Math - II (Complex Analysis)

Practice Problem Set - 1

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1. Using $|z|^2 = z\overline{z}$, show that:

   (a) $|z_1 + z_2| \leq |z_1| + |z_2|$

   (b) $|z_1 w_1 + z_2 w_2| \leq \sqrt{|z_1|^2 + |z_2|^2} \sqrt{|z_1|^2 + |z_2|^2}$

   (c) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$

   Note that (b) is a special case of **Cauchy- Schwarz Inequality** (see 5(a) below) and (c) is the usual **parallologram law.**

2. By means of an example, show that (in general)

   $$\mathrm{Arg\ z_1} + \mathrm{Arg\ z_2} \neq \mathrm{Arg}\left(\mathrm{z_1\ z_2}\right)$$

   where Arg z denotes the principal value of the argument of z. However, show that $\Re z_1 > 0$ and $\Re z_2 > 0$ then the above equality holds.

3. If $z_1 z_2 \neq 0$, then show that

   $\Re(z_1 \overline{z_2}) = |z_1|\,|z_2|$ if and only if $\arg z_1 - \arg z_2 = 2n\pi$ for some integrer n.

   In this case show further that

   (a) $|z_1 + z_2| = |z_1| + |z_2|$

   (b) $|z_1 - z_2| \leq \big|\,|z_1| - |z_2|\,\big|$.

4. If $z_1, \cdots, z_n$ are complex numbers then show that

   $$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|.$$

   Moreover, show that equality holds in this inequality if and only if for every pair $z_i(\neq 0)$, $z_j(\neq 0)$ such that $\frac{z_i}{z_j}$ is real and > 0 for $i, j = 1, 2, \cdots, n$.

5. If $z_i, w_i \in \mathbb{C}$ for $i = 1, 2, \cdots, n$. Then show that

   (a) (**Cauchy - Schwarz Inequality**)

   $$|\sum_{i=1}^{n} z_i\, w_i|^2 \leq \left(\sum_{i=1}^{n} |z_i|\right)\left(\sum_{i=1}^{n} |z_i|\right)$$

   (b) (**Lagrange's Identity**)

   $$|\sum_{i=1}^{n} z_i\, w_i|^2 \leq \left(\sum_{i=1}^{n} |z_i|\right)\left(\sum_{i=1}^{n} |z_i|\right) - \sum_{1 \leq i < j \leq n} |\, z_i\, \overline{w_j} - z_j\, \overline{w_i}\,|^2 .$$

6. Show that if $|w| < 1$, then

$$\left|\frac{z-w}{\overline{w}z-1}\right| \;<\; 1 \;\text{ if }\; |z|<1$$

and

$$\left|\frac{z-w}{\overline{w}z-1}\right| \;=\; 1 \;\text{ if }\; |z|=1.$$

7. Show that there are complex numbers z satisfying

$$|z-z_0| \;+\; |z+z_0| \;=\; 2\,|z_1| \;\text{ if and only if }\; |z_0| \le |z_1|.$$

8. Show that three distinct points z_1, z_2, z_3 in the complex plane $\mathbb{C}$ form the vertices of an equilateral triangle if and only if

$$z_1^2+z_2^2+z_3^2 \;=\; z_1z_2 \;+\; z_2z_3 \;+\; z_3z_1.$$

Deduce that if w_1, w_2, w_3 are points dividing the three sides of the triangle $\triangle(z_1, z_2, z_3)$ in the same ratio then the $\triangle(w_1, w_2, w_3)$ is equilateral if and only if the triangle $\triangle(z_1, z_2, z_3)$ is so.

9. If z_1, z_2, z_3 are three distinct complex numbers of equal moduli, then show that

$$2\,\arg\frac{z_2-z_1}{z_3-z_1} \;=\; \arg\frac{z_2}{z_3}.$$

Can you recall the theorem in school geometry which corresponds to this result ?.

10. For every $n \in \mathbb{N}$, show that

(a)

$$\left[1-\binom{n}{3}+\binom{n}{4}-\binom{n}{6}+\cdots\right]^2+\left[\binom{n}{1}-\binom{n}{3}+\binom{n}{5}-\cdots\right]^2 \;=\; 2^n$$

(b)

$$1 \;+\; \cos\theta \;+\; \cos 2\theta \;+\; \cdots \;+\; \cos n\theta \;=\; \frac{1}{2} \;+\; \frac{\sin\left(\frac{2n+1}{2}\right)\theta}{2\,\sin\left(\frac{\theta}{2}\right)} \qquad (0<\theta<2\pi)$$

(c)

$$1 \;+\; \sin\theta \;+\; \sin 2\theta \;+\; \cdots \;+\; \sin n\theta \;=\; \frac{1}{2}\cos\frac{\theta}{2} \;-\; \frac{\cos\left(\frac{2n+1}{2}\right)\theta}{2\,\sin\left(\frac{\theta}{2}\right)} \qquad (0<\theta<2\pi)$$

(d)

$$(1-z_1)(1-z_2)(1-z_3)\cdots(1-z_{n-1}) \;=\; n$$

where $z_1, z_2, \cdots, z_{n-1}$ are the nth roots of unity other than 1.

(e)

$$\sin\left(\frac{\pi}{n}\right)\,\sin\left(\frac{2\pi}{n}\right)\cdots\sin\left(\frac{(n-1)\pi}{n}\right) \;=\; \frac{n}{2^{n-1}}.$$

Hint: use (d).

11. Show that the sum of the nth roots of every nonzero complex number w is zero.

12. if $w = e^{\frac{2\pi i}{n}}$, $n \in \mathbb{N}$, $n \geq 2$, then show that

$$1 + w^h + w^{2h} + \cdots + w^{(n-1)h} = 0$$

for any integer h which is not a multiple of n.

13. Consider the equation $p(z) = a_0\, z^n \ + \ a_1\, z^{n-1} \ + \ \cdots \ + \ a_n = 0$, where $a_i \in \mathbb{C},\ i = 0.1, \cdots, n$ are given complex numbers. Show that if w is a root of this equation, then show that $\overline{w}$ is a root of the equation

$$\bar{a_0}\ (z)^n \ + \ \bar{a_1}\ (z)^{n-1} \ + \ \cdots \ + \ \bar{a_n} = 0.$$

Conclude that if w is a root of the equation $p(z) = 0$ with real coefficients, then so also is $\overline{w}$.

14. Show that if z lies on the circle $|z| = 2$ then

$$\left| \frac{1}{z^4 - 4z^2 + 3} \right| \ \leq \ \frac{1}{3}.$$

15. Show that:

   (a) $(-1 + i)^7 \ = \ -8(1 + i)$

   (b) $(1 + \sqrt{3}\, i)^{-10} \ = \ 2^{-11}(-1 + \sqrt{3}\, i).$

16. Let z_1, z_2 be distinct complex numbers and $k > 0$. Show that the set

$$\{z \in \mathbb{C} : \ |z - z_1| = k|z - z_2|\}$$

is a circle unless $k = 1$, in which the set is a straight line, the perpendicular bisector of the line joining the points z_1 and z_2.

17. Sketch the following sets and determine which are domains:

   (a) $|z - 2 + i| \ \leq 10$

   (b) $\Im(z) \ > \ \Re(z)$

   (c) $|2z + 3| \ > \ 2$

   (d) $0 \ \leq \ \arg z \ \leq \frac{\pi}{4}\ \ (z \neq 0)$

   (e) $|z - 4| \ \geq \ |z|.$

18. In each of the following cases, sketch the closure of the set.

   (a) $-\pi \ < \ \arg z \ < \ \pi \ \ (z \neq 0)$

   (b) $|\Re(z)| < |z|$

   (c) $|\Re(\frac{1}{z})| \ \leq \ \frac{1}{2}$

   (d) $\Re(z^2) \ > \ 0$

   (e) $0 \ \leq \ \arg z < \frac{\pi}{4}\ \ (z \neq 0).$

19. Determine the accumulation points of each of the following sequences.

   (a) $z_n \ = \ i^n$ (b) $z_n \ = \ \frac{i^n}{n}$ (c) $z_n \ = \ (-1)^n(1 + i)\left(\frac{n-1}{n}\right).$