On a geometric computation of the fine structure constant?

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Résumé

La constante de la structure fine α (≈ 1/137) quantifie la force électromagnétique et l'interaction entre les particules élémentaire et le champ électromagnétique. En partant de la courbe de Viviani, je calcule une courbe géométrique qui couvre la sphère et dont la longueur est proche de 2π α^{-1}. Cette courbe 3D - une clélia – est une suite de rotations infinitésimales 2D dans deux plans orthogonaux, le plan équatorial et un plan méridien en rotation. J'applique ensuite une précession de façon à maximiser la symétrie locale entre les hémisphères nord et sud. Je choisis un angle de précession qui dépend linéairement de l'angle méridien et j'utilise la pente de la précession comme variable libre. Je trouve qu'une fonction coût de forme simple (Σ ($2\, z\, r_{cyl}\, \delta\theta_{cyl}$*), la somme de surfaces multipliées par une différence d'angles) a un minimum pour une certaine pente: ce minimum correspond à une longueur de la courbe égale à 137, 035 999 820 903 8 (x 2 π). Les 9 premiers chiffres sont identiques aux valeurs expérimentales de* α^{-1}.

Abstract

The electromagnetic fine structure constant α (≈ 1/137) quantifies the electromagnetic force and the interaction between particles and the electromagnetic field. Starting from Viviani's curve, I compute a geometrical curve which covers the sphere and whose length is close to 2π α^{-1}. This 3D curve – a clelia - is built as a sequence of 2D infinitesimal rotations in two orthogonal planes, an equatorial plane and a rotating meridian plane. I then apply a precession in order to improve local symmetry between the northern and southern hemispheres. I choose a precession angle which is linearly dependent on the meridian angle and I vary the slope of the precession as a free parameter. I find a simple cost function (Σ ($2\,z\,r_{cyl}\,\delta\theta_{cyl}$*), sum of surfaces multiplied by a difference of angles) which has a minimum for a given slope: this minimum occurs for a length of the curve equal to 137, 035 999 820 903 8 (x 2* π*). The first 9 digits are identical to the experimental values of* α^{-1}.

Introduction

The fine structure constant α has puzzled physicists for the last century: its inverse value is close to 137. Calling it a "magic number", Kragh (2003) explains the role of the fine structure constant in physics in a thorough historical review. In his model of the atom, Bohr (1913) found that v, the velocity of the electron around its circular orbit, is proportional to c, the velocity of light in the vacuum: $\alpha = v/c = e^2/\hbar c$, where e is the charge of the electron and ħ is Planck constant divided by 2π. α is a dimensionless quantity. Sommerfeld (1916) incorporated relativity in Bohr's model and explained precisely the splitting of the hydrogen spectrum. Sommerfeld coined the term "fine structure constant" for α. The recent value of α^{-1}, the inverse of the fine structure constant, is 137. 035 999 07 (NIST, 2010). Slightly different values are obtained depending on the technique of measurement. α^{-1} quantifies the strength of the electromagnetic interaction, therefore it is sometimes called the "electromagnetic fine structure constant"; in quantum electrodynamics, developments in powers of α express the interaction between charged elementary particles. In the Standard Model of particle physics, the coupling constant α is not constant but changes with the energy scale (see for example Fritzsch, 2009). α has fascinated famous physicists, like Eddington, Pauli, Feynman, … α^{-1} has also been a favorite subject of numerologists who found relationships between α and natural numbers, some of them simple. Astronomers have revived the interest in α by performing tests of its temporal stability (see Uzan, 2003; Fritzsch, 2009) and by modeling the implication of its variation in time on the history of the Universe. The value of α cannot be deduced from any physical theory. It is measured from spectroscopic experiments with a remarkable precision.

Viviani's curve and its generation

Viviani's curve is presented in textbooks or websites on algebraic curves (see Ferréol, 2005 or Weisstein, 2005). It is the intersection of a sphere of radius R with a cylinder of diameter R, positioned on one radius of the sphere. It was discovered in 1692 by Vincenzo Viviani who was the last assistant of Galileo. For R=1, its Cartesian coordinates are

$\{x=\sin t \cos t, y=\cos t \cos t, z=\sin t\}$.

On a sphere, Viviani's curve has spherical coordinates such that longitude φ equals latitude λ. From a practical point of view, Viviani's curve is generated by two orthogonal rotations: a horizontal vector rotates by an angle φ in the equatorial plane and then rotates by the same angle φ in the meridian plane – perpendicular to the equatorial plane. A unitary Viviani's curve projects on the horizontal plane as a circle of radius 1/2 offset with respect to the origin and on the rotating meridian plane as a circle of radius 1. The curvature in the equatorial plane is twice that in the meridian plane. In the usual Frénet-Serret frame, the curvature and torsion of a Viviani's curve are complex trigonometric functions of the arclength; on the contrary, the two curvatures are constant (circles) in the equatorial and meridian frames.

Viviani's curve is composed of segments traveling from pole to pole. We can differentiate between 4 types of segments: upwards, downwards, clockwise and anticlockwise. The sense of rotation can be reversed without changing the length of the curves. Both the upwards and downwards segments project on the horizontal plane as two superposed circles of radius ½. A Viviani's spherical helix can be generated by introducing a phase shift on the horizontal components when the ray propagates at the pole; Viviani's spherical helix projects on the horizontal plane as two circles of radius ½ symmetrical with respect to the center of the sphere. Therefore various spins and topologies, including Moebius type, can be generated.

The number 113 in relation to π

By analogy with the computation of seismological rays by shooting, let us move along a Viviani's curve by iteration. We start at a point (for instance {1,0,0}) and we move along the curve by iterating rotation, $\varphi=\varphi+\partial\varphi$. At each step, we compute the length along the curve. We stop propagating when the distance to the starting point is minimal, without limiting the number of turns. Surprisingly, we always find that the minimum distance is obtained for a propagation length equal to $137.4 \times 2\pi$. This length does not depend on $\partial\varphi$. Knowing that the arclength of a single Viviani's curve is an elliptic integral equal to 7.64 (Ferréol, 2005), we notice that $113 \times 7.64 = 137.4 \times 2\pi$. Why 113?

If we approximate π=3.141592654 by a fraction of integers, we find that the first good approximation is 22/7=3.14285. Then, the approximation 355/113=3.14159292 is much more precise. This was found around 500 AD by Zu Chongzhi, a Chinese mathematician. 355/113 has a residual of 0.0000002667 and remains the smallest residual up to the approximation 52163/16604 which gives a residual of 0.0000002662. The next best approximation is obtained for 104348/33215 with a residual of 0.000000000332.

More simply, the role of 113 in geometry can also be illustrated by moving along a circle. We move along a circle by iteration, $\varphi=\varphi+\partial\varphi$. When φ is close to a multiple of 2π, the trajectory returns close to the origin. How many turns are needed to return closest to the origin? The smallest distance to the starting point is always obtained after 113 turns, or a multiple of 113 turns (the exception is for $\partial\varphi=2\pi/n$). Number 113 plays a similar role for waves on a cyclic trajectory; a wave of any wavelength traveling along a circle will exhibit constructive interferences after 113 turns. 113 is an important number in problems in which a discrete quantity is used on a circle.

The clelia of argument (112,113) covering the sphere

If we wish to cover the entire sphere, we can rotate slowly the basic Viviani's curve around the vertical axis, as shown by Ferréol (2005). The rate of the slow rotation, $\partial\varepsilon$, determines the density of the net covering the sphere.

Step by step, we generate a Viviani's curve (point i for $\varphi=i\Delta\varphi$) and we rotate this point by $\partial\varepsilon=i\Delta\varphi/113$ around Oz: we obtain a clelia which covers the entire sphere. Cartesian coordinates of this clelia are $\{x=\sin(112\, t/ 113) \cos(t),\ y=\cos(112\, t/ 113) \cos(t),\ z=\sin(t)\}$.

Clelia were introduced by Grandi in 1728 (Ferréol, 2005). This previous clelia returns to its origin after 113 complete quasi Viviani's cycles. The total length of the clelia of argument (112,113) is 137.028 x 2π. This clelia is also the trajectory of the tip of a vector when latitude varies from 0 to 226 π and longitude varies from 0 to 224 π. The clelia is the result of a rotation around Oz and then around Ox', the rotated Ox axis. There is no rotation around Oy', the rotated Oy axis. This clelia projects on the horizontal plane as an epicycloid (fig. 1).

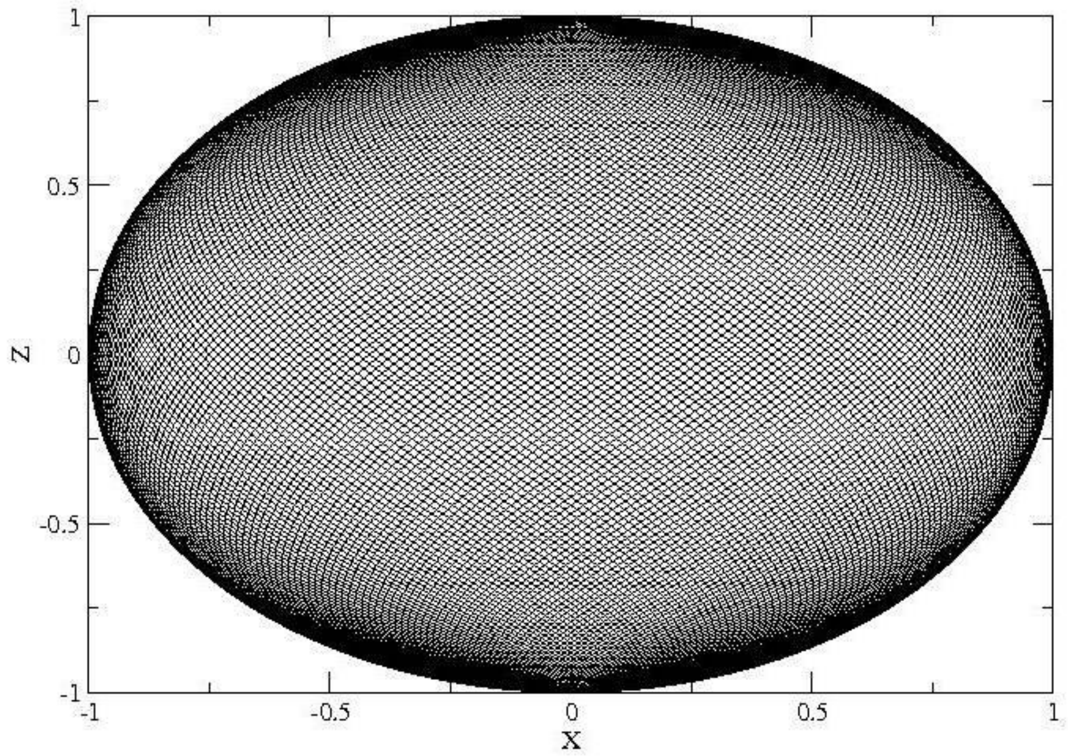

Fig. 1: Clelia of argument (112,113) covering the sphere. The sphere is transparent; we visualize both the upper hemisphere and lower hemisphere. The length of this curve is 137.028 x 2π.

Rectifying the clelia with a precession in order to improve local symmetry

For a Viviani's curve, the upward loop and downward loop are exactly symmetrical with respect to the horizontal plane. This is not the case for a clelia. This can be seen on an epicycloid, projection of a clelia on the horizontal plane; two consecutive cycles (the red and green loops on figure 2) are offset by a small rotation around Oz. Perfect local symmetry would imply the exact superposition of the two loops.

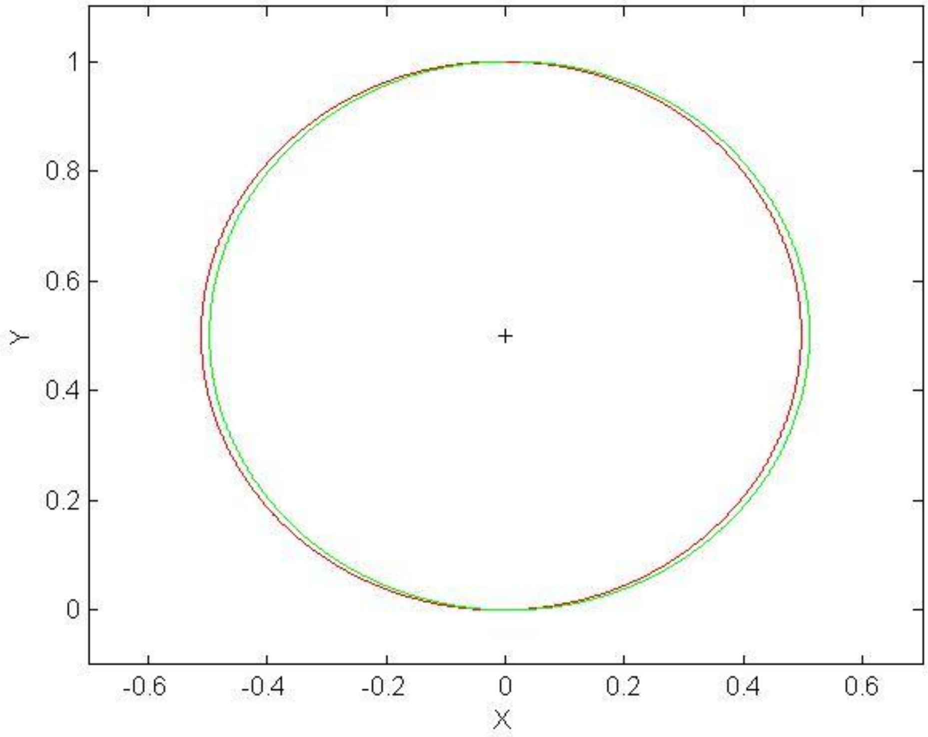

Fig. 2: Projection on the horizontal plane of one cycle of a (112,113) clelia. Purple is the upwards loop and green the downwards loop. Plus is the axis of the local cylinder. A precession or tilt of the vertical axis attempts to superpose parts of the green and purple curves.

In order to minimize local asymmetry, we add a precession of the vertical axis. We tilt Oz by rotating it around Oy' (the axis which is not used for generating the clelia). For the (112,113) clelia, the 'horizontal rotation' between two consecutive upwards loops is $\pi/113$. A rotation around Oz of $+\pi/226$ in the northern hemisphere and of $-\pi/226$ in the southern hemisphere would superpose the two loops, i-e re-establish local symmetry. Symmetry cannot be perfect because the curve has to cover the entire sphere. To obtain a clockwise then an anticlockwise rotation, how should we vary the precession angle – i-e the tilt of Oz? The precession angle should not be confused with the rotation around Oz; the precession of Oz is a rotation around Oy'. A precession angle of $+/-\pi/226 \tan(\pi/2-\varphi)$ seems appropriate, except for the infinities. The simplest dependency is that the precession angle be a linear function of latitude. A saw tooth shape is adequate; the tilt angle decreases from $a\pi/226$ for $\varphi=0$ to $-a\pi/226$ for $\varphi=\pi$ (northern hemisphere) and increases from $-a\pi/226$ for $\varphi=\pi$ to $a\pi/226$ for $\varphi=2\pi$ (southern hemisphere). The tilt of Oz should be zero at the poles, for $\varphi=\pi/2$ and $\varphi=3\pi/2$. a is the unknown constant. In order to find the best local symmetry, we need a cost function which depends on a. We made several attempts of cost functions considering different geometrical entities. We compare a point i with a point n/2 + i, where n is the total number of points of a quasi-Viviani's cycle. In a Viviani's curve, x(i) = x(n/2+i), y(i) = y(n/2+i) and z(n/2+i) = -z(i); each point in the northern hemisphere is symmetrical with a point in the southern hemisphere; the symmetry is with respect to the horizontal plane. In elementary geometry, we compute distances between points, angles between vectors, surfaces and volumes. We are dealing with the symmetry of quasi-cylinders offset from the origin; angles and vectors should originate at the center of each local cylinder, not at the center of the sphere. After various attempts, I retained a simple cost function cf related to the local cylinder: $cf = \Sigma\,(2\, z\, r_{cyl}\, \delta\theta_{cyl})$

- r_{cyl} is the average distance to the axis of the local cylinder.

  r_{cyl}(i)=sqrt(x(i)*x(i)+(y(i)-0.5)*(y(i)-0.5)) for the first quasi-cylinder and a similar expression for r(n/2+i). I started the curve with an offset in order to be symmetrical with respect to the y axis. Thus the first cylinder has coordinates (0, 0.5). The local axis of the next quasi-cylinder is rotated by $2\pi/113$ around Oz.

- z is the height.
- $\delta\theta_{cyl}$ is the angle between the horizontal vectors x(i), y(i) and x(n/2+i), y(n/2+i), as measured relative to the axis of the local quasi-cylinder.

  $\delta\theta_{cyl}$ (i)=*arcos*((x(i)*x(n/2+i)+(y(i)-0.5)*(y(n/2+i)-0.5))/(r_{cyl}(i)*r_{cyl}(n/2+i)) for the first quasi-cylinder. $\delta\theta_{cyl}$ is always 0 for a Viviani's curve, so its cost function is null.

  The length of the rectified clelia is computed for different values of a, the constant defining the tilt. For each value of a, we find a value of the cost function and a value of the total length of the curve.

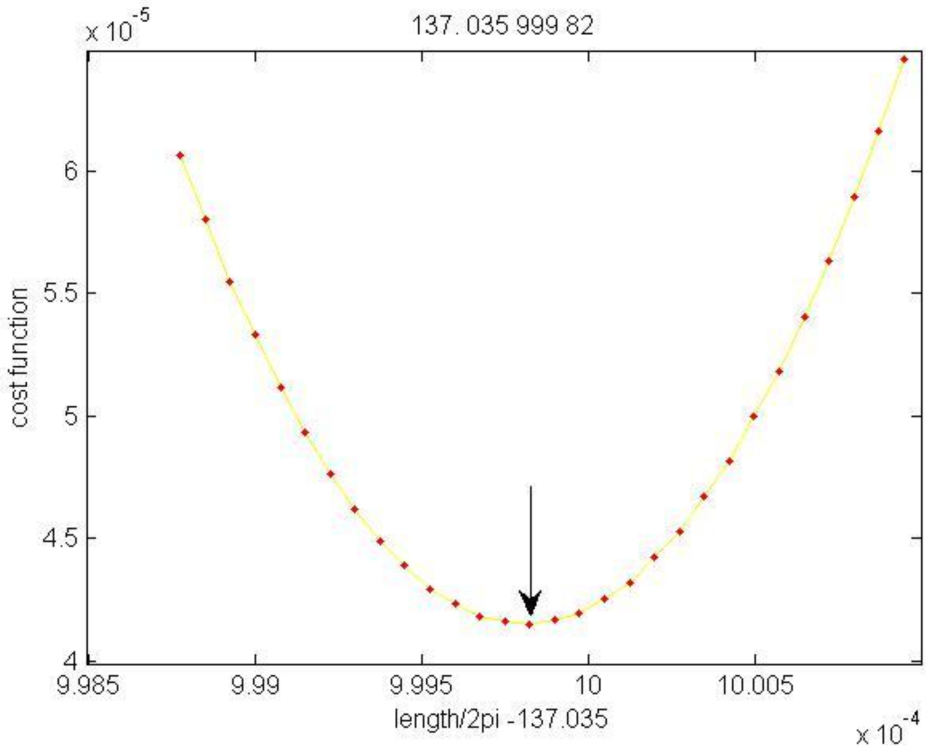

Fig. 3: Cost function measuring the local symmetry of the rectified clelia as a function of the length of the rectified clelia divided by 2π. The length which minimizes the cost function is marked by the arrow and is equal to 137.035 999 82. The cost function is plotted as y=cf-698.2373. This computation was performed with 500 000 points per lobe.

In Figure 3, we plot the cost function as a function of the length of the rectified clelia divided by 2π; this plot gives a clear minimum for a value between 137.035 999 7 and 137.035 999 9.

The computation depends on the number of points n defining the local quasi-Viviani's cycle; the tenth digit varies from 6 for 100 000 points, 8 for 500 000 to 8 for 1 000 000 points.

For 1 000 000 points, we find the minimum for 137. 035 999 820 903 8. A search for more digits is feasible but has limited interest at the present time. This value is to be compared to the experimental value of the fine structure constant which varies according to the technique of measurement. α^{-1} is listed as 137. 035 999 679(94) in the 2006 extended listing of fundamental constants of NIST. α^{-1} is given as 137. 035 999 074(44) by NIST in 2010. Gabrielse *et al*. (2006) and Kinoshita (2006) found a very precise value of 137. 035 999 710±0.000 000 096. This value was obtained after more than 800 terms of QED corrections. QED corrections are tedious and errors were found in the initial computations; so α^{-1} was later revised to 137.035 999 074, the value retained by NIST. In fact, this value of α^{-1} is dependent on the precision of QED; QED is very precise but to which limit? In a determination of α^{-1} independent of QED computations, using h/m_{Rb} measurements, Cadoret *et al* (2008) find α^{-1} = 137.035 999 45(62), with a larger error bar.

Conclusion

I computed the length of a spherical curve derived from Viviani's curve and I found that a clelia related to the approximation of π by 355/113 has a length close to α^{-1}. A linear precession of the vertical axis is added in order to improve local symmetry. I find a simple (and geometric) cost function which has a minimum for 137.035 999 820 903, extremely close to the experimental value of α^{-1}. My approach is an elementary computation and has not much to do with Physics per se. However, it is based on the first definition of the fine structure constant as v/c found in Bohr's model of the atom. Such computation would have been timely 50 or 100 years ago! Obviously the rectified clelia presented here cannot be the trajectory of an electron; it should be a curve related to some internal phenomenon within the electron. De Broglie postulated pilot waves and interferences but this was found of no practical use in Quantum Mechanics. The number 113 plays a key role in this computation; it is an indication of constructive interferences on cyclic trajectories. The fact that the cost function Σ $(2\ z\ r_{cyl}\ \delta\theta_{cyl})$ is a sum of surfaces multiplied by a difference of angles means that we are dealing with a volumetric phenomenon; this function has some resemblance with a flux (of a kind of 'virtual' rotation?). At the end of the 19th century, Lord Kelvin and J.J. Thomson modeled the atom with 'ether vortices'. A fast precession could correspond to the *zitterbewegung*, the trembling motion of the electron derived by Schrödinger from Dirac's equation.

A geometrical computation as presented here is not compatible with the present practice in Physics. Wilczek (1999), taking into consideration the Standard Model in physics, contests the primary role attributed to the fine structure constant by Pauli: 'The original idea of Pauli and others that calculating the fine structure constant was the next great item on the agenda of theoretical physics now seems misguided. We see this constant as just another running coupling, neither more nor less fundamental than many other parameters, and not likely to be the most accessible theoretically'. Pauli's intuition is recurrent among scientists and has a philosophical origin in the Platonic and Cartesian dreams of Physics as Geometry. Could Physics be derived from simple geometrical properties of Space? The present computation suggests that Space is subject to oriented rotations whose total sum nearly cancels. Local resonances of Space would be generated by suites of infinitesimal rotations closed on themselves; a resonance would be a 'particle of matter'. Does Space rotate on itself and how?

Acknowledgments

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References

Bohr, N., 1913. On the constitution of atoms and molecules, Part. 1, Philos. Mag., **26**, 1-24.

Cadoret, M., de Mirandes, E., Cladé, P., Guellati-Khélifa, S., Schwob, C., Nez, P., Julien, L. and F. Biraben, 2008. Combination of Bloch Oscillations with a Ramsey-Bordé interferometer New Determination of the Fine Structure Constant, Phys. Rev. Lett., 101, 230801.

Ferréol, R., 2005. Encyclopédies des formes mathématiques remarquables, http://www.mathcurve.com/

Fritzsch, H., 2009. The fundamental constants: a mystery in physics, World Scientific, 196 p.

Gabrielse, G., D. Hanneke,T. Kinoshita, M. Nio and B. Odom, 2006. New determination of the fine structure constant from the electron g value and QED, Phys. Rev. Lett., 97, 030802.

Kinoshita, T., 1996. The fine structure constant, Rep. Prog. Phys., **59**, 1459-1492.

Kragh, H., 2003. Magic number: a partial history of the fine-structure constant, Arch. Hist. Exact Sci., **57**, 395-431.

NIST, 2010. Fundamental physical constants, http://physics.nist.gov/

Sommerfeld, A., 1916. Zur Quantentheorie der Spektrallinien, Ann. der Physik, **356-17**, 1-94.

Uzan, J.-Ph., 2003. The fundamental constants and their variation: observational and theoretical status, Review of Modern Physics, **75**, 403-455.

Weisstein, E., 2005. Mathworld, http://mathworld.wolfram.com/

Wikipedia, The free Encyclopaedia, http://www.wikipedia.org/

Wilczek, F., 1999. Quantum Field Theory, Reviews of Modern Physics, **71**, 85-95.

Annex

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C    FORTRAN PROGRAM clelia-fsc.f – computation of alpha(-1)
c
      real*16 phi,dphi,pi,alp,dalp,x,y,z,x1,y1,z1,dd
      real*16 alp2,xr,xr1,xr2,xlon,s,ang,apr,bb,fc,cyy
      real*16 xx(1000001),yy(1000001),zz(1000001)
      open(10,form="formatted",file="vect.dat")
c
     pi=atan(1.000q+000)*4.000q+000
     n=500000
    n2=n/2
     write(6,*)n
     dphi=2.0q+000*pi/float(n)
     dalp=dphi/113.00q+000
     alp2=pi/226.0q+000
      apr=2.037670q+000
      do 15 kkk=1,40
       s=0.0q+000
       apr=apr+0.00001q+000
       do 11 k=0,n
        phi=float(k)*dphi
        alp=float(k)*dalp
        if(phi.lt.pi)bb=1.0q+000-2.0q+000*phi/pi
        if(phi.ge.pi)bb=-1.0q+000+2.0q+000*(phi-pi)/pi
        ang=apr*alp2*bb
        xlon=phi+2.0q+000*alp2-alp
        x=cos(xlon)*sin(ang)*sin(phi)-sin(xlon)*cos(phi)
        y=sin(xlon)*sin(ang)*sin(phi)+cos(xlon)*cos(phi)
        z=cos(ang)*sin(phi)
        if(k.eq.0)go to 12
         s=s+sqrt((x-x1)**2+(y-y1)**2+(z-z1)**2)
  12     x1=x
        y1=y
        z1=z
        xx(k)=x
        yy(k)=y
        zz(k)=z
  11  continue
      fc=0.0q+000
      do 14 k=0,n2
        cyy=0.5q+000
        xr1=sqrt(xx(k)**2+(yy(k)-cyy)**2)
        xr2=sqrt(xx(n2+k)**2+(yy(n2+k)-cyy)**2)
        xr=(xr1+xr2)/2.0q+000
        dd=(xx(k)*xx(n2+k)+(yy(k)-cyy)*(yy(n2+k)-cyy))/(xr1*xr2)
        fc=fc+acos(dd)*2.00q+000*zz(k)*xr
  14   continue
       write(10,'(4f21.10)')s*56.5q+000/pi,fc,apr
       write(6,'(4f21.12)')s*56.5q+000/pi,fc,apr
  15  continue
     close(10)
     stop
     end
```