Nonlinear Generation of Gaussian States

and applications

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A little about me

Hi! My name is Dharmik Patel.

I'm from Ahmedabad, India.

I recently got my Hons. BSc at the University of Toronto, in Mathematics and Physics.

My research interests are in theoretical quantum optics/photonics and applications to quantum information. I'm also interested in questions about mathematical foundations and interpretation.

Outline

Previous Work

1

Proving

$\hat{U}_N(t) = e^{i\gamma(t)}\hat{S}(z)\hat{D}(\alpha)\hat{R}(\Phi)$

[Phys. Rev. A 41, 4625 (1990).]

2

Modelling lossy generation of squeezed light via an effective $\chi^{(2)}$ process

Current work

3

Threshold detection statistics with Gaussian states

Previous Work

August 2023 - April 2024

Undergraduate Thesis

A unique representation for the unitary time evolution operator of an N-mode quadratic Hamiltonian

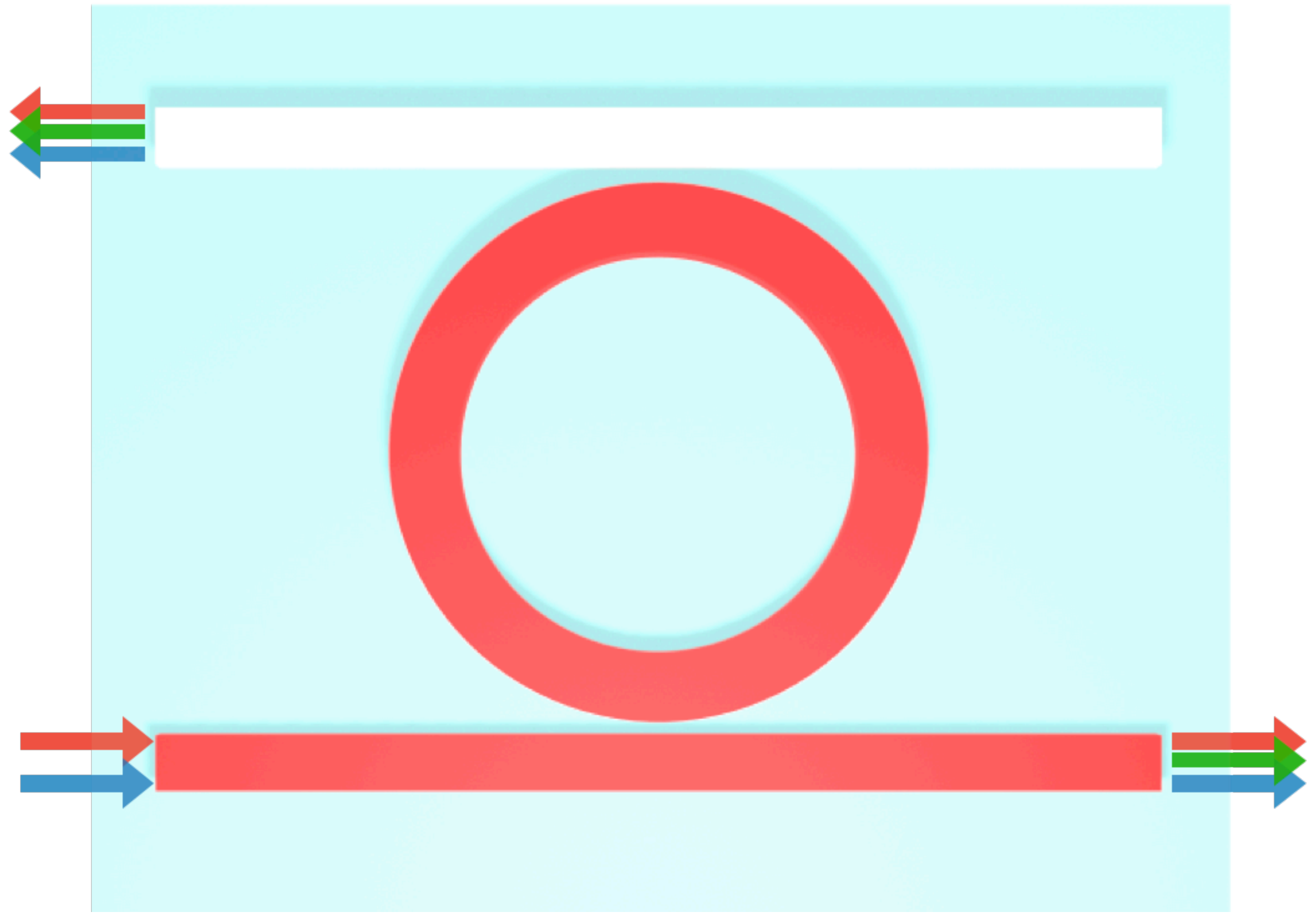
1

Proving $\hat{U}_N(t) = e^{i\gamma(t)}\hat{S}(z)\hat{D}(\alpha)\hat{R}(\Phi)$ for a quadratic Hamiltonian

[Phys. Rev. A 41, 4625 (1990).]

2

Modelling lossy generation of squeezed light via an effective $\chi^{(2)}$ process



1

Proving $\hat{U}_N(t) = e^{i\gamma(t)}\hat{S}(z)\hat{D}(\alpha)\hat{R}(\Phi)$ for a quadratic Hamiltonian

1

1.1

Quadratic Hamiltonians...

$$\hat{H}_N(t) = (\hat{a}^\dagger)^\intercal \underbrace{\omega(t)}_{N\times N\text{ Hermitian}}\hat{a} + (\hat{a}^\dagger)^\intercal \underbrace{f(t)}_{N\times N\text{ Symmetric}}\hat{a}^\dagger + (\hat{a})^\intercal f^\dagger(t)\hat{a} + \underbrace{g^\intercal(t)}_{1\times N\text{ row}}\hat{a}^\dagger + g^\dagger(t)\hat{a} + \underbrace{h(t)}_{\mathbb{R}\text{-valued}}$$

Unitary evolution formulation of the Schrödinger Equation $\xrightarrow{\text{J. Math. Phys. 4, 575–581 (1963)}}$ $\hat{U}_N(t) = \prod_{i=1}^{n} \exp\left(c_i(t)\hat{H}_i\right)$

1

1.2

... and their time evolution operators

$$\hat{U}_N(t) = \underbrace{\exp\big(i\gamma_N(t)\big)}_{\text{Global phase factor}} \underbrace{\hat{S}(z)}_{\text{Squeezing}} \underbrace{\hat{D}(\alpha)}_{\text{Displacement}} \underbrace{\hat{R}(\Phi)}_{\text{Rotation}}$$

$$\hat{S}_N(z) \equiv \exp\left[\frac{(\hat{a}^\dagger)^\intercal z \hat{a}^\dagger}{2} - \frac{\hat{a}^\intercal z^\dagger \hat{a}}{2}\right]$$

$$\hat{D}_N(\alpha) \equiv \exp\left[\alpha^\intercal \hat{a}^\dagger - \alpha^\dagger \hat{a}\right]$$

$$\hat{R}_N(\Phi) \equiv \exp\left[i(\hat{a}^\dagger)^\intercal \Phi \hat{a}\right]$$

[Phys. Rev. A 41, 4625 (1990).]

1

1.3

Disentangling operator products

Step 1

Disentangling individual operators via BCH (Glauber)

Step 2

Normal ordering via commutation

Step 3

Obtain the disentangled operator product

Lie algebraic background underpinning this:

Infinite-dimensional Lie algebra $\mathscr{L}$ & Homomorphism $\psi : \mathscr{L} \to \mathrm{su}(1,1)$

$\mathscr{L}$ defines unique disentangled form of $\hat{S}_N(z)$ and ψ lets us factorise $e^{A^\dagger(z)-A(z)}$!

1

1.4

$$\hat{U}_N(t) = \exp\left[A(t)\right]\exp\left[B^\intercal(t)\hat{a}^\dagger + (\hat{a}^\dagger)^\intercal C(t)\hat{a}^\dagger\right]\left[\sum_{n=0}^{\infty}\frac{:[(\hat{a}^\dagger)^\intercal D(t)\hat{a}]^n:}{n!}\right]\exp\left[E^\intercal(t)\hat{a} + \hat{a}^\intercal F(t)\hat{a}\right]$$

Put into Schrödinger equation, use chain rule, match coefficients on LHS/RHS

$$i\frac{\partial}{\partial t}A(t) = \mathrm{Tr}\left[f^\dagger(2C(t) + B(t)B^\intercal(t)) + g^\dagger(t)B(t) + h(t)\right],$$
$$i\frac{\partial}{\partial t}B(t) = (4C(t)f^\dagger(t) + \omega(t))B(t) + 2C(t)g^*(t) + g(t),$$
$$i\frac{\partial}{\partial t}C(t) = 4C(t)f^\dagger(t)C(t) + 2\omega(t)C(t) + f(t),$$
$$i\frac{\partial}{\partial t}D(t) = (4C(t)f^\dagger(t) + \omega(t))(D(t) + I),$$
$$i\frac{\partial}{\partial t}E(t) = (D^\intercal(t) + I)(2f^\dagger(t)B(t) + g^*(t)),$$
$$i\frac{\partial}{\partial t}F(t) = (D^\intercal(t) + I)f^\dagger(t)(D(t) + I).$$

Matrix Riccati Equation implies uniqueness of $\hat{U}_N$

You can obtain the original squeezing, displacement, and rotation parameters z, α, Φ by solving for A, B, C, D, E, and F.

2

Modelling lossy generation of squeezed light via an effective $\chi^{(2)}$ process

2

2.1

[Phys. Rev. A 110, 033709 (2024)]
[Phys. Rev. Lett. 122, 153906 (2019)]
[Phys. Rev. Applied 12, 064024 (2019)]

Quadratic Hamiltonian governing a coupled channel waveguide - ring resonator system:

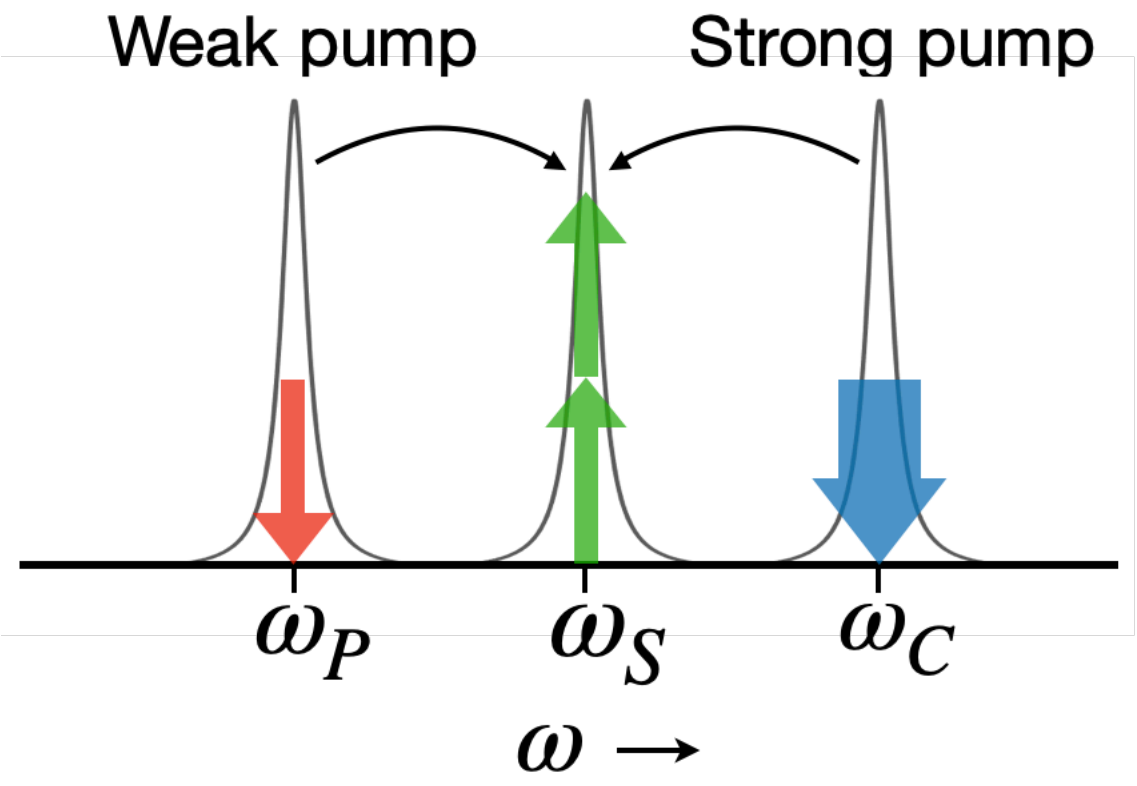
Operators for the squeezed light signal (S)

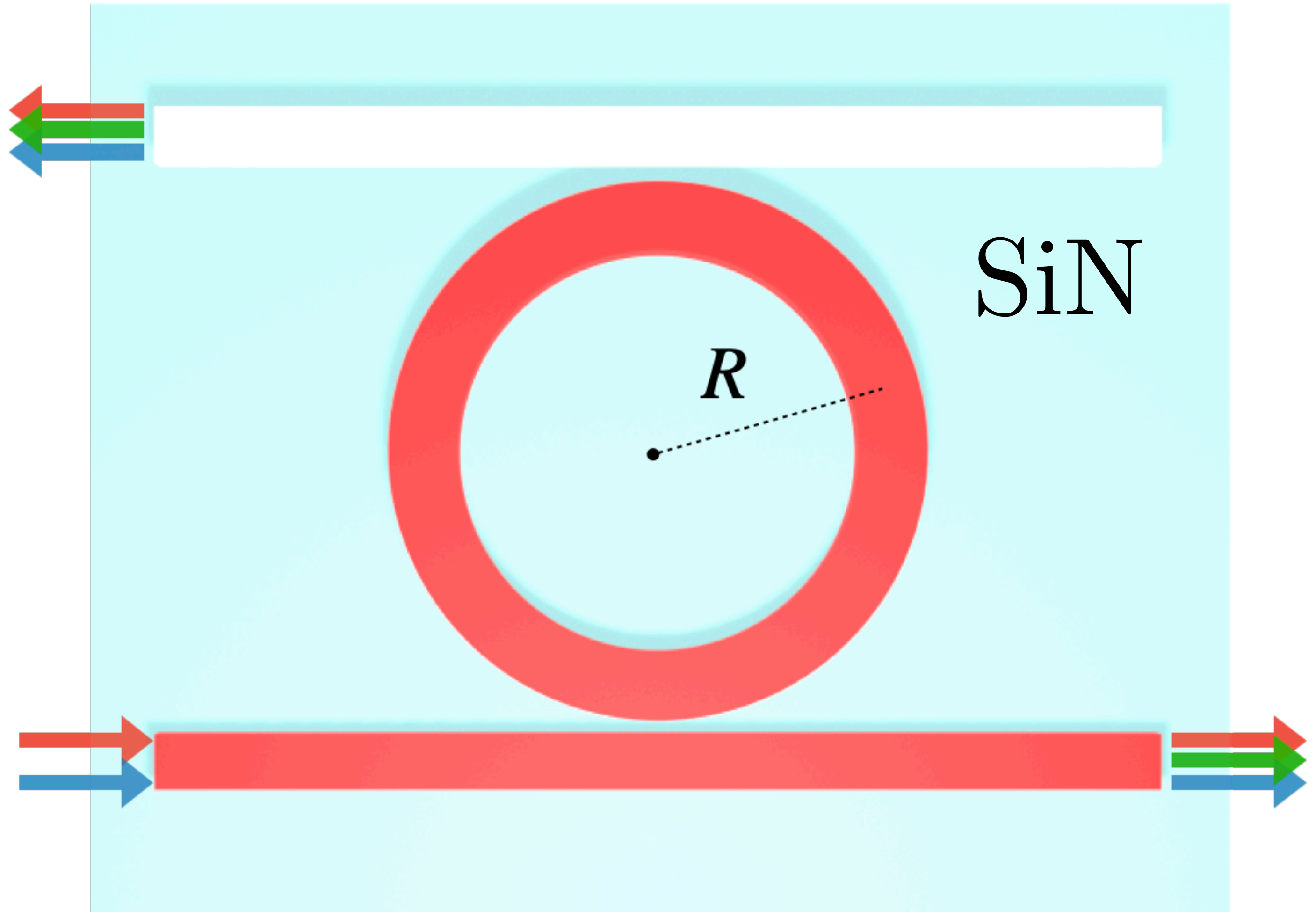
$$\hat{H}_N(t) = \hbar \sum_{k,l} \Delta_{kl}(t) a_k^\dagger a_l + \hbar \sum_{k,l} \zeta_{kl}(t) a_k^\dagger a_l^\dagger + \text{H.c.}$$

Linear coefficient functions

Nonlinear coefficient functions

Effective $\chi^{(2)}$ between P and S





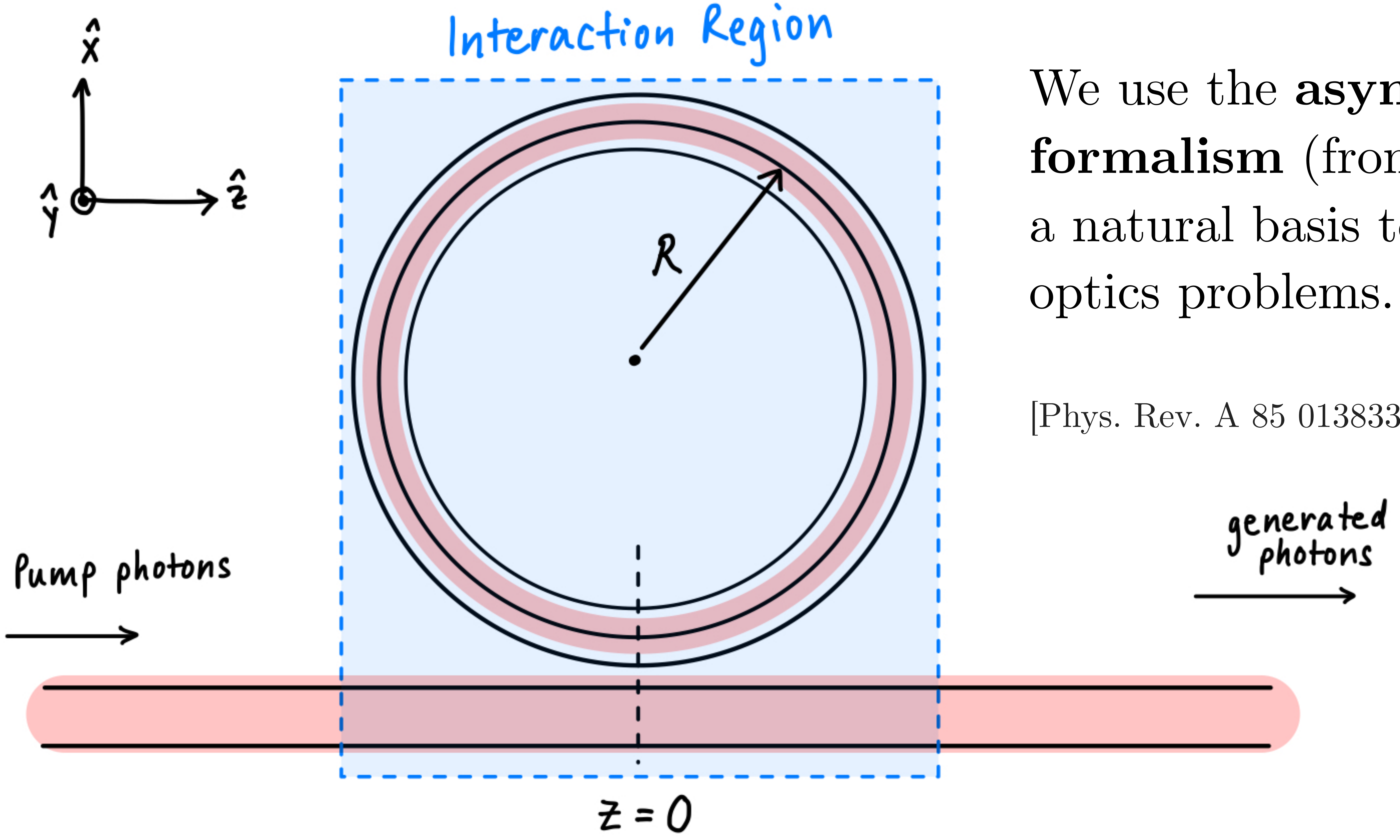
Two pumps P and C treated classically.

C is CW, P is pulsed.

We want to be able to solve for the dynamics without solving systems of 6 PDEs!

2

2.2



We use the **asymptotic in/out field formalism** (from scattering theory). It's a natural basis to treat general nonlinear optics problems.

[Phys. Rev. A 85 013833 (2012).]

We work in the **undepleted pump approximation**, which reduces higher-order Hamiltonians to quadratic ones. It is a valid approximation in most cases since a negligible fraction of the pump power is transferred to the generated fields.

2

2.3

We assume zero displacement for the squeezed state, so $\hat{U}(t) = \hat{S}(t)\hat{R}(t)e^{i\theta(t)}$ where $\hat{S}$ is determined by the **squeezing matrix J** and $\hat{R}$ by **rotation matrix $\boldsymbol{\phi}$**.

Instead of solving for 6 parameter functions in a PDE system, we determine the elements of **J** and $\boldsymbol{\phi}$ by setting up dynamical equations satisfied by auxiliary matrices **V** and **W**:

[Adv. Opt. Photon. 14, 291-403 (2022)]

Heisenberg equations for a and a†

$$\frac{d}{dt}\begin{bmatrix}\mathbf{a}\\ \mathbf{a}^\dagger\end{bmatrix} = -i\begin{bmatrix}\Delta & 2\zeta\\ -2\zeta* & -\Delta*\end{bmatrix}\begin{bmatrix}\mathbf{a}\\ \mathbf{a}^\dagger\end{bmatrix}$$

Solution to Heisenberg equations

$$\begin{bmatrix}\mathbf{a}\\ \mathbf{a}^\dagger\end{bmatrix} = \begin{bmatrix}\mathbf{V} & \mathbf{W}\\ \mathbf{W}* & \mathbf{V}*\end{bmatrix}\begin{bmatrix}\mathbf{a}(\mathbf{t_0})\\ \mathbf{a}^\dagger(\mathbf{t_0})\end{bmatrix}$$

[Phys. Rev. A 110, 033709 (2024)]

$\longrightarrow$

Reduction to coupled ODEs

$$\frac{d}{dt}\mathbf{V} = -i\Delta\mathbf{V} - 2i\zeta\mathbf{W}*$$

$$\frac{d}{dt}\mathbf{W} = -i\Delta\mathbf{W} - 2i\zeta\mathbf{V}*$$

Solved by RK4

ODE constraints

$$\mathbf{W}\mathbf{V}^\mathbf{T} - \mathbf{V}\mathbf{W}^\mathbf{T} = 0$$

$$\mathbf{V}\mathbf{V}^\dagger - \mathbf{W}\mathbf{W}^\dagger = 1$$

$$\boldsymbol{V}(t) = \cosh(\boldsymbol{u}(t))\mathrm{e}^{i\boldsymbol{\phi}(t)},$$

$$\boldsymbol{W}(t) = \sinh(\boldsymbol{u}(t))\mathrm{e}^{i\boldsymbol{\alpha}(t)}\mathrm{e}^{-i\boldsymbol{\phi}^\mathrm{T}(t)}.$$

$$\boldsymbol{J}(t) = \boldsymbol{u}(t)\mathrm{e}^{i\boldsymbol{\alpha}(t)}.$$

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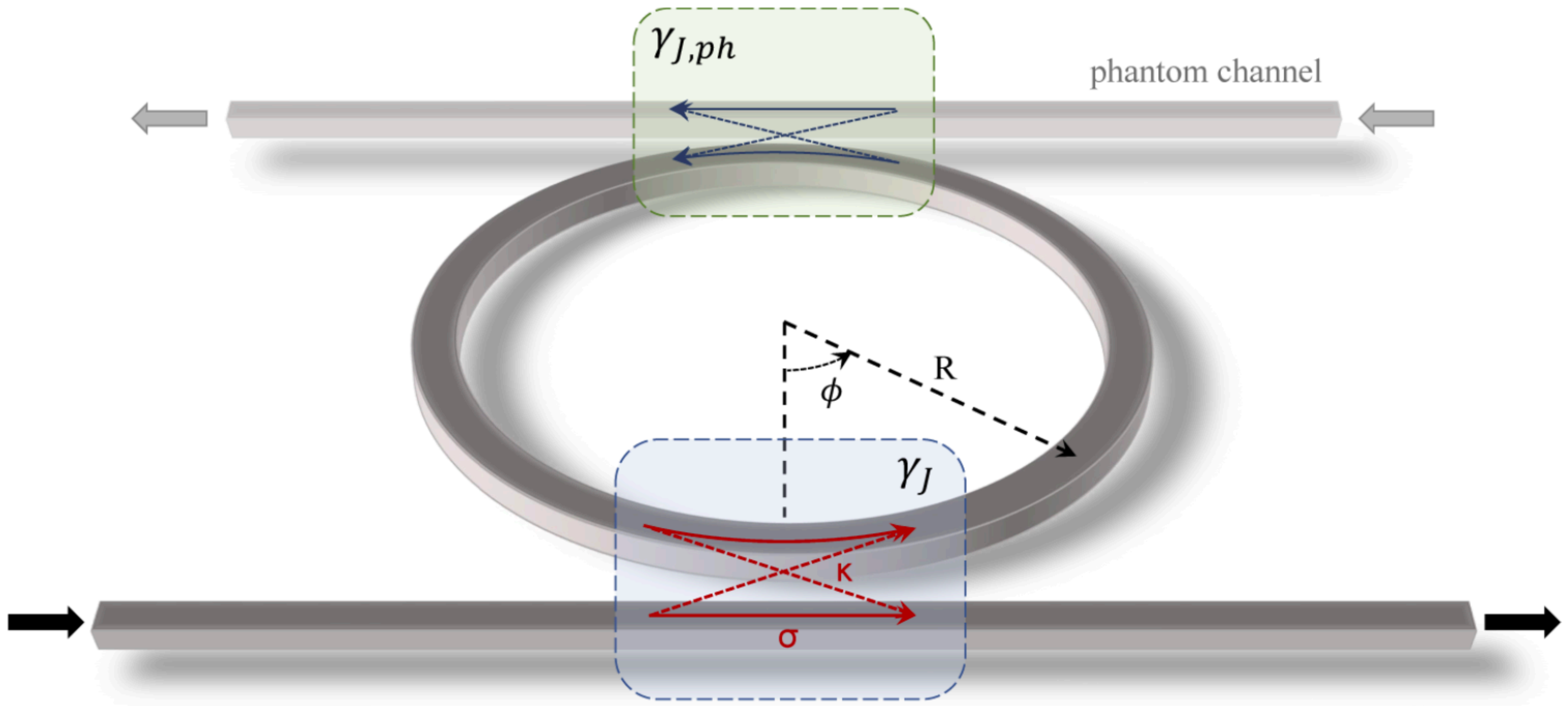
2.4

We're considering lossy generation — loss can be modelled by adding a phantom channel.

Figure 6



Schematic representation of a point-coupled ring resonator with and additional fictitious coupler and waveguide, referred to as "phantom channel."

The phantom channel is point-coupled to the ring.

Figure from [Adv. Opt. Photon. 14, 291-403 (2022)]

2

2.5

Two key statistics were examined:

The **total number of squeezed photons generated** (n_{ph}) & the **Schmidt number (K)**

K quantifies the **number of Schmidt modes in the squeezed state.**

$$K = \frac{\left(\sum_\lambda \sinh^2(r_\lambda) \right)^2}{\sum_\lambda \sinh^4(r_\lambda)}$$

[AVS Quantum Sci. 5, 011404 (2023)]

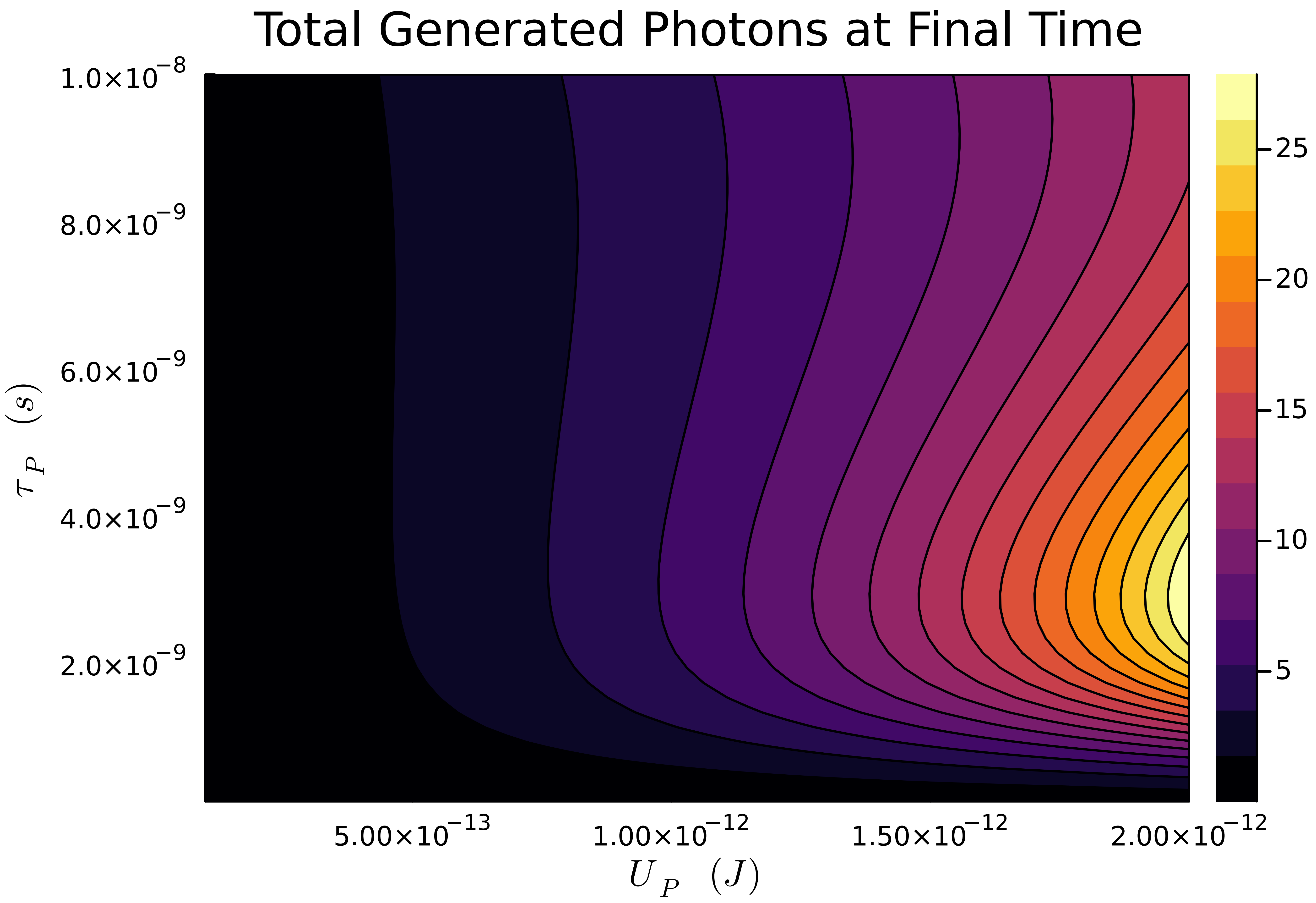
The Schmidt decomposition allows a multimode squeezed state to be written as a product of single-mode squeezed states, in each Schmidt mode.

$K = 1.0$ means the state has only a single Schmidt mode, so it is separable. Separability decreases with increasing K.

Optimising for n_{ph}

The largest n_{ph} generated was about **27.9**, for pump energy U_P about **2.0 pJ** and pulse duration τ_P about **2.93 ns**. CW pump power is 30 mW. The ring is critically coupled. This is about 20 dB of squeezing.

Very short τ_P means that the bandwidth of the pulse greatly exceeds the ring resonance width, so **only a small fraction of the pulse enters the resonator**.



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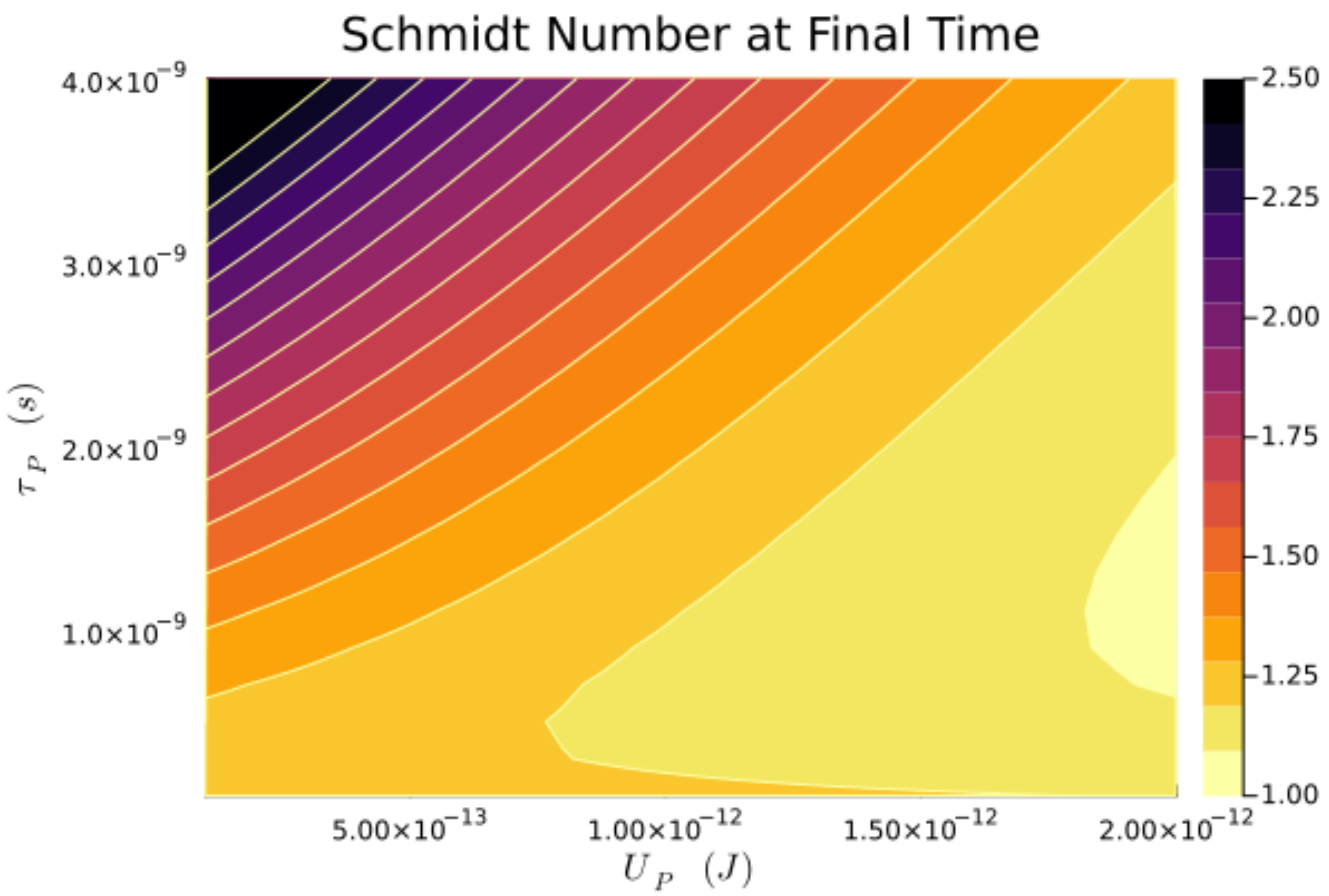
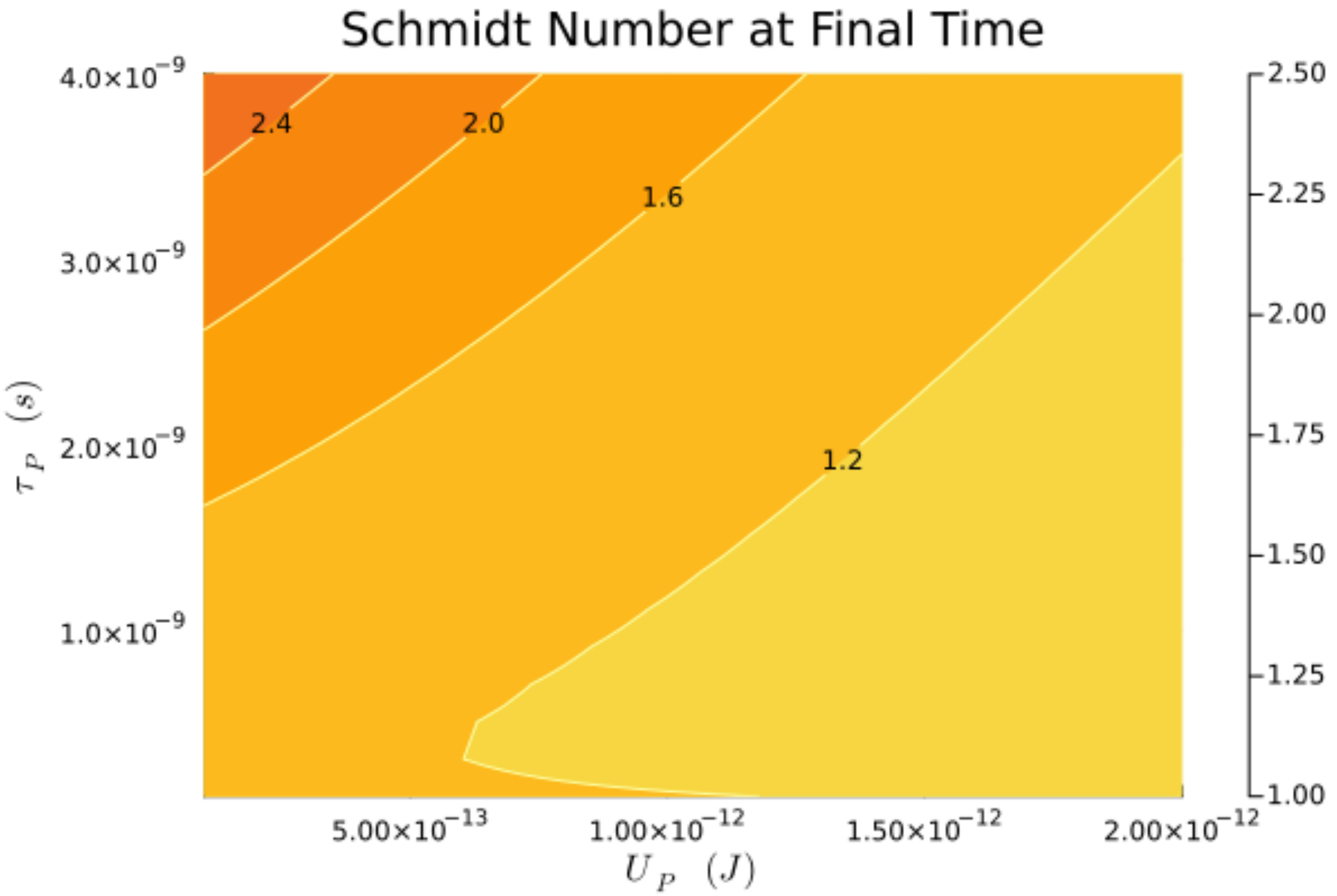
2.7

Going to Schmidt number 1.0 — returning optimal τ_P and U_P





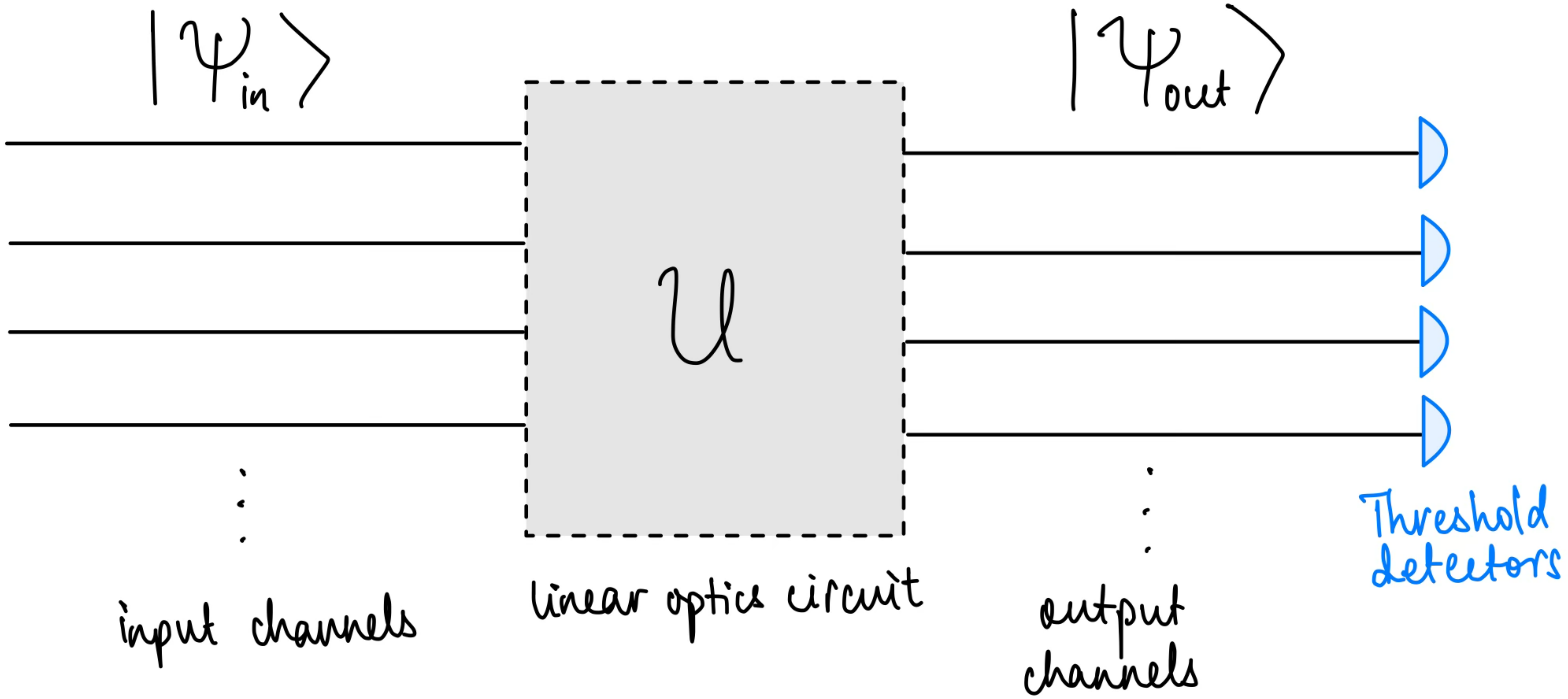
Plotting contour lines and examining the isoclines, we see that τ_P and U_P yielding a Schmidt number around 1.0 are around **1.0 ns and 2.0 pJ** respectively. When the ring resonance and the pulse bandwidth are approximately equal, we get this optimal Schmidt number.

3

Current Work

April 2024 -

Analysing linear optics circuits with threshold detectors and Gaussian states



System of interest

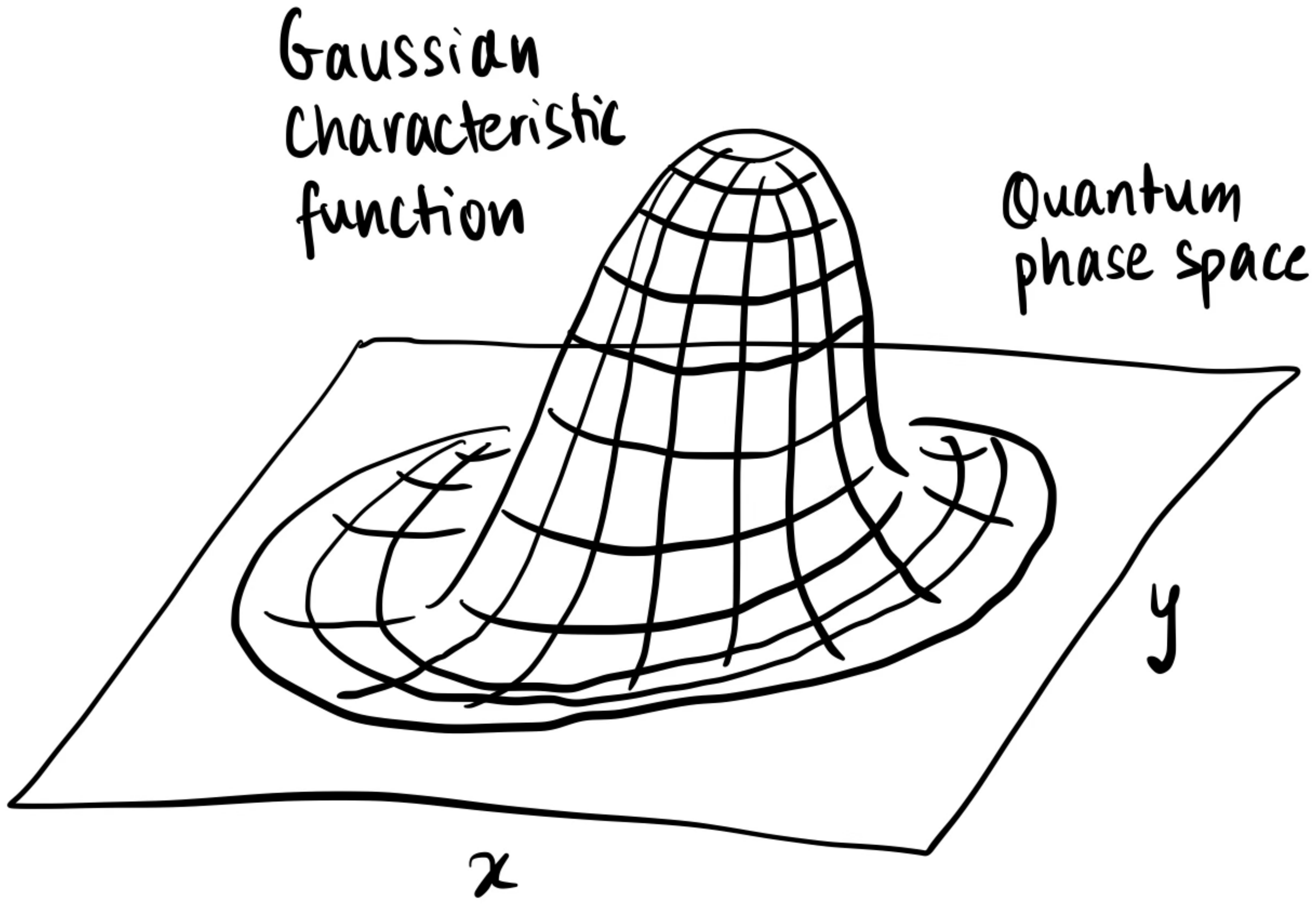
We're interested in examining detection statistics in systems such as this.

Potential applications to Gaussian boson sampling, heralded HOM using squeezed states, etc.

[Phys. Rev. A 98, 062322]

Gaussian states and the symplectic formalism

Gaussian states $\rho(\vec{\alpha}, \boldsymbol{\Sigma})$ are special; can be **completely characterised** by vectors of means $\vec{\alpha}$ and Husimi covariance matrices $\boldsymbol{\Sigma}$. [Phys. Rev. A 106, 043712].



Gaussian states have Gaussian characteristic functions.

l modes of interest,

$$\hat{\vec{\zeta}} = [\hat{a}_1, \ldots, \hat{a}_l, \hat{a}_1^\dagger, \ldots, \hat{a}_l^\dagger]$$

$$\vec{\alpha}_i = \mathrm{Tr}\left(\rho \hat{\vec{\zeta}}_i\right)$$

$$[\Sigma]_{ij} = \frac{1}{2}\mathrm{tr}\left(\left\{\hat{\vec{\zeta}}_i \hat{\vec{\zeta}}_j^\dagger\right\}\rho\right) - \cancelto{0}{\vec{\alpha}_i \vec{\alpha}_j^*} + \frac{1}{2}\delta_{ij}$$

For a squeezed vacuum state, mean displacement is 0.

3

3.3

[Phys. Rev. A 98, 062322]

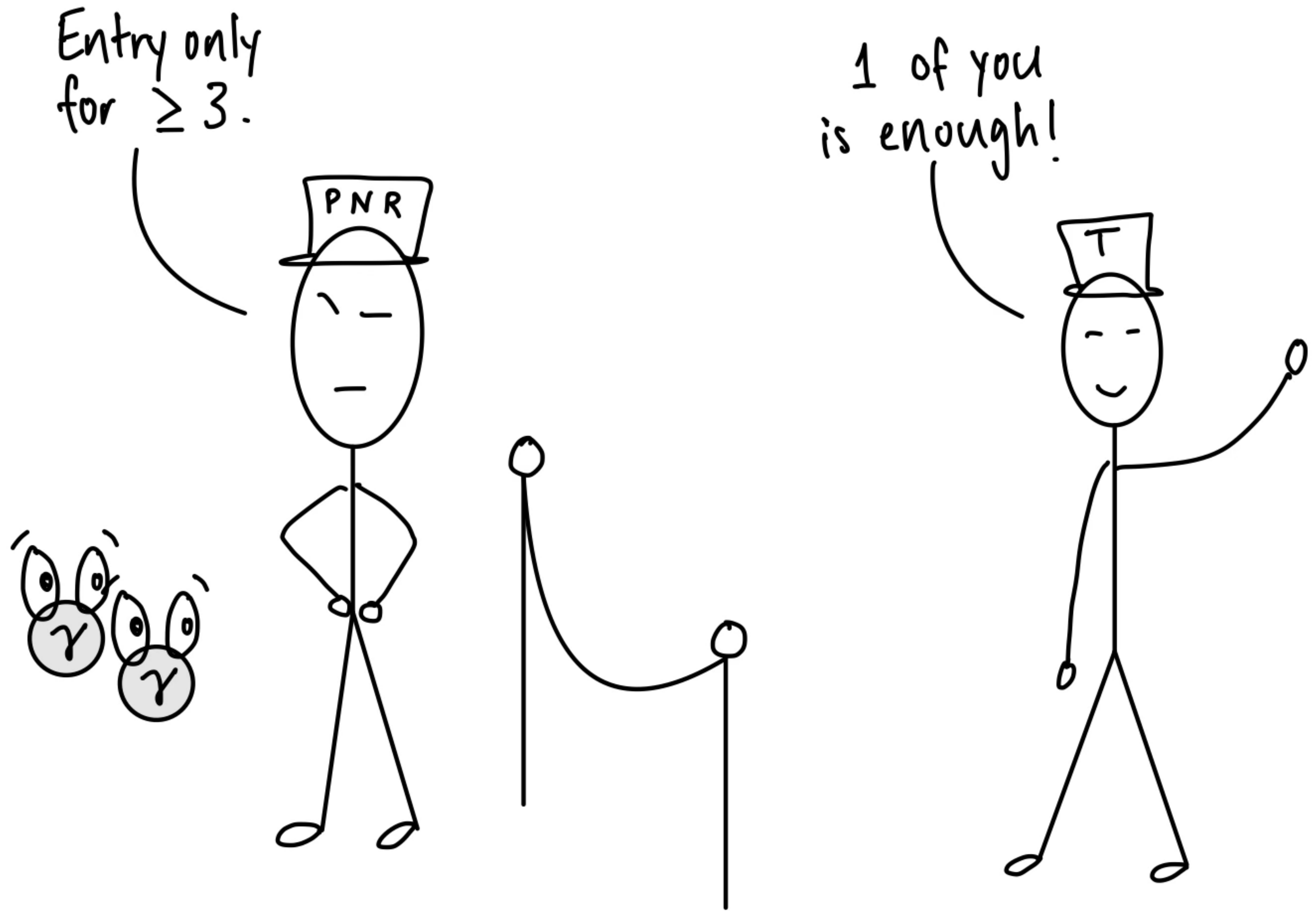
So that's how to describe Gaussian states. What about the system's detectors?

We are using threshold detectors as opposed to photon-number resolving (PNR) detectors.

Threshold detection measurements distinguish between a vacuum and a photon.

Standard in integrated photonic devices, widely available.

For classical algorithms, PNRs would make it **very, very hard** to efficiently compute statistics.



PNRs need a minimum number of photons to click; one is enough for a threshold detector.

Threshold statistics with Gaussian states

Deriving a closed-form expression for the probability

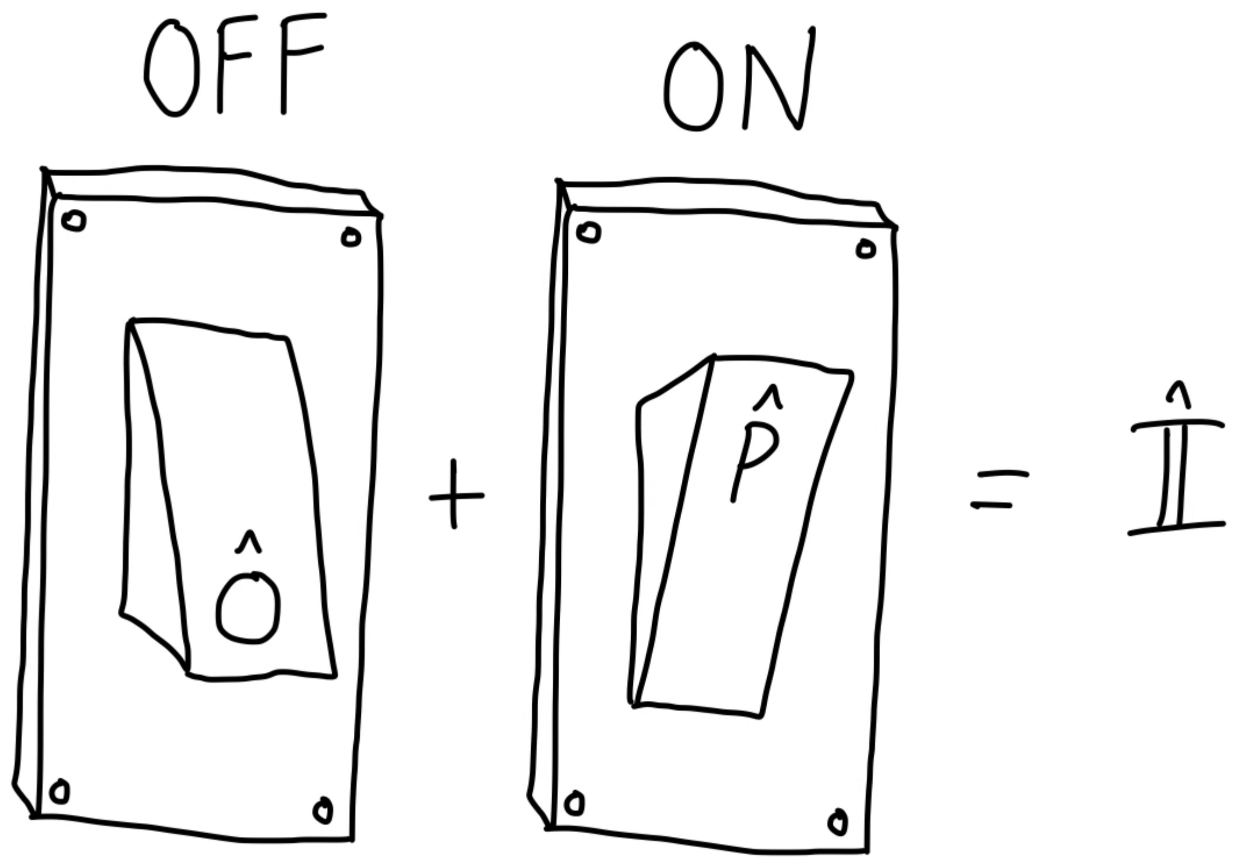
With POVM elements $\hat{P}_{nB}$, $\hat{O}_{nB}$ (output channel n, frequency bin $B \in \{S, I\}$) corresponding to click/no click respectively, we can write the projector for a desired measurement:

$$\hat{\Pi}_V = \prod_{n\in\{1,\ldots,l\}} \hat{P}_{nB}\hat{O}_{n'B'}$$

The probability that this projector is measured is given by

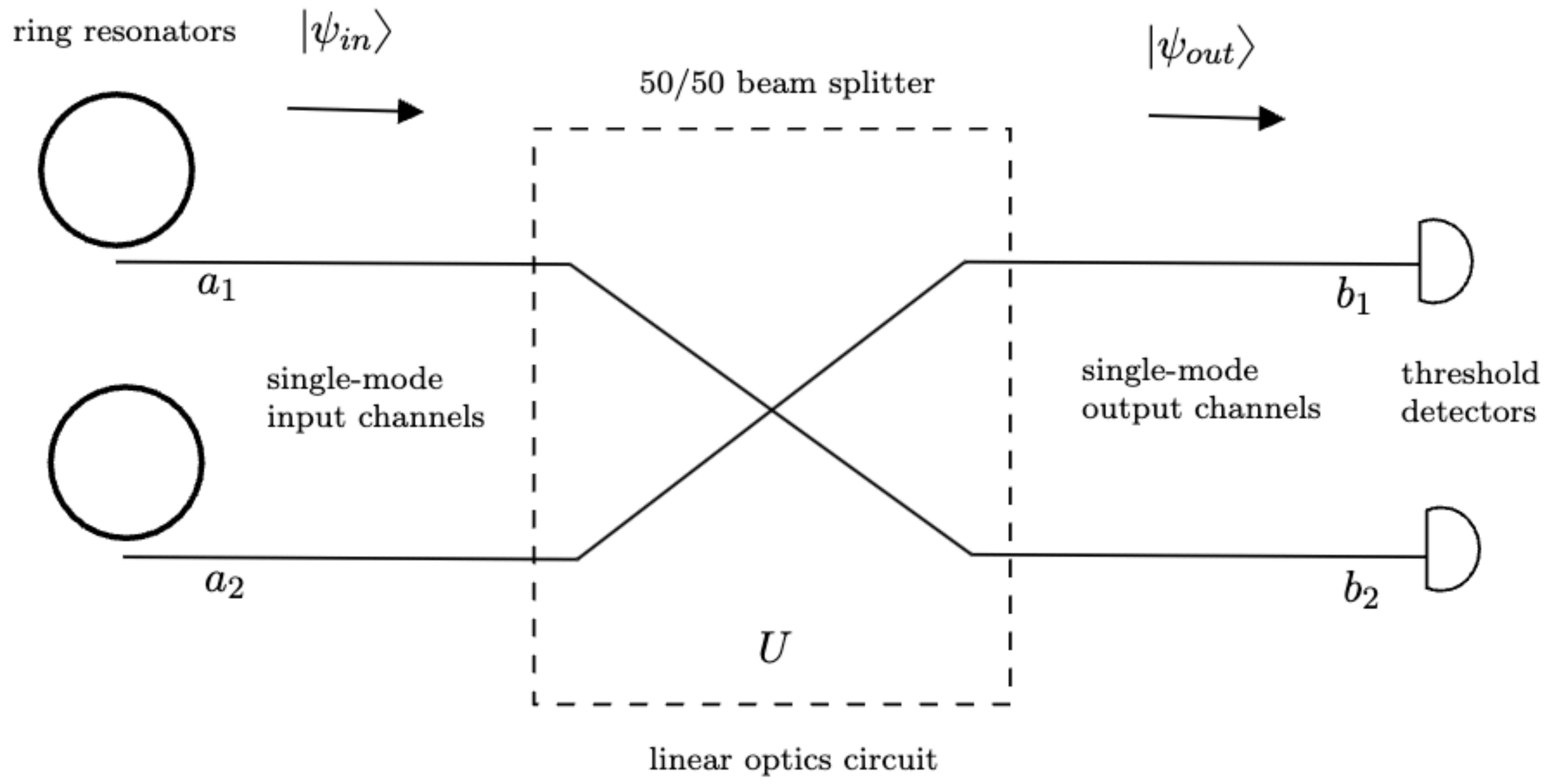
$$p_V = \mathrm{tr}\left(\hat{\Pi}_V |\Psi\rangle\langle\Psi|\right)$$

There are two methods to simplify this: one is [[Phys. Rev. A **98**, 062322]] which uses the Husimi Q-function and the other uses disentangling methods.



The threshold POVM elements can be combined for each mode to write a projector for any measurement desired.

Example: HOM with indistinguishable squeezed states

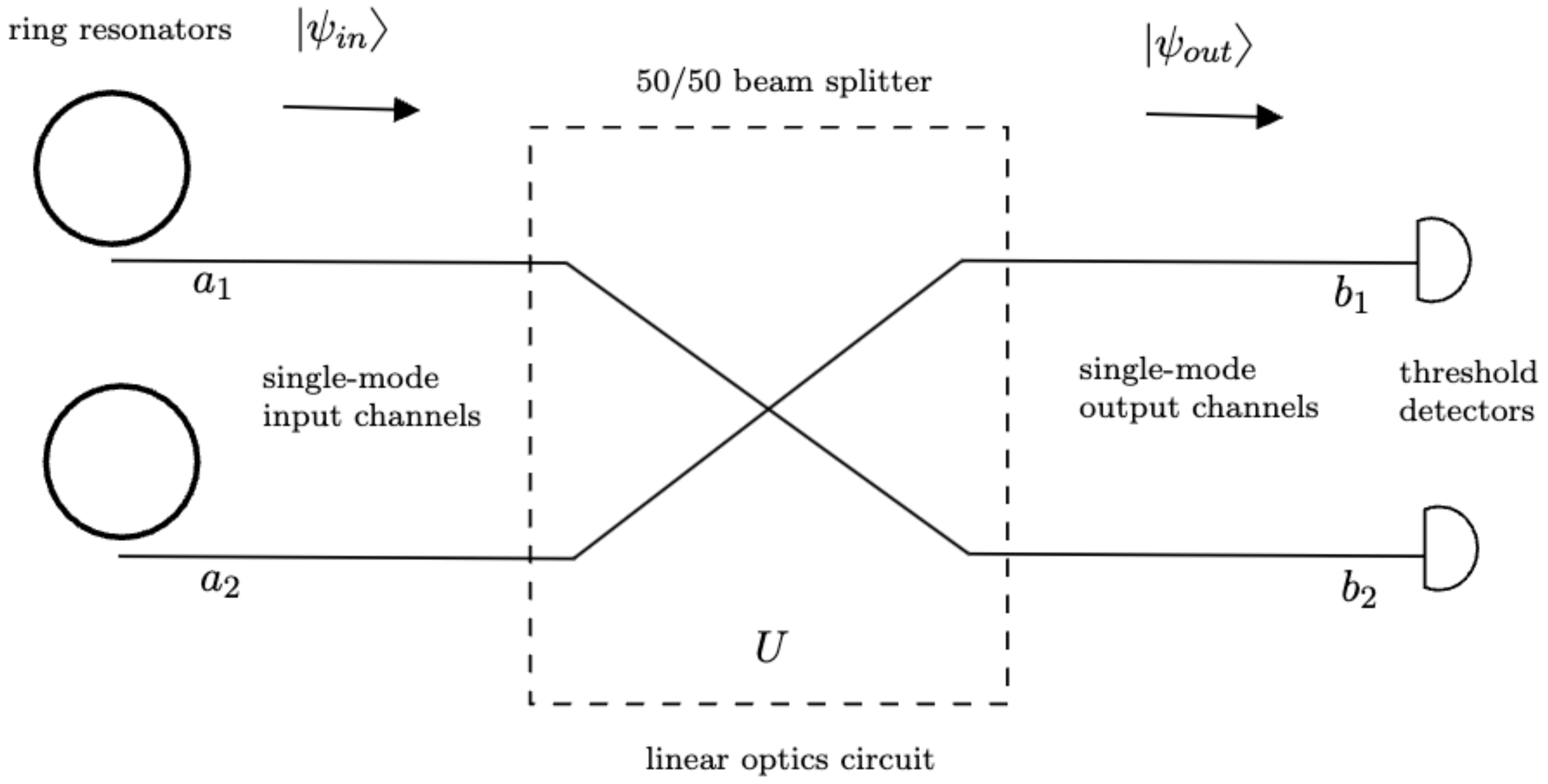


System of interest

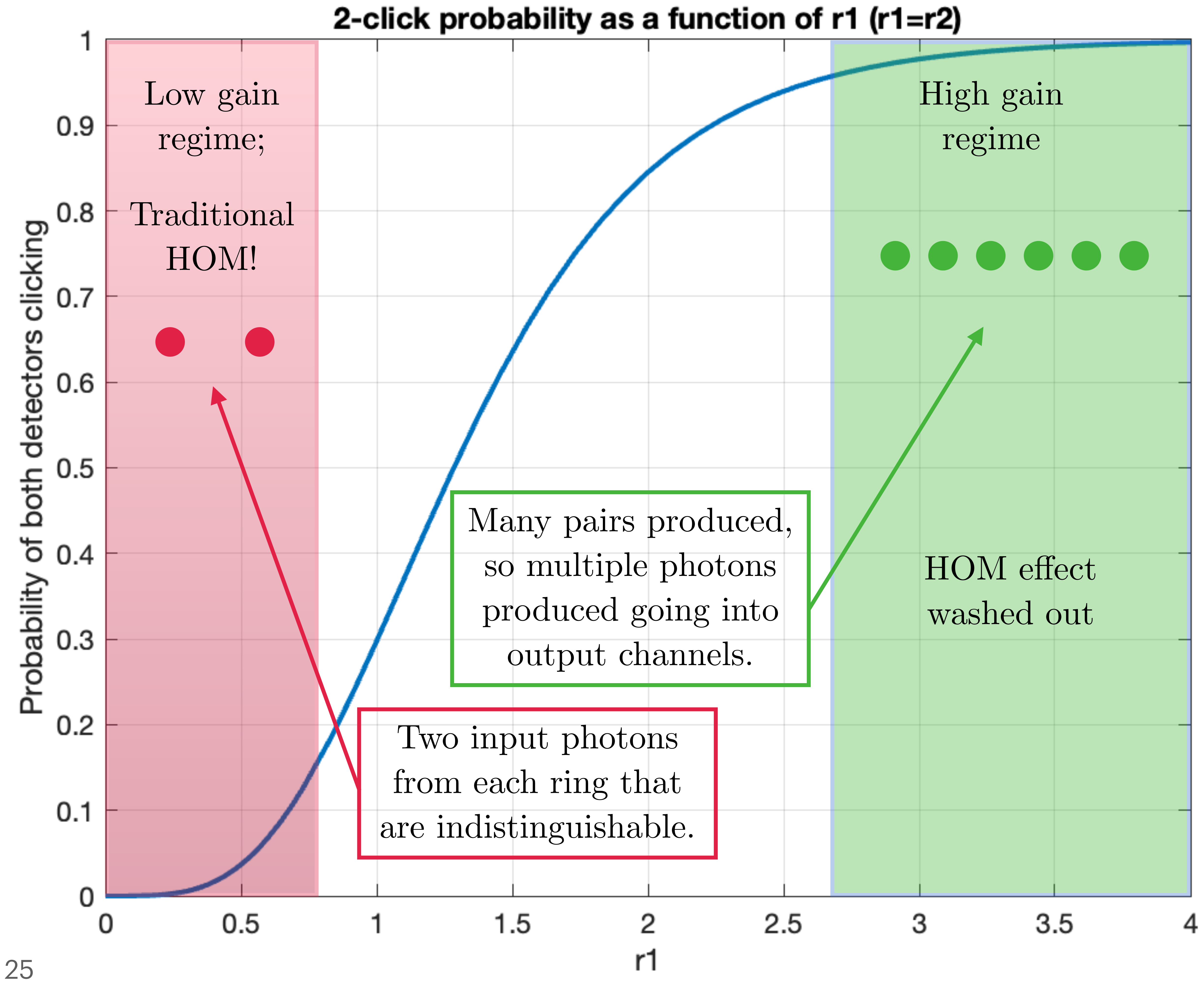
Using the asymptotic in/out field formalism, we can study realistic input Gaussian states in systems such as this — even beyond the 2-photon regime for general Gaussian input states.

In this example, we have a 50/50 beam splitter described by a unitary transformation U.

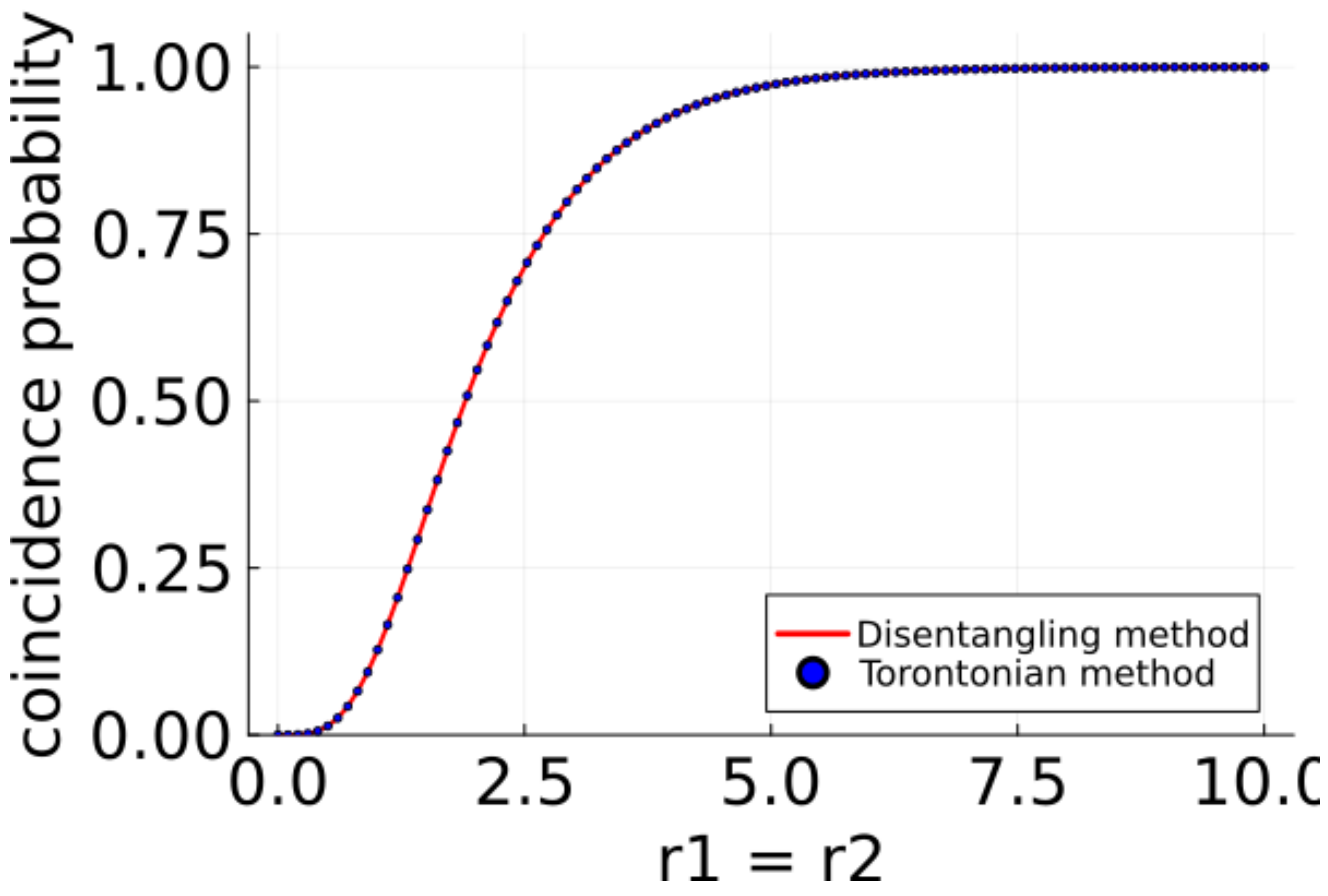
Example: HOM with indistinguishable squeezed states



Calculating coincidence click probability with respect to squeezing parameter r (identical in both input modes), we have the plot on the right:



Example: HOM with indistinguishable squeezed states



Our disentangling method agrees with the method of [Quesada 2018]!

We are no longer computing 2^N determinants, so complexity is no longer $O\left(N^3 2^N\right)$.

We expect an exponential decrease in complexity when we go from mode-wise to frequency bin-wise detectors. Edit: NO we don't

(This is very much still a work in progress.)

Future work

So, we have the theory that lets us analyse these systems with arbitrary linear optics circuits and threshold detectors sensitive to frequency bins. We now want to apply it to study more complex multimode systems.

We are also looking for more potential applications of this theory.

An interesting avenue we'd like to explore is extending my thesis work to studying non-Gaussian states generated in a nonlinear fashion, outside of the undepleted pump approximation. [Phys. Rev. A 110, 033709 (2024)]

Acknowledgements

Thank you to Prof. Sipe and Dr. Vendromin for their excellent supervision! I have learned so much from them and hope to learn much more.

Thank you to Prof. Heshami for giving me the opportunity to talk to you about my work!
Thank you for attending!

Threshold detection probabilities

The Torontonian

An l-mode Gaussian state measured, N clicks observed in modes $S = \left(i_1, i_2, \ldots, i_N\right)$.

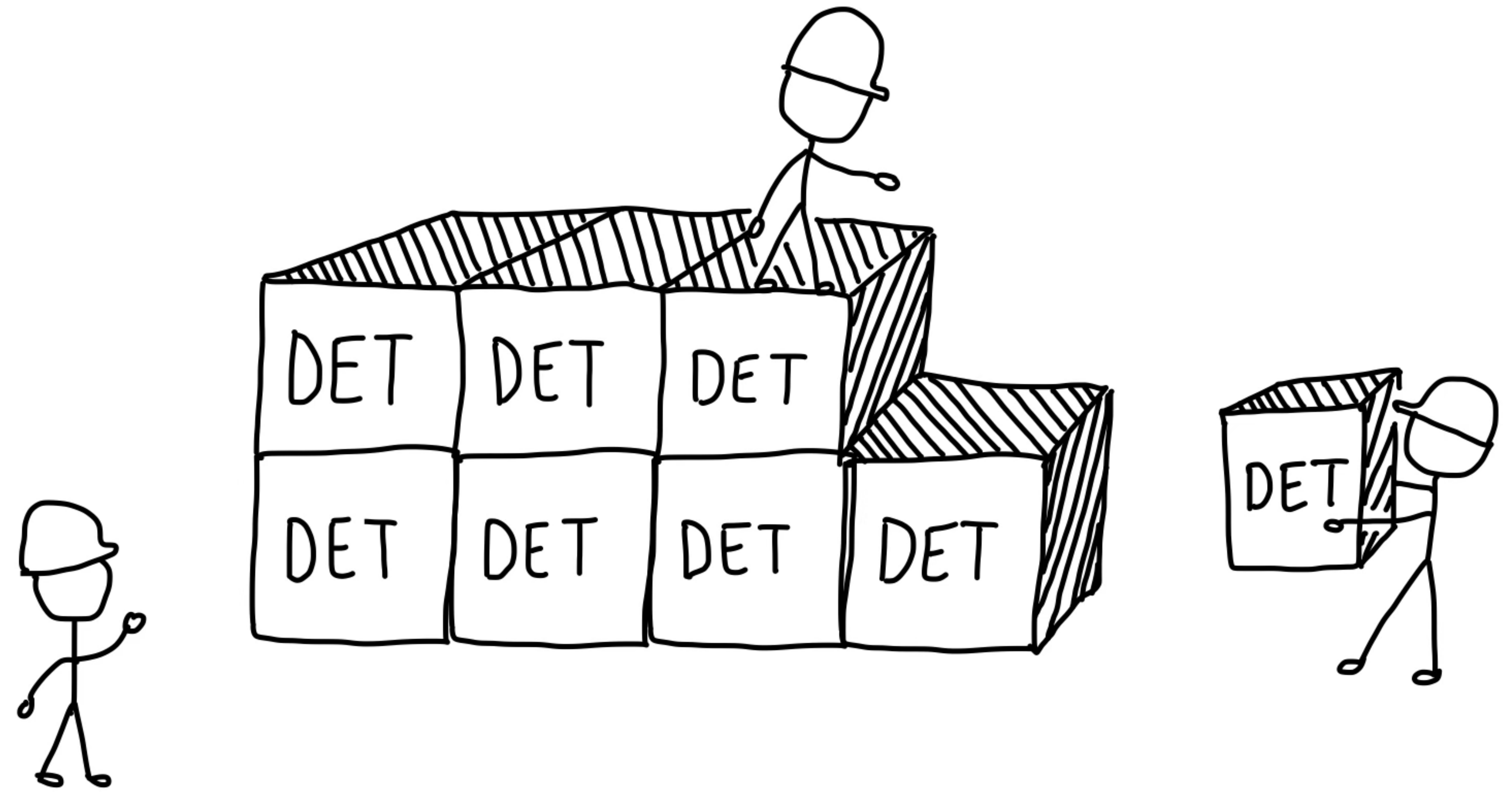
[Quesada] showed the probability can be written using the **Torontonian**:

$$p(S) = \frac{1}{\sqrt{\det \boldsymbol{\Sigma}}} \mathrm{Tor}\left(\mathbb{I} - \left(\boldsymbol{\Sigma}^{-1}\right)_{(S)}\right)$$

For any matrix of the form $\mathbf{A} \in \mathbb{C}^{2N\times 2N}$, $\mathbf{A} = \begin{bmatrix} \mathbf{W} & \mathbf{Y}^* \\ \mathbf{Y} & \mathbf{W}^* \end{bmatrix} > 0$ where $\mathbf{W}$ is Hermitian and $\mathbf{Y}$ is symmetric, we can compute its **Torontonian**.

Direct relation to the Hafnian for PNRs! [Quesada, Arrazola, Killoran (2018)].

$$\mathrm{Tor}(\mathbf{A}) = \sum_{Z \in P([N])} \frac{(-1)^{|Z|}}{\sqrt{\det\left(\mathbb{I} - \mathbf{A}_{(Z)}\right)}}$$



For direct computations of the Torontonian, the complexity is $O\left(N^3 2^N\right)$.