MM 263 (Real Analysis)

Practice Problem Set - 1

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1. Show that:

   (a) If $x,\ y$ are rational numbers then $x + y$ and xy are rational numbers.

   (b) If x is a rational number and y is an irrrational number then $x + y$ is an irrrational number. If, in addition $x \neq 0$ then show that xy is an irrrational number.

2. Prove that set $\mathbb{N} \times \mathbb{N}$ is countable.

3. Prove that set $\mathbb{Q}$ of all rational numbers is countable.

4. In the ordered field $\mathbb{Q}$, show that the set $\{r \in \mathbb{Q} : r^2 < 2\}$ is bounded above but it has no supremum.

5. Using Archimedean property, show that:

   (a) $\bigcap\limits_{n=1}^{\infty} \left(0,\ \frac{1}{n}\right) = \emptyset$,

   (b) $\bigcap\limits_{n=1}^{\infty} \left(-\frac{1}{n},\ \frac{1}{n}\right) = \{0\}$,

   (c) $\bigcap\limits_{n=1}^{\infty} \left(-\frac{1}{n},\ 1 + \frac{1}{n}\right] = [0, 1]$.

6. If $a \in \mathbb{R}$ such that $0 \leq x < \delta$ for every $\delta > 0$. Show that $x = 0$.

7. Show that supremum of the set $\left\{2 - \frac{1}{m}\ :\ m \in \mathbb{N}\right\}$ is 2.

8. If $\mathrm{S} := \left\{\frac{1}{n} - \frac{1}{m}\ :\ m, n \in \mathbb{N}\right\}$. Find $\inf\ \mathrm{S}$ and $\sup\ \mathrm{S}$.

9. Let S be a nonempty bounded set in $\mathbb{R}$. For $a \in \mathbb{R}$, show that:

   (a) For $a > 0$, $\inf\{\mathrm{aS}\} = a \inf\{\mathrm{S}\}$,

   (b) For $a < 0$, $\inf\{\mathrm{aS}\} = a \sup\{\mathrm{S}\}$,

   (c) For $a > 0$, $\sup\{\mathrm{aS}\} = a \sup\{\mathrm{S}\}$,

   (b) For $a < 0$, $\sup\{\mathrm{aS}\} = a \inf\{\mathrm{S}\}$,

10. Using Mathematical Induction, prove Bernoulli's Inequality: For $x > -1$,

$$(1+x)^n \ \geq \ 1+nx, \ \ \forall\, n \in \mathbb{N}.$$

11. Let $\{x_n\} \subset \mathbb{R}$ be an arbitrary sequence such that for every $\varepsilon > 0$, $\exists$ an $N \in \mathbb{N}$ such that $|x_N| > \varepsilon$. Show that $\{x_n\}$ is divergent.

12. Let $\{x_n\} \subset \mathbb{R}$ be an arbitrary sequence such that $\exists$ an $L \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|x_n - L| < n\varepsilon$ for all $n \geq N$. Examine whether $\{x_n\}$ is convergent / divergent / Inconclusive.

13. Let $\{x_n\}$ be sequence in $\mathbb{R}$. Show that $\lim_{n\to\infty} x_n = 0$ if and only if $\lim_{n\to\infty} |x_n| = 0$. Examine whether the convergence of $|x_n|$ imply the convergence of x_n and vice- versa.

14. Let $\{x_n\}$ and $\{y_n\}$ be any two sequence in $\mathbb{R}$ such that $\lim_{n\to\infty} x_n = 0$. Examine whether $\lim_{n\to\infty} |x_n\, y_n| = 0$.

15. Let $\{x_n\}$ and $\{y_n\}$ be sequences in $\mathbb{R}$ such that for some $N \in \mathbb{N}$, $0 \leq x_n \leq y_n$ for all $n \geq N$.

    (a) Show that if $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} a_n = 0$.

    (b) For any $r > 0$, $\lim\limits_{n\to\infty} \sqrt[n]{r} = 1$.

    (c) $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$.

16. Let $\{x_n\}$ be bounded sequence in $\mathbb{R}$. Define

$$a_k \ := \ \sup_{n \geq k} \{x_n\}$$

and

$$b_k \ := \ \inf_{n \geq k} \{x_n\}.$$

Prove that $\{a_k\}$ and $\{b_k\}$ are convergent sequence.

17. Let $\{x_n\}$ be sequence in $\mathbb{R}$. Prove that $\{x_n\}$ has no convergent subsequence if and only if $|x_n| \ \longrightarrow\ \infty$.

18. Let $\{x_n\}$ be sequence in $\mathbb{R}$ and $a \in \mathbb{R}$. Assume that each subsequence of $\{x_n\}$ has a convergent subsequence converging to a. Prove that $x_n \ \longrightarrow\ a$.

19. Let $x_1 = a$ and $x_{n+1} := x_n + \frac{1}{x_n}$ for $n \in \mathbb{N}$. Prove / Disprove: $\{x_n\}$ is convergent.

20. Let $x_0 = 1$ and $x_{n+1} \ = \ \frac{3+2x_n}{3+x_n}$ for $n \geq 0$. Prove that $\{x_n\}$ is convergent and find its value.

21. Let $\alpha \in (0,\ 1)$ and let $\{x_n\}$ be any sequence in $\mathbb{R}$ satisfying the recurence relation

$$x_{n+1} \ = \ \alpha x_n \ + \ (1-\alpha)x_{n-1}.$$

Show that $\{x_n\}$ is convergent and find its limit in terns of $\alpha,\ x_0,\ x_1$.

22. Let $\{x_n\}$ and $\{y_n\}$ be cauchy sequences in $\mathbb{R}$. Show that $\{x_n + y_n\}$ and $\{x_n y_n\}$ are cauchy sequences.

23. Let $\{x_n\}$ be sequence in $\mathbb{R}$ and $p \in \mathbb{N}$ be a given number (fixed) such that

$$\lim_{n\to\infty} |x_{n+p} - x_n| = 0.$$

Prove / Disprove: $\{x_n\}$ is cauchy sequence.

24. Let $\{x_n\}$ be decreasing sequence in $\mathbb{R}$ such that $x_n \longrightarrow 0$. Show that $\sum_{n=1}^{\infty}(-1)^{n+1}x_n$ converges.

25. Show that $\sum_{n=1}^{\infty} x_n$ converges if and only if $\sum_{n=1}^{\infty} x_{n+k}$ converges for each $k \geq 1$.

26. Let $\{x_n\}$ be sequence in $\mathbb{R}$. Show that $\exists$ a series whose partial sums are x_n.

27. Show that $\sum_{n=1}^{\infty} x_n$ converges if and only if for each $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ such that $|\sum_{k=m}^{n} x_k| < \varepsilon$ for all $n \geq m \geq N$.

28. Let $\{x_n : n \in \mathbb{N}\}$ be an arbitrary collection of **non-negative** real numbers such that $\sum\limits_{n=1}^{\infty} x_n$ converges. Examine whether following series converge / diverge / inconclusive.

(a) $\sum\limits_{n=1}^{\infty} x_n^2$

(b) $\sum\limits_{n=1}^{\infty} \sqrt{x_n}$

(c) $\sum\limits_{n=1}^{\infty} \dfrac{\sqrt{x_n}}{n}$

(d) For $x_n > 0$, $\sum\limits_{n=1}^{\infty} \left(1 - \frac{\sin x_n}{x_n}\right)$

29. Show that each of the following series converges, and determine its sum.

(a) $\sum\limits_{n=1}^{\infty} \dfrac{4n^2 - 1 + 3^{n-1}}{3^n\,(2n+1)\,(2n-1)}$

(b) $\sum\limits_{n=10}^{\infty} \dfrac{10}{n^2 - 1}$

(c) $\sum\limits_{n=1}^{\infty} \dfrac{n}{(n+1)(n+2)(n+3)}$

30. For each of the series given below, determine whether it converges or diverges.

(a) $\sum\limits_{n=1}^{\infty} \dfrac{n\cos^3(n\pi/3)}{2^n}$

(b) $\sum\limits_{n=1}^{\infty} \tan\frac{1}{n}$

(c) $\sum\limits_{n=5}^{\infty} \dfrac{\sqrt{n}+1}{(n-1)(n+2)(n-4)}$

(d) $\sum\limits_{n=1}^{\infty} \frac{n^n}{n!}$