End Term: Functional Analysis

School of Mathematics and Statistics
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Duration: 3 Hours	**Instructor: Dharmendra Kumar**
Maximum Score: 60 points	**Course Code: MM 501**

Instructions: Answer any FOUR questions. Each question carries 15 marks. You may use results proven in the lectures; however, answers without justification will receive a score of zero.

1. Prove / Disprove:

   (a) Let E be a normed linear space and let M, $N \subset E$. If both M and N are closed in E, then $M + N$ is closed in E [3]

   (b) l_∞ is Banach space with respect to $\|\cdot\|_\infty$ norm. [3]

   (c) c_{00} is Banach space with respect to $\|\cdot\|_\infty$ norm. [3]

   (d) Consider the linear space $C^1[a,b] = \{f : [a,b] \to \mathbb{K} : f^{(1)} \in C[a,b]\}$ with respect to the addition and scalar multiplication defined pointwise. If $\|f\| = \|f\|_\infty + \|f^{(1)}\|_\infty$ for all $f \in C^1[a,b]$, then $(C^1[a,b], \|\cdot\|)$ is a Banach space. [3]

   (e) Let E be normed vector space. Let $A \subset E$ and $B \subset E$ be two nonempty closed convex subsets such that $A \cap B = \phi$. Then there exists a closed hyperplane that strictly separates A and B. [3]

2. Let $\big(E, \|\cdot\|\big)$ be a normed linear space. Let $C \subset E$ be an open and convex subset containing 0.

   Let p be the gauge of C, i.e.,

   $$p(x) \;=\; p_C(x) \;= \inf\left\{\alpha > 0 \;\;:\;\; \frac{x}{\alpha} \in C\right\} \qquad \text{for} \quad x \in E.$$

   (a) Show that

   $$p(\lambda x) \;\;=\;\; \lambda\, p(x), \;\; \forall\, x \in E \;\text{ and }\; \forall\, \lambda \geq 0.$$

   [3]

   (b) Show that there is a constant M such that

   $$0 \;\leq\; p(x) \;\leq\; M\,\|x\|, \;\; \forall\, x \in E.$$

   [3]

(c) Show that
$$p(x+y) \;\leq\; p(x)+p(y), \quad \forall\, x,y \in E.$$
[3]

(d) Show that p is uniformly continuous. [3]

(e) Show that
$$\overline{C} \;=\; \Big\{x \in E \;\;:\;\; p(x) \leq 1\Big\}.$$
[3]

3. (a) Let M and N be two closed linear subspaces of Hilbert space H. Assume that
$$(u,\ v) = 0 \quad \forall\, u \in M, \;\; \forall\, v \in N.$$
Prove that $M+N$ is closed. [3]

(b) Let $K \subset H$ be a nonempty closed convex set. Then for every $f \in H$, there exists a unique element $u \in K$ such that
$$\|f-u\| \;=\; \min_{v\in K} \|f-v\| \;=\; \operatorname{dist}(f,\ K)$$
[5]

(c) Let E and F be two Banach spaces with norms $\|\cdot\|_E$ and $\|\cdot\|_F$. Let $T \in \mathcal{L}(E,\ F)$ be such that $R(T)$ is closed and dim $N(T) < \infty$. Let $|\cdot|$ denote another norm of E that is weaker than $\|\cdot\|_E$, that is, $|x| \leq M\,\|x\|_E \;\; \forall x \in E$.

Prove that there exists a constant C such that
$$\|x\|_E \;\leq\; C\,\big(\|Tx\|_F + |x|\big), \quad \forall\, x \in E.$$
[7]

4. (a) Let E be an inner product space and let $x,\, y \in E$. Show that $\|x+y\|\,\|x-y\| \leq \|x\|^2 + \|y\|^2$. [3]

(b) Let E be an inner product space and let $x,\, y \in E$. If $\lambda > 0$, then show that $|\langle x,y\rangle| \leq \lambda\|x\|^2 + \frac{1}{4\lambda}\|y\|^2$. [3]

(c) Let M be a closed subspace of a Hilbert space. Show that $M^{\perp\perp} = M$. [3]

(d) Let $E = l_p$ with $1 < p < \infty$. Let (λ_n) be a bounded sequence in $\mathbb{R}$ and consider the operator $T \in \mathcal{L}(E)$ defined by
$$Tx \;=\; (\lambda_1 x_1, \lambda_2 x_2, \cdots, \lambda_n x_n, \cdots),$$
where $x = (x_1, x_2, \cdots, x_n, \cdots)$.

Prove that T is compact operator from E into E if and only if $\lambda_n \longrightarrow 0$. [3 + 3]

5. (a) Let H be a Hilbert space and let $T : H \to H$ be a linear map such that $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in H$. Show that T is bounded. [4]

(b) Prove / Disprove: There exists infinite dimensional separable Hilbert space H such that

$$\left\{x \in H \ : \ \|x\| \leq 1\right\}$$

is compact in H. [2]

(c) Prove / Disprove: Let H be infinite-dimensional Hilbert space. Then there exists bijective compact linear map $T : H \longrightarrow H$. [3]

(d) Consider $T : l_2 \longrightarrow l_2$ such that

$$T(x) := \left(\frac{x_1}{1}, \frac{x_2}{2}, \frac{x_3}{3}, \cdots, \frac{x_n}{n}, \cdots\right).$$

(i) Find the adjoint of T . [2]

(ii) Show that $0 \in \sigma(T)$ but 0 is not an eigenvalue. [2]

(e) State the Spectral Theorem for compact self-adjoint operator. [2]

6. Let S_R and S_L be the right and left shifts operator on l_2.

(a) Prove that $\big(S_R\big)^* = S_L$. [3]

(b) Prove that $\sigma_p(S_L) = \sigma_r(S_R)$. [4]

(c) Prove that $\sigma_c(S_L) = \sigma_c(S_R)$. [4]

(d) Prove that $\sigma_r(S_L) = \sigma_p(S_R)$. [4]

All the best !