Minor- II: Functional Analysis

School of Mathematics and Statistics
University of Hyderabad

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Duration: 60 minutes
Maximum Score: 20 points

Instructor: Dharmendra Kumar
Course Code: MM 501

Instructions: You may use the result proven in the lecture but answers without justification will receive a score of zero.

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1. (a) Let $T(x) = x(1)$ for all $x \in C[0,1]$. Show that $T \in (C[0,1], \|\cdot\|_\infty)^*$ but $T \notin (C[0,1], \|\cdot\|_p)^*$ if $1 \leq p < \infty$. [1 + 3]

   (b) Let $f((x_n)) = \sum\limits_{n=1}^{\infty} \frac{x_n}{2^n}$ for all $(x_n) \in c_0$. Show that $f \in (c_0, \|\cdot\|_\infty)^*$ and that $\|f\| = 1$. Can one find some $x = (x_n) \in E := (c_0, \|\cdot\|_\infty)$ such that $\|x\| = 1$ and $f(x) = \|f\|_{E^*}$? [3 + 1]

2. If $\big(E,\ \|\cdot\|\big)$ be a normed linear space such that $E \neq \{0\}$, then show that $E^* \neq \{0\}$. [3]

3. Let E be a normed linear space and let $x \in E$. Then $\|x\| = \sup\{|f(x)| : f \in B_{E^*}\}$. [3]

4. Let $\big(E,\ \|\cdot\|\big)$ be a normed linear space. Let $C \subset E$ be an open and convex subset containing 0.

   Let p be the gauge of C, i.e.,

$$p(x) \ = \ p_C(x) \ = \inf\left\{\alpha > 0 \ \ : \ \ \frac{x}{\alpha} \in C\right\} \qquad \text{for} \quad x \in E.$$

   (a) Show that p is uniformly continuous. [3]

   (b) Show that

$$\overline{C} \ = \ \Big\{x \in E \ \ : \ \ p(x) \leq 1\Big\}.$$

   [3]