MAT 202 - Probability and Random Processes
Lecture - 5 and 6

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January 22, 2020

Outline

- **Joint and Conditional Probability**

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  - Motivation
  - How does one compute it?
  - Example

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  - Theorem 2.10:Total Probability Theorem
  - Theorem 2.11: Bayes Theorem
  - Examples
- **Independence**

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- **Random Variables**
  - Definition: RV and PMF
  - Types of RVs
  - Examples

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Introduction to Probability Theory

- Concepts of Joint Probability

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   - Calculate the probabilities of each of these outcomes.
3. Relative Frequency: Let $n_{A,B}$ be the number of times that A and B simultaneously occur in n trials:

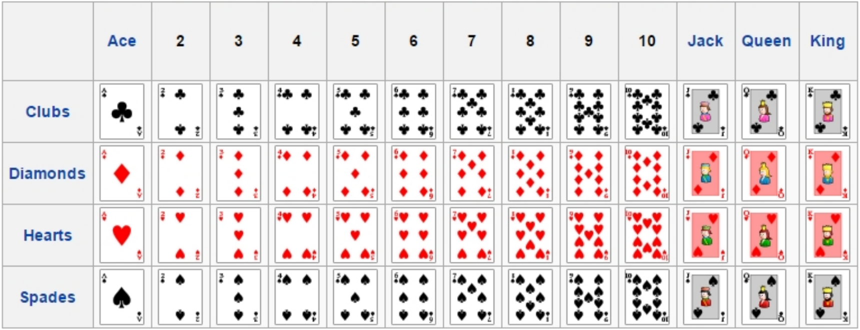
$$Pr(A, B) = \lim_{n\to\infty} \frac{n_{A,B}}{n}$$

4. Example:

Introduction to Probability Theory

- Example 1: Joint Probability

Example set of 52 playing cards; 13 of each suit clubs, diamonds, hearts, and spades



A = {Red card selected}, B = {number card selected}, C = {heart card selected}

Find: Pr(A), Pr(B), Pr(C), Pr(A,B), Pr(A,C), Pr(B,C)

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$$Pr(A_1, A_2, A_3 \cdots A_M) = P(A_M|A_1, A_2, \cdots A_{M-1})x$$
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Introduction to Probability Theory

- Example #2: Conditional Probability: Drawing cards from a deck

Suppose now that we select two cards at random from the deck. When we select the second card, we do not return the first card to the deck. In this case, we say that we are selecting cards without replacement. Compute the Joint Probability of event A: First card was a spade and B: Second card was a spade

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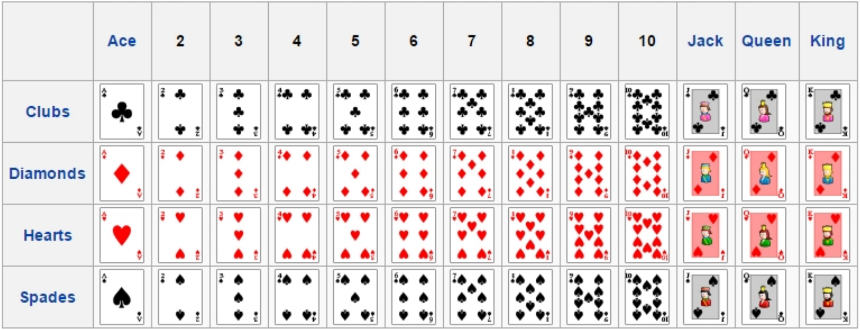

You are dealt 5 cards from a standard 52-card deck. A flush is when you are dealt all five cards of the same suit.

(1) What is the probability you are dealt a flush in spades?

(2) What is the probability of a flush in any suit?

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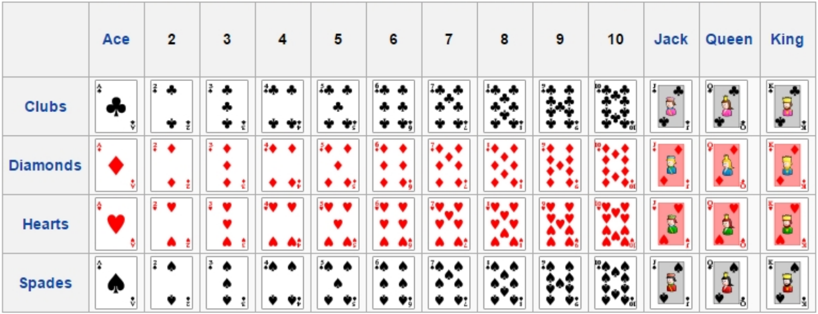

Introduction to Probability Theory

- Example #3: Conditional Probability: Game of Poker

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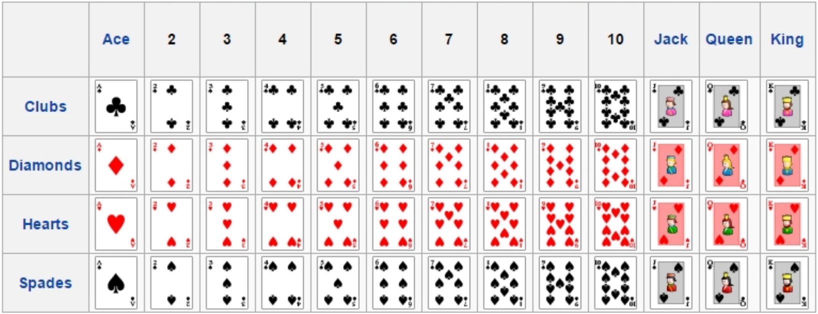

Hint: Start with defining event, $A_i = i^{th}$ card dealt to us is a spade

Special Assignment

- Example :Probabilistic Modeling - Engineering Application

- Presentation: 30 Minutes (PPT or Document Projector)
- Date: January 28, 2020 (Tuesday)
- It's an open call: Any group / student can take an initiative to present!
- Reference: Chapter-2 : Point 2.9

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- Theorem 2.10:Total Probability Theorem
  Let $B_1, B_2, \cdots B_n$ be a set of mutually exclusive and exhaustive events. That is, $B_i \cap B_j = \varnothing$ for all $i \neq j$ and

$$\bigcup_{i=1}^{n} B_i = S \Rightarrow \sum_{i=1}^{n} Pr(B_i) = 1$$

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- Theorem 2.11 (Bayes Theorem): Let $B_1, B_2, \cdots B_n$ be a set of mutually exclusive and exhaustive events. Then,

$$Pr(B_i|A) = \frac{Pr(A|B_i)Pr(B_i)}{\sum_{i=1}^{n} Pr(A|B_i)Pr(B_i)}$$

Where,
$Pr(B_i)$: a priori probability (probabilities that are formed from self-evident or presupposed models) and
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- **Examples:**

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Compute the Probability that Seat 15 was selected given that Row 20 was selected and also find the probability that Row 20 was selected given that Seat 15 was selected.

Introduction to Probability Theory

- Example #2: Communication System - Receiver (RX)

A communication system sends binary data 0 or 1 which is then detected at the RX. The RX occasionally makes mistakes and sometimes a 0 is sent and is detected as a 1 or a 1 can be sent and detected as a 0.

Introduction to Probability Theory

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3. What is the probability that the detected bit is not equal to the transmitted bit? This is the overall probability of error of the RX.

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- Mathematical:Two events A and B are independent if $Pr(A|B) = Pr(B)$ and $Pr(B|A) = Pr(A)$. So, the joint probability $Pr(A, B) = Pr(A)Pr(B)$.

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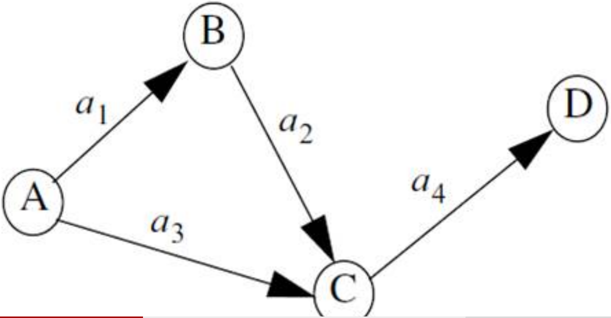
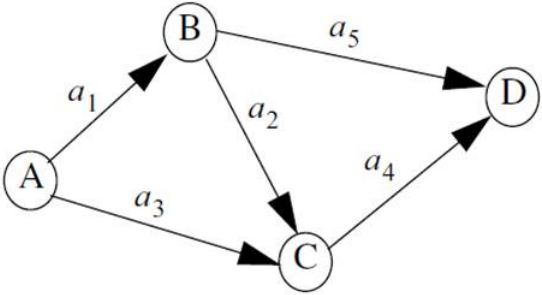
- Definition: For three Events:The events A, B and C are mutually independent if each pair of events is independent; that is
  Pr(A, B) = Pr(A)Pr(B), Pr(A, C) = Pr(A)Pr(C),
  Pr(B, C) = Pr(B)Pr(C), Pr(A, B, C) = Pr(A) Pr(B) Pr(C)
- **Examples:**

Consider a communications network with nodes A, B, C, and D and links a_1, a_2 ,a_3 and a_4 as shown in the diagram. The probability of a link being available at any time is p. In order to send a message from node A to node D we must have a path of available links from A to D. Assuming independence of link availability, What is the probability of being able to send a message?



Suppose we modify the communications network as shown in the diagram by adding a link from node B to node D. Assuming each link is available with probability p independent of any other link, what is the probability of being able to send a message from node A to D?



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Thank you