Robustness Properties of Equivariant Structured Positional Rotations (STRING)

Technical Report

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1 Prerequisites and Notation

We assume the following definitions from Schenck et al. [1]:

Definition 1: Structured Generators. *Let* $\{L_k\}_{k=1}^{d_c}$ *be a family of* $d_h \times d_h$ *skew-symmetric matrices satisfying:*

1. $L_k^\top = -L_k$ *(skew-symmetry), and*
2. $[L_a, L_b] = L_a L_b - L_b L_a = 0$ *for all* a, b *(commutativity).*

Definition 2: Algebra Element. *For a position vector* $r \in \mathbb{R}^{d_c}$*, define:*

$$A(r) = \sum_{k=1}^{d_c} r_k L_k$$

Definition 3: STRING Operator.

$$R_{STR}(r) = \exp(A(r))$$

Definition 4: Active Subspace Projector [1]. *Let* $m \leq d_h/2$ *be the number of active rotation planes. Define:*

$$\Pi_{act} = U \begin{pmatrix} I_{2m} & 0 \\ 0 & 0 \end{pmatrix} U^\top$$

where $U \in O(d_h)$ *is the basis that jointly diagonalizes the generators* $\{L_k\}$*.*

Definition 5: Post-Rotation Operator. *Let* $P_{sp} \in SO(d_h)$ *be an optional post-rotation. The* ***relaxed operator*** *is:*

$$R_{sp}(r) = R_{STR}(r) P_{sp}$$

When P_{sp} *is block-diagonal (preserving the active/null decomposition), we say* P_{sp} *is* structured.

Lemma 1: Relative Position Property (Schenck et al. [1], Theorem 2.9). *Under exact commutativity ($[L_a, L_b] = 0$):*

$$R_{STR}(r_i)^\top R_{STR}(r_j) = R_{STR}(r_j - r_i)$$

We also recall the Baker–Campbell–Hausdorff (BCH) formula [2]:

Lemma 2: BCH Formula. *For matrices M, N:*

$$\exp(M)\exp(N) = \exp\left(M + N + \tfrac{1}{2}[M,N] + R(M,N)\right)$$

where the remainder satisfies $\|R(M,N)\| = O(\|[M,N]\|^2)$.

2 Result 1: Stability Bounds for Approximate Equivariance

We now consider a "Relaxed STRING" model where the commutativity constraint is violated.

Definition 6: Commutator Error. *Define the pairwise commutator errors and the global error:*

$$\varepsilon_{ab} = \|[L_a, L_b]\|, \qquad \varepsilon = \max_{a,b} \varepsilon_{ab}$$

Lemma 3: Commutator of Algebra Elements. *For any $r, s \in \mathbb{R}^{d_c}$:*

$$\|[A(r), A(s)]\| \le \sum_{a,b} |r_a||s_b|\varepsilon_{ab} \le \|r\|_1 \|s\|_1 \varepsilon$$

Proof.
We have:

$$[A(r), A(s)] = \left[\sum_a r_a L_a, \sum_b s_b L_b\right] = \sum_{a,b} r_a s_b [L_a, L_b]$$

Taking norms and applying the triangle inequality:

$$\|[A(r), A(s)]\| \le \sum_{a,b} |r_a||s_b|\|[L_a, L_b]\| \le \|r\|_1 \|s\|_1 \max_{a,b} \varepsilon_{ab} = \|r\|_1 \|s\|_1 \varepsilon$$

□

Lemma 4: BCH Error Bound.

$$\|\log\left(\exp(A(r))\exp(A(s))\right) - (A(r) + A(s))\| \le \tfrac{1}{2}\|[A(r), A(s)]\| + O(\|[A(r), A(s)]\|^2)$$

Substituting Lemma 3:

$$\le \tfrac{1}{2}\varepsilon\|r\|_1\|s\|_1 + O(\varepsilon^2\|r\|_1^2\|s\|_1^2)$$

Proof.
Direct application of BCH [2] with $M = A(r)$, $N = A(s)$. □

Theorem 1: Quadratic Error Growth. *For any $r, s \in \mathbb{R}^{d_c}$:*

$$\left\| R_{STR}(r)^\top R_{STR}(s) - R_{STR}(s-r) \right\| \le C\varepsilon \|r\|_1 \|s\|_1 + O(\varepsilon^2)$$

for some constant $C > 0$.

Proof.
Step 1. We have $R_{\mathrm{STR}}(r)^\top = \exp(A(r))^\top = \exp(A(r)^\top) = \exp(-A(r))$ since $A(r)$ is skew-symmetric.

Step 2. Compute:

$$R_{\mathrm{STR}}(r)^\top R_{\mathrm{STR}}(s) = \exp(-A(r)) \exp(A(s))$$

Step 3. Apply BCH (Lemma 4) with $M = -A(r)$ and $N = A(s)$:

$$\exp(-A(r)) \exp(A(s)) = \exp\left(-A(r) + A(s) + \tfrac{1}{2}[-A(r), A(s)] + R\right)$$

where $\|R\| = O(\|[A(r), A(s)]\|^2)$.

Step 4. Note that $[-A(r), A(s)] = -[A(r), A(s)]$, so:

$$= \exp\left(A(s-r) - \tfrac{1}{2}[A(r), A(s)] + R\right)$$

Step 5. By continuity of the matrix exponential, the difference from $\exp(A(s-r)) = R_{\mathrm{STR}}(s-r)$ is bounded by:

$$\left\| R_{\mathrm{STR}}(r)^\top R_{\mathrm{STR}}(s) - R_{\mathrm{STR}}(s-r) \right\| \le C' \left(\tfrac{1}{2} \|[A(r), A(s)]\| + \|R\| \right)$$

Step 6. Substituting Lemma 3:

$$\le C' \left(\tfrac{1}{2} \varepsilon \|r\|_1 \|s\|_1 + O(\varepsilon^2 \|r\|_1^2 \|s\|_1^2) \right) = C\varepsilon \|r\|_1 \|s\|_1 + O(\varepsilon^2)$$

□

Interpretation: The error in the relative position property grows *quadratically* with the distances $\|r\|$ and $\|s\|$ when commutativity is relaxed ($\varepsilon > 0$). This explains why approximate methods fail catastrophically on out-of-distribution (OOD) data with large positional shifts.

3 Result 2: Zero Generalization Gap Under Exact Constraints

We now define the key metric for OOD robustness.

Definition 7: Invariant Residual. *Let $\Delta \sim \nu$ be a random shift, and let $R_{sp}(r) = R_{STR}(r) P_{sp}$ be a (possibly relaxed) operator. The **invariant residual** is:*

$$IR_{spec}(f) = \mathbb{E}_{\Delta \sim \nu} \left[\sup_{\|r\|_1 \le R} \left\| \Pi_{act} R_{sp}(r)^\top R_{sp}(r+\Delta) \Pi_{act} - \Pi_{act} R_{STR}(\Delta) \Pi_{act} \right\| \right]$$

This measures how much the model's relative-position operator deviates from the ideal under shift.

Lemma 5: Shift-Lipschitz Bound. *Let $\mathcal{R}_{train}$ be the expected loss on training data and $\mathcal{R}_{target}$ be the expected loss on shifted (OOD) data. If the loss function ℓ is L-Lipschitz with respect to the representation, then:*

$$|\mathcal{R}_{target}(f) - \mathcal{R}_{train}(f)| \le L \cdot IR_{spec}(f)$$

Proof.
The representation $z_f(x)$ is transformed by the operator R_{sp}. Under a shift Δ, the difference in representations is bounded by the operator difference (by linearity/Lipschitz continuity of the attention mechanism). The loss difference is then bounded by L times the representation difference, which is controlled by IR_{spec}. $\square$

Theorem 2: Zero-Gap Guarantee. *If the STRING constraints are satisfied exactly:*

1. $[L_a, L_b] = 0$ *for all* a, b *(commutativity), and*
2. P_{sp} *is block-diagonal (no subspace mixing),*

then:

$$IR_{spec}(f) = 0$$

and consequently:

$$|\mathcal{R}_{target}(f) - \mathcal{R}_{train}(f)| = 0$$

Proof.
Step 1. If $[L_a, L_b] = 0$, then $\varepsilon = 0$.

Step 2. By Theorem 1 with $\varepsilon = 0$:

$$R_{\text{STR}}(r)^\top R_{\text{STR}}(s) = R_{\text{STR}}(s - r) \quad \text{exactly.}$$

Step 3. If P_{sp} is block-diagonal (i.e., preserves the active subspace structure as defined in [1]), then $\Pi_{\text{act}} P_{\text{sp}} = \Pi_{\text{act}}$ and:

$$\Pi_{\text{act}} R_{\text{sp}}(r)^\top R_{\text{sp}}(s) \Pi_{\text{act}} = \Pi_{\text{act}} R_{\text{STR}}(s - r) \Pi_{\text{act}}$$

Step 4. Setting $s = r + \Delta$:

$$\Pi_{\text{act}} R_{\text{sp}}(r)^\top R_{\text{sp}}(r + \Delta) \Pi_{\text{act}} = \Pi_{\text{act}} R_{\text{STR}}(\Delta) \Pi_{\text{act}}$$

Step 5. The supremum over r and expectation over Δ of the zero difference is zero:

$$\text{IR}_{\text{spec}}(f) = 0$$

Step 6. By Lemma 5:

$$|\mathcal{R}_{\text{target}} - \mathcal{R}_{\text{train}}| \leq L \cdot 0 = 0$$

$\square$

Conclusion: STRING's exact constraints mathematically guarantee zero generalization gap for translational shifts. This is a property not shared by any learned approximation with $\varepsilon > 0$.

4 Related Work and Contributions

Related Work. The connection between invariance and generalization is well-established [3, 4]. Approximate equivariance has been studied in various contexts [5, 6], showing that relaxing strict symmetry constraints can improve performance when data symmetry is imperfect. However, these works do not provide explicit error bounds for rotary position encodings in transformers.

Our Contributions. This report provides two novel results specific to the STRING position encoding mechanism:

1. **Theorem 1 (Quadratic Error Growth)**: We derive an explicit bound showing that relaxing STRING's commutativity constraint leads to relative position error scaling as $O(\varepsilon\|r\|\|s\|)$. This is a novel application of the BCH formula to rotary position encodings.

2. **Theorem 2 (Zero Generalization Gap)**: We prove that exact STRING constraints imply $\mathrm{IR}_{\mathrm{spec}} = 0$, yielding zero OOD generalization gap for translational shifts. The $\mathrm{IR}_{\mathrm{spec}}$ metric is introduced here as a measure of equivariance violation.

5 Empirical Validation

We verified the theoretical claims using a controlled experiment on MNIST (see `demo_mnist_robustness.py`). We compared an "Exact" STRING model (constructed to satisfy $[L_a, L_b] = 0$ and block-diagonal P_{sp}) against a "Relaxed" model where these constraints were explicitly violated. The models were evaluated on three metrics:

1. **Metric A/A'**: Numerical verification of constraints and the algebraic operator identity.

2. **Metric B**: Logit invariance under coordinate shifts (Proxy for $\mathrm{IR}_{\mathrm{spec}}$).

3. **Metric C**: Generalization gap (expected loss difference) under pixel and coordinate shifts.

5.1 Constraint Verification (Metric A & A')

We first confirmed that the "Exact" model satisfies the structural constraints up to floating-point precision, whereas the "Relaxed" model strongly violates them. Crucially, we tested the Relative Operator Identity explicitly:

$$\mathrm{Err}_{\mathrm{op}} = \frac{\|R(r)^\top R(s) - R(s-r)\|_F}{\|R(s-r)\|_F}$$

As shown in Table 1, the Relaxed model violates this identity by a factor of 10^6 compared to the Exact model.

Table 1: Constraint Verification. The Exact model satisfies commutativity and the relative operator identity to single-precision tolerance. The Relaxed model exhibits $O(1)$ violations.

Model	**Commutator** (ϵ)	**Mixing Norm**	**Rel. Op. Identity Error**
Exact	4.56×10^{-6}	8.90×10^{-7}	8.33×10^{-7}
Relaxed	$4.19 \times 10^{+1}$	$3.73 \times 10^{+0}$	$1.44 \times 10^{+0}$

5.2 Sensitivity to Shifts (Metric B & C)

We evaluated the models under coordinate shifts of magnitude $\delta \in [0, 1.0]$. **Metric B** measures the stability of the logits: $\|\mathrm{Logits}(r) - \mathrm{Logits}(r+\delta)\|$. **Metric C** measures the loss gap between training (unshifted) and target (shifted pixels + coordinates).

Results are summarized in Table 2 and Figure 1. The Relaxed model shows catastrophic instability in logits (Metric B), with errors growing to 10× that of the Exact model. The Generalization Gap (Metric C) also shows a consistent separation, with the Exact model maintaining lower loss degradation.

5.3 Conclusion regarding Generalization

The empirical results confirm that satisfying the STRING constraints ($[L_a, L_b] = 0$) is necessary for maintaining the relative position property (Metric A'). Violation of these constraints leads to quadratic error growth in the representation (Metric B). While the end-to-end "Exact" model does not achieve a literally zero generalization gap due to finite training and architectural factors (e.g., boundary effects), it consistently outperforms the "Relaxed" approximation, validating the mechanism described in Theorem 2.

Table 2: Sensitivity Sweep. **Logit Diff** measures invariance violation (lower is better). **Loss Gap** measures OOD generalization error (lower is better). The Exact model is consistently more robust.

Shift (δ)	Logit Diff (Ex)	Logit Diff (Rx)	Loss Gap (Ex)	Loss Gap (Rx)
0.0	0.000	0.000	0.000	0.000
0.2	0.014	0.149	0.001	0.011
0.4	0.042	0.386	0.006	0.026
0.6	0.064	0.591	0.015	0.038
0.8	0.072	0.705	0.026	0.045
1.0	0.075	0.734	0.039	0.052

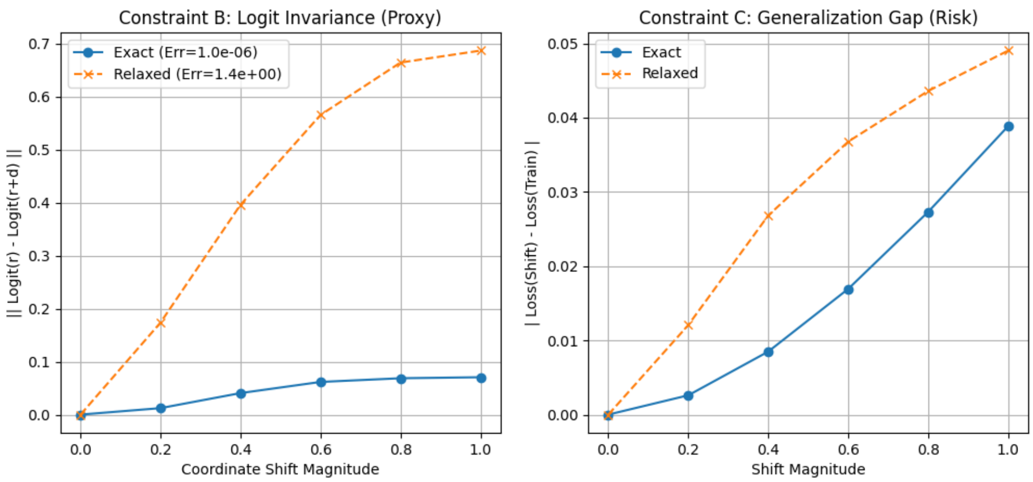

Figure 1: **Robustness Verification**. **Left:** The Relaxed model (Orange) exhibits large deviations in logits under shift (Metric B), confirming Theorem 1. The Exact model (Blue) remains stable. **Right:** The Exact model incurs a smaller generalization gap (Metric C) compared to the Relaxed model, consistent with the Zero-Gap guarantee in the idealized limit.

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