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Unit III

Unit IV

IMPORTANT FORMULAE

I. Trigonometric Formulae

1) $\sin^2 A + \cos^2 A = 1$

2) $\sec^2 A = 1 + \tan^2 A$

3) $\text{cosec}^2 A = 1 + \cot^2 A$

4) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

5) $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

6) $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \cdot \tan B}$

7) $\sin C + \sin D = 2 \sin \dfrac{C+D}{2} \cos \dfrac{C-D}{2}$

8) $\sin C - \sin D = 2 \cos \dfrac{C+D}{2} \sin \dfrac{C-D}{2}$

9) $\cos C + \cos D = 2 \cos \dfrac{C+D}{2} \cos \dfrac{C-D}{2}$

10) $\cos C - \cos D = -2 \sin \dfrac{C+D}{2} \sin \dfrac{C-D}{2}$

11) $2 \sin A \cos B = \sin(A+B) + \sin(A-B)$

12) $2 \sin A \sin B = \cos(A-B) - \cos(A+B)$

13) $2 \cos A \cos B = \cos(A-B) + \cos(A+B)$

14) $\sin 2A = 2 \sin A \cos A = \dfrac{2 \tan A}{1 + \tan^2 A}$

15) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

$$= \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

16) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

17) $\sin 3A = 3\sin A - 4\sin^3 A$

18) $\cos 3A = 4\cos^3 A - 3\cos A$

19) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

20) $\sin^2 x = \dfrac{1-\cos 2x}{2}$

21) $\cos^2 x = \dfrac{1+\cos 2x}{2}$

22) $\sin^{-1}x + \cos^{-1}x = \pi/2$

23) $\tan^{-1}x + \cot^{-1}x = \pi/2$

24) $\operatorname{cosec}^{-1}x + \sec^{-1}x = \pi/2$

Allied Angles Formulae

Angle	$-\theta$	$\pi/2-\theta$	$\pi/2+\theta$	$\pi-\theta$	$\pi+\theta$	$2\pi-\theta$	$2n\pi+\theta$
sin	$-\sin\theta$	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\sin\theta$	$\sin\theta$
cos	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$\cos\theta$	$\cos\theta$
tan	$-\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$	$-\tan\theta$	$\tan\theta$

II. Derivatives

1) $\dfrac{d}{dx}\sin x = \cos x$

2) $\dfrac{d}{dx}\cos x = -\sin x$

3) $\dfrac{d}{dx}\tan x = \sec^2 x$

4) $\dfrac{d}{dx}\cot x = -\operatorname{cosec}^2 x$

5) $\dfrac{d}{dx}\sec x = \sec x \tan x$

6) $\dfrac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cot x$

7) $\frac{d}{dx} \log x = \frac{1}{x}$

8) $\frac{d}{dx} x^n = n x^{n-1}$

9) $\frac{d}{dx} e^x = e^x$

10) $\frac{d}{dx} a^x = a^x \log a$

11) $\frac{d}{dx} [f(x)]^n = n [f(x)]^{n-1} . f'(x)$

12) $\frac{d}{dx} uv = u\frac{dv}{dx} + v\frac{du}{dx}$

13) $\frac{d}{dx} \frac{u}{v} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

14) $\frac{d}{dx}(uvw) = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$

15) $\frac{d}{dx} u = \frac{du}{ds} \cdot \frac{ds}{dx}$ (Chain rule)

III. Derivatives of Inverse Functions

1) $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

2) $\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$

3) $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$

4) $\frac{d}{dx} \cot^{-1} x = -\frac{1}{1+x^2}$

5) $\frac{d}{dx} \sec^{-1} x = \frac{1}{x\sqrt{x^2 -1}}$

6) $\frac{d}{dx} \operatorname{cosec}^{-1} x = -\frac{1}{x\sqrt{x^2 -1}}$

IV. Integration Formulae

1) $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c \; (n \neq -1)$

2) $\int \frac{1}{x} \, dx = \log x + c$

3) $\int e^x \, dx = e^x$

4) $\int a^x \, dx = \frac{a^x}{\log a}$

5) $\int \log x \, dx = x \log x - x$

6) $\int \sin x \, dx = -\cos x$

7) $\int \cos x \, dx = \sin x$

8) $\int \tan x \, dx = \log \sec x$

9) $\int \cot x \, dx = \log \sin x$

10) $\int \sec^2 x \, dx = \tan x$

11) $\int \operatorname{cosec}^2 x \, dx = -\cot x$

12) $\int \sec x \tan x \, dx = \sec x$

13) $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x$

14) $\int \sec x \, dx = \log (\sec x + \tan x)$

15) $\int \operatorname{cosec} x \, dx = \log (\operatorname{cosec} x - \cot x)$

16) $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right)$

17) $\int \frac{1}{\sqrt{x^2 - a^2}} dx = \log\left(x + \sqrt{x^2 - a^2}\right)$

18) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left(x + \sqrt{x^2 + a^2}\right)$

19) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right|$

20) $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right|$

21) $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$

22) $\int \sqrt{a^2 - x^2} \, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right)$

23) $\int \sqrt{x^2 - a^2} \, dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log\left[x + \sqrt{x^2 - a^2}\right]$

24) $\int \sqrt{x^2 + a^2} \, dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log\left[x + \sqrt{x^2 + a^2}\right]$

25) $\int e^x [f(x)+f'(x)]\,dx = e^x f(x)$

26) $\int \dfrac{f'(x)}{f(x)}\,dx = \log f(x)$

27) $\int \dfrac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)}$

28) $\int [f(x)]^n f'(x)\,dx = \dfrac{[f(x)]^{n+1}}{n+1}$

29) $\int e^{f(x)} f'(x)\,dx = e^{f(x)}$

30) $\int e^{ax} \cos bx \cdot dx = \dfrac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx)$

31) $\int e^{ax} \sin bx \cdot dx = \dfrac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$

32) $\int u \cdot v \cdot dx = u \int v\,dx - \int \left[\dfrac{du}{dx} \int v\,dx\right] dx$

33) $\int uv\,dx = u v_1 - u' v_2 + u'' v_3 \ldots$

V. Definite Integrals

1) $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$ 2) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

4) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,\ a < c < b$

5) $\int_0^{2a} f(x)\,dx = \int_0^a [f(x) + f(2a-x)]\,dx$

6) $\int_{-a}^a f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(x) \text{ is even.} \\ 0 & \text{if } f(x) \text{ is odd.} \end{cases}$

END...

Unit III

5 Statistics

5.1 : Useful Definitions

There are five methods of finding a measure of central tendency.

i) Arithmetic mean ii) Median iii) Mode

iv) Geometric mean v) Harmonic mean.

These are known as measures of central tendency.

A) Arithmetic Mean

1) For ungrouped data :

If $x_1, x_2, \cdots x_n$ are n observations

$$\text{Mean} = \bar{x} = \frac{\sum x_i}{n} = \frac{\text{Sum of the observations}}{\text{Total number of observations}}$$

2) For grouped data : If x_i are class marks and f_i are their respective frequencies for $1 \le i \le n$ then

$$\text{Mean} = \bar{x} = \frac{\sum f_i x_i}{\sum f_i}$$

Note : We can write $\sum f_i = N =$ Total frequency

3) Method of step deviation (to simplify the calculations) :

If $u_i = \dfrac{x_i - A}{h}$, where

A is the working mean or assumed mean of given data and h is the class width or gcd of all $x_i - A$

$$\bar{x} = A + h\bar{u}$$

$$= A + h\left(\frac{\sum f_i u_i}{\sum f_i}\right)$$

B) Median

1) For ungrouped data :

It divides total set of data into two equal points.

Median is value of **middle** most terms of series when arranged in **ascending** or **descending** order of magnitude.

$$\text{If n is odd then median} = \left(\frac{n+1}{2}\right)^{th} \text{ observation}$$

If n is even then there are two values in the middle so we take mean of these two values

$$\text{median} = \frac{\left(\frac{n}{2}\right)^{th} + \left(\frac{n}{2}+1\right)^{th}}{2} \text{ observation}$$

2) For group frequency distribution :

$$\boxed{\text{Median} = l + \frac{h\left(\frac{N}{2} - C\right)}{f}}$$

l = Lower limit of median class.

f = Frequency of median class.

h = Width of median class.

C = Coefficient of the class preceding the median class.

$N = \sum f_i$

Median class is the class where $\left(\frac{N}{2}\right)^{th}$ observation lies.

C) Mode

1) For ungrouped data : Mode is the most repeated observation in the given set of observation. So mode is not unique. If given data has only one mode, then it is known as unimodal otherwise multimodal.

Ex. 1) Mode of 1, 2, 3, 4, 2, 3, 2 is 2.

2) Mode of 1, 2, 3 is 1, 2, 3.

2) For grouped frequency distribution.

$$\text{Mode} = l + \frac{h(f_1 - f_0)}{2f_1 - f_0 - f_2}$$

l = Lower limit of modal class.

h = Width of the modal class.

f_1 = Frequency of the modal class.

f_0 = Frequency of the class preceding the modal class.

f_2 = Frequency of the class succeeding the modal class.

Modal class is the class with highest frequency.

D) Geometric Mean

i) For ungrouped data : Geometric mean or G.M. of n observations $x_1, x_2 \ldots x_n$ $(x_i \neq 0)$ is the n^{th} root of their product.

i.e. $\quad \text{G.M.} = (x_1 \cdot x_2 \ldots x_n)^{1/n}$

Take log on both sides

$$\log(\text{GM}) = \frac{1}{n}(\log x_1 + \log x_2 + \ldots + \log x_n)$$

$$\therefore \quad \text{GM} = \text{Antilog}\left[\frac{\sum \log x_i}{n}\right]$$

ii) For grouped data : For $x_1, x_2 \ldots x_n$ having corresponding frequencies $f_1, f_2 \ldots f_n$

$$\text{G.M.} = \left[(x_1)^{f_1} \cdot (x_2)^{f_2} \cdot (x_3)^{f_3} \ldots (x_n)^{f_n}\right]^{1/N}$$

Take log on both sides

$$\log \text{G.M.} = \frac{1}{N}[f_1 \log x_1 + f_2 \log x_2 \ldots + f_n \log x_n]$$

$$\text{G.M.} = \text{Antilog}\left[\frac{\sum f_i \log x_i}{N = \sum f_i}\right]$$

E) Harmonic Mean

Harmonic mean of a number of observations is the reciprocal of the arithmetic mean of the reciprocals of the given values.

i) For ungrouped data :

$$\text{H.M.} = \frac{n}{\left(\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}\right)}$$

i.e.

$$\text{H.M.} = \frac{1}{\left(\frac{\sum\left(\frac{1}{x_i}\right)}{n}\right)}$$

ii) For grouped data :

For $x_1, x_2 \ldots x_n$ having corresponding frequencies $f_1\ f_2 \ldots f_n$

$$\text{H.M.} = \frac{N}{\left(\frac{f_1}{x_1} + \frac{f_2}{x_2} + \ldots + \frac{f_n}{x_n}\right)} = \frac{N}{\sum_{i=1}^{n}\left(\frac{f_i}{x_i}\right)}$$

where $N = \sum f_i$

5.2 : Dispersion

[SPPU : May - 06, Dec. - 10]

Meaning of dispersion is scatteredness. To find whether measures of central tendencies are true representative of the data we calculate dispersion.

To measure the scatteredness of data from mean we use

a) Mean deviation

b) Standard deviation

a) Mean Deviation :

i) For ungrouped data :

For variates $x_1, x_2 \ldots x_n$

Deviation from average $A = d_i = x_i - A$

Deviation from mean $= d_i = x_i - \bar{x}$

$$\text{Mean deviation} = \frac{\sum |d_i|}{n}$$

ii) For grouped data :

For variate $x_1, x_2 \ldots x_n$ with corresponding frequencies $f_1, f_2 \ldots f_n$

$$\text{Mean deviation} = \frac{\sum f_i d_i}{\sum f_i} = \frac{\sum f_i |x_i - A|}{\sum f_i}$$

b) Standard Deviation (σ) :

i) For ungrouped data : $x_1, x_2, x_3 \ldots x_n$

$$\sigma = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n}}$$

Simplifying we get

$$\sigma = \sqrt{\frac{\sum (x_i)^2}{n} - \left(\frac{\sum x_i}{n}\right)^2}$$

$$\sigma = \sqrt{\frac{\sum (x_i)^2}{n} - (\bar{x})^2}$$

ii) For grouped data :

$$\sigma = \sqrt{\frac{\sum f_i (x_i - \bar{x})^2}{\sum f_i}}$$

Simplifying we get

$$\sigma = \sqrt{\frac{\sum f_i x_i^2}{N} - \left(\frac{\sum f_i x_i}{N}\right)^2} \quad \text{where } \sum f_i = N$$

i.e. $$\sigma = \sqrt{\frac{\sum f_i x_i^2}{\sum f_i} - (\bar{x})^2}$$

If we use method of step deviation for simplification of our calculations

i.e. if $u_i = \frac{x_i - A}{h}$ then

i) For ungrouped data

$$\sigma_u = \sqrt{\frac{\sum u_i^2}{n} - \left(\frac{\sum u_i}{n}\right)^2}, \quad \sigma_x = h\sigma_u$$

ii) For grouped data

$$\sigma_u = \sqrt{\frac{\sum f_i u_i^2}{\sum f_i} - \left(\frac{\sum f_i u_i}{\sum f_i}\right)^2}$$

$$\sigma_x = h\sigma_u$$

$$\bar{x} = A + h\bar{u}$$

Note : 1) The square of standard deviation is called variance given by σ^2.

2) The coefficient of variation is given by

$$\text{C.V.} = \frac{\sigma}{\text{A.M.}} \times 100$$

For comparing the variability of two series, we calculate the coefficient of variations for each series. The series having lesser C.V. is said to be more consistent.

Q.1 Goals scored by two teams A and B in a football season were as follows. Determine which team is more consistent.

[SPPU : May - 06, Dec. - 10]

Number of goals scored	Number of Matches	
	Team A	Team B
0	27	17
1	9	9
2	8	6
3	5	5
4	4	3

Ans. : Frequency distribution table for team A.

No. of goals (x_i)	Matches f_i	$d_i = x_i - 2$	$f_i d_i$	$f_i d_i^2$
0	27	− 2	− 54	108
1	9	− 1	− 9	9

2	8	0	0	0
3	5	1	5	5
4	4	2	6	12
	53		– 50	138

Thus for team A

$$\bar{x} = A + \left(\frac{\sum f_i d_i}{\sum f_i}\right)$$

$$= 2 + \frac{-50}{53} = 1.06$$

$$\sigma_A = \sqrt{\frac{\sum f_i d_i^2}{\sum f_i} - \left(\frac{\sum f_i d_i}{\sum f_i}\right)^2}$$

$$= \sqrt{\frac{138}{53} - \left(\frac{-50}{53}\right)^2}$$

$$= 1.31$$

$$\text{C.V.} = \frac{\sigma_A}{\bar{x}} \times 100$$

$$= \frac{1.31}{1.06} \times 100 = 123.6$$

Frequency distribution table for team B

No. of goals (x_i)	Matches f_i	$d_i = x_i - 2$	$f_i d_i$	$f_i d_i^2$
0	17	– 2	– 34	68
1	9	– 1	– 9	9
2	6	0	0	0
3	5	1	5	5
4	3	2	6	12
	40		– 32	94

Thus for team B

$$\bar{x} = A + \frac{\sum f_i d_i}{\sum f_i} = 2 + \left(\frac{-32}{40}\right) = 1.2$$

$$\sigma_B = \sqrt{\frac{\sum f_i d_i^2}{\sum f_i} - \left(\frac{\sum f_i d_i}{\sum f_i}\right)^2}$$

$$= \sqrt{\frac{94}{40} - \left(\frac{-32}{40}\right)^2} = 1.3$$

$$\text{C.V.} = \frac{\sigma_B}{\bar{x}} \times 100$$

$$= \frac{1.3}{1.2} \times 100 = 108.3$$

Since $(\text{C.V.})_B < (\text{C.V.})_A$

$\therefore$ Team B is more consistent.

5.3 : Moments

i) Central moments : Moments about the mean are known as central moments. The arithmetic mean of various powers of the deviation $(x_i - \bar{x})$ is called central moment of the distribution and is denoted by μ_i.

Thus

$$\mu_1 = 0$$

$$\mu_2 = \frac{\sum f_i (x_i - \bar{x})^2}{N}$$

$$\mu_3 = \frac{\sum f_i (x_i - \bar{x})^3}{N}$$

$$\mu_4 = \frac{\sum f_i (x_i - \bar{x})^4}{N}$$

are the first four moments of the distribution about mean.

ii) Raw moments : Moment about any point of observations different from mean is known as raw moments. The r^{th} moment about any number A is denoted by μ'_r and is given by

$$\mu'_r = \frac{\sum f_i (x_i - A)^r}{N}$$

Substituting r = 0, 1, 2, ... we get

$$\mu'_0 = 1$$

$$\mu'_1 = \frac{\sum f_i (x_i - A)}{N} = \frac{\sum f_i x_i}{N} - \left(\frac{\sum f_i}{N}\right) A$$

$$= \bar{x} - A$$

$$\mu'_2 = \frac{\sum f_i (x_i - A)^2}{N}$$

$$= s^2 = \text{Mean square deviation}$$

$$\mu'_3 = \frac{\sum f_i (x_i - A)^3}{N}$$

and $$\mu'_4 = \frac{\sum f_i (x_i - A)^4}{N}$$

Note : Proper choice of A can reduce that calculations of calculating μ'_r than that of μ_r.

III) Relations between μ'_r and μ_r :

Thus
$$\boxed{\begin{aligned} \mu_0 &= 1 \\ \mu_1 &= 0 \\ \mu_2 &= \mu'_2 - (\mu'_1)^2 \\ \mu_3 &= \mu'_3 - 3\mu'_2 \mu'_1 + 2(\mu'_1)^3 \\ \mu_4 &= \mu'_4 - 4\mu'_3 \mu'_1 + 6\mu'_2 (\mu'_1)^2 - 3(\mu'_1)^4 \end{aligned}}$$

5.4 : Skewness

To get the idea about the shape of the curve we study skewness. Skewness signifies departure from symmetry.

a) Positive skewness :

If the mean lies to the right of mode then the frequency curve stretches to the right then the distribution is right skewed or positively skewed.

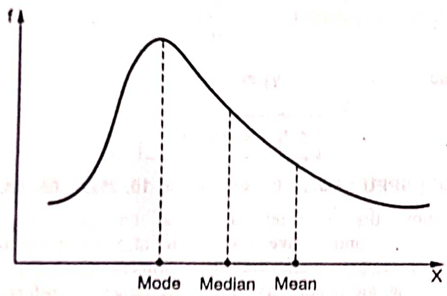


Fig. 5.1

b) Negative skewness :

If the mean lies to the left side of mode then the frequency curve stretches to the left then the distribution is left skewed or negatively skewed.

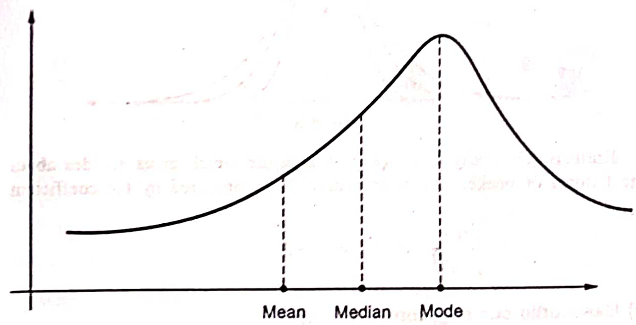


Fig. 5.2

The different measures of skewness are

i) Skewness $= \dfrac{3\,(\text{Mean} - \text{Median})}{\text{Standard deviation}}$

ii) Coefficient of skewness : $\beta_1 = \dfrac{\mu_3^2}{\mu_2^3}$

and $\gamma_1 = +\sqrt{\beta_1}$

5.5 : Kurtosis

[SPPU : Dec. - 05, 06, 07, 09, 10, May - 06, 08, 10, 13]

If we know the measures of central tendency, dispersion and skewness, we still cannot have a complete idea about the distribution. Observe the Fig. 5.3 there are three curves $C_1C_2C_3$ which are symmetrical about mean and have the same range. Therefore we should know about the flatness or peakedness of the curve.

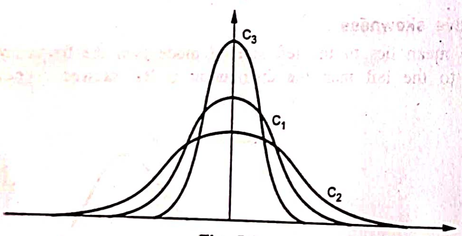


Fig. 5.3

Kurtosis (convexity of curve) is a measure which gives an idea about the flatness or peakedness of the curve. It is measured by the coefficient β_2.

$$\beta_2 = \frac{\mu_4}{(\mu_2)^2} \quad \text{or} \quad \gamma_2 = \beta_2 - 3$$

a) Mesokurtic curve : (Normal curve)

The curve C_1 which is neither flat nor peaked is called the normal curve or mesokurtic curve, for which $\beta_2 = 3$ or $\gamma_2 = 0$.

b) Platykurtic curve :

The curve (C_2) which is flatter than C_1 is platykurtic curve, for which $\beta_2 < 3$ or $\gamma_2 < 0$.

c) Leptokurtic curve :

The curve (C_3) which is more peaked than C_1 is Leptokurtic curve, for which $\beta_2 > 3$ or $\gamma_2 > 0$.

Q.1(a) If marks scored by five students in statistics test of 100 marks, are given in following table. [SPPU : June-22, Marks 5]

Student	1	2	3	4	5
Marks(/100)x	46	34	52	78	65

Find standard deviation and arithmetic mean $\bar{x}$.

Ans. : Given that marks of 5 students in statistics subject. We know that standard deviation is

$$\sigma = \left[\frac{\Sigma x_i^2}{n} - \bar{x}^2\right]^{\frac{1}{2}} ; n = 5$$

We have

x_i	x_i^2
46	2116
34	1156
52	2704
78	6084
65	4225
$\Sigma x_i = 275$	$\Sigma x_i^2 = 16285$

Now, $\bar{x} = \dfrac{\Sigma x_i}{n} = \dfrac{275}{5} = 55$

$$\sigma = \left[\frac{16285}{5} - (55)^2\right]^{\frac{1}{2}} = [3257 - 3025]^{\frac{1}{2}} = [232]^{\frac{1}{2}}$$

$\boxed{\sigma = 15.2315}$ and $\bar{x} = 55$

Q.1(b) Find coefficient of variability for following data.

☞ **[SPPU : Dec.-22, Marks 5]**

C.I.	0-10	10-20	20-30	30-40	40-50	50-60	60-70
Freq. (f)	4	7	8	12	25	18	10

Ans. :

C.I	x_i mid values	f_i	$u_i = \frac{x_i - 35}{10}$	C.F.	$f_i u_i$	$f_i u_i^2$
0-10	5	4	– 3	4	– 12	36
10-20	15	7	– 2	11	– 14	28
20-30	25	8	– 1	19	– 8	8
30-40	35	12	0	31	0	0
40-50	45	25	1	56	25	25
50-60	55	18	2	74	36	72
60-70	65	10	3	84	20	90
Total		$\sum f_i = 84$			$57 = \sum f_i u_i$	$\sum f_i u_i^2 = 259$

$$\text{mean } (\bar{x}) = A + h\left(\frac{\sum f_i u_i}{\sum f_i}\right) = 35 + 10\left(\frac{57}{84}\right) = 41.785$$

$$\sigma = \sqrt{\frac{\sum f_i u_i^2}{N} - \left(\frac{\sum f_i u_i}{N}\right)^2} \quad \text{where } \sum f_i = N$$

$$= \sqrt{\frac{259}{84} - \left(\frac{57}{84}\right)^2}$$

$$= \sqrt{3.083 - 0.4605}$$

$$= 1.619$$

Hence coefficient of variability (C.V.)

$$= \frac{\sigma}{\bar{x}} \times 100$$

$$= \frac{1.619}{41.785} \times 100 = 3.875$$

Q.2 The first four moments of a distribution about the value of 4 of the variable are – 1.5, 17, – 30 and 108. Find the moments about mean and β_1 and β_2. ☞ **[SPPU : Dec. - 06, 15, May - 11]**

Ans. : $A = 4, \mu_1' = -1.5, \mu_2' = 17,$

$\mu_3' = -30, \mu_4' = 108$

$$\therefore \mu_2 = \mu_2' - (\mu_1')^2 = 17 - (1.5)^2 = 14.75$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1)^3$$

$$= -30 - 3\,(17)\,(-1.5) + 2\,(-1.5)^3 = 39.75$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\,(\mu_1')^2 - 3(\mu_1')^4$$

$$= 108 - 4\,(-30)\,(-1.5) + 6\,(17)\,(-1.5)^2 - 3(-1.5)^4$$

$$= 108 - 180 + 229.5 - 15.1875 = 142.3125$$

$$\therefore \beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(39.75)^2}{(14.75)^3} = 0.4926$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{142.3125}{(14.75)^2} = 0.6543$$

Q.2(a) The first four moments of a distribution about 4 are – 1.4, 17, – 30 and 108. Obtain the first four central moments and coefficient of skewness and kurtosis. ☞ **[SPPU : Dec.-22, Marks 5]**

Ans. :

$A = 4, \mu_1' = -1.4, \mu_2' = 17, \mu_3' = -30, \mu_4' = 108$

$$\therefore \mu_2 = \mu_1' - (\mu_1')^2 = 17 - (1.5)^2 = 15.07$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$

$$= -30 - 3\,(17)\,(-1.4) + 2\,(-1.4)^3$$

$$= -30 + 71.4 - 5.488 = 35.912$$

$$\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2(\mu'_1)^2 - 3(\mu'_1)^4$$

$$= 108 - 4(30)(-1.4) + 6(17)(-1.4)^2 - 3(-1.4)^4$$

$$= 108 - 168 + 199.92 - 11.5248$$

$$= 128.3952$$

$$\therefore \quad \beta_1 = \text{Skewness} = \frac{\mu_3^2}{\mu_2^3} = \frac{(35.912)^2}{(15.07)^3} = 0.3768$$

$\because$ β_1 value is very small so it is negative skewness.

Now, $$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{128.3952}{(15.07)^2} = 0.5653 < 3$$

So curve is platykurtic.

Q.3 Calculate the first four moments of the following distribution about the mean and find skewness and kurtosis.

☞ **[SPPU : May - 06, 12, Dec. - 13]**

x	0	1	2	3	4	5	6	7	8
f	1	8	28	56	70	56	28	8	1

Ans. : We first calculate moments about x = 4 (Assumed mean)

x_i	f_i	$d_i = x_i - 4$	$f_i d_i$	$f_i d_i^2$	$f_i d_i^3$	$f_i d_i^4$
0	1	−4	−4	16	−64	256
1	8	−3	−24	72	−216	648
2	28	−2	−56	112	−224	448
3	56	−1	−56	56	−56	56
4	70	0	0	0	0	0
5	56	1	56	56	56	56
6	28	2	56	112	224	448
7	8	3	24	72	216	648
8	1	4	4	16	64	256
	$\sum f_i = 256$		$\sum f_i d_i = 0$	$\sum f_i d_i^2 = 512$	$\sum f_i d_i^3 = 0$	$\sum f_i d_i^4 = 2816$

We know that

$$\mu'_r = \frac{1}{N}\sum f_i (x_i - 4)^r = \frac{1}{N}\sum f_i d_i^r$$

$$\mu'_1 = \frac{1}{N}\sum f_i d_i = 0$$

$$\mu'_2 = \frac{1}{N}\sum f_i d_i^2 = \frac{512}{256} = 2$$

$$\mu'_3 = \frac{1}{N}\sum f_i d_i^3 = 0$$

$$\mu'_4 = \frac{1}{N}\sum f_i d_i^4 = \frac{2816}{256} = 11$$

using the relations between μ_r and μ'_r

$\therefore$ Moments about mean are

$$\mu_1 = 0 \text{ always}$$

$$\mu_2 = \mu'_2 - \mu_1'^2 = 2 - 0$$

$$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu_1'^3$$

$$= 0 - 3 \times 2 \times 0 + 2 \times 0 = 0$$

$$\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2\mu_1'^2 - 3\mu_1'^4$$

$$= 11 - 4(0)(0) + 6(2)(0) - 3 \times 0 = 11$$

$$\therefore \text{ Skewness } \beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{0}{2^3} = 0$$

$$\text{Kurtosis } \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{11}{4} = 2.75$$

Q.4 The first four moments of a distribution about the values 5 are 2, 20, 40 and 50. From the given information obtain the first four central moment, mean, standard deviation and coefficient of skewness and kurtosis.

☞ **[SPPU : May-08, 15, Dec.-07, June-22, Marks 5]**

Ans. : $A = 5$, $\mu_1' = 2$, $\mu_2' = 20$, $\mu_3' = 40$ and $\mu_4' = 50$.

We know that

$$\mu_1' = \bar{x} - A$$

$$\therefore \quad \bar{x} = A + \mu_1' = 5 + 2 = 7$$

To use the relations between μ_r and μ_r'.

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = 20 - (2)^2 = 16$$

$$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3 = 40 - 3(2)(20) + 2(2)^3$$

$$= 40 - 120 + 16$$

$$= -64$$

$$\mu_4 = \mu_4' - 4\mu_1'\mu_3' + 6(\mu_1')^2\mu_2' - 3(\mu_1')^4$$

$$= 50 - 4(2)(20) + 6(2)^2(2) - 3(2)^4$$

$$= 50 - 160 + 480 - 48$$

$$= 322$$

We know

$\therefore$ Variance $= \mu_2 = 16$

$\therefore$ Standard deviation $= \sqrt{\mu_2} = \sqrt{16} = 4$

Coefficient of skewness is given by,

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(-64)^2}{(16)^3} = 1$$

Since $\beta_1 = 1$, the distribution is positively skewed. Coefficient of kurtosis is given by,

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{322}{(16)^2} = 1.26$$

Since the value of β_2 is less than 3, hence the distribution is platykurtic.

Q.5 Calculate the first four moments about the mean of the following distribution. **[SPPU : May-10, Dec.-11]**

Marks	No. of students
0 - 10	06
10 - 20	26
20 - 30	47
30 - 40	15
40 - 50	06

Solution :

Class	Mid pt. x	Freq. f	$\mu = \frac{x-25}{10}$	fu	fu^2	fu^3	fu^4
0 - 10	5	6	− 2	− 12	24	− 48	96
10 - 20	15	26	− 1	− 26	26	− 26	26
20 - 30	25	47	0	0	0	0	0
30 - 40	35	15	1	15	15	15	15
40 - 50	45	6	2	12	24	48	96
Total	-	100	-	− 11	89	− 11	233

The moments about the arbitrary mean $A = 25$ are

$$\mu_r' = h^r \frac{\sum fu^r}{\sum f} \quad \text{for } r = 0, 1, 2, 3 \ldots$$

$$\mu_1' = -1.1, \; \mu_2' = 89, \; \mu_3' = -110, \; \mu_4' = 23300$$

$\therefore$ The central moments are

$$\mu_1 = 0, \qquad \mu_2 = 87.79$$

$$\mu_3 = 181.038, \qquad \mu_4 = 23457.7477$$

5.6 : Curve Fitting

Sr. No.	Name and Curve	Normal Equations
1.	The straight line $y = a + bx$	$\sum y = na + b\sum x$ $\sum xy = a\sum x + b\sum x^2$ where n = number or observed values a, b are constants
2.	$y = ab^x$ $\log y = \log a + x \log b$ $Y = A + Bx$ where $Y = \log y$, $A = \log a$, $B = \log b$	$\sum Y = nA + B\sum X$ $\sum XY = A\sum X + B\sum X^2$
3.	Second degree parabola $y = a + bx + cx^2$	$\sum y = na + b\sum x + c\sum x^2$ $\sum xy = a\sum x + b\sum x^2 + c\sum x^3$ $\sum x^2 y = a\sum x^2 + b\sum x^3 + c\sum x^4$

Examples :

Q.6 Fit a straight line to the following data :

x	0	1	2	3	4	5	6	7
y	–5	–3	–1	1	3	5	7	9

Ans. : Consider the following table for the fitting of straight line $y = a + bx$

x	y	xy	x^2
0	–5	0	00
1	–3	–3	01
2	–1	–2	04
3	1	3	09
4	3	12	16
5	5	25	25
6	7	42	36
7	9	63	49
$\sum x = 28$	$\sum x = 16$	$\sum xy = 140$	$\sum x^2 = 140$

Here n = 8 = Total number of observed points.

The normal equations are

$$\sum y = na + b\sum x$$
$$\sum xy = a\sum x + b\sum x^2$$

$$16 = 8a + 28b \quad \text{... (1)}$$
$$140 = 28a + 140b \quad \text{... (2)}$$

Dividing equation (1) by 4 equation (2) by 28, we get

$$4 = 2a + 7b \quad \text{... (3)}$$
$$5 = a + 5b \quad \text{... (4)}$$

Solving equations (3) and (4), we get

$$a = -5 \text{ and } b = 2$$

Hence the equation of straight line is

$$y = -5 + 2x$$

Q.6(a) Fit a parabola $y = ax^2 + bx + c$, by using least square method to the following data, **☞ [SPPU : June-22, Marks 5]**

x	0	1	2	3
y	2	2	4	8

Ans. : We have $y = ax^2 + bx + c$

Consider

	x	y	xy	x^2	x^2y	x^3	x^4
	0	2	0	0	0	0	0
	1	2	2	1	2	1	1
	2	4	8	4	64	8	16
	3	8	24	9	648	27	81
Sum	6	16	34	14	714	36	98

Normal equations are,

$$\Sigma y = nc + b\sum x + a\sum x^2$$

$$\Sigma xy = c\sum x + b\sum x^2 + a\sum x^3$$

$$\Sigma x^2 y = c\sum x^2 + b\sum x^3 + a\sum x^4$$

$$16 = 4c + 6b + 14a \Rightarrow 2c + 3b + 7a = 8 \qquad \text{...(Q.6(a).1)}$$

$$34 = 6c + 14b + 36a \Rightarrow 3c + 7b + 18a = 17 \qquad \text{...(Q.6(a).2)}$$

$$714 = 14c + 36b + 98a \Rightarrow 7c + 18b + 49a = 357 \qquad \text{...(Q.6(a).3)}$$

$3 \times$ Equation (Q.6(a).1) $- 2 \times$ Equation (Q.6(a).2) $\Rightarrow -5b - 15a = -10$

$$\Rightarrow \quad b + 3a = 2 \qquad \text{...(Q.6(a).4)}$$

$7 \times$ Equation (Q.6(a).1) $- 2 \times$ Equation (Q.6(a).3)

$$\Rightarrow -15b - 49a = -658$$

$$15b + 49a = 658$$

Now, ...(Q.6(a).5)

Equation (Q.6(a).5) $- 15 \times$ Equation (Q.6(a).4) $\Rightarrow 4a = 628 \Rightarrow a = 157$

$$\Rightarrow \quad b = 2 - 3a = 2 - 3(157) = -469$$

$$c = \frac{1}{2}[8 - 3b - 7a] = 158$$

$\Rightarrow a = 157,\ b = -469$ and $c = 158$

$$\therefore \quad y = 157x^2 - 469x + 158$$

Q.6(b) Fit a linear curve of the type $y = ax + b$, to following data,

[SPPU : Dec.-22, Marks 5]

x	10	15	20	25	30
y	0.75	0.935	1.1	1.2	1.3

Ans. : Consider the following table for the fitting of straight line $y = ax + b$

x	y	x^2	xy
10	0.75	100	7.5
15	0.935	225	14.025
20	1.1	400	22
25	1.2	625	30
30	1.3	900	39
$\sum x = 100$	$\sum y = 5.285$	$\sum x^2 = 2250$	$\sum xy = 112.525$

Here n = 5 The normal equation are

$$y = ax + b \qquad \text{... (Q.6(b).1)}$$

$$\sum y = a\sum x + nb \qquad \text{... (Q.6(b).2)}$$

$$\sum xy = a\sum x^2 + b\sum x$$

Computing values of $\sum x, \sum y, \sum x^2$ and $\sum xy$ in above equation we get

$$5.285 = a(100) + b(5) \qquad \text{... (Q.6(b).3)}$$

$$112.525 = a(2250) + b(100) \qquad \text{... (Q.6(b).4)}$$

Multiplying 20 in equation (Q.6(b).3) and then subtracting it from equation (Q.6(b).4) we get

$$2250a + \cancel{100b} = 112.525$$

$$2000a + \cancel{100b} = 105.7$$

$$- \quad - \quad -$$

$$250a = 6.825$$

$a = 0.0273$

So after computing a value in equation (3). We have $b = 0.511$

Hence equation of straight line is

$$\boxed{y = 0.0273\,x + 0.511}$$

Q.7 Find the least squares fit of the form $y = a + bx^2$ to the following data.

x	-1	0	1	2
y	2	5	3	0

Ans. : Given that $y = a + b x^2$, Put $x^2 = z$

$\therefore y = a + bz$

Consider the following table for the fitting of curve.

x	$z = x^2$	y	z	z^2
-1	1	2	2	1
0	0	5	0	0
1	1	3	3	1
2	4	0	0	16
0	$\sum z = 6$	$\sum y = 10$	$\sum yz = 5$	$\sum z^2 = 18$

Here n = 4,

Normal equations are $\sum y = na + b\sum z$

$$\sum yz = a\sum z + b\sum z^2$$

$\therefore$ We get $10 = 4a + 6b$

$$5 = 6a + 18\,b$$

Solving above equations, we get $a = \frac{25}{6}$, $b = -\frac{10}{9}$

Hence $\quad y = \frac{25}{6} - \frac{10}{9}x^2$

$\therefore \quad 18\,y = 75 - 20\,x^2$

Q.7(a) Fit a straight line of the form $y = ax + b$ to the following data by the least square method : **[SPPU : May-18, Marks 4]**

x	0	6	8	10	14	16	18	20
y	3	12	15	18	24	27	30	33

Ans. :

Let, $\quad y = ax + b = b + ax$

Normal equations are

$$\sum y = nb + a\sum x$$

and $\quad \sum xy = b\sum x + a\sum x^2$

Consider the following table

x	y	xy	x^2
0	3	0	0
6	12	72	36
8	15	120	64
10	18	180	100
14	24	336	196
16	27	432	256
18	30	540	324
20	33	660	400
$\sum x = 92$	$\sum y = 162$	$\sum xy = 2340$	$\sum x^2 = 1376$

$\therefore$ By normal equations, n = 8

$$162 = 8\,b + 92\,a \qquad \text{...(Q.7(a).1)}$$

$$2340 = 92\,b + 1376\,a \qquad \text{...(Q.7(a).2)}$$

Dividing equation (Q.7(a).1) by 2 and (Q.7(a).2) by 4, we get

$$\Rightarrow \quad 81 = 4b + 46a \qquad \text{...(Q.7(a).3)}$$

$$585 = 23b + 334a \qquad \text{...(Q.7(a).4)}$$

Multiplying equation (Q.7(a).3) by 23 and equation (Q.7(a).4) by 4, we get

$$1863 = 92b + 1058a \qquad \text{...(Q.7(a).5)}$$

$$\underline{{}_{-}2340 = {}_{-}92b \pm 1376a} \qquad \text{...(Q.7(a).6)}$$

$$477 = 318a$$

$$a = \frac{3}{2} \text{ and } b = \frac{1}{8}\left(162 - 92\left(\frac{3}{2}\right)\right) = 3$$

Thus $\quad y = ax + b = \frac{3}{2}x + 3$

Q.7(b) Fit a straight line of the form $Y = aX + b$ to the following data by the least square method : [SPPU : May-19, Marks 4]

X	1	3	4	5	6	8
Y	- 3	1	3	5	7	11

Ans. : Consider the following table

x	y	xy	x^2
1	-3	-3	1
3	1	3	9
4	3	12	16
5	5	25	25
6	7	42	36
8	11	88	64
$\sum x = 27$	$\sum y = 24$	$\sum xy = 167$	$\sum x^2 = 151$

Here, n = 6

The normal equations are

$$\sum y = na + b\sum x$$

$$\sum xy = a\sum x + b\sum x^2$$

$$24 = 6a + 27b \Rightarrow 8 = 2a + 9b \qquad \text{...(Q.7(b).1)}$$

$$167 = 27a + 151b \Rightarrow 167 = 27a + 151b \qquad \text{...(Q.7(b).2)}$$

Equation (Q.7(b).1) $\times 27$ – Equation (Q.7(b).2) $\times 2 \Rightarrow$

$$-118 = -59b \Rightarrow b = 2$$

$\therefore$ Equation (Q.7(b).1) $\Rightarrow 2a = 8 - 9b = -10$

$$a = -5$$

$$\therefore \quad y = ax + b \Rightarrow \boxed{y = -5x + 2}$$

Q.7(c) Fit a law of the form y = ap + b by least square method for the data, [SPPU : June-22, Marks 5]

p	100	120	140	160	180	200
y	0.9	1.1	1.2	1.4	1.6	1.7

Ans. : Normal equations are use p = x, m = b

	x	y	xy	x^2
	100	0.9	90	10000
	120	1.1	132	14400
	140	1.2	168	19600
	160	1.4	224	25600
	180	1.6	288	32400
	200	1.7	340	40000
Sum	900	7.9	1242	142000

$\Sigma y = mb + a\,\Sigma x$, $\Sigma yx = b\,\Sigma x + a\,\Sigma x^2$

Therefore, we get,

$$7.9 = 6b + 900a \qquad \text{...(Q.7(c).1)}$$

$$1242 = 900b + 142000a \qquad \text{...(Q.7(c).2)}$$

Equation (Q.7(c).1) × 150

$$1185 = 900b + 135000a \quad \text{...(Q.7(c).3)}$$

Equation (Q.7(c).2) - Equation (Q.7(c).3)

$\Rightarrow$ $57 = 7000\,a$

$\Rightarrow$ $a = 0.00814$

and $b = 0.0957$

$\therefore$ $y = ax + b = 0.00814\,x + 0.0957$

Q.7(d) Fit a linear curve y = ax + b, by least square method to the data, **☞ [SPPU : Dec.-22, Marks 5]**

x	100	120	140	160	180	200
y	0.9	1.1	1.2	1.4	1.6	1.7

Ans. : Given straight line curve

$$y = ax + b \quad \text{...(Q.7(d).1)}$$

Normal equations are

$$\sum y = a\sum x + nb \quad \text{...(Q.7(d).2)}$$

$$\sum xy = a\sum x^2 + b\sum x \quad \text{...(Q.7(d).3)}$$

Here n = 6

x	y	x^2	xy
100	0.9	10000	90
120	1.1	14400	132
140	1.2	19600	168
160	1.4	25600	224
180	1.6	32400	288
200	1.7	40000	340
$\sum x = 900$	$\sum y = 7.9$	$\sum x^2 = 142000$	$\sum xy = 1242$

Substituting values in equation (2) and (3) we have

$$7.9 = 900\,a + 6\,b$$

$$1242 = 142000\,a + 900\,b$$

Solving above equation a = 0.0081, b = 0.095.

Hence required straight line is

$$y = 0.0081\,x + 0.095$$

Q.8 Fit a curve $y = ax^b$ to the following data

x	2000	3000	4000	5000	6000
y	15	15.5	16	17	18

Ans. : We have $y = ax^b$

Taking log on both sides

$$\log y = \log a + b \log x$$

Put log y = Y, log x = X, log a = c

$$\therefore \quad Y = c + bX \quad \text{... (Q.8.1)}$$

$\therefore$ Normal equations are

$$\sum Y = nc + b\sum X \quad \text{... (Q.8.2)}$$

$$\sum XY = c\sum X + b\sum X^2 \quad \text{... (Q.8.3)}$$

Consider the following table for curve fitting :

x	y	X = log x	Y = log y	XY	X^2
2000	15	3.30103	1.17609	3.88498	10.8967
3000	15.5	3.47712	1.19033	4.13892	12.09037
4000	16	3.60206	1.20412	4.33731	12.97483
5000	17	3.69897	1.23044	4.55139	13.682379
6000	18	3.77815	1.2552	4.7426	14.27442
		$\sum X = 17.85$	$\sum Y = 6.056$	$\sum XY = 21.652$	$\sum X^2 = 63.9188$

Substituting these values in normal equations, we get

$$6.056 = 5C + 17.85\ b$$

$$21.652 = 17.85\ C + 63.9188\ b$$

Solving above equations, we get $c = 1.117,\ b = 0.026$

$\therefore \quad a = 13.09$

Hence $y = (13.09)\ (x)^{0.026}$

Q.9 Fit a curve of the form $y = ab^x$ to the following data

x	2	3	4	5	6
y	144	172.3	207.4	248.8	298.5

Ans. : We have $y = ab^x$

Taking log on both sides, we get

$$\log y = \log a + x \log b$$

$Y = A + Bx$, where $Y = \log y$, $A = \log a$, $B = \log b$

Normal equations are $\sum Y = An + B\sum X$ ...(Q.9 1)

$$\sum XY = A\sum X + B\sum X^2 \quad \text{... (Q.9 2)}$$

Consider the following table for curve fitting

x	y	Y = log y	xY	X^2
2	144	2.1584	4.3167	4
3	172.3	2.2363	6.7089	9
4	207.4	2.3168	9.2672	16
5	248.8	2.3959	11.9795	25
6	298.5	2.4749	14.8494	36
$\sum x = 20$		$\sum Y = 11.5823$	$\sum xY =$ 47.1217	$\sum X^2 = 90$

Here $n = 5$,

Normal equations become, $11.5823 = 5A + 20\ B$

$$47.1217 = 20\ A + 90\ B$$

Solving above equations, we get

$A = 2,\ B = 0.07925$

$A = \log a \Rightarrow 2 = \log b \Rightarrow \ a = \text{Antilog}\ (2) = 100$

$B = \log b \Rightarrow b = 1.2$

Hence $y = 100\ (1.2)^x$

Q.10 Fit a parabola to the following data

x	1	2	3	4	5
y	1090	1220	1390	1625	1915

Ans. : Let the equation of parabola be $y = a + bx + cx^2$

Consider the following table for the fitting of parabola.

x	y	xy	x^2	x^2y	x^3	x^4
1	1090	1090	1	1090	1	1
2	1220	2440	4	4880	8	16
3	1390	4170	9	12510	27	81
4	1625	6500	16	26000	64	256
5	1915	9575	25	47875	125	625
$\sum x = 15$	$\sum y =$ 7240	$\sum xy =$ 23775	$\sum x^2 =$ 55	$\sum x^2 y =$ 92355	$\sum x^3 =$ 225	$\sum x^4 =$ 979

Here $n = 5$

Normal equations are

$$\sum y = na + b\sum x + c\sum x^2$$

$$\sum xy = a\sum x + b\sum x^2 + c\sum x^3$$

$$\sum x^2 y = a\sum x^2 + b\sum x^3 + c\sum x^4$$

Substituting values, we get

$$7240 = 5a + 15b + 55c$$

$$23775 = 15a + 55b + 225c$$

$$92355 = 55a + 225b + 979c$$

Solving we get $a = 1024$, $b = \frac{81}{2}$, $c = \frac{55}{2}$

Hence

$$y = 1024 + \frac{81}{2}x + \frac{55}{2}x^2$$

$$2y = 2048 + 81x + 55x^2$$

Q.11 Fit the curve $y = ax + \frac{b}{x}$ to the following data

x	1	2	3	4	5	6	7	8
y	5.43	6.28	8.23	10.32	12.63	14.86	17.27	19.51

Ans. :

We have $y = ax + \frac{b}{x}$

By least square method, we have $s = \sum\left(y - ax - \frac{b}{x}\right)^2$

$$\frac{\partial s}{\partial a} = 2\sum\left(y - ax - \frac{b}{x}\right)(-x) \Rightarrow \sum(-yx + ax^2 + b) = 0$$

$$\sum xy = a\sum x^2 + nb \qquad \dots (Q.11.1)$$

$$\frac{\partial s}{\partial b} = 0 \Rightarrow 2\sum\left(y - ax - \frac{b}{x}\right)\left(-\frac{1}{x}\right) = 0$$

$$\Rightarrow \sum\left(\frac{y}{-x} + a + \frac{b}{x^2}\right) = 0$$

$$\Rightarrow \sum\frac{y}{x} = na + b\sum\frac{1}{x^2} \qquad \dots (Q.11.2)$$

Equations (Q.11.1) and (Q.11.2) are normal equations. Consider the following table for the fitting of the required curve.

x	y	x^2	xy	$\frac{1}{x}$	$\frac{1}{x^2}$	$\frac{y}{x}$
1	5.43	1	5.43	1	1	5.43
2	6.28	4	12.56	0.5	0.25	3.14
3	8.23	9	24.69	0.3333	0.1111	2.743
4	10.32	16	41.28	0.25	0.0625	2.58
5	12.63	25	63.15	0.2	0.04	2.526
6	14.86	36	89.16	0.1666	0.0278	2.476
7	17.27	49	120.89	0.1429	0.0204	2.467
8	19.51	64	156.08	0.125	0.0156	2.438
$\sum x = 36$		$\sum x^2 =$ 204	$\sum xy = 513.24$		$\sum\frac{1}{x^2} =$ 1.5274	$\sum\frac{y}{x} =$ 23.8

$h = 8$

$\therefore$ Normal equations become

$$513.24 = 204a + 8b$$

$$23.8 = 8a + 1.5274\,b$$

Solving above equations, we get

$$a = 2.3972,\ b = 3.02629$$

Hence $\qquad y = 2.3972\,x + \frac{3.02629}{x}$

5.7 : Correlation and Regression

[SPPU : Dec. - 06, 08, 09, 10, May - 10, 13]

I) Karl Pearson's Coefficient of Correlation

To measure the intensity or degree of linear relationship between two variables, Karl Pearson developed a formula called correlation coefficient.

a) Correlation coefficient between two variables x and y is denoted by $r(x, y)$ and is defined as

$$\boxed{r(x, y) = \frac{\text{cov}(x, y)}{\sigma_x \sigma_y}}$$

where $\text{cov}(x, y)$ = co-variance of (x, y)

$$= \frac{1}{n} \sum (x_i - \bar{x})(y_i - \bar{y})$$

where,

$$\bar{x} = \frac{\sum x_i}{n}$$

$$\bar{y} = \frac{\sum y_i}{n}$$

$$\text{cov}(x, y) = \frac{1}{n} \sum x_i y_i - (\bar{x})(\bar{y})$$

i.e.

$$\text{cov}(x, y) = \frac{1}{n} \sum x_i y_i - \left(\frac{\sum x_i}{n}\right)\left(\frac{\sum y_i}{n}\right)$$

$$\sigma_x^2 = \frac{1}{n} \sum x_i^2 - (\bar{x})^2$$

i.e.

$$\sigma_x^2 = \frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n}\right)^2$$

Similarly

$$\boxed{\sigma_y^2 = \frac{\sum y_i^2}{n} - \left(\frac{\sum y_i}{n}\right)^2}$$

b) Method of step deviation

If $u_i = \frac{x_i - A}{n}$ and $v_i = \frac{y_i - B}{k}$

$$\text{cov}(u, v) = \frac{\sum u_i v_i}{n} - \bar{u} \cdot \bar{v}$$

then

$$\sigma_u^2 = \frac{\sum u_i^2}{n} = \bar{u}, \quad \sigma_v^2 = \frac{\sum v_i^2}{n} - \bar{v}$$

$$\bar{u} = \frac{\sum u_i}{n}, \quad \bar{v} = \frac{\sum v_i}{n}$$

and

$$r(u, v) = \frac{\text{cov}(u, v)}{\sigma_u \sigma_v}$$

Note that $r(x, y) = r(u, v)$

> **Note**
> 1) If $r = 0$ then there is lack of relationship between x and y.
> 2) If $r = \pm 1$ then the relationship between x and y is very strong.

c) Correlation coefficient for bivariate frequency distribution

When a data is presented in bivariate frequency distribution then also

$$r(x, y) = \frac{\text{cov}(x, y)}{\sigma_x \sigma_y}$$

where

$$\boxed{\text{cov}(x, y) = \frac{\sum f_i x_i y_i}{\sum f_i} - \bar{x}\,\bar{y}}$$

$$= \frac{\sum f_i x_i y_i}{\sum f_i} - \left(\frac{\sum f_i x_i}{\sum f_i}\right)\left(\frac{\sum f_i y_i}{\sum f_i}\right)$$

i.e. $\text{cov}(x, y) = \dfrac{\sum f_i x_i y_i}{N} - \left(\dfrac{\sum f_i x_i}{N}\right)\left(\dfrac{\sum f_i y_i}{N}\right)$

Also $\boxed{\sigma_x^2 = \dfrac{\sum f_i x_i^2}{\sum f_i} - (\bar{x})^2}$

and $\boxed{\sigma_y^2 = \dfrac{\sum f_i y_i^2}{\sum f_i} - (\bar{y})^2}$

d) Method of step deviation

If $\quad u_i = \dfrac{x_i - A}{n}, \quad v_i = \dfrac{y_i - B}{k}$

$$\boxed{\begin{gathered} \text{cov}(u, v) = \frac{\sum f_i u_i v_i}{\sum f_i} - \bar{u} \cdot \bar{v} \\ \sigma_u^2 = \frac{\sum f_i u_i^2}{\sum f_i} - (\bar{u})^2 \\ \sigma_v^2 = \frac{\sum f_i v_i^2}{\sum f_i} - (\bar{v})^2 \end{gathered}}$$

where $\quad \bar{u} = \dfrac{\sum f_i u_i}{\sum f_i} \qquad \bar{v} = \dfrac{\sum f_i v_i}{\sum f_i}$

and $\quad r(x, y) = r(u, v) = \dfrac{\text{cov}(u, v)}{\sigma_u \cdot \sigma_v}$

Substituting all the above values we can write

$$r(x, y) = \frac{\dfrac{\sum f_i u_i v_i}{N} - \left(\dfrac{\sum f_i u_i}{N}\right)\left(\dfrac{\sum f_i v_i}{n}\right)}{\sqrt{\dfrac{\sum f_i u_i^2}{N} - \left(\dfrac{\sum f_i u_i}{N}\right)^2}\sqrt{\dfrac{\sum f_i v_i^2}{N} - \left(\dfrac{\sum f_i v_i}{N}\right)^2}}$$

i.e. $$\boxed{r(x, y) = \frac{N \cdot \sum f_i u_i v_i - (\sum f_i u_i)(\sum f_i v_i)}{\sqrt{N \sum f_i u_i^2 - (\sum f_i u_i)^2}\sqrt{N \sum f_i v_i^2 - (\sum f_i v_i)^2}}}$$

Q.12 From a group of 10 students marks obtained by each in papers of Mathematics and Applied Mechanics are given as.

x - marks in maths	23	28	42	17	26	35	29	37	16	46
y - marks in ap mech	25	22	38	21	27	39	24	32	18	44

Calculate Karl Pearson's coefficient of correlation.

[SPPU : May - 13, Dec. - 12, 13]

Ans. :

x_i	y_i	$u_i = x_i - 35$	$v_i = y_i - 39$	u_i^2	v_i^2	$u_i v_i$
16	18	− 19	− 21	361	441	399
17	21	− 18	− 18	324	324	324
23	25	− 12	− 14	144	196	168
26	27	− 9	− 12	81	144	108
28	22	− 7	− 17	49	289	119
29	24	− 6	− 15	36	225	90
35	39	0	0	0	0	0
37	32	2	− 7	4	49	− 14
42	38	7	− 1	49	1	− 7
46	44	11	5	121	25	55
		$\sum u = -51$	$\sum v = -100$	$\sum y^2 = 1169$	1694	1242

$$\bar{u} = \frac{\sum u_i}{n}$$

$$\therefore \quad \bar{u} = \frac{-51}{10} = -5.1 \qquad \bar{u}^2 = 26.01$$

$$\therefore \quad \bar{v} = -10 \qquad \bar{v}^2 = 100$$

Now $\text{cov}(u, v) = \frac{1}{n}\sum u_i v_i - \bar{u}\bar{v}$

$$= \frac{1}{10}(1242) - 51 = 73.2$$

$$\sigma_u^2 = \frac{1}{n}\sum u_i^2 - (\bar{u})^2$$

$$= \frac{1169}{10} - 26.01 = 90.89$$

$$\sigma_u = \sqrt{90.89} = 9.5336$$

$$\sigma_v^2 = \frac{1}{n}\sum v_i^2 - \bar{v}^2$$

$$= \frac{1694}{10} - 100 = 69.4$$

$$\sigma_v = \sqrt{69.4} = 8.33$$

$$r(x, y) = r(u, v) = \frac{\text{cov}(u, v)}{\sigma_u \cdot \sigma_v}$$

$$= \frac{73.2}{9.534 \times 8.33}$$

$$= 0.9217$$

Q.12(a) Calculate the coefficient of correlation from the following information.

$n = 10$, $\Sigma x = 40$, $\Sigma x^2 = 190$, $\Sigma y^2 = 200$, $\Sigma xy = 150$, $\Sigma y = 40$.

☞ [SPPU : June-22, Marks 5]

Ans. : From given data $\bar{x} = \frac{\Sigma x}{n} = \frac{40}{10} = 4$, $\bar{y} = \frac{40}{10} = 4$

$$\text{cov}(x, y) = \frac{\Sigma xy}{n} - \bar{x}\bar{y} = \frac{150}{10} - 4 \times 4 = 15 - 16 = -1$$

$$\sigma_x = \sqrt{\frac{\Sigma x^2}{n} - \bar{x}^2} = \sqrt{\frac{190}{10} - 16} = \sqrt{3}$$

$$\sigma_y = \sqrt{\frac{\Sigma y^2}{n} - \bar{y}^2} = \sqrt{\frac{200}{10} - 16} = \sqrt{4} = 2$$

$$r = \frac{\text{cov}(x, y)}{\sigma_x \sigma_y} = \frac{-1}{\sqrt{3}\sqrt{4}} = -\frac{1}{2\sqrt{3}} = -0.2886$$

Q.12(b) Find the correlation coefficient for the following data.

☞ [SPPU : Dec.-22, Marks 5]

Population density	200	500	400	700	800
Death rate	12	18	16	21	10

Ans. : Let x = Population density

y = Death rate

Consider $u = x - A$ and $v = y - B$

$= x - 500$ $\quad = y - 15$

x	y	u = x − 500	v	u^2	v^2
200	12	− 300	− 3	90000	9
500	18	0	3	0	9
400	16	− 100	1	10000	1
700	21	200	6	40000	36
800	10	300	− 5	90000	25
Total	-	$\sum u = 100$	$2 = \sum v$	$230000 = \sum u^2$	$80 = \sum v^2$

Here given, $n = 5$

$$\bar{u} = \frac{\sum u}{n} = \frac{100}{5} = 20$$

$$\bar{v} = \frac{\sum v}{n} = \frac{2}{5} = 0.4$$

$$r(u, v) = \frac{\sum uv - n\bar{u}\bar{v}}{\sqrt{\left(\sum u^2 - n\bar{u}^2\right)\left(\sum v^2 - n\bar{v}^2\right)}}$$

$$= \frac{500 - 5(20)(0.4)}{\sqrt{[(230000 - 5(20)^2][(80 - 5(0.4)^2]}}$$

$$= 0.1082$$

5.8 : Regression

☞ [SPPU : Dec. - 03, 05, 06, 07, 08, May - 95 , 06, 07, 09, 10]

If x and y are correlated. If the points in scatter diagram lies on some curve then that curve is called curve of regression.

1) The equation of line of regression of y on x is given by

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

where $\bar{x}, \bar{y}$ are means of distributions for x and y respectively.

2) The equation of line of regression of x on y is given by

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

where, $r\frac{\sigma_y}{\sigma_x}$ = regression coefficient of y on x

$= b_{yx}$

$r\frac{\sigma_x}{\sigma_y}$ = regression coefficient of x on y

$= b_{xy}$

Thus $b_{yx} \cdot b_{xy} = r^2$

Q.13 Obtain regression lines for the following data.

☞ [SPPU : May - 09, Dec. - 06, 07, 08,12]

x	6	2	10	4	8
y	9	11	5	8	7

Ans. : To find regression coefficient b_{xy} and b_{yx} prepare the following table.

x_i	y_i	x_i^2	y_i^2	$x_i y_i$
6	9	36	81	54
2	11	4	121	22
10	5	100	25	50
4	8	16	64	32
8	7	64	49	56
$\sum x_i = 30$	$\sum y_i = 40$	$\sum x_i^2 = 220$	$\sum y_i^2 = 340$	$\sum x_i y_i = 214$

Here $n = 5$

$$\bar{x} = \frac{\sum x_i}{n} = \frac{30}{5} = 6 \quad \text{and}$$

$$\bar{y} = \frac{\sum y_i}{n} = \frac{40}{5} = 8$$

$$\sigma_x^2 = \frac{\sum x_i^2}{n} - (\bar{x})^2 = \frac{220}{5} - (6)^2$$

$$= 44 - 36 = 8$$

$$\sigma_y^2 = \frac{\sum y_i^2}{n} - (\bar{y})^2 = \frac{340}{5} - (8)^2$$

$$= 68 - 64 = 4$$

$$\text{cov}(x, y) = \frac{\sum (x_i y_i)}{n} - \bar{x}\bar{y}$$

$$= \frac{214}{5} - 6 \times 8$$

$$\text{cov}(x, y) = 42.8 - 48 = -5.2$$

$$b_{yx} = \frac{\text{cov}(x, y)}{\sigma_x^2} = \frac{-5.2}{8} = -0.65$$

$$b_{xy} = \frac{\text{cov}(x, y)}{\sigma_y^2} = \frac{-5.2}{6} = -1.3$$

Regression line of Y on X is

$$y-\bar{y} = b_{yx}(x-\bar{x})$$

$$y - 8 = -0.65\ (x - 6)$$

$$y = -0.65\ x + 3.9 + 8$$

$$y = -0.65\ x + 11.9$$

Regression line of X on Y is

$$x-\bar{x} = b_{xy}(y-\bar{y})$$

$$x - 6 = -1.3\ (y - 8)$$

$$x - 6 = -1.3\ y + 10.4$$

$$x = -1.3\ y + 10.4 + 6$$

$$x = -1.3\ y + 16.4$$

Q.13(a) Obtain the line of regression of y on x for the following data. Also, estimate the value of y for x = 10.

☞ [SPPU : May-18, Marks 4]

x	2	4	5	6	8	11
y	18	12	10	8	7	5

Ans. : Consider the following table

$u = x - 6,\ v = y - 8$

x	y	u = x − 6	v = y − 8	u^2	v^2	uv
2	18	−4	10	16	100	−40
4	12	−2	4	4	16	−8
5	10	−1	2	1	04	−2
6	8	0	0	0	00	0
8	7	2	−1	4	1	−2
11	5	5	−3	25	9	−15
		$\sum u = 0$	$\sum v = 12$	$\sum u^2 = 50$	$\sum v^2 = 130$	$\sum uv = -67$

Now $$\bar{u} = \frac{\sum u}{6} = 0,\ \bar{v} = \frac{\sum v}{6} = 2$$

$$\text{cov}(u, v) = \frac{\sum uv}{n} - \bar{u}\,\bar{v} = \frac{-67}{6} = -11.167$$

$$\sigma_u^2 = \frac{\sum u^2}{n} - \bar{u}^2 = \frac{50}{6} = 8.333$$

$$\sigma_v^2 = \frac{\sum v^2}{n} - \bar{v}^2 = \frac{130}{6} - 4 = \frac{106}{6} = 17.667$$

$$r(x, y) = r(u, v) = \frac{\text{cov}(u,v)}{\sigma_u\,\sigma_v} = \frac{-11.167}{\sqrt{8.333}\sqrt{17.667}}$$

$$r(u, v) = -0.9203$$

$$\bar{x} = A + \bar{u} = 6 + 0 = 6$$

$$\bar{y} = B + \bar{v} = 8 + 2 = 10$$

$$\sigma_x = \sigma_u \text{ and } \sigma_y = \sigma_v$$

$$\therefore \quad b_{yx} = r\frac{\sigma_y}{\sigma_x} = \frac{(-0.9203)(4.203)}{(2.8867)} = 1.3399$$

∴ The regression line of y on x is

$$y-\bar{y} = b_{yx}(x-\bar{x})$$

$$y - 10 = 1.399(x - 6)$$

$$\therefore \quad y = 1.3399\ x - 1.9606$$

∴ At $x = 10$

$$y = 13.399 - 1.9606 = 11.4384$$

Q.14 From record of analysis of correlation data the following results are available variance of x = 9 and lines of regression are given by

$8x - 10y + 66 = 0$

$40\ x - 18y = 214$

Find out a) Mean values for x and y services. b) Standard deviation of y services. c) Coefficient of correlation between x and y services.

☞ [SPPU : May - 06, Dec. - 11, 15, 22]

Ans. : Let $\bar{x}, \bar{y}$ be the means of x and y series then as $(\bar{x}, \bar{y})$ satisfy equations of lines of regression we have

$$8\bar{x} - 10\bar{y} + 66 = 0 \quad \text{... (Q.14.1)}$$
$$40\bar{x} - 18\bar{y} - 214 = 0$$

Solving the above equation we get

$$\bar{x} = 13, \bar{y} = 17$$

Expressing equation (Q.14.1) of lines of regression y in terms of x and x in terms of y, we have

$$8x - 10y + 66 = 0 \text{ as } \begin{cases} y = \frac{8}{10}x + \frac{66}{10} & \text{...(Q.14.2)} \\ x = \frac{10}{8}y - \frac{66}{8} & \text{...(Q.14.3)} \end{cases}$$

and

$$40x - 18y - 214 = 0 \text{ as } \begin{cases} y = \frac{40}{18}x - \frac{214}{18} & \text{...(Q.14.4)} \\ x = \frac{18}{40}y + \frac{214}{40} & \text{...(Q.14.5)} \end{cases}$$

Now we take combination of a line y on x with the other line x on y from equation (Q.14.2), (Q.14.3), (Q.14.4) and (Q.14.5) thus

$$y = \frac{8}{10}x + \frac{66}{10} \text{ and } x = \frac{18}{40}y + \frac{214}{40} \quad \text{... (Q.14.6)}$$

$$x = \frac{10}{8}y - \frac{66}{8} \text{ and } y = \frac{40}{18}x - \frac{214}{18} \quad \text{... (Q.14.7)}$$

For pair of lines given by equation (Q.14.6), we have coefficient of regression as

$$b_{yx} = \frac{8}{10} \quad b_{xy} = \frac{18}{40}$$

$$r^2 = b_{yx}\ b_{xy} = \frac{8 \times 18}{10 \times 40} = 0.3600$$

Hence as $r > 0$, we consider the equation (Q.14.6) as the lines of regression for x, y distribution

$$r^2 = 0.36$$
$$\therefore \quad r = \pm 0.6$$

For these lines equation (Q.14.6) $b_{yx} = \frac{8}{10}$ and $b_{xy} = \frac{18}{40}$ are positive we take r as positive.

$\therefore r = 0.6 =$ correlation coefficient between x and y.

Now $\quad b_{yx} = r\frac{\sigma_y}{\sigma_x} \quad$ As $\quad \sigma_x^2 = 9 \quad \therefore \sigma_x = 3$

$$\therefore \quad \frac{8}{10} = 0.6\frac{\sigma_y}{3}$$

$$\therefore \quad \sigma_y = \frac{24}{6} = 4$$

Q.14(a) For a bivariate data, the regression equation of Y on X is $4x + y = \mu$ and the regression equation of X on Y is $9x + y = \lambda$. Find the values of μ and λ. Also, find the correlation coefficient between X and Y, if the means of X and Y are 2 and – 3 respectively. ☞ **[SPPU : Dec.-18, Marks 4]**

Ans. : $\bar{x} = 2$ and $\bar{y} = -3$

The lines of regression are $9x + y = \lambda$ and $4x + y = \mu$. The point of intersection of two regression lines is (x, y) i.e. $(\bar{x}, \bar{y})$ lies on both the regression lines.

$$9\bar{x} + \bar{y} = \lambda \quad \text{...(Q.14(a).1)}$$
$$4\bar{x} + \bar{y} = \mu \quad \text{...(Q.14(a).2)}$$

Substituting values of $\bar{x}$ and $\bar{y}$ we get,

$$9(2) + (-3) = \lambda$$
$$\lambda = 18 - 3 = 15$$

and

$$4(2) + (-3) = \mu$$
$$\mu = 8 - 3 = 5$$

Thus, the regression lines are,

$$9x + y = 15 \text{ and } 4x + y = 15$$

Let $\quad 9x + y = 15$ be the regression line of x and y.

$$\therefore \quad x = \frac{15}{9} - \frac{y}{9}$$

$$b_{xy} = -\frac{1}{9} = -0.11$$

Let 4x + y = 5 be the regression line of y on x.

So $y = 5 - 4x$

$\therefore$ $b_{yx} = -4$

Correlation coefficient between x and y is

$$r = \sqrt{b_{yx} b_{xy}} = \sqrt{(-4)\times(-0.11)}$$

$$= \sqrt{0.44} = 0.663$$

Q.14(b) Find the regression equation of Y on X for a bivariate data with the following details. n = 25, $\sum_{i=1}^{n} x_i = 75$, $\sum_{i=1}^{n} y_i = 100$, $\sum_{i=1}^{n} x_i^2 = 250$, $\sum_{i=1}^{n} y_i^2 = 500$, $\sum_{i=1}^{n} x_i y_i = 325$.

[SPPU : Dec.-19, Marsk 4]

Ans. :

Given : $\sum x_i = 30$, $\sum y_i = 40$, $\sum x_i^2 = 220$, $n = 5$,

$\sum y_i^2 = 340$, $\sum x_i y_i = 214$, $\therefore \bar{x} = \frac{\sum x_i}{n} = \frac{30}{5} = 6$

$$\bar{y} = \frac{40}{5} = 8$$

$$cov(x, y) = \frac{\sum xy}{n} - \bar{x}\bar{y} = \frac{214}{5} - 6\times 8 = -5.2$$

$$\sigma_x^2 = \frac{\sum x_i^2}{n} - \bar{x}^2 = \frac{220}{5} - 36 = 44 - 36 = 8$$

$\therefore$ $\sigma_x = \sqrt{8}$

$$\sigma_y^2 = \frac{\sum y^2}{n} - \bar{y}^2 = \frac{340}{5} - 64 = 68 - 64 = 4$$

$\therefore$ $\sigma_y = 2$

$\therefore$ $$r = \frac{cov(x,y)}{\sigma_x \sigma_y} = \frac{-5.2}{\sqrt{8}\times 2} = -0.9192$$

Q.14(c) If the two lines of regression are $9x + y - \lambda = 0$ and $4x + y = \mu$ and the means of x and y are 2 and − 3 respectively. Find values of λ, μ and correlation coefficient between x and y.

[SPPU : June-22, Marks 5]

Ans. : $\bar{x} = 2$ and $\bar{y} = -3$

The lines of regression are $9x + y = \lambda$ and $4x + y = \mu$.

The point of intersection of two regression lines is (x, y) i.e. $(\bar{x}, \bar{y})$ lies on both the regression lies.

$$9\bar{x} + \bar{y} = \lambda \quad \text{... (Q.14(c).1)}$$

$$4\bar{x} + \bar{y} = \mu \quad \text{... (Q.14(c).2)}$$

Substituting values of $\bar{x}$ and $\bar{y}$ we get,

$$9(2) + (-3) = \lambda$$

$$\lambda = 18 - 3 = 15$$

and $4(2) + (-3) = \mu$

$\therefore$ $\mu = 8 - 3 = 5$

Thus, the regression lies are,

$$9x + y = 15 \quad \text{and} \quad 4x + y = 15$$

Let 9x + y = 15 be the regression line of x and y, so it can be written as

$$x = \frac{15}{9} - \frac{y}{9}$$

$\therefore$ $$b_{xy} = -\frac{1}{9} = -0.11$$

Let 4x + y = 5 be the regression line of y on x. So it can be written as y = 5 − 4x.

$\therefore$ $b_{yx} = -4$

Correlation coefficient between x and y is given as,

$$r = \sqrt{b_{yx}\, b_{xy}} = \sqrt{(-4)\times(-0.11)}$$

$$= \sqrt{0.44}$$

$$= 0.663$$

Alternate Method : Since the given equations (Q.14(c).1) and (Q.14(c).2) pass through $(\bar{x}, \bar{y})$

$\therefore$ we have

$$\bar{x} = 19.13 - 0.87\,\bar{y}$$

and

$$\bar{y} = 11.64 - 0.50\,\bar{x}$$

Solving these we get

$$\bar{x} = 15.79$$

$$\bar{y} = 3.74$$

These are the required mean of x and y respectively.

Also from equation (1) and (2), we have

$$b_{xy} = -0.87$$

and

$$b_{yx} = -0.50$$

$\therefore$ Coefficient of correlation

$$r = \pm\sqrt{(b_{xy})\times(b_{yx})}$$

$$\Rightarrow \quad r = \pm\sqrt{(0.87)(-0.50)}$$

$$= -0.66$$

We choose here negative sign, because b_{xy} and b_{yx} are both having negative signs.

END...

Unit IV

6 Probability and Probability Distributions

6.1 : Probability

1) When an experiment is conducted and each outcome of the experiment has the same chance of appearing as any other then we call the outcomes as equally likely.

$$P(A) = \text{Probability of occurrence of any event A}$$

$$= \frac{\text{Number of favourable outcomes}}{\text{Total number of outcomes}}$$

When an event succeeds in "S" ways and fails in "F" ways (n = F + S = Total number of ways) then

$$P\text{ (Success)} = \frac{S}{S+F} \quad \text{and} \quad P\text{ (Failure)} = \frac{F}{S+F}$$

Generally the probability of success is denoted by p and the probability of failure is denoted by q.

Obviously $p + q = 1$ i.e. $q = 1 - p$.

2) Theorems on Probability

a) Addition theorem

1) If A and B are mutually exclusive events then Prob (A or B)

i.e. $P(A \cup B) = P(A) + P(B)$

If A, B, C are mutually exclusive then $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

2) When the events are not mutually exclusive then probability that atleast one of the two events A and B will occur is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

i.e. $P(A \text{ or } B) = P(A) + P(B) - P(A \,\&\, B)$

In case of three events

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B)$$

$$= -P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

b) Multiplication theorem

If A and B are two independent events, then the probability that both will occur is equal to the product of their individual probabilities.

$$P(A \& B) = P(A) \times P(B)$$

Similarly $P(A, B \& C) = P(A) \times P(B) \times P(C)$

c) Conditional probability

Multiplication theorem is not applicable when events are dependent.

e.g. when we are computing probability of a particular event A. When given information about occurrence of B. Such a probability is referred to as conditional probability.

$\therefore$ for two dependent events A and B probability of B, given A has occurred is denoted by

$$P(B|A) = \frac{P(A, B)}{P(A)}$$

Similarly, probability of A given B has occurred is

$$P(A/B) = \frac{P(A, B)}{P(B)}$$

$$\therefore \quad P(A \& B) = P(A) \times P(B/A)$$

$$P(A \& B) = P(B) \times P(A/B)$$

for three events A, B and C.

$$P(A, B \& C) = P(A) \times P(B/A) \times P(C/A, B)$$

Q.1 What is the probability that a leap year will contain 53 Mondays ?

Ans. : A leap year has 366 days.

This contains complete 52 weeks and two more days. These two days may take following combinations.

(i) Monday - Tuesday (ii) Tuesday - Wednesday (iii) Wednesday - Thursday (iv) Thursday - Friday (v) Friday - Saturday (vi) Saturday - Sunday and (vii) Sunday - Monday.

Out of these 7 combinations only 2 contain Monday.

$\therefore$ Required probability $= \frac{2}{7}$

Q.2 Prof. X and Madam Y appear for an interview for two posts. The probability of Prof. X's selection is $\frac{1}{7}$ and that of Madam Y's selection is $\frac{1}{5}$. Find the probability that only one of them is selected. What is probability that at least one of them is selected ?

[SPPU : Dec.-11]

Ans. : $P(X) = \frac{1}{7}$ $\qquad P(Y) = \frac{1}{5}$

$$P(\overline{X}) = 1 - \frac{1}{7} = \frac{6}{7} \qquad P(\overline{Y}) = 1 - \frac{1}{5} = \frac{4}{5}$$

As only one of this is selected

$\Rightarrow$ If X is selected, Y is not selected (Case A)

$\Rightarrow$ If Y is selected, X is not selected (Case B)

$$P(A) = P(X) \times P(\overline{Y}) = \frac{1}{7} \times \frac{4}{5} = \frac{4}{35}$$

$$P(B) = P(\overline{X}) \times P(Y) = \frac{6}{7} \times \frac{1}{5} = \frac{6}{35}$$

$\therefore$ Required probability

$$P(A \text{ or } B) = P(A) + P(B) = \frac{4}{35} + \frac{6}{35} = \frac{10}{35} = \frac{2}{7}$$

$$P(\overline{X}) = \frac{6}{7}, P(\overline{Y}) = \frac{4}{5}$$

$\therefore$ Probability that none is selected $= P(\overline{X}) \times P(\overline{Y}) = \frac{6}{7} \times \frac{4}{5}$

$$= \frac{24}{35}$$

$\therefore$ P (at least one is selected) $= 1 - \frac{24}{35} = \frac{11}{35}$

Q.3 A bag contains 3 red and 5 black balls and a 2^{nd} bag contains 6 red and 4 black balls. A ball is drawn from each bag. Find the probability that one is red and other is black.

Ans. : For 1^{st} bag

$$P(r_1) = \frac{3}{8} \text{ and } P(b_1) = \frac{5}{8}$$

For 2^{nd} bag

$$P(r_2) = \frac{6}{10} \quad \text{and } P(b_2) = \frac{4}{10}$$

$\therefore$ A ball is drawn from each bag 1 red from 1^{st} and 1 black from 2^{nd} or 1 black from 1^{st} and 1 red from 2^{nd}.

From first case

$$P(r_1 \text{ and } b_2) = P(r_1) \times P(b_2) = \frac{3}{8} \times \frac{4}{10} = \frac{12}{80}$$

Second case $P(b_1 \text{ and } r_2) = \frac{5}{8} \times \frac{6}{10} = \frac{30}{80}$

$\therefore$ Required probability $= \frac{12}{80} + \frac{30}{80} = \frac{42}{80}$

Mutually exclusive case.

Q.4 Three coins are tossed once. Find the Probability of getting exactly 2 heads. Exactly two heads are possible in how many ways. HTH, HHT, THH.

Ans. : Probability (H) $= \frac{1}{2}$, Probability (T) $= \frac{1}{2}$

$\therefore$ Probability (HTH) $= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$

Probability (HHT) $= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$

Probability (THH) $= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$

There are mutually exclusive events.

$\therefore$ P(A or B or C) $= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$

Q.5 A, B play a game of alternate tossing a coin one who gets head first wins the game. Find the probability that B wins the game if A has a start. **[SPPU : Dec.-07]**

Ans. : Following are the cases where B wins the game :

1) TH 2) TTTH 3) TTTTTH ....

We know $P(T) = \frac{1}{2} \quad P(H) = \frac{1}{2}$

$$\therefore \quad P((1)) = P(T) \cdot P(H) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2^2}$$

$$P((2)) = P(T)\,P(T)\,P(T)\,P(T) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2^4}$$

$$P((3)) = P(T) \cdot P(T) \cdot P(T) \cdot P(T) \cdot P(T) \cdot P(H) = \frac{1}{2^6}$$

$\therefore$ Required probability $= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6}$ ... which is a geometric series

$a + a_r + a_r^2 \text{ ....} + \frac{a}{1-r}$ with $a = \frac{1}{2^2}$ and $r = \frac{1}{2^2}$

$\therefore$ Required probability $= \frac{1/4}{1-1/4} = \frac{1}{3}$

Q.6 A student takes his examination in four subjects α, β, γ, δ He estimates his chances of passing α as $\frac{4}{5}$, in β as $\frac{3}{4}$ in γ as $\frac{5}{6}$ in δ as $\frac{2}{3}$. To qualify, he must pass in α and at least two other subjects. What is the probability that he qualifies ?

Ans. : Here

$$P(\alpha) = \frac{4}{5}, \quad P(\beta) = \frac{3}{4},$$

$$P(\gamma) = \frac{5}{6}, \text{ and } P(\delta) = \frac{2}{3}$$

$$P(\bar{\alpha}) = 1 - \frac{4}{5} = \frac{1}{5}, \quad P(\bar{\beta}) = 1 - \frac{3}{4} = \frac{1}{4}$$

$$P(\gamma) = 1 - \frac{5}{6} = \frac{1}{6}, \quad P(\bar{\delta}) = 1 - \frac{2}{3} = \frac{1}{3}$$

There are four possibilities of passing at least two subjects,

I) Passing in β, γ and failing in δ

$$= P(\beta) \times P(\gamma) \times P(\bar{\delta}) = \frac{3}{4} \times \frac{5}{6} \times \frac{1}{3} = \frac{5}{24}$$

II) Passing in γ, δ and failing in β

$$= P(\gamma)\times P(\delta)\times P(\bar{\beta}) = \frac{5}{6}\times\frac{2}{3}\times\frac{1}{4} = \frac{5}{36}$$

III) Passing in δ,β and failing in γ

$$= P(\delta)\times P(\beta)\times P(\bar{\gamma}) = \frac{2}{3}\times\frac{3}{4}\times\frac{1}{6} = \frac{1}{12}$$

IV)Passing in β,γ,δ

$$= P(\beta)\times P(\gamma)\times P(\delta) = \frac{3}{4}\times\frac{5}{6}\times\frac{2}{3} = \frac{5}{12}$$

$\therefore$ Probability of passing in at least two other subjects

$$= \frac{5}{24}+\frac{5}{36}+\frac{1}{12}+\frac{5}{12} = \frac{61}{72}$$

$\therefore$ Probability of passing α and at least two other subjects

$$= \frac{4}{5}\times\frac{61}{72} = \frac{61}{90}$$

Q.6(a) A die is tampered in such a way that the probability of observing an even number is twice as likely to observe an odd number. Find the expected value of the upper most face obtained after rolling the die.

☞ [SPPU : Dec.-22, Marks 5]

Ans. : Sample space S = {1, 1, 2, 3, 3, 4, 5, 5, 6}

So n (S) = 9

P (greater than 3) need to occur

$\therefore$ Number of required outcomes

n (E) = {4, 5, 5, 6}

$\therefore$ Required probability $= \frac{n(E)}{n(S)} = \frac{4}{9}$.

6.2 : Baye's Theorem

If $B_1, B_2, B_3 ... B_n$ are mutually exclusive events with $P(B_i) \neq 0$ (i = 1, 2, 3, 4,...n) of a random experiment then for any arbitrary event A of the sample space of the above experiment with P(A) > 0 we have,

$$P(B_i/A) = \frac{P(B_i)\ P(A/B_i)}{\sum_{i=1}^{n} P(B_i)\ P(A/B_i)}$$

For n = 3

$$P(B_1/A) = \frac{P(B_1)P(A/B_1)}{P(B_1)\ P(A/B_1)+P(B_2)P(A/B_2)+P(B_3)P(A/B_3)}$$

And

$$P(B_2/A) = \frac{P(B_2)P(A/B_2)}{P(B_1)\ P(A/B_1)+P(B_2)P(A/B_2)+P(B_3)P(A/B_3)}$$

Q.7 Three urns contain 6 red, 4 black ; 4 red, 6 black ; 5 red, 5 black balls respectively. One of the urns is selected at random and a ball is drawn from it. If the ball drawn is red find the probability that it drawn from the first urn.

Ans. :

Let B_1 : The ball is drawn from urn I

B_2 : The ball is drawn from urn II

B_3 : The ball is drawn from urn III

A : The ball is red.

We have to find P (B_1/A)

$\therefore$By Baye's theorem

$$P(B_1/A) = \frac{P(B_1)P(A/B_1)}{P(B_1)\ P(A/B_1)+P(B_2)P(A/B_2)+P(B_3)P(A/B_3)} \quad ...(Q.7.1)$$

As the three urns are equally likely to be selected

$$\therefore \quad P(B_1) = P(B_2) = P(B_3) = \frac{1}{3}$$

and $P(A/B_1)$ = P (a red ball is drawn from urn I) $= \frac{6}{10} = \frac{3}{5}$

$P(A/B_2)$ = P (a red ball is drawn from urn II) $= \frac{4}{10} = \frac{2}{5}$

$P(A/B_3)$ = P (a red ball is drawn from urn III) $= \frac{5}{10} = \frac{1}{2}$

From (Q.7.1)

$$P(B_1/A) = \frac{\frac{1}{3}\times\frac{6}{10}}{\frac{1}{3}\times\frac{6}{10}+\frac{1}{3}\times\frac{4}{10}+\frac{1}{3}\times\frac{5}{10}} = \frac{2}{5}$$

Q.7(a) A series of five one day matches is to be played between India and Sri Lanka. Assuming that the probability of India's win in each match as 0.6 and results of all the five matches independent of each other, find the probability that India wins the series.

[SPPU : Dec.-18, Marks 4]

Ans. : Let p be the probability that India wins the match.

$\therefore$ $p = 0.6$ and $q = 1 - p = 0.4$, $n = 5$

The probability that India wins the series

$$= p(r = 3) + p(r = 4) + p(r = 5)$$

$$= {}^nC_3 p^3 q^2 + {}^nC_4 p^4 q + {}^nC_5 p^5 q^0$$

$$= {}^5C_3 (0.6)^3 (0.4)^2 + {}^5C_4 (0.6)^4 (0.4) + {}^5C_5 (0.6)^5$$

$$= 10(0.6)^3 (0.4)^2 + 5(0.6)^4 (0.4) + (0.6)^5 = 0.68256$$

Q.7(b) A riddle is given to three students whose probabilities of solving it are $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$ respectively. Find the probability that the riddle is solved.

[SPPU : May-19, Marks 4]

Ans. : Let P(A), P(B) and P(C) be probability that A, B and C can solve the problem respectively.

$\therefore$

$$P(A) = \frac{1}{2}, \quad P(\bar{A}) = \frac{1}{2}$$

$$P(A) = \frac{1}{3}, \quad P(\bar{B}) = 1 - \frac{1}{3} = \frac{2}{3}$$

$$P(C) = \frac{1}{4}, \quad P(\bar{C}) = 1 - \frac{1}{4} = \frac{3}{4}$$

The probability that the riddle (problem) is solved means probability that at least one student solved the riddle.

$\therefore$ Required Probability $= 1 -$ Probability that riddle is not solved

$$= 1 - P(\bar{A})P(\bar{B})P(\bar{C})$$

$$= 1 - \frac{1}{2}\left(\frac{2}{3}\right)\left(\frac{3}{4}\right) = 1 - \frac{1}{4} = \frac{3}{4}$$

Q.7(c) Bag 1 contains 2 white and 3 red balls. Bag 2 contains 4 white and 5 red balls. One ball is drawn randomly from bag 1 and is placed in bag 2. Later, one ball is draw randomly from bag 2. Find the probability that it is red. **[SPPU : June-22, Marks 5]**

Ans. :

For bag 1 : $P(w_1) = \frac{2}{5}$, $P(r_1) = \frac{3}{5}$

For bag 2 : $P(w_1) = \frac{4}{9}$, $P(r_1) = \frac{5}{9}$

One ball is transferred to bag 2, A ball can be I) White II) Red

One ball from bag 1 can be selected by $5c_1 = 5$ ways

I) Total number of ball in bag 2 are 10,

White = 5 and Red = 5

Probability of selecting red ball is $\frac{5}{10} = \frac{1}{2}$

$\therefore$ The required probability $= \frac{3}{5}\times\frac{1}{2} = \frac{3}{10}$

II) White = 4, Red = 6

Probability of selecting red ball is $\frac{6}{10} = \frac{3}{5}$

$\therefore$ The required probability $= \frac{3}{5}\times\frac{3}{5} = \frac{9}{25}$

Q.8 In a bolt factory, machines A,B and C manufactures respectively 25%, 35% and 40% of the total of their output 5, 4 and 2 percent are defective bolts. A bolt is drawn at random from the product and is found to be defective. What is the probability that it was manufactured by machine B.

Ans. :

Let B_1 : Bolt is manufactured by machine A

B_2 : Bolt is manufactured by machine B

B_3 : Bolt is manufactured by machine C

D : The bolt is defective

$P(B_1) = 0.25,\ P(B_2) = 0.35,\ P(B_3) = 0.40$

Given that

$P(D/B_1) = 0.05,\ P(D/B_2) = 0.04,\ P(D/B_3) = 0.02$

∴ By Baye's theorem

$$P(B_2/D) = \frac{P(B_2)P(D/B_2)}{P(B_1)P(D/B_1)+P(B_2)P(D/B_2)+P(B_3)P(D/B_3)}$$

$$= \frac{0.35\times 0.04}{0.25\times 0.05+0.35\times 0.04+0.4\times 0.02}$$

$$= 0.41$$

Q.8(a) Three factories A, B and C produce light bulbs. 20 %, 50 % and 30 % of the bulbs are available in the market by factories A, B and C respectively. Among these, 2 %, 1 % and 3 % of the bulbs produced by factories A, B and C are defective. A bulb is selected at random in the market and found to be defective. Find the probability that this bulb was produced by factory B.

☞ [SPPU : Dec.-22, Marks 5]

Ans. : Let e_1, e_2, e_3 be the events of the bulb is produced by machines A, B, C respectively.

∴ $P(e_1) = \frac{20}{100} = \frac{2}{10},\ P(e_2) = \frac{50}{100} = \frac{5}{10},$

$P(e_3) = \frac{30}{100} = \frac{3}{10}$

Let e be the event of bulb being defective.

∴ $P(e|e_1) = \frac{2}{100},\ P(e|e_2) = \frac{1}{100},\ P(e|e_3) = \frac{3}{100}$

Required probability = $P(e_2|e)$

$$= \frac{P(e_2)\cdot P(e|e_2)}{P(e_1)\cdot P(e|e_1)+P(e_2)\cdot P(e|e_2)+P(e_3)\cdot (e|e_3)}$$

$$= \frac{\left(\frac{5}{10}\right)\left(\frac{1}{100}\right)}{\left(\frac{2}{10}\right)\left(\frac{2}{100}\right)+\left(\frac{5}{10}\right)\left(\frac{1}{100}\right)+\left(\frac{3}{10}\right)\left(\frac{3}{100}\right)}$$

$$= \frac{\frac{5}{1000}}{\frac{4}{1000}+\frac{5}{1000}+\frac{9}{1000}} = \frac{5}{1000}\times\frac{1000}{18}$$

$$= \frac{5}{18}$$ ... **Ans.**

6.3 : Mathematical Expectation

I) Let X be a random variable with all possible values $x_1, x_2, \ldots x_n$ and probability functions $f(x_1), f(x_2), \ldots f(x_n)$ respectively. Then the mathematical expectation of X is denoted by E(X) and defined as

$$E(X) = \sum_{i=1}^{n} x_i\ f(x_i) \text{ where } \sum_{i=1}^{n} f(x_i) = 1$$

E(X) is also called expected value or mean value of X.

Note : We may write E(X) as

$$E(X) = \sum x_i\ P(x_i) = \sum X\,P(X)$$

II) Let X be a continuous random variable in internal I. (i.e. X is defined in some internal). A function f(x) is called the probability density function (pdf) if it satisfies the following conditions.

i) $f(x) \geq 0$ for all $x \in I$

ii) $\int_I f(x)dx = 1$

Note : For any interval a < x < b

$$P(a < x < b) = \int_a^b f(x)dx$$

Q.9 Three coins are tossed together, X the random variable denotes the number of heads with the distribution give,

x	0	1	2	3
P(x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Find the mathematical expectation of X.

[SPPU : June-22, Marks 8]

Ans. : We have

$$E(X) = \sum_{i=1}^{n} x_i \, P(x_i)$$

Here $n = 4,\ P(x_1) = P(0) = \frac{1}{8},\ P(x_2) = P(1) = \frac{3}{8}$

$$P(x_3) = P(2) = \frac{3}{8},\ P(x_3) = P(3) = \frac{1}{3}$$

$$\therefore \quad E(X) = 0\times\frac{1}{8}+1\times\frac{3}{8}+2\times\frac{3}{8}+3\times\frac{1}{8} = \frac{12}{8} = \frac{3}{2} = 1.5$$

Q.10 If a die is thrown twice and X denotes the sum of digits in two throws. Find E(X).

Ans. :

We have,

X	2	3	4	5	6	7	8	9	10	11	12
P(X)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$\therefore \quad E(X) = \sum_{i=1}^{11} x_i \, P(x_i)$$

$$= 2\times\frac{1}{36}+3\times\frac{2}{36}+\frac{12}{36}+\frac{20}{36}+\frac{30}{36}+\frac{42}{36}+\frac{40}{36}+\frac{36}{36}+\frac{30}{36}+\frac{22}{36}+\frac{12}{36}$$

$$= \frac{252}{36} = 7$$

$$E(X) = 7$$

Q.11 If

x	1	2	3	4
P (x)	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$	0

Find $E(x^2)$

Ans. :

Consider the table.

X^2	1^2	2^2	3^2	4^2
P (x)	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$	0

$$\therefore \quad E(X^2) = \sum x^2 P(x)$$

$$E(X^2) = 1^2\times\frac{1}{6}+2^2\times\frac{2}{3}+3^2\times\frac{1}{6}+4^2\times 0 = \frac{13}{3}$$

Q.12 Given the density function $f(x) = k\ e^{-ax}$; $x \geq 0$ and $a \geq 0$

$= 0$; otherwise.

Find k

Ans. : We have

$$\int_{-\infty}^{\infty} f(x,a)dx = \int_{-\infty}^{0} f(x,a)dx + \int_{0}^{\infty} f(x,a)dx = 1$$

$$\therefore 0 + \int_{0}^{\infty} f(x,a)dx = 1$$

$$\therefore \quad \int_{0}^{\infty} ke^{-ax}\,dx = 1 \Rightarrow k\left[\frac{e^{-ax}}{-a}\right]_0^{\infty} = 1$$

$$-\frac{k}{a}(0-1) = 1 \Rightarrow \frac{k}{a} = 1 \Rightarrow \boxed{k = a}$$

Q.13 If the pdf f(x) is defined as

$$f(x) = \begin{cases} \dfrac{A}{x^3} & ; 5 \le x \le 10 \\ 0 & ; \text{Otherwise} \end{cases}$$ **Find the value of A.**

Ans. :

We have,

$$P(a \le x \le b) = \int_a^b f(x)dx$$

$$P(5 \le x \le 10) = \int_5^{10} \frac{A}{x^3} dx = 1$$

$$\Rightarrow A\left[\frac{x^{-3+1}}{-2}\right]_5^{10} = 1$$

$$\Rightarrow \frac{A}{-2}\left[\frac{1}{x^2}\right]_5^{10} = 1$$

$$\Rightarrow \frac{A}{-2}\left[\frac{1}{100} - \frac{1}{25}\right] = 1$$

$$\Rightarrow \frac{A}{-2}\left[\frac{1-4}{100}\right] = 1$$

$$\Rightarrow \frac{A}{-2}\left[\frac{-3}{100}\right] = 1 \Rightarrow A = \frac{200}{3}$$

Q.14 If X is random variable with distribution given by

x	0	1	2	3
P (x)	k	3 k	3 k	k

Find k.

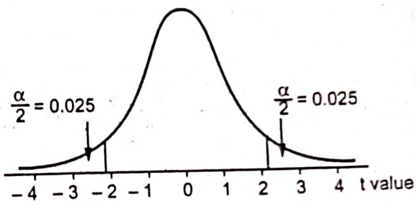
Ans. :

We have,

$$E(x) = \sum x\,P(x) \text{ and } \sum P(x) = 1$$

$$\therefore\ k + 3k + 3k + k = 1 \Rightarrow 8k = 1 \Rightarrow k = \frac{1}{8}$$

Student's t distribution table



For example, the t value for 18 degrees of freedom is 2.101 for 95 % confidence interval (**2-Tail** α = 0.05).

Table 6.1

df							
	90%	95%	97.5%	99%	99.5%	99.5%	1-Tail Confidence Level
	80%	90%	95%	98%	99%	99.9%	2-Tail Confidence Level
	0.100	0.050	0.025	0.010	0.005	0.0005	3-Tail Alpha
df	0.20	0.10	0.05	0.02	0.01	0.001	2-Tail Alpha
1	3.0777	6.3138	12.7062	31.8205	63.6567	636.6192	
2	1.8856	2.9200	4.3027	6.9646	9.9248	31.5991	
3	1.6377	2.3534	3.1824	4.5407	5.8409	12.9240	
4	1.5332	2.1318	2.7764	3.7469	4.6041	8.6103	
5	1.4759	2.0150	2.5706	3.3649	4.0321	6.8688	
6	1.4398	1.9432	2.4469	3.1427	3.7074	5.9588	
7	1.4149	1.8946	2.3646	2.9980	3.4995	5.4079	
8	1.3968	1.8595	2.3060	2.8965	3.3554	5.0413	
9	1.3830	1.8331	2.2622	2.8214	3.2498	4.7809	
10	1.3722	1.8125	2.2281	2.7638	3.1693	4.5869	
11	1.3634	1.7959	2.2010	2.7181	3.1058	4.4370	
12	1.3562	1.7823	2.1788	2.6810	3.0545	4.3178	
13	1.3502	1.7709	2.1604	2.6503	3.0123	4.2208	
14	1.3450	1.7613	2.1448	2.6245	2.9768	4.1405	

15	1.3406	1.7531	2.1314	2.6025	2.9467	4.0728
16	1.3368	1.7459	2.1199	2.5835	2.9208	4.0150
17	1.3334	1.7396	2.1098	2.5669	2.8982	3.9651
18	1.3304	1.7341	2.1009	2.5524	2.8784	3.9216
19	1.3277	1.7291	2.0930	2.5395	2.8609	3.8834
20	1.3253	1.7247	2.0860	2.5280	2.8453	3.8495
21	1.3232	1.7207	2.0796	2.5176	2.8314	3.8193
22	1.3212	1.7171	2.0739	2.5083	2.8188	3.7921
23	1.3195	1.7139	2.0687	2.4999	2.8073	3.7676
24	1.3178	1.7109	2.0639	2.4922	2.7969	3.7454
25	1.3163	1.7081	2.0595	2.4851	2.7874	3.7251
26	1.3150	1.7056	2.0555	2.4786	2.7787	3.7066
27	1.3137	1.7033	2.0518	2.4727	2.7707	3.6896
28	1.3125	1.7011	2.0484	2.4671	2.7633	3.6739
29	1.3114	1.6991	2.0452	2.4620	2.7564	3.6594
30	1.3104	1.6973	2.0423	2.4573	2.7500	3.6460

6.4 : Binomial Distribution

I) Consider the experiment in which we perform a series of n independent trials. Each trial has only two outcomes or two mutually exclusive possibilities, a success or a failure.

Let p = Probability of getting a success

q = Probability of getting a failure

and $p + q = 1$

As r successes and $(n-r)$ failures can occur in ${}^nC_r$ mutually exclusive cases

$\therefore$ P [r successive in n trials] = ${}^nC_r \cdot p^r \cdot q^{n-r}$

Substituting r = 0, 1, 2, 3 ..... n we get the following table.

r	0	1	2	3	......	n
p (r)	${}^nC_0p^0q^n$	${}^nC_1p^1q^{n-1}$	${}^nC_2p^2q^{n-2}$	${}^nC_3p^3q^{n-3}$	......	${}^nC_np^nq^{n-n}$

$${}^nC_0 = 1, \quad {}^nC_n = 1$$

Consider now the Binomial expansion of

$$(q+p)^n = q^n + {}^nC_1q^{n-1}p + {}^nC_2q^{n-2}p^2 + ... + p^n$$

Terms of R.H.S. of this expansion give probability of r = 0, 1, 2, ..., n success. This is the reason for above probability distribution to be called Binomial probability distribution. It is denoted by B (n, p, r).

Thus $B(n, p, r) = {}^nC_r\, p^r\, q^{n-r}$

II) Mean and Variance of the Binomial Distribution

☞ **[SPPU : Dec. - 04]**

1) Mean $\therefore$ $\mu = np$ as $p + q = 1$

$\therefore$ Mean of binomial distribution is np.

2) Variance = npq and S.D. = $\sqrt{npq}$

Q.15 An average box containing 10 articles is likely to have 2 defectives. If we consider a consignment of 100 boxes, how many of them are expected to have three or less defectives.

☞ **[SPPU : May - 01, 09, 16, Dec. - 02, 06]**

Ans. : Let p = Probability of box containing defective articles

$= 2/10 = 1/5$

q = Probability on non defective item

$= 4/5$

Probability of box containing three or less defective articles.

$= P(r \le 3) = P(r = 0) + P(r = 1) + P(r = 2) + P(r = 3)$

r = Number of defective items

$$P(r = 0) = {}^{10}C_0\left(\frac{1}{5}\right)^0\left(\frac{4}{5}\right)^{10} = 0.10738$$

$$P(r=1) = {}^{10}C_1\left(\frac{1}{5}\right)^1\left(\frac{4}{5}\right)^9 = 0.2684$$

$$P(r=2) = {}^{10}C_2\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^8 = 0.302$$

$$P(r=3) = {}^{10}C_3\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^7 = 0.2013$$

$P(r \le 3) = 0.10738 + 0.2684 + 0.302 + 0.2013 = \mathbf{0.8791}$

$100 \times 0.8791 = 87.91$

88 boxes are expected to contain three or less defectives.

Q.15(a) The expected number of matches those will be won by India in a series of five one day matches between India and England is three. If the probability of India's win in each match remains the same and the results of all the five matches are independent of each other, find the probability that India wins the series, using Binomial distribution. Assume that each match ends with a result.

☞ [SPPU : June-22, Marks 5]

Ans. : Let P be the probability that India wins the match

$\therefore \quad P = \frac{3}{5} = 0.6,\ q = 1 - p = 0.4,\ n = 5$

The probability that India wins the series is

$$\begin{aligned} P &= P(r=3) + P(r=4) + P(r=5) \\ &= {}^nC_3p^3q^2 + {}^nC_4p^4q^1 + {}^nC_5p^5q^0 \\ &= {}^5C_3(0.6)^3(0.4)^2 + {}^5C_4(0.6)^4(0.4) + {}^5C_5(0.6)^5 \\ &= 10(0.6)^3 + 5(0.6)^4(0.4) + (0.6)^5 \\ &= 0.68256 \end{aligned}$$

Q.16 Mean and variance of binomial distribution are 6 and 2 respectively. Find $P(r \ge 1)$. ☞ [SPPU : Dec. - 03, 11, May - 13]

Ans. : Mean $= np = 6$

Variance $= npq = 2$

$6 \times q = 2$

$$q = 1/3$$

$$P = 1 - \frac{1}{3} = \frac{2}{3}$$

As $np = 6 \Rightarrow n\cdot\frac{2}{3} = 6 \Rightarrow n = 9$

$$\therefore \quad P(r \ge 1) = 1 - \left\{{}^nC_0\, p^r \cdot q^{n-r}\right\}$$

$$= 1 - \left[{}^9C_0\left(\frac{2}{3}\right)^0 \times \left(\frac{1}{3}\right)^9\right] = 0.9999$$

Q.17 The incidence of a certain disease is such that on the average 20 % of workers suffer from it. If 10 workers are selected at random, find the probability that

i) Exactly 2 worker suffer from disease.

ii) Not more than 2 workers suffer. ☞ [SPPU : Dec.-14]

Ans. : Let $P = \frac{20}{100} = \frac{1}{5},\ q = 1 - P = \frac{4}{5}$

$n = 10$

By Binomial distribution

$$P(r) = {}^nC_r\, p^r q^{n-r} = {}^{10}C_r\left(\frac{1}{5}\right)^r\left(\frac{4}{5}\right)^{n-r}$$

i) $P(r=2) = {}^{10}C_2\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^{10-2} = 0.302$

ii) $P(r \le 2) = P(r=0) + P(r=1) + P(r=2)$

$$= {}^{10}C_0\left(\frac{1}{5}\right)^0\left(\frac{4}{5}\right)^{10} + {}^{10}C_1\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^9 + 0.302$$

$$= \mathbf{0.678}$$

Q.17(a) On an average, 20 % of the computers in a firm are virus infected. If 10 computers are chosen at random from this firm, find the probability that at least one computer is virus infected, using Binomial distribution. ☞ [SPPU : Dec.-22, Marks 5]

Ans. : Given, $n = 10, P = \frac{20}{100} = \frac{1}{5}$

$\therefore \quad q = 1 - \frac{1}{5} = \frac{4}{5}$

Let X denote the number of infected virus computer chosen.

$\therefore \quad X = 1$

$$P(X \geq 1) = P(1) + P(2) + P(3) + P(4) + P(5) + P(6) + P(7) + P(8) + P(9) + P(10)$$

$$= 1 - P(r < 1)$$

$$= 1 - P(r < 0)$$

$$= 1 - {}^{10}C_0 \left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^{10}$$

$$= 1 - \frac{10!}{0!(10-0)!} \times 1 \times \frac{1048576}{9765625}$$

$$= 1 - \frac{10 \times \cancel{9!}}{\cancel{9!}} \times 0.10737$$

$$= 1 - 1.0737$$

$$= -0.0737$$

6.5 : Hypergeometric Distribution

If an experiment is sampling without replacement then we can not apply Binomial distribution. In this case the hypergeometric distribution is used.

The probability that we select a sample of size n containing r defective items from a population of N items known to contain M defective items is

$$P(X = r) = \frac{\binom{M}{r}\binom{N-M}{n-r}}{\binom{N}{n}}; \quad r = 0, 1, 2, \ldots n.$$

p (x) is called the pmf of the hypergeometric distribution.

Note : Hypergeometric distribution is applicable only if a random sample is taken without replacement from a population consisting of two classes only.

Q.18 In the manufacture of car tyres, a particular production process is know to yield 10 tyres with defective walls in every batch of 100 tyres produced. From a production batch of 100 tyres a sample of 4 is selected for testing to destruction. Find

i) The probability that the sample contains 1 defective tyre.

ii) Find the expectation of the number of defectives in a sample of size 4.

Ans. : The sampling is clearly without replacement so use Hypergeometric distribution

Here $N = 100, M = 10, n = 4, r = 1, P = \frac{10}{100} = 0.1$

i) $$P(X = r) = \frac{\binom{M}{r}\binom{N-M}{n-r}}{\binom{N}{n}}$$

$$P(X = 1) = \frac{\binom{10}{1}\binom{90}{3}}{\binom{100}{4}} = \frac{10 \times 117480}{3921225} = 0.3$$

ii) $$E(X) = nP = 4 \times 0.1 = 0.4$$

Q.19 Among the 200 employees of a company, 160 are union members and the others are non union. If four employees are to be chosen to serve on the staff welfare committee find the probability that two of them committee find the probability that two of them will be union member and the others non-union.

Ans. : Let X denote the number of union members selected in the sample.

$\therefore$ By hypergeometric distribution

$$N = 200, M = 160, n = 4, r = 2$$

$$P(X = 2) = \frac{\binom{M}{r}\binom{N-M}{n-r}}{\binom{N}{n}} = \frac{\binom{160}{2}\binom{40}{2}}{\binom{200}{4}} = 0.1534.$$

6.6 : Poisson Distribution

1) The distribution with frequencies given by, $\sum_{r=0}^{\infty} \frac{e^{-z} \cdot z^r}{r!}$ corresponding to 0, 1, 2, ... r, success is called Poisson distribution.

2) Variance $\sigma^2 = \lambda = nP$

3) Mean and Variance of Poisson's distribution = nP

Q.20 The accidents per shift in factory are given by,

Acc / Shift	0	1	2	3	4	5
Frequency	142	158	67	27	5	1

Find a Poisson distribution.

Ans. :

x_i	f_i	$f_i x_i$	
0	142	0	Mean = $\frac{\sum f_i x_i}{\sum f_i}$
1	158	158	
2	67	134	$= \frac{398}{400}$
3	27	81	
4	5	20	z = 0.995
5	1	5	
	400	398	

x_i	f_i	$f_i x_i$	$P(r) = \frac{z^r e^{-z}}{r!}$	Exp ≠ 400 × P (r)
0	142	0	0.3697	148
1	158	158	0.3678	147
2	67	134	0.183	73
3	27	81	0.607	24
4	5	20	0.0157	6.0
5	1	5	0.003	1

Q.21 If the probability that a concrete cube fails is 0.001. Determine the probability that out of 1000 cubes i) Exactly two ii) More than one cubes will fail. **[SPPU : Dec.-15]**

Ans. : Here P = 0.001, n = 1000, np = z = 1

Thus the poisson's distribution is

$$p(r) = \frac{z^r e^{-z}}{r!} = \frac{e^{-1}}{r!}$$

i) Probability that exactly 2 cubes fail

$$= P(r = 2) = \frac{e^{-1}}{r!} = \frac{1}{2e}$$

ii) Probability that more than one cube fail = P (r ≥ 2)

$$= 1 - P(r < 2) = 1 - P(0) - P(1)$$

$$= 1 - e^{-1} - \frac{e^{-1}}{1} = 1 - \frac{2}{e}$$

$$= \frac{e-2}{e}$$

Q.22 A manufacturer of electronic goods has 4 % of his product defective. He sells the articles in packets of 300 and guarantees 90 % good quality. Determine the probability that a particular packet will violet the quarantee. **[SPPU : May-15]**

Ans. : We have P = 0.04, n = 300

$$z = np = 12$$

By Poisson distribution

$$P(r) = \frac{e^{-z} z^r}{r!} = \frac{e^{-12} 12^r}{r!}$$

The probability that a particular packet violet the quarantee is that

$$= P(r \geq 1) = 1 - \sum_{r=1}^{r} \frac{e^{-12} 12^r}{r!}$$

Q.22(a) In a factory manufacturing razor blades, there is a small chance of $\frac{1}{500}$ for any blade to be defective. The blades are supplied in a packet of 10. Use Poisson distribution to calculate the approximate number of packets containing at least one defective blade in a consignment of 10,000 packets.

[SPPU : Dec.-18, Marks 4]

Ans. : We have $p = \frac{1}{500}$, $n = 10$, $z = np = 10 \times \frac{1}{500} = \frac{1}{50}$. The probability that a packet contains at least one defective $= p(r \geq 1) = 1 - p(0)$

$$= 1 - \frac{e^{-0.02}(0.02)^0}{0!} = 0.0198$$

Q.22(b) During working hours, on an average 3 phone calls are coming into a company within an hour. Using Poisson distribution, find the probability that during a particular working hour, there will be at the most one phone call. **[SPPU : May-19, Marks 4]**

Ans. : The probability that phone call coming into a company is

$$= \frac{3}{60} = 0.05$$

$\therefore$ $z = 0.05$

The probability of getting at most one phone call is

$$= P(r \geq 1) = P(r = 0) + P(r = 1)$$

$$= \frac{e^{-z}z^0}{0!} + \frac{e^{-z}z^1}{1!} = e^{-z}[1 + z] = 0.9987$$

Q.22(c) On an average, 180 cars per hour pass a specified point on a particular road. Using Poisson distribution, find the probability that at least two cars pass the point in any one minute.

[SPPU : June-22, Marks 5]

Ans. : $\lambda = \frac{180}{60} = 3 = \text{Mean}$

Probability that at least one car pass the point in any one minute is $P(r \geq 1)$

$$= 1 - P(r < 1) = 1 - P(r = 0)$$

$$= 1 - \frac{\lambda^0 e^{-\lambda}}{0!} = 1 - e^{-\lambda} = 1 - e^{-3} = 0.9502$$

Q.22(d) The number of industrial injuries per working week in a factory is known to follow a Poisson distribution with mean 0.5. Find the probability that during a particular week, at least two accidents will take place.

[SPPU : Dec.-22, Marks 5]

Ans. : Suppose the number of industrial injuries per working week in a factory is X then $X \sim P(\lambda)$ because the mean in 0.5, then $\lambda = 0.5$

$$P(X = k) = \frac{e^{-\lambda} \cdot \lambda^k}{k!} = \frac{e^{-0.5}(0.5)^k}{k!}$$

1) $P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$

$$= e^{0.5}\left(1 + \frac{0.5}{1} + \frac{(0.5)^2}{2}\right) \approx 0.95$$

2) $P(X > 2) = 1 - P(X \leq 2) = 1 - 0.95 \approx 0.05$

Q.23 If the probability that an individual suffers a bad reaction from a certain injection is 0.001, determine the probability out of 2000 individuals. I) Exactly 3 II) More than 2 will suffer a bad reaction.

[SPPU : May-13]

Ans. : Here $P = 0.001$

$$n = 2000$$

$\therefore$ $\lambda = nP = 0.001 \times 2000 = 2$

$\therefore$ $P(r) = \frac{e^{-\lambda}\lambda^r}{r!} = \frac{e^{-2}\, 2^r}{r!}$

$\therefore$ I) $P(3) = \frac{e^{-2}(2)^3}{3!} = 0.136 \times \frac{8}{6}$

II) Prob (more than 2) = P(3) + P(4) .... + P(2000)

$$= 1 - [P(0) + P(1) + P(2)]$$

$$= 1 - \left[\frac{e^{-2} \times 2^0}{0!} + \frac{e^{-2} \times 2^1}{1!} + \frac{e^{-2} \times 2^2}{2!}\right]$$

$$= 1 - e^{-2}[1 + 2 + 2]$$

$$= 1 - 0.136 \times 5$$

$$= 0.32$$

Q.23(a) In a factory manufacturing razor blades, there is a small chance of $\frac{1}{500}$ for any blade to be defective. The blades are supplied in a packet of 10. Use Poisson distribution to calculate the approximate number of packets containing at least one defective blade in a consignment of 10,000 packets. **[SPPU : Dec.-18, Marks 4]**

Ans. : We have $p = \frac{1}{500}$, $n = 10$, $z = np = 10 \times \frac{1}{500} = \frac{1}{50}$. The probability that a packet contains at least one defective $= p(r \geq 1) = 1 - p(0)$

$$= 1 - \frac{e^{-0.02}(0.02)^0}{0!} = 0.0198$$

6.7 : Normal Distribution

I) The general equation of the normal distribution is given by

$$y = f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

where the variable 'x' assumes all values from $-\infty$ to ∞ and σ, called the parameters of the distribution respectively are known as mean and standard deviation of the distribution and $-\infty < \mu < \infty$, $\sigma > 0$ x is called the normal variate and f (x) is probability density function of the normal distribution.

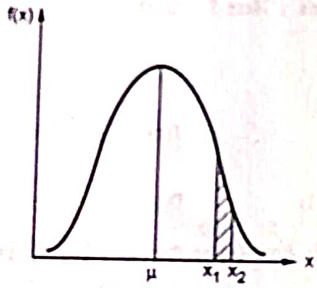


Fig. 6.1

The graph of the normal distribution is called the normal curve (some times known as normal probability curve or normal curve of errors). It is bell - shaped and symmetrical about the mean 'μ' as shown in the figure. The two tails of the curve extend to $+\infty$ and $-\infty$ towards the positive and negative directions of the X-axis respectively and gradually approach the X-axis without ever meeting it. The line $x = \mu$ divides the area under the normal curve above, X-axis into two equals parts. The area under the normal curve between any two given ordinates

$x = x_1$ and $x = x_2$ represents the probability of values failing into the given interval. The total area under the normal curve above the x-axis is '1' i.e.

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\text{Thus } P(x_1 < x < x_2) = \int_{x_1}^{x_2} f(x)\, dx = \int_{x_1}^{x_2} \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

II) Standard Form of the Normal Distribution

If 'X' is a normal random variable with mean 'μ' and standard deviation σ, then the random variable $z = \frac{X - \mu}{\sigma}$ has the normal distribution with mean '0' and standard deviation 1. The random variable z is called the Standard normal random variable.

Thus probability density function or the normal distribution in standard form is given by

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \quad (-\infty < z < \infty)$$

It is free from any parameter. This is useful to compute areas under the normal probability curve by making use of standard tables.

Area under the Normal Curve :

The area under the normal curve is divided into two equal parts by $z = 0$. Left hand side area and right hand side area to $z = 0$ is 0.5.

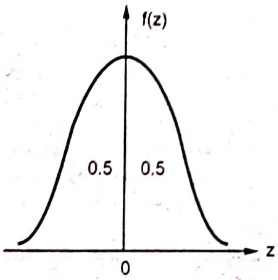


Fig. 6.2

III) Table of Area

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5259
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7793	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8906	0.8925	0.8943	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9494	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9653	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9708
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9883	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9809	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Table 6.1 In each row and each column 0.5 to be substracted

IV) Important Formulae

1) The total area under the curve

= Sum of the probabilities = 1 i.e. $\int_{-\infty}^{\infty} y\,dx$

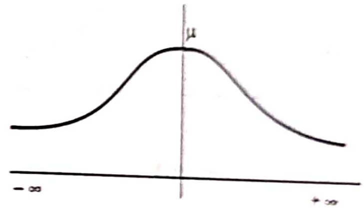


Fig. 6.3

2) $P(x_1 < x < x_2) = \int_{x_1}^{x_2} y\,dx$

= Area under the curve from x_1 to x_2

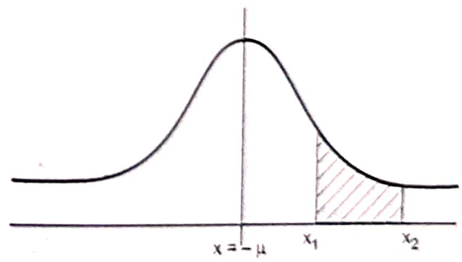


Fig. 6.4

3) $P(\mu < x < x_1) = \int_{\mu}^{x_1} y\,dx$

Put $\dfrac{x-\mu}{\sigma} = z, \quad \dfrac{dx}{\sigma} = dz$

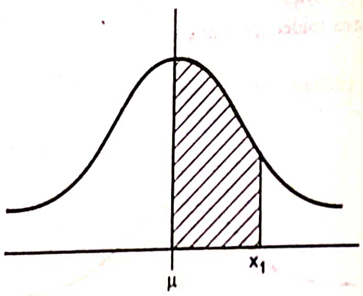


Fig. 6.5

$$P(0 < z < z_1) = \int_0^{z_1} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

$$= \int_0^{z_1} (z)\, dz$$

is known as normal integral gives the area under the standard normal curve between $z = 0$ and $z = z_1$.

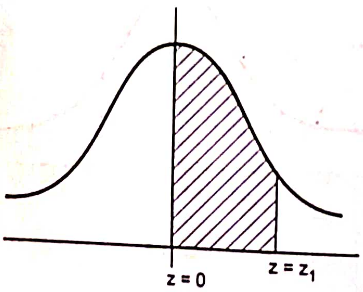


Fig. 6.6

4) $P(z_1 < z < z_2) = A(z_2) - A(z_1)$

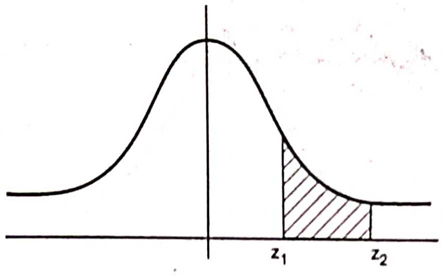


Fig. 6.7

5) $P(z > z_1) = 0.5 - A(z_1)$

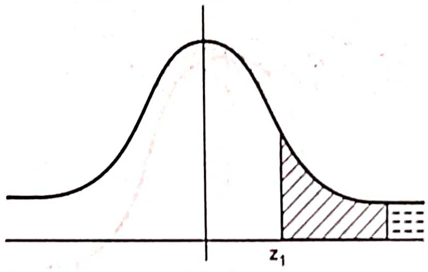


Fig. 6.8

6) $P(z < -z_1) = 0.5 - A(z_1)$

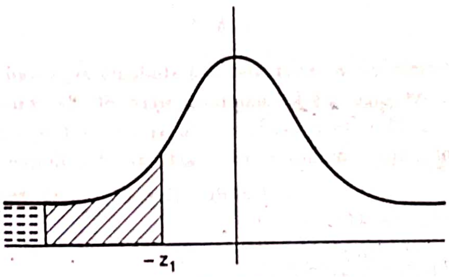


Fig. 6.9

7) $P(-z_1 < z < -z_2) = A(z_1) - A(z_2)$

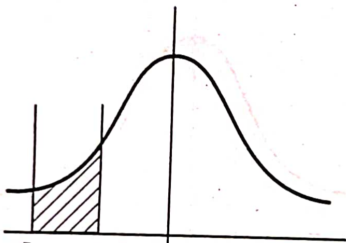


Fig. 6.10

8) $P(-z_1 < z < -z_2) = A(z_1) + A(z_2)$

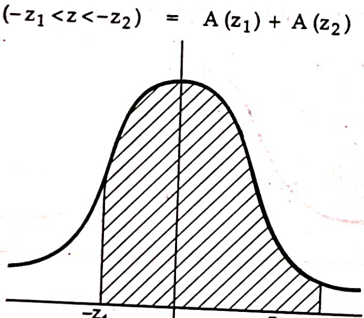


Fig. 6.11

Q.24 In a certain examination test 200 students appeared in subject of statistics. Average marks obtained were 50 % with standard deviation 5 %. How many students do you expect to obtain more than 60 % of marks, supposing that marks are distributed normally.

☞ **[SPPU : Dec.-05, 06, 12, May-10, 14]**

Ans. : $\mu = 0.5$, $\sigma = 0.05$, $x_1 = 0.6$

$$z_1 = \frac{0.6 - 0.5}{0.05} = 2$$

A corresponding to $z = 2$ is 0.4772

$$P(x \geq 6) = 0.5 - 0.4772 = 0.0228$$

Number of students expected to get more than 60 % marks

$$= 0.0228 \times 200$$

$$= 46 \text{ students approximately.}$$

Q.24(a) The lifetime of an article has a normal distribution with mean 400 hours and standard deviation 50 hours. Find the expected number of articles out of 2,000 whose lifetime lies between 335 hours to 465 hours. (Given : Z = 1.3, A = 0.4032)

☞ **[SPPU : June-22, Marks 5]**

Ans. : Given that $\mu = 400$, $\sigma = 50$, $n = 2000$

$$Z = \frac{x-\mu}{\sigma} = \frac{x-400}{50}$$

When $x_1 = 3.35$, $z_1 = \frac{335-400}{50} = \frac{-65}{50} = -1.3$

$x_2 = 465$, $z_2 = \frac{465-400}{50} = 1.3$

$$\therefore \quad P(335 < x < 465) = P(-1.3 < z < 1.3)$$

$$= 2P(z < 1.3) = 2 \times 0.4032$$

$$= 0.8064$$

Q.25 In a distribution exactly normal 7 % of the items are under 35 and 89 % are under 63. Find the mean and standard deviation of the distribution.

☞ **[SPPU : Dec.-10]**

Ans. : From Fig. Q.25.1 it is clear that 7 % of items are under 35 means area under 35 is 0.07. Similarly area for $x \geq 63$ is 0.11.

$$P(x < 35) = 0.07, \quad P(x > 63) = 0.11$$

$x = 35$, $x = 63$ are located as shown in Fig. Q.25.1

When $x = 35$, $z = \frac{35 - \mu}{\sigma} = -z_1$ (say)

(Negative sign because $x = 35$ is to the left of $x = \mu$)

When $x = 63$, $z = \frac{63 - \mu}{\sigma} = z_2$ (say)

Area A_2 for $P(0 < z < z_2) = 0.39$

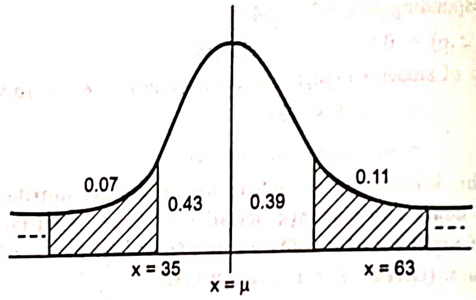


Fig. Q.25.1

$\therefore$ By Table 6.1, $z_2 = 1.23$

Area A_1 for $P(0 < z < z_1) = 0.43$

Corresponding $z_1 = 1.48$

Thus we get two simultaneous equation

$$\frac{35-\mu}{\sigma} = -1.48 \quad \text{... (Q.25.1)}$$

$$\frac{63-\mu}{\sigma} = 1.23 \quad \text{... (Q.25.2)}$$

Solving these two equations we get.

$\therefore \qquad \sigma = 10.33 \quad \mu = 50.3$ (approx)

Q.26 In an intelligence test administered to 1000 students the average score was 42 and standard deviation is 24. Find the number of students with score lying between 30 to 54. ☞ **[SPPU : May-15]**

Ans. : Here $n = 1000$, $\bar{x} = u = 42$, $\sigma = 24$

$$z = \frac{x-u}{\sigma} = \frac{x-42}{24}$$

When $x_1 = 30$, $z_1 = \frac{30-42}{24} = \frac{-12}{24} = -\frac{1}{2}$

When $x_2 = 54$, $z_2 = \frac{54-42}{24} = \frac{1}{2}$

Given that for $z = 0.5$, Area $= 0.1915$

$$\therefore P(30 < x < 15) = P\left(-\frac{1}{2} < z < \frac{1}{2}\right)$$

$$= A\left(\frac{1}{2}\right) + A\left(-\frac{1}{2}\right)$$

$$= 0.1915 + 0.1915 = \mathbf{0.3830}$$

$\therefore$ The number of students with score lying between 30 to 54 is

$= 0.3830 \times 1000 = 383$ students

Q.27 A random sample of 200 screws is drawn from a population which represents the size of screws. If a sample is distributed normally with a mean 3.15 cm of S.D. 0.025 cm. Find the expected number of screws whose size falls between 3.12 cm and 3.2 cm.

☞ **[SPPU : May - 05, Dec. - 14]**

Ans. : Given that $\mu = 3.15$, $\sigma = 0.025$

$$X_1 = 3.12 \text{ and } X_2 = 3.2$$

$$z_1 = \frac{X_1-\mu}{\sigma} = -1.2, \quad z_2 = \frac{X_2-\mu}{\sigma} = 2$$

$$\therefore P(3.12 < X < 3.2) = P(-1.2 < z < 2)$$

$$= P(0 < z < 1.2) + P(0 < z < 2)$$

$$= 0.3849 + 0.4772 = 0.8621$$

$\therefore$ Expected number of screws $= 200 \times 0.8621 \cong \mathbf{172}$

Q.27(a) In a sample of 1,000 cases, the mean of a certain examination is 14 and standard deviation is 2.5. Assuming the distribution to be normal. Find the number of students scoring between 12 and 15.

[Given : $Z_1 = 0.4$, $A_1 = 0.1554$, $Z_2 = 0.8$, $A_2 = 0.2881$]

☞ **[SPPU : May-19, Marks 4]**

Ans. : Here $n = 1000$, $\mu(\bar{x}) = 14$, $\sigma = 2.5$

$$z = \frac{x-\mu}{\sigma} = \frac{x-14}{2.5}$$

When $x = 12$ then $\quad z_2 = \frac{12-14}{2.5} = -0.8$

And when $x = 15$, then

$$z_1 = \frac{15-14}{2.5} = 0.4$$

Given that

For $z_1 = 0.4$, $A_1 = 0.1554$ and $z_2 = -0.8 \Rightarrow A = 0.2881$

The required probability $P(12<x<15)$

$$= A_1 + A_2 = 0.1554 + 0.2881 = 0.4433$$

Therefore, the required number of students $= 0.4433 \times 1000 = 443$

Q.27(b) Find the directional derivative of $\phi = xy^2 + yz^2 + zx^2$ at (1,1,1) along the line $2(x-2) = y+1 = z-1$.

[SPPU : May-19, Marks 4]

Ans. : We have $\phi = xy^2 + yz^2 + zx^2$

$$\nabla\phi = (y^2 + 2xz)\bar{i} + (2xy + z^2)\bar{j} + (2yz + x^2)\bar{k}$$

$$\nabla\phi_{(1,1,1)} = 3\bar{i} + 3\bar{j} + 3\bar{k}$$

Now, to find $\bar{u}$, compare given equation of line with

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

We have $2(x-2) = y+1 = z-1$

$$\frac{x-2}{1} = \frac{y+1}{2} = \frac{z-1}{2}$$

$\therefore$ $\alpha = x_2 - x_1 = 1$

$\beta = y_2 - y_1 = 2$

And, $\gamma = z_2 - z_1 = 2$

Therefore, $\bar{u} = \alpha\bar{i} + \beta\bar{j} + \gamma\bar{k} = \bar{i} + 2\bar{j} + 2\bar{k}$

$\therefore$ $$\hat{u} = \frac{\bar{u}}{|\bar{u}|} = \frac{1}{3}(\bar{i} + 2\bar{j} + 2\bar{k})$$

$$\text{D.D.} = (\nabla\phi)_P \cdot \hat{u} = \frac{1}{3}(3+6+6) = 5$$

Q.27(c) The height of a student in a school follows a normal distribution with mean 190 cm and variance 80 cm^2. Among the 1,000 students from the school, how many are expected to have height above 200 cm ? (Given : z = 1.118, A = 0.3686)

[SPPU : Dec.-22, Marks 5]

Ans. : Let, H = Height of students and $H \rightarrow n\,(190, 80)$

Proportion of students having height above 20 cm.

$$= P(h > 200) = P\left(\frac{h-\mu}{\sigma} > \frac{200-190}{\sqrt{80}}\right)$$

$$= P(Z > 1.1180)$$

$$= 0.5 - 3686$$

$$= 0.1314$$

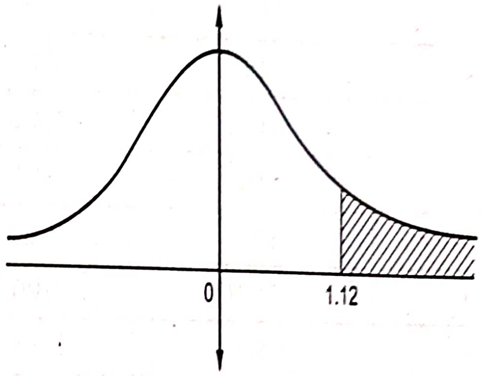


Fig. Q.27(c).1

$\therefore$ Number of students out of 1000 having height above 200 cm

= 1000 × Proportion of students having height above 200 cm

$$= 1000 \times 0.1314$$

$$= 1.31 \text{ Student}$$

6.8 : Chi-square Distribution

1) Level of significance :

Probability of rejecting the null hypothesis H_0 when it is true is called as level of significance. It is denoted by α. Thus it is the probability of committing error of Type I.

If we try to minimize level of significance, the probability of error of Type II increases.

So level of significance cannot be made zero. However we can fix it in advance as 0.01 or 0.05 i.e. (1 % or 5 %). In most of the cases it is 5!.

Degrees of freedom	Distribution of χ^2	
	5 %	1 %
1	3.841	6.635
2	5.991	9.210
3	7.815	11.345
4	9.488	13.277
5	11.070	15.086
6	15.592	16.812
7	15.067	18.475
8	15.507	20.090
9	16.919	21.666
10	18.307	23.209
11	19.675	24.725
12	21.026	26.217
13	22.362	27.668
14	23.685	29.141
15	24.996	30.578
16	26.296	32.000
17	27.587	33.409
18	28.869	34.191
19	30.144	36.191
20	31.410	37.566
21	32.671	38.932
22	33.924	40.289
23	35.172	41.638

24	36.415	42.980
25	37.652	44.314
26	38.885	45.642
27	40.113	46.963
28	41.337	48.278
29	42.557	49.588
30	43.773	50.892
40	55.759	63.691
60	79.082	88.379
∞	–	–

Table 6.2

2) Test for goodness of fit of χ^2 distribution :

Consider a frequency distribution, we try to fit some probability distribution.

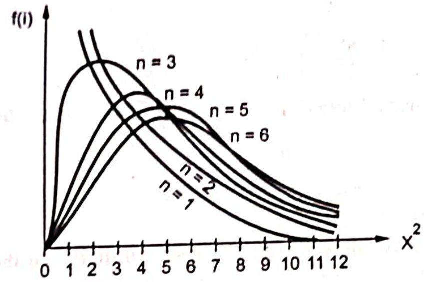


Fig. 6.12

Let H_0 : Fitting of the probability distribution to given data is proper.

As the test is based on χ^2 distribution. $\therefore$ known as χ^2 test of goodness of fit.

Suppose $O_1\ O_2\ \ldots.\ O_i\ \ldots.\ O_k$ be the observed frequencies and $e_1\ e_2\ \ldots\ e_i\ \ldots\ e_k$ be the expected frequencies or theoretical frequencies. There is no significant difference between observed and theoretical (expected) frequencies.

Let P = Number of parameters estimated for fitting the probability distribution.

$$N = \sum_{i=1}^{k} O_i = \sum_{i=k}^{k} e_i$$

If H_0 is true then the static

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - e_i)^2}{e_i} = \sum_{i=1}^{k} \frac{O_i^2 - 2e_i O_i + e_i^2}{e_i}$$

$$= \sum_{i=1}^{k} \frac{O_i^2}{e_i} - 2O_i + e_i$$

$$= \sum_{i=1}^{k} \frac{O_i^2}{e_i} - \sum_{i=1}^{k} 2O_i + \sum_{i=1}^{k} e_i$$

$$= \sum_{i=1}^{k} \frac{O_i^2}{e_i} - 2N + N$$

$$\chi^2 = \sum_{i=1}^{k} \frac{O_i^2}{e_i} - N$$

has Chi-square distribution with (k–p–1) degrees of freedom. If

$\chi^2_{k-p-1} \geq \chi^2_{k-p-1,\ \alpha}$

(calculated) (expected or table value)

Thus we reject H_0

Q.28 A set of five similar coins is tossed 210 times and the result is

No. of heads	*0*	*1*	*2*	*3*	*4*	*5*
Frequency	*2*	*5*	*20*	*60*	*100*	*31*

Test the hypothesis that the data follow a binomial distribution.

☞ [SPPU : May-05, Dec.-07]

Ans. : Let H_0 : The data follows binomial distribution

Here $N = 210$

p_i = Probability of getting head = $\frac{1}{2}$

q_i = Probability of getting tail = $\frac{1}{2}$

Now $p(r) = \beta(n, p, r) = {}^nC_r\ p^r\ q^{n-r}$

$$= {}^5C_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{5-r} = {}^5C_r \left(\frac{1}{2}\right)^5$$

$\therefore$

$$p(0) = {}^5C_0 \left(\frac{1}{2}\right)^5 = \frac{1}{32}$$

$$p(1) = {}^5C_1 \left(\frac{1}{2}\right)^5 = \frac{5}{32}$$

$$p(2) = {}^5C_2 \left(\frac{1}{2}\right)^5 = \frac{10}{32}$$

$$p(3) = {}^5C_3 \left(\frac{1}{2}\right)^5 = \frac{10}{32}$$

$$p(4) = {}^5C_4 \left(\frac{1}{2}\right)^5 = \frac{5}{32}$$

$$p(5) = {}^5C_5 \left(\frac{1}{2}\right)^5 = \frac{1}{32}$$

As expected frequency $E(x_i) = N \cdot p_i$ thus we prepare the table.

x_i	O_i	P_i	$e_i = N \cdot P_i$	$O_i - e_i$
0	2	1/32	7	– 5
1	5	5/32	35	– 30
2	20	10/32	70	– 50
3	60	10/32	70	– 10
4	100	5/32	35	65
5	37	1/32	7	30
	224			

$k = 6,\ p = 0$

$$\therefore \quad \chi^2_{k-p-1} = \sum_{i=1}^{k} \frac{(O_i - e_i)^2}{e_i}$$

$$\chi^2_5 = \frac{(-5)^2}{7} + \frac{(-30)^2}{35} + \frac{(-50)^2}{70} + \frac{(-10)^2}{70} + \frac{(+65)^2}{35} + \frac{(30)^2}{7}$$

$$= \frac{25}{7} + \frac{900}{35} + \frac{2500}{70} + \frac{100}{70} + \frac{4225}{35} + \frac{900}{7}$$

$$= 315.7142 \text{ (Calculated)}$$

$$\chi^2_{5,\,0.05} = 11.07 \text{ (Table value)}$$

$$\therefore \quad \chi^2_5 > \chi^2_{5,\,0.05}$$

$\therefore H_0$ is rejected.

Q.28(a) The proportions of blood types O, A, B and AB in the general population of a country are known to be in the ratio 49:38:9:4 respectively. A research team observed the frequencies of the blood types as 88, 80, 22 and 10 respectively in a community of that country. Test the hypothesis at 5 % level of significance that the proportions for this community are in accordance with the general population of that country. (Given : $\chi^2_{tab} = 7.815$)

[SPPU : June-22, Marks 5]

Ans. :

Blood Type	O	A	B	AB
Observed frequency (O_i)	44	40	11	5
Expected frequency (e_i)	49	38	9	4
$(O_i - e_i)^2$	25	4	4	1

$$K = 4,\ P = 0,\ \chi^2_{K-P-1} = \sum_{i=1}^{k} \frac{(O_i - e_i)^2}{e_i}$$

$$\chi^2_3 = \frac{25}{49} + \frac{4}{38} + \frac{4}{9} + \frac{1}{4} = 1.309$$

Table value $\chi^2_{3,0.05} = 7.815$

$$\chi^2_3 < \chi^2_{3,0.05}$$

$\therefore$ We accept this hypothesis.

Q.28(b) A pea cultivating experiment was performed. 219 round yellow peas, 81 round green peas, 61 wrinkled yellow peas and 31 wrinkled green peas were noted. Theory predicts that these phenotypes should be obtained in the ratios 9:3:3:1. Test the compatibility of the data with theory, using 5 % level of significance. 2 (Given : χ^2_{tab} =7.815)

[SPPU : Dec.-22, Marks 5]

Ans. : Given data

Round and yellow	Round and green	Wrinkled and yellow	Wrinkled and green	Total
219	81	61	31	392

From the given data the corresponding frequencies are

Expected frequencies (e_i)	$\frac{9}{16}\times 392 = 220.5$	$\frac{3}{16}\times 392 = 73.5$	$\frac{3}{16}\times 392 = 73.5$	$\frac{1}{16}\times 392 = 24.5$

$$X^2_{k-P-1} = X^2_3 = \sum \frac{(O_i - e_i)}{e_i}$$

$$X^2_3 = \frac{(219-220.5)^2}{220.5} + \frac{(81-73.5)^2}{73.5} + \frac{(61-73.5)^2}{73.5} + \frac{(31-24.5)^2}{24.5}$$

$$= 0.10204 + 0.765306 + 2.12585 + 1.7244898$$

$$= 4.6258498 < 7.815$$

$$X^2_{tab} = 7.815$$

The calculated value of X^2 is less than $X^2_{tab} = 7.815$, So accept H_0 at 5 % level of significance.

Conclusion : There is a height degree of agreement between the theory and experiment.

6.9 : Student's T-Distribution

It is used when sample size is ≤ 30 and the population standard deviation is unknown.

Student's t-distribution is defined as

$$t_{n-1} = \frac{\bar{x}-\mu_0}{\frac{s}{\sqrt{n}}} \rightarrow \text{t-distribution with } n-1 \text{ degrees of freedom}$$

$$\text{where } s = \left(\frac{\sum (x-\bar{x})^2}{n-1}\right)^{\frac{1}{2}}$$

Q.29 A random sample of size 16 has 53 as mean. The sum of squares of the deviation from mean is 135. Can this sample be regarded as taken from the population having 56 as mean ?

Ans. : Let H_0 : there is no significant difference between the sample mean and the hypothetical mean

$$H_0 : \mu = 56, H_1 = \mu \neq 56$$

$$t_{n-1} = \frac{\bar{x}-\mu}{\frac{s}{\sqrt{n}}}$$

Given that $\bar{x} = 53$, $u = 56$, $n = 16$

$$\sum (x-\bar{x})^2 = 135$$

$$s = \sqrt{\frac{\sum (x-\bar{x})^2}{n-1}} = \sqrt{\frac{135}{15}} = 3$$

$$\therefore \quad t_{n-1} = \frac{53-56}{3/\sqrt{16}} = \frac{-3\times 4}{3} = -4$$

$\therefore$ $|t| = 4$ degree of freedom $= 16 - 1 = 15$

Table value of t is $t_{0.05} = 1.753$.

$\therefore$ $|t| = 4 > t_{0.05} = 1.753$, the calculated value of t is more than the table value of t so the hypothesis is rejected.

Hence the sample mean has not come from a population having 56 as mean.

Q.30 Suppose that sweets are sold in packages of fixed weight of the contents the producer of the packages is interested in testing that average weight of contents in package is 1 kg. Hence a random sample of 12 packages is drawn and their contents found in kg as follows :

1.05, 1.01, 1.04, 0.98, 0.96, 1.01, 0.97, 0.99, 0.98 0.95, 0.97, 0.95

Using the above data what should he conclude about the average weight of contents in the packets ?

Ans. : Let μ denotes the average weight of contents in packages. Producer wants to test $H_0 : \mu = \mu_0 = 1$ against $H_1 : \mu \neq 1$

By t-test

$$t_{n-1} = \frac{\bar{x}-\mu_0}{\frac{s}{\sqrt{n}}}$$

$$\text{where } \bar{x} = \text{Sample mean} = \frac{\sum x_i}{n}$$

$$s^2 = \frac{\sum x_i^2 - n\bar{x}^2}{n-1}, \quad \bar{x} = 0.9883$$

$$s^2 = \frac{11.7376 - 11.7208}{11} = 0.001164$$

$$s = 0.034112$$

$$t_{11} = \frac{(0.9883-1)\sqrt{12}}{0.034112} = -1.188145$$

$$|t_{11}| = 1.188145$$

By Table 6.1 $t_{11, 0.05}$, $= 2.201$

Calculated value < table value

$\therefore$ We accept H_0 at 5 % level of significance. Hence the producer should conclude that the average weight of contents of package may be taken as 1 kg.

... END

UNIT V

7 Numerical Methods

Q.1 Explain bisection method.

Ans. :

1) If $f(x)$ is continuous in the interval (a, b) and f(a) and f(b) have different signs then the equation $f(x) = 0$ has at least one root between $x = a$ to $x = b$

2) Bisection method : This method is based on the repeated application of intermediate value property.

Let $f(x)$ be continuous in the interval [a, b] such that f(a) and f(b) have opposite signs. Without loss of generality, assume that $f(a) < 0$ and $f(b) > 0$,

then the first approximation to the root is

$$x_1 = \frac{1}{2}(a+b).$$

Now there are three possibilities :

i) If $f(x_1) = 0$, then x_1 is the root of $f(x) = 0$

ii) If $f(x_1) < 0$, then root lies between x_1 and b.

iii) If $f(x_1) > 0$, then root lies between a and x_1.

Suppose $f(x_1) > 0$ then root lies in the interval $[a, x_1]$

$\therefore$ The second approximation to the root is $x_2 = \frac{1}{2}(a+x_1)$.

Again there are three possibilities.

i) If $f(x_2) = 0$, then x_2 is the root of $f(x) = 0$

ii) If $f(x_2) < 0$, then root lies between x_2 and x_1.

iii) If $f(x_2) > 0$ then root lies between a and x_2.

Assume that $f(x_2) > 0$, then the root lies in the interval $[a, x_2]$

$\therefore$ The third approximation to the root is

$$x_3 = \frac{1}{2}(a+x_2) \text{ and so on.}$$

Repeat this procedure till we reached to required root.

Q.2 Find a real root of the equation $x^3 - x = 1$ between 1 and 2 by bisection method. Compute give iterations.

Ans. : Given : $x^3 - x - 1 = 0$

Let $f(x) = x^3 - x - 1 = 0$

$f(1) = 1^3 - 1 - 1 = -1\ (-ve)$

$f(2) = 2^3 - 2 - 1 = 5\ (+ve)$

$\because$ $f(1) < 0$ and $f(2)$ 0,

$\therefore$ By intermediate value theorem root lies between 1 and 2.

$$\therefore \quad x_1 = \frac{1+2}{2} = \frac{3}{2} = 1.5$$

$$f(1.5) = (1.5)^3 - 1.5 - 1 = 0.875 > 0\ (+ve)$$

Hence, root lies between 1 and 1.5.

$$x_2 = \frac{1+1.5}{2} = 1.25$$

$$f(1.25) = -0.2968 < 0\ (-ve)$$

$\therefore$ Root lies between 1.25 and 1.5

$$x_3 = \frac{1.25+1.5}{2} = 1.375$$

$$f(1.375) = 0.2246 > 0\ (+ve)$$

$\therefore$ Root lies between 1.25 and 1.375

$$x_4 = \frac{1.25+1.375}{2} = 1.3125$$

$$\therefore \quad f(1.3125) = -0.0515 < 0\ (-ve)$$

$\therefore$ Root lies between 1.3125 and 1.375.

$$x_5 = \frac{1.3125+1.375}{2} = 1.3437$$

After 5th iteration the required root is 1.3437.

Q.3 Find the real root of equation $x\log_{10} x = 1.2$ by bisection method correct to four decimal places.

Ans. :

Let $f(x) = x\log_{10}x - 1.2 = 0$

$f(2) = -0.59794$

$f(3) = 0.23136$

Hence a root lies between 2 and 3.

∴ First approximation to the root is

$$x_1 = \frac{1}{2}(2+3) = 2.5$$

∴ $f(x_1) = f(2.5) = -0.205$ (–)ve

Hence root lies between 2.5 and 3.

Second approximation to the root is

$$x_2 = \frac{1}{2}[2.5+3]$$

$$x_2 = 2.75$$

Now $f(x_2) = 0.008$ (+)ve

Hence root lies between 2.5 and 2.75.

∴ Third approximation to the root is

$$x_3 = \frac{1}{2}(2.5+2.75)$$

$$x_3 = 2.625$$

Now $f(x_3) = -0.099$ (– ve).

Hence root lies between 2.625 and 2.75.

∴ Fourth approximation to the root is :

$$x_4 = \frac{1}{2}(2.625+2.75)$$

$$x_4 = 2.6875$$

Now $f(x_4) = f(2.6875) = -0.046$ (–)ve

Hence root lies between 2.6875 and 2.75.

∴ Fifth approximation :

$$x_5 = \frac{1}{2}(2.6875+2.75) = 2.71875$$

$f(x_5) = f(2.71875) = -0.019$ (–ve)

Hence root lies between 2.71875 and 2.75

∴ Sixth approximation :

$$x_6 = \frac{1}{2}(2.71875+2.75)$$

$$x_6 = 2.734375$$

$f(x_6) = f(2.734375) = -0.00546$ (–)ve

Hence roots lies between 2.73475 and 2.75

∴ Seventh approximation :

$$x_7 = \frac{1}{2}(2.73475+2.75)$$

$$x_7 = 2.74218175$$

$f(x_7) = f(2.74218175) = 0.0013$ (+)ve

Hence root lies between 2.734375 and 2.7421875

∴ Eighth approximation :

$$x_8 = \frac{1}{2}(2.734375+2.7421875)$$

$$x_8 = 2.73828125$$

$f(x_8) = f(2.73828125) = -0.002$ (–ve)

Hence root lies between 2.73828125 and 2.7421875.

∴ Ninth approximation :

$$x_9 = \frac{1}{2}(2.73828125+2.7421875)$$

$$x_9 = 2.740234$$

$f(x_9) = f(2.740234) = -0.00035$ (–)ve

Hence root lies between 2.7402 and 2.7421

∴ Tenth approximation :

$$x_{10} = \frac{1}{2}(2.7402+2.7421)$$

$$x_{10} = 2.74115$$

$$f(x_{10}) = f(2.74115) = -0.00043 \text{ (+)ve}$$

Hence the root lies between 2.7402 and 2.74115.

∴ Eleventh approximation :

$$x_{11} = \frac{1}{2}(2.7402 + 2.74115)$$

$$x_{11} = 2.740675$$

$$f(x_{11}) = f(2.740675) = 0.000025 \text{ (+)ve}$$

Hence root lies between 2.7402 and 2.740675.

∴ Twelfth approximation :

$$x_{12} = \frac{1}{2}(2.7402 + 2.740675)$$

$$x_{12} = 2.7404$$

$$f(x_{12}) = f(2.7404) = -0.00021 \text{ (−)ve}$$

Hence root lies between 2.7404 and 2.70675.

∴ Thirteen approximation :

$$x_{13} = \frac{1}{2}(2.7404 + 2.70675)$$

$$x_{13} = 2.7405$$

$$f(x_{13}) = f(2.7405) = -0.00012 \text{ (−)ve}$$

Hence root lies between 2.7405 and 2.740675.

∴ Fourteenth approximation :

$$x_{14} = \frac{1}{2}(2.7405 + 2.740675)$$

$$x_{14} = 2.74058$$

$$f(x_{14}) = f(2.74058) = -0.00005 \text{ (−)ve}$$

Hence root lies between 2.74058 and 2.740675

∴ Fifteenth approximation :

$$x_{15} = \frac{1}{2}(2.74058 + 2.740675)$$

$$x_{15} = 2.7406$$

Since x_{14} and x_{15} are same upto four decimal places.

Hence the approximate real root is 2.7406.

Q.4 Find root of equation $x^3 - 4x - 9 = 0$ using bisection method correct to 3 decimal places.

Ans. :

$$f(x) = x^3 - 4x - 9 = 0$$

$$f(0) = -9$$

$$f(1) = -12$$

$$f(2) = -9 \text{ (−ve)}$$

$$f(3) = 6 \text{ (+ve)}$$

Root lies between 2 and 3.

So, $a = 2, b = 3$

1st approximation :

$$x_1 = \frac{2+3}{2} = 2.5$$

$$f(2.5) = -3.375 \text{ (−ve)}$$

Root lies between 2.5 and 3.

2nd approximation :

$$x_2 = \frac{2.5+3}{2} = 2.75$$

$$f(2.75) = 0.79 \text{ (+ve)}$$

Root lies between 2.5 and 2.75.

3rd approximation :

$$x_3 = \frac{2.5+2.75}{2} = 2.625$$

$$f(2.625) = -1.412 \text{ (−ve)}$$

Root lies between 2.625 and 2.75.

4th approximation :

$$x_4 = \frac{2.75+2.625}{2} = 2.6875$$

$$f(2.6875) = -0.339 \text{ (−ve)}$$

Root lies between 2.75 and 2.6875.

5^{th} approximation :

$$x_5 = \frac{2.75+2.6875}{2} = 2.7185$$

$f(2.7185) = 0.22$ (+ve)

Root lies between 2.6875 and 2.7185.

6^{th} approximation :

$$x_6 = \frac{2.6875+2.7185}{2} = 2.703$$

$f(2.703) = -0.063$

Root lies between 2.703 and 2.7185.

7^{th} approximation :

$$x_7 = \frac{2.703+2.7185}{2} = 2.7094$$

$f(2.7094) = 0.051$ (+ve)

Root lies between 2.703 and 2.7094.

8^{st} approximation :

$$x_8 = \frac{2.703+2.7094}{2} = 2.7062$$

$f(2.7062) = -5.89\times10^{-3}$ (–ve)

Root lies between 2.7062 and 2.7094.

9^{th} approximation :

$$x_9 = \frac{2.7062+2.7094}{2} = 2.7078$$

$f(2.7078) = 0.022$ (+ve)

Root lies between 2.7062 and 2.7078.

10^{th} approximation :

$$x_{10} = \frac{2.7062+2.7078}{2} = 2.707$$

$f(2.707) = 4.48\times10^{-3}$ (+ve)

Root lies between 2.7062 and 2.707.

11^{th} approximation :

$$x_{11} = \frac{2.7062+2.707}{2} = 2.7066$$

Q.4(a) Find the root of the equation $x^4+2x^3-x-1=0$, lying in the interval [0, 1] using the bisection method at the end of fifth iteration.

[SPPU : June-22, Marks 5]

Ans. : Given that $f(x) = x^4+2x^3-x-1 = 0$

Root lies between 0 and 1, $x_1 = \frac{0+1}{2} = \frac{1}{2} = 0.5$

$$f(0.5) = (0.5)^4+2(0.5)^3-(0.5)-1 = -1.1875 < 0$$

and $\quad f(1) = 1+2-1-1 = 1 > 1$

$\therefore$ Root lies between 0.5 and 1

$\therefore \quad x_2 = \frac{0.5+1}{2} = 0.75,\ f(0.75) = -0.589 < 0$

$\therefore$ Root lies between 0.75 and 1

$$x_3 = \frac{0.75+1}{2} = \frac{1.75}{2} = 0.875$$

$\therefore \quad f(0.875) = 0.0570 > 1$

$\therefore$ Root lies between 0.75 and 0.875

$\therefore \quad x_4 = \frac{0.75+0.875}{2} = \frac{1.625}{2} = 0.8125$

$f(0.8125) = -0.3039 < 0$

$\therefore$ Root lies between 0.8125 and 0.875

$$x_5 = \frac{0.8125+0.875}{2} = 0.84375$$

$f(0.84375) = -0.1355 < 0$

$\therefore$ Root lies between 0.84375 and 0.875

$$x_6 = 0.859375$$

Q.5 Explain Regula Falsi method.

Ans. : The bisection method guarantees that the iterative process will converge. It is however slow. Thus attempt have been made to speed up bisection method retaining its guaranteed convergence. A method of doing this is called the method of false position.

It is sometimes known as method of linear interpolation.

This is oldest method for finding the real root of numerical equation and closely resembles the bisection method. In this method.

Let $\quad f(x) = 0 \quad$ ...(Q.5.1)

Let $y = f(x)$ be represented by the curve AB cuts the X - axis at P.

The real root of (1) is OP.

The false position of curve AB is taken as the chord AB.

The chord AB cuts the X-axis at Q. The approximates root of $f(x) = 0$ is OQ. By this method, we find OQ.

Let A[a, f(a)], B[b, f(b)] be the extremities of the chord AB. The equation of the chord AB is

$$y - f(a) = \frac{f(b)-f(a)}{b-a}(x-a)$$

To find OQ, put $y = 0$

$$0 - f(a) = \frac{f(b)+f(a)}{b-a}(x-a)$$

$$(x - a) = -\frac{(b-a)f(a)}{f(b)-f(a)}$$

$$x = a + \frac{(a-b)f(a)}{f(b)-f(a)}$$

$$\boxed{x = \frac{a\,f(b)-b\,f(a)}{f(b)-f(a)}}$$

We continue the literation till the root is found to the desired accuracy. This method is also known as method of linear interpolation.

Q.6 Find the real root of the equation $x\log_{10}x-1.2 = 0$ by the method of false position method correct to four decimal places.

Ans. : Here $f(x) = x\log_{10}x-1.2 = 0$

$$f(2) = 2\log_{10}2-1.2 = -0.59794 < 0$$

$$f(3) = 3\log_{10}3-1.2 = 0.23136 > 0$$

One root lies between 2 and 3.

Taking $a = 2$, $b = 3$

$$f(2) = -0.59794, \quad f(3) = 0.23136$$

By method of false position; we have

$$x_1 = \frac{a\,f(b)-b\,f(a)}{f(b)-f(a)}$$

$$= \frac{2\,f(3)-3\,f(2)}{f(3)-f(2)}$$

$$= \frac{2\,(0.23136)-3(-0.59794)}{0.23136-(-0.59794)}$$

$$x_1 = 2.72102$$

$$f(2.72102) = 2.72102\log_{10}2.72102-1.2$$

$$= -0.01709 < 0$$

Since f(2.72102) and f(3) are of opposite sign. So the root lies between 2.72102 and 3.

$$x_2 = \frac{2.72102\,f(3)-3\,f(2.72102)}{f(3)-f(2.72102)}$$

$$= \frac{2.72102\,(0.23136)-3(-0.01709)}{0.23136-(-0.01709)}$$

$$x_2 = 2.74021$$

Now $f(2.74021) = -0.00038$

Since f(2.74021) and f(3) are of opposite sign. So the root lies between 2.74021 and 3.

$$x_3 = \frac{2.74021\,f(3)-3\,f(2.74021)}{f(3)-f(2.74021)}$$

$$x_3 = \frac{2.74021(0.23136)-3(-0.00038)}{0.23136-(-0.00038)}$$

$$x_3 = 2.74064$$

Again $\quad f(x_3) = 2.74064\log_{10}2.74064-1.2$

$$f(3) = -0.00001$$

Root lies between 2.74064 and 3.

$$x_4 = \frac{2.74064\,f(3)-3\,f(2.74064)}{f(3)-f(2.74064)}$$

$$= \frac{2.74064(0.23136)-3(-0.00001)}{0.23136-(-0.00001)}$$

$$x_4 = 2.74065$$

Here the root correct to four decimal place is 2.7406.

Q.7 Find the root of the equation $x^3 - 5x - 7 = 0$ which lies between 2 and 3 by method of false position.

Ans. : Here we have

$$f(x) = x^3 - 5x - 7 = 0$$

$$f(2) = 8 - 10 - 7 = -9$$

$$f(3) = 27 - 15 - 7 = +5$$

As f(2) and f(3) are of opposite sign so the root lies between 2 and 3

First Iteration :

$$x_1 = \frac{af(b) - bf(a)}{f(b) - af(a)}$$

Here $a = 2,\ b = 3$

$$f(2) = -9,\ f(3) = 5$$

$$x_1 = \frac{2f(3) - 3f(2)}{f(3) - f(2)}$$

$$= \frac{2(5) - 3(-9)}{5 - (-9)} = \frac{37}{14} = 2.6429$$

$$x_1 = 2.6429$$

Now $f(2.6429) = (2.6429)^3 - 5(2.6429) - 7$

$$= -1.7541$$

Again f(2.6429) and f(3) are of opposite sign. Root lies between 2.6429 and 3.

$$a = 2.6429,\ b = 3$$

Second Iteration :

$$x_2 = \frac{2.6429f(3) - 3f(2.6429)}{f(3) - f(2)}$$

$$= \frac{2.6429(5) - 3(-1.7541)}{5 - (-1.7541)}$$

$$= \frac{18.4768}{6.7541} = 2.7356$$

Now $f(2.7356) = (2.7356)^3 - 5(2.7356) - 7$

$$= -0.2061$$

Root lies between 2.7356 and 3

Third Iteration :

$$a = 2.7356,\ b = 3$$

$$x_3 = \frac{2.7356(f)3 - 3f(2.7356)}{f(3) - f(2.7356)}$$

$$= \frac{2.7356(5) - 3(-0.2061)}{5 - (-0.2061)}$$

$$= \frac{14.2963}{5.2061} = 2.7461$$

Now $f(2.7461) = (2.7461)^3 - 5(2.7461) - 7$

$$= -0.02198$$

Roots lies between 2.7461 and 3

Four Iteration :

$$x_4 = \frac{2.7461f(3) - 3f(2.7461)}{f(3) - f(2.7461)}$$

$$= \frac{2.7461(5) - 3(-0.02198)}{5 - (-0.02198)}$$

$$= \frac{13.79644}{5.02198} = 2.7472$$

Hence root of given equation is 2.7472

Q.8 Explain the Secant method.

Ans. : The secant method requires two initial approximations x_0 and x_1, preferably both reasonably close to the solution x^*. From x_0 and x_1, we can determine that the points $(x_0, y_0) = f(x_0)$ and $(x_1, y_1 = f(x_1))$ both lie on the graph of f. Connecting these points given the (secant) line.

$$y - y_1 = \frac{y_1 - y_0}{x_1 - x_0}(x - x_1)$$

Since, we want $f(x) = 0$, we set $y = 0$, solve for x and use that as our next approximation. Repeating this process gives us the iteration.

$$x_{i+1} = x_i - \frac{x_i - x_{i-1}}{y_i - y_{i-1}} y_i \quad \text{...(Q.8.1)}$$

with $y_i = f(x_i)$. Below Fig. Q.8.1 is for an illustration.

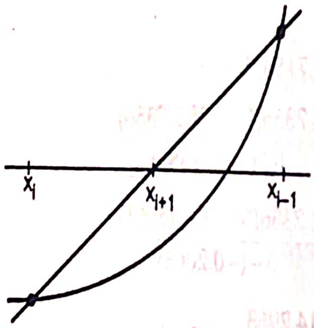


Fig. Q.8.1 The secant method in the case when root is bracketed

- For example suppose $f(x) = x^4 - 5$, which has a solution $x^* = \sqrt[4]{5} \approx 1.5$. Choose $x_0 = 1$ and $x_1 = 2$ as initial approximation, we have, $y_0 = f(1) = -4$ and $y_1 = f(2) = 11$. We may then calculate x_2 from the formulae.

$$x_2 = 2 - \frac{2-1}{11-(-4)} 11 = \frac{19}{15} \approx 1.2666$$

Plugging $x_2 = \frac{19}{15}$ into f(x) we obtain $y_2 = f\left(\frac{19}{15}\right) \approx -2.425$.

In the next step we would use $x_1 = 2$ and $x_2 = \frac{19}{15}$ in the formula (Q.8.1) find x_3 and so on.

Q.8 a) Find a real root of the equation $x^3 - 2x - 5 = 0$ by the method of false position at the end of fifth iteration.

[SPPU : Dec.-22, Marks 5]

Ans. : Suppose $f(x) = x^3 - 2x - 5 = 0$

$\because \quad f(0) = -5, f(1) = -6, f(2) = -1, f(3) = 16$

So that Root lies between 2 and 3.

Taking $x_0 = 2, f_0 = -1$ and $x_1 = 3, f_1 = 16$ by method formulae, we have

$$x_2 = x_1 - \frac{(x_1 - x_0)}{(f_1 - f_0)} \times f_1$$

$$= 3 - \frac{(3-2)}{16-(-1)} \times 16 = 2.0588$$

$$f(x_2) = -0.3908$$

$\because \; f_2 \times f_1 < 0$ so

$x_2 = 2.0588$, $f_2 = -0.3908$ and suppose $x_1 = 3$, $f_1 = 16$

$$x_3 = x_2 - \frac{(x_2 - x_1)}{(f_2 - f_1)} \times f_2$$

$$= 2.0588 - \frac{(2.0588-3)}{(-0.3908-16)} \times (-0.3908)$$

$$= 2.0812$$

$$f(x_3) = -0.1479$$

Now again $\because \; f_3 \times f_1 < 0$

$$x_3 = 2.0812, \; f_3 = -0.1479$$

Let $x_2 = 3$ so that $f_2 = 16$

$$x_4 = x_3 - \frac{(x_3 - x_2)}{(f_1 - f_2)} \times f_3$$

$$= 2.0812 - \frac{(2.0812-3)}{(-0.1479-16)} \times (-0.1479)$$

$$= 2.0896$$

$$f(x_4) = -0.0511$$

Now again we have as $f_4 \times f_1 < 0$

$$x_4 = 2.0896, \; f_4 = -0.0577$$

and Let $x_3 = 3, \; f_3 = 16$

$$x_5 = x_4 - \frac{(x_4 - x_3)}{(f_4 - f_3)} \times f_4$$

$$= 2.0896 - \frac{(2.0896-3)}{(-0.0511-16)} \times (-0.0511)$$

$$= 2.0925$$

$$f(x_5) = -0.0229$$

Hence the root at the end of 5^{th} iteration is 2.0925.

Q.9 Explain Newton Raphson method.

Ans. : • The solution to an equation f(x) = 0 may often be found by a simple procedure called Newton-Raphson method.

- Let $x = x_0$ be the initial or starting value of the root. This method is generally used to improve the result obtained by the previous method.
- This method consists of refacing the part of the curve between points $[x_0, f(x_0)]$ and the x-axis by means of the tangent to the curve at the point and is described graphically in Fig. Q.9.1.

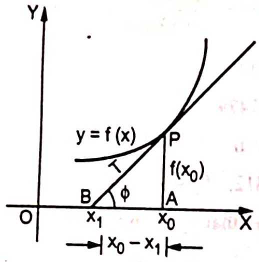


Fig. Q.9.1

- The interrupt OT on the x-axis of the tangent to the curve at the point P is taken as the first approximation. From the figure,

$$\tan\phi = \frac{AD}{BA}$$

$$\tan\phi = \frac{f(x_0)}{x_0 - x_1}$$

But $\tan\phi = \frac{dy}{dx} = f'(x_0)$ $\quad\therefore y = f(x)$

This Gives $f'(x_0) = \frac{f(x_0)}{x_0 - x_1}$

$$(x_0 - x_1)f'(x_0) = f(x_0)$$

$$x_0 - x_1 = \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Repeating the process replacing x_0 by x_1 we get second iteration as.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \text{ and so on}$$

In general, after (n + 1) iterations, we get

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

where $\quad n = 0, 1, 2, 3 \ldots$

Q.9(a) Obtain the root of the equation $x^3 - 4x - 9 = 0$ that lies between 2 and 3 by Newton-Raphson method correct to four decimal places. ☞ **[SPPU : Dec.-22, Marks 5]**

Ans. :

Let $\quad f(x) = x^3 - 4x - 9$

Since $\quad f(2) = -9$ and $f(3) = 6$

$\therefore$ A root lies between 2 and 3

Consider

$$f'(x) = 3x^2 - 4$$

$$f''(x) = 6x$$

$$f''(3) = 18$$

$\because$ f(3) = 6 and $f''(3) = 18$ are same in sign, we choose $x_0 = 3$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{f(3)}{f'(3)} = 3 - \frac{6}{23} = 2.739$$

$$f(x_1) = f(2.739) = (2.739)^3 - 4(2.739) - 9$$

$$= 0.59231$$

$$f'(x_1) = f'(2.739) = 3(2.739)^2 - 4$$

$$= 18.5064$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.739 - \frac{0.59231}{18.5064}$$

$$= 2.707$$

$$f(x_2) = f(2.707) = (2.707)^3 - 4(2.707) - 9$$

$$= 0.008487$$

$$f'(x_2) = f'(2.707) = 3(2.707)^2 - 4 = 17.983547$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.707 - \frac{0.008487}{17.983547}$$

$$= 2.706528$$

Root correct to fourth decimal place = 2.7065

Q.10 Find a real root of equation $x^4 - x - 10 = 0$ correct to three decimal place by using Newton Raphson method.

Ans. : $f(x) = x^4 - x - 10 = 0 \quad f(0) = -10$

$$\left.\begin{array}{l} f(1) = -10 \ (-ve) \\ f(2) = 4 \ (+ve) \end{array}\right\} \text{Root lies between 1 and 2}$$

$$x_0 = \frac{1+2}{2} = 1.5$$

1^{st} approximation,

(n = 0)

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 1.5 - \left[\frac{(1.5)^4 - 1.5 - 10}{4(1.5)^3 - 1}\right] = 2.015$$

Note : Continue finding root until value upto 3 decimal will be equal.

2^{nd} approximation

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 2.015 - \left[\frac{(12.015)^4 - 2.015 - 10}{4(2.015)^3 - 1}\right]$$

$$x_2 = 1.87409$$

3^{rd} approximation

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = 1.87409 - \left[\frac{(1.87409)^4 - 1.87409 - 10}{4(1.87409)^3 - 1}\right]$$

$$x_3 = 1.85586$$

4^{th} approximation

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$x_4 = 1.85586 - \left[\frac{(1.85586)^4 - 1.85586 - 10}{4(1.85586)^3 - 1}\right]$$

$$x_4 = 1.85586$$

3^{rd} and 4^{th} approximation are same.

Hence, one root is 1.8558.

Q.10(a) Find a real root of the equation $x^3 + 2x - 5 = 0$ by applying Newton-Raphson method at the end of fifth iteration.

[SPPU : June-22, Marks 5]

Ans. : $f(x) = x^3 + 2x - 5$

$$f'(x) = 3x^2 + 2$$

$$f(1.5) = 1.4 > 0 \text{ and } f(1) = -2 < 0$$

$\therefore$ Take $x_0 = 1.3$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.3 - \frac{f(1.3)}{f'(1.3)} = 1.3 - \frac{(-0.2)}{7.1} = 1.328$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.328 - \frac{-0.0019}{7.3} = 1.328 + 0.000268$$

$$x_2 = 1.328268$$

This is correct upto 5 decimal places.

Q.11 Apply Newton Raphson method to solve $3x - \cos x - 1 = 0$ correct to three decimal place.

Ans. : $f(x) = 3x - \cos x - 1 = 0$

$$f(0) = -2 \ (-\text{ve})$$

$$f(1) = 1.459 \ (+\text{ve}) \Rightarrow x_0 = 0.5$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 0.5 - \frac{3(0.5) - \cos(0.5) - 1}{3 + \sin(0.5)}$$

$$x_1 = 0.6085$$

$$x_2 = 0.6085 - \frac{3(0.6085) - \cos(0.6085) - 1}{3 + \sin(0.6085)}$$

$$x_2 = 0.6071$$

$$x_3 = 0.6071 - \frac{3(0.6071) - \cos(0.6071) - 1}{3 + \sin(0.6071)}$$

$$x_3 = 0.6071$$

2^{nd} and 3^{rd} approximation is same.

Hence, one root is 0.6071.

Q.12 Using Newton Raphson method find real root of $x \log_{10} x = 1.2 = 0$ correct to five decimal places.

Ans. :

$$f(x) = x \log_{10} x - 1.2 = 0$$

$$= x(0.4343)\log_e x - 1.2$$

$$f(2) = -0.597940 \; (-\text{ve})$$

$$f(3) = 0.2313 \; (+\text{ve})$$

Root lies between 2 and 3.

$$f(x) = x \log_{10} x - 1.2 = 0$$

$$= x\left(\frac{\log_e 10}{\log e}\right) - 1.2$$

$$f(x) = x[0.4343 \log_e x] - 1.2$$

$$f'(x) = 0.4343\left[x \cdot \frac{1}{x} + \log_e x\right] - 0$$

$$f'(x) = 0.4343 + 0.4343 \log_e x$$

$$= 0.4343(1 + \log_e x)$$

$$x_0 = \frac{2+3}{2} = 2.5$$

1^{st} approximation,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2.5 - \frac{[(2.5)\log_{10}(2.5) - 1.2]}{[0.4343(1 + \log_e 25)]}$$

$$x_1 = 2.74650$$

2^{nd} approximation,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.74650 - \left(\frac{(274650)\log_{10} 2.74650 - 1.2}{0.4343(1 + \log_e 2.74650)}\right)$$

$$x_2 = 2.740649$$

3^{rd} approximation,

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 2.740649 - \left(\frac{(2.740649)\log_{10} 2.740649 - 1.2}{0.4343(1 + \log_e 2.740649)}\right)$$

$$x_3 = 2.74064$$

Root locus 2.74064.

Q.13 Evaluate $\sqrt{12}$ to four decimal place by Newton - Raphson method.

Ans. : Let

$$x = \sqrt{12}$$

$$x^2 = 12$$

$$f(x) = x^2 - 12 = 0$$

$$f(3) = -3$$

$$f(4) = 4$$

Root lies between + 3 and 4

$$x_0 = \frac{3+4}{2} = 3.5$$

1^{st} approximation,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3.5 - \left(\frac{(3.5)^2 - 12}{2(3.5)}\right)$$

$$x_1 = 3.46429$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 3.46429 - \left(\frac{(3.464290)^2 - 12}{2(3.46429)}\right)$$

$$= 3.46410$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 3.46410 - \left(\frac{(3.46410)^2 - 12}{2(3.46410)}\right)$$

$$= 3.4640$$

Q.13(a) Solve $2x - \cos x - 3 = 0$ by using the method of successive approximations correct of three decimal places.

[SPPU : Dec.-22, Marks 5]

Ans. : Let $f(x) = 2x - \cos x - 3 = 0$

Given equation can be written as

$$x = \frac{\cos x + 3}{2}$$

Hence successive approximation method can be applied.

Let $x_0 = \frac{\pi}{2}$

$$\therefore \quad x_1 = \frac{1}{2}\left(\cos\frac{\pi}{2} + 3\right) = 1.5$$

$$x_2 = \frac{1}{2}(\cos 1.5 + 3) = 1.535$$

Similarly,

$$x_3 = \frac{1}{2}(\cos 1.535 + 3) = 1.518$$

$$x_4 = \frac{1}{2}(\cos 1.518 + 3) = 1.526$$

$$x_5 = \frac{1}{2}(\cos 1.526 + 3) = 1.522$$

$$x_6 = \frac{1}{2}(\cos 1.522 + 3) = 1.524$$

$$x_7 = \frac{1}{2}(\cos 1.524 + 3) = 1.523$$

$$x_8 = \frac{1}{2}(\cos 1.523 + 3) = 1.524$$

$\therefore$ The required $x_0 t$ is 1.524

Q.14 Explain Gauss elimination method.

Ans. : This is the elementary elimination method and it reduces the given system of linear equations to an equivalent upper triangular system of linear equations which is then solved by backward substitution. To describe the method, we consider the system of three equations in three unknowns for the sake of clarity and simplicity.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \quad \text{...(Q.14.1)}$$

The **Gauss elimination** method consists of the following steps.

Step 1 : Elimination of x_1 : Assuming $a_{11} \neq 0$ we divide coefficients of the first equation of system (1) by a_{11}, we get

$$x_1 + a'_{12}x_2 + a'_{13}x_3 = b'_1 \quad \text{... (Q.14.2)}$$

Where $a'_{12} = \frac{a_{12}}{a_{11}}, a'_{13} = \frac{a_{13}}{a_{11}}$ and $b'_1 = \frac{b_1}{a_{11}}$

Using equation (Q.14.2), we now eliminate x_1 from the remaining equations [i.e. From second and third equations of (Q.14.1)] by subtracting a_{21} times equation (Q.14.2) from second equation of (Q.14.1), and a_{31} times equation (Q.14.2) from third equation of (Q.14.1). We thus get system consisting of two equations in two unknowns, namely x_2 and x_3 as,

$$\begin{aligned} a'_{22}x_2 + a'_{23}x_3 &= b'_2 \\ a'_{32}x_2 + a'_{33}x_3 &= b'_3 \end{aligned} \quad \text{... (Q.14.3)}$$

where $a'_{22} = a_{22} - \frac{a_{21}}{a_{11}}a_{12}, \; a'_{23} = a_{23} - \frac{a_{21}}{a_{11}}a_{13},$

$b'_2 = b_2 - \frac{a_{21}}{a_{11}}b_1, \; a'_{32} = a_{32} - \frac{a_{31}}{a_{11}}a_{12},$

$$a'_{33} = a_{33} - \frac{a_{31}}{a_{11}} a_{13},$$

$$b'_3 = b_3 - \frac{a_{31}}{a_{11}} b_1,$$

Step 2 : Elimination of x_2 : Assuming $a'_{22} \neq 0$, we divide the coefficients of the first equation of system (Q.14.3), we get

$$x_2 + a''_{23} x_3 = b''_2 \quad \text{...(Q.14.4)}$$

Where $$a''_{23} = \frac{a'_{23}}{a'_{22}}, \; b''_2 = \frac{b'_2}{a'_{22}}$$

Using equation (Q.14.4), we next eliminate x_2 from second equation of (Q.14.3) by subtracting a'_{32} times equation (Q.14.4) from second equation of (Q.14.3), we thus get

$$a''_{33} x_3 = b''_3 \quad \text{...(Q.14.5)}$$

Where $$a''_{33} = a'_{33} - \frac{a'_{32}}{a'_{22}} a'_{23}, \; b''_3 = b'_3 - \frac{a'_{32}}{a'_{22}} b'_2$$

Step 3 : Determination of unknowns x_3, x_2, x_1 : Collecting the first equation from each stage i.e. from equations (Q.14.2), (Q.14.4) and (Q.14.5), we obtain

$$\begin{aligned} x_1 + a'_{12} x_2 + a'_{13} x_3 &= b_1 \\ x_2 + a''_{23} x_3 &= b''_3 \\ x_3 &= \frac{b''_3}{a''_{33}} \end{aligned} \quad \text{...(Q.14.6)}$$

The system (Q.14.4) is an upper triangular system and can be solved by a process called back substitution. The last equation determine x_3, which is then substituted in the next last equation to determine x_2 and finally on substituting x_3 and x_2 in the first equation, we obtain x_1.

Remark 1 : The elements a_{11}, a'_{22} and a''_{33} which have been assumed to be non-zero are called **pivot elements** and the equations containing these pivot elements are called **pivotal equations.**

Remark 2 : The process of dividing by a_{11} to first equation of system (1) for making coefficient of x_1 unity is called **normalization.**

Remark 3 : A necessary and sufficient condition for using Gauss elimination method is that all leading elements (pivot elements) be non-zero.

In the elimination process, if any one of the pivot elements a_{11}, a'_{22} ... vanishes or becomes very small compared to other elements in that equation then we attempt to rearrange the remaining equation so as to obtain a non-vanishing pivot or to avoid the division by small pivot element (equivalently multiplication by large number) to every coefficient in the pivotal equation. This procedure is a called **partial pivoting.** If the matrix A is diagonally dominant or real, symmetric and positive definite then no pivoting is necessary.

We note that, if the numerators of any fractions contain rounding errors, the effects of such errors are diminished when the denominator is large pivot.

I) Partial pivoting : In the first step of elimination, the numerically largest coefficient of x_1 is chosen from all the equations and brought as first pivot by interchanging the first equation with equation having the largest coefficient of x_1. In the second step of elimination the largest coefficient of x_2 in magnitude is chosen from the remaining equations (leaving the first equation) and brought as second pivot by interchanging the second equation with the equation having the largest coefficient of x_2. This process is continued till we arrive at the equation with the single variable. In other words, partial pivoting involves searching for largest coefficient of an unknown quantity amongst a systems of equations at each step of the elimination.

Complete pivoting : A slightly better result may be obtain by disregarding the order of elimination of x_1, x_2, x_3 and solving at each step of elimination process, the equation in which largest coefficient in the entire set (or in the entire matrix of coefficient) occurs. This requires not only an interchange of equations but also interchange of the position of the variables. The equation is then solved for that unknown to which the largest coefficient is attached. This process is called as **complete pivoting** and is rarely employed as it changes the order of unknowns and consequently adds complexity.

Remark 4 : The Gauss elimination method can be interpreted in matrix form as reducing the augmented matrix to upper triangular from whose leading diagonal elements are unity by using row operations only. Thus.

$$[A \mid B] \xrightarrow[\text{Elimination}]{\text{Gauss}} [U \mid B']$$

Where $[A \mid B]$ is the augmented matrix.

II) Ill-conditioned linear systems : In practical applications, one encounters systems in which small change in the coefficients of the system or right-hand side terms result in very large changes in the solution. Such systems are said to be **Ill-conditioned.** If the corresponding changes in the solution are also small, then the system is **well-conditioned.**

Q.15 Use Gauss elimination method to solve the following system of equations.

$x_1+4x_2-x_3=-5$

$x_1+x_2-6x_3=-12$

$3x_1-x_2-x_3=4$

[SPPU : Dec.-16]

Ans. : Given system is

$$\left.\begin{aligned} x_1+4x_2-x_3 &= -5 \\ x_1+x_2-6x_3 &= -12 \\ 3x_1-x_2-x_3 &= 4 \end{aligned}\right\} \quad \ldots (Q.15.1)$$

Step 1 : Since coefficient of x_1 is unity in the first equation, we eliminate x_1 from second and third equations of (Q.15.1) by subtracting first equation from second equation and 3 times first equation from third equation of (Q.15.1), we obtain

$$-3x_2-5x_3 = -7$$
$$-13x_2+2x_3 = 19 \quad \ldots (Q.15.2)$$

Step 2 : Dividing the first equation (2) by -3, we get

$$x_2+\frac{5}{3}x_3 = \frac{7}{3} \quad \ldots (Q.15.3)$$

Using equation (Q.15.3), we next eliminate x_2 from second equation of (Q.15.2) by adding 13 times equation (Q.15.3) in second equation of (Q.15.2), we get

$$\frac{71}{3}x_3 = \frac{148}{3}$$

Or

$$x_3 = \frac{148}{71} \quad \ldots (Q.15.4)$$

Step 3 : Collecting first equation of (Q.15.1) and equations (Q.15.3) and (Q.15.4), we have

$$x_1+4x_2-x_3 = -5$$
$$x_2+\frac{5}{3}x_3 = \frac{7}{3}$$
$$x_3 = \frac{148}{71} \quad \ldots (Q.15.5)$$

The system (Q.15.5) is equivalent upper triangular system. Using backward substitution, the solution is

$$x_1 = \frac{117}{71},\ x_2 = \frac{-81}{71} \text{ And } x_3 = \frac{148}{71}$$

Q.15(a) Solve by Gauss elimination method, the system of equations :

$2x_1+x_2+x_3=10$

$3x_1+2x_2+3x_3=18$

$x_1+4x_2+9x_3=16$

[SPPU : June-22, Marks 5]

Ans. : Given that,

$$2x_1+x_2+x_3 = 10 \quad \ldots (Q.15(a).1)$$
$$3x_1+2x_2+3x_3 = 18 \quad \ldots (Q.15(a).2)$$
$$x_1+4x_2+9x_3 = 16 \quad \ldots (Q.15(a).3)$$

3 × equation (Q.15(a).1) − 2 × equation (Q.15(a).2) $\Rightarrow -x_2-3x_3=-6$

$$= x_2+3x_3 = 6 \quad \ldots (Q.15(a).4)$$

equation (Q.15(a).1) − 2 × equation (Q.15(a).3) $\Rightarrow = -2x_2-17x_3=-22$

$$= 2x_2+17x_3 = 22 \quad \ldots (Q.15(a).5)$$

equation (Q.15(a).5) − 2 × equation (Q.15(a).4)

$$11x_3 = 10 \Rightarrow x_3 = \frac{10}{11}$$

equation (Q.15(a).4) $\Rightarrow x_2 = 6-3x_3 = 6-\frac{30}{11} = \frac{36}{11}$

equation (Q.15(a).1) $2x_1 = 10-x_2-x_3 = 10-\frac{30}{11}-\frac{10}{11} = \frac{110-46}{11} = \frac{64}{11}$

$$\Rightarrow x_1 = \frac{32}{11}$$

$$\therefore \quad x_1 = \frac{32}{11},\ x_2 = \frac{36}{11},\ x_3 = \frac{10}{11}$$

Q.15(b) Solve by Gauss elimination method, the system of equations :

$4x_1 + x_2 + x_3 = 4$

$x_1 + 4x_2 - 2x_3 = 4$

$3x_1 + 2x_2 - 4x_3 = 6$ **☞[SPPU : Dec.-22, Marks 5]**

Ans. : Given system is

$$4x_1 + x_2 + x_3 = 4 \quad \text{...(Q.15(b).1)}$$
$$x_1 + 4x_2 - 2x_3 = 4 \quad \text{...(Q.15(b).2)}$$
$$3x_1 + 2x_2 - 4x_3 = 6 \quad \text{...(Q.15(b).3)}$$

1st iteration : Dividing 1st equation by (Q.15(b).4), we get,

$$x_1 + \frac{1}{4}x_2 + \frac{1}{4}x_3 = 1 \quad \text{...(Q.15(b).4)}$$

Now by use of equation (Q.15(b).4), eliminate x_1 from equation (Q.15(b).2) and (Q.15(b).3), we get,

$$\frac{15}{4}x_2 - \frac{9}{4}x_3 = 3 \quad \text{...(Q.15(b).5)}$$
$$\frac{5}{4}x_2 - \frac{19}{4}x_3 = 3 \quad \text{...(Q.15(b).6)}$$

2nd iteration : Dividing the 5th equation by $\frac{15}{4}$, we get,

$$x_2 - \frac{3}{5}x_3 = \frac{4}{5} \quad \text{...(Q.15(b).7)}$$

Using equation (Q.15(b).7) to eliminate x_2 from equation (Q.15(b).6), we get,

$$-4x_3 = 2$$
$$\Rightarrow \quad x_3 = \frac{-1}{2}$$

Collecting (Q.15(b).1), (Q.15(b).4) and (Q.15(b).7), we obtain,

$$x_1 + \frac{1}{4}x_2 + \frac{1}{4}x_3 = 1$$
$$x_2 - \frac{3}{5}x_3 = \frac{4}{5}$$
$$x_3 = \frac{-1}{2}$$

After using backward substitution, we get solutions

$$x_1 = 1,\ x_2 = \frac{1}{2} \text{ and } x_3 = \frac{-1}{2}$$

Q.16 Solve the following system of equations by Gauss elimination method :

$10x + 2y + z = 9$

$2x + 20y - 22 = -44$

$-2x + 3y + 10z = 22$

☞ [SPPU : May-16]

Ans. : Step 1 : Given that

$$10x + 2y + z = 9 \quad \text{... (Q.16.1)}$$
$$2x + 20y - 22 = -44 \quad \text{... (Q.16.2)}$$
$$-2x + 3y + 10z = 22 \quad \text{... (Q.16.3)}$$

Dividing equation (Q.16.1) by 10, we get

$$x + \frac{2}{10}y + \frac{1}{10}z = \frac{9}{10} \quad \text{... (Q.16.4)}$$

Eequation (Q.16.2) – Equation (Q.16.4) × 2

$$\Rightarrow \quad \frac{196}{10}y - \frac{22}{10}z = \frac{-458}{10} \quad \text{... (Q.16.5)}$$

Eequation (Q.16.3) + Equation (Q.16.4) × 2

$$\Rightarrow \quad \frac{32}{10}y + \frac{102}{10}z = \frac{238}{10} \quad \text{... (Q.16.6)}$$

Dividing equation (Q.16.5) by $\frac{196}{10}$, we get

$$y - \frac{22}{196}z = -\frac{458}{196} \quad \text{... (Q.16.7)}$$

$\therefore$ Equation (Q.16.6) $- \frac{32}{10}$ Equation (Q.16.7)

$$\Rightarrow \quad \left(\frac{102}{10} + \frac{22 \times 32}{1960}\right)z = \frac{238}{10} + \frac{458 \times 32}{1960}$$
$$\frac{20696}{1960}z = \frac{61304}{1960}$$

$$\therefore \quad z = \frac{61304}{20696} = 2.9621$$

Equation (Q.16.6) $\Rightarrow y = \frac{238}{10} - \frac{102}{10} z = -6.41342$

Equation (Q.16.4) $\Rightarrow x = -\frac{2}{10} y - \frac{1}{10} z + \frac{9}{10}$

$= 1.886474$

Thus the solution is $x = 1.8864$, $y = -6.41342$,

$z = 2.9621$

Q.17 Explain Gauss Seidel iteration method.

Ans. : The Gauss-Seidel method is a modification of the Jacobi's iteration method. As in Jacobi's iteration method, consider a system of equations in which each equation is first solved for unknown having large coefficient, thereby expressing it explicitly in terms of other unknowns as

$$\left.\begin{array}{l} x_1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 \ldots\ldots\ldots\ldots - a_{1n}x_n) \\ x_2 = \frac{1}{a_{22}}(b_2 - a_{11}x_1 - a_{13}x_3 \ldots\ldots\ldots\ldots - a_{2n}x_n) \\ x_3 = \frac{1}{a_{33}}(b_3 - a_{31}x_1 - a_{32}x_2 - a_{34}x_4 \ldots\ldots - a_{3n}x_n) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_4 = \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 \ldots\ldots - a_{nn-1}x_{n-1}) \end{array}\right\} \ldots \text{(Q.17.1)}$$

We again start with initial approximations $(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, \ldots x_n^{(0)} = 0)$. However, this time we substitute these only in the first equation on the right hand side of system (Q.17.1), we substitute $(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, \ldots, x_n^{(0)})$ and denote the improved results $x_2^{(1)}$. In the third equation of (Q.17.1), we

substitute $(x_1^{(1)}, x_2^{(1)}, x_3^{(0)}, \ldots, x_n^{(0)})$ and call the result $x_3^{(1)}$. Proceeding like this we find first iteration values as $x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \ldots, x_n^{(1)}$. This completes the first stage of iteration and the entire process is repeated till the values of $x_1, x_2, \ldots x_n$ are obtained to desired accuracy.

Thus, if the values of the variables in k^{th} iteration are $x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}$ then the values in the next $(k+1)^{th}$ iteration are given by

$$\left.\begin{array}{l} x_1^{(k+1)} = \frac{1}{a_{11}}(b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} \ldots\ldots\ldots\ldots - a_{1n}x_n^{(k)}) \\ x_2^{(k+1)} = \frac{1}{a_{22}}(b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)} \ldots\ldots\ldots - a_{2n}x_n^{(k)}) \\ x_3^{(k+1)} = \frac{1}{a_{33}}(b_3 - a_{31}x_1^{(k+1)} - a_{32}x_2^{(k+1)} - a_{34}x_4^{(k)} \ldots\ldots - a_{3n}x_n^{(k)}) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n^{(k+1)} = \frac{1}{a_{nn}}(b_n - a_{n1}x_1^{(k+1)} - a_{n2}x_2^{(k+1)} \ldots\ldots\ldots - a_{nn-1}x_{n-1}^{(k+1)}) \end{array}\right\} \ldots \text{(Q.17.2)}$$

Above system of approximation (Q.17.2) can be considered as a general formula for Gauss-Seidel iterative method.

Remark (1) : For any choice of the first (initial) approximation, Gauss–Seidel iterative method converges. Condition for convergence of iteration process is same as discussed in Jacobi's iteration method. That is "if in each equation of the system, the absolute value of the largest coefficient is greater than the sum of the absolute values of all the remaining coefficients in that equation." In this method of iteration, the result of any stage within a step is used in succeeding stages of the same step, the method is also called method of **successive correction.**

Q.18 Solve the following system of equations by Gauss Seidel method

$20x + 4y - z = 32$

$2x + 17y + 4z = 35$

$x + 3y + 10z = 24$ **[SPPU : Dec.-14, 15]**

Ans. : The given linear system is diagonally dominant

$$\therefore \quad x = \frac{1}{20}(32-4y+z) \quad \ldots (Q.18.1)$$

$$y = \frac{1}{17}(35-2x-4z) \quad \ldots (Q.18.2)$$

$$z = \frac{1}{10}(24-x-3y) \quad \ldots (Q.18.3)$$

First Iteration : Let us start iteration with initial approximation,

$x^{(0)} = y^{(0)} = z^{(0)} = 0$

$$\therefore \quad x^{(1)} = \frac{1}{20}(32) = 1.6$$

Substitute $x^{(1)}$, $z^{(0)}$ in equation (Q.18.2), we get

$$y^{(1)} = \frac{1}{17}(35-3.2) = 1.8705$$

Substitute $x^{(1)}$, $y^{(1)}$, in equation (Q.18.3)

$$\therefore \quad z^{(1)} = \frac{1}{10}(24-1.6-5.6117) = 1.6788$$

Second Iteration : Substitute $y^{(1)}$, $z^{(1)}$ in equation (Q.18.1)

$$\therefore \quad x^{(2)} = \frac{1}{20}(32-4y^{(1)}+z^{(1)}) = 1.3098$$

$$y^{(2)} = \frac{1}{17}(35-2x^{(2)}-4z^{(1)}) = 1.5097$$

$$z^{(2)} = \frac{1}{10}(24-x^{(2)}-3y^{(2)}) = 1.8161$$

By continuing in this way, the successive iterations can be tabulated as

n	x	y	z
1	1.6	1.8705	1.6788
2	1.3098	1.5097	1.8161
3	1.3888	1.4681	1.8206
4	1.3974	1.4660	1.8204
5	1.3978	1.4660	1.8204

Thus, the solution is x = 1.3978, y = 1.4660, z = 1.8204.

Q.18(a) Solve by Gauss-Seidel method, the system of equations :

$$20x_1 + x_2 - 2x_3 = 17$$
$$3x_1 + 20x_2 - x_3 = -18$$
$$2x_1 - 3x_2 + 20x_3 = 25$$

[SPPU : June-22, Marks 5]

Ans. : Given linear system is diagonally dominant.

$$\therefore \quad x_1 = \frac{1}{20}(17-x_2+2x_3) \quad \ldots (Q.18(a).1)$$

$$x_2 = \frac{1}{20}(-18-20x_2+x_3) \quad \ldots (Q.18(a).2)$$

$$x_3 = \frac{1}{20}(25+3x_2-2x_1) \quad \ldots (Q.18(a).3)$$

First Iteration : Put $x_2^{(0)}, x_3^{(0)} = 0$

$$x_1^{(1)} = \frac{1}{20}(17) = 0.85$$

Substitute $x_1^{(1)}$ and $x_3^{(0)}$ in equation (Q.18(a).2)

$$x_1^{(1)} = \frac{1}{20}(-18-3x_1+x_3)$$

$$= \frac{1}{20}(-18-2.55) = -1.0275$$

Substituting $x_1^{(1)}$ and $x_2^{(1)}$ in equation (Q.18(a).3), we get,

$$x_3^{(1)} = \frac{1}{20}(25-2x_1+3x_2)$$

$$= \frac{1}{20}(25-2(0.85)+3(-1.0275))$$

$$x_3^{(1)} = 1.0109$$

Second Iteration : $x_1^{(1)} = 0.85$, $x_2^{(1)} = -1.0275$, $x_3^{(1)} = 1.0109$

$$x_1^{(2)} = \frac{1}{20}(17-x_2^{(1)}+2x_3^{(1)})$$

$$= \frac{1}{20}(17-(-1.0275)+2(1.0109)) = 1.0025$$

$$x_2^{(2)} = \frac{1}{20}(-18-3x_1+x_3)$$

$$= \frac{1}{20}(-18-3(1.0025) - (1.0109))$$

$$= -0.9998$$

$$x_3^{(2)} = \frac{1}{20}(25 - 2x_1 + 3x_2)$$

$$= \frac{1}{20}(25-2(1.0025)+3(-0.9998)) = 0.9998$$

Third Iteration : $x_1^{(2)} = 1.0025$, $x_2^{(2)} = -0.9998$

$$x_3^{(2)} = 0.9998$$

$$x_1^{(3)} = \frac{1}{20}(17-x_2^{(2)}+2x_3^{(2)})$$

$$= \frac{1}{20}(17-(-0.9998)+2(0.9998)) = 1.0000$$

$$x_2^{(3)} = \frac{1}{20}(-18-3x_1+x_3)$$

$$= \frac{1}{20}(-18-3(1.0000)+(0.9998))$$

$$= -1.0000$$

$$x_3^{(3)} = \frac{1}{20}(25-2x_1+3x_2)$$

$$= \frac{1}{20}(25-2(1.0000)+3(-1.0000))$$

$$= 1.0000$$

By continuing in this way, the successive iterations can be tabulated as

n	x_1	x_2	x_3
1	0.8500	- 1.0275	1.0109
2	1.0025	- 0.9998	0.9998
3	1.0000	- 1.0000	1.0000

Thus, the solution is $x_1 = 1$, $x_2 = -1$ and $x_3 = 1$.

Q.18(b) Solve by Gauss - Seidel method, the system of equations :

$$2x_1 + x_2 + 6x_3 = 9$$
$$8x_1 + 3x_2 + 2x_3 = 13$$
$$x_1 + 5x_2 + x_3 = 7$$

[SPPU : Dec.-22, Marks 5]

Ans. : First we arrange given system in diagonally dominant. Hence we get,

$$8x_1 + 3x_2 + 2x_3 = 13$$
$$x_1 + 5x_2 + x_3 = 7$$
$$2x_1 + x_2 + 6x_3 = 9$$

Now we can write given system as below

$$x_1 = \frac{1}{8}[13-3x_2-2x_3]$$

$$x_2 = \frac{1}{5}[7-x_1-x_3]$$

$$x_3 = \frac{1}{6}[9-2x_1-x_2]$$

1st iteration : We will start process by taking $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 0$, we get,

$$x_1^{(1)} = \frac{1}{8}[13-3(0)-2(0)] = 1.625$$

$$x_2^{(1)} = \frac{1}{5}[7-1.625-0] = 1.075$$

$$x_3^{(1)} = \frac{1}{6}[9-2(1.625)-1.075] = 0.779$$

2nd iteration : Now set of values $x_1^{(1)}, x_2^{(1)}, x_3^{(1)}$ now becomes 2nd trial solution, we get

$$x_1^{(2)} = \frac{1}{8}[13-3(1.075)-2(0.779)] = 1.0271$$

$$x_2^{(2)} = \frac{1}{5}[7-1.0271-0.779] = 1.03878$$

$$x_3^{(2)} = \frac{1}{6}[9-2(1.0271)-1.03878] = 0.9845$$

3rd iteration : Now again use new set of values $x_1^{(2)}, x_2^{(2)}, x_3^{(2)}$ to get 3rd trial solutions.

$$x_1^{(3)} = \frac{1}{8}[13-3(1.03878)-2(0.9845)] = 0.9893$$

$$x_2^{(3)} = \frac{1}{5}[7-0.9893-0.9845] = 1.00524$$

$$x_3^{(3)} = \frac{1}{6}[9-2(1.00524)-0.9845] = 1.00083$$

4th iteration : Now again take new $x_1^{(3)}, x_2^{(3)}, x_3^{(3)}$ values, for further trial, we get,

$$x_1^{(4)} = \frac{1}{8}[13-3(1.00524)-2(1.00083)] = 0.9978$$

$$x_2^{(4)} = \frac{1}{5}[7-0.9978-1.00083] = 1.000274$$

$$x_3^{(4)} = \frac{1}{6}[9-2(0.9978)-1.00083] = 1.000595$$

Here $x_1 = 1, y_1 = 1, z_1 = 1$

Q.19 Explain Cholesky method.

Ans. : Let AX = B be a Linear System.

If the coefficient matrix A is symmetric and positive definite then matrix A can be expressed as

$A = LL^t$ where L is the lower triangular matrix

or $A = UU^t$ where U is the upper triangular matrix

Putting $A = LL^t$ in AX = B, we get

$$LL^tX = B \quad \text{... (Q.19.1)}$$

Let

$$L^tX = Z \quad \text{... (Q.19.2)}$$

$\therefore$

$$L(L^tX) = LZ = B \quad \text{... (Q.19.3)}$$

Solving equation (Q.19.3), we get Z and using it to solve equation (Q.19.2), we get X.

Q.20 Solve the following system of equations by Cholesky's method

$2x_1 - x_2 = 1$

$-x_1 + 3x_2 + x_3 = 0$

$x_2 + 2x_3 = 0$

[SPPU : Dec.-14]

Ans. : Given linear system can be written in matrix form as

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

i.e.

$$AX = B \quad \text{... (Q.20.1)}$$

We can express matrix A as $A = LL^t$

$$\therefore \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

Multiplying the right hand side matrices and equating corresponding elements of equation

we get

$$l_{11}^2 = 2 \Rightarrow l_{11} = \sqrt{2}$$

$$l_{11}\, l_{21} = -1 \Rightarrow l_{21} = -\frac{1}{\sqrt{2}}$$

$$l_{11}\, l_{31} = 0 \Rightarrow l_{31} = 0$$

$$l_{21}^2 + l_{22}^2 = 3 \Rightarrow l_{22} = \sqrt{\frac{5}{2}}$$

$$l_{21}\, l_{31} + l_{22}\, l_{32} = 1 \Rightarrow l_{32} = \sqrt{\frac{2}{5}}$$

$$l_{31}^2 + l_{32}^2 + l_{33}^2 = 2 \Rightarrow l_{33} = \sqrt{\frac{8}{5}}$$

Thus,

$$L = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \sqrt{\frac{5}{2}} & 0 \\ 0 & \sqrt{\frac{2}{5}} & \sqrt{\frac{8}{5}} \end{bmatrix}$$

Now, we have

$$AX = B \Rightarrow L(L^tX) = B$$

Let $\quad L^tX = Z \quad$ ... (Q.20.2)

$\therefore \quad LZ = B \quad$ ... (Q.20.3)

Now solve equation (Q.20.3)

$$\begin{bmatrix} \sqrt{2} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \sqrt{\frac{5}{2}} & 0 \\ 0 & \sqrt{\frac{2}{5}} & \sqrt{\frac{8}{5}} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$R_1 \Rightarrow \sqrt{2}\, z_1 = 1 \Rightarrow z_1 = \frac{1}{\sqrt{2}}$$

$$R_2 \Rightarrow -\frac{1}{\sqrt{2}} z_1 + \sqrt{\frac{5}{2}}\, z_2 = 0 \Rightarrow z_2 = \frac{1}{\sqrt{10}}$$

$$R_3 \Rightarrow \sqrt{\frac{2}{5}}\, z_2 + \sqrt{\frac{8}{5}}\, z_3 = 0 \Rightarrow z_3 = -\frac{1}{\sqrt{40}}$$

Now, the solution of $L^tX = z$ is

$$\begin{bmatrix} \sqrt{2} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \sqrt{\frac{5}{2}} & \sqrt{\frac{2}{5}} \\ 0 & 0 & \sqrt{\frac{8}{5}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{10}} \\ -\frac{1}{\sqrt{40}} \end{bmatrix}$$

$$\therefore \quad R_3 \Rightarrow \sqrt{\frac{8}{5}}\, x_3 = -\frac{1}{\sqrt{40}} \Rightarrow x_3 = -\frac{1}{8}$$

$$R_2 \Rightarrow \sqrt{\frac{5}{2}}\, x_2 + \sqrt{\frac{2}{5}}\, x_3 = \frac{1}{\sqrt{10}} \Rightarrow x_2 = \frac{1}{4}$$

$$R_1 \Rightarrow \sqrt{2}\, x_1 - \frac{1}{\sqrt{2}} x_2 = \frac{1}{\sqrt{2}} \Rightarrow x_1 = \frac{5}{8}$$

Thus, the required solution is

$$x_1 = \frac{5}{8},\ x_2 = \frac{1}{4},\ x_3 = -\frac{1}{8}$$

Q.21 Explain Guass Jacobi's method.

Ans. : Consider the equations,

$$\left.\begin{aligned} a_{11}x + a_{12}y - a_{13}z = d_1 \\ a_{21}x + a_{22}y - a_{23}z = d_2 \\ a_{31}x + a_{32}y - a_{33}z = d_3 \end{aligned}\right\} \quad \text{...(Q.21.1)}$$

If a_1, b_2, c_3 are large as compared to other coefficients, then solving these for x, y, z respectively, the system can be written in the form

$$\left.\begin{aligned} x = k_1 - l_1 y - m_1 z \\ y = k_2 - l_2 x - m_2 z \\ z = k_3 - l_3 x - m_3 y \end{aligned}\right\} \quad \text{...(Q.21.2)}$$

Let us start with the initial approximations $x_0 = 0$, $y_0 = 0$, $z_0 = 0$. Substituting these values on the right sides of (2) we get the first approximations $x^{(1)} = k_1$, $y^{(1)} = k_2$, $z^{(1)} = k_3$.

Now substituting values of $x^{(1)}, y^{(1)}, z^{(1)}$ in equation (7.15) we will get the second approximation say $x^{(2)}$, $y^{(2)}$ and $z^{(2)}$,

Again, substituting $x^{(2)}$, $y^{(2)}$ and $z^{(2)}$ in equation (7.15), we will get the third approximations,

$\therefore$ say $x^{(3)}$, $y^{(3)}$ and $z^{(3)}$.

This process is repeated till the difference between two consecutive approximations is negligible.

Q.22 Solve Gauss Jacobi method

$27x + 6y - z = 85$

$6x + 15y - 2z = 72$

$x + y + 54z = 110$

Ans. : We have

$$\left.\begin{aligned} |a_{11}| > |a_{12}| + |a_{13}| \\ |a_{22}| > |a_{21}| + |a_{23}| \\ |a_{23}| > |a_{31}| + |a_{32}| \end{aligned}\right\} \quad \text{...(Q.22.1)}$$

$$\left.\begin{aligned} x &= \frac{1}{27}[85-6y+z] \\ y &= \frac{1}{15}[72-6x-3z] \\ z &= \frac{1}{54}[110-x-y] \end{aligned}\right\} \quad \ldots(Q.22.2)$$

Starting with initial approximation

$x = y = z = 0$

i) Substituting $x = y = 0$ in equation (Q.22.2), we get the first approximations

$$x^{(1)} = \frac{85}{27} = 3.148$$

$$y^{(1)} = \frac{72}{15} = 4.8$$

$$z^{(1)} = \frac{110}{54} = 2.037$$

ii) Substituting $x^{(1)}$, $y^{(1)}$ and $z^{(1)}$ in equation (Q.22.2), We will get the second approximations

$$x^{(2)} = \frac{1}{27}[85-6(4.8)+2.037]$$

$$= 2.1569$$

$$y^{(2)} = \frac{1}{15}[72-6\times3.148-2\times2.037]$$

$$= 3.2692$$

$$z^{(2)} = \frac{1}{54}[110-3.148-4.8]$$

$$= 1.889$$

iii) Substituting $x^{(2)}$, $y^{(2)}$ and $z^{(2)}$ in equation (Q.22.2), We will get the third approximations,

$$x^{(3)} = \frac{1}{27}[85-6\times3.2622+1.889] = 2.4915$$

$$y^{(3)} = \frac{1}{15}[72-6\times21569-2\times1.839]$$

$$= 3.68537.$$

$$z^{(3)} = \frac{1}{54}[110-2.1565-3.2692]$$

$$= 1.93655$$

iv) $$x^{(4)} = \frac{1}{27}[85-6\times3.68537+1.93655]$$

$$= 2.4009$$

$$y^{(4)} = \frac{1}{15}[72-6\times2.4915-2\times1.93655]$$

$$= 3.54519.$$

$$z^{(4)} = \frac{1}{54}[110-2.4915-3.68534]$$

$$= 1.92265$$

v) $$x^{(5)} = \frac{1}{27}[85-6\times3.54519+1.92265]$$

$$= 2.4315$$

$$y^{(5)} = \frac{1}{15}[72-6\times2.4009-2\times1.92265]$$

$$= 3.58328.$$

$$z^{(5)} = \frac{1}{54}[110-2.4009-3.54519]$$

$$= 1.92692$$

vi) $$x^{(6)} = \frac{1}{27}[85-6\times3.5325+1.92692]$$

$$= 2.42323$$

$$y^{(6)} = \frac{1}{15}[72-6\times2.4315-2\times1.92692]$$

$$= 3.5704$$

$$z^{(6)} = \frac{1}{54}[110-2.431-3.58328]$$

$$= 1.92763$$

vii) $x^{(7)} = \frac{1}{27}[85-6\times 3.5704+1.9258]$

$= 2.426$

$y^{(7)} = \frac{1}{15}[72-6\times 2.42323-2\times 1.9258]$

$= 3.573$

$z^{(7)} = \frac{1}{54}[110-2.42323-3.5704] = 1.926$

Here, 6^{th} and 7^{th} approximation are same

so, $x = 2.42$

$y = 3.57$

$z = 1.926$

Q.23 Solve by Jacobi's iteration method, the system of equations :

$4x_1 + 2x_2 + x_3 = 14$

$x_1 + 5x_2 - x_3 = 10$

$x_1 + x_2 + 8x_3 = 20$

[SPPU : June-22, Marks 5]

Ans. : We have,

$$x_1 = \frac{1}{4}[14-2x_2-x_3] \quad ...(Q.23.1)$$

$$x_2 = \frac{1}{5}[10-x_1+x_3] \quad ...(Q.23.2)$$

$$x_3 = \frac{1}{8}[20-x_1-x_2] \quad ...(Q.23.3)$$

Start with initial approximation $x_1 = x_2 = x_3 = 0$

I) $x_1^{(1)} = \frac{14}{4} = 3.5,\ x_2^{(1)} = \frac{10}{5} = 2,\ x_3^{(1)} = \frac{20}{8} = 2.5$

II) $x_1^{(2)} = \frac{1}{4}(14-2x_2^{(1)}-x_3^{(1)})$

$= \frac{1}{4}(14-2(2)-2.5)$

$= 1.875$

$x_2^{(2)} = \frac{1}{5}[10-x_1^{(1)}+x_3^{(1)}]$

$= \frac{1}{5}[10-3.5+2.5]$

$= 1.8$

$x_3^{(2)} = \frac{1}{8}(20-x_1^{(1)}-x_2^{(1)})$

$= \frac{1}{8}(20-3.5-2)$

$= 1.8125$

Similarly, $x_1^{(3)}$, $x_2^{(3)}$, $x_3^{(3)}$ calculation.

III) $x_1^{(3)} = \frac{1}{4}(14-2x_2^{(2)}-x_3^{(2)})$

$= \frac{1}{4}(14-2(1.8)-1.8125)$

$= 2.1469$

$x_2^{(3)} = \frac{1}{5}(10-x_1^{(2)}+x_3^{(3)})$

$= \frac{1}{5}(10-1.875+1.8125)$

$= 1.9875$

$x_3^{(3)} = \frac{1}{8}(20-x_1^{(2)}-x_2^{(2)})$

$= \frac{1}{8}(20-1.875-1.8)$

$= 2.041$

Same way $x_1^{(4)}$, $x_2^{(4)}$, $x_3^{(4)}$ we can calculate.

IV) $x_1^{(4)} = \frac{1}{4}(14-2x_2^{(3)}-x_3^{(3)})$

$= \frac{1}{4}(14-2(1.9875)-2.041)$

$= 1.99$

$$x_2^{(4)} = \frac{1}{5}(10 - x_1^{(3)} + x_3^{(3)})$$
$$= \frac{1}{5}(10 - 2.1469 + 2.041)$$
$$= 1.99$$
$$x_3^{(4)} = \frac{1}{8}(20 - x_1^{(3)} - x_2^{(3)})$$
$$= \frac{1}{8}(20 - 2.1469 - 1.9875)$$
$$= 1.98$$

Hence, the solution is $x_1 = 2$, $x_2 = 2$ and $x_3 = 2$

Q.24 Solve by Jacobi's iteration method, the system of equations :
$20x_1 + x_2 - 2x_3 = 17$
$3x_1 + 20x_2 - x_3 = -18$
$2x_1 - 3x_2 - 20x_3 = 25$ **[SPPU : Dec.-22, Marks 5]**

Ans. : Given system of equation are :

$$20x_1 + x_2 - 2x_3 = 17$$
$$3x_1 + 20x_2 - x_3 = -18$$
$$2x_1 - 3x_2 + 20x_3 = 25$$

by above system, we get,

$$\left.\begin{aligned} x_1 &= \frac{1}{20}[17 - x_2 + 2x_3] \\ x_2 &= \frac{1}{20}[-18 - 3x_1 + x_3] \\ x_3 &= \frac{1}{20}[25 - 2x_1 + 3x_2] \end{aligned}\right\} \quad \text{...(Q.24.1)}$$

1st iteration : Let initial approximation $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 0$, substitute these values in system (Q.24.1), we get,

$$x_1^{(1)} = \frac{1}{20}[17 - 0 + 2(0)] = 0.85$$
$$x_2^{(1)} = \frac{1}{20}[-18 - 3(0) + (0)] = -0.9$$

$$x_3^{(1)} = \frac{1}{20}[25 - 2(0) + 3(0)] = 1.25$$

2nd iteration : Now putting above values of $x_1^{(1)}, x_2^{(1)}, x_3^{(1)}$ in system (Q.24.1), then we get,

$$x_1^{(2)} = \frac{1}{20}[17 - (-0.9) + 2(1.25)] = 1.02$$
$$x_2^{(2)} = \frac{1}{20}[-18 - 3(0.85) + (1.25)] = -0.965$$
$$x_3^{(2)} = \frac{1}{20}[25 - 2(0.85) + 3(-0.9)] = 1.1515$$

3rd iteration :

$$x_1^{(3)} = \frac{1}{20}[17 - (-0.965) + 2(1.1515)] = 1.0134$$
$$x_2^{(3)} = \frac{1}{20}[-18 - 3(1.02) + (1.1515)] = -0.9954$$
$$x_3^{(3)} = \frac{1}{20}[25 - 2(1.02) + 3(-0.965)] = 1.0032$$

4th iteration :

$$x_1^{(4)} = \frac{1}{20}[17 - (-0.9954) + 2(1.0032)] = 1.0009$$
$$x_2^{(4)} = \frac{1}{20}[-18 - 3(1.0134) + 1.0032] = -1.0018$$
$$x_3^{(4)} = \frac{1}{20}[25 - 2(1.0134) + 3(-0.9954)] = 0.999$$

Hence, solution is $x_1 = 1$, $x_2 = -1$ and $x_3 = 1$

END...

Unit VI

8 Interpolation, Numerical Differentiation and Integration

8.1 : Finite Differences

Consider the function $y = f(x)$. The values of y corresponding to different values of x are obtained by substituting the various values of x. Let the consecutive values of x differing by h be $x_0, x_0 + h = x_1$, $x_0 + 2h = x_2, \ldots x_0 + nh = x_n$ and corresponding values of $y = f(x)$ be $y_0 = f(x_0)$, $y_1 = f(x_1)$, $y_2 = f(x_2) \ldots y_n = f(x_n)$. The values of the independent variable x are called arguments and corresponding values of dependent variable y are called entries.

8.1.1 : Forward Differences

The first forward difference is defined by $\Delta f(x) = f(x+h) - f(x)$ where Δ is called the forward or descending difference operator. (Δ = Delta)

In particular $\Delta f(x_0) = f(x_0 + h) - f(x_0)$

$$\Delta y_0 = y_1 - y_0$$

$$\Delta f(x_0 + h) = f(x_0 + 2h) - f(x_0 + h)$$

$$\Delta y_1 = y_2 - y_1$$

....

$$\Delta y_{n-1} = y_n - y_{n-1}$$

The differences of first forward differences are called the second forward differences.

Thus

$$\Delta^2 f(x) = \Delta(\Delta f(x)) = \Delta(f(x+h) - f(x))$$

$$= \Delta f(x+h) - \Delta f(x)$$

$$= f(x+2h) - f(x+h) - f(x+h) + f(x)$$

$$\Delta^2 f(x) = f(x+2h) - 2f(x+h) + f(x)$$

In particular

$$\Delta^2 f(x_1) = f(x_1 + 2h) - 2f(x_1 + h) + f(x_1)$$

i.e. $$\Delta^2 y_1 = y_3 - 2y_2 + y_1$$

and $$\Delta^2 y_2 = y_4 - 2y_3 + y_2$$

The third forward difference is given by

$$\Delta^3 f(x) = \Delta(\Delta^2 f(x)) = \Delta[f(x+2h) - 2f(x+h) + f(x)]$$

$$= f(x+3h) - f(x+2h) - 2[f(x+2h) - f(x+h)] + f(x+h) - f(x)$$

$$= f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)$$

In particular

$$\Delta^3 f(x_1) = f(x_1 + 3h) - 3f(x_1 + 2h) + 3f(x_1 + h) - f(x_1)$$

$$\Delta^3 y_1 = y_4 - 3y_3 + 3y_2 - y_1$$

In general, n^{th} forward difference is defined by

$$\Delta^n f(x) = \Delta^{n-1} f(x+h) - \Delta^{n-1} f(x)$$

These forward differences are shown in the following table.

Forward difference table

Argument x	Entry f(x)	First forward differences $\Delta f(x)$	2^{nd} forward differences $\Delta^2 f(x)$	3^{rd} forward differences $\Delta^3 f(x)$
x_0	y_0			
		$\Delta y_0 = y_1 - y_0$		
$x_0 + h$	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$	
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$
$x_0 + 2h$	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$	
		$\Delta y_2 = y_3 - y_2$		$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$
$x_0 + 3h$	y_3		$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$	
		$\Delta y_3 = y_4 - y_3$		
$x_0 + 4h$	y_4			

Q.1 Using Newton's forward interpolation formula, find y at x = 8 from the following data. ☞ **[SPPU : June-22, Marks 5]**

x	0	5	10	15	20	25
y	7	11	14	18	24	32

Ans. : Consider the following forward different table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	7					
		4				
5	11		–1			
		3		2		
10	14		1		–1	
		4		1		0
15	18		2		–1	
		6		0		
20	24		2			
		8				
25	32					

Here h = 5, x = 8, $x_0 = 0$, $y_0 = 7$

$$u = \frac{x - x_0}{h} = \frac{8-0}{5} = \frac{8}{5} = 1.6$$

By Newton's forward interpolation formula

$$y = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{6}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{24}\Delta^4 y_0 + \ldots$$

$$y = 7 + 1.6(4) + \frac{1.6\ (0.6)}{2}(-1) + \frac{(1.6)\ (0.6)\ (-0.4)}{6}(2) + \frac{(1.6)\ (0.6)\ (-0.4)\ (-1.4)}{24}(-1) + 0$$

$$y = 7 + 6.4 - 0.48 + (-1.28) + (-0.224)$$

$$\boxed{y = 11.416}$$

8.1.2 : Backward Differences

The first backward difference is defined by

$$\nabla f(x) = f(x) - f(x-h)$$

where ∇ is called the backward or ascending difference operator and h is interval difference.

(∇ = Del)

In particular

$$\nabla y_1 = y_1 - y_0$$
$$\nabla y_2 = y_2 - y_1$$
$$\vdots$$
$$\nabla y_n = y_n - y_{n-1}$$

The second backward difference is defined by

$$\nabla^2 f(x) = \nabla(\nabla f(x)) = \nabla(f(x) - f(x-h))$$
$$= f(x) - 2f(x-h) + f(x-2h)$$

In particular

$$\nabla^2 y_2 = y_2 - 2y_1 + y_0$$
$$\nabla^2 y_3 = y_3 - 2y_2 + y_1$$
$$\vdots$$
$$\nabla^2 y_n = y_n - 2y_{n-1} + y_{n-2}$$

These backward differences are shown in the following table.

Argument x	Entry f(x)	1st backward differences ∇ f(x)	2nd backward differences ∇^2 f(x)	3rd backward differences ∇^3 f(x)
x_0	y_0			
		$\nabla y_1 = y_1 - y_0$		
$x_0 + h$	y_1		$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$	
		$\nabla y_2 = y_2 - y_1$		$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$

$x_0 + 2h$	y_2		$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$	
		$\nabla y_3 = y_3 - y_2$		$\nabla^3 y_4 = \nabla^2 y_4 - \nabla^2 y_3$
$x_0 + 3h$	y_3		$\nabla^2 y_4 = \nabla y_4 - \nabla y_3$	
		$\nabla y_4 = y_4 - y_3$		
$x_0 + 4h$	y_4			

Properties of differences :

1) If C is a constant then $\Delta C = 0$.

2) $\Delta\,[f(x) \cdot g(x)] = f(x+h)\,\Delta g(x) + g(x)\,\Delta f(x)$

3) $\Delta \left\{ \dfrac{f(x)}{g(x)} \right\} = \dfrac{g(x)\,\Delta f(x) - f(x)\,\Delta g(x)}{g(x+h)\,g(x)}$

4) If $\Delta f(x) = 0$ then it does not mean that either $\Delta = 0$ or $f(x) = 0$.

Q.1(a) Using Newton's backward difference formula, find y at x = 4.5 for the following data. [SPPU : Dec.-22, Marks 5]

x	1	2	3	4	5
y	3.47	6.92	11.25	16.75	22.94

Ans. : We first construct backward difference table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
$1 = x_0$	$3.47 = y_0$				
		$3.45 = \nabla y_1$			
$2 = x_1$	$6.92 = y_1$		$0.88 = \nabla^2 y_2$		
		$4.33 = \nabla y_2$		$0.29 = \nabla^3 y_3$	
$3 = x_2$	$11.25 = y_2$		$1.17 = \nabla^2 y_3$		$0.77 = \nabla^4 y_4$
		$5.5 = \nabla y_3$		$-0.48 = \nabla^3 y_4$	
$4 = x_3$	$16.75 = y_3$		$0.69 = \nabla^2 y_4$		
		$6.19 = \nabla y_4$			
$5 = x_4$	$22.94 = y_4$				

Newton's backward difference formula is given by -

$$y = f(x) = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_n + \ldots$$

$$\text{where} \Rightarrow u = \frac{x - x_n}{h} = \frac{4.5-5}{1} = -0.5$$

So,

$$y = 22.94 + (-0.5)(6.19) + \frac{(-0.5)(-0.5+1)}{2!}(0.69) + \frac{(-0.5)(-0.5+1)(-0.5+2)}{3!}(-0.48) + \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)}{4!}(0.77)$$

$$= 22.94 - 3.095 - 0.08625 + 0.03 - 0.03078$$

$$= \mathbf{19.75797}$$

Q.1(b) Using Newton's backward difference formula, find the value of $\sqrt{155}$ from the following data. [SPPU : June-22, Marks 5]

x	150	152	154	156
$y = \sqrt{x}$	12.247	12.329	12.410	12.490

Ans. : We have,

x	$y = \sqrt{x}$	∇y	$\nabla^2 y$	$\nabla^3 y$
150	12.247			
		0.082		
152	12.329		− 0.001	
		0.081		0
154	12.410		− 0.001	
		0.080		
156	12.490			

Here $x = 155$, $u = \frac{x-156}{h} = \frac{155-156}{2} = -0.5$, $h = 2$

$$\therefore \quad f(x) = f(a+nh) + u\nabla f(a+nh) + \frac{u(u+1)}{2}\nabla^2 f(a+nh) + \ldots$$

$$= 12.490 + (-0.5(0.080) + \frac{-(0.5)(0.5)}{2}(-0.001) + 0$$

$$= 12.490 - 0.04 + 0.00125$$

$$f(x) = 12.450125 = \sqrt{155}$$

8.1.3 : The Central Difference

The first order central difference is defined by

$$\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$$

where δ is the central difference operator.

In particular

$$\delta y_{1/2} = y_1 - y_0$$

$$\delta y_{3/2} = y_2 - y_1$$

The second order central difference is given by

$$\delta^2 f(x) = \delta\left(f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)\right)$$

$$= f\left(x+\frac{h}{2}+\frac{h}{2}\right) - f\left(x+\frac{h}{2}-\frac{h}{2}\right)$$

$$-f\left(x-\frac{h}{2}+\frac{h}{2}\right) + f\left(x-\frac{h}{2}-\frac{h}{2}\right)$$

$$= f(x+h) - 2f(x) + f(x-h)$$

In particular

$$\delta^2 y_1 = y_2 - 2y_1 + y_0 = \delta y_{3/2} - \delta y_{1/2}$$

$$\delta^2 y_2 = y_3 - 2y_2 + y_1$$

These central differences are shown in the following table.

Argument x	Entry y = f(x)	1st central differences δy	2nd central differences $\delta^2 y$	3rd central differences $\delta^3 y$
x_0	y_0			
		$\delta y_{1/2} = y_1 - y_0$		
$x_0 + h$	y_1		$\delta^2 y_1 = \delta y_{3/2} - \delta y_{1/2}$	
		$\delta y_{3/2} = y_2 - y_1$		$\delta^3 y_{3/2} = \delta^2 y_2 - \delta^2 y_1$
$x_0 + 2h$	y_2		$\delta^2 y_2 = \delta y_{5/2} - \delta y_{3/2}$	
		$\delta y_{5/2} = y_3 - y_2$		$\delta^3 y_{5/2} = \delta^2 y_3 - \delta^2 y_2$
$x_0 + 3h$	y_3		$\delta^2 y_3 = \delta y_{7/2} - \delta y_{5/2}$	
		$\delta y_{7/2} = y_4 - y_3$		
$x_0 + 4h$	y_4			

8.1.4 : The Shift Operator E

The shift operator E is defined by

$$E\,f(x) = f(x+h)$$

$$\therefore \quad E^2 f(x) = E(E\,f(x)) = E(f(x+h)) = f(x+2h)$$

$$E^3 f(x) = f(x+3h)$$

$$\vdots$$

$$E^n f(x) = f(x+nh)$$

The inverse shift operator E^{-1} is defined as

$$E^{-1}f(x) = f(x-h)$$

$$E^{-2}f(x) = f(x-2h)$$

$$\vdots$$

$$E^{-n}f(x) = f(x-nh)$$

8.1.5 : The Average Operator μ

The average operator μ is defined as

$$\mu\, f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right)+f\left(x-\frac{h}{2}\right)\right]$$

and

$$\mu^2 f(x) = \mu\,[\mu\, f(x)] = \mu\left[\frac{f\left(x+\frac{h}{2}\right)+f\left(x-\frac{h}{2}\right)}{2}\right]$$

$$= \frac{1}{4}\left\{f\left(x+\frac{h}{2}+\frac{h}{2}\right)-f\left(x+\frac{h}{2}-\frac{h}{2}\right)\right.$$

$$\left.+f\left(x-\frac{h}{2}+\frac{h}{2}\right)-f\left(x-\frac{h}{2}-\frac{h}{2}\right)\right\}$$

$$= \frac{1}{4}\,[f(x+h)-f(x-h)]$$

8.1.6 : Symbolic Relations

1) Show that $E[f(x)] = (1+\Delta)\, f(x)$

We have $E[f(x)] = f(x+h)$

and $(1+\Delta)\, f(x) = f(x)+f(x+h)-f(x)$

$$= f(x+h) = E\,[f(x)]$$

Thus $\boxed{E \equiv 1+\Delta}$ or $\Delta \equiv E-1$

$\therefore$ $E^2 \equiv (1+\Delta)^2$ and $\boxed{E^n \equiv (1+\Delta)^n}$

2) $E\nabla\,[f(x)] = \nabla\, E\,[f(x)] = \Delta\, f(x)$

We have $E\,\{\nabla\,[f(x)]\} = E\,\{f(x)-f(x-h)\}$

$$= f(x+h)-f(x) = \Delta\, f(x)$$

and $\nabla\,\{E\, f(x)\} = \nabla\, f(x+h)$

$$= f(x+h)-f(x) = \Delta\, f(x)$$

Thus $E\nabla\,[f(x)] = \nabla\, E\,[f(x)] = \Delta\, f(x)$

$\therefore$ $\boxed{E\,\nabla \equiv \nabla\, E \equiv \Delta}$

3) Relation between Δ and ∇

$\{(1+\Delta)\,(1-\nabla)\}\,[f(x)] = f(x)$

$\Rightarrow$ We have $\{(1+\Delta)\,(1-\nabla)\}\, f(x) = (1+\Delta)\,\{(1-\nabla)f(x)\}$

$= (1+\Delta)\,\{f(x)-f(x)+f(x-h)\}$

$= (1+\Delta)\, f(x-h)$

$= E\, f(x-h) = f(x) = 1\cdot f(x)$

$\therefore$ $\boxed{(1+\Delta)(1-\nabla) \equiv 1}$

4) Relation between the operator E and D

From differential calculus, we have

$$\frac{d}{dx} f(x) = D\, f(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \qquad \text{... (8.1)}$$

Now by Taylor's series, we have

$$f(x+h) = f(x)+h\, f'(x)+\frac{h^2}{2!}\, f''(x)+\ldots$$

$$E\, f(x) = f(x)+h\, D\, f(x)+\frac{h^2}{2!}\, D^2 f(x)+\ldots$$

$$= \left\{1+h\,D+\frac{h^2}{2!}\,D^2+\ldots\right\} f(x)$$

$$E\,f(x) = e^{hD} f(x)$$

$$E \equiv e^{hD}$$

$$\therefore \quad hD \equiv \log E \equiv \log(1+\Delta)$$

$$D \equiv \frac{1}{h}\left\{\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \ldots\right\}$$

5) Relation between δ and E and μ and E

$\delta \equiv E^{\frac{1}{2}} - E^{-\frac{1}{2}}$ and $\mu \equiv \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)$

$\Rightarrow$ We have

$$\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right) = E^{\frac{1}{2}} f(x) - E^{-\frac{1}{2}} f(x)$$

$$\delta f(x) = \left[E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right] f(x)$$

$$\therefore \quad \delta \equiv E^{\frac{1}{2}} - E^{-\frac{1}{2}}$$

Now

$$\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right]$$

$$= \frac{1}{2}\left[E^{\frac{1}{2}} f(x) + E^{-\frac{1}{2}} f(x)\right]$$

$$= \left(\frac{E^{1/2} + E^{-1/2}}{2}\right) f(x)$$

$$\therefore \quad \boxed{\mu \equiv \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)}$$

6) Relation between μ and δ : $\mu = \left[1 + \frac{\delta^2}{4}\right]^{1/2}$

We have $\delta^2 = \left[E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right]^2 = E + E^{-1} - 2$

$$\therefore \quad \left[1 + \frac{\delta^2}{4}\right]^{\frac{1}{2}} = \left[1 + \frac{1}{4}\left(E + E^{-1} - 2\right)\right]^{\frac{1}{2}}$$

$$= \frac{1}{2}\left[E + E^{-1} + 2\right]^{\frac{1}{2}} = \frac{1}{2}\left[E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right] = \mu$$

7) Prove that $\Delta\nabla \equiv \nabla\Delta = \delta^2$

Consider $\Delta\nabla\, f(x) = \Delta[\nabla f(x)]$

$$= \Delta[f(x) - f(x-h)]$$

$$= \Delta f(x) - \Delta f(x-h)$$

$$= f(x+h) - f(x) - f(x) + f(x-h)$$

$$\Delta\nabla f(x) = f(x+h) - 2f(x) + f(x-h) \quad \ldots (8.2)$$

$$\nabla\Delta f(x) = \nabla[\Delta f(x)] = \nabla[f(x+h) - f(x)]$$

$$= \nabla f(x+h) - \nabla f(x)$$

$$= f(x+h) - f(x) - f(x) + f(x-h)$$

$$\nabla\Delta f(x) = f(x+h) - 2f(x) + f(x-h) \quad \ldots (8.3)$$

Now

$$\delta^2 f(x) = \delta[\delta f(x)] = \delta\left[f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)\right]$$

$$= \delta f\left(x+\frac{h}{2}\right) - \delta f\left(x-\frac{h}{2}\right)$$

$$= f(x+h) - f(x) - f(x) + f(x-h)$$

$$= f(x+h) - 2f(x) + f(x-h) \quad \ldots (8.4)$$

From (8.2), (8.3), (8.4), we get

$$\Delta\nabla f(x) = \nabla\Delta f(x) = \delta^2 f(x)$$

$$\boxed{\Delta\nabla \equiv \nabla\Delta \equiv \delta^2}$$

Relationship between the operators

	Δ	∇	E
Δ	Δ	$(1-\nabla)^{-1}-1$	$E-1$
∇	$1-(1+\Delta)^{-1}$	∇	$1-E^{-1}$
E	$\Delta+1$	$(1-\nabla)^{-1}$	E
δ	$\Delta(1+\Delta)^{-\frac{1}{2}}$	$\nabla(1-\nabla)^{-\frac{1}{2}}$	$E^{\frac{1}{2}}-E^{-\frac{1}{2}}$
μ	$\left(1+\frac{1}{2}\Delta\right)(1+\Delta)^{\frac{1}{2}}$	$\left(1-\frac{1}{2}\nabla\right)(1-\nabla)^{-\frac{1}{2}}$	$\frac{1}{2}\left(E^{\frac{1}{2}}+E^{-\frac{1}{2}}\right)$

8.1.7 : Generalized Power or Factorial Function

The first factor is x and the successive factors decrease by a constant difference h, such a product of n consecutive factors is known as a factorial function and it is denoted by $x^{(n)}$ where n is a positive integer.

$$\therefore \quad x^{(n)} = x(x-h)(x-2h)\ldots(x-(n-1)h)$$

For h = 1, $\quad x^{(n)} = x(x-1)(x-2)\ldots(x-n+1)$

and for h = 0, $\quad x^{(n)} = x(x-0)(x-0)\ldots(x-0)=x^n$

Fundamental theorem : The n^{th} difference of a polynomial function f(x) of n^{th} degree is constant i.e. $\Delta^n f(x)$ = constant and $\Delta^{n+1} f(x)=0$.

Q.1(c) Prove that

$$\Delta \log f(x) = \log\left\{1+\frac{\Delta f(x)}{f(x)}\right\}$$

Ans. : We have

$$\text{L.H.S.} = \Delta \log f(x) = \log f(x+h) - \log f(x)$$

$$= \log\left[\frac{f(x+h)}{f(x)}\right] = \log\left[\frac{E\,f(x)}{f(x)}\right]$$

$$= \log\left[\frac{(1+\Delta)\,f(x)}{f(x)}\right] = \log\left[\frac{f(x)+\Delta f(x)}{f(x)}\right]$$

$$= \log\left[1+\frac{\Delta f(x)}{f(x)}\right]$$

$$= \text{R.H.S.}$$

Q.2 Evaluate i) $\Delta^n (e^{3x+4})$

ii) $\left(\frac{\Delta^2}{E}\right)x^4$

iii) $\Delta^{14}[(1-ax^5)(1-bx^4)(1-cx^3)(1-dx^2)]$

iv) $\Delta[e^{ax}\sin bx]$ **(Take h = 1)**

Ans. : i) We have

$$\Delta(e^{3x+4}) = e^{3(x+1)+4} - e^{3x+4} = e^{3x+4}(e^3-1)$$

$$\Delta^2(e^{3x+4}) = (e^3-1)\Delta e^{3x+4} = (e^3-1)^2 e^{3x+4}$$

By induction

$$\Delta^n(e^{3x+4}) = (e^3-1)^n e^{3x+4}$$

ii) $$\left(\frac{\Delta^2}{E}\right)x^4 = \left[\frac{(E-1)^2}{E}\right]x^4 = \left[\frac{E^2-2E+1}{E}\right]x^4$$

$$= [E-2+E^{-1}]x^4 = E[x^4]-2x^4+E^{-1}[x^4]$$

$$= (x+1)^4-2x^4+(x-1)^4 = 12x^2$$

iii) Here $f(x) = (1-ax^5)(1-bx^4)(1-cx^3)(1-dx^2)$

is a polynomial of degree 14 and coefficient of x^{14} is abcd.

∴ By fundamental theorem, we have

$$\Delta^{14} f(x) = abcd\,(14\,!)$$

iv) Let $\quad f(x) = e^{ax}$ and $g(x) = \sin bx$

$$\Delta f(x) = e^{ax+a} - e^{ax} = e^{ax}(e^a-1)$$

$$\Delta g(x) = \sin(bx+b) - \sin bx$$

$$= 2\cos\left(bx+\frac{b}{2}\right)\sin\frac{b}{2}$$

$$\therefore \Delta[f(x)\cdot g(x)] = f(x+1)\Delta g(x) + g(x)\Delta f(x)$$

$$= e^{ax+a}\, 2\cos\left(bx+\frac{b}{2}\right)\sin\frac{b}{2} + \sin bx\, e^{ax}\,(e^a - 1)$$

$$= e^{ax}\left[2\cos\left(bx+\frac{b}{2}\right)\sin\frac{b}{2} + (\sin bx)\,(e^a - 1)\right]$$

Q.3 Find $f(6)$ and $f(7)$ if $f(x) = x^3 + 5x - 7$ for $x = -1, 0, 1, 2, 3, 4, 5$.

Ans. : We construct the following table.

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
-1	-13				
		6			
0	-7		0		
		6		6	
1	-1		6		0
		12		6	
2	11		12		0
		24		6	
3	35		18		0
		42		6	
4	77		24		
		66			
5	143				

Now

$$f(6) = E^7 f(-1) = (1+\Delta)^7 f(-1)$$

$$= (1 + {}^7C_1\Delta + {}^7C_2\Delta^2 + {}^7C_3\Delta^3 + {}^7C_4\Delta^4 + {}^7C_5\Delta^5 + {}^7C_6\Delta^6 + {}^7C_7\Delta^7) f(-1)$$

$$= f(-1) + 7\Delta f(-1) + 21\Delta^2 f(-1) + 35\Delta^3 f(-1)$$

$$= -13 + 7\times 6 + 21\times 0 + 35\times 6$$

$$f(6) = 239$$

Now

$$f(7) = f(8-1) = E^8 f(-1) = (1+\Delta)^8 f(-1)$$

$$= (1 + {}^8C_1\Delta + {}^8C_2\Delta^2 + {}^8C_3\Delta^3) f(-1)$$

$$= f(-1) + 8\Delta f(-1) + 28\Delta^2 f(-1) + 56\Delta^3 f(-1)$$

$$= -13 + 8(6) + 28(0) + 56(6)$$

$$f(7) = 371$$

Q.4 Given that

$u_0 + u_8 = 80,\ u_1 + u_7 = 10,\ u_2 + u_6 = 5,\ u_3 + u_5 = 10$ find u_4.

Ans. : As 8 values of u are given $\therefore \Delta^8 u_0 = 0$

$$\therefore (E-1)^8 u_0 = 0$$

$$E^8 u_0 - {}^8C_1 E^7 u_0 + {}^8C_2 E^6 u_0 - {}^8C_3 E^5 u_0 + {}^8C_4 E^4 u_0 - {}^8C_5 E^3 u_0 + {}^8C_6 E^2 u_0 - {}^8C_7 E u_0 + {}^8C_8 u_0 = 0$$

$$u_8 - 8u_7 + 28u_6 - 56u_5 + 70u_4 - 56u_3 + 28u_2 - 8u_1 + u_0 = 0$$

$$\therefore (u_8 + u_0) - 8(u_1 + u_7) + 28(u_6 + u_2) - 56(u_3 + u_5) + 70u_4 = 0$$

$$\Rightarrow 70u_4 = 420$$

$$\Rightarrow u_4 = 6$$

Q.5 Find the a and b in the following table

x	1	2	3	4	5	6	7	8
f(x)	1	8	a	64	b	216	343	512

Ans. : As six values of f(x) are given

$$\therefore \Delta^6 f(x) = 0 \quad \forall x$$

$$\therefore (E-1)^6 f(x) = 0$$

$$\therefore [E^6 - 6E^5 + 15E^4 - 20E^3 + 15E^2 - 6E + 1] f(x) = 0$$

$$\therefore f(x+6) - 6f(x+5) + 15f(x+4) - 20f(x+3) + 15f(x+2) - 6f(x+1) + f(x) = 0 \quad \text{... (Q.5.1)}$$

Substituting x = 1 in (1) we get

$$f(7) - 6f(6) + 15f(5) - 20f(4) + 15f(3) - 6f(2) + f(1) = 0$$

$$\Rightarrow 15f(5) + 15f(3) = 2280$$

$$\therefore a + b = 152 \quad \text{... (Q.5.2)}$$

Substituting x = 2 in equation (1) we get

$$f(8) + 6f(7) + 15f(6) - 20f(5) + 15f(4) - 6f(3) + f(2) = 0$$

$$\Rightarrow -20f(5) - 6f(3) = -2662$$

$$3a + 10b = 1331 \quad \text{... (Q.5.3)}$$

Solving equation (3) and equation (2) we get

$a = f(3) = 27$ and $b = f(5) = 125$

8.2 : Newton's Formulae for Interpolation

I) Newton-Gregory Formula for Forward Interpolation

Consider the set of $(n+1)$ equidistant values of the function $y = f(x)$ viz $[a, f(a)], [a+h, f(a+h)], [a+2h, f(a+2h)], \dots [a+nh, f(a+nh)]$.

Let $P_n(x)$ be a polynomial in x of degree n.

We have

$$P_n(x) = A_0 + A_1 (x-a) + A_2 (x-a)(x-a-h) + A_3 (x-a)(x-a-h)(x-a-2h) + \dots + A_n (x-a)(x-a-h)\dots(x-a-(n-1)h) \quad \text{... (8.5)}$$

where $A_0, A_1, A_2, \dots A_n$ are constants and are chosen such that

$P_n(a) = f(a)$, $P_n(a+h) = f(a+h)$

$P_n(a+2h) = f(a+2h) \dots P_n(a+nh) = f(a+nh)$

Substituting $x = a$ in equation (8.5) we get

$$P_n(a) = A_0 \Rightarrow A_0 = f(a)$$

Substituting $x = a+h$ in equation (8.5) we get

$$P_n(a+h) = A_0 + h A_1$$

$$\therefore \quad f(a+h) = f(a) + h A_1$$

$$\therefore \quad A_1 = \frac{f(a+h)-f(a)}{h} = \frac{\Delta f(a)}{h}$$

Again substituting $x = a+2h$ in equation (8.5), we get

$$P_n(a+2h) = A_0 + 2h A_1 + 2h^2 A_2$$

$$\therefore \quad f(a+2h) = f(a) + 2(f(a+h)-f(a)) + 2h^2 A_2$$

$$\therefore \quad 2h^2 A_2 = f(a+2h) - 2f(a+h) + f(a) = \Delta^2 f(a)$$

$$\therefore \quad A_2 = \frac{\Delta^2 f(a)}{2!\,h^2}$$

Similarly, we get $A_3 = \dfrac{\Delta^3 f(a)}{3!\,h^3} \dots A_n = \dfrac{\Delta^n f(a)}{n!\,h^n}$

Now, substituting values of $A_0, A_1, A_2, \dots A_n$ in equation (8.5) we get

$$P_n(x) = f(a) + \frac{\Delta f(a)}{h}(x-a) + \frac{\Delta^2 f(a)}{2!\,h^2}(x-a)(x-a-h) + \frac{\Delta^3 f(a)}{3!\,h^3}(x-a)(x-a-h)(x-a-2h) + \dots + \frac{\Delta^n f(a)}{n!\,h^n}(x-a)(x-a-h)\dots(x-a-(n-1)h) \quad \text{... (8.6)}$$

which is Newton-Gregory formula for forward interpolation

Substituting $\dfrac{x-a}{h} = u$ or $x = a + hu$ in equation (8.6) We get,

$$f(a+uh) = P_n(x) \equiv f(x)$$

$$= f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \dots + \frac{u(u-1)(u-2)\dots(u-n+1)}{n!}\Delta^n f(a)$$

In factorial notation, we get,

$$P_n(x) = f(a) + u^{(1)}\Delta f(a) + u^{(2)}\frac{\Delta^2 f(a)}{2!} + \dots + u^{(n)}\frac{\Delta^n f(a)}{n!}$$

where $u^{(n)} = u(u-1)(u-2)\dots(u-(n-1))$

Note : This formula is useful for interpolation near the beginning of a set of tabular values.

II) Newton-Gregory Formula for Backward Interpolation

Consider the set of $n+1$ equidistant values of the function $y = f(x)$ viz $[a, f(a)], [a+h, f(a+h)], [a+2h, f(a+2h)], \dots [a+nh, f(a+nh)]$. Let $P_n(x)$ be a polynomial in x of degree n.

We have

$$P_n(x) = A_0 + A_1(x-a-nh) + A_2(x-a-nh)$$
$$(x-a-nh+h) + A_3\,(x-a-nh)$$
$$(x-a-nh+h)\,(x-a-nh+2h) + \ldots$$
$$+ A_n(x-a-nh)\,(x-a-nh+h)\ldots(x-a-h) \quad \ldots (8.7)$$

where $A_0, A_1, A_2, \ldots A_n$ are constants and are chosen such that $P_n(a+nh) = f(a+nh) \ldots$

$\ldots\ldots\; P_n(a) = f(a)$

Substituting $x = a+nh$ in equation (8.7) we get

$$P_n(a+nh) = A_0 \quad \therefore A_0 = f(a+nh)$$

Substituting $x = a+nh-h$ in equation (1)

we get

$$P_n(a+nh-h) = A_0 - hA_1$$
$$\therefore f(a+nh-h) = f(a+nh) - hA_1$$
$$\therefore \quad A_1 = \frac{f(a+nh)-f(a+nh-h)}{h}$$
$$= \frac{\Delta f(a+nh)}{h}$$

Substituting $x = a+nh-2h$ in equation (1)

we get

$$P_n(a+nh-2h) = A_0 + A_1(-2h) + A_2(-2h)(-h)$$
$$\therefore \quad 2h^2 A_2 = -f(a+nh) + 2\,[f(a+nh)$$
$$-f(a+nh-h)] + f(a+nh-2h)$$
$$\therefore \quad 2h^2 A_2 = f(a+nh) - 2f(a+nh-h) + f(a+nh-2h)$$
$$A_2 = \frac{\nabla^2 f(a+nh)}{2!\,h^2}$$

Similarly we get

$$A_3 = \frac{1}{3!\,h^3}\nabla^3 f(a+nh)\ldots$$

$$\ldots\ldots\; A_n = \frac{1}{n!\,h^n}\nabla^n f(a+nh)$$

Substituting values of $A_0, A_1, A_2, \ldots A_n$ in equation (1) we get

$$P_n(x) = f(a+nh) + \frac{\nabla f(a+nh)}{h}(x-a-nh)$$
$$+ \frac{\nabla^2 f(a+nh)}{2!\,h^2}(x-a-nh)(x-a-nh+h)$$
$$+ \ldots + \frac{\nabla^n f(a+nh)}{n!\,h^n}(x-a-nh)\ldots(x-a-h) \quad \ldots (8.8)$$

which is the Newton-Gregory formula for backward interpolation

Substituting $u = \dfrac{x-(a+nh)}{h}$ i.e. $x = a+nh+uh$ in equation (8.8) we get,

$$P_n(x) = P_n(a+nh+uh) = f(a+nh) + u\nabla f(a+nh)$$
$$+ \frac{u(u+1)}{2!}\nabla^2 f(a+nh) + \ldots$$
$$\ldots + \frac{u(u+1)(u+2)\ldots(u+n-1)}{n!}\nabla^n f(a+nh)$$

Note : This formula is useful for interpolation near the end of a set of tabular values.

Q.6 From the following table find

i) y when x = 7 ii) y when x = 17 iii) y when x = 19

x :	8	10	12	14	16	18
f(x) = y :	10	19	32.5	54	89.5	15.4

Ans. : Consider the following forward difference table.

x	y = f(x)	Δ f(x)	Δ^2 f(x)	Δ^3 f(x)	Δ^4 f(x)	Δ^5 f(x)
8	10					
		9				
10	19		4.5			
		13.5		3.5		
12	32.5		8		2.5	

		21.5		6		6.5
14	54		14		9	
		35.5		15		
16	89.5		29			
		64.5				
18	15.4					

i) To find y at x = 7 :

As x = 7 is at the beginning of the table but x = 7 is the outside point of the given range of x.

∴ Use Newton-Gregory forward interpolation formula for extrapolation at x = 7.

We have

$$f(x) = f(a) + u\,\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(a) + \frac{u(u-1)(u-2)(u-3)(u-4)}{5!}\Delta^5 f(a) + \ldots \quad \ldots (Q.6.1)$$

Here a = 8 and $u = \frac{x-a}{h} = \frac{x-8}{2} = \frac{7-8}{2} = -\frac{1}{2}$

∴ Equation (1) becomes

$$f(7) = 10 - \frac{9}{2} + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{1}{2!}\right)(4.5) + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\frac{3.5}{6} + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)\frac{2.5}{24} + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)\left(-\frac{9}{2}\right)\frac{6.5}{120}$$

$$f(7) = 5.1777$$

ii) To find y at x = 17

As x = 17 is at the end of the given table, we use N.G. Backward interpolation formula at x = 17.

We have

$$f(x) = f(a+nh) + u\nabla f(a+nh) + \frac{u(u+1)}{2!}\nabla^2 f(a+nh) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(a+nh) + \frac{u(u+1)(u+2)(u+3)}{4!}\Delta^4 f(a+nh) + \frac{u(u+1)(u+2)(u+3)(u+4)}{5!}\nabla^4 f(a+nh) + 0 \quad \ldots (Q.6.2)$$

Here x = 17, $u = \frac{x-(a+nh)}{h} = \frac{17-18}{2} = -\frac{1}{2}$

Equation (Q.6.2) becomes

$$\therefore \quad f(17) = 154 + \left(-\frac{1}{2}\right)(64.5) + \left(-\frac{1}{2}\right)\left(-\frac{1}{2}+1\right)\frac{29}{2} + \left(-\frac{1}{2}\right)\left(-\frac{1}{2}+1\right)\left(-\frac{1}{2}+2\right)\frac{15}{6} + \left(-\frac{1}{2}\right)\left(-\frac{1}{2}+1\right)\left(-\frac{1}{2}+2\right)\left(-\frac{1}{2}+3\right)\frac{9}{24} + \left(-\frac{1}{2}\right)\left(-\frac{1}{2}+1\right)\left(-\frac{1}{2}+2\right)\left(-\frac{1}{2}+3\right)\left(-\frac{1}{2}+4\right)\frac{6.5}{120}$$

$$\therefore \quad f(17) = 126.841$$

iii) To find y at x = 19

As x = 19 is at the end of the table but x = 19 is the outside point of the given range of x.

∴ We use NGBIF for extrapolation at x = 19

$$\therefore \quad u = \frac{19-18}{2} = \frac{1}{2}$$

$\therefore$ Equation (Q.6.2) becomes

$$f(19) = 154+\left(\frac{1}{2}\right)(64.5)+\frac{1}{2}\left(\frac{3}{2}\right)\frac{29}{2}$$

$$= +\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\left(\frac{15}{6}\right)+\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\left(\frac{7}{2}\right)\frac{9}{24}$$

$$= +\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\left(\frac{7}{2}\right)\left(\frac{9}{2}\right)\frac{6.5}{120}$$

$\therefore$ $f(19) = 219.208$

Q.7 From the following data, find the number of students who obtained less than 45 marks.

Marks	30-40	40-50	50-60	60-70	70-80
No. of students	31	42	51	35	31

Ans. : We construct the following table.

Marks x	No. of students f(x)	Δ f(x)	Δ^2 f(x)	Δ^3 f(x)	Δ^4 f(x)
below 40	31				
		42			
below 50	73		9		
		51		– 25	
below 60	124		– 16		37
		35		12	
below 70	159		– 4		
		31			
below 80	190				

As 45 lies at the beginning of the given table.

$\therefore$ We use N.G.F. for forward interpolation.

We have

$$f(x) = f(a)+u\Delta f(a)+\frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(a)+0 \quad ...(Q.7.1)$$

Here $u = \frac{x-a}{h} = \frac{45-40}{10} = \frac{1}{2}$

$\therefore$ Equation (Q.7.1) becomes

$$f(45) = f(40)+\frac{1}{2}\Delta f(40)+\frac{1}{2}\left(\frac{1}{2}-1\right)\frac{\Delta^2 f(40)}{2} + \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\frac{\Delta^3 f(40)}{6} + \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)\frac{\Delta^4 f(40)}{120}$$

$$f(45) = 47.868 \simeq 48$$

Thus approximately 48 students obtained marks less than 45.

Q.8 From the tabulated values of x and y given below, prepare forward difference table. Find polynomial passing through the points and find it's slope at x = 1.5

x	0	2	4	6	8
y	5	29	125	341	725

Ans. : Consider the following forward difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	5				
		24			
			72		
2	29			48	
		96			
			120		
4	125				0
		216		48	
			168		
6	341				
		384			
8	725				

We have $u = \dfrac{x-x_0}{h} = \dfrac{x-0}{2} = \dfrac{x}{2}$

By Newton's forward difference formula we have

$$y = f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots$$

$$y = f(x) = 5 + \frac{x}{2}(24) + \frac{x}{2}\left(\frac{x}{2}-1\right)\frac{72}{2} + \frac{x}{2}\left(\frac{x}{2}-1\right)\left(\frac{x}{2}-2\right)\frac{68}{6}$$

$$y = x^3 + 3x^2 + 2x + 5$$

$\therefore$ At $x = 1.5$, $y = 18.125$

And $\dfrac{dy}{dx} = 3x^2 + 6x + 2$

$\therefore \left(\dfrac{dy}{dx}\right)_{x=1.5}$ = Slope at (1.5) = 17.75

Q.9 From the following data find the cubic polynomial and hence find f(4).

x	0	1	2	3
f(x)	1	2	1	10

Ans. : We construct the following difference table.

x	f(x)	Δ f(x)	Δ^2 f(x)	Δ^3 f(x)
0	1			
		1		
1	2		– 2	
		– 1		12
2	1		10	
		+ 9		
3	10			

Take $a = 0$, $\therefore$ $u = \dfrac{x-a}{h} = \dfrac{x-0}{1} = x$

$\therefore$ Using NGFIF we get

$$f(x) = f(0) + x\Delta f(0) + \frac{x(x-1)}{2}\Delta^2 f(0) + \frac{x(x-1)(x-2)}{6}\Delta^3 f(0) + 0$$

$$= 1 + x + (x^2 - x)(-1) + (x^3 - 3x^2 + 2x)(2)$$

$$f(x) = 2x^3 - 7x^2 + 6x + 1$$

which is the required polynomial.

We have, if a tabulated function is a polynomial then interpolation and extrapolation are obtained by using polynomial.

$$\therefore \quad f(4) = 2(4)^3 - 7(4)^2 + 6(4) + 1$$

$$= 128 - 112 + 25 = 16 + 25 = 37$$

Note : We get same value of f(4) by using Newton-Gregory backward interpolation formula.

III) (A) Lagrange's Interpolation Formula

The Newton-Gregory backward or forward interpolation formulae are applicable only when argument x is equally spaced. But the Lagranges interpolation formula is more general and can be applied for unequally spaced argument.

Let $y = f(x)$ be continuous and differentiable $(n+1)$ times in the interval (a, b). Given $n+1$ values $[x_0, f(x_0)], [x_1, f(x_1)], [x_2, f(x_2)] \ldots [x_n, f(x_n)]$ where arguments $x_0, x_1, x_2, \ldots x_n$ are not necessarily equally spaced. As $(n+1)$ values of f(x) are considered so $(n+1)^{th}$ differences are zero. Let $P_n(x)$ be a polynomial in x of degree n. We write $P_n(x)$ in the following form.

$$P_n(x) = A_0(x-x_1)(x-x_2)\ldots(x-x_n) + A_1(x-x_0)(x-x_2)\ldots(x-x_n) + \ldots + A_n(x-x_0)(x-x_1)\ldots(x-x_{n-1}) \quad \ldots (8.9)$$

where $A_0, A_1, A_2, \ldots A_n$ are constants and are chosen such that $P_n(x_0) = f(x_0), P_n(x_1) = f(x_1) \ldots P_n(x_n) = f(x_n)$

Substituting $x = x_0$ in equation (8.9), we get

$$P_n(x_0) = f(x_0) = A_0(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)$$

$$\therefore \quad A_0 = \frac{1}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} f(x_0)$$

Substituting $x = x_1$ in equation (8.9), we get

$$f(x_1) = A_1(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)$$

$$\therefore \quad A_1 = \frac{1}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} f(x_1)$$

Similarly, we get

$$A_2 = \frac{1}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)} f(x_2)$$

$$\vdots$$

$$A_n = \frac{1}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})} f(x_n)$$

Substituting values of $A_0, A_1, A_2, \ldots A_n$ in equation (1), we get

$$f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} f(x_0)$$

$$+ \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} f(x_1)$$

$$+ \ldots + \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})} f(x_n)$$

Which is the Lagranges interpolation formula. We may write above formula as

$$P_n(x) = f(x) = L_0 f(x_0) + L_1 f(x_1) + \ldots + L_i f(x_i) + \ldots + L_n f(x_n)$$

where $L_i = \frac{(x-x_0)(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)}$

For $0 \le i \le n$

(B) Inverse Lagranges Interpolation formula

In this formula we have to find x for given value of y by interchanging roles of x and y in the Lagranges interpolation formula.

Thus

$$x = \frac{(y-y_1)(y-y_2)\ldots(y-y_n)}{(y_0-y_1)(y_0-y_2)\ldots(y_0-y_n)} x_0 + \ldots + \frac{(y-y_0)(y-y_1)\ldots(y-y_{n-1})}{(y_n-y_0)(y_0-y_1)\ldots(y_n-y_{n-1})} x_n$$

i.e. to find x at $y = y^*$, we substitute $y = y^*$ in above equation and then simplify.

Q.10 Find f(5) by using Lagranges interpolation formula, given that $f(1) = 2$, $f(2) = 4$, $f(3) = 8$, $f(4) = 16$, $f(7) = 128$.

Ans. : The given data will be tabulated as follows

x	1	2	3	4	7
f(x)	2	4	8	16	128

Here $x_0 = 1, x_1 = 2, x_2 = 3, x_3 = 4, x_4 = 7$ and $x = 5$

By Lagranges formula we have

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} f(x_0)$$

$$+\frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)}f(x_1)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)}f(x_2)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)}f(x_3)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}f(x_4)$$

$$f(5) = \frac{(-12)}{36}(2)+\frac{(-16)}{(-10)}(4)+\frac{(-24)}{8}(8)+\frac{(-48)}{-18}(16)+\frac{24}{360}(128)$$

$$f(5) = 33.13$$

Q.11 Using Lagranges interpolation formula to evaluate y for x = 1.07 for the following set of values.

x	1.0	1.2	1.3	1.5
y	1.0	1.728	2.197	3.375

Ans. : We have

$x_0 = 1, \quad x_1 = 1.2, \quad x_2 = 1.3, x_3 = 1.5$

$y_0 = 1, \quad y_1 = 1.728, \quad y_2 = 2.197, \quad y_3 = 3.375$

By Lagranges interpolation formula

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

Putting values of x_0, x_1, x_2, x_3 and y_0, y_1, y_2, y_3 and $x = 1.07$, we get,

$$\boxed{y = 0.35816}$$

Q.12 The velocity distribution of fluid near a flat surface is given below.

x	0.1	0.3	0.6	0.8
v	0.72	1.81	2.73	3.47

where x is the distance from the surface (mm) and v is the velocity (mm/sec). Use Lagrangues interpolation formula to obtain velocity at x = 0.4

Ans. : We have

$x_0 = 0.1, \ x_1 = 0.3, \ x_2 = 0.6, \ x_3 = 0.8, \ v_0 = 0.72, \ v_1 = 1.81,$
$v_2 = 2.73, \ v_3 = 3.47$

By Lagranges interpolation formula

$$v = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}v_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}v_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_1)}v_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}v_3$$

$$= \frac{0.72(0.4-0.3)(0.4-0.6)(0.4-0.8)}{(0.1-0.3)(0.1-0.6)(0.1-0.8)} + \frac{1.81(0.4-0.1)(0.4-0.6)(0.4-0.8)}{(0.3-0.1)(0.3-0.6)(0.3-0.8)}$$

$$+\frac{2.73(0.4-0.1)(0.4-0.3)(0.4-0.8)}{(0.6-0.1)(0.6-0.3)(0.6-0.8)} + \frac{3.47(0.4-0.1)(0.4-0.3)(0.4-0.6)}{(0.8-0.1)(0.8-0.3)(0.8-0.6)}$$

$$= 2.16028$$

8.3 : Numerical Differentiation

I) Derivatives using Newton-Gregory Forward Interpolation Formula

Consider the Newton forward interpolation formula

$$y = f(x) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(a) + \ldots \quad \ldots (8.10)$$

where $u = \frac{x-a}{h}$ where $a = x_0$ and h is interval difference

$$\therefore \quad y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{6}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{24}\Delta^4 y_0 + \ldots \quad \ldots (8.11)$$

where $u = \frac{x-x_0}{h}$

Here $y \to u \to x$

$\therefore$ Differentiating equation (8.11) with respect to x, we get

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{h}\frac{dy}{du}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{h}\left\{\Delta y_0 + \frac{(2u-1)}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{24}\Delta^4 y_0 + \ldots\right\} \quad \ldots (8.12)$$

At $x = x_0$, $u = \frac{x-x_0}{h} = 0$

$$\therefore \quad \left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\right]$$

Differentiating equation (8.12) with respect to x, we get

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{du}\left[\frac{dy}{dx}\right]\cdot\frac{du}{dx} = \frac{1}{h}\frac{d}{du}\left[\frac{dy}{dx}\right]$$

$$= \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{6u-6}{6}\Delta^3 y_0 + \frac{12u^2-36u+22}{24}\Delta^4 y_0 + \ldots\right] \quad \ldots (8.13)$$

$$\therefore \quad \left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \ldots\right]$$

Now differentiating equation (8.11) with respect to x we get

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left[\frac{d^2y}{dx^2}\right] = \frac{d}{du}\left[\frac{d^2y}{dx^2}\right]\cdot\frac{du}{dx} = \frac{1}{h}\frac{d}{du}\left[\frac{d^2y}{dx^2}\right]$$

$$\therefore \quad \frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\Delta^3 y_0 + \frac{24u-36}{24}\Delta^4 y_0 + \ldots\right]$$

$$\therefore \quad \left[\frac{d^3y}{dx^3}\right]_{x=x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \ldots\right]$$

II. Derivatives using Newton-Gregory Backward Interpolation Formula

Consider the Newton's backward interpolation formula

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \frac{u(u+1)(u+2)(u+3)}{4!}\Delta^4 y_n + \ldots$$

$$\therefore \quad y = y_n + u\nabla y_n + \frac{u^2+u}{2}\nabla^2 y_n + \frac{u^3+3u^2+2u}{6}\nabla^3 y_n + \frac{u^4+6u^3+11u^2+6u}{24}\nabla^4 y_n + \ldots \quad \ldots (8.14)$$

where $u = \frac{x-x_n}{h}$ Here $y \to u \to x$

Differentiating equation (8.14) with respect to x, we get

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{h}\frac{dy}{du}$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2u+1}{2}\nabla^2 y_n + \frac{3u^2+6u+2}{6}\nabla^3 y_n + \frac{4u^3+18u^2+22u+6}{24}\nabla^4 y_n + \ldots\right] \quad \ldots (8.15)$$

At $x = x_n$, $u = 0$

$$\therefore \left[\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \ldots\right]$$

Differentiating equation (8.15) with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{du}\left[\frac{dy}{dx}\right]\cdot\frac{du}{dx} = \frac{1}{h}\frac{d}{dx}\left[\frac{dy}{dx}\right]$$

$$= \frac{1}{h^2}\left[\nabla^2 y_n + \frac{6u+6}{6}\Delta^3 y_n + \frac{12u^2+36u+22}{24}\nabla^4 y_n + \ldots\right] \quad \ldots (8.16)$$

At $x = x_n$, $u = 0$

$$\therefore \left[\frac{d^2y}{dx^2}\right]_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \ldots\right]$$

Differentiating equation (8.16) with respect to x, we get

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{du}\left(\frac{d^2y}{dx^2}\right)\cdot\frac{du}{dx} = \frac{1}{h}\frac{d}{du}\left(\frac{d^2y}{dx^2}\right)$$

$$\therefore \quad \frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{24u+36}{24}\nabla^4 y_n + \ldots\right]$$

At $x = x_n$, $u = 0$

$$\therefore \left[\frac{d^3y}{dx^3}\right]_{x=x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \ldots\right]$$

III. Applications of Derivatives to Find Maxima and Minima of a Tabulated Function

Consider Newton's forward interpolation formula

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots$$

Differentiating with respect to u, we get

$$\frac{dy}{du} = \Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + \ldots$$

Consider $\frac{dy}{du} = 0$

$$\therefore \Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 = 0$$

Solving above equation for u and then find x by using $x = x_0 + uh$ at which y is maximum or minimum.

Q.13 Given that

x	1	1.1	1.2	1.3	1.4	1.5	1.6
y	7.989	8.403	8.781	9.129	9.451	9.750	10.031

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at i) $x = 1.1$ ii) $x = 1$ iii) $x = 1.6$

Ans. : Consider the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1	7.989						
		0.414					
1.1	8.403		– 0.036				
		0.378		0.006			
1.2	8.781		– 0.030		– 0.002		
		0.348		0.004		0.002	
1.3	9.129		– 0.026		0.000		– 0.003
		0.322		0.004		– 0.001	
1.4	9.451		– 0.023		– 0.001		
		0.299		0.005			
1.5	9.750		– 0.018				
		0.281					
1.6	10.031						

i) We have

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\right]$$

Here $h = 0.1$ and $x_1 = 1.1$

$$\therefore \left(\frac{dy}{dx}\right)_{x_1 = 1.1} = \frac{1}{h}\left[\Delta y_1 - \frac{1}{2}\Delta^2 y_1 + \frac{1}{3}\Delta^3 y_1 - \frac{1}{4}\Delta^4 y_1 + \frac{1}{5}\Delta^5 y_1\right]$$

$$\left(\frac{dy}{dx}\right)_{1.1} = \frac{1}{0.1}\left[0.378 - \frac{1}{2}(-0.03) + \frac{1}{3}(0.004) - \frac{1}{4}(0) + \frac{1}{5}(-0.001)\right] = 3.941$$

$$\left(\frac{d^2y}{dx^2}\right)_{x_1} = \frac{1}{h^2}\left[\Delta^2 y_1 - \Delta^3 y_1 + \frac{11}{12}\Delta^4 y_1 - \frac{5}{6}\Delta^5 y_1 + \ldots\right]$$

$$= \frac{1}{(0.1)^2}\left[-0.03 - (0.004) + \frac{11}{12}(0) - \frac{5}{6}(-0.001)\right]$$

$$= -3.3167$$

ii)

$$\left(\frac{dy}{dx}\right)_{x_0 = 1} = \frac{1}{0.1}\left[0.414 - \frac{1}{2}(-0.036) + \frac{1}{3}(0.006) - \frac{1}{4}(-0.002) + \frac{1}{5}(0.002) + \frac{1}{6}(-0.003)\right]$$

$$= +2.9$$

$$\left(\frac{d^2y}{dx^2}\right)_{x_0 = 1} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \ldots\right]$$

$$= \frac{1}{(0.1)^2}\left[-0.036 - 0.006 + \frac{11}{12}(-0.002) - \frac{5}{6}(0.002) + \ldots\right] = -2.9827$$

iii) By using backward difference table, we have

$$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^5 y_n + \frac{1}{5}\nabla^6 y_n + \ldots\right]$$

Here $h = 0.1$, $x_n = 1.6$, $\nabla y_6 = 0.281$, $\nabla^2 y_6 = -0.018$ etc.

$$\therefore \left(\frac{dy}{dx}\right)_{1.6} = \frac{1}{0.1}\left[0.281 + \frac{1}{2}(-0.018) + \frac{1}{3}(0.005) + \frac{1}{4}(-0.001) + \frac{1}{5}(-0.001)\right] = 2.732$$

$$\left(\frac{d^2y}{dx^2}\right)_{1.6} = \frac{1}{(0.1)^2}\left[-0.018+0.005+\frac{11}{12}(-0.001)\right.$$
$$\left.+\frac{5}{6}(-0.001)\right] = -1.475$$

Q.13(a) Determine the value of $y = \sqrt{151}$, using Newton's forward difference formula, from the following data.

[SPPU : Dec.-22, Marks 5]

x	150	152	154	156
$y=\sqrt{x}$	12.247	12.329	12.410	12.490

Ans. : We consider following forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
150	12.247			
		0.0820		
152	12.329		− 0.001	
		0.0810		0
154	12.410		− 0.001	
		0.0800		
156	12.490			

Here Given $h = 2$, $x_0 = 150$, $x = 151$ and $u = 0.5$

$\therefore$
$$x = x_0 + uh$$
$$u = \frac{x-x_0}{h} = \frac{151-150}{2} = \mathbf{0.5}$$

By Newton's forward difference formula,

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots$$
$$= 12.247 + 0.5(0.0820) + \frac{0.5(0.5-1)}{2!}(-0.001) + 0$$

$$\boxed{y(151) = 12.288}$$

Q.14 The population of a certain city is shown in the following table

Year (x)	1931	1941	1951	1961	1971
Population (y) (in Lakhs)	46.62	60.80	79.95	103.56	132.65

Find the rate of growth of the population in 1961.

Ans. : Here h = 10, the rate of growth of population is dy/dx.

Here find $\frac{dy}{dx}$ at x = 1961 which lies near to the end of tabular values so we use Newton's backward interpolation formula for derivatives.

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2u+1}{2}\nabla^2 y_n + \frac{3u^2+6u+2}{6}\nabla^3 y_n\right.$$
$$\left.+\frac{4u^3+18u^2+22u+6}{24}\nabla^4 y_n + \ldots\right]$$

Here $u = \frac{x-x_n}{h} = \frac{1961-1971}{10} = -1$

Consider the backward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1931	40.62				
		20.18			
1941	60.80		− 1.03		
		19.15		5.49	
1951	79.95		4.46		− 4.47
		23.61		1.02	
1961	103.56		5.48		
		29.09			
1971	132.65				

Thus

$$\left(\frac{dy}{dx}\right)_{x=1961} = \frac{1}{10}\left[29.09-\frac{1}{2}(5.48)+\frac{3(-1)^2+6(-1)+2}{6}\right.$$

$$\left.(1.02)+\frac{2(-1)^3+9(-1)^2+11(-1)+3}{12}(-4.47)\right]$$

$$= 2.6553$$

$\therefore$ The rate of growth of population in the year 1961 is 2.6553 Lakhs.

8.4 : Numerical Integration

8.4.1 : A General Quadrature Formula

Let $I = \int_a^b y\,dx$ where $y = f(x)$ be given for certain equidistant values of arguments say $x_0, x_0+h, \ldots$ Let the range b-a be divided into n equal parts each of width h.

$$\therefore \quad h = \frac{b-a}{n} \quad \therefore \; b-a = nh$$

Let $x_0 = a,\ x_1 = a+h,\ x_2 = a+2h, \ldots x_n = a+nh = b$

Put $u = \dfrac{x-a}{h} \quad \therefore \; dx = h\,du$

and $x = a+uh$

At $x = a,\ u = 0$ and at $x = a+nh \quad u = n$

$$\therefore \quad I = \int_a^b y\,dx = \int_0^n y(u)\,h\,du$$

By using Newton's forward difference formula in the integrand, we get

$$I = h\int_0^n \left[y_0 + u\Delta y_0 + \frac{u^2-u}{2}\Delta^2 y_0 \right.$$

$$\left. + \frac{u^3-3u^2+2u}{6}\Delta^3 y_0 + \ldots\right] du$$

$$= h\left[u y_0 + \frac{u^2}{2}\Delta y_0 + \left(\frac{u^3}{3}-\frac{u^2}{2}\right)\frac{\Delta^2 y_0}{2}\right.$$

$$\left. + \left(\frac{u^4}{4}-\frac{3u^3}{3}+\frac{2u^2}{2}\right)\frac{\Delta^3 y_0}{6} + \ldots\right]_0^n$$

$$= h\left[n y_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3}-\frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2}\right.$$

$$\left. + \left(\frac{n^4}{4}-n^3+n^2\right)\frac{\Delta^3 y_0}{6} + \ldots \text{ upto } (n+1) \text{ terms}\right] \quad \ldots (8.17)$$

which is known as a General quadrature formula.

We can obtain number of quadrature formulae from (8.17) by substituting n = 1, 2, 3, ...

8.4.2 : Trapezoidal Rule

Substitute n = 1 in General quadrature formula.

$\therefore$ The interval of integration will be from x_0 to x_0+h. As there are only two functional values y_0 and y_1. So there are no differences of orders two and higher than two. $\therefore$ we get

$$\int_{x_0}^{x_0+h} y\,dx = h\left[y_0+\frac{1}{2}\Delta y_0\right]$$

$$= h\left[y_0+\frac{y_1-y_0}{2}\right] = h\left[\frac{y_0+y_1}{2}\right]$$

Similarly

$$\int_{x_0+h}^{x_0+2h} y\,dx = h\left[\frac{y_1+y_2}{2}\right]$$

$$\vdots$$

$$\int_{x_0+(n-1)h}^{x_0+nh} y\,dx = h\left[\frac{y_{n-1}+y_n}{2}\right]$$

Thus

$$\left(\frac{dy}{dx}\right)_{x=1961} = \frac{1}{10}\left[29.09 - \frac{1}{2}(5.48) + \frac{3(-1)^2+6(-1)+2}{6}(1.02) + \frac{2(-1)^3+9(-1)^2+11(-1)+3}{12}(-4.47)\right]$$

$$= 2.6553$$

$\therefore$ The rate of growth of population in the year 1961 is 2.6553 Lakhs.

8.4 : Numerical Integration

8.4.1 : A General Quadrature Formula

Let $I = \int_a^b y\,dx$ where $y = f(x)$ be given for certain equidistant values of arguments say $x_0, x_0+h, \ldots$ Let the range b-a be divided into n equal parts each of width h.

$\therefore \quad h = \frac{b-a}{n} \quad \therefore \; b-a = nh$

Let $x_0 = a,\ x_1 = a+h,\ x_2 = a+2h, \ldots x_n = a+nh = b$

Put $u = \frac{x-a}{h} \quad \therefore \; dx = h\,du$

and $x = a+uh$

At $x = a,\ u = 0$ and at $x = a+nh\ \ u = n$

$$\therefore \quad I = \int_a^b y\,dx = \int_0^n y(u)\,h\,du$$

By using Newton's forward difference formula in the integrand, we get

$$I = h\int_0^n \left[y_0 + u\Delta y_0 + \frac{u^2-u}{2}\Delta^2 y_0 + \frac{u^3-3u^2+2u}{6}\Delta^3 y_0 + \ldots\right] du$$

$$= h\left[u y_0 + \frac{u^2}{2}\Delta y_0 + \left(\frac{u^3}{3} - \frac{u^2}{2}\right)\frac{\Delta^2 y_0}{2} + \left(\frac{u^4}{4} - \frac{3u^3}{3} + \frac{2u^2}{2}\right)\frac{\Delta^3 y_0}{6} + \ldots\right]_0^n$$

$$= h\left[n y_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{6} + \ldots \text{upto } (n+1) \text{ terms}\right] \quad \ldots (8.17)$$

which is known as a General quadrature formula.

We can obtain number of quadrature formulae from (8.17) by substituting n = 1, 2, 3, ...

8.4.2 : Trapezoidal Rule

Substitute n = 1 in General quadrature formula.

$\therefore$ The interval of integration will be from x_0 to x_0+h. As there are only two functional values y_0 and y_1. So there are no differences of orders two and higher than two. $\therefore$ we get

$$\int_{x_0}^{x_0+h} y\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$

$$= h\left[y_0 + \frac{y_1-y_0}{2}\right] = h\left[\frac{y_0+y_1}{2}\right]$$

Similarly

$$\int_{x_0+h}^{x_0+2h} y\,dx = h\left[\frac{y_1+y_2}{2}\right]$$

$$\vdots$$

$$\int_{x_0+(n-1)h}^{x_0+nh} y\,dx = h\left[\frac{y_{n-1}+y_n}{2}\right]$$

Adding these n integrals, we get

$$I = \int_{x_0}^{x_0+nh} y\,dx$$

$$= h\left[\frac{1}{2}(y_0+y_n)+(y_1+y_2+y_3+\ldots+y_{n-1})\right]$$

= Distance between 2 consecutive ordinates

[Mean of first and last ordinate + Sum of all the intermediate ordinates]

This is known as Trapezoidal rule.

8.4.3 : Simpson's $\left(\frac{1}{3}\right)^{rd}$ Rule

Substitute n = 2 in general quadrature formula ∴ the interval of integration will be from x_0 to x_0+2h. As there are only three functional values y_0, y_1, y_2. So there are no differences of orders 3 and higher than 3.

∴ We get

$$\int_{x_0}^{x_0+2h} y\,dx = h\left[2y_0+2\Delta y_0+\left(\frac{8}{3}-2\right)\frac{\Delta^2 y_0}{2}+0\right]$$

$$= h\left[2y_0+2y_1-2y_0+\frac{1}{3}(y_2-2y_1+y_0)\right]$$

$$= \frac{h}{3}[y_0+4y_1+y_2]$$

Similarly

$$\int_{x_0+2h}^{x_0+4h} y\,dx = \frac{h}{3}[y_2+4y_3+y_4]$$

$$\int_{x_0+(n-2)h}^{x_0+nh} y\,dx = \frac{h}{3}[y_{n-2}+4y_{n-1}+y_n]$$

Adding all these n integrals, we get

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}[(y_0+y_n)+4(y_1+y_3+\ldots+y_{n-1})$$

$$+2(y_2+y_4+\ldots+y_{n-2})]$$

$$= \left(\frac{1}{3}\right)^{rd} \text{ of h [(Sum of extreme ordinates)}$$

+ 4 (Sum of odd ordinates)

+ 2 (Sum of even ordinates)]

This is known as Simpson's $\left(\frac{1}{3}\right)^{rd}$ rule.

8.6.4 : Simpson's $\left(\frac{3}{8}\right)^{rd}$ Rule

Substitute n = 3 in general quadrature formula ∴ The interval of integration will be from x_0 to x_0+3h. As there are only 4 functional values y_0, y_1, y_2, y_3. So there are no differences of orders 4 and higher than 4.

∴ We get

$$\int_{x_0}^{x_0+3h} y\,dx = h\left[3y_0+\frac{9}{2}\Delta y_0+\left(\frac{27}{3}-\frac{9}{2}\right)\frac{\Delta^2 y_0}{2}+\left(\frac{81}{4}-27+9\right)\frac{\Delta^3 y_0}{6}\right]$$

$$= h\left[3y_0+\frac{9}{2}(y_1-y_0)+\frac{9}{4}(y_2-2y_1+y_0)\right.$$

$$\left.+\frac{3}{8}(y_3-3y_2+3y_1-y_0)\right]$$

$$= \frac{3h}{8}[y_0+3y_1+3y_2+y_3]$$

Similarly

$$\int_{x_0+3h}^{x_0+6h} y\,dx = \frac{3h}{8}[y_3 + 3y_4 + 3y_5 + y_6]$$

$$\vdots$$

$$\int_{x_0+(n-3)h}^{x_0+nh} y\,dx = \frac{3h}{8}[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n]$$

Adding all these integrals, we get

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{8}[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\ldots+y_{n-1})$$

$$+2(y_3+y_6+y_9+\ldots+y_{n-3})]$$

$$= \left(\frac{3}{8}\right)^{th} \text{ of h [(Sum of extreme ordinates)}$$

+ 2 (Sum of multiple of 3 ordinates)

+ 3 (Sum of all remaining ordinates)]

This is known as Simpson's $\left(\frac{3}{8}\right)^{th}$ rule.

Note :

1) There is no restriction for the number of intervals in Trapezoidal rule.

2) In Simpson's $\left(\frac{1}{3}\right)^{rd}$ rule, the number of subintervals must be even.

3) In Simpson's $\left(\frac{3}{8}\right)^{th}$ rule, the number of subintervals must be multiple of 3.

4) To get more accuracy, divide the given interval into maximum number of subintervals.

Q.15 Evaluate $\int_0^1 \frac{1}{1+x}dx$ **by Simpson's** $\left(\frac{1}{3}\right)^{rd}$ **and** $\left(\frac{3}{8}\right)^{th}$ **rule. Compare the result with actual value.**

Ans. : To apply Simpson's $\left(\frac{1}{3}\right)^{rd}$ and $\left(\frac{3}{8}\right)^{th}$ rules, number of subintervals must even and multiple of 3 so we take 6 subintervals

$$\therefore \quad h = \frac{1-0}{6} = \frac{1}{6}$$

The values of y corresponding to each value of x are given as

x	0	1/6	2/6	3/6	4/6	5/6	1
y	1	6/7	6/8	6/9	6/10	6/11	6/12

i) By Simpson's $\left(\frac{1}{3}\right)^{rd}$ rule

$$I = \int_0^1 \frac{1}{1+x}dx = \frac{h}{3}[(y_0+y_6)+4(y_2+y_3+y_5)+2(y_2+y_4)]$$

$$= \frac{1}{18}\left[\left(1+\frac{6}{12}\right)+4\left(\frac{6}{7}+\frac{6}{9}+\frac{6}{11}\right)+2\left(\frac{6}{8}+\frac{6}{10}\right)\right]$$

$$= \mathbf{0.6931}$$

ii) By Simpson's $\left(\frac{3}{8}\right)^{th}$ rule

$$\int_0^1 \frac{dx}{1+x} = \frac{3h}{8}[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3]$$

$$= \frac{3}{8}\left(\frac{1}{6}\right)\left[\left(1+\frac{6}{12}\right)+3\left(\frac{6}{7}+\frac{6}{8}+\frac{6}{10}+\frac{6}{11}\right)+2\left(\frac{6}{9}\right)\right]$$

$$= \mathbf{0.6931}$$

iii) $I = \int_0^1 \frac{1}{1+x} dx = [\log(1+x)]_0^1 = \log 2 - \log 1$

$= 0.6930$

The actual value is very near to the values obtained by Simpson's rules.

Q.16 Evaluate :

$\int_0^1 \frac{1}{1+x^2} dx$ **taking** $h = \frac{1}{6}$ **by using Simpson's** $\left(\frac{3}{8}\right)^{th}$ **rule. Compare result with actual value.**

Ans. : Let $y = \frac{1}{1+x^2}$, $h = \frac{1}{6}$

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6} = 1$
y	1	0.97297	0.9	0.8	0.69231	0.59016	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's $\left(\frac{3}{8}\right)^{th}$ rule

$$I = \int_0^1 \frac{1}{1+x^2} dx = \frac{3h}{8} [(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5 + 2y_3]$$

$$= \frac{1}{16} [(1+0.5)+3(0.97297+0.9+0.69231+0.59016+2(0.8)]$$

$I = 0.785395$

Actual value :

$$I = \int_0^1 \frac{1}{1+x^2} dx = [\tan^{-1} x]_0^1 = \frac{\pi}{4} = 0.7853981$$

Result is correct upto five decimal places.

Q.16(a) Evaluate $\int_0^1 \frac{dx}{x^2+1}$ **using Simpson's** $\frac{1}{3}^{rd}$ **rule. (Take h = 0.2)**

[SPPU : June-22, Marks 5]

Ans. : Let $y = \frac{1}{1+x^2}$, $h = \frac{1}{6}$

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6} = 1$
y	1	0.97297	0.9	0.8	0.69231	0.59016	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's $\left(\frac{3}{8}\right)^{th}$ rule

$$I = \int_0^1 \frac{1}{1+x^2} dx = \frac{3h}{8} [(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5 + 2y_3]$$

$$= \frac{1}{16} [(1+0.5)+3(0.97297+0.9+0.69231+0.59016+2(0.8)]$$

$I = 0.785395$

Actual value :

$$I = \int_0^1 \frac{1}{1+x^2} dx = [\tan^{-1} x]_0^1 = \frac{\pi}{4} = 0.7853981$$

Result is correct upto five decimal places.

Q.17 Evaluate :

$\int_0^\pi \frac{\sin^2\theta}{5+4\cos\theta} d\theta$ **by Simpson's** $\left(\frac{3}{8}\right)^{th}$ **rule, taking** $h = \frac{\pi}{6}$.

Ans. : Let $y = \frac{\sin^2\theta}{5+4\cos\theta} d\theta$, $h = \frac{\pi}{6}$

θ	0	π/6	π/3	π/2	2π/3	5π/6	π
y	0	0.0295	0.1071	0.2	0.25	0.1627	0

By Simpson's $\left(\frac{3}{8}\right)^{th}$ rule, we have

$$I = \int_0^{\pi} y\, d\theta \frac{3h}{8} [(y_0 + y_6) + 2y_3 + 3(y_1 + y_2 + y_4 + y_5)]$$

$$= \frac{3}{8}\frac{\pi}{6} [0 + 2(0.2) + 3(0.0295 + 0.1071 + 0.25 + 0.1627)]$$

$$= \frac{3}{8}\frac{\pi}{6} [2.04835] = 0.40219$$

$\therefore \quad I = 0.40219$

Q.18 Use Simpson's 3/8th rule, to estimate $\int_1^7 f(x)\,d(x)$ from the following data. **☞ [SPPU : Dec.-22, Marks 5]**

x	1	2	3	4	5	6	7
f(x)	81	75	80	83	78	70	60

Ans. :

x	1	2	3	4	5	6	7
f(x)	$y_0 = 81$	$y_1 = 75$	$y_2 = 80$	$y_3 = 83$	$y_4 = 78$	$y_5 = 70$	$y_6 = 60$

Here, h = 1, by Simposon's $\frac{3^{th}}{8}$ method.

$$= \frac{3h}{8}\{(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)]$$

$$= \frac{3}{8}\{(81 + 60) + 3(75 + 80 + 78 + 70) + 2(83)]$$

$$= \mathbf{456.0000}$$

... *END* ✍

Unit VI

9 Numerical Solutions of Ordinary Differential Equations

9.1 : Finite Differences

Q.1 Explain Euler's method and modified Euler's method.

Ans. : Euler's Method

Consider the differential equation

$$\frac{dy}{dx} = f(x,y) \qquad \text{... (Q.1.1)}$$

with $y(x_0) = y_0$

Suppose we want to solve equation (Q.1.1) at $x = x_1 = x_0 + h$, $x = x_2 = x_0 + 2h$, ... $x = x_0 + nh$.

Integrating equation (Q.1.1) within $[x_0, x_1]$, we get

$$\int_{y_0}^{y_1} dy = \int_{x_0}^{x_1} f(x,y)\,dx$$

$$\therefore \quad y_1 - y_0 = \int_{x_0}^{x_1} f(x,y)\,dx$$

$$\therefore \quad y_1 = y_0 + \int_{x_0}^{x_1} f(x,y)\,dx$$

Assume that $f(x,y) \simeq f(x_0, y_0) \; \forall \; x \in [x_0, x_1]$

$\therefore$ We have $y_1 \simeq y_0 + h\,f(x_0, y_0)$ ... (Q.1.2)

Thus value y at $x = x_1$ is calculated by using equation (Q.1.2).

Similarly for the interval $[x_1, x_2]$

We have $y_2 \simeq y_1 + \int_{x_1}^{x_2} f(x,y)\,dx$

$$y_2 \simeq y_1 + h\,f(x_1, y_1)$$

Proceeding in this way, we obtain general formula as

$$y_{n+1} = y_n + h f(x_n, y_n) \quad \because n = 0, 1, 2, \ldots$$

To obtain the solution with desired accuracy, we have to take a smaller value of h. Hence the solution is obtained very slowly. Due to this Euler's method is rarely used. The more accurate results will be obtained by the modified method.

Modified Euler's Method

To start we use Euler's formula to find value of y at $x = x_1$

$$\therefore \quad y_1^{(0)} = y_0 + h f(x_0, y_0)$$

To find more correct approximation at $x = x_1$, we use Euler's Modified iteration formula as follows.

$$y_1^{(n+1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(n)})] \quad \ldots (Q.1.3)$$

$$\because n = 0, 1, 2$$

In particular $y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

By using formula (1), we can obtain approximations $y_1^{(1)}, y_1^{(2)}, \ldots, y_1^{(n)}$. This process is repeated till no significant change occurs. Suppose $y_1^{(n-1)} \simeq y_1^{(n)}$ we call this as y_1. To find the value of y at $x = x_2$, the above procedure is repeated in the interval $[x_1, x_2]$.

$$\therefore \quad y_2^{(n+1)} = y_n + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(n)})]$$

Q.2 Using Euler's method solve $\frac{dy}{dx} = 1 + y^2$ given y(0) = 0. Take h = 0.05 and obtain y(0.05), y(0.1), y(0.15).

Ans. : We have

$$\frac{dy}{dx} = 1 + y^2 = f(x, y),$$

$$h = 0.05, \; x_0 = y_0 = 0$$

Iteration (1) Euler's formula for y at x = 0.05

$$y_1 = y_0 + h f(x_0, y_0)$$

$$= 0 + (0.05)(1 + 0^2) = 0.05$$

Iteration (2) Approximate value of y at $x_1 = 0.1$ is given by,

$$y_2 = y_1 + h f(x_1, y_1)$$

$$= 0.05 + 0.05\,(1.0025) = 0.1001$$

Iteration (3) Approximate value of y at $x_2 = 0.15$ is given by Euler's formula as,

$$y_3 = y_2 + h f(x_2, y_2) = 0.1506$$

Thus $y(0.05) = 0.05$, $y(0.1) = 0.1001$ and $y(0.15) = 0.1506$.

Q.2(a) Use Euler's method, to solve $\frac{dy}{dx} = x + y$, y(0) = 1 Tabulate values of y for x = 0 to x = 0.3 (Take h = 0.1)

[SPPU : June-22, Marks 5]

Ans. : We have

$$\frac{dy}{dx} = x + y = f(x, y) \text{ and } x_0 = 0, y_0 = 1$$

$$h = 0.1, \; x_1 = x_0 + h = 0.1$$

$$\therefore \quad y_1 = y_0 + h(x_0, y_0) = 1 + 0.1\,[0 + 1] = 1.1$$

$$y(0.1) = 1.1$$

$$y_2 = y(x_2) = y(0.2) = y_1 + h(x_1, y_1)$$

$$= 1.1 + 0.1\,[0.1 + 1.1] = 1.22$$

$$y(0.2) = 1.22$$

$$y_3 = y_2 + h f(x_2, y_1)$$

$$= 1.22 + 0.1\,[0.2 + 1.22]$$

$$y_3 = 1.362$$

$$\therefore \quad y(x_3) = 1.362 = y(0.3)$$

Q.2(b) Use Euler's method to solve $\frac{dy}{dx} = x^2 + y$, y(0) = 1. Tabulate values of y for x = 0 to x = 0.3. (Take h = 0.1)

[SPPU : Dec.-22, Marks 5]

Ans. :

Here $f(x, y) = x^2 + y$, $h = 0.1$, $x_0 = 0$, $y_0 = 1$

Step I : $f(x_0, y_0) = x_0^2 + y_0 = 0^2 + 1 = 1$

$\therefore$ $y_1 = y_0 + h\,f(x_0, y_0) = 1 + (0.1) \times (1) = 1.1$

Thus $y_1 = \mathbf{1.1}$

Step II : $f(x_1, y_1) = x_1^2 + y_1 = (0.1)^2 + 1.1 = 1.11$

$\therefore$ $y_2 = y_1 + h\,f(x_1, y_1) = 1.1 + (0.1)\,(1.11)$

$\Rightarrow$ $y_2 = \mathbf{1.211}$

Step III : $f(x_2, y_2) = x_2^2 + y_2 = (0.2)^2 + 1.211 = 1.251$

$\therefore$ $y_3 = y_2 + h\,f(x_2, y_2) = 1.211 + (0.1)\,(1.251)$

$\Rightarrow$ $y_3 = \mathbf{1.3361}$

Tabulated solution is,

x	0	0.1	0.2	0.3
y	1	1.1	1.211	1.3361

Q.3 Given that $\frac{dy}{dx} = \frac{y-x}{y+x}$ with initial condition y(0) = 1. Find y for x = 0.1. Also find the better approximation at x = 0.1 by dividing the interval [0, 0.1] into five steps.

Ans. : We have

$$\frac{dy}{dx} = \frac{y-x}{y+x} \text{ with } x_0 = 0,\ y_0 = 1$$

$$= f(x, y)$$

i) If we take h = 0.1 then the approximate value of y at x = 0.1 is given by Euler's theorem as

$$y_1 = y_0 + h\,f(x_0, y_0) = 1 + 0.1\,f(0, 1)$$

$$= 1 + 0.1\left(\frac{1-0}{1+0}\right) = 1.1$$

ii) For better approximation we divide the interval into five steps.

$\therefore$ $h = \frac{0.1-0}{5} = 0.02$

Iteration (1) The approximate value of y at x = 0.02 is given by Euler's formula as

$$y_1 = y_0 + h\,f(x_0, y_0)$$

$$= 1 + (0.02)\left(\frac{1-0}{1+0}\right) = 1.02$$

Iteration (2) The approximate value of y at x = 0.04 is given by Euler's formula as ($x_1 = 0.02$, $y_1 = 1.02$)

$$y_2 = y_1 + h\,f(x_1, y_1)$$

$$= 1.02 + (0.02)\left\{\frac{1.02 - 0.02}{1.02 + 0.02}\right\} = 1.0392$$

Iteration (3) The approximate value of y at x = 0.06 is given by Euler's formula as

$y_3 = y_2 + h\,f(x_2, y_2)$. Here $x_2 = 0.04$, $y_2 = 1.0392$

$= 1.0392 + (0.02)\ f(0.04,\ 1.0392) = 1.0577$

Iteration (4) The approximate value of y at x = 0.08 is given by Euler's formula as

$y_4 = y_3 + h\,f(x_3, y_3)$ (Here $x_3 = 0.06$, $y_3 = 1.0577$)

$= 1.0577 + (0.02)\ f(0.06,\ 1.0577) = 1.0755$

Iteration (5) The approximate value of y at x = 1 is given by Euler's formula as

$$y_5 = y_4 + h\,f(x_2, y_4)$$

(Here $x_4 = 0.08$, $y_4 = 1.0755$)

$y_5 = 1.0755 + h\ f(0.08,\ 1.0755)$

$= 1.0927$

Thus the value of y at x = 0.1 is 1.0927.

Q.3(a) Use Runge-Kutta method of 4^{th} order to solve $\frac{dy}{dx} = \frac{y-x}{y+x}$, y(0) = 1 at x = 0.2 with h = 0.2. ☞ **[SPPU : Dec.-22, Marks 5]**

Ans. : We have $x_0 = 0$, $y_0 = 1$ and $h = 0.2$. Now to obtain y at $x = 0.2$, we do following steps,

$$k_1 = h f(x_0, y_0) = h\left(\frac{y_0 - x_0}{y_0 + x_0}\right) = 0.2\left(\frac{1-0}{1+0}\right) = 0.2$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= h f\left(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}\right) = h\ f\ (0.1,\ 1.1)$$

$$= 0.2\left(\frac{1.1 - 0.1}{1.1 + 0.1}\right) = 0.1667$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= h f\left(0 + \frac{0.2}{2}, 1 + \frac{0.1667}{2}\right)$$

$$= 0.2\ f\ (0.1,\ 1.08335)$$

$$= 0.2\left(\frac{1.08335 - 0.1}{1.08335 + 0.1}\right) = 0.1661$$

$$k_4 = h f(x_0 + h, y_0 + k_3)$$

$$= h\ f\ (0 + 0.2,\ 1 + 0.1661)$$

$$= 0.2 \times \left(\frac{1.661 - 0.2}{1.661 + 0.2}\right) = 0.1414$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.2 + 2\ (0.1667) + 2\ (0.1661) + 0.1414]$$

$$= \mathbf{0.1678}$$

Giving $(y)_{t\ x=0.2} = y_0 + k = 1 + 0.1678 = \mathbf{1.1678}$

Q.4 Using Euler's modified method, find value of y when x = 0.1 given that $\frac{dy}{dx} = x^2 + y$, y(0) = 1.

Ans. : We have

$$\frac{dy}{dx} = x^2 + y = f(x, y) \quad \therefore x_0 = 0,\ y_0 = 1$$

Take $h = 0.05$, $f(x_0, y_0) = 1$

By Euler's formula $y_1 = y_0 + h\ f(x_0, y_0) = 1.05$

$$f(x_1, y_1) = x_1^2 + y_1 = (0.05)^2 + 1.05 = 1.0525$$

Step a) Using Euler's modified method, the improved values of y are given by,

Iteration (1)

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$$

$$y_1^{(1)} = 1 + \frac{0.05}{2}[x_0^2 + y_0 + x_1^2 + y_1]$$

$$= 1 + \frac{0.05}{2}[1 + 1.0525] = 1.0513$$

Iteration (2)

Now $$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$y_1^{(2)} = 1 + \frac{0.05}{2}[1 + 1.0538] = 1.0513$$

Hence we take $y_1 = 1.0513$ which is correct upto 4 decimal places.

Step b) To obtain y_2 i.e. y at x = 0.1, first we use Euler's formula, here $x_1 = 0.05$. $y_1 = 1.0513$

$$y_2 = y_1 + h\ f(x_1, y_1) = y_1 + h\ (x_1^2 + y_1)$$

$$y_2 = 1.0513 + (0.05)\ ((0.05)^2 + 1.0513)$$

$$= 1.1040$$

$$f(x_2, y_2) = x_2^2 + y_2 = 1.114$$

Iteration (1) By modified Euler's formula, we have

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(0)})]$$

$$= 1.1055$$

Iteration (2)

$$y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})]$$

$$= 1.1055$$

$\therefore$ The value of $y_2 = 1.1055$ correct upto 4 decimal.

Thus the value of y at x = 0.1 is 1.1055.

Q.4(a) Using Modified-Euler's method, find y(0, 2), given $\frac{dy}{dx} + xy^2 = 0$, y(0) = 2. Take h = 0.2 (Two iterations only)

[SPPU : June-22, Marks 5]

Ans. : We have, $f(x, y) = -xy^2$, $y(0) = 2$, $y_0 = 2$, $x_0 = 0$,

$h = 0.2$

$f(x_0, y_0) = -0 \times 4 = 0$

By Euler's formula $y_1 = y_0 + h f(x_0, y_0) = 2 + 0 = 2$

By Euler's modified method,

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$$

$$= 2 + 0.1[0 + (-x_1, y_1^2)] = 2 + 0.1[-0.2 \times 4]$$

$$y_1^{(1)} = 2 - 0.08 = 1.92$$

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)^{(1)}]$$

$$= 2 + 0.1[0 + f(0.2, 1.92)]$$

$$= 2 + 0.1[-0.2 \times (1.92)^2]$$

$$\boxed{y_1^{(2)} = 1.926272}$$

Q.4(b) Using modified Euler's method, find y(1.1). Given $\frac{dy}{dx} = 2 + \sqrt{xy}$, y(1) = 1. Take h = 0.1. (Two iterations only)

[SPPU : Dec.-22, Marks 5]

Ans. : Here $f(x, y) = 2 + \sqrt{xy}$

$h = 0.1$, $x_0 = 1$, $y_0 = 1$

$$f(x_0, y_0) = 2 + \sqrt{x_0 y_0} = 2 + \sqrt{1 \times 1} = 2 + 1 = 3$$

$$\therefore \quad y_1 = y_0 + h f(x_0, y_0)$$

$$= 1 + 0.1(3) = 1 + 0.3 = 1.3$$

$$\therefore \quad f(x_1, y_1) = 2 + \sqrt{x_1 y_1} = 2 + \sqrt{1.1 \times 1.3}$$

$$= 3.3635$$

Using modified Euler's method, improved values of y_1 are given by,

1st iteration : $y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$

$$= 1 + \frac{0.1}{2}[3 + 3.3635]$$

$$= \mathbf{1.3181}$$

2nd iteration : $y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_0, y_1^{(1)})\right]$

here $f\left(x_0, y_1^{(1)}\right) = 2 + \sqrt{1 \times 1.3181} = 3.3181$

So, $y_1^{(2)} = 1 + \frac{0.1}{2}[3 + 3.3181] = 1.315905$

Q.5 Using modified Euler's method solve equation $\frac{dy}{dx} = x - y^2$; y(0) = 1 to calculate y at x = 0.2 take h = 0.2.

Ans. : We have,

$$\frac{dy}{dx} = x - y^2 = f(x, y),\ x_0 = 0 \text{ and } y_0 = 1$$

$h = 0.2$, $x_1 = 0.2$, $f(x_0, y_0) = 0 - 1 = -1$

Euler's formula is

$$y_1 = y_0 + hf(x_0, y_0)$$

$$y_1 = 1 + 0.2(-1) = 0.8$$

Iteration (1)

$$y_1 = 1 + 0.2(-1) = 0.8$$

Euler's modified formula is

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$$

$$= 1 + \frac{0.2}{2}[-1 + x_1 - y_1^2]$$

$$y_1^{(1)} = 1 + \frac{0.2}{2}[-1 + 0.2 - (0.8)^2]$$

$$= 0.8560$$

Iteration (2)

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$= 1 + \frac{0.2}{2}[-1 - 0.5327] = 0.8467$$

Iteration (3)

$$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$$

$$= 1 + \frac{0.2}{2}[-1 - 0.5169] = 0.8483$$

Iteration (4)

$$y_1^{(4)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(3)})]$$

$$= 1 + \frac{0.2}{2}[-1 - 0.5196] = 0.8480$$

Iteration (5)

$$y_1^{(5)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(4)})]$$

$$= 1 + \frac{0.2}{2}[-1 - 0.5191] = 0.8481$$

Iteration (6)

$$y_1^{(6)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(5)})]$$

$$= 1 + \frac{0.2}{2}[-1 - 0.5193] = 0.8481$$

Here, $y_1^{(5)} = y_1^{(6)}$

$\therefore$ $[y]_{x=0.2} = y(0.2) = 0.8481$

Q.6 Explain Runge Kutta methods.

Ans. : The Runge Kutta methods do not required the calculations of higher order derivatives like Taylor's series and it is designed to give greater accuracy with the advantage of requiring only the function values at some selected points on the subinterval. Thus this method is widely used as compare with Taylor's and Euler's methods.

A) Consider the Euler's modified formula in the following form.

$$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$$

We put $x_1 = x_0 + h$, $y_1 = y_0 + h f(x_0, y_0)$ in the right side of above equation.

$$\therefore \quad y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_0 + h, y_0 + h f(x_0, y_0))]$$

$$y_1 = y_0 + \frac{h}{2}[f_0 + f(x_0 + h, y_0 + hf_0)] \quad \text{... (Q.6.1)}$$

where $f_0 = f(x_0, y_0)$

Now, put $k_1 = h f_0$ and $k_2 = h f(x_0 + h, y_0 + k_1)$

$\therefore$ The equation (Q.6.1) becomes,

$$y_1 = y_0 = \frac{1}{2}(k_1 + k_2) = y_0 + k$$

where $k = \frac{1}{2}(k_1 + k_2)$ and $k_1 = hf_0$

$$k_2 = h f(x_0 + h, y_0 + k_1)$$

This is known as the second order Runge-Kutta formula.

B) The most commonly used fourth order Runge Kutta formula is given as,

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= y_0 + k$$

where $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$k_1 = h f(x_0, y_0)$$

$$k_2 = h f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$$

$$k_3 = h f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)$$

$$k_4 = h f(x_0 + h, y_0 + k_3)$$

C) Runge-Kutta method for solution of simultaneous first order differential equations :

Consider the simultaneous first order D.E. as

$$\frac{dy}{dx} = f(x, y, z) \text{ and } \frac{dz}{dx} = g(x, y, z)$$

with initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$. Here x is independent variable and y, z are dependent variables.

The starting point is (x_0, y_0, z_0) and increments (step sizes) for x, y, z be h, k, l respectively.

i) Runge Kutta formula of second order :

$$k_1 = h f(x_0, y_0, z_0) \quad l_1 = h g(x_0, y_0, z_0)$$

$$k_2 = h f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$

$$l_2 = h g\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$

Then $y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$

and $z_1 = z_0 + \frac{1}{2}(l_1 + l_2), \; x_1 = x_0 + h$

ii) Runge Kutta formula of fourth order

$$k_1 = h f(x_0, y_0, z_0), \; l_1 = hg(x_0, y_0, z_0)$$

$$k_2 = h f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$

$$l_2 = h g\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$

$$k_3 = h f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)$$

$$l_3 = h g\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)$$

$$k_4 = h f(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$l_4 = h g(x_0 + h, y_0 + k_3, z_0 + l_3)$$

then $y_1 = y_0 + \frac{h}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

and $z_1 = z_0 + \frac{h}{6}[l_1 + 2l_2 + 2l_3 + l_4]$

Q.7 Use Runge-Kutta formulae of 2^{nd} and 4^{th} orders to find y when x = 0.1 and x = 0.2 given that $\frac{dy}{dx} = x + y$ with y(0) = 1.

Ans. : I) By Range Kutta formula of 2^{nd} order :

Runge Kutta 2^{nd} order formula is

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

where $k_1 = h f(x_0, y_0)$

and $k_2 = h f(x_0 + h, y_0 + k_1)$

To determine y(0.1) we take h = 0.1

Here $x_0 = 0$, $y_0 = 1$

$$\therefore \quad k_1 = h f_0 = h f(x_0, y_0)$$

$$= 0.1\,(0+1) = 0.1$$

$$k_2 = 0.1\,[f(0 + 0.1, 1 + 0.1)]$$

$$= 0.1\,(0.1 + 1.1) = 0.12$$

$$\therefore \quad y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

$$= 1 + \frac{1}{2}(0.1 + 0.12) = 1.11$$

To determine $y_2 = y(0.2)$, we take

$x_0 = 0.1$, $y_0 = 1.11$ and h = 0.1

$$\therefore \quad k_1 = (0.1)\ f(0.1,\ 1.11)$$
$$= 0.1\ (0.1 + 1.11) = 0.121$$
$$k_2 = (0.1)\ f(0.1 + 0.1,\ 1.11 + 0.121)$$
$$= 0.1\ (0.2 + 1.231) = 0.1431$$
$$\therefore \quad y_2 = y(0.2) = 1.11 + \frac{1}{2}(0.121 + 0.1431)$$
$$\therefore \quad y_2 = \mathbf{1.24205}$$

ii) By Runge Kutta formula of 4^{th} order :

This formula is

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where $k_1 = h\, f(x_0, y_0)$

$$k_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$k_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$k_4 = h\, f(x_0 + h, y_0 + k_3)$$

To find $y(0.1) = y_1$ we take $h = 0.1$

$$\therefore \quad k_1 = h\, f(x_0, y_0) = (0.1)\ f(0, 1) = 0.1$$
$$k_2 = (0.1)\ f(0.05,\ 1.05) = 0.11$$
$$k_3 = (0.1)\ f(0.05,\ 1.055) = 0.1105$$
$$k_4 = (0.1)\ f(0.1,\ 1.1105) = 0.12105$$
$$\therefore \quad y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= 1 + \frac{1}{6}[0.1 + 0.22 + 0.2210 + 0.12105]$$
$$= 1.11034$$

To find $y(0.2) = y_2$ we take

$x_0 = 0.1$, $y_0 = 1.11034$, and $h = 0.1$

$$\therefore \quad k_1 = (0.1)\ f(0.1,\ 1.11034) = 0.12103$$
$$k_2 = (0.1)\ (0.15 + 1.17085) = 0.13208$$
$$k_3 = (0.1)\ (0.15 + 1.17638) = 0.13263$$
$$k_4 = (0.1)\ (0.2 + 1.24297) = 0.14429$$
$$\therefore \quad y_2 = y(0.2) = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$
$$= 1.11034 + \frac{1}{6}[0.12103 + 2\ (0.13208) + 2\ (0.13263) + 0.14429]$$
$$y_2 = 1.24279$$

Thus $y(0.2) = 1.24279$

Q.8 Using fourth order Runge Kutta method solve the equation $\frac{dy}{dx} = \sqrt{x+y}$ with $y(0) = 1$ and find y (0.2) taking h = 0.2.

Ans. : The 4^{th} order Runge Kutta formula is

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where $k_1 = h\, f(x_0, y_0)$

$$k_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$k_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$k_4 = h\, f(x_0 + h, y_0 + k_3)$$

Here

$$f(x, y) = \sqrt{x+y},\quad x_0 = 0,\quad y_0 = 1,\quad h = 0.2$$
$$\therefore \quad k_1 = 0.2(\sqrt{0+1}) = 0.2$$
$$k_2 = (0.2)\sqrt{0.1 + 1.1} = 0.2191$$

$$k_3 = (0.2)\sqrt{0.1+1.10955} = 0.2120$$

$$k_4 = (0.2)\sqrt{0.2+1.2120} = 0.2377$$

$$\therefore \quad y_1 = 1+\frac{1}{6}[0.2+2(0.2191)+2(0.2120)+0.2377]$$

$$y(0.2) = 1.2167$$

Q.8(a) Use Runge-Kutta method of 4^{th} order to solve $\frac{dy}{dx} = xy$, y(1) = 2 at x = 1.2 with h = 0,2. **[SPPU : June-22, Marks 5]**

Ans. : Given that,

$$\frac{dy}{dx} = xy = f(x, y),\ h = 0.2,\ y(1) = 2$$

We have $x_0 = 1, y_0 = 2$

$$k_1 = h f(x_0, y_0) = 0.2(1\times 2) = 0.4$$

$$k_2 = h f\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right) = 0.2\,[f(1.1, 2.2)]$$

$$= 0.2\,[1.1\times 2.2] = 0.484$$

$$k_3 = h f\left[x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right] = 0.2\,(f(1.1, 2.242)$$

$$= 0.2\,[1.1\times 2.242] = 0.49324$$

$$k_4 = h f(x_0+h, y_0+k_3) = 0.2\,[f(1.2, 2.44324)$$

$$= 0.2\,[1.2\times 2.49324] = 0.5485128$$

$$\therefore \quad y_1 = y_0+\frac{1}{6}[k_1+2k_2+2k_3+k_4]$$

$$= 2+\frac{1}{6}[0.4+0.968+0.98648+0.5485128]$$

$$y_1 = 2.483832$$

Q.9 Use R.K. method of 4^{th} order to obtain the numerical solution of $y' = x^2+y^2$ with y(1) = 1.5 in the interval (1, 1.2) with h = 0.1.

Ans. : We have

$$y' = x^2+y^2 \quad \therefore f(x, y) = x^2+y^2$$

and $x_0 = 1,\ y_0 = 1.5$ and $h = 0.1$

Iteration (1) To find y(1.1)

$$f(x_0, y_0) = f(1, 1.5) = 1+2.25 = 3.25$$

$$k_1 = h f(x_0, y_0) = (0.1)(3.25) = 0.325$$

$$k_2 = h f\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right)$$

$$= (0.1)\,[(1.05)^2+(1.6625)^2]$$

$$k_2 = 0.38664$$

$$k_3 = h f\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right)$$

$$= (0.1)\,[(1.05)^2+(1.6933)^2]$$

$$k_3 = 0.39698$$

$$k_4 = h f(x_0+h, y_0+k_3)$$

$$= (0.1)\,[(1.1)^2+(1.89698)^2]$$

$$= 0.48086$$

$$\therefore \quad y_1 = y(1.1) = y_0+\frac{1}{6}[k_1+2k_2+2k_3+k_4]$$

$$= 1.5+\frac{1}{6}[0.325+2(0.38664)+2(0.39698)+0.48086]$$

$$y_1 = 1.8955$$

Iteration (2) To find $y_1 = y(1.2)$ we take

$$x_0 = 1.1,\ y_0 = 1.8955,\ h = 0.1$$

$$\therefore \quad k_1 = h f(x_0, y_0)$$

$$= (0.1)\,[(1.1)^2+(1.8955)^2]$$

JUNE-2022 [5869]-286	Solved Paper

Course 2019

Time : $2\frac{1}{2}$ Hours] **[Maximum Marks : 70**

Instructions to the candidates :

1) *Q.1 is compulsory.*
2) *Attempt Q.2 or Q.3, Q.4 or Q.5, Q.6 or Q.7, Q.8 or Q.9.*
3) *Neat diagrams must be drawn wherever necessary.*
4) *Figures to the right indicate full marks.*
5) *Use of electronic pocket calculator is allowed.*
6) *Assume suitable data, if necessary.*

Q.1 *Write the correct option for the following multiple choice questions.*

a) *For a given set of bivariate data, $\bar{x} = 2$, $\bar{y} = -3$. The regression coefficient of x on y is -0.11. By using the regression equation of x on y, the most probable value of x when y = 0 is* ________. **[2]**

i) 0.57 *ii) 0.87*

iii) 0.77 *iv) 1.77*

Ans. : Use $x-\bar{x} = b_{xy}(y-\bar{y})$. At y = 0,

$x = \bar{x} - b_{xy}\bar{y} = 2 - 0.33 = 1.77$ **[Ans. : iv]**

b) *If probability density function f(x) of a continuous random variable x is defined by*

$$f(x) = \begin{cases} \frac{1}{4}, & -2 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

then $P(x \le 1)$ is ________. **[2]**

i) $\frac{1}{4}$ *ii)* $\frac{1}{2}$

iii) $\frac{1}{3}$ *iv)* $\frac{3}{4}$

[Ans. : i]

c) Lagrange's polynomial through the points

x	0	1	2
y	4	0	6

is given by ________. [2]

i) $y = 5x^2 - 3x + 4$ ii) $y = 5x^3 + 3x + 4$

iii) $y = 5x^2 - 9x + 4$ iv) $y = x^2 - 9x + 4$ **[Ans. : iii]**

d) Using Gauss elimination method, the solution of system of equations $x + \frac{1}{4}y + \frac{1}{4}z = 1$, $\frac{15}{4}y - \frac{9}{4}z = 3$, $\frac{5}{4}y - \frac{19}{4}z = 3$ is ________. [2]

i) $x = 1, y = 2, z = 3$ ii) $x = \frac{1}{2}, y = 1, z = \frac{1}{2}$

iii) $x = 2, y = \frac{1}{2}, z = 2$ iv) $x = 1, y = \frac{1}{2}, z = -\frac{1}{2}$

[Ans. : iv]

e) The first four central moments of a distribution are 0, 16, – 64 and 162. The coefficient of Kurtosis β_2 is ________. [1]

i) 1.20 ii) 0.6328

iii) 1 iv) 0.3286 **[Ans. : ii]**

f) If f(x) is continuous on [a, b] and f(a) f(b) < 0 then to find a root of f(x) = 0, initial approximation x_0 by bisection method is ________. [1]

i) $x_0 = \frac{a-b}{2}$ ii) $x_0 = \frac{f(a)+f(b)}{2}$

iii) $x_0 = \frac{a+b}{2}$ iv) $x_0 = \frac{a-b}{a+b}$ **[Ans. : iii]**

Q.2 a) If marks scored by five students in statistics test of 100 marks, are given in following table. [5]

Student	1	2	3	4	5
Marks(/100)x	46	34	52	78	65

Find standard deviation and arithmetic mean $\bar{x}$.

(Refer Q.1(a) of Chapter - 5)

b) Fit a law of the form $y = ap + b$ by least square method for the data, **(Refer Q.7(c) of Chapter - 5)** [5]

p	100	120	140	160	180	200
y	0.9	1.1	1.2	1.4	1.6	1.7

c) If the two lines of regression are $9x + y - \lambda = 0$ and $4x + y = \mu$ and the means of x and y are 2 and – 3 respectively. Find values of λ, μ and correlation coefficient between x and y.

(Refer Q.14(c) of Chapter - 5) [5]

OR

Q.3 a) The first four moments of a distribution about 5 are 2, 20, 40 and 50. Find first four moments about mean and β_1, β_2.

(Refer Q.4 of Chapter - 5) [5]

b) Fit a parabola $y = ax^2 + bx + c$, by using least square method to the following data, **(Refer Q.6(a) of Chapter - 5)** [5]

x	0	1	2	3
y	2	2	4	8

c) Calculate the coefficient of correlation from the following information.

$n = 10$, $\Sigma x = 40$, $\Sigma x^2 = 190$, $\Sigma y^2 = 200$, $\Sigma xy = 150$, $\Sigma y = 40$. [5]

(Refer Q.12(a) of Chapter - 5)

Q.4 a) *Bag 1 contains 2 white and 3 red balls. Bag 2 contains 4 white and 5 red balls. One ball is drawn randomly from bag 1 and is placed in bag 2. Later, one ball is draw randomly from bag 2. Find the probability that it is red.* **(Refer Q.7(b) of Chapter - 6)** **[5]**

b) *The expected number of matches those will be won by India in a series of five one day matches between India and England is three. If the probability of India's win in each match remains the same and the results of all the five matches are independent of each other, find the probability that India wins the series, using Binomial distribution. Assume that each match ends with a result.* **(Refer Q.15(a) of Chapter - 6)** **[5]**

c) *The lifetime of an article has a normal distribution with mean 400 hours and standard deviation 50 hours. Find the expected number of articles out of 2,000 whose lifetime lies between 335 hours to 465 hours. (Given : Z = 1.3, A = 0.4032)* **(Refer Q.24(a) of Chapter - 6)** **[5]**

OR

Q.5 a) *Find the expected value of the number of heads obtained when three fair coins are tossed simultaneously.* **(Refer Q.9 of Chapter - 6)** **[5]**

b) *On an average, 180 cars per hour pass a specified point on a particular road. Using Poisson distribution, find the probability that at least two cars pass the point in any one minute.*
(Refer Q.22(c) of Chapter - 6) **[5]**

c) *The proportions of blood types O, A, B and AB in the general population of a country are known to be in the ratio 49:38:9:4 respectively. A research team observed the frequencies of the blood types as 88, 80, 22 and 10 respectively in a community of that country. Test the hypothesis at 5 % level of significance that the proportions for this community are in accordance with the general population of that country. (Given : χ^2_{tab} = 7.815)* **(Refer Q.28(a) of Chapter - 6)** **[5]**

Q.6 a) *Find the root of the equation $x^4 + 2x^3 - x - 1 = 0$, lying in the interval [0, 1] using the bisection method at the end of fifth iteration.*
(Refer Q.4(a) of Chapter - 7) **[5]**

b) *Find a real root of the equation $x^3 + 2x - 5 = 0$ by applying Newton-Raphson method at the end of fifth iteration.*
(Refer Q.10(a) of Chapter - 7) **[5]**

c) *Solve by Gauss-Seidel method, the system of equations :*

$$20x_1 + x_2 - 2x_3 = 17$$
$$3x_1 + 20x_2 - x_3 = -18$$
$$2x_1 - 3x_2 + 20x_3 = 25$$

(Refer Q.18(a) of Chapter - 7) **[5]**

OR

Q.7 a) *Solve by Gauss elimination method, the system of equations :*

$$2x_1 + x_2 + x_3 = 10$$
$$3x_1 + 2x_2 + 3x_3 = 18$$
$$x_1 + 4x_2 + 9x_3 = 16$$

(Refer Q.15(a) of Chapter - 7) **[5]**

b) *Solve by Jacobi's iteration method, the system of equations :*

$$4x_1 + 2x_2 + x_3 = 14$$
$$x_1 + 5x_2 - x_3 = 10$$
$$x_1 + x_2 + 8x_3 = 20$$

(Refer Q.23 of Chapter - 7) **[5]**

c) *Use Regula-Falsi method to find a real root of the equation $e^x - 4x = 0$ correct to three decimal places.* **[5]**

Ans. :

We have $f(x) = e^x + 4x$

$$f(2) = e^2 - 8 = -0.6109 < 0$$

$$f(2.2) = e^{2.2} - 4(22) = 0.2250 > 0$$

$\therefore$ Root Lies between [2, 2.2], a = 2, b = 2.2

$$\therefore \quad x_1 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(b)} = \frac{2f(2.2) - 2.2f(2)}{f(2.2) - f(2)} = \frac{1.79398}{0.8359}$$

$$x_1 = 2.14616$$

$$f(x_1) = -0.03268 < 0$$

$\therefore$ Root lies between [2.14616, 2.2], a = 2.14616, b = 2.2

$$\therefore \quad x_2 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = \frac{0.554782}{0.25768} = 2.15298$$

$$f(x_2) = -0.00144 < 0$$

$\therefore$ Root lies between [2.15298, 2.2]

a b

$$x_3 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = \frac{0.4875885}{0.22644} = 2.153279$$

$\therefore \quad x_3 = 2.153279$ is correct upto 3 decimal places.

Q.8 a) *Using Newton's forward interpolation formula, find y at x = 8 from the following data.* **(Refer Q.1 of Chapter - 8)** **[5]**

x	0	5	10	15	20	25
y	7	11	14	18	24	32

b) *Evaluate* $\int_0^1 \frac{dx}{x^2+1}$ *using Simpson's* $\frac{1}{3}^{rd}$ *rule. (Take h = 0.2)*

(Refer Q.16(a) of Chapter - 8) **[5]**

c) *Use Euler's method, to solve* $\frac{dy}{dx} = x + y,\ y(0) = 1$

Tabulate values of y for x = 0 to x = 0.3 (Take h = 0.1)

(Refer Q.2(a) of Chapter - 9) **[5]**

OR

Q.9 a) *Use Runge-Kutta method of* 4^{th} *order to solve* $\frac{dy}{dx} = xy,\ y(1) = 2$ *at x = 1.2 with h = 0,2.*

(Refer Q.8(a) of Chapter - 9) **[5]**

b) *Using Modified-Euler's method, find y(0, 2), given* $\frac{dy}{dx} + xy^2 = 0,\ y(0) = 2$. *Take h = 0.2 (Two iterations only)*

(Refer Q.4(a) of Chapter - 9) **[5]**

c) *Using Newton's backward difference formula, find the value of* $\sqrt{155}$ *from the following data* **(Refer Q.1(b) of Chapter - 8)** **[5]**

x	150	152	154	156
$y = \sqrt{x}$	12.247	12.329	12.410	12.490

DECEMBER-2022 [5925]-260 Solved Paper

Course 2019

Time : $2\frac{1}{2}$ Hours] **[Maximum Marks : 70**

Q.1 *Write the correct option for the following multiple choice questions :* **[2]**

i)

y	:	1	2	3
x	:	1	5	9

The least square fit of the form x = ay + b to the above data is ______.

a) x = 2y − 5 *b) x = 4y + 4*

c) x = 4y + 1 *d) x = 4y − 3* **[Ans. : d]**

ii) For two events A and B, $P(A) = \frac{2}{3}$, $P(B) = \frac{3}{8}$ *and* $P(A \cap B) = \frac{1}{4}$, *then the events A and B are ______.* **[2]**

a) mutually exclusive and independent

b) not mutually exclusive and not independent

c) independent, but not mutually exclusive

d) mutually exclusive, but not independent **[Ans. : c]**

iii) Using Gauss elimination method, the solution of system of equations $x + 4y - z = -5$, $y + \frac{5}{3}z = \frac{7}{3}$ and $-13y + 2z = 19$ is ______. [2]

a) $x = \frac{117}{71}$, $y = -\frac{81}{71}$, $z = \frac{148}{71}$

b) $x = \frac{71}{117}$, $y = -\frac{71}{81}$, $z = \frac{71}{148}$

c) $x = -\frac{117}{71}$, $y = \frac{81}{71}$, $z = -\frac{148}{71}$

d) $x = 1, y = 2, z = 0$ **[Ans. : a]**

iv) If Lagrange's polynomial passes through

x	0	1
y	1	2

then $\int_0^1 y\,dx$ ______. [2]

a) $\frac{2}{3}$ b) $\frac{3}{2}$

c) $\frac{1}{2}$ d) 3 **[Ans. : b]**

v) If $\sum xy = 2638$, $\bar{x} = 14$, $\bar{y} = 17$, $n = 10$, then cov (x, y) = ______. [1]

a) 25.8 b) 23.9

c) 20.5 d) 24.2 **[Ans. : a]**

vi) If x_0, x_1 are two initial approximations to the root of $f(x) = 0$, by secant method the next approximation x_2 is given by ______. [1]

a) $x_2 = \frac{x_0 + x_1}{2}$ b) $x_2 = x_1 - \frac{f(x_1)}{f^1(x_1)}$

c) $x_2 = x_1 - \frac{(x_1 - x_0)}{(f_1 - f_0)} f_1$ d) $x_2 = x_1 + \frac{(x_1 + x_0)}{(f_1 + f_0)} f_1$ **[Ans. : c]**

Q.2 a) The first four moments of a distribution about 4 are – 1.4, 17, – 30 and 108. Obtain the first four central moments and coefficient of skewness and kurtosis. **(Refer Q.2(a) of Chapter - 5)** [5]

b) Fit a linear curve of the type $y = ax + b$, to following data, [5]

x	10	15	20	25	30
y	0.75	0.935	1.1	1.2	1.3

(Refer Q.6(b) of Chapter - 5)

c) Find the correlation coefficient for the following data. [5]

Population density	200	500	400	700	800
Death rate	12	18	16	21	10

(Refer Q.12(b) of Chapter - 5)

OR

Q.3 a) Find coefficient of variability for following data. [5]

C.I.	0-10	10-20	20-30	30-40	40-50	50-60	60-70
Freq. (f)	4	7	8	12	25	18	10

(Refer Q.1(b) of Chapter - 5)

b) Fit a linear curve $y = ax + b$, by least square method to the data, [5]

x	100	120	140	160	180	200
y	0.9	1.1	1.2	1.4	1.6	1.7

(Refer Q.7(d) of Chapter - 5)

c) *The regression equations are* $8x - 10y + 66 = 0$ *and* $40x - 18y = 214$. *The value of variance of x is 9. Find* [5]

i) *The mean values of x and y*

ii) *The correlation x and y and*

iii) *The standard deviation of y*

(Refer Q.14 of Chapter - 5)

Q.4 a) *Three factories A, B and C produce light bulbs. 20 %, 50 % and 30 % of the bulbs are available in the market by factories A, B and C respectively. Among these, 2 %, 1 % and 3 % of the bulbs produced by factories A, B and C are defective. A bulb is selected at random in the market and found to be defective. Find the probability that this bulb was produced by factory B.* **(Refer Q.8(a) of Chapter - 6)** [5]

b) *On an average, 20 % of the computers in a firm are virus infected. If 10 computers are chosen at random from this firm, find the probability that at least one computer is virus infected, using Binomial distribution.* **(Refer Q.17(a) of Chapter - 6)** [5]

c) *The height of a student in a school follows a normal distribution with mean 190 cm and variance 80* cm^2. *Among the 1,000 students from the school, how many are expected to have height above 200 cm ? (Given : z = 1.118, A = 0.3686)*

(Refer Q.27(c) of Chapter - 6) [5]

OR

Q.5 a) *A die is tampered in such a way that the probability of observing an even number is twice as likely to observe an odd number. Find the expected value of the upper most face obtained after rolling the die.* **(Refer Q.6(a) of Chapter - 6)** [5]

b) *The number of industrial injuries per working week in a factory is known to follow a Poisson distribution with mean 0.5. Find the probability that during a particular week, at least two accidents will take place.* **(Refer Q.22(d) of Chapter - 6)** [5]

c) *A pea cultivating experiment was performed. 219 round yellow peas, 81 round green peas, 61 wrinkled yellow peas and 31 wrinkled green peas were noted. Theory predicts that these phenotypes should be obtained in the ratios 9:3:3:1. Test the compatibility of the data with theory, using 5 % level of significance. (**Given :** $\chi^2_{tab} = 7.815$)*

(Refer Q.28(b) of Chapter - 6) **[5]**

Q.6 a) *Obtain the root of the equation $x^3 - 4x - 9 = 0$ that lies between 2 and 3 by Newton-Raphson method correct to four decimal places.* **(Refer Q.9(a) of Chapter - 7)** **[5]**

b) *Solve $2x - \cos x - 3 = 0$ by using the method of successive approximations correct of three decimal places.*

(Refer Q.13(a) of Chapter - 7) **[5]**

c) *Solve by Gauss - Seidel method, the system of equations :* **[5]**

$$2x_1 + x_2 + 6x_3 = 9$$
$$8x_1 + 3x_2 + 2x_3 = 13$$
$$x_1 + 5x_2 + x_3 = 7$$ **(Refer Q.18(b) of Chapter - 7)**

OR

Q.7 a) *Solve by Gauss elimination method, the system of equations :* **[5]**

$$4x_1 + x_2 + x_3 = 4$$
$$x_1 + 4x_2 - 2x_3 = 4$$
$$3x_1 + 2x_2 - 4x_3 = 6$$ **(Refer Q.15(b) of Chapter - 7)**

b) *Solve by Jacobi's iteration method, the system of equations :* **[5]**

$$20x_1 + x_2 - 2x_3 = 17$$
$$3x_1 + 20x_2 - x_3 = -18$$
$$2x_1 - 3x_2 - 20x_3 = 25$$ **(Refer Q.24 of Chapter - 7)**

c) *Find a real root of the equation $x^3 - 2x - 5 = 0$ by the method of false position at the end of fifth iteration.*

(Refer Q.8(a) of Chapter - 7) **[5]**

Q.8 a) *Using Newton's backward difference formula, find y at x = 4.5 for the following data.* **(Refer Q.1(a) of Chapter - 8)** **[5]**

x	1	2	3	4	5
y	3.47	6.92	11.25	16.75	22.94

b) *Use Simpson's 3/8th rule, to estimate* $\int_1^7 f(x)d(x)$ *from the following data.* **(Refer Q.18 of Chapter - 8)** **[5]**

x	1	2	3	4	5	6	7
f(x)	81	75	80	83	78	70	60

c) *Use Euler's method to solve* $\frac{dy}{dx} = x^2 + y,\ y(0) = 1$. *Tabulate values of y for x = 0 to x = 0.3. (Take h = 0.1)*
(Refer Q.2(b) of Chapter - 9) **[5]**

OR

Q.9 a) *Use Runge-Kutta method of 4th order to solve* $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0) = 1$ *at x = 0.2 with h = 0.2.* **(Refer Q.3(a) of Chapter - 9)** **[5]**

b) *Using modified Euler's method, find y(1.1). Given* $\frac{dy}{dx} = 2+\sqrt{xy}$, $y(1) = 1$. *Take h = 0.1. (Two iterations only)*

(Refer Q.4(b) of Chapter - 9) **[5]**

c) *Determine the value of* $y = \sqrt{151}$, *using Newton's forward difference formula, from the following data.*
(Refer Q.13(a) of Chapter - 8) **[5]**

x	150	152	154	156
$y = \sqrt{x}$	12.247	12.329	12.410	12.490

... *END*