

## Deutsch Algorithm: Full Derivation

We begin with two qubits initialized as

$$|\psi_0\rangle = |0\rangle |1\rangle.$$

### 1. Apply Hadamard Gates

Applying the Hadamard to each qubit gives

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

Thus

$$|\psi_1\rangle = (H|0\rangle) \otimes (H|1\rangle) = \frac{1}{2}(|0\rangle + |1\rangle)(|0\rangle - |1\rangle).$$

Expanding:

$$|\psi_1\rangle = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle).$$

### 2. Apply the Oracle $U_f$

The oracle acts as follows:

$$U_f|x, y\rangle = |x, y \oplus f(x)\rangle.$$

Because the second qubit is in the state  $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ , the oracle adds a phase  $(-1)^{f(x)}$  depending on  $x$ . Thus the state becomes

$$|\psi_2\rangle = \frac{1}{\sqrt{2}} \left( (-1)^{f(0)} |0\rangle + (-1)^{f(1)} |1\rangle \right) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

We only care about the *top* qubit for the final measurement, so define

$$|\psi_{\text{top}}\rangle = \frac{1}{\sqrt{2}} \left( (-1)^{f(0)} |0\rangle + (-1)^{f(1)} |1\rangle \right).$$

### 3. Apply the Final Hadamard

Applying  $H$  to the top qubit:

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

Thus

$$H|\psi_{\text{top}}\rangle = \frac{1}{\sqrt{2}} \left[ (-1)^{f(0)} H|0\rangle + (-1)^{f(1)} H|1\rangle \right].$$

Substituting the Hadamard results:

$$H|\psi_{\text{top}}\rangle = \frac{1}{2} \left[ (-1)^{f(0)}(|0\rangle + |1\rangle) + (-1)^{f(1)}(|0\rangle - |1\rangle) \right].$$

Group the  $|0\rangle$  and  $|1\rangle$  terms:

$$H|\psi_{\text{top}}\rangle = \frac{1}{2} \left[ \left( (-1)^{f(0)} + (-1)^{f(1)} \right) |0\rangle + \left( (-1)^{f(0)} - (-1)^{f(1)} \right) |1\rangle \right].$$

#### 4. Evaluate All 4 Possible Functions

There are four Boolean functions  $f : \{0, 1\} \rightarrow \{0, 1\}$ .

$$\begin{aligned} f(0) = 0, f(1) = 0 &\Rightarrow H |\psi_{\text{top}}\rangle = |0\rangle, \\ f(0) = 1, f(1) = 1 &\Rightarrow H |\psi_{\text{top}}\rangle = |0\rangle, \\ f(0) = 0, f(1) = 1 &\Rightarrow H |\psi_{\text{top}}\rangle = |1\rangle, \\ f(0) = 1, f(1) = 0 &\Rightarrow H |\psi_{\text{top}}\rangle = |1\rangle. \end{aligned}$$

#### 5. Final Measurement

$$\boxed{\begin{cases} |0\rangle, & \text{if } f \text{ is constant,} \\ |1\rangle, & \text{if } f \text{ is balanced.} \end{cases}}$$

Thus the Deutsch algorithm determines whether  $f$  is constant or balanced using a single oracle query.