

Quantum Algebra: A Visual Guide to Bra-Ket Notation

Study Sheet

Introduction

In quantum computing, notation can be tricky because symbols are often dropped for brevity. This guide defines the three major products using visual cues.

1 The Inner Product (“Bra-Ket”)

- **Notation:** $\langle a|b\rangle$
- **Visual Cue:** Pointy brackets facing **inward** like a diamond $\langle \dots \rangle$.
- **Meaning:** Calculates the **Overlap** or similarity between two states.
- **Result:** A **Scalar Number** (e.g., 0, 1, 0.5).

Example from Grover’s Algorithm: In the Grover example, you encountered $\langle w|s\rangle = \frac{1}{2}$.

This asks: “How much of the target $|w\rangle$ is hiding inside the superposition $|s\rangle$?” The answer is the number 0.5.

2 The Outer Product (“Ket-Bra”)

- **Notation:** $|a\rangle\langle b|$
- **Visual Cue:** Pointy brackets facing **outward** like two funnels back-to-back $|\dots\rangle\langle\dots|$.
- **Meaning:** Creates an **Operator** (a Matrix) that transforms vectors. Read it as “A machine that accepts state b and transforms it into state a ”.
- **Result:** A **Matrix**.

Example from Grover’s Algorithm: The operator $2|s\rangle\langle s|$ is an outer product.

- $\langle s|$ is the “sensor”: measures how much input is parallel to $|s\rangle$.
- $|s\rangle$ is the “builder”: reconstructs that amount in the direction of $|s\rangle$.
- Together, they act as a **Projector**.

3 The Tensor Product (“Ket-Ket”)

- **Notation:** $|a\rangle \otimes |b\rangle$ (often written as $|a\rangle |b\rangle$ or $|ab\rangle$).
- **Visual Cue:** Two vertical bars or Kets side-by-side $|\dots\rangle |\dots\rangle$.
- **Meaning:** Glues smaller systems together to make a larger system.
- **Result:** A Larger Vector.

Example from Grover’s Algorithm: The starting state $|0\rangle \otimes |0\rangle$ (or $|00\rangle$) describes “Qubit 1 is 0 AND Qubit 2 is 0”.

The Cheat Sheet: Look at the Brackets

Notation	Name	Visual Shape	Result	Meaning
$\langle A B\rangle$	Inner Product	$\langle \rangle$ (Closed)	Number	“How similar are A and B?”
$ A\rangle\langle B $	Outer Product	$\rangle \langle$ (Open)	Matrix	Transformation / Projection
$ A\rangle B\rangle$	Tensor Product	$ \rangle \rangle$ (Parallel)	Vector	Combined System

A Concrete Math Example

Let's define standard bit states as vectors:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

1. Inner Product (Scalar)

Calculates overlap. Since $|0\rangle$ and $|1\rangle$ are orthogonal:

$$\langle 0|1\rangle = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = (1)(0) + (0)(1) = \mathbf{0}$$

2. Outer Product (Matrix)

Transforms $|1\rangle$ into $|0\rangle$:

$$|0\rangle\langle 1| = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 0 & 1 \cdot 1 \\ 0 \cdot 0 & 0 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{1} \\ 0 & 0 \end{bmatrix}$$

3. Tensor Product (Larger Vector)

Expands the system state space:

$$|0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$