

Minor- I: Real Analysis - II

School of Mathematics and Statistics
University of Hyderabad

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Duration: 60 minutes
Maximum Score: 20 points

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Course Code: MM 451

Instructions: You may use results proven in the lectures; however, answers without justification will receive a score of zero.

1. Examine whether the following limits exist (in \mathbb{R}) and find their values if they exist (in \mathbb{R}).

(a) $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 - y^3}{x^2 + y^2}$ [3]

(b) $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 + y^2}{x^2 + y}$ [3]

2. Examine the continuity of $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ at $(0, 0)$, where for all $(x, y) \in \mathbb{R}^2$,

$$f(x, y) = \begin{cases} \frac{x^2 y^3}{x^6 + y^4} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

[3]

3. Let $\Omega \subseteq \mathbb{R}^2$, $(x_0, y_0) \in \mathbb{R}^2$ and $r > 0$ be such that $(B_r(x_0) \times B_r(y_0)) \setminus \{(x_0, y_0)\} \subseteq \Omega$. Let $\lim_{x \rightarrow x_0} f(x, y)$ exist (in \mathbb{R}) for each $y \in B_r(y_0) \setminus \{y_0\}$, $\lim_{y \rightarrow y_0} f(x, y)$ exist (in \mathbb{R}) for each

$x \in B_r(x_0) \setminus \{x_0\}$ and $\lim_{(x,y) \rightarrow (x_0, y_0)} f(x, y) = \ell \in \mathbb{R}$.

Show that $\lim_{x \rightarrow x_0} \left(\lim_{y \rightarrow y_0} f(x, y) \right) = \lim_{y \rightarrow y_0} \left(\lim_{x \rightarrow x_0} f(x, y) \right) = \ell$. Using this fact, show that $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{x^2 + y^2}$ does not exist (in \mathbb{R}). [4 + 1]

4. Let $f(x, y) = (x + y^2, xy)$ for all $(x, y) \in \mathbb{R}^2$. Using directly the definition of differentiability, show that $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is differentiable and also find $f'(x_0, y_0)$, where $(x_0, y_0) \in \mathbb{R}^2$. [3]

5. Let $f(x, y, z) = (x^3 y + y^2 z, xyz)$ and $g(x, y) = (x^2 y, xy, x - 2y, x^2 + 3y)$ for all $x, y, z \in \mathbb{R}$. Use chain rule to find $(g \circ f)'(\mathbf{a})$, where $\mathbf{a} = (1, 2, -3)$. [3]

All the best !