Grover's Algorithm: 2-Qubit Derivation

Overview

We demonstrate Grover's search algorithm for a database of $N = 4$ items using 2 qubits. The goal is to find a specific target state $|w\rangle$ (the "winner") with a single query.

1 Step 1: Initialization (Uniform Superposition)

We begin with 2 qubits initialized to $|00\rangle$. We apply the Hadamard gate to both to create an equal superposition of all states.

$$|\psi_0\rangle = |0\rangle\,|0\rangle$$

$$|s\rangle = (H \otimes H)\,|00\rangle = \frac{1}{2}\big(\,|00\rangle + |01\rangle + |10\rangle + |11\rangle\,\big)$$

This state $|s\rangle$ acts as our uniform background. Note that the amplitude of every state is $\frac{1}{2}$.

$$|s\rangle = \sum_{x \in \{0,1\}^2} \frac{1}{2}\,|x\rangle$$

2 Step 2: The Oracle U_w

The Oracle marks the target state $|w\rangle$ by flipping its phase (sign). Mathematically, the operator is defined as:

$$U_w = I - 2\,|w\rangle\,\langle w|$$

Applying this to our superposition $|s\rangle$:

$$|\psi_{oracle}\rangle = U_w\,|s\rangle = |s\rangle - 2\,|w\rangle\,\langle w|s\rangle$$

Since $|s\rangle$ is a uniform superposition, the inner product (overlap) is $\langle w|s\rangle = \frac{1}{2}$.

$$|\psi_{oracle}\rangle = |s\rangle - 2\,|w\rangle\left(\frac{1}{2}\right) = |s\rangle - |w\rangle$$

Expanding this for a specific example target $|w\rangle = |11\rangle$:

$$|\psi_{oracle}\rangle = \frac{1}{2}\big(\,|00\rangle + |01\rangle + |10\rangle - |11\rangle\,\big)$$

The target now has a negative amplitude, while others remain positive.

3 Step 3: The Diffuser (Amplitude Amplification)

The diffusion operator D performs an "inversion about the mean". It is defined using the initial superposition state $|s\rangle$:

$$D = 2\,|s\rangle\,\langle s| - I$$

We apply D to the current state $|\psi_{oracle}\rangle$:

$$|\psi_{final}\rangle = D\,|\psi_{oracle}\rangle = (2\,|s\rangle\,\langle s| - I)(|s\rangle - |w\rangle)$$

Expanding the terms:

$$|\psi_{final}\rangle = 2\,|s\rangle\,\underbrace{\langle s|s\rangle}_{1} - 2\,|s\rangle\,\underbrace{\langle s|w\rangle}_{1/2} - \underbrace{|s\rangle + |w\rangle}_{-(|s\rangle - |w\rangle)}$$

Simplifying:

$$|\psi_{final}\rangle = 2\,|s\rangle\,(1) - 2\,|s\rangle\left(\frac{1}{2}\right) - |s\rangle + |w\rangle$$

$$|\psi_{final}\rangle = 2\,|s\rangle - |s\rangle - |s\rangle + |w\rangle$$

Canceling out the $|s\rangle$ terms:

$$\boxed{|\psi_{final}\rangle = |w\rangle}$$

4 Step 4: Measurement

The final state vector is exactly the target state $|w\rangle$.

$$P(w) = |\,\langle w|\psi_{final}\rangle\,|^2 = |1|^2 = 100\%$$

For $N = 4$, exactly one iteration of the Grover operator rotates the state directly to the solution.