L-11

Outline :

* Motivation
* Expected value of a RV
* Expected value of Functions of RVs
* Moments and Central Moments
* Examples.

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* Motivation :

- Complete Statistical Description of Random Variable :

PDF OR PMF ↙ ↘ CDF } Derive information about RV.

- Not Sufficient to evalute an entire charectristics of RVs.

  ↳ Exception : Gaussian R.V. : $X \sim N(\mu, \sigma^2)$

  μ : mean ↙ ↘ σ^2 : variance

- Different operation on R.V helps us to derive other parameters : Skewness & Kurtosis

Moments

The n^{th} moment of any RV : X is defined as,

X : Continous R.V. : $\boxed{E[x^n] = \int x^n f_X(x)\, dx}$

X : Discrete R.V. : $\boxed{E[x^n] = \sum_k x_k^n P_X(x_k)}$

$\boxed{n=0}$:

$\boxed{n=1}$: Mean = Average = Expected value = Expectation

= 1st Moment.

X : Continous R.V. $\boxed{E[X] = \int_{-\infty}^{\infty} x \cdot \underbrace{f_X(x)}_{PDF}\, dx} = \mu_X$

X : Discrete R.V. $\boxed{E[X] = \sum_k x_k \underbrace{P_X(x_k)}_{PMF}}$

$\boxed{n=2}$: Mean Square Value.

C.R.V : $E[X^2] = \int x^2 \cdot f_X(x)\, dx \longrightarrow E[X^n]$ <u>n^{th} moment</u>

D.RV : $E[X^2] = \sum_k x_k^2 \cdot P_X(x_k) \longrightarrow E[X^n]$

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Central Moments

There is a limitation with Moment when evaluting an effect of Randomness with some scaler/dominant value.

Example : Consider a RV : $Y = a + X$

Y: R.V. (Random); a: Deterministic (not Random); X: R.V. (Random)

Assume : $a >>$ Randomness contributed by X

i.e : $\boxed{\text{Y tends to take small fluctuations about a constant value } a.}$ → Fixed Signal corrupted by noise.

$$Y^n = (a+X)^n$$
$$\approx a^n$$

i.e: n^{th} moment of Y : $E[Y^n]$ would be dominated by the fixed part a. So it is difficult to characterize the randomness in Y by looking at the moments

So, we can use the Concept of $\boxed{\text{Central Moment}}$

Defination: The n^{th} central moment of any R.V. X is defined as,

X : C.R.V. : $E[(X-\mu_X)^n] = \int (x-\mu_X)^n f_X(x)\,dx$

X : D.R.V. : $E[(X-\mu_X)^n] = \sum_k (x_k-\mu_X)^k P_X(x_k)$

Justify: " The lowest central moment of any real intrest is the second central moment. "

$\rightarrow E[(X-\mu_X)^0] = E[1] = 1$

$\rightarrow E[(X-\mu_X)] = E[X] - E[\mu_X]$
$= E[X] - \mu_X$
$= \mu_X - \mu_X$
$= 0$

Special Name:

" Variance " of R.V.

$$\sigma_X^2 = E[(X-\mu_X)^2]$$
$$= E[X^2 - 2X\mu_X + \mu_X^2]$$
$$= E[X^2] - 2\mu_X E[X] + \mu_X^2$$
$$= E[X^2] - 2\mu_X^2 + \mu_X^2$$

$$\boxed{\sigma_X^2 = E[X^2] - \mu_X^2}$$

: 1^{st} moment squared is subsstracted from the second moment.

(Standard Deviation)

S.T.D :

$$\boxed{\sigma_X = \sqrt{E[(X-\mu_X)^2]} = \sqrt{E[X^2]-\mu_X^2}}$$

...se : A measure of the width of the PDF of R.V.

Higher Order Central Moments :

$n = 3$: 3rd Central Moment : $\boxed{\text{Skewness}}$

It is a measure of symmetry of the PDF about the mean

It is asymmetry in a statistical distribution

$$C_s = \frac{E[(X-\mu_x)^3]}{\sigma_X^3}$$

$= +$ value : If R.V has PDF skewed to the RIGHT

$= -$ value : LEFT

$n=4$: 4^{th} Central Moment : $\boxed{\text{Kurtosis}}$

$$C_K = \frac{E[(X-\mu_x)^4]}{\sigma_X^4}$$

= Large value ; i.e : The R.V X will have a large peak near the mean.

* Expected value of F^n of R.V. : X : with pdf $f_X(x)$ & function, $g(x)$

C.R.V : $\boxed{E[g(x)] = \int g(x)\, f_X(x)\, dx}$

D.R.V : $\boxed{E[g(x)] = \sum_K g(x_K)\, P_X(x_K)}$

Theorem: For any constant a and b

$$E[aX+b] = aE[X] + b.$$

Furthermore, for any f^n $g(x)$ = sum of several other f^ns

$= g_1(x) + g_2(x) + \dots g_N(x)$,

$$E\left[\sum_{k=1}^{N} g_k(X)\right] = \sum_{k=1}^{N} E[g_k(X)]$$

" Expectation is a linear operation and the expectation operator can be exchanged with any other linear operation "

Proof:

* $E[aX+b] = \int_{-\infty}^{\infty} (ax+b) f_X(x)\, dx$

$$= a\int_{-\infty}^{\infty} x f_X(x)\, dx + b\int_{-\infty}^{\infty} f_X(x)\, dx$$

$$= aE[X] + b \quad (\because \int \text{PDF}\, dx = 1)$$

* $E\left[\sum_{k=1}^{N} g_k(X)\right] = \int_{-\infty}^{\infty} \left[\sum_{k=1}^{N} g_k(x)\right] f_X(x)\, dx$

$$= \sum_{k=1}^{N} \underbrace{\int_{-\infty}^{\infty} g_k(x) f_X(x)\, dx}_{E[\ ]}$$

$$= \sum_{k=1}^{N} E[g_k(X)]$$

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