LINEAR ALGEBRA

Lecture-2

August 3, 2019

Elimination using Matrices

Why do you need this?
For computer implementation
Consider the system of equations

$$\begin{aligned} 2x + 4y - 2z &= 2 \\ 4x + 9y - 3z &= 8 \\ -2x - 3y + 7z &= 10 \end{aligned}$$

This is same as

$$\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

$A\boldsymbol{v}$ is a combination of the columns of A. Components of $\boldsymbol{v}$ multiply those columns
$A\boldsymbol{v} = x \times (\mathtt{col1}) + y \times (\mathtt{col2}) + z \times (\mathtt{col3})$.

Method of Elimination using Elementary Matrices

The method of Elimination discussed so far will be implemented on the coefficient matrix.
This is needed for computer implementation and further discussions
The general step in elimination was for the i^{th}row,

$$a'_{ij} = a_{ij} + \ell_{ij} * a_{pivot\_row,j}$$

The remaining rows are unaffected. This can be accomplished by the following matrix multiplication

Method of Elimination using Elementary Matrices

$$\begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -l_{ij} & \dots & 1 & .. & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0\dots & 0 & 0 & \dots & 1 \end{bmatrix} \times$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13}\dots & a_{1n-1} & a_{1n} \\ a_{21} & a_{22} & a_{23}\dots & a_{2n-1} & a_{2n} \\ \dots & \dots & \dots & \dots & \\ a_{i1} & a_{i2} & a_{i3}\dots & a_{in-1} & a_{in} \\ \dots & \dots & \dots & \dots & \\ a_{n1} & a_{n2} & a_{n3}\dots & a_{nn-1} & a_{nn} \end{bmatrix}$$

where $\ell_{ij} = a_{ij}/a_{pivot\_row,j}$

Elementary matrices are obtained from the Identity matrix by replacing the zero in the ij^{th} location by -ℓ_{ij}.

We call this the Elementary matrix E_{ij}

Method of Elimination using Matrices

The first step involves making all the elements in the first column from row 2 to row n as zeroes. This means, we have to premultiply the given matrix by corresponding Elementary matrices as given below:

$A_1 = E_{n1}E_{(n-1)1}E_{(n-2)1}...E_{31}E_{21} \times A.$

The next step is to make all elements in second column zero below row 2. This will call for multiplication by elementary matrices, $E_{n2}E_{(n-1)2}E_{(n-2)2}...E_{31}E_{22}$. Continuing this way, we will end up with the Upper Triangular matrix.
Thus, U = Product of Elementary Matrices $\times A$

Example of A to U with Elementary Matrices

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 7 & 9 \end{bmatrix}$$

Clearly in the first step, we make a_{21}=0. For this the required Elementary Matrix is

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example of A to U with Elementary Matrices

$E_{21} \times A$ gives

$$A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 4 & 7 & 9 \end{bmatrix}$$

To reduce a_{31} to zero, the corresponding Elementary Matrix is

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

and $E_{31} \times A_1$ gives

$$A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -3 \end{bmatrix}$$

Example of A to U with Elementary Matrices

Now to make a_{32}=0, the Elementary Matrix is

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

and $E_{32} \times A_2$ gives

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix}$$

Example of A to U with Elementary Matrices

Thus,

$$U = E_{32} \times E_{31} \times E_{21} \times A$$

Now we need to multiply these matrices. All of them were lower triangular - zeroes above the diagonal and one more special feature was that just one element was non-zero. Prove that the product of two such matrices will be lower triangular.

A to U with Elementary Matrices

What does E_{ij} do? It does a multiplication of A to achieve
$\text{row'}_i = \text{row}_i - \ell_{ij}.\text{row}_j$
Hence the inverse of this operation will be
$\text{row'}_i = \text{row}_i + \ell_{ij}.\text{row}_j$
In terms of matrices this will be given as

$$E_{ij}^{-1} = \begin{bmatrix} 1 & 0 & ... & 0 & ... & 0 & ... & 0 \\ 0 & 1 & ... & 0 & ... & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & ... & l_{ij} & ... & 1 & .. & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & ... & 0... & 0 & 0 & ... & 1 \end{bmatrix}$$

A to U with Elementary Matrices

It can be shown that

$$E_{ij}^{-1} \times E_{ij} = I$$

From equation (11), multiplying on the left by E_{32}^{-1}, E_{31}^{-1} and E_{21}^{-1} in this order, we will get

$$A = E_{21}^{-1} \times E_{31}^{-1} \times E_{32}^{-1} \times U$$

Since all the E's are Lower Triangular Matrices, their product will also be Lower Triangular and hence, we denote it by L. Hence, we have reduced A to the form

$$A = LU$$

A=LU

In our example

$$\mathsf{E}_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \mathsf{E}_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}, \mathsf{E}_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, L = product of these three matrices and is given by

$$\mathsf{L} = \mathsf{E}_{21}^{-1} \times \mathsf{E}_{31}^{-1} \times \mathsf{E}_{32}^{-1}$$

Carrying out these multiplications, we obtain

$$\mathsf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 1 & 1 \end{bmatrix}$$

$A = LU$

Hence,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 7 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix}$$

L will always have 1's in the diagonal elements. However, sometimes, U might not have 1's in the diagonal. In such a case, we can write $U = DU'$, where D is the diagonal matrix with elements same as the diagonal elements of U, and U' will have only 1's along the diagonal.

Permutation Matrices

Whatever we have studied so far, assumed that the pivotal elements did not have zeros. If however, the pivotal element is zero at any time, we try to exchange this row with another row below, in which the element in the same column is not zero. In terms of Elementary matrices, this is achieved by premultiplying by a *Permutation Matrix*.

Permutation Matrices

A Permutation matrix is obtained from the identity matrix by interchanging the rows which we want to interchange in our elimination process.
Example of Permutation Matrices of size 3x3 are

$$P_{123} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, P_{213} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, P_{321} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P_{132} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, P_{321} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, P_{231} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Solution of Linear Equations

These permutation matrices form a *Group*. Since we will be dealing only with Permutation matrices in which only two rows are exchanged, the remaining rows being fixed, we shall use the notation P_{ij} to indicate interchange of rows i and j. Thus, Rows i and j can be exchanged during the elimination process, by premultipying with P_{ij}. This can be verified by performing $P_{ij} \times I$. Inverse of a Permutation Matrix is its transpose.

The given system of linear equations $AX = b$ is thus reduced to $LUX = b$. If we substitute $UX = Y$, then the equation is reduced to $LY = b$. This can be easily solved by back-substitution and then $UX = Y$ can be solved again by forward substitution.

Augmented Matrix

Instead of working with the Matrix A, we shall consider the augmented matrix $B = A|b$, which means B is a matrix in which the vector b is introduced as the last column. All other elements are same as in A. If we now perform all the operations on B and make A an upper triangular matrix, the same operations are performed on b, and hence we will get the resulting triangular system of equations, which can be easily solved by back subsitution. This method of solving linear system of equations is called **GAUSSIAN ELIMINATION**

Complete Example of Gaussian Elimination

Consider the system of equations

$$\begin{aligned} x + 2y + 3z + 4u &= 10 \\ x + 2y + 4z + 5u &= 12 \\ 2x + 3y + 2z + 5u &= 12 \\ 3x + 4y + 5z + 6u &= 18 \end{aligned}$$

In Augmented Matrix form, this will be

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 10 \\ 1 & 2 & 4 & 5 & 12 \\ 2 & 3 & 2 & 5 & 12 \\ 3 & 4 & 5 & 6 & 18 \end{array}\right]$$

Gaussian Elimination

$A\|b = \left[\begin{array}{cccc\|c} 1 & 2 & 3 & 4 & 10 \\ 1 & 2 & 4 & 5 & 12 \\ 2 & 3 & 2 & 5 & 12 \\ 3 & 4 & 5 & 6 & 18 \end{array}\right]$	$E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$R_2 = R_2 - R_1$	$A_1\|\mathbf{b} = E_{21} * A\|\mathbf{b}$
$A_1\|b = \left[\begin{array}{cccc\|c} 1 & 2 & 3 & 4 & 10 \\ 0 & 0 & 1 & 1 & 2 \\ 2 & 3 & 2 & 5 & 12 \\ 3 & 4 & 5 & 6 & 18 \end{array}\right]$	$E_{31} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$R_3 = R_3 - 2R_1$	$A_2\|\mathbf{b} = E_{31} * A_1\|\mathbf{b}$
$A_2\|\mathbf{b} = \left[\begin{array}{cccc\|c} 1 & 2 & 3 & 4 & 10 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & -1 & -4 & -3 & -8 \\ 3 & 4 & 5 & 6 & 18 \end{array}\right]$	$E_{41} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix}$
$R_4 = R_4 - 3R_1$	$A_3\|\mathbf{b} = E_{41} * A2\|\mathbf{b}$

Gaussian Elimination

$A_3\vert\mathbf{b} = \begin{bmatrix} 1 & 2 & 3 & 4 & 10 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & -1 & -4 & -3 & -8 \\ 0 & -2 & -4 & -6 & -12 \end{bmatrix}$	$P_{23} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
Interchange Rows 2 and 3	$A_4\vert b = P_{23} * A_3\vert\mathbf{b}$
$A_4\vert b = \begin{bmatrix} 1 & 2 & 3 & 4 & 10 \\ 0 & -1 & -4 & -3 & -8 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & -2 & -4 & -6 & -12 \end{bmatrix}$	$E_{42} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$R_4 = R_4 - 2R_2$	$A_5\vert\mathbf{b} = E_{42} * A_4\vert\mathbf{b}$
$A_5\vert\mathbf{b} = \begin{bmatrix} 1 & 2 & 3 & 4 & 10 \\ 0 & \text{-1} & -4 & -3 & -8 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 4 & 0 & 4 \end{bmatrix}$	$P_{34} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
Interchange Rows 3 and 4	$A_6\vert\mathbf{b} = P_{34} * A_5\vert\mathbf{b}$

Gaussian Elimination

$A_6\|b = \begin{bmatrix} 1 & 2 & 3 & 4 & 10 \\ 0 & \text{-1} & -4 & -3 & -8 \\ 0 & 0 & 4 & 0 & 4 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$	$E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -0.25 & 1 \end{bmatrix}$
$R_4 = R_4 - 0.25R_3$	$A_7\|\mathbf{b} = E_{43} * A_6\|\mathbf{b}$
$A_8\|b = \begin{bmatrix} 1 & 2 & 3 & 4 & 10 \\ 0 & \text{-1} & -4 & -3 & -8 \\ 0 & 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	$A_{Ech} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -4 & -3 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Circled elements are the *pivot elements.*
Solution of the given system of equations can be obtained by method of back substitution from A_8 as $x = y = z = u = 1$.
But we proceed further to get the Row Reduced Echelon form.

Gaussian Elimination

$A_8\|\mathbf{b} = \begin{bmatrix} 1 & 2 & 3 & 4 & 10 \\ 0 & -1 & -4 & -3 & -8 \\ 0 & 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	$E_{24} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$R_2 = R_2 + 3R_4$	$A_9\|\mathbf{b} = E_{24} * A_8\|\mathbf{b}$
$A_9\|\mathbf{b} = \begin{bmatrix} 1 & 2 & 3 & 4 & 10 \\ 0 & -1 & -4 & 0 & -5 \\ 0 & 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	$E_{14} = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$R_1 = R_1 - 4R_4$	$A_{10}\|\mathbf{b} = E_{14} * A_9\|\mathbf{b}$
$A_{10}\|\mathbf{b} = \begin{bmatrix} 1 & 2 & 3 & 0 & 6 \\ 0 & -1 & -4 & 0 & -5 \\ 0 & 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	$E_{13} = \begin{bmatrix} 1 & 0 & -0.75 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$R_1 = R_1 - 0.75R_3$	$A_{11}\|\mathbf{b} = E_{13} * A_{10}\|\mathbf{b}$

Gaussian Elimination

$A_{11}\|\mathbf{b} = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	$E_{12} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$R_1 = R_1 + 2R_2$	$A_{12}\|\mathbf{b} = E_{12} * A_{11}\|\mathbf{b}$
$A_{12}\|\mathbf{b} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	$A_{rref} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

The *row reduced Echelon form* $A_{12}|\mathbf{b}$ gives the solution to the original system of equations as $x = y = z = u = 1$.

No Solution/Infinite No of solutions

Consider the systems of equations

$$\begin{array}{ccccccc|c} x & + & 2y & + & z & = 1 & 1 \\ 2x & + & 4y & + & 2z & = 2 & 2 \\ 3x & + & 6y & + & 3z & = 3 & 4 \end{array}$$

In short form, we can write this as $AX = \mathbf{b_1}|\mathbf{b_2}$
The augmented matrices for these two systems are

$$A|\mathbf{b_1}|\mathbf{b_2} = \left[\begin{array}{ccc|c|c} 1 & 2 & 1 & 1 & 1 \\ 2 & 4 & 2 & 2 & 2 \\ 3 & 6 & 3 & 3 & 4 \end{array}\right]$$

Gaussian Elimination

The echelon form of these augmented matrices are

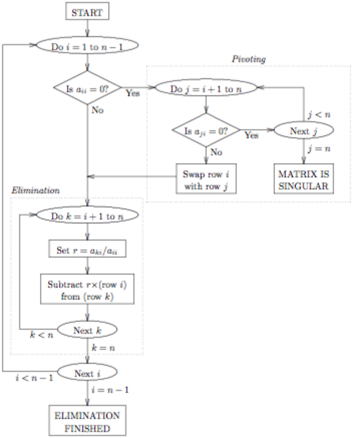
$$A|\mathbf{b_1}|\mathbf{b_2} = \left[\begin{array}{ccc|c|c} 1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Clearly, the first system of equations $A\mathbf{X} = \mathbf{b_1}$ is consistent but has an infinite number of solutions. However, the second system of equations $A\mathbf{X} = \mathbf{b_2}$ is inconsistent as can be seen from the third row.

Gaussian Elimination

- The E matrices are called Elementary Matrices, which when multiplies the given matrix from left, does the desired row operation on the given matrix.
- E matrices to obtain the Echelon form are all *Lower Triangular Matrices*, while those to obtain the Row Reduced Echelon form are all *Upper Triangular Matrices*.
- Each of the Elementary matrices is obtained from the Identity matrix by replacing one zero element by a non-zero element.
- P matrices are permutation matrices, which when premultplies a matrix, intechanges two rows.

Gaussian Elimination - Flow Chart



CONCLUSIONS

- A**x**=**b** becomes upper triangular after elimination
- We do row_i-ℓ_{ij} *row_j to make a_{ij} zero
- $\ell_{ij} = a_{ij}/a_{jj}$, where j^{th} row is the pivot row
- if $a_{jj} = 0$, then we interchange j^{th}row with k^{th} row, $a_{kj} \neq 0$
- Upper Triangular system solved by back substitution
- If no a_{kj}=0, interchange not possible, and no solution
- If Upper triangular matrix is inconsistent, no solution
- If consistent, unique or infinite number of solutions

Summary of Gauss Elimination

Given system of equations $A\mathbf{X} = \mathbf{b}$
The matrix A is reduced to a triangular matrix by elementary row operations. The same operations are done on $\mathbf{b}$ as well. So we end up with a triangular system of equations, which can then be easily solved by back substitution

Invertible Matrices

If A is a square matrix, then its inverse defined as A^{-1} is the matrix such that

$$A^{-1}A = AA^{-1} = I$$

Where do you need this?
In solving $A\mathbf{X} = \mathbf{b}$ if A^{-1} exists, then multiplying both sides by A^{-1} on the left, we obtain,

$$\begin{aligned} A^{-1}(A\mathbf{X}) &= A^{-1}\mathbf{b} \\ (A^{-1}A)\mathbf{X} &= A^{-1}\mathbf{b} \\ \mathbf{X} &= A^{-1}\mathbf{b} \end{aligned}$$

Invertible Matrices

Hence existence of A^{-1} determines whether a system of equations will have a solution or not. If A^{-1} does not exist, then unique solution does not exist. In such a scenario, there are two possibilities:

1. Infinite Number of solutions
2. No solution

Geometrical interpretation of nature of solutions

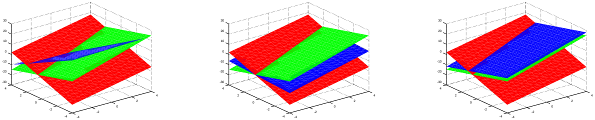
Consider the system of equations

$$Ax = \mathbf{b}$$

where A is a 3×3 matrix. Then, each of the equations represents a plane. Following possibilities exist:

- If no pair of planes is parallel to each other, then the system will have a unique solution
- If the planes pass through a straight line, then they will have an entire line of solution (single infinite solutions)
- If two of the planes are parallel, then there is no solution
- If all the planes are coincident, then the entire plane is a solution (that is there is a double infinite number of solutions)

Nature of solutions



(a) Unique Solution (b) Single Inf Sol (c) No Solutions

When all the planes are same, then the whole plane gives double infinity of solutions.

Existence and Determination of A^{-1}

We shall follow Gauss Jordan method. In this method, the given matrix A is augmented by the Identity matrix of the same order. Let this augmented matrix be B. On B, the same transformations as in Gaussian elimination is carried out. Once A is reduced to an Upper Triangular form, the operation is not stopped but carried on till A is reduced to Identity matrix. The reduced Identity matrix at this level is the inverse of A.

Existence and Determination of A^{-1}

This is not magic but simple mathematical logic.
Let $B = [A\ I]$
Since to reduce A to an identity matrix, we use only row transformations, it is equivalent to premultiplying by Elementary matrices. Let the product of all these elementary matrices, which might include Permutation matrices also, be E.
Then, we obtain $EA = I$
Hence E is left inverse of A. Since E is a product of elementary matrices, each of which has an inverse, hence, E^{-1} exists.
Hence, $E^{-1}(EA) = E^{-1} \Rightarrow A = E^{-1} \Rightarrow AE = I$.
Hence E is right inverse also. Hence, by definition of inverse, $A^{-1} = E = EI$
Hence the resulting matrix on the second half of the augmented matrix gives A^{-1}

Determination of A^{-1}

To find the inverse of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 5 \\ 2 & 3 & 2 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}$$

The augmented matrix is

$$B = \left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 1 & 2 & 4 & 5 & 0 & 1 & 0 & 0 \\ 2 & 3 & 2 & 5 & 0 & 0 & 1 & 0 \\ 3 & 4 & 5 & 6 & 0 & 0 & 0 & 1 \end{array}\right]$$

Determination of A^{-1}

The product of all the elementary matrices that reduce A to Identity is

$$E = A^{-1} = \begin{bmatrix} -4.25 & 2 & 0.5 & 0.75 \\ 4.75 & -3 & -0.5 & -0.25 \\ 0.25 & 0 & -0.5 & 0.25 \\ -1.25 & 1 & 0.5 & -0.25 \end{bmatrix}$$

When can this fail?

If the process of reducing the given matrix to A fails due to pivots being zeroes, then the process of evaluating the inverse also fails. Thus, a necessary and sufficient condition for the existence of A^{-1} is that the pivot elements should be non-zero. (Exchanges are allowed in getting the pivots)
More on this later.