CHAPTER 3

REVIEW OF SIGNAL SYNTHESIS ALGORITHMS AND IMPLEMENTATION ASPECTS OF TFDs

3.1. INTRODUCTION

In this chapter we review various signal synthesis algorithms based on short-time Fourier transform (STFT), Wigner-Ville distribution (WVD) and linear signal space. Later, we consider the discretization issues of the time-frequency distributions. The aliasing problems in bilinear representations, the definitions of generalized time-frequency distribution (TFDs) and the computational aspects of WVD and other TFDs are also considered.

3.2. SIGNAL SYNTHESIS ALGORITHMS

Signal synthesis is concerned with the estimation of a signal whose time-frequency characteristics are in the desired fashion. The advantage of specifying the behavior of the signal to be synthesized in time-frequency plane is that its characteristics can be time-variant or nonstationary, which is not available either in the time or frequency-domain representation. This additional information motivates us to process the signal in the joint time-frequency domain. The classical method of signal synthesis is the STFT approach, since the description of a signal jointly in time and frequency can be better understood by Fourier transforming the short segments of the signal. However, TFDs that offer better resolution than STFT can give flexibility in designing the time-variant filters because of some interesting properties like marginals, instantaneous energy, etc. Review of some of these approaches will follow.

3.2.1. STFT-Based Synthesis

The concepts of STFT analysis and synthesis have been widely used in analyzing and modeling of quasi-stationary signals, such as speech. The viewpoint can either be thought of as a filter-bank model or by a block-by-block analysis. In the filter bank model, the input signal is filtered by a bank of band-pass filters, which span the frequency range of interest (Crochiere, 1980). In the synthesis procedure, a signal can be reconstructed from its STFT spectra by summing the outputs of the band-pass filters. This method of analysis and synthesis is referred to as the filter-bank summation method (FBS). The FBS method is motivated by the following relation between a sequence and its STFT:

$$\hat{x}(n) = \left[\frac{1}{N\,w(0)}\right]\sum_{k=0}^{N-1} X(n,k)\,e^{j\frac{2\pi nk}{N}} \quad , \tag{3.1}$$

where, without loss of generality, we assume that *w(0)* is non-zero. The FBS method is shown in Fig. 3.1. Perfect reconstruction is possible by constraining the analysis window as:

$$w(n)N\sum_{r=-\infty}^{\infty}\partial(n-rN) = Nw(0)\partial(n). \tag{3.2}$$

This will be clearly satisfied for any causal analysis window whose length is less than the number of analysis filters *N*. The above equation is often referred to as the FBS constraint because this requirement of the analysis window ensures the exact signal recovery.

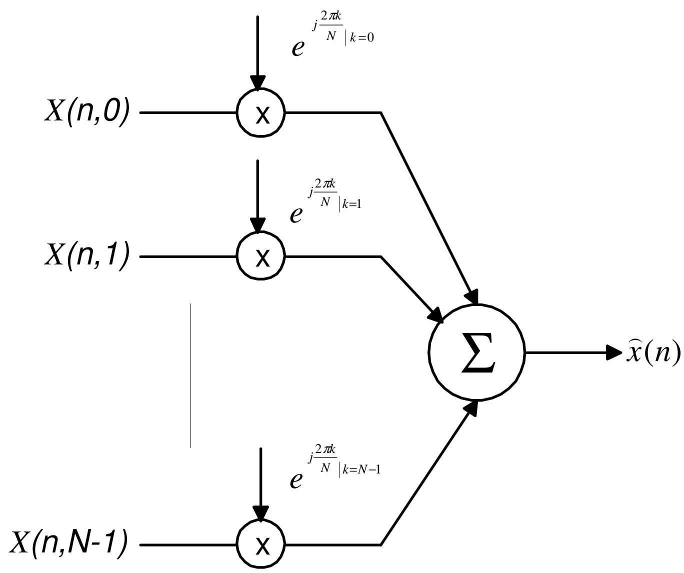


Fig. 3.1. Filter bank summation method of synthesis

Another classical approach, the overlap-and-add method (OLA), synthesizes signals from its STFT spectra by inverse transforming the STFT spectra to recover the short-time segments of the signal in time (Allen *et al*, 1977). These overlapped signal segments are then appropriately summed to reproduce the time signal. Just as the FBS method has been motivated from the filtering viewpoint of that STFT, the OLA method is motivated from the Fourier transform view point of the STFT and can be understood by the following relation:

$$\hat{x}(n) = \left[\frac{1}{W(0)}\right]\sum_{p=-\infty}^{\infty}\frac{1}{N}\left[\sum_{k=0}^{N-1}X(p,k)e^{j\frac{2\pi kn}{N}}\right], \tag{3.3}$$

where $W(0) = \sum_{n=-\infty}^{\infty} w(n)$.

In the OLA method, we take the inverse discrete Fourier transform (IDFT) for each fixed time in the STFT. However, instead of dividing out the analysis window from each of the resulting short-time series sections, we perform an overlap-and-add operation between the short-time sections. This method works provided the analysis window is designed such that the sections overlap-and-add operation effectively eliminates the analysis window from the synthesized sequence. A unique representation of the signal by the magnitude of its STFT under restrictions on the signal and the analysis window of the STFT is considered in (Nawab *et al*, 1983). When a signal's STFT is modified in a time variant or invariant fashion, the resulting spectra may not be a valid STFT. In such cases, the signal is approximated to the modified STFT in a least-squares sense. The necessary and sufficient conditions on the analysis filter under which perfect reconstruction of the input signal is possible have been considered in (Dembo *et al*, 1988). They have also dealt with the synthesis of an optimal signal (optimality criteria being the minimum mean-squared error) from the modified spectra.

3.2.2. WVD-Based Synthesis

WVD has received considerable attention in the recent years as an analysis tool for nonstationary or time-varying signals. Besides this, it satisfies most of the properties of the Cohen's class, which makes it very attractive for time-frequency analysis. Hence, WVD is considered to be a better representation in the time-frequency plane and often the signals are modified in the WVD domain. The signal synthesis problem from this modified or specified WVD was first formulated and studied by Boudrex-Bartels and Parks using the least-squares procedure (Boudreaux *et al*, 1986). It involves an approximation of the modified distribution, which makes use of the even or odd-indexed samples only. They included applications in time-varying filtering and signal separation. Another application mentioned has been an algorithm that can be used to reduce the group delay of digital filters and the design of analysis windows for the STFT. The outer-product approximation (OPA) is based on the fact that WVD representation is intimately tied to the outer-product operation (Yu *et al*, 1987). The OPA synthesizes the signal in segments in order to remove any length restriction. The synthesis procedure involves an approximation of a 2-D function as a product of two 1-D functions. However, the adequate combination of the synthesized signals seems to be a problem. The OPA interpretation is important when the testing of a time-frequency function as a valid WVD has to be investigated. The "overlapping method" (OM) of signal synthesis has a recursive structure and has no constraint on the length of the signal unlike the OPA method. Another suboptimal approach similar to OM has been considered in (Krattenthaler *et al*, 1991). The basis function approximation (BFA) approach involves expressing a time-frequency function as a bilinear combination of basis and cross-WD. A least-squares approximation (LSA) leads to an eigenvalue-eigenvector decomposition of a symmetric matrix. It has been observed that there are two significant eigen values that correspond to the even and the odd -indexed sequences. If the modified or specified WVD is a valid representation, then the above methods can be used for the exact signal recovery. If the modified or specified WVD is not a valid representation, then the LSA and the BFA involve least-squares estimation.

3.2.3. Bilinear Signal Synthesis

Any bilinear signal representation (BSR) is given by

$$T_{x,y}(\sigma,\varepsilon) = \iint u_T(\sigma,\varepsilon;t_1,t_2)\, q_{x,y}(t_1,t_2)\, dt_1\, dt_2\,, \tag{3.4}$$

where $q_{x,y}(t_1,t_2) = x(t_1)\, y^*(t_2)$ is the outer product of the individual signals $x(t)$ and $y(t)$, and $u_T(\sigma,\varepsilon;t_1,t_2)$ is a kernel function specifying the BSR. $T_{x,y}(\sigma,\varepsilon)$ is the cross BSR of the two individual signals $x(t)$ and $y(t)$; the corresponding auto BSR is then defined as $T_{x,x}(\sigma,\varepsilon)$. The generalized synthesis problem can be stated as follows (Hlawatsch, 1992b):

Let T be a given modified / specified BSR and $\widehat{T}$ be the model function. The objective is to find a signal $s(t)$ such that the model function $\widehat{T}$ approximates the specified/ modified BSR. The model function, in general, will not be the exact BSR unless the specified BSR is a valid representation and in which case it is natural to look for the model function $\widehat{T}$ in the sense that it minimizes the synthesis error $\varepsilon_x = \underset{x}{arg\,min} \left\|\widehat{T} - T_x\right\|$, where

$$\varepsilon_x^2 = \left\|\widehat{T} - T_x\right\|^2 \triangleq \iint_{\sigma\,\varepsilon} \left|\widehat{T}(\sigma,\varepsilon) - T(\sigma,\varepsilon)\right|^2 d\sigma\, d\varepsilon\,. \tag{3.5}$$

The WVD-based synthesis is a special case of the above stated problem, where BSR is replaced by WVD. A constrained synthesis problem in the framework of BSR has been considered in (Hlawatsch, 1993), which encompasses the WVD and the ambiguity function as special cases. Two methods based on the orthogonal projection operators and the orthogonal bases to characterize the signal space have been developed. It has been shown that constraining the signal space prior to synthesis gives a greater flexibility in designing a time-variant filter with desirable properties.

The design of a linear, time-invariant filter with a specific pass region was considered in (Hlawatsch, 1994). By defining a time-frequency subspace, a unified approach to the time-frequency (TF) signal expansion and TF filtering has been proposed. It was further shown that the optimal TF space is the eigen space of the TF region. Time-frequency projection filters derived by projecting the TF region onto the TF subspace and by a linear combination of the signals in the subspace can be used to synthesize the signal. The computational aspects have been also given .

3.3. IMPLEMENTATION OF TFDs

In this Section, we consider the implementation aspects of the time-frequency representations. As any TFD in the Affine class or the Cohen's class can be expressed as smoothening of WVD, the transition from continuous to discrete cases will be explained based on WVD. We review the interpretation of the Cohen's class as smoothening of the WVD that justifies this approach. However, the definition of the discrete case is not evident and a lot of investigation is going on to define the continuous counterparts that will preserve most of the properties as well as simple from the implementation point of view. The generalization of the discrete Cohen's class can then be dealt with in a similar fashion as is done for the WVD case.

3.3.1. Aliasing in the TF Plane

We define the following terms to understand and interpret the problem of aliasing:

a) Ambiguity function (AF):

$$A_x(\theta,\tau) = \int x(t+\frac{\tau}{2})\, x^*(t-\frac{\tau}{2})\, e^{-j\theta t}\, dt. \tag{3.6}$$

b) Temporal autocorrelation function (TACF):

$$R_x(t,\tau) = x(t+\frac{\tau}{2})\, x^*(t-\frac{\tau}{2}). \tag{3.7}$$

c) Spectral autocorrelation function (SACF):

$$R_x(\theta,\omega) = X(\omega+\frac{\theta}{2})\, X^*(\omega-\frac{\theta}{2}). \tag{3.8}$$

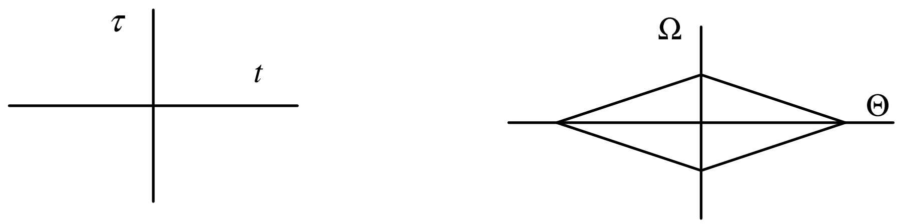
Discretization of the continuous time WVD requires sampling the TACF. However, by recalling Eqn. (3.7), the existence of noninteger samples makes the sampling process ambiguous and has pulled much of the controversy (Bergmann, 1991). Claasen and Mecklenbrauker have proposed a scheme to compute the discrete time WVD. The definition of WVD for a sampled signal is (Claasen, 1980a):

$$W_x(n,\omega) = 2\sum_{k=-\infty}^{\infty} x(n+k)\, x^*(n-k)\, e^{-j2\omega k}. \tag{3.9}$$

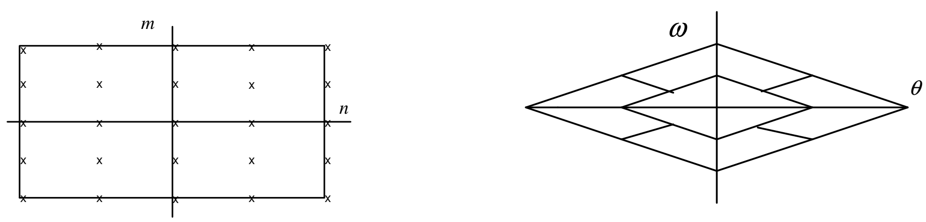
It can be easily seen from the above equation that the resulting spectrum is periodic in ω with period π. On the other hand, the spectrum $X(\omega)$ of the signal $x(n)$ is periodic in ω with 2π. If the analog signal is sampled at the Nyquist rate, then it appears that the contributions in the range $\pi/2 < |\omega| < \pi$ of the signal's spectrum is folded in the range $|\omega| < \pi/2$ in the WVD spectrum. Therefore, distortion can be avoided only by oversampling the signal atleast by a factor of two (Claasen *et al*, 1980b). The aliasing is caused by the fact that only the even or odd samples are used in computing the distribution. Information is thereby lost, unless the odd or even samples fully describe the signal or the signal is oversampled. The utilization of the information provided by the Nyquist sampled signal has motivated further research in this direction.

3.3.2. Sampling the TACF

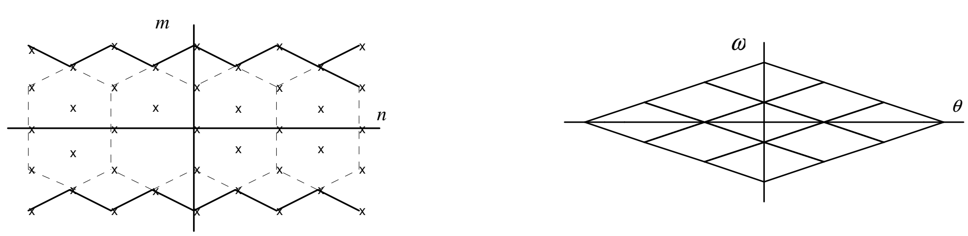
Various techniques exist for sampling the TACF that finally lead to discretization of the underlying distribution. To compute samples of the TACF, one must work with the samples of the underlying continuous signal. From this signal, there are two common ways of computing it. The first method is called the "half outer-product" sampling scheme. The "Half outer-product" sampling requires only even or odd-indexed samples to compute the TACF (O'Neill, 1997). It is defined for: $R_x^h\ (n,\ m)\ \forall\ n\ Z\ \ \ and\ \ even\ m.$ The "full outer product" scheme is defined for: $R_x^f\,(n,m)\ \forall\ \ n \pm \frac{m}{2}\ \ \in\ Z\ and\ \ m\ \in\ Z$. As is evident from the definition, this occurs for $n\ \in\ Z\ and\ m\ even$ and also for $n\ \in\ Z + \frac{1}{2}\ and\ m\ odd$. Hence, it uses all the available samples and results in a nonaliased SACF. The continuous case of TACF, "half outer-product" sampling, "full outer-product "and "double outer-product" sampling schemes for TACF and the spectral repetition in SACF domain are all depicted in Fig. 3.2. All the definitions for generalizing the discrete versions of the Cohen's class in one way or the other use these sampling schemes. The generalized TFDs (GTFDs) can now be computed in the same way as is done for the WVD case. This approach was followed to compute the discrete versions which was first investigated by Boashash and Black (Boashash *et al*, 1987). He used the "half outer product" sampling scheme on an over sampled signal.



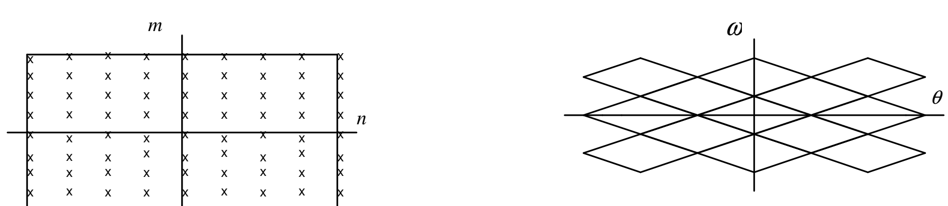
(a) Continuous case



(b) Half outer product sampling



(c) Full outer product sampling



(d) Double outer product sampling

Fig. 3.2 Different sampling schemes for the TACF and the resulting repetitions in the

SACF. On the left are the pictorial representations of the continuous, half, full and double outer product sampling schemes, and on the right are the resulting SACFs

The GTFDs can be represented by:

$$C_x(n,\omega) = 2\int_0^{2\pi} e^{-\theta t}\, e^{j\theta u} \sum_{u=-\infty}^{\infty} \sum_{\tau=-\infty}^{\infty} x(u+\tau)\, x^*(u-\tau)\, \phi(\theta,2\tau)\, e^{-j2\omega k}\, d\theta\,. \quad (3.10)$$

It is shown in (Morris *et al*, 1996) that even the TACF computed from the oversampled signal causes the GTFD to have a mirror spectra distanced by π; this requires pre-processing of the signal for identifying the "true" time-varying spectrum of the given sequence. This has motivated Jeong *et al* to consider a more appropriate definition that resembles the continuous Cohen's class called the alias-free GTFDs (AF-GTFDs) using the "full outer-product" sampling (Jeong *et al*, 1992b). However, the properties of either GTFDs or AF-GTFDs are different from the Cohen's class. A detailed analysis of the properties of the AF-GTFDs can be found in (O'Neil, 1997).

3.4. COMPUTATIONAL ASPECTS

In this Section, we consider the computation of different distributions in the Cohen's class. The different interpretations of the GTFDs, e.g., smoothening of the WVD, two-dimensional Fourier transform of the generalized autocorrelation function (GACF), Fourier transform of the two-dimensional convolution of kernel, TACF, etc., lead to different approaches. Very little effort has been made to specific distributions, except the

Choi-Williams distribution (CWD) and the WVD. Hence, we concentrate on computing the WVD, the CWD and the different approaches to computing the GTFDs.

3.4.1. Computation of WVD

Because of the optimal time-frequency concentration in the joint plane for analytic and quadratic signals, WVD has been considered to be the most reliable estimate of the "true spectrum". Since the sampling of WVD in the discrete case leads to a generalized definition of the GTFDs, it is natural to expect that the implementation issues can, in general, be carried to GTFDs as well, particularly the data flow. Boashash has first addressed the problem of computing the WVD with the fast Fourier transform (FFT) technique that we now consider (Boashash *et al*, 1987).

The sampled WVD can be expressed as:

$$W_x(n,\omega) \;=\; 2\,T \sum_{k=-\infty}^{\infty} x(n+k)\, x^*(n-k)\, w(n+k)\, w^*(n-k) e^{-j2\omega k}\,, \tag{3.11}$$

where w is a window function of length $2L+1$ that satisfies $w(n)=0\ for\,|n|>L$.

From the above equation, it is evident that

$$W_x(n,\omega) \;=\; W_x(n,\omega+\pi). \tag{3.12}$$

Hence, the WVD will be periodic in π. If the signal were real, this would imply that to avoid aliasing, sampling rate constraint should be

$$f_s \geq 4B,$$

where B is the bandwidth of the signal.

In this case, we must transform the real signal into analytic signal *z(n)* corresponding to *x(n),* and is defined in the time domain as:

$$z(n) = x(n) + jH[x(n)] \tag{3.13}$$

where *H[.]* represents the Hilbert transform. Further discussion on computing the analytic signal can be found in (Reilly *et al*, 1994). If the signal is replaced by its analytic part, the periodicity is unchanged, however, the absence of negative frequencies does not cause aliasing which would occur if the signal were sampled at the Nyquist rate. Hence, the sampling rate constraint becomes the well-known Nyquist rate as:

$$f_s \geq 2B.$$

To apply the FFT technique the frequency variable must be sampled. In his work, Boashash has ignored the scaling of twiddle factor by a factor of two. Claasen and Mecklenbrauker have noted that since the spectra is periodic in π, rather than sampling the frequency axis in the range $[0 - 2\pi]$, sampling the spectra in the interval $[0 - \pi]$ is sufficient which results in the kernel equivalent to the FFT kernel and can be computed without any modifications or post-processing (Claasen *et al*, 180a). It can be shown that:

$$W_x(n, \frac{m\pi}{N}) = 2T \sum_{k=-\infty}^{\infty} x(n+k)\, x^*(n-k)\, w(n+k)\, w^*(n-k) e^{-j\frac{2mk}{N}}. \tag{3.14}$$

Standard Fourier transform techniques require the time sequence to be indexed from *0* to *N-1*. Some pre-processing is required to re-arrange the data so that the above equation can be used for applying the FFT techniques. An efficient way of computing the above equation by cosine part and sine part of the DFT was considered in (Pei *et al*, 1992). We follow this sampling scheme in Chapter 6, when we deal with the implementation issues of GTFDs.

3.4.2. Computation of CWD

The distribution of interest in the Cohen's class other than WVD is the Choi-Williams distribution. Because of its variable trade-offs in auto term resolution to cross term suppression, CWD has been widely used in many applications, the example being in moving target detection in radar signal processing (Giridhar, 1998). Since the kernel is known specifically, instead of computing Eqn. (3.14) using a brute force technique, integrating the expression with respect to θ reduces to a double summation. This can be expressed mathematically using the kernel $\phi(\theta,\tau) = e^{-\frac{\theta^2\tau^2}{\sigma}}$ as:

$$CW_x(n,\omega) = 2\sum_{\tau=-\infty}^{\infty} e^{-j2\omega\tau} \sum_{u-\infty}^{\infty} \frac{1}{\sqrt{4\pi\tau^2/\sigma}} e^{-\frac{(u-n)^2\sigma}{4\tau^2}} x(u+\tau)x^*(u-\tau). \qquad (3.15)$$

The method presented in (Barry, 1992) uses matrix calculation to compute the above equation. Instead of computing Eqn. (3.15) directly over all indices, pre-processing of the data to make it a summation over non-zero sampling points greatly reduces the number of computations. He have observed that any time frequency representation can be calculated by performing a weighed element mapping over the outer product matrix and then Fourier transforming the resulting function of τ. The weighed matrix elements are determined by the kernel function. However, it requires the regular butterfly structure which in turn requires global communication and the order of complexity is directly proportional to the signal length, i.e., number of multipliers in the pre-processing stage earlier to the FFT stage directly depends on the signal length. We will follow the data alignment in terms of matrix in our subsequent discussion on implementing GTFDs using the time-recursive approach outlined in Chapter 6.

3.4.3. Generalized Autocorrelation Function Approach

There exist various schemes for computing the generalized TFDs. This is followed from the interpretation of the Cohen's class in various ways. For example, the Cohen's class can be expressed in terms of AF as:

$$C_x(t,\omega;\phi) = \iint A_x(\theta,\tau)\, \phi(\theta,\tau)\, e^{-j\tau\omega}\, e^{j\theta t}\; d\tau\; d\theta, \tag{3.16}$$

and in terms of WVD as:

$$C_x(t,\omega;\phi) = \iint W_x(t',\omega')\, \psi(t-t',\omega-\omega')\; d\tau'\; d\omega', \tag{3.17}$$

where $\psi(t,\omega) = \iint \phi(\theta,\tau)\, e^{-j\tau\omega}\, e^{j\theta t}\; d\tau\; d\theta$.

The most attractive method of computation stems from the Fourier transform viewpoint of the GTFDs. If we define the generalized ACF as:

$$R_x(t,\tau) = \int_{-\pi}^{\pi}\int x(u+\frac{\tau}{2})\, x^*(u-\frac{\tau}{2})\, \phi(\theta,\tau)\, e^{j\theta u}\, e^{-j\theta t} du\; d\theta\ , \tag{3.18}$$

then it is evident that any time-frequency distribution can be represented as a Fourier transform of the generalized ACF, given by:

$$C_x(t,\omega) = \int_{-\infty}^{\infty} R_x(t,\tau)\, e^{-j\omega\tau}\, d\tau . \tag{3.18}$$

However, when the kernel is real and symmetric or conjugate symmetric, then the generalized ACF would be real-valued. Morris and Quan have exploited this fact to compute the real-valued TFDs, especially WVD, CWD and spectrogram (Quian *et al*, 1990). The direct evaluation of the FFT over the generalized ACF does not take into account the redundancy in calculations. Hence, they computed the discrete Fourier transform (DFT) using the discrete sine transform (DST) and the discrete cosine transform (DCT) modules to obtain the resulting spectrum. The CWD can be computed using this approach, while the GACF can be obtained by the method discussed in the previous Section.

3.4.4. Spectrogram Decomposition

A new interpretation of the GTFDs as a weighed sum of spectrograms was provided by (Cunningham *et al*, 1994a). Spectrogram decomposition and weighed reversal correlator decomposition (WRCD) were investigated in the context of linear signal synthesis algorithms They extended the concept to represent arbitrary TFD as a weighed combination of spectrograms with orthonormal windows. Based on this fact the spectral representation of the linear operator $\breve{\psi}$, associated with each real-valued TFD, they have represented arbitrary TFD as a weighed combination of the spectrograms. The TFDs were analyzed and the resolution comparisons of different TFDs were made based on the above approximations. Since the spectrogram is a nonnegative distribution, while representing a negative distribution, the weights of the corresponding spectrograms will be negative. In such a case, the interpretation of the linear operator becomes hard to understand. WRCD has been derived to represent high-resolution TFDs that are in general negative. A fast algorithm that mainly works on the above principle that any arbitrary TFDs can be expressed as a weighed combination of spectrograms was proposed in (Cunnigham *et al*, 1994b). It is claimed that the parallel stages of the spectrogram implementation blocks lead to faster implementation.