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Introduction to Statistical Methods
Probability

Revision-3.0

Prof Vineet Garg

BITS Pilani Work Integrated Learning Programmes (WILP)
Bangalore Professional Development Center

Introduction

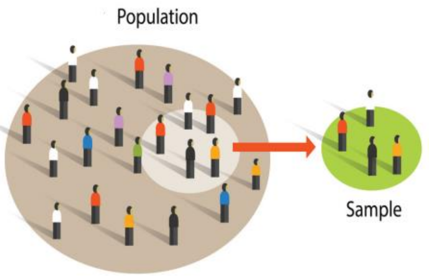
Why do we need to learn Probability? Are there any business reasons?

Equal Employment Opportunity Commission (EEOC) in a country audits the employment record and analyze the data:

- If a manager is randomly selected, what is the probability that it is a woman on that managerial position?
- A special bonus is awarded in a year in a company to the eligible employees. What is the probability that it will go a to woman?
- If an employee is physically challenged. Does he/she get the similar opportunities? How to find that?

QA engineer in an LED bulb manufacturing company selects a random sample of 150 bulbs from the total manufactured bulbs in a week to test the luminance. Why only 150?

- It is infeasible to test all the bulbs.
- The QA engineer sample data will be used by the management to ascertain the luminance of the weekly lot.
- Is sample correctly telling abut the weekly lot? Yes...Of Course. How are you sure?



- **We have reviewed that *inferential statistics* involves taking a sample from a population, computing a *statistic* on the sample, and then infer from the statistic the value of the corresponding *parameter* of the population.**
- **But...there is an uncertainty in this inference.**
- **This module will help us to learn the ways to establish the *confidence level or the probability* on this uncertain inference.**

Structure of Probability

Experiment:

Experiment is a process that produces outcomes. Examples:

- Surveying 20 randomly selected consumers.
- Data Collection of NIFTY for the first Monday of every month for the 5 years.
- Rolling a die.

NIFTY 50

Event:

Event is a specific outcome of an experiment. Examples:

- How many consumers give negative feedback in the survey?
- Which Mondays in 2022 NIFTY crossed 17,000 mark?
- Getting an even number in the die.

Elementary Events:

Events that cannot be further broken down into other events. Example:

- Getting an even number in the die is an event. But getting one of {2, 4, 6} is an elementary event.

Sample Space:

Enumeration of all elementary outcome events of an experiment. Example:

- When a die is rolled, sample space is {1, 2, 3, 4, 5, 6}.

Classical Probability: Foundation

- Probabilities are assigned based on laws and rules. This method involves an experiment, which produces the outcomes. Outcomes has an event (E) for which the probability has to be found out.
- Let us say, ***N*** times an experiment is performed and ***m*** times the desired result (***E***) occurs. So the probability (***P***) that event to occur is **P(E) = m/N.**

Example: In a particular plant, three machines (M1, M2 and M3) manufacture a given product. M1 always produces 30% of the total count of this product. However, 10% products produced by M1 are defective. If all products produced by M1, M2 and M3 are mixed. What is the probability that a product when randomly selected is produced by M1?

- Let us say total products = x
- Products produced by M1 = 0.30x
- So the probability that a randomly selected product would be produced by M1 = 0.30x/x = **0.30**

What is the probability that this randomly selected product will be manufactured by M1 and is defective?

- 10% products of M1 are always defective, so defective products by M1 = 0.10*0.30x
- So the probability that a randomly selected products would be produced by M1 and it is defective also = (0.10*0.30x)/x = **0.03**

- The probability can be determined ***a priori***; that is, it can be determined prior to the experiment (even before the actual selection takes place).
- The range of P(E) will be always 0 to 1 that is **0 ≤ P(E) ≤ 1**

Historical Data Based Probability

Probability P(H) is assigned based on the historical data.

$$P(H) = \frac{\textit{Number of Times an Event Occurred}}{\textit{Total Number of Past Happenings}}$$

Example: A retail chain data shows that when suppliers sent 80 shipments, 9 of them were rejected by the quality inspectors. What is the probability that the inspectors would reject the next shipment?

P(H) = 9/80 = 0.11

Assume this next shipment is actually rejected. What is the probability that the following shipment will be rejected?

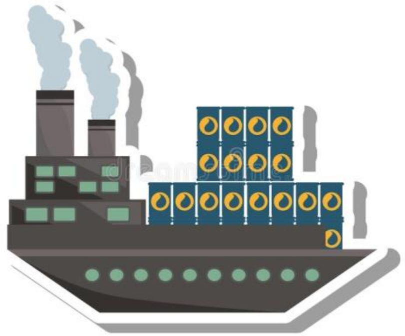
P(H) = 10/81 = 0.12

Subjective Probability

- The subjective method of assigning probability is based on the ***insights*** of the person determining the probability.
- Although not a scientific approach to probability, the subjective method often is based on the accumulation of knowledge, understanding and experience stored and processed in the human mind.

Example: A director of transportation for an oil refinery is asked the probability of getting a shipment of oil out of a Gulf country to India within three weeks. A director who has scheduled many such shipments, has the knowledge of current political, climatological and economic conditions. He may be able to give an accurate probability that the shipment can be made on time.

This is a domain specific approach.
This course does not deal with these approaches.

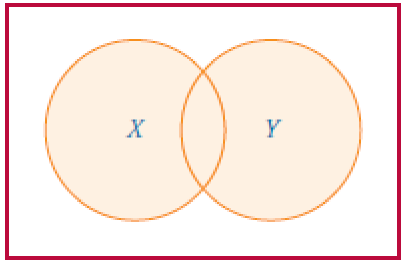


Set Theory Aspects: 1/2

Union:

The union of two sets X, Y is formed by combining elements from each of the sets and is represented by X∪Y.

If X = {1, 4, 7, 9} and Y = {2, 3, 4, 5, 6} then X∪Y = {1, 2, 3, 4, 5, 6, 7, 9}
All the values of X and Y form the union. No value is listed more than once. The complete shaded portion represents the union of X and Y.



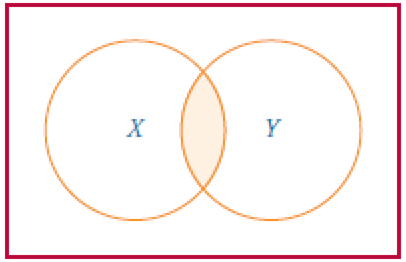
Union

Intersection:

The intersection of two sets X, Y is the common element set and is represented by **X∩Y**.

If X = {1, 4, 7, 9} and Y = {2, 3, 4, 5, 6} then X∩Y = {4}

The shaded portion represents the intersection of X and Y.



Intersection

Mutually Exclusive Events:

If the occurrence of one event prevents the occurrence of the other event, such events are called mutually exclusive events. The probability of two mutually exclusive events occurring at the same time is zero. E.g. a manufactured part cannot be defective and perfect at the same time. For two mutually exclusive events X and Y: **X∩Y = {ϕ}.**

Set Theory Aspects: 2/2

Independent Events:

If the occurrence or non-occurrence of one of the events does not affect the occurrence or non-occurrence of the other event.

P(X|Y) denotes the probability of X occurring given that Y has occurred. If X and Y are independent, then **P(X|Y) = P(X)**........*This concept comes under Conditional Probability. Will be revisited.*

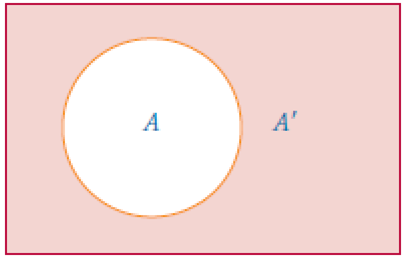
A quality inspector randomly draws an automobile part from a lot that contains 10% defective parts.

The part selected is defective.
He keeps it back in the lot.
Makes a second draw.
Second draw event is independent of the first one.

The part selected is defective.
He keeps it separate.
Makes a second draw.
Second draw event is not independent of the first one because now there are lesser defective parts.

Complementary Events:

The complement of event A is denoted by A' or $\bar{A}$ (used interchangeably in these slides). It is pronounced as "not A" or "A prime" or "A bar". For example, if in rolling one die, event A is getting an even number, the complement of A is getting an odd number. It is denoted as **P(A') = 1- P(A).**



Shaded Portion: The Complement of A

Counting Possibilities: *m*n* Rule

For an operation that can be done in ***m*** ways and a second operation that can be done ***n*** ways, the two operations then can occur, in order, in ***m*n*** ways. This rule can be extended to cases with three or more operations.

Example: A customer decides to buy a certain brand of a new car. Options for the car include 3 different engines, 8 different paint colors, and 4 interior packages. If each of these options is available with each of the others, how many different options are present?

Total different options = 3x8x4 = **96**

Example: A scientist wants to do a research design to study the effects of gender (M, F), marital status (single, divorced, married), and economic class (lower, middle, and upper) on the frequency of airline ticket purchases per year. How many possible groups are present?

Total possible groups = 2x3x3 = **18**

Counting Possibilities: w and w/o Replacement

With Replacement:

When an event occurs and the previous sample that was drawn is mixed back to the population, it is called *With Replacement*.

Example: A die is rolled. It has 6 possible outcomes. When the second time the same die is rolled, the previous outcome is also available to re-occur.

Sampling **n** items from a population of size **N** with replacement would provide $\mathbf{N^n}$ possibilities.

Example: A die is rolled. It has 6 possible outcomes. If the die is rolled 4 times, how many possibilities are there?

6^4 = **1296** possibilities

Example: Suppose in a lottery six numbers are drawn from the digits 0 through 9, with replacement (digits can be reused). How many different groupings of six numbers can be drawn?

10^6 = **1000000**

Without Replacement:

When an event occurs and the previous sample that was drawn is not mixed back to the population (or it is not possible to do so), it is called *Without Replacement*.

Sampling **n** items from a population of size **N** without replacement would provide $\mathbf{{}^{N}C_{n}}$ possibilities.

$${}^{N}C_{n} = \frac{N!}{n!(N-n)!}$$

Example: An organization has 24 employees. This year budget permits any and only 4 random employees to go for a fully reimbursed training. How many possibilities are there for any 4 employees to get selected?

${}^{24}C_4$ = 24! / 4! x (24-4)! = **10,626**

What is Permutation then?

First let us revisit Combination: An unordered subset selection is called the combination.

Example: An organization has 24 employees. This year budget permits any and only 4 random employees to go for a fully reimbursed training. How these four employees will be selected when there is no other constraint.

We have reviewed it on the previous slide that it can be done in ${}^{N}C_{n}$ ways, i.e. ${}^{24}C_{4} = 10626$ ways.

Permutation: An ordered subset selection is called the permutation.

Example: An organization has 24 employees. This year budget permits any and only 4 randomly selected employees to go for a fully reimbursed cruise vacation. The first ticket available is Platinum, the second is Gold, the third is Silver and the fourth is Bronze. The employees are to be selected in this seat order. In how many way four employees will be selected?

The employee for the:

Platinum seat can be selected in 24 ways.

Gold seat can be selected in 23 ways.

Silver seat can be selected in 22 ways.

Bronze seat can be selected in 21 ways.

Total Ways = 24x23x22x21 = **255024**

This is also represented by:

$$^{N}P_{n} = \frac{N!}{(N-n)!} = N.(N-1).(N-2)\dots.(N-n+1)$$

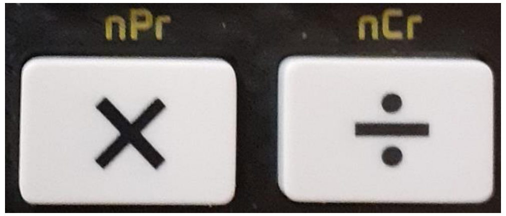
Example

I am giving a dinner party to my close friends. I have 8 bottles of Wine-A, 10 bottles of Wine-B, and 12 bottles of Wine-C:

1) If 6 bottles of wine are to be randomly selected from the whole lot, how many ways are there to do this?

Total 6 bottles are to be selected out of 30.

So total ways $^{N}C_{n} = {}^{30}C_{6} = \frac{N!}{n!(N-n)!} = \frac{30!}{6!(30-6)!} = 593775$



2) If 6 bottles are randomly selected, how many ways are there to obtain 2 bottles of each variety?

Total ways = $^{8}C_{2}$ x $^{10}C_{2}$ x $^{12}C_{2}$ = 83160

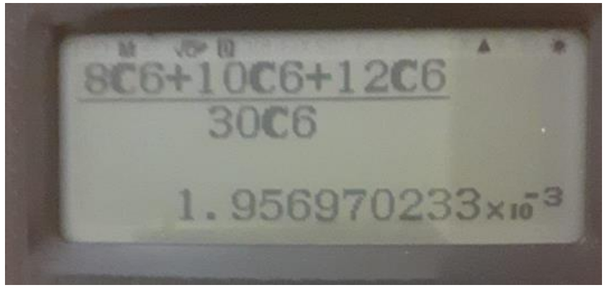
3) If 6 bottles are to be randomly selected, what is the probability that this results in two bottles of each variety being chosen?

Probability for 2 bottles of each type = 83160/593775 = 0.14

4) Wine-A bottles are uniquely numbered. If I want to serve 3 bottles of Wine-A, how many ways are there to select 3 bottles of Wine-A out of the lot of 8 if ordering of selection is to be considered?

There are 8 bottles of Wine-A which are uniquely numbered.

So, total ways to select 3 bottles of Wine-A = $^{N}P_{n} = \frac{N!}{(N-n)!} = \frac{8!}{(8-3)!} = 336$



5) If 6 bottles are randomly selected, what is the probability that all of them are the same variety?

Probability for the all the 6 bottles of the same type = $\frac{^{8}C_{6} + {}^{10}C_{6} + {}^{12}C_{6}}{^{30}C_{6}} = 0.002$

Exercise

1. X = {1, 3, 5, 7, 8, 9}, Y = {2, 4, 7, 9}, and Z = {1, 2, 3, 4, 7}, solve the following:
   a. $X \cap Y \cap Z$
   b. $(X \cup Y) \cap Z$
   c. $(X \cap Y) \cup (Y \cap Z)$

2. A bin contains 6 parts. 2 parts are defective and 4 are acceptable. If 3 of the 6 parts are selected from the bin without replacement, how large is the sample space? Which counting rule did you use, and why?

Answer:

2 defective and 4 are acceptable part
3 are to be selected without replacement.

Possibilities:

1 defective + 2 acceptable = $^2C_1 * {}^4C_2$ = 2*6 = **12**
2 defective + 1 acceptable = $^2C_2 * {}^4C_1$ = 1*4 = **4**
0 defective + 3 acceptable = $^4C_3$ = **4**

Total **20** possibilities

For this sample space, what is the probability that exactly one of the three sampled parts is defective?
12/20 = **0.60**

Examples

Marginal Probability:

- What is the probability P(H) that a person owns a Hyundai car?

$$P(H) = \frac{\text{Count of Hyundai Car Owners}}{\text{Count of Total Car Owners}}$$

Union Probability:

- What is the probability (P) that a person owns a Hyundai or a Mahindra car?

$$P(H \cup M) = \frac{\text{Count of Hyundai or Mahindra or both Car Owners}}{\text{Count of Total Car Owners}}$$

Joint Probability:

- What is the probability (P) that a person owns a Hyundai as well a Mahindra car?

$$P(H \cap M) = \frac{\text{Count of Owners with both Hyundai and Mahindra Cars}}{\text{Count of Total Car Owners}}$$

Conditional Probability:

- What is the probability that a person owns a Hyundai given that he owns a Mahindra car?

$$P(H|M) = \frac{P(H \cap M)}{P(M)}$$

We will revisit

General Law of Addition

Indian Industry Association conducted a survey in few public factories and found that 70% workers agreed that it would increase their productivity if **Free Transportation (T)** is provided. 67% of the workers mentioned if **Free Cafeteria (C)** is provided, it would increase their productivity. 56% workers were in favor of both and all worker participated in the survey.

Probability Matrix		Free Cafeteria: Yes	Free Cafeteria: No	
Free Transportation	Yes	0.56	0.14	0.70
	No	0.11	0.19	0.30
		0.67	0.33	

General Law of Addition:

$$P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$$

Which matrix cells need to be added to get the same answer?

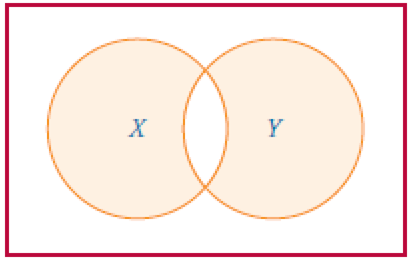
Why 0.56 is subtracted? Justify from the matrix cells.

What is the probability that a worker responded to Free Transportation or Free Cafeteria or both?

$P(T \cup C) = P(T) + P(C) - P(T \cap C) = 0.70 + 0.67 - 0.56 = \mathbf{0.81}$

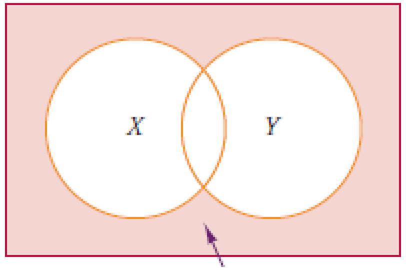
Law of Addition: Few Cases

$$P(X \text{ or } Y \text{ but not both}) = P(X) + P(Y) - \underbrace{P(X \cap Y) - P(X \cap Y)}_{\textit{Intersection subtracted twice}}$$



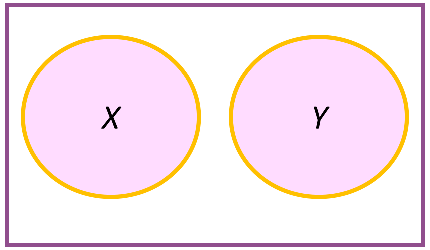
X or Y but not both

$$P(\text{Neither } X \text{ nor } Y) = 1 - P(X \cup Y)$$



Neither X nor Y

If X and Y are mutually exclusive, $P(X \cup Y) = P(X) + P(Y)$ because $P(X \cap Y) = 0$



X or Y when they are exclusive

Exercise

From the probability matrix shown for region and profession below:

1. What is the probability that the professional is from the East?

Answer: 0.21

2. What is the probability that the professional is from the Marketing or from the North?

Answer: 0.64

3. What is the probability that the professional is from the South or from the IT?

Answer: 0.40

Probability Matrix	North	South	East	West
IT	0.12	0.05	0.04	0.07
Finanace	0.15	0.03	0.11	0.06
Marketing	0.14	0.09	0.06	0.08

Joint Probability: Multiplication Law

- The *intersection* of two events X and Y is denoted as (X∩Y) and their probability of happening together is called the *Joint Probability* denoted by **P**(X∩Y).
- The joint probability can be calculated as:

$$P(X \cap Y) = P(X) \,.\, P(Y|X) = P(Y) \,.\, P(X|Y)$$

- Here the following terms are called the conditional probability:
  - P(Y|X) = Probability for Y to occur when X has occurred.
  - P(X|Y) = Probability for X to occur when Y has occurred.

Example: According to the ITES Association Statistics, 46% of the call center work force is female. In addition, 25% of the women in the call centre work part time. What is the probability that a randomly selected member is a woman and also works part-time?

W: Woman, T: Part Time
Given that:
P(W) = 0.46
P(T|W) = 0.25.......Note that a worker is a woman and then she works part time.
We have to find out the joint probability:
P(W ∩ T) = P(W) . P(T|W) = 0.46 x 0.25 = **0.115**

Joint Probability: Independent Events

$P(X \cap Y) = P(X) . P(Y|X) = P(Y) . P(X|Y)$

We have reviewed from the previous slides:

Mutually Exclusive Events:

If the occurrence of one event precludes the occurrence of the other event, such events are called mutually exclusive events. The probability of two mutually exclusive events occurring at the same time is zero. E.g. a manufactured part cannot be defective and perfect at the same time. For two mutually exclusive events X and Y: $\mathbf{X \cap Y = \{\phi\}}$

and

Independent Events:

If the occurrence or non-occurrence of one of the events does not affect the occurrence or non-occurrence of the other event.

$P(X|Y)$ denotes the probability of X occurring given that Y has occurred. If X and Y are independent, then **$P(X|Y) = P(X)$** and **$P(Y|X) = P(Y)$**

When two events X and Y are independent, $P(X|Y) = P(X)$ and $P(Y | X) = P(Y)$

The intersection of two independent events, $P(X \cap Y) = P(X) . P(Y | X)$ becomes:

$$\mathbf{P(X \cap Y) = P(X) . P(Y)}$$

Example: From the probability matrix, identify if events A and D are independent.

$P(A \cap D) = 8 / 85 = 0.094$

$P(A) = 20 / 85$ and $P(D) = 34 / 85$

$P(A).P(D) = (20/85).(34/85) = 0.094$

Since, $P(A \cap D) = P(A).P(D)$, A and D are independent.

	D	E	
A	8	12	20
B	20	30	50
C	6	9	15
	34	51	**85**

Exercise

Remember: $P(X \cap Y) = P(X).P(Y \mid X) = P(Y).P(X \mid Y)$

1. A company has 140 employees, of which 30 are supervisors. 80 of the employees are married, and 20% of the married employees are supervisors. (i) If a company employee is randomly selected, what is the probability that the employee is married as well as a supervisor? (ii) If a married employee is randomly selected what is the probability that he/she is a supervisor?

   Given that: P(S) = 30/140, P(M) = 80/140,

   (i) P(M ∩ S) = 16/140 = 0.1143

   (ii) Since, P (M ∩ S) = P(M). P (S | M), so P (S | M) = (16/140)/(80/140) = 0.20

2. From the probability matrix shown below and <u>using the multiplication law</u> find out the following:

   a) P(Finance ∩ South)

      Solution: P(Finance ∩ South) = P(Finance) . P(South | Finance) = (0.15+0.03+0.11+0.06) x (0.03) / (0.15+0.03+0.11+0.06) = 0.03

   b) P(IT ∩ West)

      Answer = 0.07

   c) P(Finance ∩ Marketing)

      Answer = 0 (mutually exclusive)

Probability Matrix	North	South	East	West
IT	0.12	0.05	0.04	0.07
Finanace	0.15	0.03	0.11	0.06
Marketing	0.14	0.09	0.06	0.08

Conditional Probability

- Conditional probability P(X|Y) is defined as the probability of X given Y has occurred.
- Joint Probability (Multiplication Law) term can be re-arranged as:

$$P(X \cap Y) = P(X).P(Y \mid X) = P(Y).P(X \mid Y), or$$

$$P(X \mid Y) = \frac{P(X \cap Y)}{P(Y)}, or$$

$$\boldsymbol{P(X \mid Y) = \frac{P(X).P(Y \mid X)}{P(Y)}}$$

Example: what is the probability that a randomly selected worker believes **Free Transportation (T)** would not improve productivity given that the worker does believe **Free Cafeteria (C)** would improve productivity?

Probability Matrix		**Free Cafeteria**		
		Yes	**No**	
Free Transportation	**Yes**	0.56	0.14	0.70
	No	0.11	0.19	0.30
		0.67	0.33	

The questions asks $\boldsymbol{P(\bar{T} \mid C)}$

$$P(\bar{T} \mid C) = \frac{P(\bar{T} \cap C)}{P(C)}$$

$$= \frac{0.11}{0.67} = 0.16$$

Example

Remember: $P(X \cap Y) = P(X).P(Y \mid X) = P(Y).P(X \mid Y)$

Equal Employment Opportunity Commission (EEOC) in a country audited the employment record of a company and collected the following data for their employee count:

Position Type	Male (M)	Female (F)
Managerial (M)	8	3
Professional (Lawyers, Doctors, Accountants) **(P)**	31	13
Technical (T)	52	17
Clerical (C)	9	22

Total Male employees = 8+31+52+9 = 100
Total Female employees = 3+13+17+22 = 55
Total Employees = 100+55 = 155

1) What is the probability that a female would be selected given that she is a technical employee?

Answer: $P(F|T) = P(F \cap T) / P(T) = 17/(52+17) = 17/69 = 0.246$

2) What is the probability that the employee is either a male or is a clerical worker?

Answer: $P(M \cup C) = P(M) + P(C) – P(M \cap C) = (100/155) + (31/155) – (9/155) = 122/155 = 0.787$

3) What is the probability that the employee is from the technical group if it is known that the employee is a male?

Answer: $P(T|M) = P(T \cap M) / P(M) = 52/(8+31+52+9) = 52/100 = 0.52$

Denominator 155 will nullify each other so not shown in the calculations in questions 1 and 3.

Exercise

Remember: $P(X \cap Y) = P(X).P(Y|X) = P(Y).P(X|Y)$

Start-up India Scheme conducted a survey and found that 46% of the start-up owners found the fluctuating economy (E) as the main challenge. The second challenge was to find the relevant skill-set (S) by 37% start-up owners. 15% owners selected both. All start-up owners participated. A start-up owner is randomly selected:

	E	E'	
S	0.15	0.22	0.37
S'	0.31	0.32	0.63
	0.46	0.54	

A. What is the probability that the owner believes the economy is a challenge for growth if the owner already believes that finding skill set is a challenge?

Answer: $P(E|S) = P(E \cap S) / P(S) = 0.15/0.37 = 0.405$

B. What is the probability that the owner believes that finding skill set is a challenge for growth if the owner already believes that the economy is a challenge?

Answer: $P(S|E) = P(E \cap S) / P(E) = 0.15/0.46 = 0.326$

C. Given that the owner does not select the economy as a challenge for growth, what is the probability that the owner believes that finding the skillset is a challenge for growth?

Answer: $P(S|E') = P(S \cap E') / P(E') = 0.22/0.54 = 0.407$

D. What is the probability that the owner believes neither that the economy is a challenge for growth nor that finding the skillset is a challenge for growth?

Answer: $P(E' \cap S') = 0.32$

Or, $1 – P(E \cup S) = 1 – \{P(E) + P(S) – P(E \cap S)\} = 1 – \{0.46 + 0.37 – 0.15\} = 0.32$

Exercise

At a fuel pump, 40% of the customers use Regular fuel, 35% use Plus fuel and 25% use Supreme fuel. Of those customers who use Regular, only 30% fill their tanks full. Of those who use Plus, only 60% fill their tanks full whereas who use Supreme, only 50% fill their tanks full.

1) What is the probability that the next customer will request Plus fuel and will fill the tank full?

Answer: 0.21

2) What is the probability that the next customer will fill the tank full?

Answer: 0.455

3) If the next customer fills the tank full, what is the probability that he would request:

   a) Regular Fuel?

   Answer: 0.264

   a) Plus Fuel?

   Answer: 0.462

   a) Supreme Fuel?

   Answer: 0.274

Bayes' Theorem

Thomas Bayes: English Statistician 1702-1761

- From the general law of addition, we know that if X and Y are two sets:

  $(X \cup Y) = X + Y - (X \cap Y)$

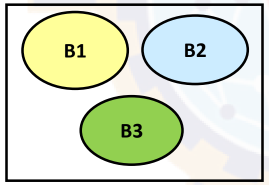
- If X and Y are mutually exclusive, then:

  $(X \cup Y) = X + Y$, because $X \cap Y = \{\phi\}$

- Let us say, an automobile part is manufactured by three suppliers.
- Let B1, B2 and B3 are the events that a randomly selected part came from these suppliers respectively.
- Let A is the event that the selected part is acceptable (not defective):

  $= A \cap (B1 \cup B2 \cup B3)$

  $= (A \cap B1) \cup (A \cap B2) \cup (A \cap B3)$



- Let P(A) is the probability that the part came from one of these suppliers and it is acceptable (mutually exclusive: U = +), then:

  $P(A) = P(A \cap B1) + P(A \cap B2) + P(A \cap B3)$

- From the Joint Probability (Multiplication Law), we know that:

  $P(X \cap Y) = P(X) \,.\, P(Y|X) = P(Y) \,.\, P(X|Y)$----------------------------*(i)*

- So: $P(A) = P(B1).P(A|B1) + P(B2).P(A|B2) + P(B3).P(A|B3)$
- Or in general, $P(A) = \sum_{i=1}^{3} P(Bi).P(A\,|Bi) \;------- -(ii)$

- From (i), we can also find the probability that the part is from B3 given that it is acceptable (A):

$$P(B3|A) = \frac{P(B3).P(A|B3)}{P(A)} \;---- -(iii)$$

- Replacing (ii) in (iii)

$$P(B3|A) = \frac{P(B3).P(A|B3)}{\sum_{i=1}^{3} P(Bi).P(A\,|Bi)}$$

- In general if there are n mutually exclusive events out of which rth must occur given A has occurred:

$$\boxed{P(Br|A) = \frac{P(Br).P(A|Br)}{\sum_{i=1}^{n} P(Bi).P(A\,|Bi)}}$$

- This is called the Bayes' Theorem Formula that helps to find the **revised** or **posterior probability** of an event (Br) out of n mutually exclusive events in the effect of an event A.

Example

Remember: $P(X \cap Y) = P(X).P(Y|X) = P(Y).P(X|Y)$

A leading printer manufacturing company outsourced two companies for its ink cartridges : **Altana (B1)** and **Jacobs (B2)**. Altana produces 65% and Jacobs produces 35% of the cartridges. 8% percent of the cartridges produced by Altana are defective and 12% of the Jacobs cartridges are defective. A customer purchases a new cartridge. The cartridge is **defective (D)**. Now what is the probability that Altana produced it?

Given that:

P(B1) = 0.65

P(B2) = 0.35

P(D | B1) = 0.08

P(D | B2) = 0.12

We need to find the value of **P(B1|D)**

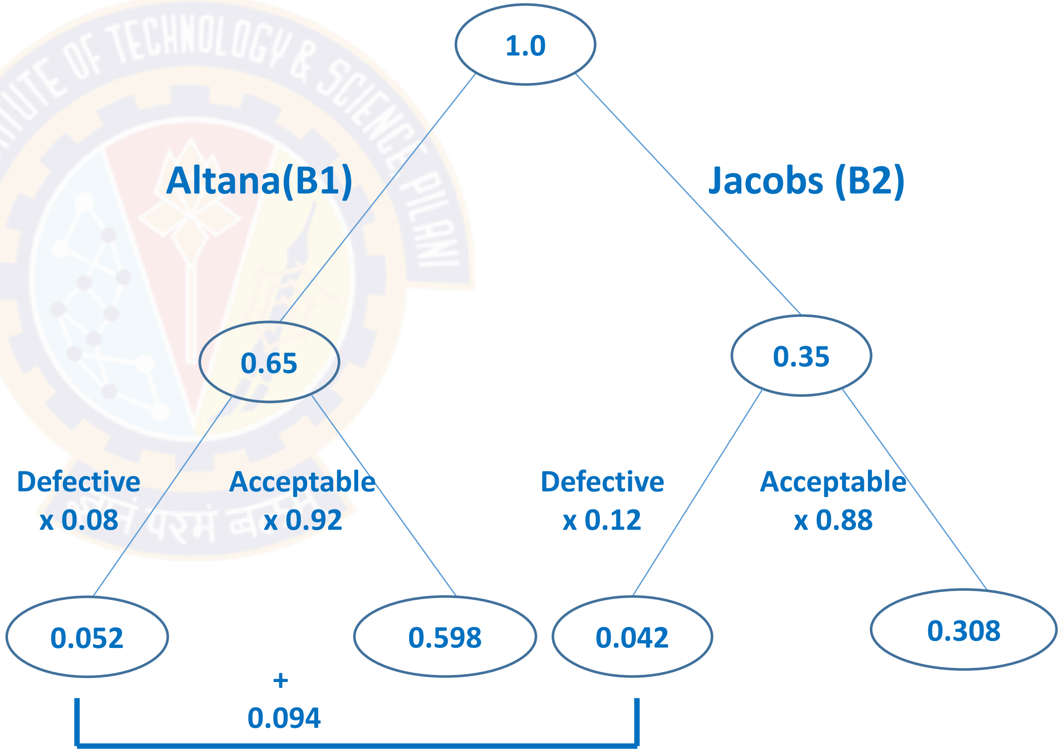
We know that, **P(D). P(B1|D) = P(B1). P(D|B1)**

From the Bayes' Theorem:

$$P(B1|D) = \frac{P(B1).P(D|B1)}{P(B1).P(D|B1) + P(B2).P(D|B2)}$$

$$= \frac{(0.65).(0.08)}{(0.65).(0.08) + (0.35).(0.12)}$$

$$= \mathbf{0.5531}$$



P(B1|D) = 0.052/0.094 = 0.5531   P(B2|D) = 0.042/0.094 = 0.4469

Tree Diagram

The probability that the defective cartridge came from Altana reduced (65% to 55.31%) and Jacobs increased (35% to 44.69%), because a higher % of defective cartridges came from Jacobs.

Observation

Exercise

1. Prepare a Tree Diagram for the Regular/Plus/Premium Fuel question and the solve using the diagram.
2. A LED lamp produced by Mega Electronics was found to be defective. There are three factories (A, B, C) where such lamps are manufactured. A Quality Manager (QM) is responsible for investigating the source of the found defects. QM wants to know if a randomly selected lamp is defective (D), what is the probability that the lamp was manufactured in factory C based on the data available in the given table.

Factory	% of total production	Probability of defective lamps
A	0.35 = P(A)	0.015 = P(D|A)
B	0.35 = P(B)	0.010 = P(D|B)
C	0.30 = P(C)	0.020 = P(D|C)

Answer:

$$P(C|D) = \frac{P(C).P(D|C)}{P(A).P(D|A) + P(B).P(D|B) + P(C).P(D|C)}$$

We need to find P(C|D):

$$= \frac{0.30x0.020}{0.35x0.015 + 0.35x0.010 + 0.30x0.020}$$

= 0.41

Bayes' Theorem for Classification

Naïve Bayes Classifier!

Age Group (A)	Income (I)	Student ? (S)	Credit Rating (R)	Buys Computer (C)?
Young	High	No	Fair	No
Young	High	No	Excellent	No
Middle	High	No	Fair	Yes
Senior	Medium	No	Fair	Yes
Senior	Low	Yes	Fair	Yes
Senior	Low	Yes	Excellent	No
Middle	Low	Yes	Excellent	Yes
Young	Medium	No	Fair	No
Young	Low	Yes	Fair	Yes
Senior	Medium	Yes	Fair	Yes
Young	Medium	Yes	Excellent	Yes
Middle	Medium	No	Excellent	Yes
Middle	High	Yes	Fair	Yes
Senior	Medium	No	Excellent	No

- Classification Training data is given in the table with the class of interest as Buys Computer **(C)**? It is decided on several attributes like Age Group (A), Income (I), Student (S), and Credit Ratings (R).
- The assumption is that attributes has probabilistic relationship with the class but the attributes are independent among themselves.
- We need to predict the class of a test record **X (A=Young, I=Medium, S=Yes, R=Fair).**
- *How the formulation will look like from the Bayes' Theorem perspective?*
- We need to calculate P (C | X) twice. Once for the value of C as Yes and then No. The bigger of these two will decide in which class X belongs to.

From the Bayes' Theorem:

$$P(C = Yes \mid X) = \frac{P(C = Yes).P(X|C = Yes)}{P\ (X)}$$

- **For the Yes Class:**

  P(C = Yes) = 9/14 = 0.64
  P (Age = Young| C = Yes) = 2/9
  P (Income = Medium| C = Yes) = 4/9
  P(Student = Yes | C = Yes) = 6/9
  P (Credit Rating = Fair | C= Yes) = 6/9

Similarly, for the No Class:

  P(C = No) = 5/14 = 0.36
  P (Age = Young| C = No) = 3/5
  P (Income = Medium| C = No) = 2/5
  P(Student = Yes | C = No) = 1/5
  P (Credit Rating = Fair | C= No) = 2/5

- So, P (X | C=Yes) = (2/9)x(4/9)x(6/9)x(6/9) = 0.044
- So, **P (C = Yes | X) = (0.64 x 0.044) / P(X) = 0.028/P(X)**

So, P (X | C=No) = (3/5)x(2/5)x(1/5)x(2/5) =0.019

So, **P (C = No| X) = (0.36 x 0.019) / P(X) = 0.007/P(X)**

Because:
P (C = Yes | X) > P (C = No | X)
So:
X is predicted to buy the Computer.

Why P(X) is not required?

Thank You