Engineering Application - An Optical Communication System

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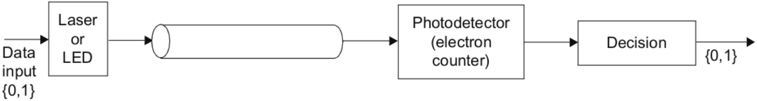
Block diagram



[1]

- Binary data is transmitted by pulsing a laser / LED coupled to an optical fiber.
- To transmit binary 1, turn on the light source for T seconds.
- The receiver converts the signal back into a string of binary numbers using a photo-detector.

Random Variable

- We define a random variable X to be the **number of electrons counted during a T second interval**.
- We describe this random variable in terms of two conditional probability mass functions:

$$P_{X|0}(k) = Pr(X = k|0\ sent)$$
$$P_{X|1}(k) = Pr(X = k|1\ sent)$$

- These probability mass functions should be those of Poisson random variables.

Poisson random variables

- Let the two probability mass functions be given by:

$$P_{X|0}(k) = \frac{R_0^k}{k!} e^{-R_0}$$
$$P_{X|1}(k) = \frac{R_1^k}{k!} e^{-R_1}$$

- R_0 and R_1 are interpreted as the "average" number of electrons observed when 0 is sent or 1 is sent respectively.
- $R_0 < R_1$

How to decide?

- Suppose that during a certain bit interval k electrons are emitted.
- A logical decision would be to calculate $Pr(0\ sent|X = k)$ and $Pr(1\ sent|X = k)$ and choose the larger one.
- This is referred to as a *maximum a posteriori* decision rule.
- Binary 1 was sent if:

$$Pr(1\ sent|X = k) > Pr(0\ sent|X = k)$$

Threshold electrons for 1 sent

- We know $Pr(X = k|1\ sent)$ and want to deduce $Pr(1\ sent|X = k)$.
- Using total probability theorem we get:

$$\begin{aligned} P_X(k) &= Pr(X = k) \\ &= P_{X|0}(k)Pr(0\ sent) + P_{X|1}(k)Pr(1\ sent) \end{aligned}$$

- Applying Bayes's theorem we get :

$$Pr(1\ sent|X = k) = \frac{P_{X|1}(k)Pr(1\ sent)}{P_X(k)}$$
$$Pr(0\ sent|X = k) = \frac{P_{X|0}(k)Pr(0\ sent)}{P_X(k)}$$

Using Poisson random variable

- We get:

$$Pr(0\ sent|X = k) = \frac{\frac{R_O^k}{2k!}e^{-R_O}}{\frac{R_O^k}{2k!}e^{-R_O} + \frac{R_1^k}{2k!}e^{-R_1}}$$

$$Pr(1\ sent|X = k) = \frac{\frac{R_1^k}{2k!}e^{-R_1}}{\frac{R_O^k}{2k!}e^{-R_O} + \frac{R_1^k}{2k!}e^{-R_1}}$$

- For signal sent $= 1$

$$Pr(1\ sent|X = k) > Pr(0\ sent|X = k)$$

$$\frac{1R_1^k}{2k!}e^{-R_1} > \frac{1R_0^k}{2k!}e^{-R_0}$$

$$k > \frac{R_1 - R_0}{\ln(R_1/R_0)}$$

- If the number of emitted electrons is greater than k then 1 is sent otherwise 0 is sent.

Error Quantification

- There will be situations when signal sent is 0 but observed emitted electrons will be greater than threshold and vice versa and hence causing the error.
- Though these situations are very rare but it is important to quantify them.
- Using conditional probability and total probability theorem, we can write error of probability as:

$$P(error) \;=\; P(error|0\ sent)P(0\ sent) + P(error|1\ sent)P(1\ sent)$$

- Here, threshold is taken as $x_0 = \lfloor \frac{R_1 - R_0}{\ln(R_1/R_0)} \rfloor$ for the sake of calculation simplicity.

Error when 0 sent

- Calculating error when 0 sent and electrons emitted are $X > x_0$.

$$
\begin{aligned}
P(error|0\ sent) \ = \ P(X > x_0|0\ sent) &= \sum_{k=x_0+1}^{\infty} P_{X|0}(k) \\
&= \sum_{k=x_0+1}^{\infty} \frac{R_0^k}{k!} e^{-R_0} \\
&= 1 - \sum_{k=0}^{x_0} \frac{R_0^k}{k!} e^{-R_0}
\end{aligned}
$$

Error when 1 sent

- Emitted electrons are $X \leq x_0$ and signal sent is 1.

$$
\begin{aligned}
P(error|1\ sent) = P(X \leq x_0|1\ sent) &= \sum_{k=0}^{x_0} P_{X|1}(k) \\
&= \sum_{k=0}^{x_0} \frac{R_1^k}{k!} e^{-R_1}
\end{aligned}
$$

Total Error

- Here P(1 sent) = P(0 sent) = 1/2 i.e. are considered equi-probable.
- Total error will be:

$$P(error) = P(error|1\ sent)P(1\ sent) + P(error|0\ sent)P(0\ sent)$$

$$\therefore P(error) = \frac{1}{2} - \frac{1}{2}\sum_{k=0}^{x_0} \frac{R_0^k e^{-R_0} - R_1^k e^{-R_1}}{k!}$$

Reference

S. L. Miller and D. G. Childers, *Probability and random processes with applications to signal processing and communications.*
Elsevier/Acad. Press, AP, 2 ed., 2012.