MM 212 (Probability and Statistics)

Practice Problem Set - 5

(Continuous Random Variables)

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1. Let X be a random variable with probability density function

$$f(x) = \begin{cases} c(1-x^2) & -1 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) What is the value of c ?

   (b) What is the cumulative distribution function of X ?

2. A system consisting of one original unit plus a spare can function for a random amount of time X. If the density of X is given(in units of months) by

$$f(x) = \begin{cases} cxe^{-x/2} & x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

   what is the probability that the system functions for at least 5 months ?

3. Consider the function

$$f(x) = \begin{cases} C(2x-x^3) & 0 < x < \frac{5}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

   Could f be a probability density function ? If so, determine C. Repeat if $f(x)$ were given by

$$f(x) = \begin{cases} C(2x-x^2) & 0 < x < \frac{5}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

4. The probability density function of $X :=$ the lifetime of a certain type of electronic device (measured in hours) is given by

$$f(x) = \begin{cases} \frac{10}{x^2} & x > 10, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find $P\{X > 20\}$.

   (b) What is the cumulative distribution function of X ?

   (c) What is the probability that of 6 such types of devices, at least 3 will function for at least 15 hours ? What assumptions are you making ?

5. A filling station is supplied with gasoline once a week. If its weekly volume of sales in thousands of gallons is a random variable with probability density function

$$f(x) = \begin{cases} 5(1-x)^4 & 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

what must the capacity of the tank be so that the probability of the supply being exhausted in a given week is .01 ?

6. The density function of X is given by

$$f(x) = \begin{cases} a + bx^2 & 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

If $E[X] = \frac{3}{5}$. Find a and b.

7. The lifetime in hours of an electronic tube is a random variable having a probability density function given by

$$f(x) = \begin{cases} xe^{-x} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Compute the expected lifetime of such a tube.

8. Trains headed for destination A arrive at the train station at 15-minute intervals starting at 7 A.M., whereas trains headed for destination B arrive at 15-minute intervals starting at $7:05$ AM.

   (a) If a certain passenger arrives at the station at a time uniformly distributed between 7 and 8 A.M. and then gets on the first train that arrives, what proportion of time does he or she go to destination A ?

   (b) What if the passenger arrives at a time uniformly distributed between $7:10$ and $8:10$ A.M. ?

9. A point is chosen at random on a line segment of length L. Interpret this statement and find the probability that the ratio of the shorter to the longer segment is less than $\frac{1}{4}$.

10. You arrive at a bus stop at 10 A.M., knowing that the bus will arrive at some time uniformly distributed between 10 and $10:30$ AM.

    (a) What is the probability that you will have to wait longer than 10 minutes ?

    (b) If, at 10:15, the bus has not yet arrived, what is the the probability that you will have to wait at least an additional 10 minutes ?

11. If X is a normal random variable with parameters $\mu = 10$ and $\sigma^2 = 36$. Find the following:

    (a) $P(X > 5)$,

    (b) $P(4 < X < 16)$,

    (c) $P(X < 8)$,

    (d) $P(X < 20)$,

    (e) $P(X > 16)$.

12. Suppose that X is a normal random variable with mean 5. If $P(X > 9) = .2$ (approximately). What is $\text{Var}(X)$?

13. Suppose that X is a normal random variable with mean 12 and variance 4. Find the value of c such that $P(X > c) = .10$.

14. (a) A fire station is to be located along a road of length L, $L < \infty$. If fires occur at points uniformly chosen on $(0,\ L)$, where should the station be located so as to minimize the expected distance from the fire ? That is, choose a so as to: minimize $E\big[|X-a|\big]$, where X is uniformly distributed over $(0,\ L)$.

    (b) Now suppose that the road is of infinite length stretching from point 0 outward to ∞. If the distance of a fire from point 0 is exponentially distributed with rate λ,, where should the fire station now be located ? That is. we want to minimize $E\big[|X-a|\big]$, where X is now exponential with rate λ.

15. If X is uniformly distributed over $(-1,\ 1)$, find

    (a) $P\big(|X| > \frac{1}{2}\big)$,

    (b) the density function of the random variable $|X|$.

16. If X is an exponential random variable with parameter $\lambda = 1$. Compute the probability density function of the random variable Y defined by $Y = \log X$.

17. If X is uniformly distributed over $(0,\ 1)$, find the density function of $Y = e^X$.

18. For any real number t, we define t^+ by

$$t^+ = \begin{cases} t & t \geq 0, \\ 0 & t < 0. \end{cases}$$

    Let c be a constant.

    (a) Show that

$$E\big[(Z-c)^+\big] = \frac{1}{\sqrt{2\pi}}\, e^{-c^2/2} - c\big(1-\Phi(c)\big),$$

    where Z is a standard normal random variable.

    (b) Find $E\big[(X-c)^+\big]$, where X is normal with mean μ and variance σ^2.

19. With $\Phi(x)$ being the probability that a normal random variable with mean 0 and variance 1 is less than x, which of the following are true:

    (a) $\Phi(-x) = \Phi(x)$,

    (b) $\Phi(-x) + \Phi(x) = 1$,

    (c) $\Phi(-x) = \frac{1}{\Phi(x)}$.