Lecture - 13

Transformation of Random Variables :

R. V. X → $f_X(x)$ pdf, $F_X(x)$ CDF

Y → $Y = g(X)$ New RV. → ?? → $f_Y(y)$, $F_Y(y)$

{Study of PHY. System}

Assumption : ① PDF of X is known or pdf of input variable X is known.

: ② Input variable has undergone a transformation

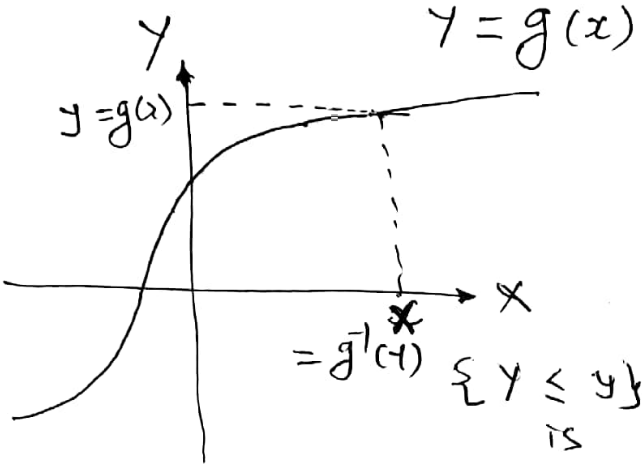
Objective : To find CDF $F_Y(y)$ or $f_Y(y)$ for given R.V. X.

Example : Monotonic Function

Increasing f^n: $\{Y \le y\} \approx X \le g^{-1}(y)$

Decreasing f^n: $\{Y \le y\} \approx X \geq g^{-1}(y)$

Monotonically Increasing F^n



$\{Y \le y\}$ is equi, $X \le g^{-1}(y)$

$Y = g(X)$

$\Rightarrow X = g^{-1}(Y)$ exist & well behaved

$\Rightarrow$ Objective : To find the pdf of Y

Step-1 Calculate the CDF of Y

$$F_Y(y) = Pr(Y \le y)$$
$$= Pr(g(X) \le y)$$
$$= Pr(X \le g^{-1}(y))$$
$$= F_X(g^{-1}(y))$$

Step-2

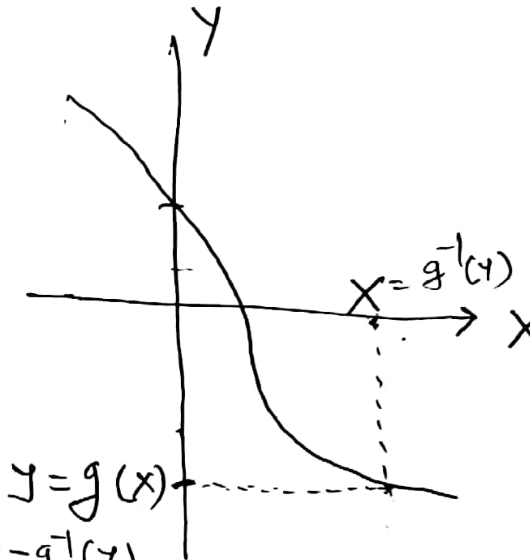
$\Rightarrow$ Differentiating w.r.t. y,

$$f_Y(y) = \frac{d}{dy}\left(F_X(g^{-1}(y))\right)$$
$$= f_X(g^{-1}(y)) \cdot \frac{dg^{-1}(y)}{dy} = f_X(x)\frac{dx}{dy}\Bigg|_{x = g^{-1}(y)}$$

Monotonically Decreasing F^n.

$\{Y \le y\}$ is equivalent to the event,

$$X \ge g^{-1}(y)$$



$\Rightarrow x = g^{-1}(y)$

Step-1 $F_Y(y) = Pr\{Y \le y\}$
$$= Pr(X \ge g^{-1}(y))$$
$$= 1 - F_X(g^{-1}(y))$$

Step-2 Differentiating w.r.t y

$$f_Y(y) = 0 - f_X(g^{-1}(y))\frac{d}{dy}[g^{-1}(y)]$$

$$\boxed{f_Y(y) = -f_X(x)\frac{dx}{dy}\Bigg|_{x = g^{-1}(y)}}$$

$$F_X(x) = F_Y(g(x))$$

$$f_X(x) = \frac{d}{dx}[F_Y(g(x))]$$

$$= f_Y(g(x))\,\frac{d}{dx}[g(x)]$$

$$= f_Y(g(x))\,\frac{d}{dx}[g(x)]$$

$$= f_Y(y)\cdot\frac{dy}{dx}$$

$$\Rightarrow f_X(x) = f_Y(y)\cdot\frac{dy}{dx}$$

$$\Rightarrow f_Y(y) = \left.\frac{f_X(x)}{\frac{dy}{dx}}\right|_{x=g^{-1}(y)}$$

OR

$$\boxed{f_Y(y) = f_X(x)\cdot\left.\frac{dx}{dy}\right|_{x=g^{-1}(y)}}$$

OR

$\boxed{\text{Step-2}}$ Differentiating w.r.t. x,

$$\boxed{f_Y(y) = -\left.\frac{f_X(x)}{dy/dx}\right|_{x=g^{-1}(y)}}$$

General, for any monotonic ↑ or ↓ f^n,

$$\boxed{f_Y(y) = \frac{f_X(x)}{\left|\frac{dy}{dx}\right|_{x=g^{-1}(y)}}}$$

To $\boxed{\text{find the PDF of the RV } Y}$ given the PDF of X : [Steps - 1 to 4]

(i) Write Y in terms of X and write the same relation in terms of y

Ex: $Y = X^2 \Rightarrow X = \pm\sqrt{Y}$

(ii) Find : $\dfrac{dx}{dy}$

(iii) Write : $f_Y(y) = f_X(x) \cdot \left|\dfrac{dx}{dy}\right|$

in terms of Y

(iv) Using the given range of X, find the range of Y.

Ex: $f_X(x) = x^2$, $\boxed{-1 < x < 2}$

and $\boxed{Y = X^3}$ The range of Y is as follow :

$x = -1 \Rightarrow Y = (-1)^3 = -1.$

$x = 2 \Rightarrow Y = (2)^3 = 8$

The range of Y : $\boxed{-1 < y < 8}$

Example : 1

$X \sim N(0, \sigma^2)$

$Y = e^X.$

Find the PDF of Y.

Sol^n: 1. $y = e^x$ which is a monotonic f^n.

$\therefore \boxed{x = \log y}$

2. $\frac{dx}{dy} = \frac{1}{y}$

3. $f_Y(y) = f_X(y) \cdot \frac{dx}{dy}$

Now, $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\left[\frac{(x-0)^2}{2\sigma^2}\right]}$, $-\infty < x < \infty$

$= \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}$

$\therefore f_X(\overset{\log y}{y}) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(\log y)^2/2\sigma^2}$, $y \geq 0$ $\quad [\because e^{-\infty} = 0$, $e^{\infty} = \infty]$

So, $f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(\log y)^2/2\sigma^2} \cdot \left|\frac{1}{y}\right|$

$= \frac{1}{y\sqrt{2\pi\sigma^2}} e^{-(\log y)^2/2\sigma^2}$, $y \geq 0$

4.

$x = -\infty \Rightarrow y = 0$

$x = \infty \Rightarrow y = \infty$

Example : 2

$$X \sim N(0, \sigma^2)$$

$$Y = X^2$$

Find, pdf of Y, $f_Y(y)$

Sol^n:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

1. $y = x^2 \Rightarrow x = \pm\sqrt{y}$

2. $F_Y(y) = P(Y \leq y)$

$$= P(X^2 \leq y)$$

$$= P(-\sqrt{y} \leq X \leq \sqrt{y})$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

($\because$ Defination of $F_X(x)$)

3. Differentiating above w.r.t. Y,

$$f_Y(y) = \frac{d}{dy}[F_X(\sqrt{y})] - \frac{d}{dy}[F_X(-\sqrt{y})]$$

$$= f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} - f_X(-\sqrt{y}) \cdot \left(\frac{-1}{2\sqrt{y}}\right)$$

$$= [f_X(\sqrt{y}) + f_X(-\sqrt{y})] \frac{1}{2\sqrt{y}}$$

$\downarrow$? $\quad\quad \downarrow$?

- Now,

$$f_X(\sqrt{y}) = \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{(\sqrt{y})^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2}$$

$$f_X(-\sqrt{y}) = \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-y^2/2\sigma^2}$$

- Limit:

$Y = X^2 \Rightarrow x = -\infty, Y = \infty$

$\Rightarrow x = \infty, Y = \infty$

$$f_Y(y) = 2 \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-y/2\sigma^2} \cdot \frac{1}{2\sqrt{y}}$$

$$= \frac{1}{\sqrt{2\pi y}\,\sigma} \cdot e^{-y/2\sigma^2}, \quad Y > 0$$

Example : 3

If X is uniformly distributed in $(-1, 1)$. Find the pdf of $Y = \sin\left(\frac{\pi x}{2}\right)$.

Sol^n:

$$f_X(x) = \begin{cases} \frac{1}{2}, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$y = \sin\left(\frac{\pi x}{2}\right)$$

$$\therefore \boxed{x = \frac{2}{\pi} \sin^{-1} y}$$

$$\frac{dx}{dy} = \frac{2}{\pi} \cdot \frac{1}{\sqrt{1-y^2}}$$

$$f_X(y) = \frac{1}{2}$$

$$\therefore f_Y(y) = f_X(y) \cdot \left|\frac{dx}{dy}\right|$$

$$= \frac{1}{2} \cdot \frac{2}{\pi} \cdot \frac{1}{\sqrt{1-y^2}}$$

$$= \frac{1}{\pi\sqrt{1-y^2}}$$

Range :

$$x = -1 \Rightarrow y = \sin\left(-\frac{\pi}{2}\right) = -1$$

$$x = 1 \Rightarrow y = \sin \pi/2 = +1$$

$$\therefore f_Y(y) = \begin{cases} \frac{1}{\pi\sqrt{1-y^2}}, & -1 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Example: Two Random Variables are X & Y.

Let $Z = X + Y$. : one Function of two RVs.

Find :

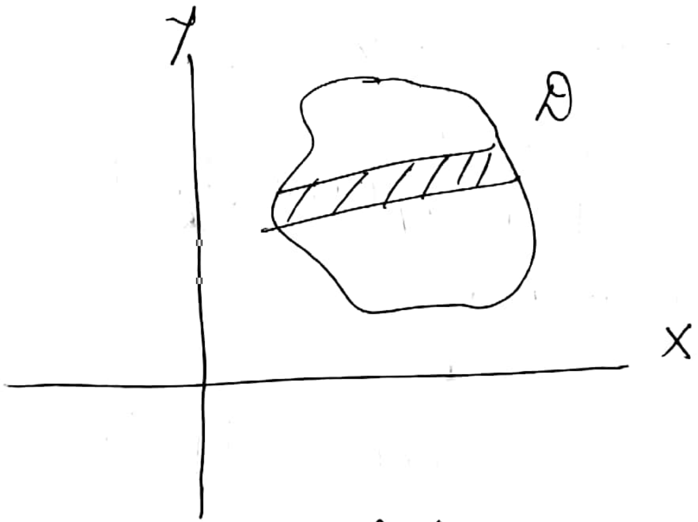
(i) pdf of Z, $f_Z(z)$

(ii) $f_Z(z)$, if X & Y are independent.

(iii) Let $X \sim N(0,1)$ and $Y \sim N(0,1)$ are independent RVs. Prove that $Z \sim N(0,2)$

(iv) If X & Y are exponential RVs with para λ, then find $f_Z(z)$.

<u>Hint</u> :



2-D Plane

$Z = g(X, Y)$

X & Y are conti/Disc. RVs

$$P[(X,Y) \in D] = \iint_{(x,y) \in D} f_{XY}(x,y)\, dx\, dy$$

Sol^n:

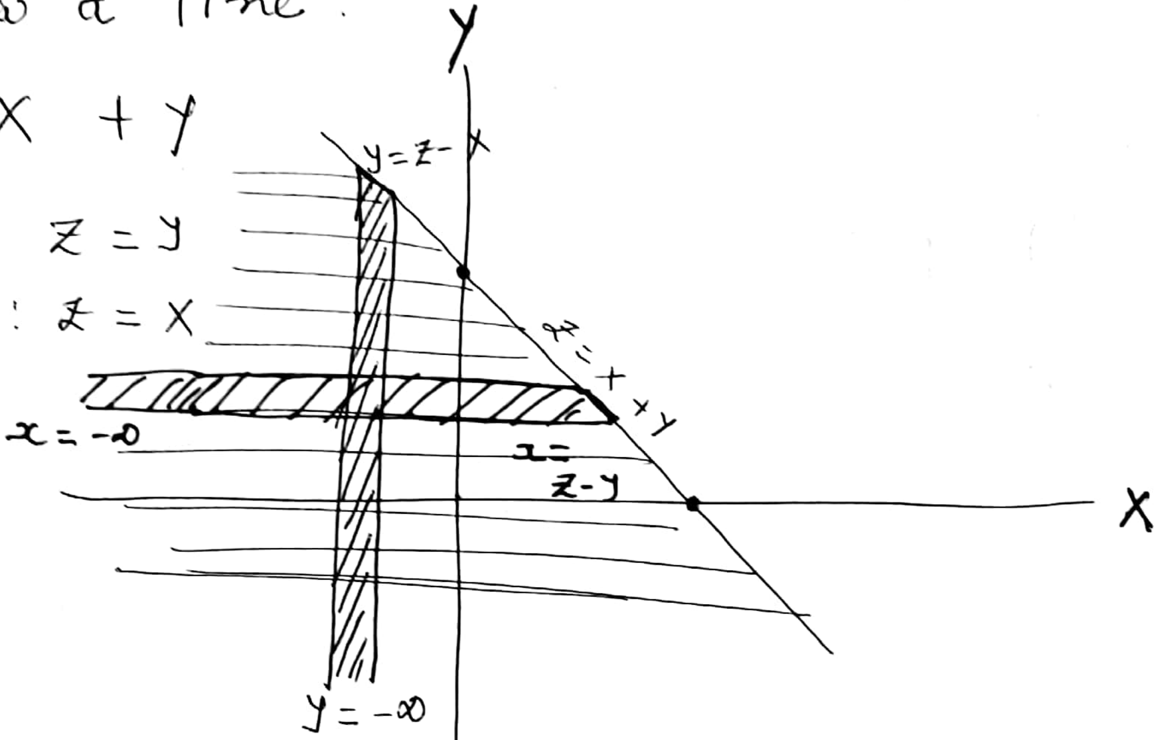
(i)

— Draw a line:

$Z = X + Y$

Put $X = 0$: $Z = Y$

$Y = 0$: $Z = X$



Start with Distribution f^n, Unknown

— CDF, $F_Z(z) = Pr(Z \le z)$

$= Pr(X + Y \le z)$

Decide V-Strip / H-Strip :

$$= \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{x=z-y} f_{XY}(x,y)\,dx\,dy \quad \text{H. Strip} \rightarrow (1)$$

$$= \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{y=z-x} f_{XY}(x,y)\,dy\,dx \quad \text{V. Strip} \rightarrow (2)$$

— Use of Leibnitz Rule :

Leibnitz Rule:

Let $G(x) = \int_{a(x)}^{b(x)} h(x, y)\, dy$

$$\frac{d}{dx}[G(x)] = \underbrace{\frac{d}{dx} b(x)}_{\text{Upper limit Derivative w.r.t. } x \ (1)} \cdot h(x, b(x)) - \underbrace{\frac{d}{dx} a(x)}_{\text{Lower limit Derivative w.r.t. } x \ (2)} h(x, a(x))$$

$$+ \underbrace{\int_{a(x)}^{b(x)} \frac{\partial}{\partial x} h(x, y)\, dy}_{\text{partial derivative}}$$

(if $h(x, y)$ is not a f^n of x then term will be zero) (3)

* Let apply to eq^n (1) & eq (2):

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

- when the limits are constant, take the (1) & (2) terms are zero. and take the derivative inside.

$$f_Z(z) = \int_{-\infty}^{\infty} \left[\frac{d}{dz} \int_{-\infty}^{z-y} f_{XY}(x, y)\, dx \right] dy$$

$$= \int_{-\infty}^{\infty} \left[\underbrace{\frac{d}{dz}(z-y)}_{1} \cdot f_{XY}(z-y, y) - \frac{d}{dz}(-\infty) \underbrace{f_{XY}(-\infty, y)}_{\text{Zero}} + \underbrace{\int_{-\infty}^{z-y} \frac{\partial}{\partial z} f_{XY}(x, y)}_{\text{Zero}} \right] dy$$

$$= \int_{-\infty}^{\infty} 1 \cdot f_{XY}(z-y, y)\, dy = \int_{-\infty}^{\infty} f_{XY}(z-y, y)\, dy \longrightarrow (3)$$

For, eqⁿ (2),

$$\cancel{f_Z(z) = \int_{-\infty}^{\infty} \left[\frac{d}{dz} \right]}$$

$$f_Z(z) = \int_{-\infty}^{\infty} \left[\frac{d}{dz} \int_{-\infty}^{z-x} f_{XY}(x,y)\,dy\right] dx$$

$$= \int_{-\infty}^{\infty} \Big[\underbrace{\frac{d}{dz}(z-x)}_{1} \cdot \underline{f_{xy}(x, z-x)} - \frac{d}{dz}(-\infty)\underbrace{f_{XY}(x,-\infty)}_{\text{zero}} + \int_{-\infty}^{z-x} \underbrace{\frac{\partial}{\partial z} f_{XY}(x,y)}_{\text{zero}}\,dx \Big] dx$$

$$= \int_{-\infty}^{\infty} 1 \cdot f_{XY}(x, z-x)\,dx$$

$$= \int_{-\infty}^{\infty} \underline{f_{XY}(x, z-x)}\,dx \longrightarrow (4)$$

[Convolution of two fⁿ if X & Y are independent]

- we can go forward with Any one solution (3) or (4).

- Let's go ahead with (4)

$$\boxed{f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x, z-x)\,dx}$$

(ii)

Find $f_Z(z)$, if X & Y are independent.

$$f_Z(z) = \int_{-\infty}^{\infty} \underbrace{f_X(x) \cdot f_Y(z-x)}_{\text{Convolution b/w two } f^n} \, dx \longrightarrow (5)$$

OR

$$f_Z(z) = \int_{-\infty}^{\infty} f_Y(y) \cdot f_X(z-y) \, dy$$

(iii) $\left.\begin{array}{l} X \sim N(0,1) \\ Y \sim N(0,1) \end{array}\right\} \Rightarrow Z \sim N(0,2)$

- write pdf of X & Y

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

From eq^n (5),

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-x)^2}{2}} \, dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{(z^2 - 2zx + 2x^2)}{2}} \, dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{\left(\sqrt{2}x - \frac{z}{\sqrt{2}}\right)^2 + \frac{z^2}{2}}{2}} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{4}} \cdot e^{-\frac{\left(\sqrt{2}x - \frac{z}{\sqrt{2}}\right)^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{4}} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\left(\sqrt{2}x - \frac{z}{\sqrt{2}}\right)^2}{2}} dx$$

Take $u = \sqrt{2}x - \frac{z}{\sqrt{2}}$ $\quad x = -\infty \Rightarrow u = -\infty$

$\Rightarrow du = \sqrt{2}\, dx$ $\quad x = \infty \Rightarrow u = +\infty$

$$= \frac{1}{\sqrt{2\pi}} e^{-z^2/4} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} \frac{du}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-z^2/4} \cdot \frac{1}{\sqrt{2}} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du}_{=1}$$

$$= \frac{1}{\sqrt{2\pi} \cdot \sqrt{2}} e^{-\frac{z^2}{2(\sqrt{2})^2}}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{z^2}{2\sigma^2}}$$

$$\boxed{\begin{aligned} \therefore Z &\sim N(0, \sigma^2) \\ &\sim N(0, 2) \end{aligned}}$$

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v) X, Y : exponential R.Vs. with parameter λ

- First write PDF of X & Y.

$$f_X(x) = \lambda \cdot e^{-\lambda x}, \quad x > 0$$

$$f_Y(y) = \lambda e^{-\lambda y}, \quad y > 0$$

We know that, from eqn (4),

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z-x)\, dx$$

$$= \int_{-\infty}^{\infty} \lambda \cdot e^{-\lambda x} u(x) \cdot \lambda e^{-\lambda(z-x)} u(z-x)\, dx$$

Now, $u(x) \cdot u(z-x) = \begin{cases} 1, & 0 < x < z \\ 0, & \text{otherwise} \end{cases}$

$$\therefore f_Z(z) = \lambda^2 e^{-\lambda z} \int_0^z dz$$

$$\boxed{f_Z(z) = \lambda^2 \cdot z \cdot e^{-\lambda z} u(z)}$$