CHAPTER 2

TIME-FREQUENCY ANALYSIS METHODS

2.1. INTRODUCTION

The distribution of signal energy in the time or frequency-domain is very straightforward. The distribution of energy in time is defined as the squared magnitude of the signal, $|x(t)|^2$, and the energy distribution in frequency is defined as the magnitude of the Fourier transform, $|X(\omega)|^2$. The Fourier transform, being a unitary operator, provides a different but equivalent representation of the signal. However, neither the signal nor its Fourier transform indicates how the energy is distributed simultaneously in time and frequency. For example, consider time-frequency distributions of two synthetic signals shown in Fig. 2.1. The two signals appear identical only if the time or the frequency energy distribution is considered. However, the two signals are clearly not identical from its time-frequency distribution. Hence, to distinguish such signals and to provide a more revealing picture of the signal's characteristics, a joint time-frequency representation is necessary. Time-frequency distributions (TFDs) are two-dimensional functions that indicate the joint time-frequency energy content of a signal. They have been utilized in a wide range of signals, including speech, music and other acoustic signals, biological signals, radar and sonar signals, and geographical signals. Most TFDs of interest are members of Cohen's class. However, the current representations are adaptive and offer more advantageous properties like Affine class, L-Wigner distribution, etc. In this chapter we review TFDs of Cohen's class, the Affine class and some adaptive signal representations and form a sufficient background to understand Chapters three and four.

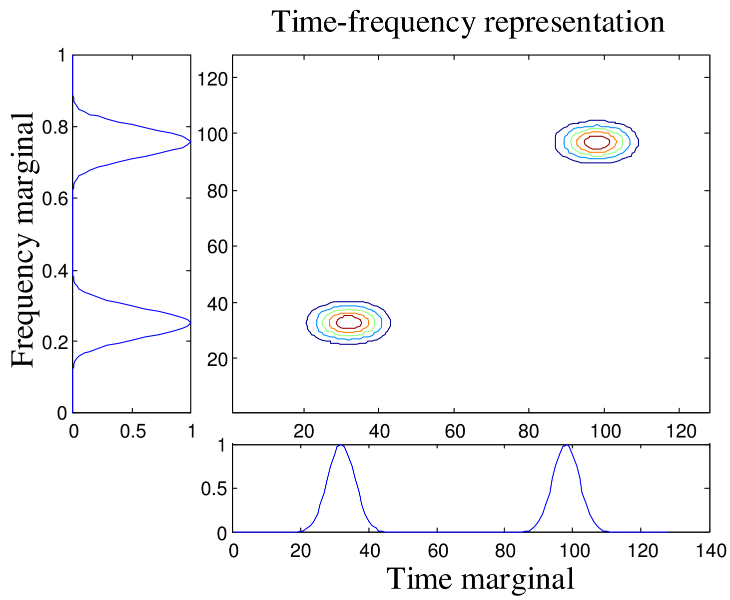


Fig. 2.1. (a) Frequency marginal, time-frequency representation and time marginal of a low frequency signal followed by a high frequency signal

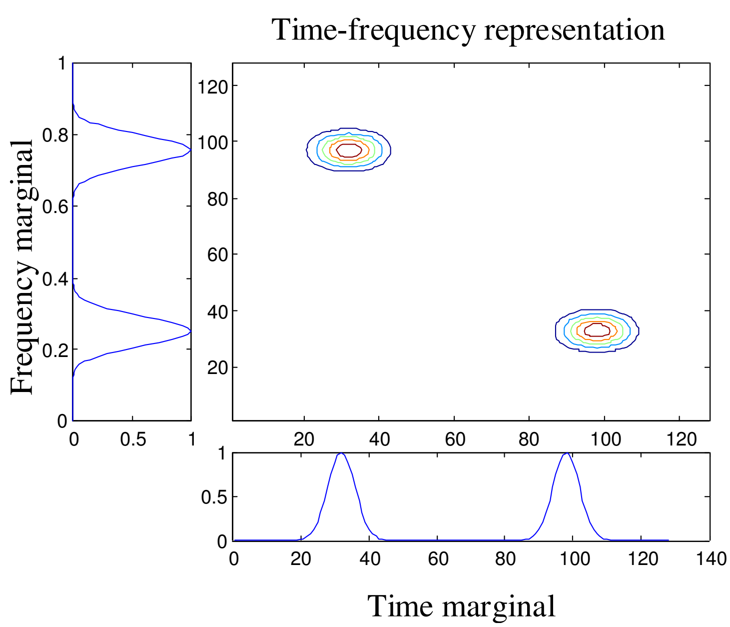


Fig. 2.1. (b) Frequency marginal, time-frequency representation and time marginal of a high frequency signal followed by a low frequency signal

2.1.1. Brief Historical Perspective

The origin of time-frequency analysis goes back to 1940s when Gabor's instrumental work on signal representation based on elementary Gaussian elements for a proper description of the signal in combined time and frequency domains took place. Later, the work done by Wigner in the field of quantum mechanics had been applied to signal processing by Ville. Page had developed the concept of instantaneous power spectrum as the rate of change of energy spectrum in the range $-\infty$ to T. Levin used the same definition for the segment T and ∞**,** and defined a new function as the average of both the types of instantaneous power spectra (Cohen, 1989). TFDs for the nonstationary process were considered by (Flandrin *et al*, 1985). In 1968, a fundamental result was published by Rihackzak that was stemmed from physical observations, now known as the Rihaczek distribution (Rihaczek, 1968). The existing results were given a mathematical treatment and an insight had been provided into their properties by Claasen and Mecklenbrauker in their series of papers (Claasen *et al*, 1980a) and in particular, the Wigner-Ville distribution (WVD) was investigated. Cohen had generalized the concept of time-frequency analysis by unifying the definition of TFDs having different properties. Adaptive signal representations have been given a lot of attention to overcome some of the difficulties associated with TFDs. These representations were mainly investigated by Jones and Baranuik (Jones *et al*, 1993b), Choi and Williams (Choi *et al*, 1989) and Jones and Parks (Jones *et al*, 1992a). The recent focus is towards extending the analysis domain beyond time and frequency to obtain more redundant representations (Mann, 1995). We will discuss some of these methods in the subsequent Chapters.

2.1.2. Objectives of TFDs

Before we present what the TFDs should reflect, it would be appropriate to define the instantaneous frequency (IF) and the group delay, as:

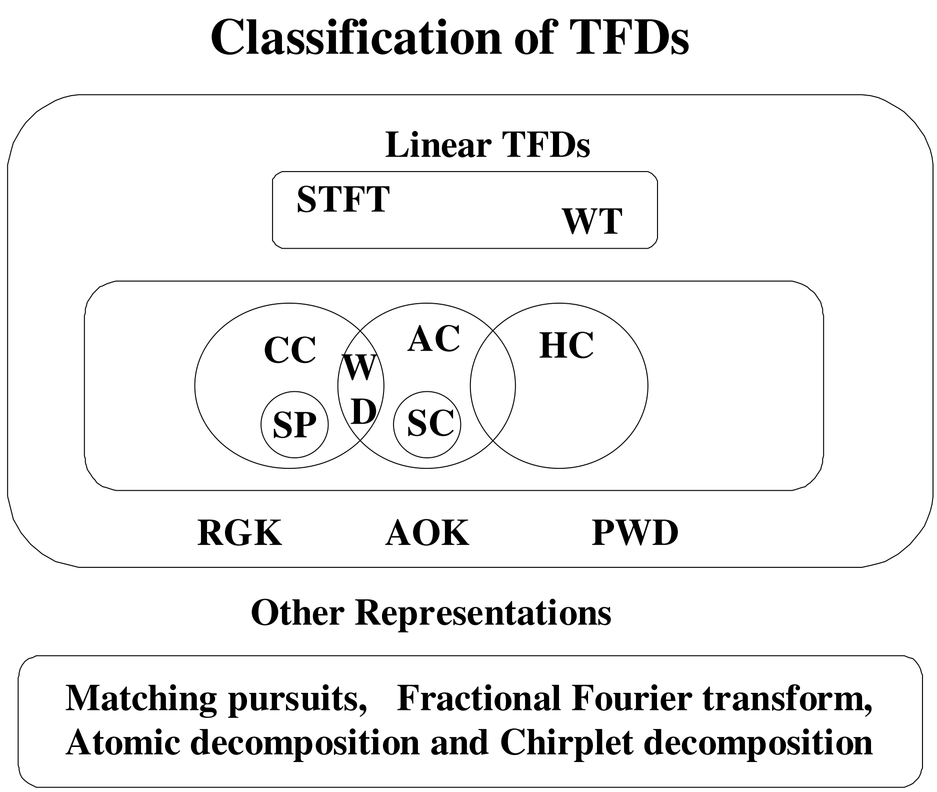
$$f_s(t) = \frac{1}{2\pi}\frac{d}{dt}\,arg[s(t)] \text{ and } t_s(f) = \frac{1}{2\pi}\frac{d}{df}\,arg[S(f)]\text{, respectively.} \quad (2.1)$$

The IF which represents the energy concentration in the frequency domain as a function of time describes the signal's true characteristics. However, the concept of IF is meaningless for multicomponent and nonanalytic signals, where IF is an ambiguous representation. Hence the TFDs are expected to represent the true energy along the path of the instantaneous frequency even when the constraints are lifted. Thus the TFDs are required to attain the following goals:

- Discriminate multicomponents signal from monocomponent signals.
- Facilitate the separation of multicomponents signal from monocomponent signals.
- Track the IF as accurately as possible.
- Existence of an inversion method to uniquely reconstruct the signal.

2.2. GENERAL CLASSES OF TFDs

The TFDs can be classified according to their properties. Two types of time-frequency distributions are those that are linear or quadratic functions of the signal. Examples of linear time-frequency distributions are short-time Fourier transform (STFT) and wavelet transform (WT). Examples of quadratic time-frequency distributions are the spectrogram, scalogram, WVD, etc. A general classification is shown in Fig. 2.2 (O’Neill, 1997). They are also classified according to their behavior when an operator is applied to a signal.



STFT: Sort time Fourier transform, WT: Wavelet transform, CC: Cohen’s class, AF: Affine class, HC: Hyperbolic class, WD: Wigner distribution, SP: Spectrogram, SC: Scalogram, RGK: Radially Gaussian kernel, AOK: Adaptive optimal kernel, PWD: Polynomial Wigner distribution

Fig. 2.2. Classification of time-frequency representations

Three prominent examples of operators are the time-shift operator, the frequency shift operator and the scale operator. We review some general classes of quadratic distributions, Affine class and shift-covariant class. Finally, we present the adaptive signal representations that give flexibility in analyzing the signal by depicting them in domains other than time and frequency.

2.2.1. Cohen's Class

To distribute the energy of the signal over time and frequency, several authors have proposed different methods with each of them having unique properties. A unified approach proposed by Cohen can be expressed as:

$$C(t,\omega) = \frac{1}{2\pi}\iiint s(u+\frac{\tau}{2})\, s^{*}(u-\frac{\tau}{2})\, \phi(\theta,\tau)\, e^{-j\omega\tau}\, e^{j\theta(u-t)}\, du\, d\tau\, d\theta, \tag{2.2}$$

where $\phi(\theta,\tau)$ is an arbitrary function called the kernel by Classsen and Meclenbrauker (Claasen *et al*, 1980a). The kernel can be a function of time and frequency. In general it is preferred to be of low pass in nature because it acts as a filtering means in the ambiguity function (AF) domain. Kernel determines the properties of the TFDs build upon them. Stated otherwise, the desired properties get reflected as constraints on the kernel. We have mentioned some desired properties of the kernel at a later stage. The kernels can be time and frequency dependent but they are not considered to be of Cohen's class (Cohen, 1995). Some adaptive representations vary the kernel in time and/or frequency dependent fashion to match the signal's characteristics. We will now briefly review some of the TFDs belonging to the Cohen's class.

2.2.2. Wigner-Ville Distribution

The Wigner[†] distribution is the prototype of distribution that is qualitatively different from spectrogram (magnitude squared of the STFT). The WVD has been successfully used in analyzing nonstationary signals, i.e., signals whose frequency behavior varies with time. The WVD of a signal $s(t)$ is given by

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[†] Wigner and Wgner-Ville terms are interchangeably used

$$W(t,\omega) = \frac{1}{2\pi}\int s(t+\frac{\tau}{2})\, s^*(t-\frac{\tau}{2})\, e^{-j\omega\tau}\, d\tau. \tag{2.3}$$

This equation can be obtained by setting the kernel equal to one in Eqn. (2.2). Most of the properties of WVD can be obtained with this interpretation. Perhaps the most remarkable property of the WVD is that for a Gaussian windowed linear chirp signal, defined as:

$$s(t) = e^{-\alpha t^2} e^{j(a_0 + a_1 t + a_2 t^2)} \tag{2.4}$$

WVD concentrates the energy of the signal along the instantaneous frequency of the signal, given by:

$$W_s(t,\omega) = e^{-(\omega - a_1 - 2a_2 t)^2}\, e^{-2\alpha t^2}. \tag{2.5}$$

The distribution and the instantaneous frequency are shown in Fig. 2.3. The WVD of the sum of two signals is not the sum of individual WVDs. Instead, it will be the sum of their WVDs plus another component that is the cross WVD of the two signals:

$$W_{x+y}(t,\omega) = W_x + W_y + 2Re\{W_{x,y}(t,\omega)\}, \tag{2.6}$$

where the cross Wigner distribution of the two signals is defined as:

$$W_{x,y}(t,\omega) = \int x(t+\frac{\tau}{2})\, y^*(t-\frac{\tau}{2})\, e^{-j\omega t}\, d\tau. \tag{2.7}$$

The cross Wigner distribution of the two signals is commonly called a cross term. All quadratic time-frequency distributions, including spectrogram, will contain cross terms.

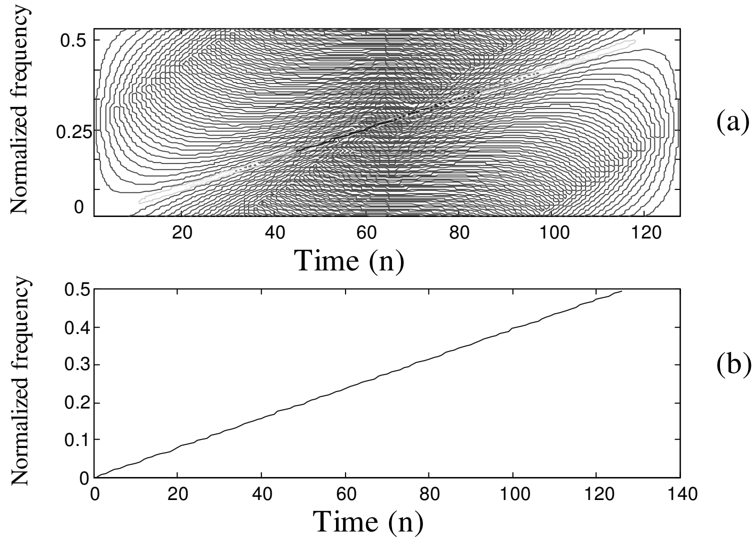


Fig. 2.3. (a) Wigner distribution and (b) Instantaneous frequency of a linear FM signal

Cross terms can also occur within a single component signal, e.g., non-linear frequency modulated signals. An example of cross terms within a signal is shown in Fig. 2.4. An auto term is defined, rather vaguely, as parts of the Wigner distribution that corresponds to the true spectrum of the signal.

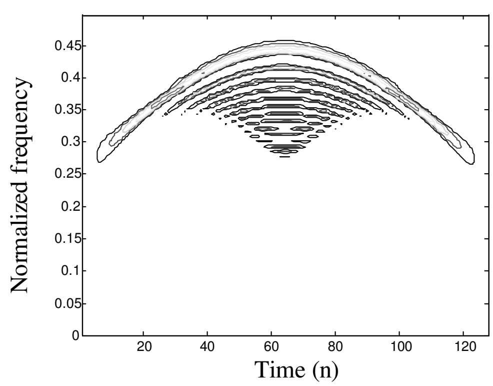


Fig. 2.4. Illustration of cross terms in a monocomponent signal

The cross terms do not actually represent the signal's energy and are hence undesirable. The structure of cross terms has been investigated and well understood by many researchers (Bikadash *et al*, 1993). Suppose the two auto terms are separated in time by Δt and in frequency by Δf, as shown in Fig. 2.5, there will be a cross term centered

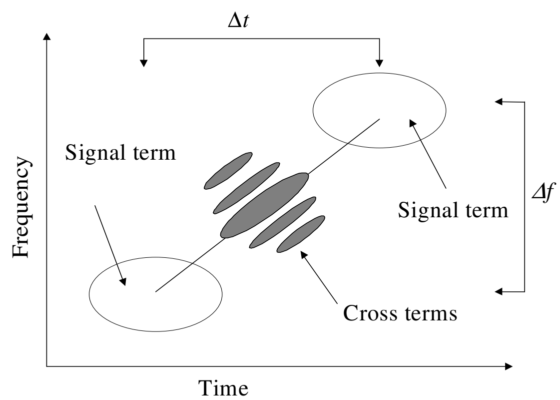


Fig. 2.5. The interference geometry in the time-frequency plane

between the two auto terms in the time-frequency plane and it oscillates in the time direction with a rate Δf and in the frequency direction with a Δt rate. Later, we will see how we can employ kernel as a filtering means to suppress these cross terms at the expense of the degradation in auto term resolution. Inspite of these spurious cross terms, the Wigner distribution has been often employed because of its capability to resolve multicomponent signals that are time-frequency disjoint. Of all the quadratic representations, Wigner distribution alone attains simultaneous resolution in time and frequency. Besides, it satisfies most of the properties that a TFD has to satisfy. Some distributions belonging to Cohen's class are tabulated in Table 2.1. A more detailed discussion can be found in (Hlawatsch *et al*,1992a).

Table 2.1: Different time-frequency distributions belonging to the Cohen's class

S. No.	Time-Frequency Distribution	$\phi(\theta,\tau)$	$C_s(t,f)$
1	Page distribution	$e^{-j\pi\lvert\tau\rvert\theta}$	$2\,\mathrm{Re}\left\{s^*(t)\,e^{j2\pi ft}\int_{-\infty}^{t} s(t')e^{-j2\pi ft'}dt'\right\}$
2	Levin distribution	$e^{-j\pi\lvert\tau\rvert\theta}$	$2\,\mathrm{Re}\left\{s^*(t)\,e^{j2\pi ft}\int_{t}^{\infty} s(t')\,e^{-j2\pi ft'}dt'\right\}$
3	Wigner distribution	1	$\int_\tau s\left(t+\tau/2\right)s^*\left(t-\tau/2\right)e^{-j2\pi f\tau}d\tau$
4	Choi-Williams Distribution	$exp(-\tau^2\theta^2/\sigma)$	$\int\int\phi(\theta,\tau)A_s(\theta,\tau)e^{j2\pi(tv-f\tau)}d\tau d\theta$ here $A_s(\theta,\tau)$ is the Ambiguity function.
5	Spectrogram	$\int h^*(v-\frac{\tau}{2})h(v+\frac{\tau}{2})$ $exp(-j\theta v)\,dv$ where h is the window function	$\left\lvert\int e^{-j2\pi f}s(\tau)h(\tau-t)d\tau\right\rvert^2$
6	Rihaczek Distribution	$e^{-j\pi\tau\theta}$	$\int_\tau x(t+\tau)\,x^*(t)\,e^{-j2\pi f\tau}$
7	Generalized Exponential Distribution	$exp\left(-\left(\frac{\tau}{\tau_0}\right)^{2M}\left(\frac{\theta}{\theta_0}\right)^{2N}\right)$	$\int\int\phi(\theta,\tau)A_s(\theta,\tau)e^{j2\pi(t\theta-f\tau)}d\tau d\theta$ here $A_s(\theta,\tau)$ is the Ambiguity function.
8	Butterworth Distribution	$\frac{1}{1+\left(\frac{\tau}{\tau_0}\right)^{2M}\left(\frac{\theta}{\theta_0}\right)^{2N}}$	$\int\int\phi(\theta,\tau)A_s(\theta,\tau)e^{j2\pi(t\theta-f\tau)}d\tau d\theta$ here $A_s(\theta,\tau)$ is the Ambiguity function.

2.2.3. Choi-Williams Distribution

The inherently associated cross terms in all bilinear distributions are of major concern in spectral estimation and in multicomponent signal separation. By looking at the kernel as filtering function in the ambiguity function (AF) domain, there have been many kernels which reduce these cross terms. However, kernels constrained to construct distributions of desired properties are of utmost importance. In the AF domain, the locus of the auto terms falls around the origin while those of cross terms lies away from origin. Choi and Williams have identified this property of the AF and have chosen a kernel that has larger weights in the vicinity of the auto terms and smaller weights farther away from the origin in the AF domain, given as: $\phi(\theta,\tau) = e^{-\frac{\theta^2\tau^2}{\sigma}}$

The CWD can be expressed as (Choi *et al*, 1989):

$$CWD(t,f) = \int e^{-j2\pi f\tau} \int \frac{1}{\sqrt{4\pi\tau^2/\sigma}}\, s(u+\frac{\tau}{2})\, s^*(u-\frac{\tau}{2})\, e^{-\frac{(u-t)^2}{4\tau^2/\sigma}}\, du\, d\tau \qquad (2.8)$$

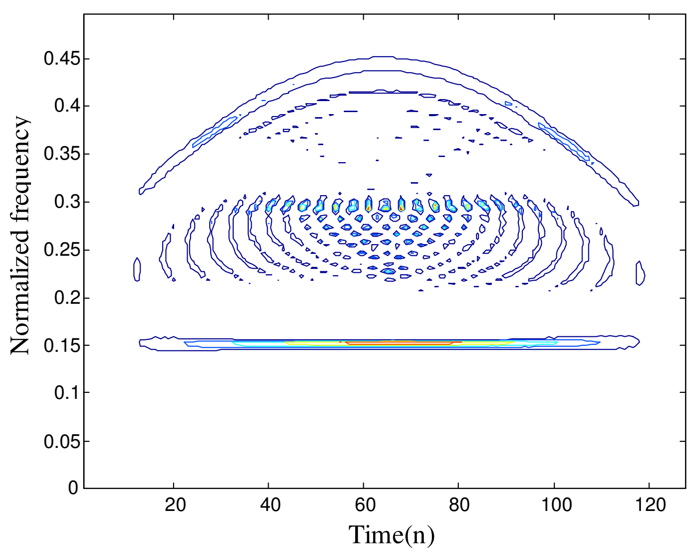
The capability of CWD in suppressing the cross terms without much degradation in auto term can be controlled by a proper choice of the kernel spread (i.e., σ). A resolution comparison of several distributions and a detailed discussion of CWD can be found in (Jones *et al*, 1992a).

2.2.4. Spectrogram

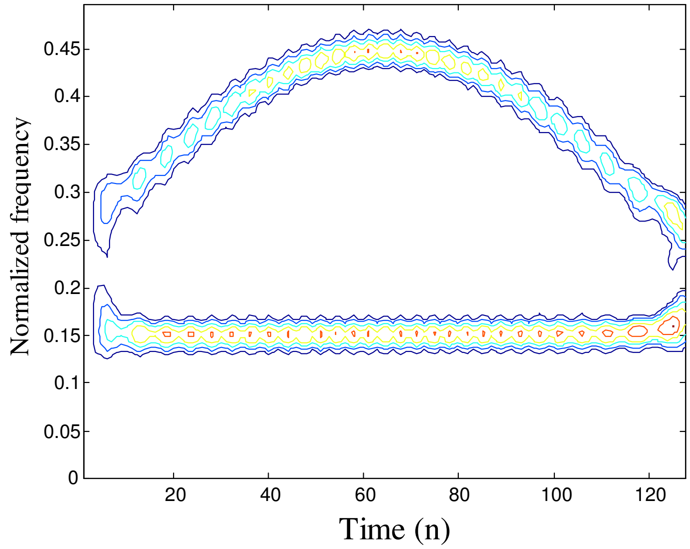
The classical definition of spectrogram can be considered as the squared magnitude of the short-time Fourier transform (Nawab *et al*, 1988). However, spectrogram can be considered as a member of Cohen's class with the kernel being the ambiguity function of the analysis window, and it is given by:

$$S(t,\omega;h) = \left| \int s(\tau)h(t-\tau)e^{-j\omega\tau}d\tau \right|^2. \tag{2.9}$$

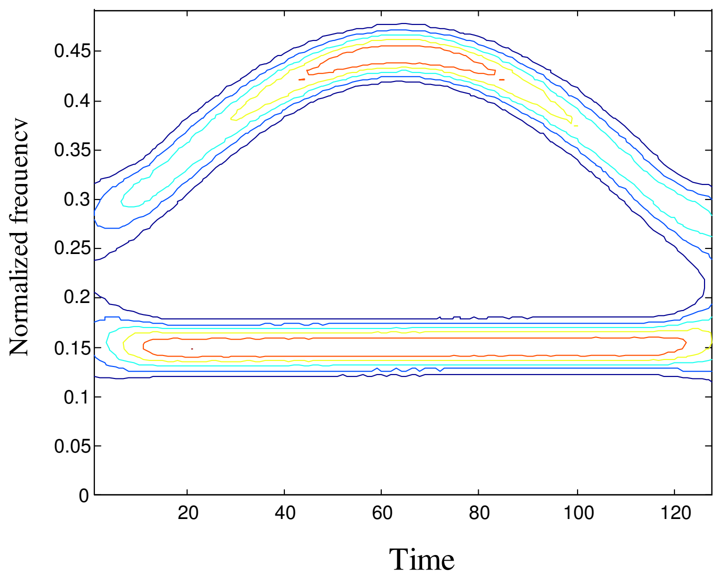
The properties of the spectrogram obviously change with a change in the window function. The interesting property of the spectrogram is that, unlike other bilinear TFDs, it is always nonnegative. Apparently it seems that the spectrogram does not suffer from cross terms, but strictly speaking the cross terms in this representation exactly fall in the auto term region and interfere with them. Hence, spectrogram can be considered as a smoothened version of the WVD to suppress cross terms and as a natural consequence the auto term resolution reduces. A comparison of CWD, WVD and spectrogram for a synthetic signal in their cross term suppression and auto term resolution is shown in Fig. 2.6.



(a)



(b)



(c)

Fig. 2.6. Resolution and cross terms comparison of (a) Wigner distribution, (b) Choi-Williams distribution and (c) Spectrogram of multicomponent signal

2.2.5. Properties of TFDs in Cohen's Class

As indicated earlier, the properties of the distributions constructed in the Cohen's class reflect as constraints on the kernel. We look at the kernel's behavior to get the desired property. We briefly mention some properties now. Many other properties like finite time support, strong time support, inversion and realizability have been discussed with proofs in (Giridhar, 1998).

a) Marginals: Instantaneous Energy and Energy Density Spectrum

Integrating the distribution along one axis gives the energy density in the other domain. For the time marginal to give instantaneous energy the kernel must be constrained as:

$\phi(\theta,0) = 1$ and for the frequency marginal to give the energy density spectrum $\phi(0,\tau) = 1$.

b) Total Energy

If the marginals are given, then the total energy will be the energy of the signal. Evaluating the integral in the expression for distribution with respect to time and frequency shows that for the total energy to be preserved, the kernel should satisfy: $\phi(0,0) = 1$.

c) Uncertainty Principle

Any joint distribution that satisfies the marginals will yield the uncertainty principle. Thus the condition for the uncertainty principle is that both marginals must be correctly given.

d) Reality

Since time-frequency distributions are usually considered to be energy distributions, they should be real and positive. For the distribution to be real, the kernel has to satisfy the constraint:

$$\phi(\theta,\tau) = \phi^*(-\theta,-\tau).$$

e) Positivity

The constraint on the kernel is difficult to evaluate and the characteristic function approach is used to check for positivity. In general, we are interested in distributions satisfying marginals. Wigner had shown that distributions which simultaneously satisfy marginals and positivity cannot exist. Loughlin-Pitton-Atlas have devised a scheme to construct positive distributions. Eventhough one is interested in distribution with marginals, it would be more appealing for an energy function to be positive-valued. Unfortunately such is not the case.

f) Time and Frequency Shifts

If we translate a signal shifted in time by t_0, we expect the distribution to be translated in time by the same amount. This can happen only if the kernel is time independent. Similarly, if the kernel is frequency independent the distribution would be shift invariant in frequency.

That is, $\phi(\theta,\tau\ ;\ t,\omega) = \phi(\theta,\tau)$.

g) Scale Invariance

If a signal is linearly scaled, then the spectrum is inversely scaled. Hence, when a signal $s(t)$ is scale d by a factor of a, the requirement on the distribution is that:

$$C_{sc}(t,\omega) = C(at,\ \omega / a) \quad \text{for } s_{sc}(t) = \sqrt{\mathrm{a}}\ s(at). \tag{2.10}$$