COUNTING SPACES

Consider a permutation of a finite set. If it is written as the product of cycles, say $(1\ 4\ 5\ 6)\,(4\ 3\ 1)\,(9\ 5)\,(2)$, of course what we actually write on the page are elements of our set surrounded by parentheses. Within a cycle, we separate elements of the set by leaving a gap. Let us call each of these *spaces*. In this short article we show how counting the number of spaces is useful in understanding permutations. We begin with a simple observation.

Proposition. *A permutation can be written with m spaces if and only if it can be written as the product of m swaps (i.e., cycles of length two).*

To see this, first notice that the product of m swaps $(a_1\ b_1)\,(a_2\ b_2)\cdots(a_m\ b_m)$ is already written with m spaces. For the opposite implication, consider that a single cycle written with i spaces, say $(c_0\ c_1\ \cdots\ c_i)$, can be written as the product of i swaps as $(c_0\ c_1)\,(c_0\ c_2)\cdots(c_0\ c_i)$. Thus an arbitrary permutation written with m spaces can be written as the product of m swaps. For example,

$$(1\ 4\ 5\ 6)\,(4\ 3\ 1)\,(9\ 5)\,(2) = (1\ 4)\,(1\ 5)\,(1\ 6)\,(4\ 3)\,(4\ 1)\,(9\ 5)\,.$$

Now, it is often useful to write a permutation as the product of disjoint cycles, and our main result tells us the number of spaces that appear.

Theorem 1. *If a permutation σ can be written as a product of disjoint cycles using n spaces, then n is the smallest number of swaps whose product is σ.*

To begin, let us show that when writing a permutation as the product of disjoint cycles, the number of spaces is unique for the permutation. To see this, consider the differences in two such ways of writing a permutation. Although the same cycles of length at least two appear, each cycle might be written starting with a different element. For example, $(1\ 7\ 6)$ can be written as $(6\ 1\ 7)$. Also, trivial cycles, such as (8), may appear or be omitted. Finally, the cycles can be written in any order. Since these differences do not affect the number of spaces, we can unambiguously refer to the number of spaces when a permutation is written as the product of disjoint cycles. Next, let us look at how this number changes when we multiply by a swap.

Lemma. *Take a permutation σ' and a swap $(a\ b)$. Define $\sigma = \sigma'\,(a\ b)$. If we write σ' and σ each as the product of disjoint cycles, then the number of spaces for σ is one more or one less than the number of spaces for σ'.*

Proof. Write σ' as the product of disjoint cycles, where we include the trivial cycles (a) or (b) in case σ' fixes a or b. If there are cycles that do not contain a or b, let τ be their product.

We distinguish two possibilities. Either a and b are in different cycles, say $(a\ c_1\ c_2\ \cdots)$ and $(b\ d_1\ d_2\ \cdots)$, or they are in the same cycle, say $(a\ c_1\ c_2\ \cdots\ b\ d_1\ d_2\ \cdots)$. In either case, we allow the possibility that there are no elements c_i or no elements

d_i. Since a, b, the c_i, and the d_i are all distinct,

$$(a\ c_1\ c_2\ \cdots)\,(b\ d_1\ d_2\ \cdots)\,(a\ b) = (a\ c_1\ c_2\ \cdots\ b\ d_1\ d_2\ \cdots) \text{ and}$$
$$(a\ c_1\ c_2\ \cdots\ b\ d_1\ d_2\ \cdots)\,(a\ b) = (a\ c_1\ c_2\ \cdots)\,(b\ d_1\ d_2\ \cdots)\,.$$

We see then that multiplication by the swap $(a\ b)$ either replaces parentheses)(by a space or vice versa. If there are no other cycles in σ', then one of the expressions on the right gives σ as the product of disjoint cycles. Otherwise we multiply by τ to get this form.

Therefore when we write σ as the product of disjoint cycles, the number of spaces goes up by one if a and b are in different cycles of σ', and it goes down by one if a and b are in the same cycle of σ'. □

Next, we claim the following. Let n be the number of spaces when we write σ as the product of disjoint cycles. If σ can be written as the product of m swaps, then $m \geq n$ and $m - n$ is even. This claim immediately implies Theorem 1, because if m_0 is the smallest number of swaps whose product is σ, then $m_0 \geq n$ by the claim and $m_0 \leq n$ by our proposition above, so $m_0 = n$. The fact that $m - n$ is even will be used in the proof of Theorem 2.

To prove our claim, we induct on m. For $m = 0$, σ is the identity permutation and $n = 0$. For $m = 1$, σ is a single swap and $n = 1$.

For $m \geq 2$, write $\sigma = (a_1\ b_1) \cdots (a_m\ b_m)$, and assume that our claim holds for $m-1$. Let $\sigma' = (a_1\ b_1) \cdots (a_{m-1}\ b_{m-1})$ and let n' be the number of spaces when σ' is written as the product of disjoint cycles. By our inductive hypothesis, $m-1 \geq n'$ and $(m-1)-n'$ is even. Our lemma tells us $n = n' \pm 1$, so $n = n' \pm 1 \leq n'+1 \leq m$. Furthermore, $m - n = m - \big(n' \pm 1\big)$, which is even since $(m-1) - n'$ is even. Therefore for all m, we get $m \geq n$ and $m - n$ is even. The fact that $m \geq n$ completes Theorem 1.

Another application of counting spaces is the following corollary to the above proposition. Recall that an *odd* or *even* permutation is one that can be written as the product of an odd or even number of swaps, respectively.

Corollary. *A permutation is odd if it can be written with an odd number of spaces and even if it can be written with an even number of spaces.*

For example, $(1\ 4\ 5\ 6)\,(4\ 3\ 1)\,(9\ 5)\,(2)$ is written with six spaces, so it is even. Finally, counting spaces also gives us a nice proof of the following well known fact.

Theorem 2. *No permutation can be both odd and even.*

To see this, suppose a permutation σ is written as the product of both m_1 and m_2 swaps. Let n be the number of spaces when we write σ as the product of disjoint cycles. By our claim, $m_1 - n$ and $m_2 - n$ are both even. Therefore $(m_1 - n) - (m_2 - n) = m_1 - m_2$ is even, which means m_1 and m_2 are both odd or both even. This proof, since it ultimately relies on our lemma, is essentially equivalent to the one attributed to David M. Bloom in [1, Theorem 9.15].

Summary. When writing down a permutation as the product of cycles, we can count the number of spaces used. This tells us the smallest number of swaps whose product is the permutation, and also whether the permutation is odd or even.

References

[1] J. B. Fraleigh, *A First Course in Abstract Algebra*, Seventh Edition. Addison Wesley, 2003.