Deutsch-Jozsa Algorithm: N-qubits

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Overview

We are implementing the initialization phase of the Deutsch-Jozsa algorithm. We will define N as the number of qubits in the *input register* (query register). The system also requires 1 *ancilla* qubit.

- **Total Qubits:** $N + 1$
- **General Case:** N input qubits.
- **Our Example:** $N = 2$ (Total 3 qubits: 2 input, 1 ancilla).
- **Comparison Case:** $N = 1$ (Total 2 qubits: 1 input, 1 ancilla - Original Deutsch Algo).

1 Step 1: The Initial State ($|\psi_0\rangle$)

We start with all input qubits in the state $|0\rangle$ and the ancilla qubit prepared in the state $|1\rangle$.

1.1 General Case (N inputs)

The state of the system is the tensor product of N zeros and one one.

$$|\psi_0\rangle = |0\rangle^{\otimes N} \otimes |1\rangle = |0\rangle_0 |0\rangle_1 \dots |0\rangle_{N-1} |1\rangle_{anc}$$

1.2 Specific Example ($N = 2$, Total 3)

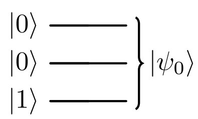
Here, we have 2 input qubits ($|x_0\rangle, |x_1\rangle$) and 1 ancilla.

$$|\psi_0\rangle = |0\rangle |0\rangle |1\rangle$$

1.3 Similarity with Basic Deutsch ($N = 1$, Total 2)

In the simplest case, we only have 1 input.

$$|\psi_0\rangle = |0\rangle |1\rangle$$



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2 Step 2: Creating Superposition ($|\psi_1\rangle$)

We apply a Hadamard gate (H) to every qubit in the system to create a uniform superposition.

2.1 General Case (N inputs)

We apply $H^{\otimes N}$ to the input register and H to the ancilla.

$$|\psi_1\rangle = (H^{\otimes N} \otimes H) |\psi_0\rangle$$

Expanding this:

$$|\psi_1\rangle = \sum_{x \in \{0,1\}^N} \frac{|x\rangle}{\sqrt{2^N}} \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

Note: The input register becomes an equal superposition of all possible 2^N bitstrings.

2.2 Specific Example ($N = 2$, Total 3)

We apply H to $|0\rangle |0\rangle |1\rangle$.

$$|\psi_1\rangle = (H |0\rangle) \otimes (H |0\rangle) \otimes (H |1\rangle)$$

$$|\psi_1\rangle = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

If we expand the input register part:

$$|\psi_1\rangle = \frac{1}{2} (|00\rangle + |01\rangle + |10\rangle + |11\rangle) \otimes |-\rangle$$

Notice how the input register is now a superposition of 00, 01, 10, and 11.

2.3 Similarity with Basic Deutsch ($N = 1$, Total 2)

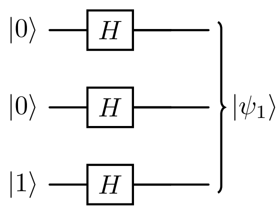
For the single bit case:

$$|\psi_1\rangle = (H |0\rangle) \otimes (H |1\rangle)$$

$$|\psi_1\rangle = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

Observation: The structure is identical. The ancilla always goes to the $|-\rangle$ state, and the input qubits always go to the $|+\rangle$ state.

Circuit Diagram for Initialization:



3 Step 3: The Oracle Query (U_f)

The core of the algorithm is the "Black Box" or Oracle (U_f). This unitary operator evaluates the function $f(x)$ without us knowing the internal mechanics.

The operation is defined as:

$$U_f |x\rangle |y\rangle = |x\rangle |y \oplus f(x)\rangle$$

where $\oplus$ denotes addition modulo 2 (XOR).

3.1 Phase Kickback Mechanism

Because our ancilla qubit is initialized to the state $|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$, the action of the Oracle creates a "Phase Kickback."

$$U_f |x\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) = |x\rangle \left(\frac{|0 \oplus f(x)\rangle - |1 \oplus f(x)\rangle}{\sqrt{2}} \right)$$

This simplifies to:

$$|x\rangle (-1)^{f(x)} |-\rangle$$

The function $f(x)$ is now encoded in the *phase* of the input qubits.

3.2 General Case (N inputs)

Applying U_f to the superposition $|\psi_1\rangle$:

$$|\psi_2\rangle = \sum_{x \in \{0,1\}^N} \frac{(-1)^{f(x)} |x\rangle}{\sqrt{2^N}} \otimes |-\rangle$$

The ancilla remains unchanged ($|-\rangle$), but the signs of the input states $|x\rangle$ are flipped if $f(x) = 1$.

3.3 Specific Example ($N = 2$, Total 3)

Our input register is in superposition of 4 states: $|00\rangle, |01\rangle, |10\rangle, |11\rangle$.

$$|\psi_2\rangle = \frac{1}{2} \left[(-1)^{f(00)} |00\rangle + (-1)^{f(01)} |01\rangle + (-1)^{f(10)} |10\rangle + (-1)^{f(11)} |11\rangle \right] \otimes |-\rangle$$

Depending on the function $f(x)$, some terms will have a positive sign (+) and others a negative sign (-).

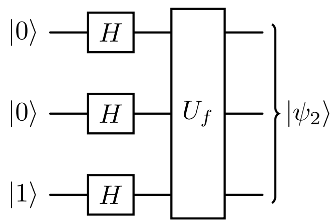
3.4 Similarity with Basic Deutsch ($N = 1$, Total 2)

For the 1-bit case, we only have two terms:

$$|\psi_2\rangle = \frac{1}{\sqrt{2}} \left[(-1)^{f(0)} |0\rangle + (-1)^{f(1)} |1\rangle \right] \otimes |-\rangle$$

Observation: In both cases, the oracle does not "collapse" the superposition; it merely marks the solutions with a negative phase.

Circuit Diagram with Oracle:



4 Step 4: Final Interference and Detailed Derivation

We apply a final layer of Hadamard gates ($H^{\otimes N}$) to the input register. This creates interference, canceling out wrong answers and amplifying the correct one.

4.1 The State Before Final Hadamard ($|\psi_{top}\rangle$)

Let's focus strictly on the input register (ignoring the ancilla for a moment). After the oracle, our state is a superposition of all 4 basis states, weighted by the function $f(x)$.

$$|\psi_{top}\rangle = \frac{1}{2} \left[(-1)^{f(00)} |00\rangle + (-1)^{f(01)} |01\rangle + (-1)^{f(10)} |10\rangle + (-1)^{f(11)} |11\rangle \right]$$

4.2 How Hadamard Affects Each Term

We now apply $H^{\otimes 2}$ to this state. The 2-qubit Hadamard transforms every single basis state into a superposition of *all* states. **Crucially, every basis state contributes to the $|00\rangle$ output with a positive sign.**

Let's look at the expansion for each basis state:

$$
\begin{aligned}
H^{\otimes 2}|00\rangle &= \frac{1}{2}(|\mathbf{00}\rangle + |01\rangle + |10\rangle + |11\rangle) \\
H^{\otimes 2}|01\rangle &= \frac{1}{2}(|\mathbf{00}\rangle - |01\rangle + |10\rangle - |11\rangle) \\
H^{\otimes 2}|10\rangle &= \frac{1}{2}(|\mathbf{00}\rangle + |01\rangle - |10\rangle - |11\rangle) \\
H^{\otimes 2}|11\rangle &= \frac{1}{2}(|\mathbf{00}\rangle - |01\rangle - |10\rangle + |11\rangle)
\end{aligned}
$$

Notice the bolded part: Every term starts with $+\frac{1}{2}|00\rangle$.

4.3 Calculating the Amplitude of $|00\rangle$

To find the probability of measuring $|00\rangle$, we sum up the contributions from all 4 terms.

We take the coefficient from our state $|\psi_{top}\rangle$ (which is $\frac{1}{2}(-1)^{f(x)}$) and multiply it by the contribution from the Hadamard expansion ($+\frac{1}{2}$).

$$
\text{Amplitude}_{00} = \underbrace{\frac{1}{2}}_{\text{From } |\psi_{top}\rangle} \times \left[\underbrace{\frac{1}{2}(-1)^{f(00)}}_{\text{from } |00\rangle} + \underbrace{\frac{1}{2}(-1)^{f(01)}}_{\text{from } |01\rangle} + \underbrace{\frac{1}{2}(-1)^{f(10)}}_{\text{from } |10\rangle} + \underbrace{\frac{1}{2}(-1)^{f(11)}}_{\text{from } |11\rangle} \right]
$$

Factoring out the $\frac{1}{2}$ terms:

$$
\text{Amplitude}_{00} = \frac{1}{4}\left[(-1)^{f(00)} + (-1)^{f(01)} + (-1)^{f(10)} + (-1)^{f(11)}\right]
$$

4.4 Testing the Cases

The probability of measuring $|00\rangle$ is $|\text{Amplitude}_{00}|^2$. Let's test this sum:

Case A: Constant Function (e.g., $f(x) = 0$ for all x)
All exponents are 0, so $(-1)^0 = 1$.

$$\text{Sum inside brackets} = (1 + 1 + 1 + 1) = 4$$

$$\text{Amplitude}_{00} = \frac{1}{4}(4) = 1$$

$$\textbf{Probability: } |1|^2 = 100\%$$

Case B: Balanced Function (Two outputs are 0, two are 1)
Two terms will be $(-1)^0 = +1$, and two terms will be $(-1)^1 = -1$.

$$\text{Sum inside brackets} = (1 + 1 - 1 - 1) = 0$$

$$\text{Amplitude}_{00} = \frac{1}{4}(0) = 0$$

$$\textbf{Probability: } |0|^2 = 0\%$$

5 Step 5: Conclusion

By calculating the sum of the signed terms, we proved that:

- If f is **Constant**, the terms add up constructively $\rightarrow$ You measure $|00\rangle$.
- If f is **Balanced**, the terms cancel out exactly $\rightarrow$ You measure **anything but** $|00\rangle$.

Complete Circuit Diagram: