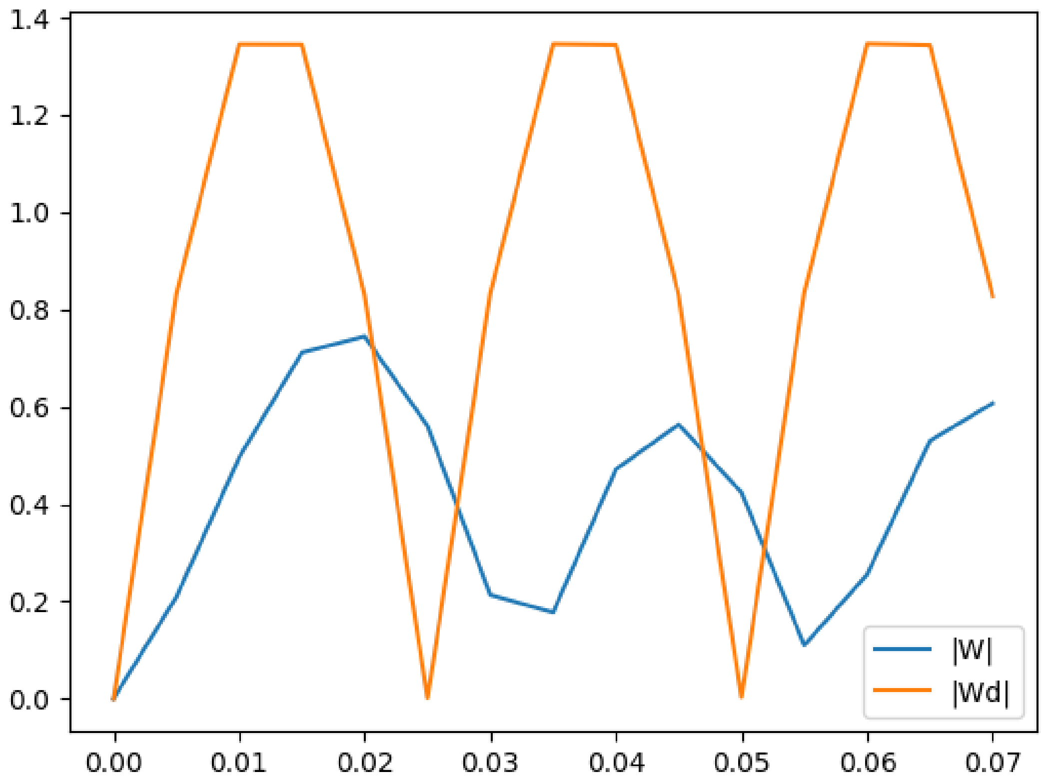
f = 1 Hz



f = 5 Hz

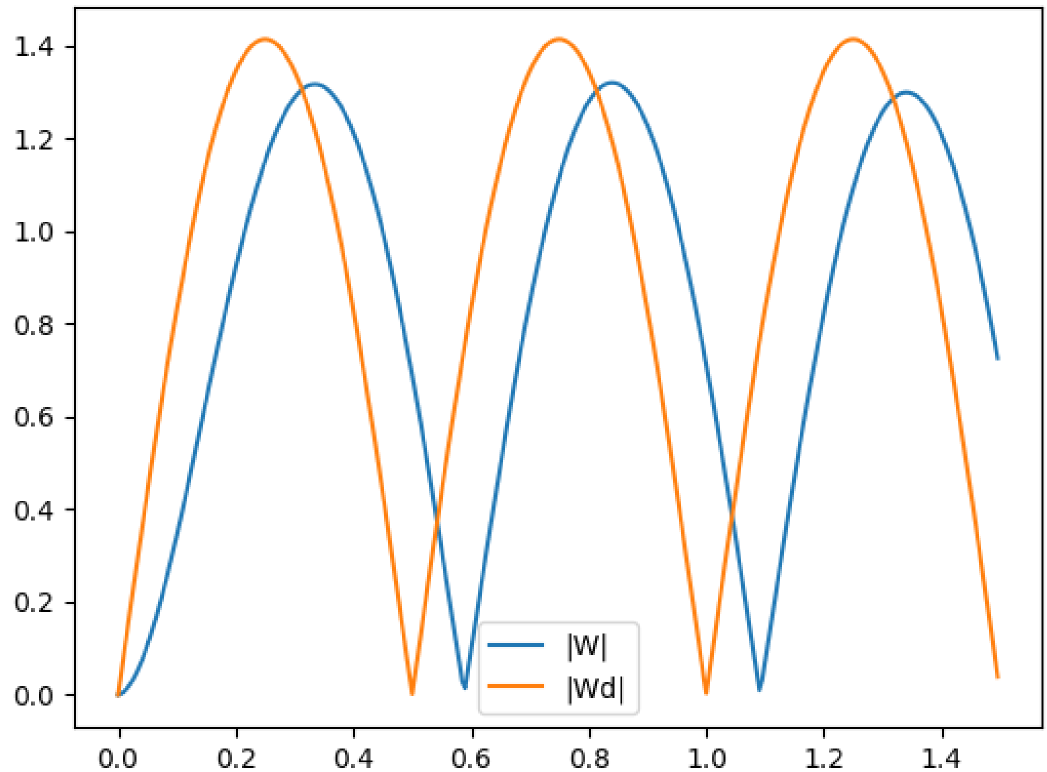


f = 10 Hz

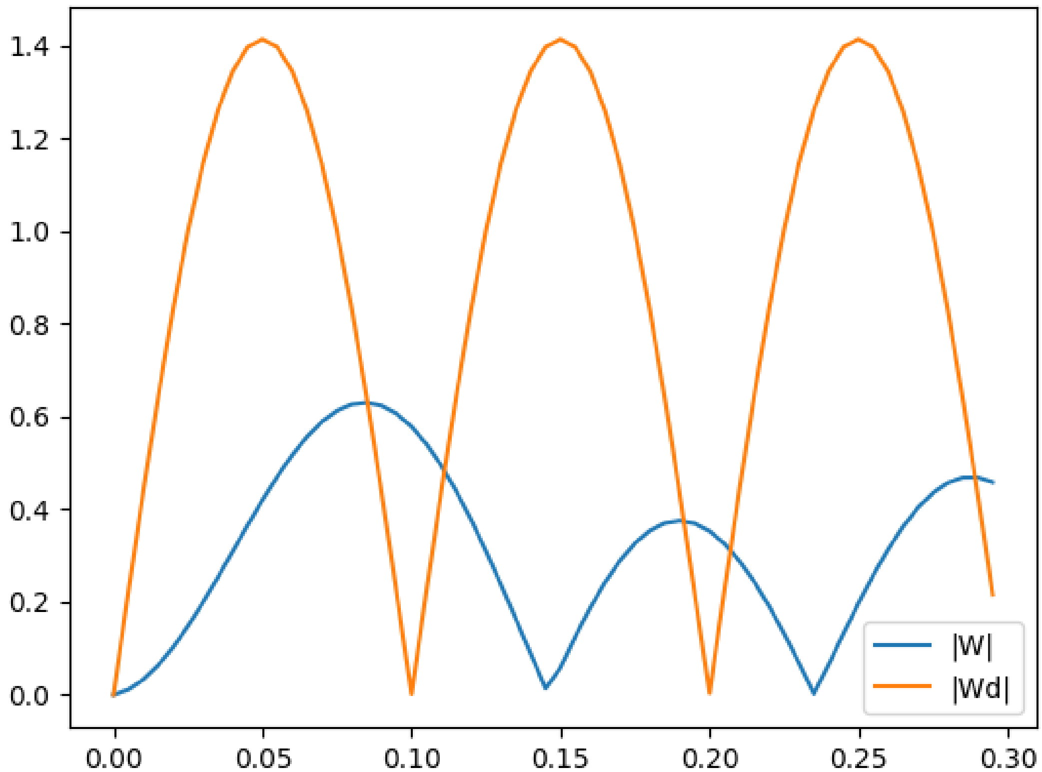


f = 20 Hz

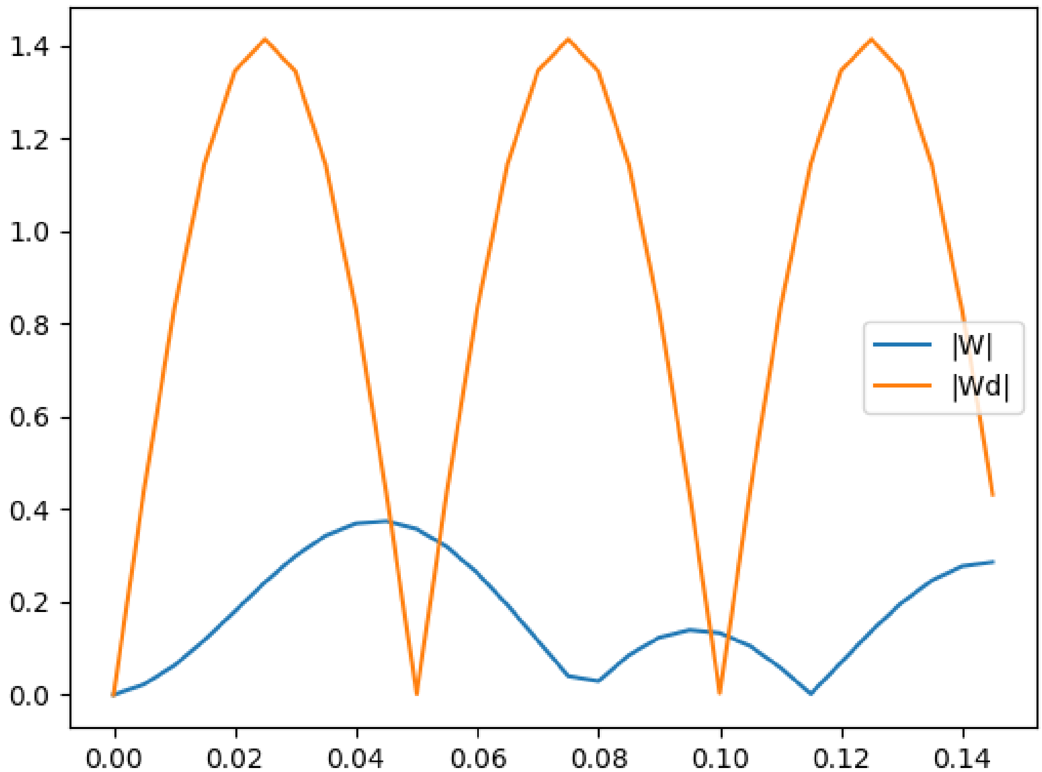
Case2) Kp = Kd = 10



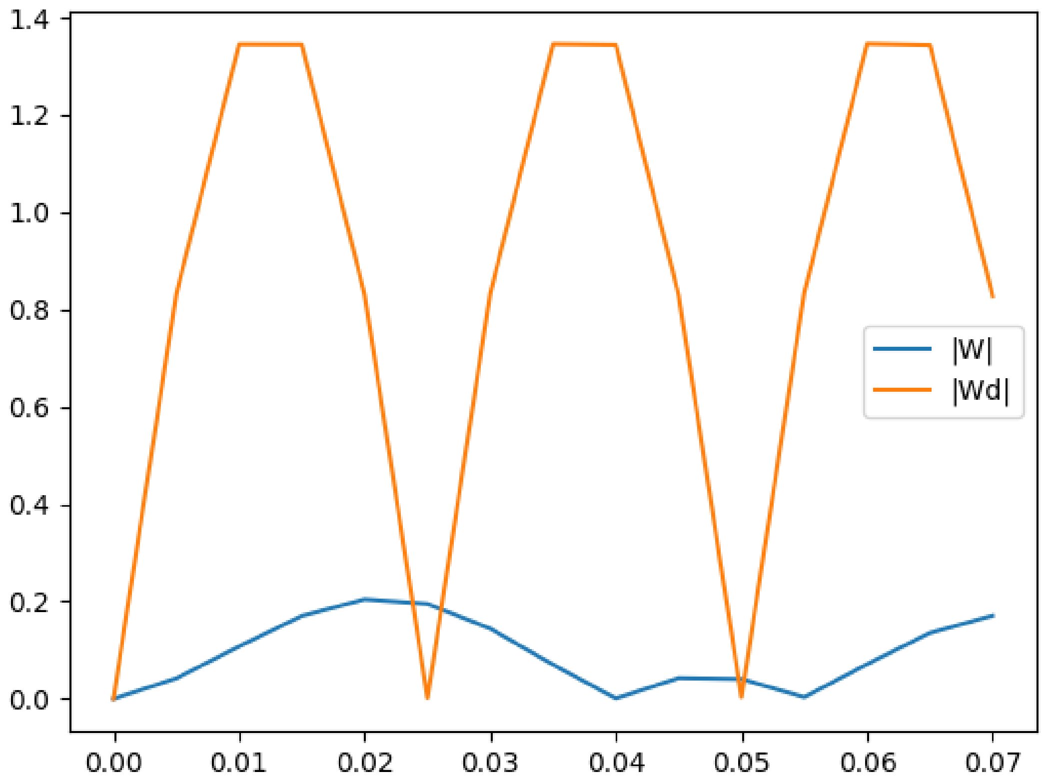
f = 1 Hz



f = 5 Hz



f = 10 Hz



f = 20 Hz