ENPM667 Final Project Report

LQR and LQG Design and Simulation

Final Project Report

ENPM667-RO01
Dustin Hartnett
Kyle DeGuzman

First Component

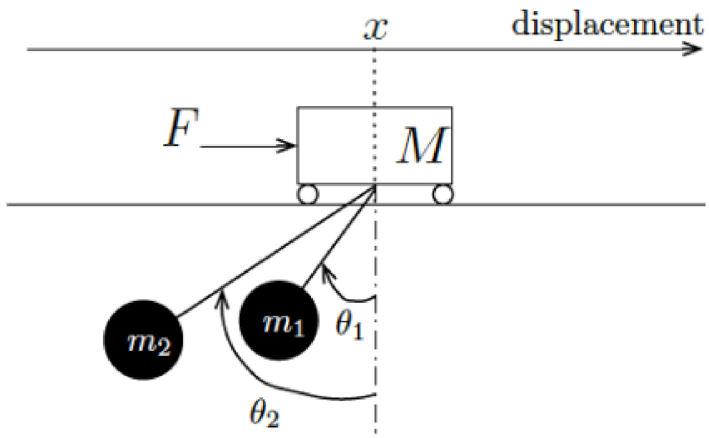


Fig. 1: Crane visualization

This project defines a crane moving horizontally along a one-dimensional track (left and right movements). The cart is assumed to be *frictionless* and has two masses attached to it by cables (see Figure 1). The following quantities are defined below:

- M: The mass of the cart.
- F: The external force applied to the cart.
- m_1: The mass of load 1.
- m_2: The mass of load 2.
- l_1: The length of the cable connecting m_1 to the cart.
- l_2: The length of the cable connecting m_2 to the cart.
- θ_1: The angle made between the cable of m_1 and the vertical line passing through the middle of the cart.
- θ_2: The angle made between the cable of m_2 and the vertical line passing through the middle of the cart.

Part A

To obtain the equations of motion for the system to be used in the state-space representation, the equations for kinetic and potential energies are first derived.

$$\dot{x}_{cart} = \dot{x}$$
$$\dot{x}_{pend_1} = \dot{x} - \dot{x}_{m_1}$$
$$\dot{x}_{pend_2} = \dot{x} - \dot{x}_{m_2}$$

The kinetic energy of the system is derived below. First, the kinetic energy of the cart itself is found; this is given by the following:

$$T_{cart} = \frac{1}{2}M\dot{x}^2$$

The kinetic energy of one pendulum is given by the following equation:

$$T_{pendulum} = \frac{1}{2}m\dot{x}_m^2$$
$$\dot{x}_{m_1} = (\dot{x} - \dot{\theta}_1 l_1 cos(\theta_1))\hat{\imath} + (\dot{\theta}_1 l_1 sin(\theta_1))\hat{\jmath}$$
$$\dot{x}_{m_1}^2 = (\dot{x} - \dot{\theta}_1 l_1 cos(\theta_1))^2 + (\dot{\theta}_1 l_1 sin(\theta_1))^2$$
$$= \dot{x}^2 - 2\dot{x}\dot{\theta}_1 l_1 cos(\theta_1) + \dot{\theta}_1^2 l_1^2 cos^2(\theta_1) + \dot{\theta}_1^2 l_1^2 sin^2(\theta_1)$$
$$= \dot{x}^2 - 2l_1 cos(\theta_1)\dot{x}\dot{\theta}_1 + \dot{\theta}_1^2 l_1^2 (cos^2(\theta_1) + sin^2(\theta_1))$$
$$\dot{x}_{m_1}^2 = \dot{x}^2 - 2l_1 cos(\theta_1)\dot{x}\dot{\theta}_1 + \dot{\theta}_1^2 l_1^2$$

The kinetic energy equations for m_1 and m_2 are given by:

$$T_{m_1} = \frac{1}{2}m_1\dot{x}^2 - m_1 l_1 cos(\theta_1)\dot{x}\dot{\theta}_1 + \frac{1}{2}m_1\dot{\theta}_1^2 l_1^2$$
$$T_{m_2} = \frac{1}{2}m_2\dot{x}^2 - m_2 l_2 cos(\theta_2)\dot{x}\dot{\theta}_2 + \frac{1}{2}m_2\dot{\theta}_2^2 l_2^2$$

The total kinetic energy is the sum of the kinetic energy equations of the cart and the two pendulum masses.

$$T = T_{cart} + T_{m_1} + T_{m_2}$$
$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m_1\dot{x}^2 - m_1 l_1 cos(\theta_1)\dot{x}\dot{\theta}_1 + \frac{1}{2}m_1\dot{\theta}_1^2 l_1^2 + \frac{1}{2}m_2\dot{x}^2 - m_2 l_2 cos(\theta_2)\dot{x}\dot{\theta}_2 + \frac{1}{2}m_2\dot{\theta}_2^2 l_2^2$$

Simplifying, the final equation for kinetic energy is given by:

$$T = \frac{1}{2}(M + m_1 + m_2)\dot{x}^2 - m_1 l_1 cos(\theta_1)\dot{x}\dot{\theta}_1 - m_2 l_2 cos(\theta_2)\dot{x}\dot{\theta}_2 + \frac{1}{2}m_1\dot{\theta}_1^2 l_1^2 + \frac{1}{2}m_2\dot{\theta}_2^2 l_2^2$$

For potential energy, the cart height is treated as a reference point. Using this, the potential energy of one pendulum below the cart is given by:

$V = -mgh = -mgLcos(\theta)$

So for both pendulum masses, the equations are given by:

$$V_{m_1} = -m_1 g l_1 cos(\theta_1)$$
$$V_{m_2} = -m_2 g l_2 cos(\theta_2)$$

As with kinetic energy, the total potential energy is the sum of the potential energy equations of the two pendulum masses:

$$V = V_{m_1} + V_{m_2} = -m_1 g l_1 cos(\theta_1) - m_2 g l_2 cos(\theta_2)$$

With both kinetic and potential energy equations for the whole system derived, the Lagrangian can now be obtained.

$$\mathcal{L} = T - V$$
$$\mathcal{L} = [\frac{1}{2}(M+m_1+m_2)\dot{x}^2 - m_1 l_1 cos(\theta_1)\dot{x}\dot{\theta}_1 - m_2 l_2 cos(\theta_2)\dot{x}\dot{\theta}_2 + \frac{1}{2}m_1\dot{\theta}_1^2 l_1^2 + \frac{1}{2}m_2\dot{\theta}_2^2 l_2^2] - ...$$
$$...[-m_1 g l_1 cos(\theta_1) - m_2 g l_2 cos(\theta_2)]$$

The partial derivatives are taken with respect to $\dot{x}$ and x first. The general Lagrangian equation form is:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = F$$

Using the equation for $\mathcal{L}$, the derivatives are given by:

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = (M + m_1 + m_2)\dot{x} - m_1 l_1 cos(\theta_1)\dot{\theta}_1 - m_2 l_2 cos(\theta_2)\dot{\theta}_2$$
$$\frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{x}}) = (M+m_1+m_2)\ddot{x} - m_1 l_1 cos(\theta_1)\ddot{\theta}_1 + m_1 l_1 sin(\theta_1)\dot{\theta}_1^{\ 2} - m_2 l_2 cos(\theta_2)\ddot{\theta}_2 + m_2 l_2 sin(\theta_2)\dot{\theta}_2^{\ 2}$$
$$\frac{\partial \mathcal{L}}{\partial x} = 0$$

Plugging these values into the Lagrangian equation results in

$$(M+m_1+m_2)\ddot{x} - m_1 l_1 cos(\theta_1)\ddot{\theta}_1 + m_1 l_1 sin(\theta_1)\dot{\theta}_1^{\ 2} - m_2 l_2 cos(\theta_2)\ddot{\theta}_2 + m_2 l_2 sin(\theta_2)\dot{\theta}_2^{\ 2} = F$$

The partial derivatives are now taken with respect to $\dot{\theta}_1$, θ_1, $\dot{\theta}_2$, and θ_2. The general Lagrangian form is

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1} - \frac{\partial\mathcal{L}}{\partial\theta_1} = 0$$

Note that the right-hand side here is 0 since there are no external loads being applied to these system states. The derivatives are given by:

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1} = -m_1 l_1 cos(\theta_1)\dot{x} + m_1 l_1^2\dot{\theta}_1$$
$$\frac{d}{dt}(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1}) = m_1 l_1 sin(\theta_1)\dot{x}\dot{\theta}_1 - m_1 l_1 cos(\theta_1)\ddot{x} + m_1 l_1^2\ddot{\theta}_1$$
$$\frac{\partial\mathcal{L}}{\partial\theta_1} = m_1 l_1 sin(\theta_1)\dot{x}\dot{\theta}_1 - m_1 g l_1 sin(\theta_1)$$

Plugging these values into the Lagrangian equation results in:

$$m_1 l_1 sin(\theta_1)\dot{x}\dot{\theta}_1 - m_1 l_1 cos(\theta_1)\ddot{x} + m_1 l_1^2\ddot{\theta}_1 - m_1 l_1 sin(\theta_1)\dot{x}\dot{\theta}_1 + m_1 g l_1 sin(\theta_1) = 0$$

Simplifying,

$$m_1 g l_1 sin(\theta_1) + m_1 l_1^2\ddot{\theta}_1 - m_1 l_1 cos(\theta_1)\ddot{x} = 0$$
$$g sin(\theta_1) + l_1\ddot{\theta}_1 - cos(\theta_1)\ddot{x} = 0$$
$$cos(\theta_1)\ddot{x} - l_1\ddot{\theta}_1 - g sin(\theta_1) = 0$$

The exact same steps are followed for θ_2, resulting in

$$cos(\theta_2)\ddot{x} - l_2\ddot{\theta}_2 - g sin(\theta_2) = 0$$

The following equations derived earlier are now used to determine the nonlinear state-space representation:

$$(M+m_1+m_2)\ddot{x} - m_1 l_1 cos(\theta_1)\ddot{\theta}_1 - m_2 l_2 cos(\theta_2)\ddot{\theta}_2 + m_1 l_1 sin(\theta_1)\dot{\theta}_1^{\,2} + m_2 l_2 sin(\theta_2)\dot{\theta}_2^{\,2} = F$$
(1)

$cos(\theta_1)\ddot{x} - l_1\ddot{\theta}_1 - g sin(\theta_1) = 0$ (2)

$cos(\theta_2)\ddot{x} - l_2\ddot{\theta}_2 - gsin(\theta_2) = 0$ (3)

Equations (2) and (3) can be rearranged:

$$l_1\ddot{\theta}_1 = cos(\theta_1)\ddot{x} - gsin(\theta_1)$$
$$\Rightarrow \ddot{\theta}_1 = \frac{1}{l_1}cos(\theta_1)\ddot{x} - \frac{g}{l_1}sin(\theta_1)$$
$$l_2\ddot{\theta}_2 = cos(\theta_2)\ddot{x} - gsin(\theta_2)$$
$$\Rightarrow \ddot{\theta}_2 = \frac{1}{l_2}cos(\theta_2)\ddot{x} - \frac{g}{l_2}sin(\theta_2)$$

These two equations for $\ddot{\theta}_1$ and $\ddot{\theta}_2$ are substituted into equation (1):

$$(M+m_1+m_2)\ddot{x} - m_1l_1cos(\theta_1)(\frac{1}{l_1}cos(\theta_1)\ddot{x} - \frac{g}{l_1}sin(\theta_1)) - m_2l_2cos(\theta_2)(\frac{1}{l_2}cos(\theta_2)\ddot{x} - \frac{g}{l_2}sin(\theta_2)) + ...$$
$$...m_1l_1sin(\theta_1)\dot{\theta}_1^{\ 2} + m_2l_2sin(\theta_2)\dot{\theta}_2^{\ 2} = F$$

Rearranging terms to solve for $\ddot{x}$, the following is obtained:

$$\ddot{x} = \frac{F - m_1(gcos(\theta_1)sin(\theta_1) + l_1sin(\theta_1)\dot{\theta}_1^2) - m_2(gcos(\theta_2)sin(\theta_2) + l_2sin(\theta_2)\dot{\theta}_2^2)}{M + m_1sin^2(\theta_1) + m_2sin^2(\theta_2)}$$

This result is then plugged into the earlier equations for $\ddot{\theta}_1$ and $\ddot{\theta}_2$:

$$\ddot{\theta}_1 = \frac{1}{l_1}cos(\theta_1)[\frac{F - m_1(gcos(\theta_1)sin(\theta_1) + l_1sin(\theta_1)\dot{\theta}_1^2) - m_2(gcos(\theta_2)sin(\theta_2) + l_2sin(\theta_2)\dot{\theta}_2^2)}{M + m_1sin^2(\theta_1) + m_2sin^2(\theta_2)}] - \frac{g}{l_1}sin(\theta_1)$$
$$\ddot{\theta}_2 = \frac{1}{l_2}cos(\theta_2)[\frac{F - m_1(gcos(\theta_1)sin(\theta_1) + l_1sin(\theta_1)\dot{\theta}_1^2) - m_2(gcos(\theta_2)sin(\theta_2) + l_2sin(\theta_2)\dot{\theta}_2^2)}{M + m_1sin^2(\theta_1) + m_2sin^2(\theta_2)}] - \frac{g}{l_2}sin(\theta_2)$$

The nonlinear state space representation of the system is given by

$$\dot{\vec{X}} = F(\vec{X}, \vec{U}) = \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix}$$

$$= \begin{bmatrix} \dot{x} \\ \frac{F - m_1(gcos(\theta_1)sin(\theta_1) + l_1 sin(\theta_1)\dot{\theta}_1^2) - m_2(gcos(\theta_2)sin(\theta_2) + l_2 sin(\theta_2)\dot{\theta}_2^2)}{M + m_1 sin^2(\theta_1) + m_2 sin^2(\theta_2)} \\ \dot{\theta}_1 \\ \frac{1}{l_1}cos(\theta_1)[\frac{F - m_1(gcos(\theta_1)sin(\theta_1) + l_1 sin(\theta_1)\dot{\theta}_1^2) - m_2(gcos(\theta_2)sin(\theta_2) + l_2 sin(\theta_2)\dot{\theta}_2^2)}{M + m_1 sin^2(\theta_1) + m_2 sin^2(\theta_2)}] - \frac{g}{l_1}sin(\theta_1) \\ \dot{\theta}_2 \\ \frac{1}{l_2}cos(\theta_2)[\frac{F - m_1(gcos(\theta_1)sin(\theta_1) + l_1 sin(\theta_1)\dot{\theta}_1^2) - m_2(gcos(\theta_2)sin(\theta_2) + l_2 sin(\theta_2)\dot{\theta}_2^2)}{M + m_1 sin^2(\theta_1) + m_2 sin^2(\theta_2)}] - \frac{g}{l_2}sin(\theta_2) \end{bmatrix}$$

Part B

The equilibrium point is specified by $x = 0$ and $\theta_1 = \theta_2 = 0$. To linearize the system, the following small angle approximation assumptions are made:

$$cos(\theta) = 1$$
$$sin(\theta) = \theta$$
$$\dot{\theta}^2 = 0$$

With these assumptions, equation (1) from earlier (derived from the Lagrangian) reduces to:

$$\ddot{x}(M + m_1 + m_2) - m_1 l_1 \ddot{\theta}_1 - m_2 l_2 \ddot{\theta}_2 = F$$

Equations (2) and (3) (also from the Lagrangian) reduce to:

$$\ddot{x} - l_1\ddot{\theta}_1 - g\theta_1 = 0$$
$$\ddot{x} - l_2\ddot{\theta}_2 - g\theta_2 = 0$$

Solving for $\ddot{\theta}_1$ and $\ddot{\theta}_2$ results in:

$$\ddot{\theta}_1 = \frac{1}{l_1}\ddot{x} - \frac{g}{l_1}\theta_1$$

$$\ddot{\theta}_2 = \frac{1}{l_2}\ddot{x} - \frac{g}{l_2}\theta_2$$

Once again, these two equations are substituted into the reduced Lagrangian equation then rearranged to solve for $\ddot{x}$:

$$\ddot{x}(M + m_1 + m_2) - m_1 l_1(\frac{1}{l_1}\ddot{x} - \frac{g}{l_1}\theta_1) - m_2 l_2(\frac{1}{l_2}\ddot{x} - \frac{g}{l_2}\theta_2) = F$$

The result is again plugged back into the equations for $\ddot{\theta}_1$ and $\ddot{\theta}_2$:

$$\ddot{\theta}_1 = \frac{1}{l_1}[-\frac{m_1 g}{M}\theta_1 - \frac{m_2 g}{M}\theta_2 + \frac{1}{M}F] - \frac{g}{l_1}\theta_1$$
$$\ddot{\theta}_2 = \frac{1}{l_2}[-\frac{m_1 g}{M}\theta_1 - \frac{m_2 g}{M}\theta_2 + \frac{1}{M}F] - \frac{g}{l_2}\theta_2$$

Rearranging, the two equations become:

$$\ddot{\theta}_1 = -\frac{m_2 g}{M l_1}\theta_2 - \frac{g}{l_1}(\frac{m_1}{M} + 1)\theta_1 + \frac{1}{M l_1}F$$
$$\ddot{\theta}_2 = -\frac{m_1 g}{M l_2}\theta_1 - \frac{g}{l_2}(\frac{m_2}{M} + 1)\theta_2 + \frac{1}{M l_2}F$$

The linearized system for the given equilibrium point is given by the form:

$$\dot{\vec{X}} = A\vec{X} + B\vec{U}$$
$$\vec{X} = \begin{bmatrix} x \\ \dot{x} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix}$$

The linearized state space representation of the system is given below:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{m_1 g}{M} & 0 & -\frac{m_2 g}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{g}{l_1}(\frac{m_1}{M}+1) & 0 & -\frac{m_2 g}{M l_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{m_1 g}{M l_2} & 0 & -\frac{g}{l_2}(\frac{m_2}{M}+1) & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{M l_1} \\ 0 \\ \frac{1}{M l_2} \end{bmatrix} F$$

Part C

To check if the system is controllable, the controllability matrix $\mathcal{C}$ is found and its rank is checked:

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & A^3B & A^4B & A^5B \end{bmatrix}$$
$$rank(\begin{bmatrix} B & AB & A^2B & A^3B & A^4B & A^5B \end{bmatrix}) = n$$

A system (A, B) is controllable if its controllability matrix is full rank and nonsingular. A square matrix is full rank if its determinant is not zero. For this system, the determinant is calculated using MATLAB and is given by:

$$determinant = \frac{-g^6(l_1^2 - l_2^2)^2}{M^6 l_1^6 l_2^6}$$

The above equation is set equal to zero to determine the conditions that will make this system uncontrollable:

$$\frac{-g^6(l_1^2 - l_2^2)^2}{M^6 l_1^6 l_2^6} = 0$$
$$l_1^2 - l_2^2 = 0$$
$$l_1 = l_2$$

The system is deemed to be uncontrollable if $l_1 = l_2$. Therefore, to ensure stability in the system, it is determined that $l_1 \neq l_2$.

Part D

To implement an LQR controller, it is necessary to introduce a forcing control input, K, such that the system can be controlled when defined with non-equilibrium point initial conditions or fed a forcing input. The state-space representation with a feedback controller is given by:

$$\dot{X}(t) = AX(t) + B_K U_K(t) + B_D U_D(t)$$

and the feedback controller equation is defined by:

$U_K = KX(t)$, where $K = -R^{-1}B_K^T P$

P is the symmetric positive definite solution to the following Ricatti equation:

$$A^T P + PA - PB_K R^{-1} B_K^T P = -Q$$

R and Q are positive definite gain matrices belonging to the minimized cost function:

$$J = \int_{t_0}^{\infty} X^T(t)QX(t) + U_K(t)^T R U_K(t)\ dt$$

After rearranging and substituting components, the state-space representation for this system with zero disturbance noise becomes:

$\dot{X}(t) = (A + B_K K)X(t) + B_D U_D(t)$, $B_D = 0$.

The system is defined as having the following variable values:

M = 1000 kg
m_1 = 100 kg
m_2 = 100kg
l_1 = 20m
l_2 = 10m

The gain matrices Q and R were defined as follows for optimal system response:

$$
Q = \begin{bmatrix} 20 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 7500 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
R = 0.00001
$$

The matrix C defining the output vector of the system was defined as follows to ensure all three system state variables can be visualized:

$$
C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}
$$

After plugging in the numerical values for each variable, defining the Q and R gain matrices, solving for the controller K, and simulating the system in MATLAB, the following response to initial conditions (x, θ_1, θ_2) = (0, 3°, -5°) for the linearized system around the equilibrium point is given in Figure 2 below.

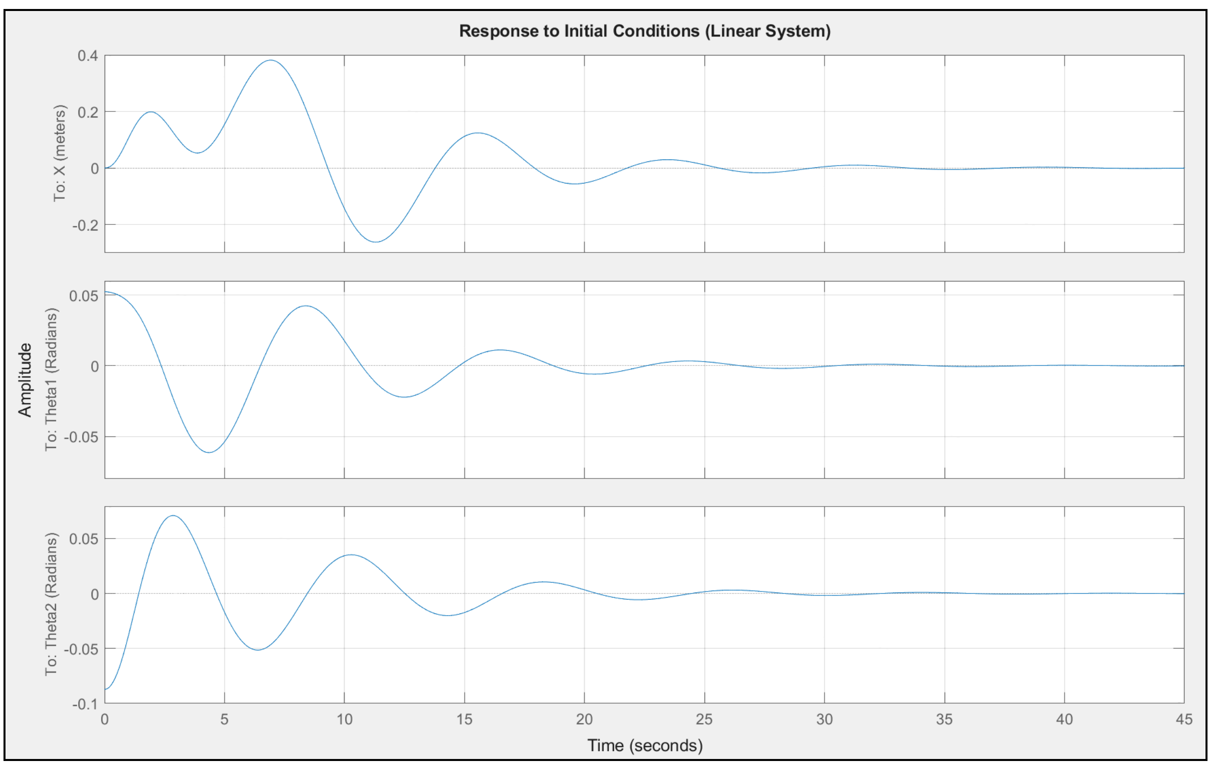


Fig. 2: Response to Initial Conditions (Linear System)

The controller K was also applied to the nonlinear system to test the effect of the controller on a non-linearized version of the same system. The nonlinear system response can be seen in Figure 3 below.

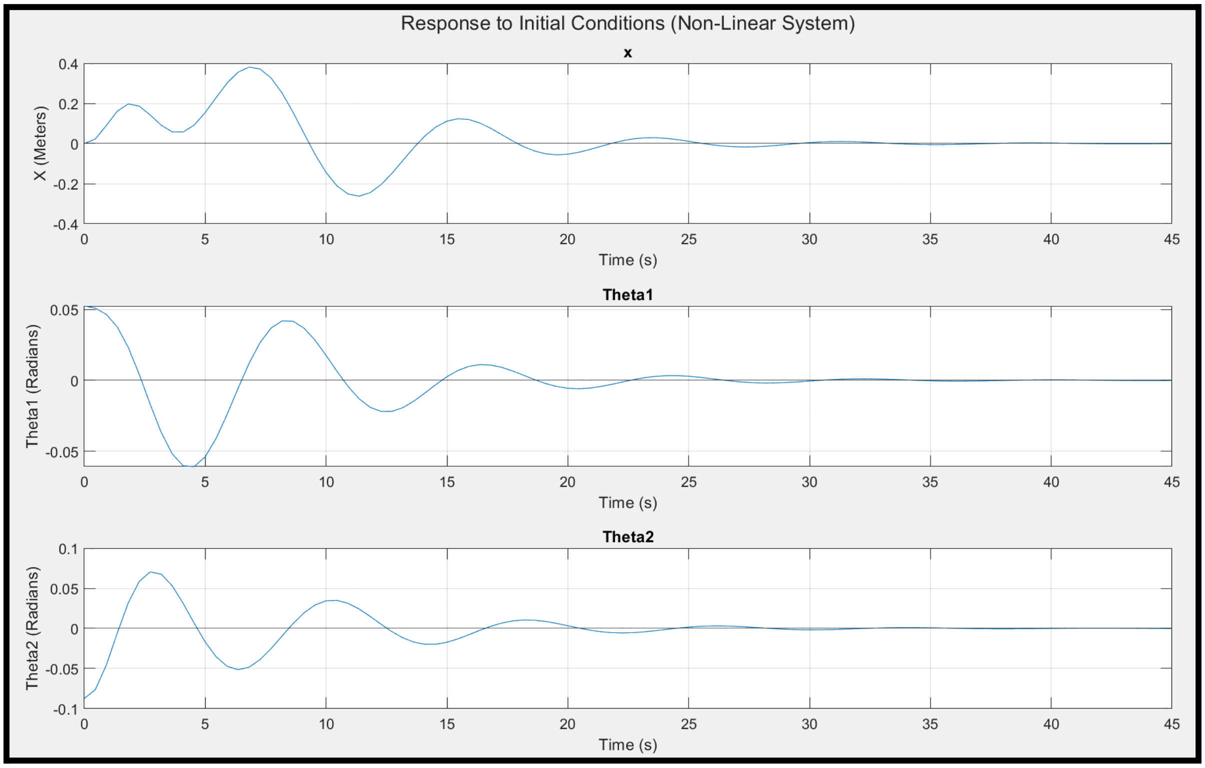


Fig. 3: Response to Initial Conditions (Nonlinear System)

The matrix $A + B_K K$ is calculated as:

$$(A + B_K K) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1.41 & 3.95 & 24.86 & 8 & 15.62 & -13.99 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0.07 & 0.2 & 0.75 & 0.4 & 0.78 & -0.7 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0.14 & 0.4 & 2.49 & 0.8 & 0.58 & -1.4 \end{bmatrix}$$

To test the stability of this system with the controller implemented with Lyapunov's Indirect Method, the eigenvalues of the matrix need to be examined. If all of the eigenvalues of the state matrix (A + BK) have negative real parts and no imaginary parts lying on the imaginary axis, then the controlled system can be deemed stable. The eigenvalues of this matrix are found by setting the determinant of the matrix minus the identity matrix multiplied by the unknown eigenvalues to zero and solving for the eigenvalues λ:

$det((A + B_K K) - \lambda I) = 0$

The identity matrix I is a 6 by 6 matrix with one's on the diagonal:

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The six eigenvalues of this matrix were computed in MATLAB and are found as follows:

$$\lambda_1 = -1.04 + 1.38i$$
$$\lambda_2 = -1.04 - 1.38i$$
$$\lambda_3 = -0.28 + 0.51i$$
$$\lambda_4 = -0.28 - 0.51i$$
$$\lambda_5 = -0.15 + 0.8i$$
$$\lambda_6 = -0.15 - 0.8i$$

All the eigenvalues have *negative real parts*. Therefore, by means of Lyapunov's Indirect Method for stability, the system is deemed to be locally stable around the equilibrium point.

Second Component

Part E

The following output vectors of the system were examined for observability:

1) *{x(t)}*
2) *{θ_1(t), θ_2(t)}*
3) *{x(t), θ_2(t)}*
4) *{x(t), θ_1(t), θ_2(t)}*

The matrices defining the output vector for each observable system, C_1, C_2, C_3, and C_4, are given by:

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$C_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$C_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The system (A, C) is an observable system (A^T, C^T) is a controllable, and (A^T, C^T) is controllable if its controllability matrix is full rank. For this state space representation, the matrix needs to be of rank 6 in order to be deemed full rank. Therefore, (A, C) is observable if, for *n* = 6:

$$rank(\begin{bmatrix} C^T & A^T C^T & (A^T)^2 C^T & (A^T)^3 C^T & (A^T)^4 C^T & (A^T)^5 C^T \end{bmatrix}) = n.$$

The controllability matrices were formed and evaluated using MATLAB. The ranks for each system with their corresponding output vectors are as follows:

- $\begin{bmatrix} X(t) \end{bmatrix}$: $rank = 6$, therefore the system is observable for this variable.
- $\begin{bmatrix} \theta_1(t), \theta_2 \end{bmatrix}$: $rank = 4$, therefore the system is *not* observable for these variables.
- $\begin{bmatrix} x(t), \theta_2(t) \end{bmatrix}$: $rank = 6$, therefore the system is observable for these variables.
- $\begin{bmatrix} x(T), \theta_1(t), \theta_2(t) \end{bmatrix}$: $rank = 6$, therefore the system is observable for these variables.

This concludes that three of the four output vectors are observable for the linearized system.

Part F

An observer is used in cases when the state of a system cannot be directly measured by sensors or by other means, and a state estimate is required to calculate the proper control response. A Luenberger Observer is defined by the following equation:

$$\dot{\hat{X}}(t) = A\hat{X}(t) + B_K U_K(t) + L(Y(t) - C\hat{X}(t))$$

$\hat{X}(t)$ is the state estimator for the observer, $(Y(t) - C\hat{X}(t))$ is the correction term used to guide the observer based on it error between its state estimate and the output of the real system, and $Y(t) = CX(t)$ is the output from the real system.

L is an observer gain matrix defined by:

$$L = PC^T \Sigma_V^{-1}$$

P is the symmetric positive definite solution to the following Ricatti equation:

$$AP + PA^T + B_D \Sigma_D B_D^T - PC^T \Sigma_V^{-1} CP = 0$$

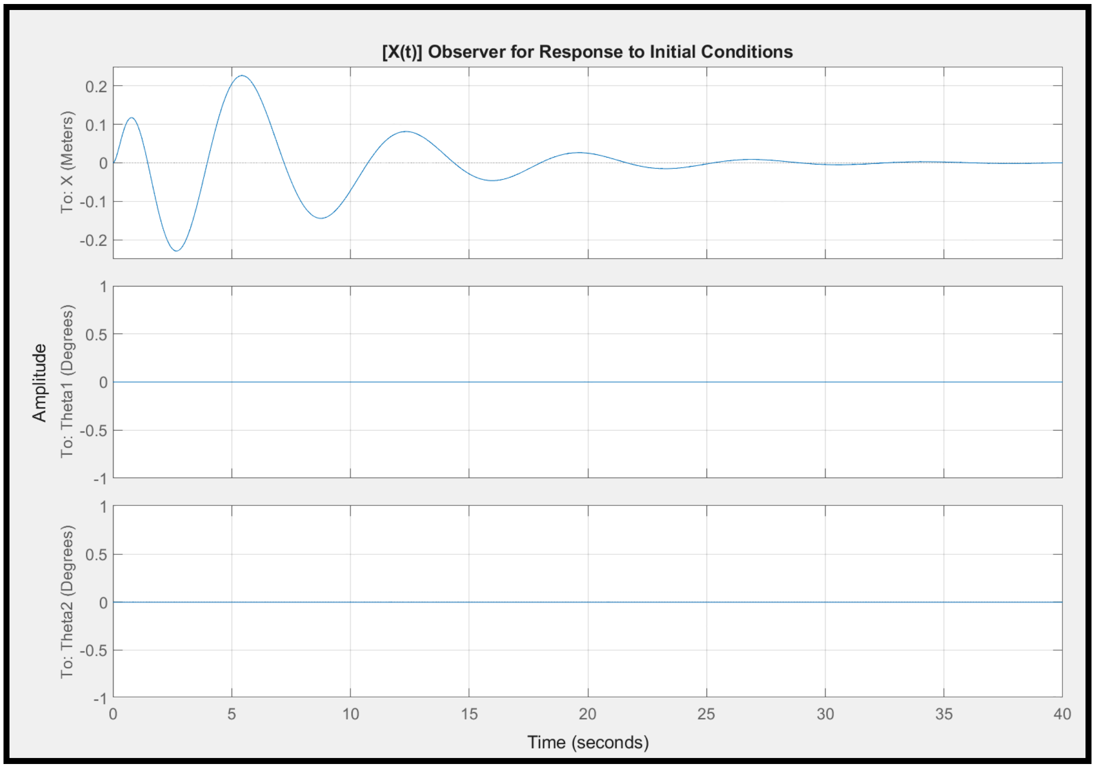
Σ_D and Σ_V are covariances associated with the independent zero mean white Gaussian processes $U_D(t)$ and $V(t)$, which are the simulated sensor and environmental noise. For this simulation, the following covariance values were selected:

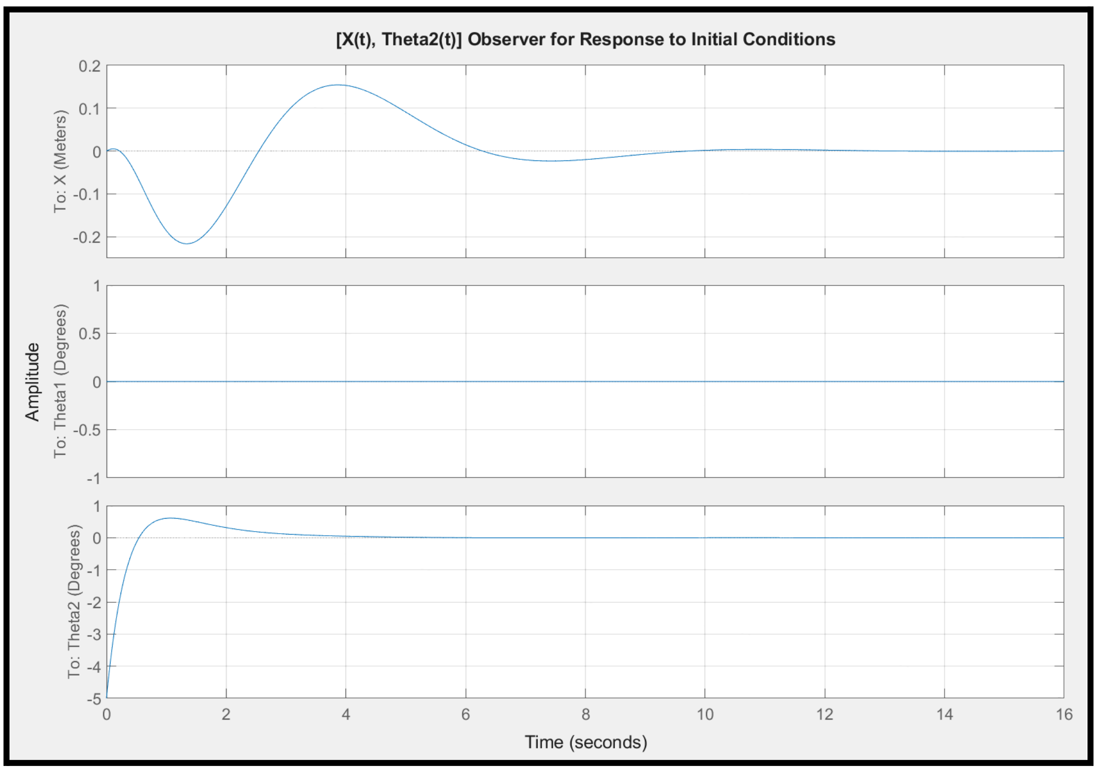
$$\Sigma_D = 0.01$$
$$\Sigma_V = 0.1$$

A Luenberger observer was calculated for each of the three observable output vectors and applied to both the linear and nonlinear state space representations of the system.

Linear System

The best Luenberger observations as a response to both initial conditions and a step input were found for all three observable output vectors independently, and each observer was simulated using MATLAB. The observers of the linear system for a response to initial conditions are seen in Figure 4 on the next two pages.


[X(t)] Observer for Response to Initial Conditions
Amplitude
To: X (Meters)
To: Theta1 (Degrees)
To: Theta2 (Degrees)
Time (seconds)


[X(t), Theta2(t)] Observer for Response to Initial Conditions
Amplitude
To: X (Meters)
To: Theta1 (Degrees)
To: Theta2 (Degrees)
Time (seconds)

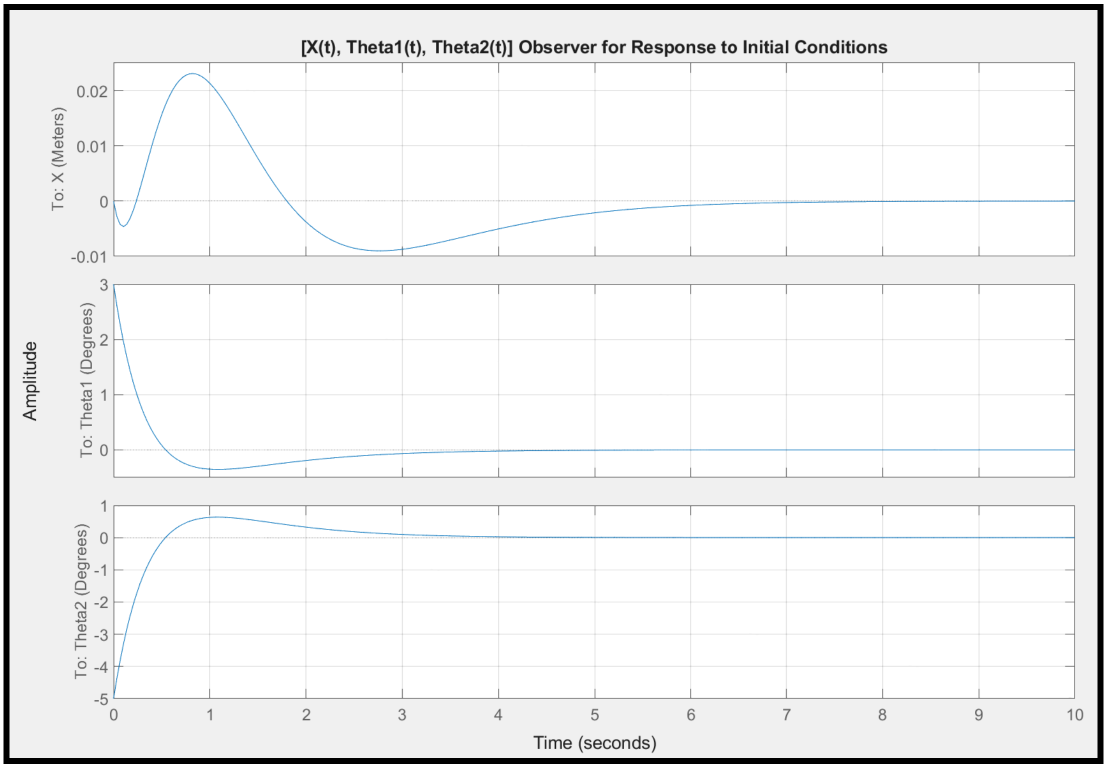
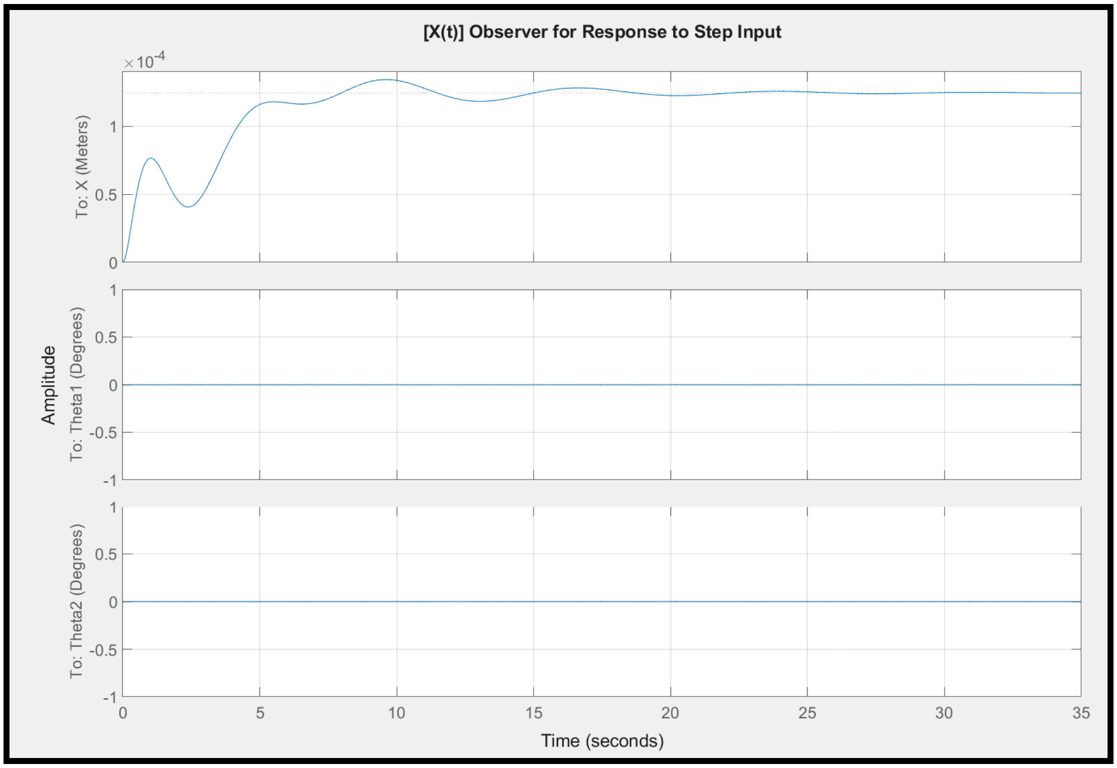
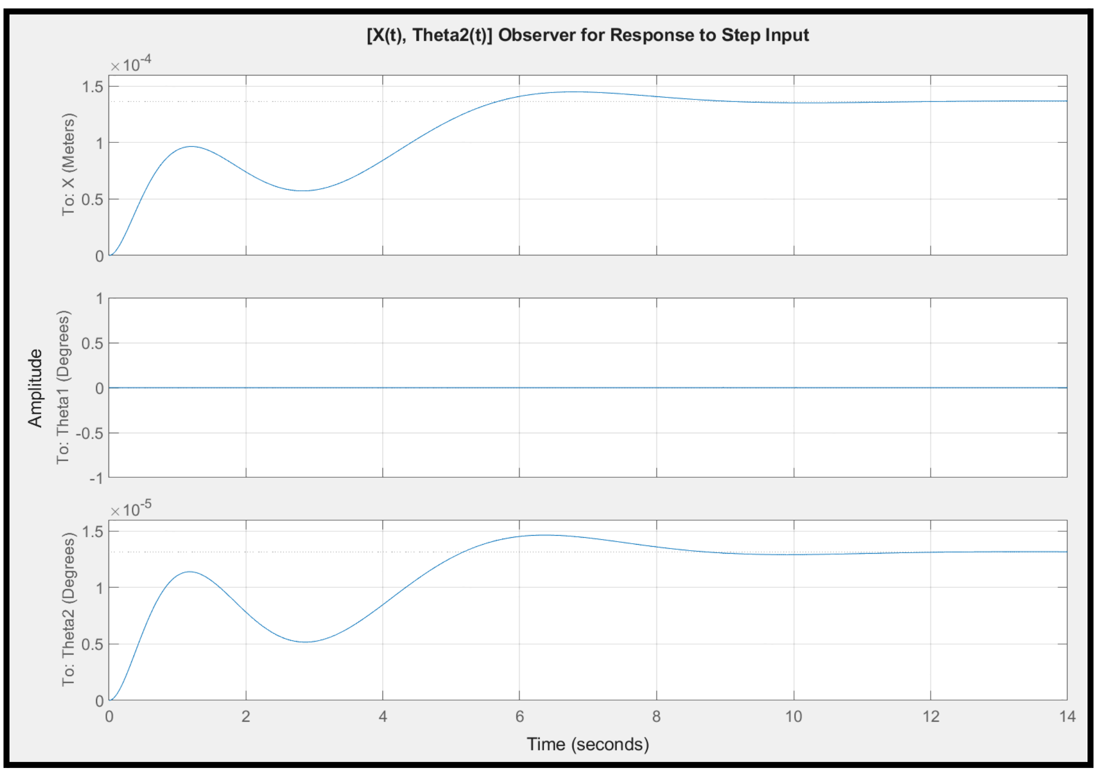


Fig. 4: Observers for response to initial conditions (linear system)

The observers of the linear system for a response to a step input are as seen in Figure 5 in the next two pages.


[X(t)] Observer for Response to Step Input
×10^-4
To: X (Meters)
Amplitude
To: Theta1 (Degrees)
To: Theta2 (Degrees)
Time (seconds)


[X(t), Theta2(t)] Observer for Response to Step Input
×10^-4
To: X (Meters)
Amplitude
To: Theta1 (Degrees)
×10^-5
To: Theta2 (Degrees)
Time (seconds)

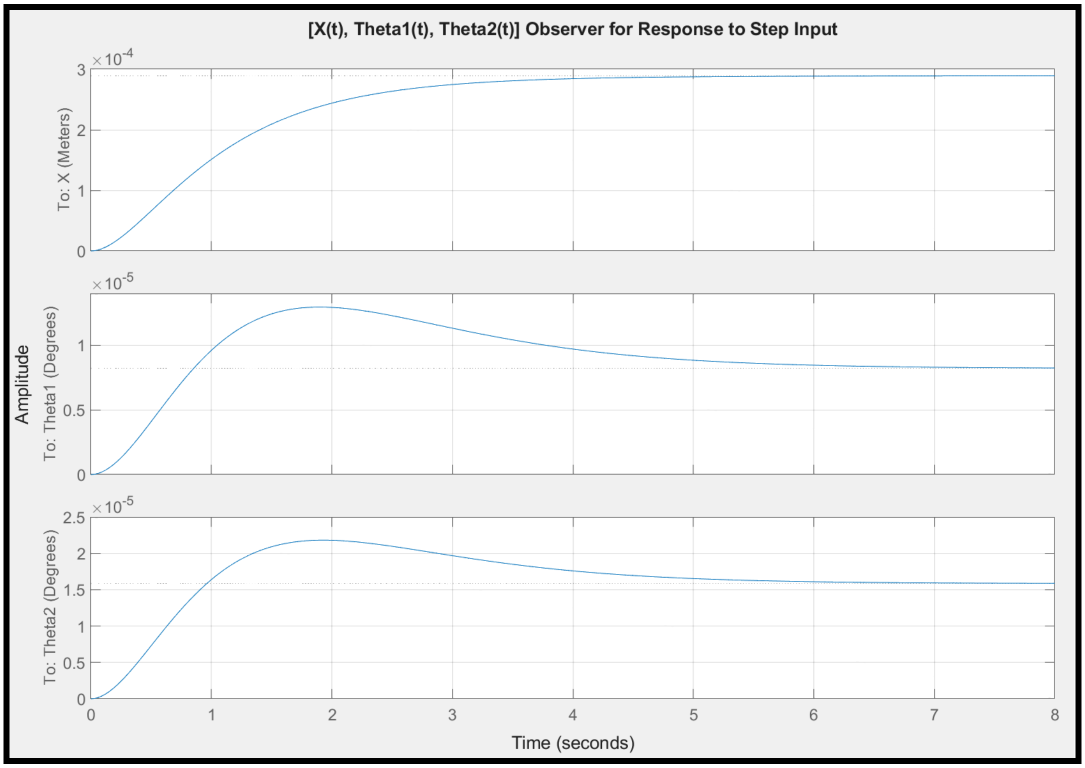
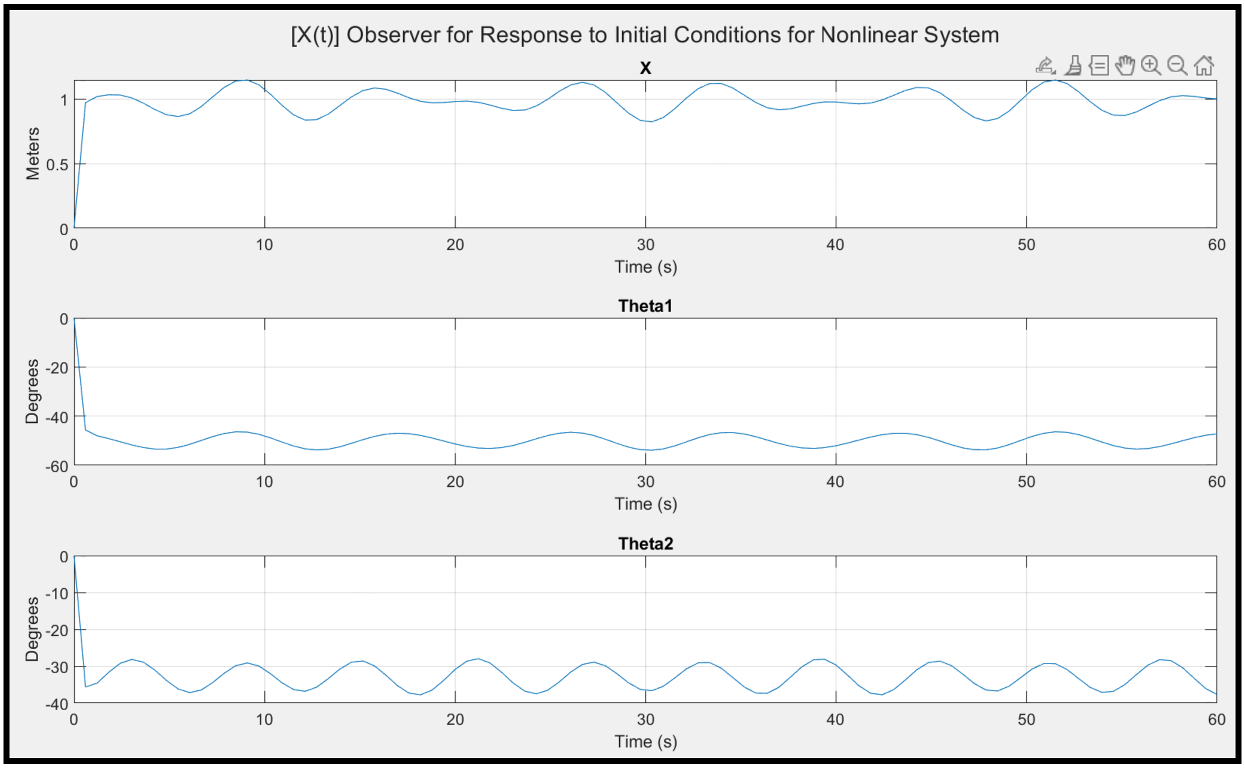
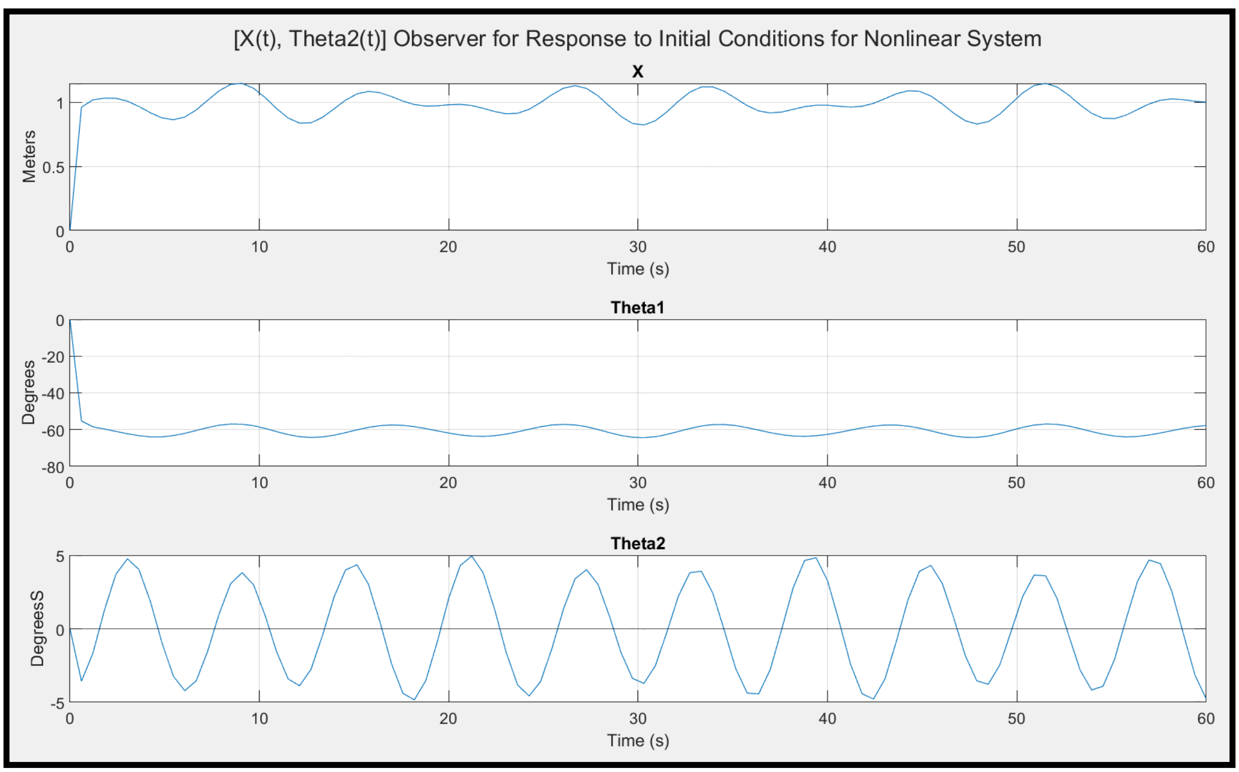


Fig. 5: Observers for response to initial conditions (linear, step input system)

Nonlinear System

The same observers were applied to the nonlinear state space representation of the system for a response to both initial conditions and a step input. The observers for the nonlinear system did not perform as well as they did with the linearized system. The observers of the nonlinear system for a response to initial conditions are seen in Figure 6 on the next two pages.


[X(t)] Observer for Response to Initial Conditions for Nonlinear System
X
Meters
Time (s)
Theta1
Degrees
Time (s)
Theta2
Degrees
Time (s)


[X(t), Theta2(t)] Observer for Response to Initial Conditions for Nonlinear System
X
Meters
Time (s)
Theta1
Degrees
Time (s)
Theta2
DegreesS
Time (s)

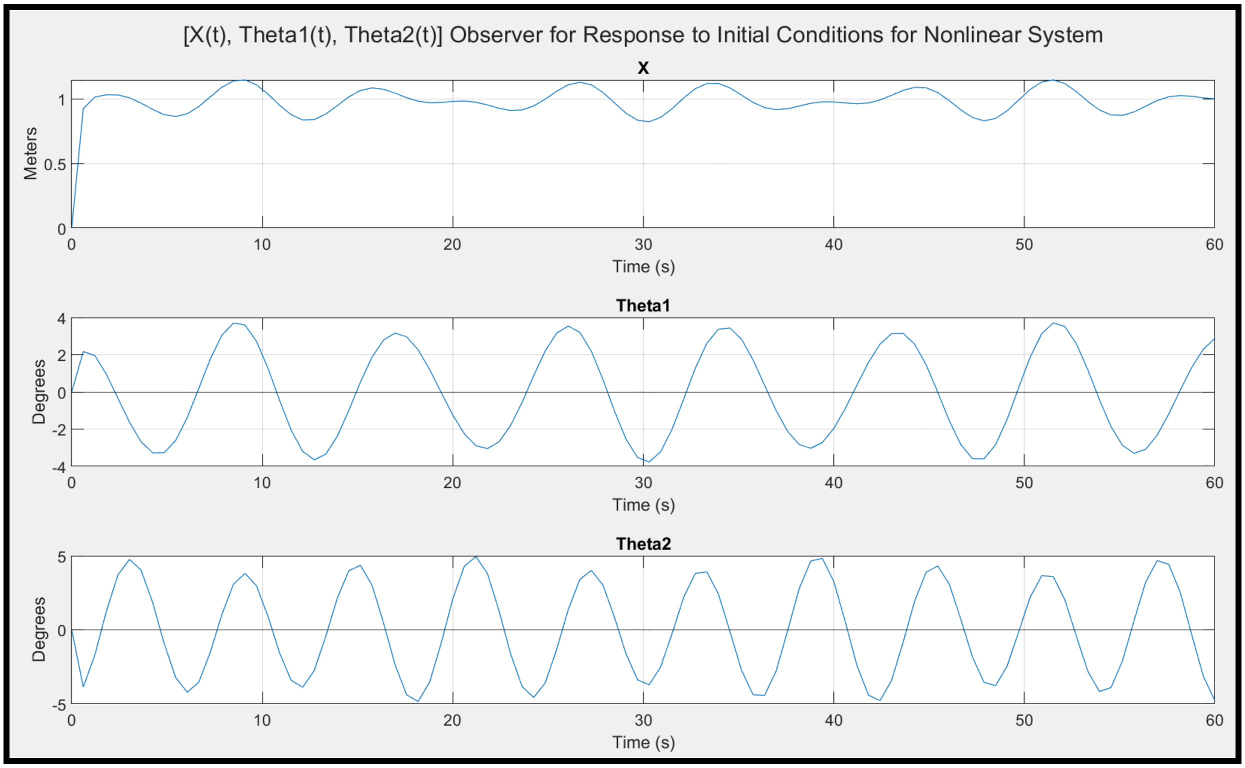
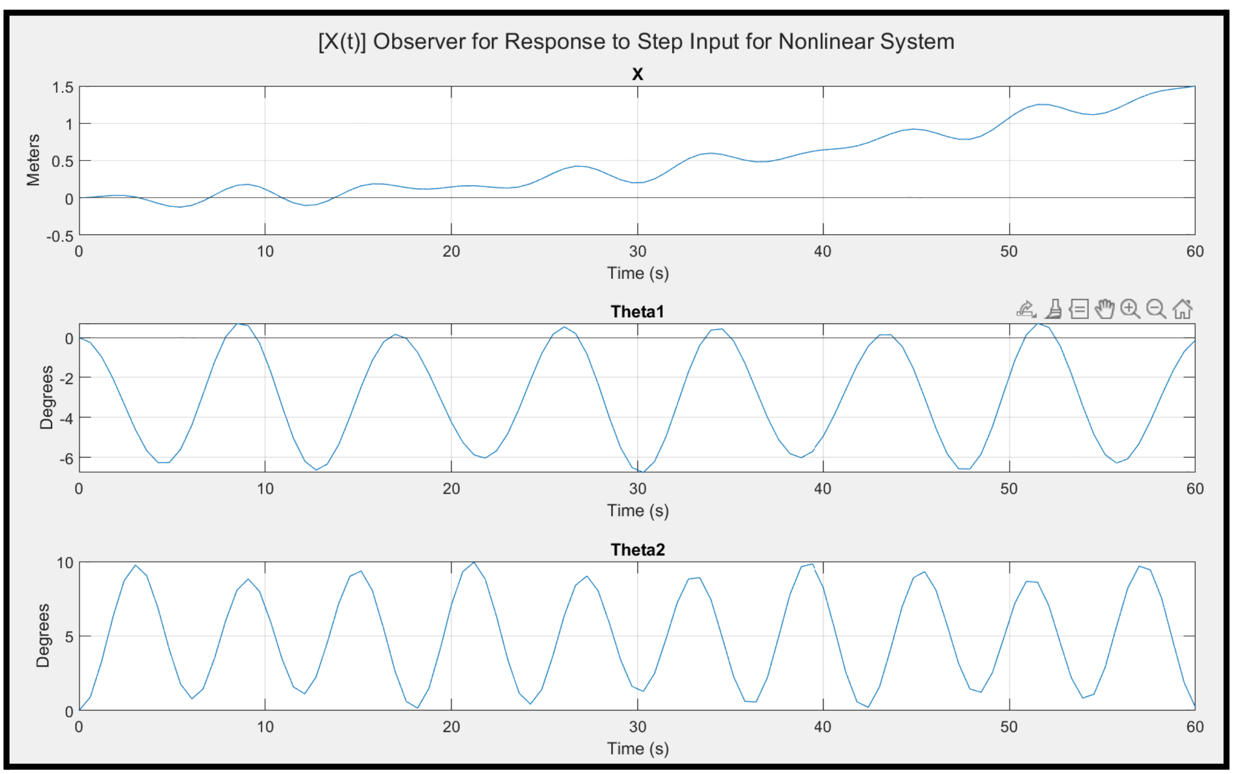


Fig. 6: Observers for response to initial conditions (nonlinear system)

The observers of the nonlinear system for a response to a step input are seen in Figure 7 below and on the next page.



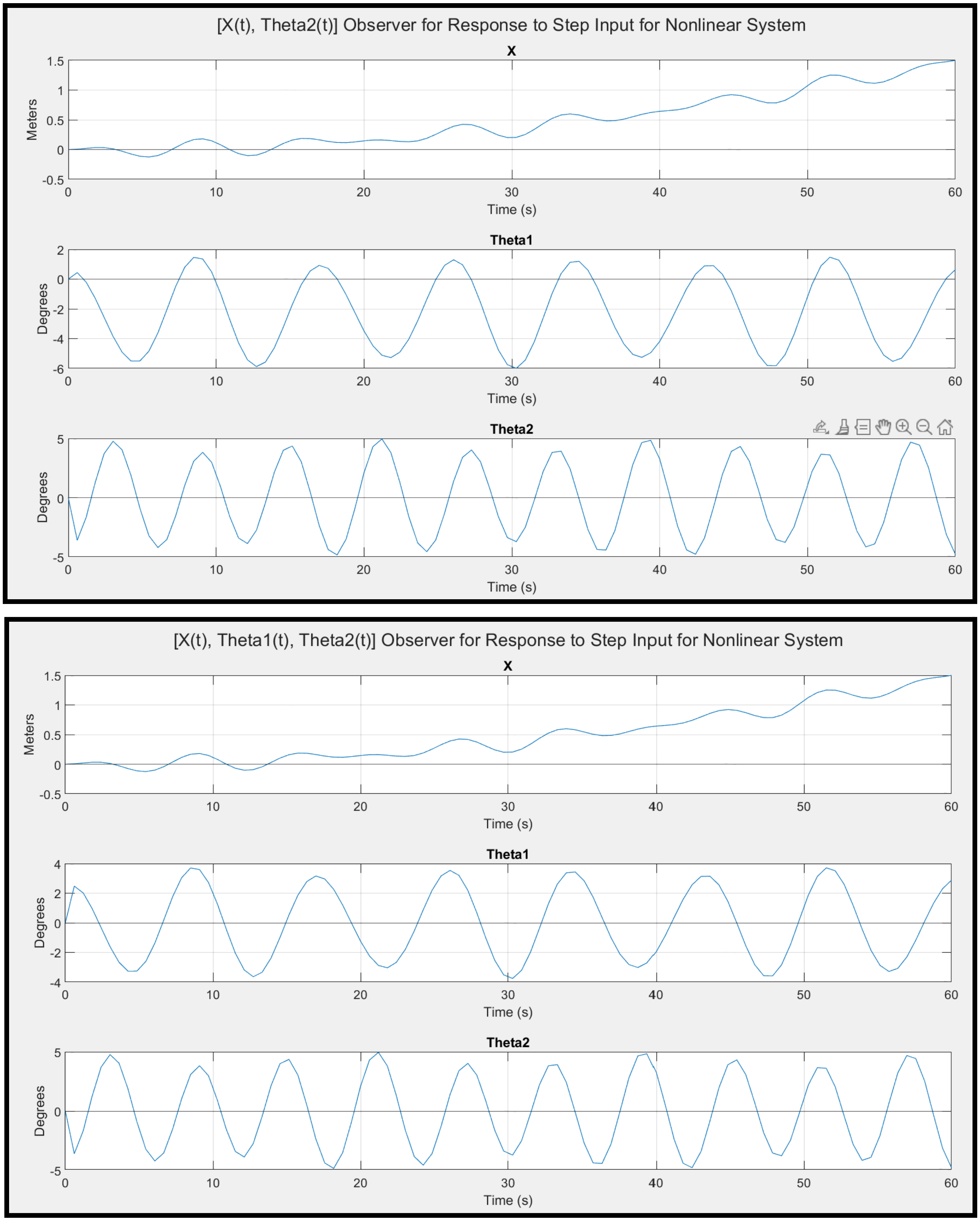


Fig. 7: Observers for response to initial conditions (nonlinear, step input system)

Part G

The smallest output vector in this case is the output vector *{x(t)}*. To design an LQG controller for this output vector and system, the controller from the LQR controller method and the Luenberger observer are both necessary pieces. Because of the separation principle, the system state with the LQG controller applied is known to be locally stable around the defined equilibrium point if the system with the LQR controller applied, (A, B), is stable and the system state (A, C) is observable, both of which have been proven. Therefore, the system with the LQG controller applied will be stable.

By defining the following equations

$$\dot{X}(t) = AX(t) + B_K U_K(t) + B_D U_D(t)$$
$$Y(t) = CX(t) + V(t)$$
$$\dot{\hat{X}}(t) = A\hat{X}(t) + B_K U_K(t) + L(Y(t) - C\hat{X}(t))$$
$$U(t) = KX(t)$$

the closed-loop state-space representation of the LQG becomes:

$$\begin{bmatrix} \dot{X}(t) \\ \dot{\hat{X}}(t) \end{bmatrix} = \begin{bmatrix} A & B_K K \\ LC & A - LC + B_K K \end{bmatrix} \begin{bmatrix} X(t) \\ \hat{X}(t) \end{bmatrix} + \begin{bmatrix} B_D \\ 0 \end{bmatrix} U_D(t)$$
$$Y(t) = CX(t) + V(t)$$

For this simulation, the noise covariance values were decreased slightly to the following values:

$$\Sigma_D = 0.001$$
$$\Sigma_V = 0.0001$$

And the resultant gain matrices that resulted in an optimal response to initial conditions were selected as:

$$Q = \begin{bmatrix} 1500 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 25000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R = 0.001$

The noise $U_D(t)$ was only applied to the *x(t)* state variable, simulating noise in the motion of the cart. The controller matrix K and the Luenberger observer were applied to the original nonlinear state space representation of the system using only a *x(t)* output vector. The simulated results from MATLAB are as shown in Figure 8 below.

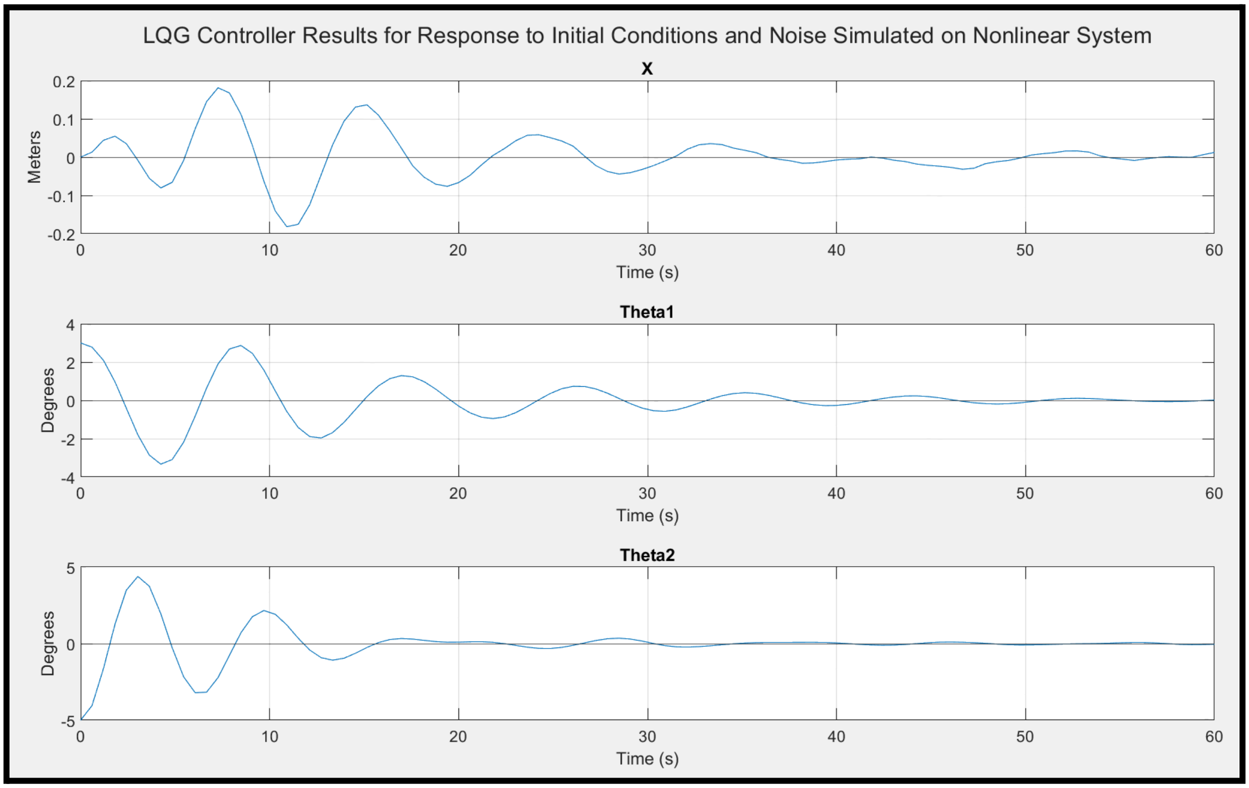


Fig. 8: LQG Controller Results

To reconfigure the controller to asymptotically track a constant reference on *x(t)*, the state space representation of the system needs to be modified to track error with regards to a constant reference. The controller parameters change slightly, with the minimized cost function of the controller becoming:

$$J = \int_{t_0}^{\infty} (X(t) - X_d)^T Q (X(t) - X_d) + (U_K(t) - U_\infty)^T R (U(t) - U_\infty)\ dt$$

X_d and U_∞ are the desired system states and the continuous input, or references, such that:

$$AX_d + BU_\infty = 0$$

$\tilde{X}(t)$ and $\tilde{U}_K(t)$ are then defined to be the reference tracking error terms such that:

$$\tilde{X}(t) = X(t) - X_d,\ \tilde{U}_K(t) = U_K(t) - U_\infty$$

$$\dot{\tilde{X}}(t) = A\tilde{X}(t) + B_K\tilde{U}_K(t) + B_D U_D(t),\ \tilde{U}_K = K\tilde{X}(t)$$

After rearranging and substituting components, the governing equation for the state space representation of the system that can asymptotically track a instant reference on *x(t)* becomes:

$$\dot{\tilde{X}}(t) = (A + B_K K)\tilde{X}(t) + B_D U_D(t).$$

In order to reject constant force disturbances applied to the cart, the integral component of state variables is needed as an additional state variable. This is a variation of the LQR controller, known as the *LQRI controller*, that can be used to track a reference with constant disturbances applied to the system. The integral component state variable is defined as follows:

$$X_I(t) = \int_{t_0}^{t} \tilde{X}(t)\ dt,\ X_I(0) = 0$$

The new state variable is added to the state space representation of the system such that:

$$\bar{X}(t) = \begin{bmatrix} \tilde{X}(t) \\ X_I(t) \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} A & 0 \\ I & 0 \end{bmatrix}$$

$$\bar{B}_K = \begin{bmatrix} B_K \\ 0 \end{bmatrix}$$

$$\bar{B}_D = \begin{bmatrix} B_D \\ 0 \end{bmatrix}$$

The controller K changes to match the modified state space equation such that:

$$\tilde{U}_K(t) = \bar{K}\bar{X}(t)$$

where $\bar{K}$ is the controller that minimizes the new cost function

$$J = \int_{t_0}^{\infty} \bar{X}^T(t)\bar{Q}\bar{X}(t) + \tilde{U}_K(t)^T\bar{R}\tilde{U}_K(t)\ dt$$

The resultant modified state space representation of the system able to reject constant force disturbances then becomes:

$$\dot{\bar{X}}(t) = (\bar{A} + \bar{B}\bar{K})\bar{X}(t) + \bar{B}_D U_D(t)$$

Therefore, the current controller design is *unable* to reject constant force disturbances applied to the cart.