Minor- III: Functional Analysis

School of Mathematics and Statistics
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Duration: 60 minutes
Maximum Score: 20 points
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Course Code: MM 501

Instructions: Each question carries equal marks: 5. Attempt any FOUR questions. You may use results proven in the lectures; however, answers without justification will receive a score of zero.

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1. Show that Banach space cannot have a countable Hamel Basis. [3]

   Using this fact, show that:

   (a) the space $\mathcal{P}[a,\ b]$ cannot be complete with respect to any norm. [1]

   (b) c_{00} cannot be complete with respect to any norm. [1]

2. Let E and F be two Banach spaces and let $\big(T_n\big)$ be a sequence in $\mathcal{L}\big(E,\ F\big)$. Assume that for each $x \in E$, $T_n x$ converges as $n \longrightarrow \infty$ to a limit denoted by Tx. Show that if $x_n \longrightarrow x$ in E, then $T_n x_n \longrightarrow Tx$ in F. [5]

3. Let E and F be Banach spaces and let $T : E \longrightarrow F$ be a bounded linear map, which is surjective. Show that T is injective if and only if there exists a positive constant M such that

$$\|T(x)\| \;\geq\; M\|x\|, \quad \forall\, x \in E.$$

[5]

4. Let $\alpha = (\alpha_n)$ be a given sequence of real numbers. Assume that $\sum_{n=1}^{\infty} |x_n|\ |\alpha_n| \;<\; \infty$ for every $x = (x_n) \in l_2$. Show that $\alpha \in l_2$.

[5]

5. Let $E = \Big(C[0,1],\ \|\cdot\|_\infty\Big)$ and $F = \Big(\big\{u \in C^1[0,1],\quad u(0) = 0\big\},\quad \|\cdot\|_\infty\Big)$ be normed vector space. Consider the linear operator $T : E \longrightarrow F$ such that:

$$\big(Tu\big)(t) \;=\; \int_0^t u(s)\, ds.$$

(a) Show that T is continous linear map. [1]

(b) Find T^{-1}. [1]

(c) Show that T^{-1} is not continuous. [1]

(d) Does this contradict the Open mapping theorem ? Please Justify it ! [2]