The signal can be reconstructed from any one of these axes since the synthesis is not unique. This redundency in signal representation can be used to isolate components and precisely represent them. For example, let a dictionary *A* contain the words *go, and* and *bring,* and let another dictionary *B* be redundant, containing the words *go, and, bring* and *fetc*h. If we use *A* to form a sentence using *go and bring*, we require three words. Where as if use B, the same idea can be conveyed with either three words or one word *fetch*. Hence, we can convey the meaning more precisely with dictionary *B,* whereas it is possible only in one way with dictionary *A*.

4.5. APPLICATIONS

Signal analysis in the transform domain is of importance, since it paves the way for application of various spectral estimation techniques and the interpretation of the signal in the transform domain leads to different applications. Spectral estimation based on WVD and optimized STFT are considered and system identification that demonstrates the chirp transform as a denoising tool will also be considered.

4.5.1. Spectral Estimation

Spectral estimation is concerned with estimating the true spectrum of the desired component of the signal in a multicomponent scenario or under noisy conditions. The objective of any TFR to concentrate the energy along the instantaneous frequency cannot be attained unless some processing of the signal is done. In order to estimate the true spectrum of a signal, we can utilize the information in the chirp transform to isolate individual components and then apply any TFD to the separated component. Since the transformation is redundant, it is extremely difficult to separate the individual components at the same time. We need to look for either a local maxima or a global maxima to detect that component of the signal which generates it. Then, the component is estimated by the synthesis procedure and subtracted from the signal. Now the effect of the isolated component is eliminated and the transform gives a less complex representation. This divide and conquer approach, similar to the idea behind matching pursuits, can be continued until all the components are identified or the residue left out does not give rise to a local maxima, which implies that the residue does not match with the expansion set and cannot be represented precisely. To demonstrate this idea, we take up analyzing the test given in (Jones *et al*, 1994a) that consists of two impulses, two rectangular gated sinusoids of different frequencies and a Gaussian signal. We add two more components: two parallel Gausssian windowed chirps with one component intersecting in time and frequency with that of the gated sinusoid of higher frequency. The test signal used and its WVD are shown in Fig. 4.5.

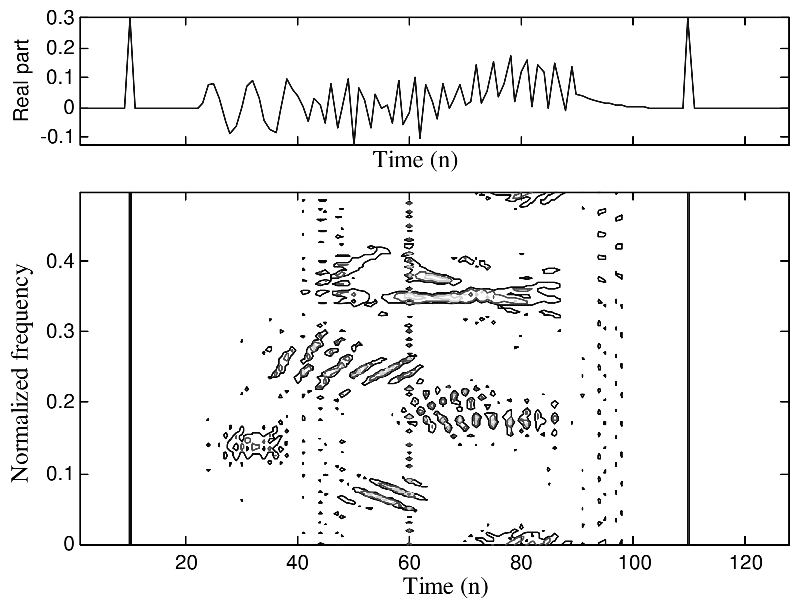

Fig. 4.5. A test signal consisting of two impulses, two rectangular windowed sinusoids, two Gaussian windowed chirps and a Gaussian signal and the WVD of the test signal

Apparently it is visible that signal has many components and separation into individual components is difficult using conventional techniques. The magnitude of the chirp transform of the test signal at two local maxima ($\beta = 0.01, 1$) and at $\beta = 20$ is shown in Fig. 4.6.

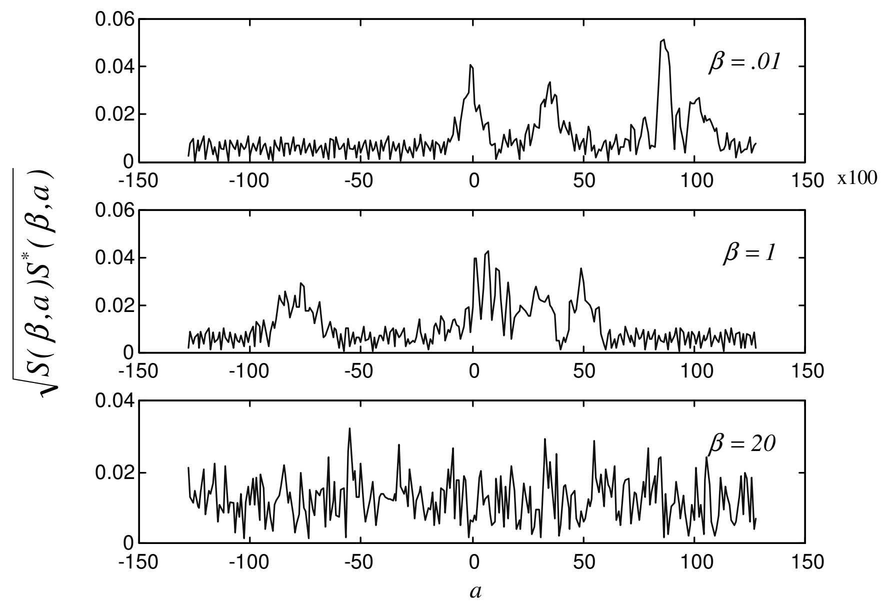

Fig. 4.6. Chirp transform of the test signal

The components are clearly visualized in the transform domain and can be identified by windowing. We can estimate the component we are interested in by using these modified coefficients along the axis corresponding to the component and by substituting in the synthesis equation. The Gaussian signal (orthogonal along $\beta = 0.01$) estimated by windowing the transform coefficients at $\beta = 0.01$ is shown in Fig. 4.7, together with the test signal. The magnitude of the transform, after the estimated signal has been removed, is depicted in Fig. 4.8.

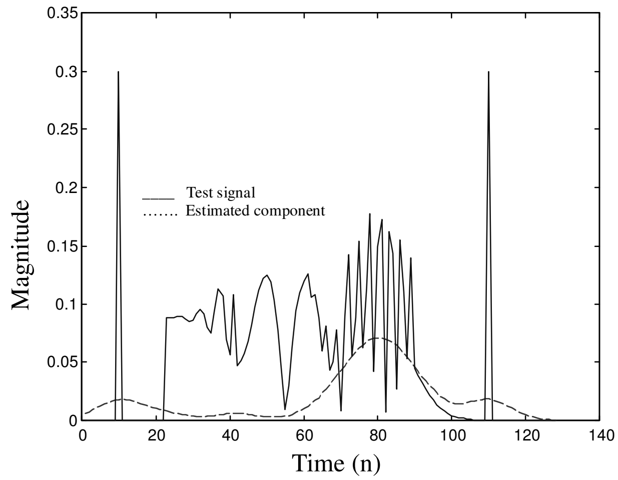

Fig. 4.7. Magnitude of the test signal and the estimated component

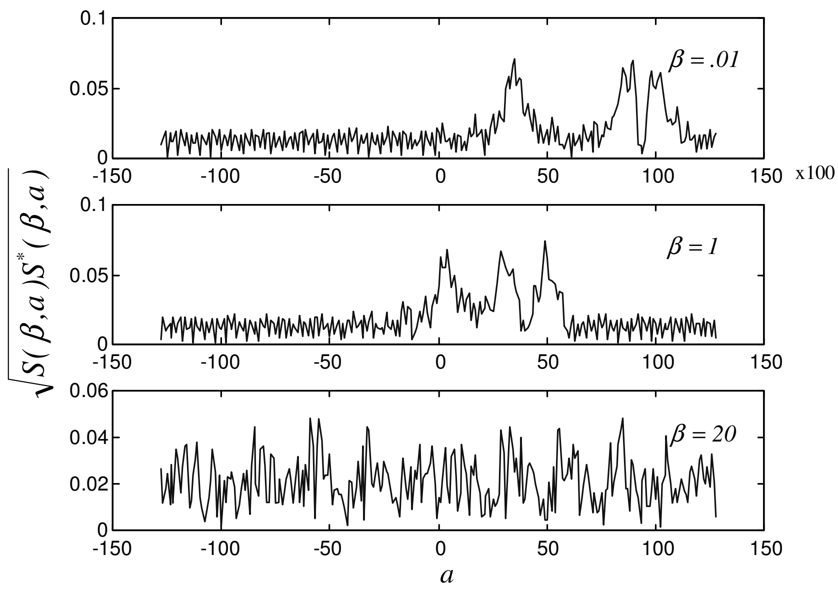

Fig. 4.8. Chirp transform after the estimated component is removed

It can be clearly observed from the above figure that the eliminated component simplifies the analysis procedure. The processed signal is again transformed and the same procedure is repeated until all the components are identified. After four iterations, the magnitude of the transform along the same axis is depicted in Fig. 4.9.

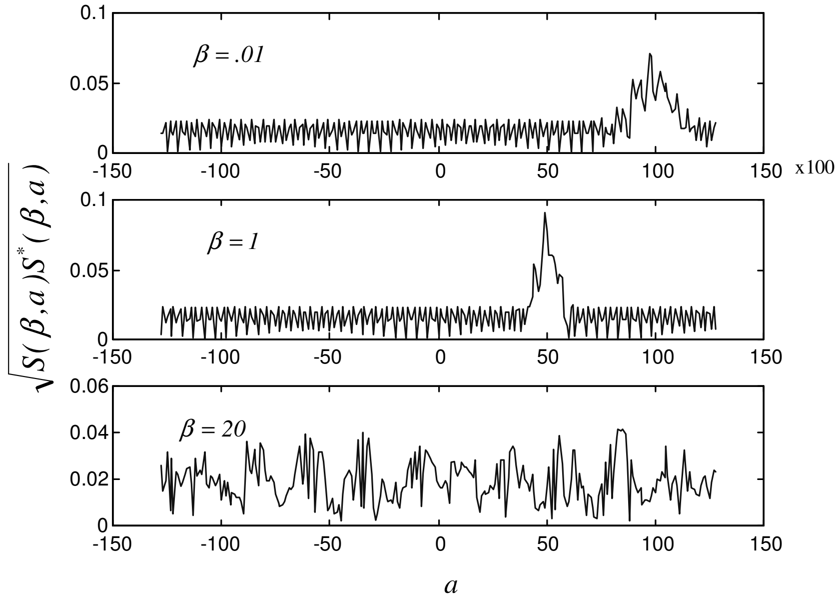

Fig. 4.9 Chirp transform after four iterations

Since it neither gives a local nor a global maxima, we stop the iterative analysis followed by synthesis and approximate the signal as a linear combination of the estimated components and the residue. The WVD of the residue that essentially consists of two impulses after five iterations (which means that five components are identified) is shown in Fig. 4.10. Applying the WVD to each isolated component gives the spectrum of the signal with maximum auto term resolution and almost no cross terms, as the WVD attains maximum resolution in the time and the frequency domains simultaneously among all bilinear distributions. Applying WVD to a signal having *N* components, we get

$$WVD(s_\beta) = \sum_{i=1}^{N} WVD(s_i, s_i) + \sum_{i=1}^{N} \sum_{j=1i \quad j \neq i}^{N} WVD(s_i, s_j). \tag{4.23}$$

The auto terms exactly represent the true spectrum (cross terms can be completely neglected because the components are identified) and the resulting simple scheme could be a non-invertable representation given by

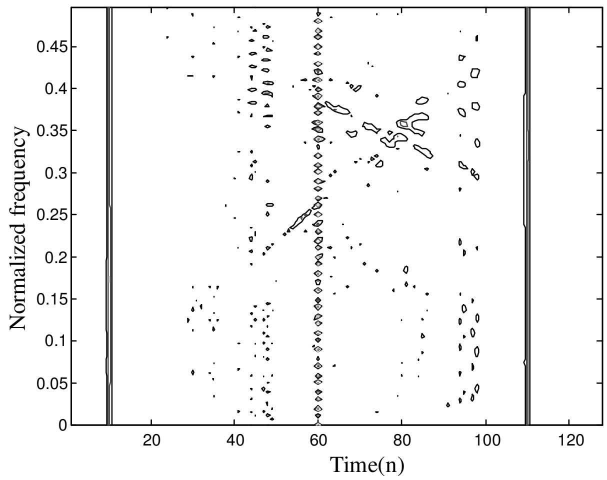

Fig. 4.10. WVD of the residue

$$WVD(s_\beta) = \sum_{i=1}^{N} WVD(s_i) . \tag{4.24}$$

However, such a procedure may be misleading while analyzing nonlinear FM signals which result in cross terms even in the absence of other components. To alleviate this problem, we may use short time Fourier transform with optimized window parameter. It directly follows from Eqn. (4.24) that

$$STFT(s_\beta) = \sum_{i=1}^{N} STFT(s_i) \tag{4.25}$$

To find the optimal window parameter, we use the concentration measure (Jones *et al*, 1994a)

$$M(t,a) = \frac{\iint \left| STFT_a(\tau,\omega)\, w(\tau - t) \right|^4 d\tau\, d\omega}{\left(\iint \left| STFT_a(\tau,\omega)\, w(\tau - t) \right|^2 d\tau\, d\omega \right)^2}, \tag{4.26}$$

that essentially searches for the window parameter which results in concentrated representation in the frequency direction, where $STFT_a(\tau,\omega)$ is the $STFT$ of the signal; the analysis window with the parameter γ is given by:

$h(t) = (\frac{\gamma}{\pi})^{\frac{1}{4}} e^{-\frac{\gamma t^2}{2}}$ and $w(t)$ is a symmetric window function to measure the concentration in a specified region (it is not the analysis window used in STFT). This window parameter is searched over all time instants. In our problem, we have considered the parameter as the amplitude modulation law, which is indirectly the window length (noting that the tails of a Gaussian pulse can be neglected outside the interval spanning a length of thrice the standard deviation). We wish to compute the integral in Eqn. (4.26) for the expansion set. Instead of computing the above equation directly, we can further simplify the concentration measure by observing the following characteristics of the expansion set:

The expansion set at a particular β has the same time dependent frequency spread and it does not vary with time. To illustrate this point, let us consider a signal of the form

$$s(t) = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} e^{-\frac{\alpha t^2}{2} + j\frac{\beta t^2}{2}}.$$

Then the spread in frequency at a given time is given by:

$$\sigma^2_{\omega/t} = \frac{1}{2}(\alpha + a) + \frac{1}{2}\frac{\beta^2}{\alpha + a}. \tag{4.27}$$

Since it does not vary with over time and is also invariant in frequency shift in the signal spectrum, we can confine our search for the window parameter to a single time instant t_0. Thus, the modified concentration measure can be given as:

$$M(a) = \frac{\int | STFT_a(t_0,\omega)|^4 \; d\omega}{\left(\int | STFT_a(t_0,\omega)|^2 \;\; d\omega\right)^2}. \tag{4.28}$$

The parameter that maximizes this measure for various values of chirp rates have been found experimentally and are shown in Fig. 4.11.

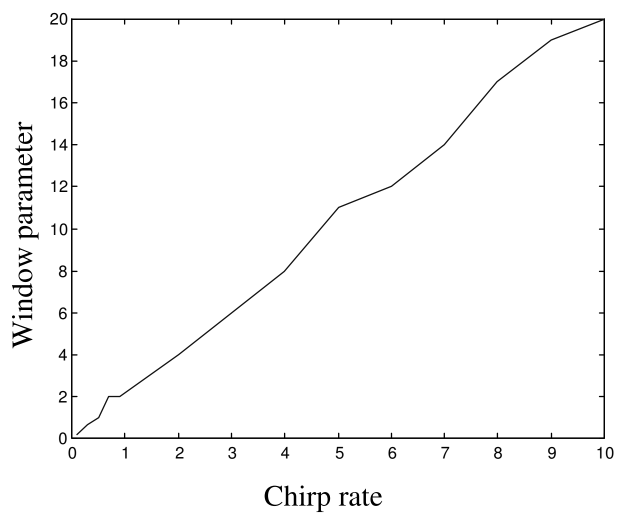

Fig. 4.11 Window parameter estimated from Eqn. (4.28)

It can be observed from the figure that the window parameter is almost varying linearly with the chirp rate. It can be analytically justified as follows:

For slowly varying signals, i.e., $\beta \approx 0$, the signal is almost stationary and the window can be of infinite length, i.e., $a \approx 0$. For highly nonstationary signals, like an impulse, where $\beta \approx \infty$, the window should be very small to localize the signal in time, i.e., $a \approx \infty$. Now, we will mathematically validate the result by recalling that the optimal window parameter can be obtained by minimizing the time dependent frequency spread. For purely frequency modulated signals, the optimal window parameter is given by (Cohen, 1995):

Optimal window length $\approx \dfrac{1}{2|\varphi''(t)|}$, where $\varphi(t)$ is the phase of the signal. The above method of computing the optimal window also results in the same value that the parameter a is linearly varying with β. STFT of a multicomponent signal and chirp transform of the signal are shown in Fig. 4.12.

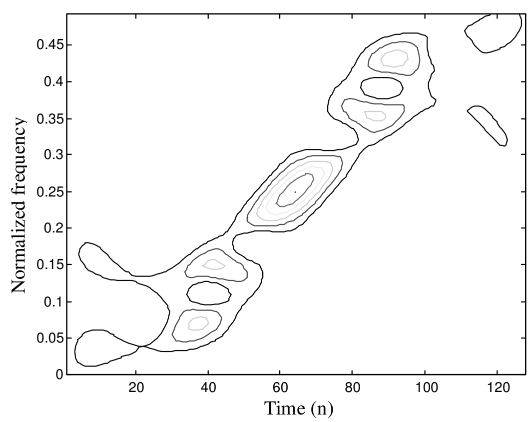

Fig. 4.12. (a) STFT of a multicomponent signal consisting of a sinusoidal FM and a linear FM components

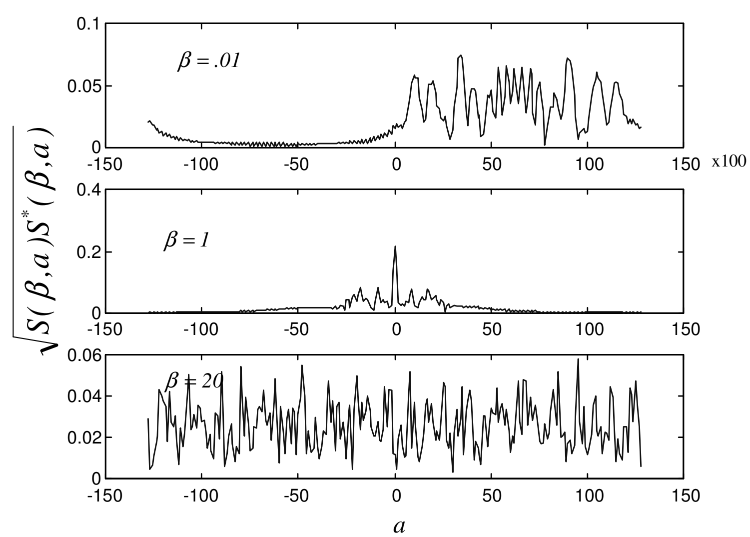

Fig. 4.12 (b) Chirp transform of the multicomponent signal

Using the iterative estimation of the component, we can isolate the linear FM component that generates the local maxima, as is clearly visible from the above figure. We can apply WVD to the estimated component but applying WVD to the residue may result in cross terms. Hence we use the optimized STFT to the residue. The WVD of the estimated component and optimized STFT of the residue are shown in Fig. 4.13.

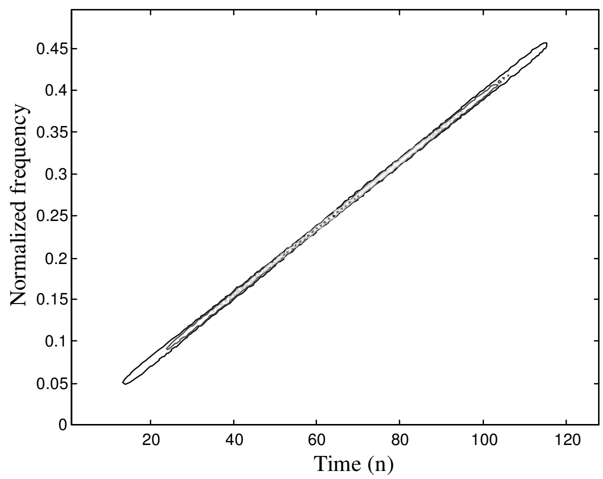

Fig. 4.13. (a) WVD of the estimated component

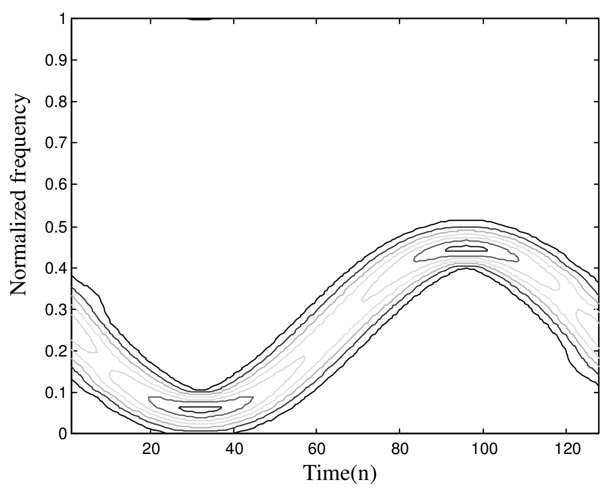

Fig. 4.13. (b) Optimized STFT of the residue

4.5.2. System Identification

System identification is a classical problem in signal processing which has many applications in many fields, including channel estimation in wireless communications. In general, the problem can be stated as follows:

$y(n) = \sum_{k} x(k)\, h(n-k) + v(n)$, where

$x(n)$: Transmitted signal.

$h(n)$: Impulse response of an LTI channel.

$v(n)$: Zero-mean additive Gaussian noise.

$y(n)$: Received signal with noise.

$\breve{y}(n)$: Received signal without noise (true signal).

The problem is to identify the linear time-invariant (LTI) system transfer function $H(\omega)$ of $h(n)$, given the input and output signals. The conventional method of solving this problem is the cross spectral method, which is equivalent to the least squared method in stationary cases, and can be formulated as

$H(\omega) = \dfrac{S_{xy}(\omega)}{S_{xx}(\omega)}$, where $S_{xx}(\omega)$ is the auto spectrum of $x(n)$ and $S_{xy}(\omega)$ is the cross spectrum of $x(n)$ and $y(n)$. If the additive noise is a zero-mean Gaussian process and statistically independent of the signal $x(n)$, then the estimate is asymptotically unbiased and its error variance approaches the Cramer-Rao lower bound. However, noise removal is not possible if the known transmitted signal were the pseudo random signal, which is correlated with the noise. Xia (Xia, 1997) has considered chirp signals instead of pseudo random signals for their wideband characteristics and localization in the joint time-frequency plane making the uncorrelated additive Gaussian noise removal possible and observed an improvement of 15dB over conventional system identification techniques. In the present context, we try to use chirp transform as a denoising tool, which is schematically shown in Fig. 4.14

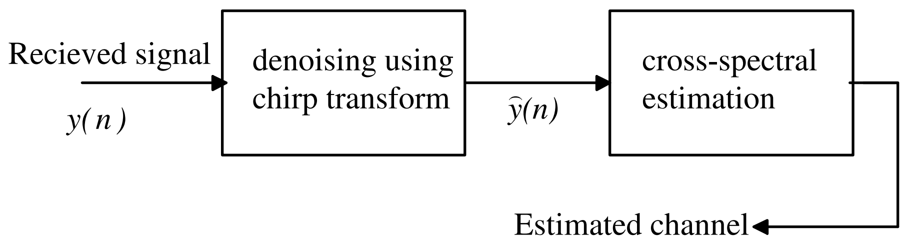

Fig. 4.14. System identification using chirp transform as a denoising tool

Mask design: If we attempt to remove the noise in the joint time-frequency plane, it has to be done by observing the pattern of the received signal and noise. As noise occupies the whole spectrum and any received signal spectrum is dominated by the known transmitted signal, we can design a mask to eliminate the noise components. In the chirp transform domain, we concentrate in the region corresponding to the transmitted signal for the following reason (Xia, 1997):

Assume the chirp signal $x(n) = exp(j\beta n^r)$ for some constants $r \geq 2$ and $\beta \neq 0$, then

$$\begin{aligned}
\breve{y}(n) &= \sum_k h(k)\, x(n-k) \\
&= \sum_k h(k)\, exp(j\beta (n-k)^r) \qquad (4.29) \\
&= exp(j\beta n^r) \sum_k h(k)\, exp\left(j\beta \sum_{l=0}^{r-1} \binom{r\text{-}1\text{-}l}{l} n^l k^{r-l}\right)
\end{aligned}$$

which is dominated by the original chirp $x(n)$ for finite tap LTI systems $h(k)$. As a special case, when r =2, $\breve{y}(n) = x(n)G(2\beta n)$, where $G(\omega)$ is the Fourier transform of the signal $h(n)x(n)$. When the channel $h(n)$ has only a finite tap, the function $G(\omega)$ is usually smooth. Since the transmitted signal was known to both the transmitter and the receiver, by the above property, its pattern in the transform domain may help in designing the mask for filtering noise. The mask cannot be designed based on the characteristics of the input signal, because the expansion set is strongly correlated with the input signal and the resulting mask that concentrates the transform in the vicinity of the input signal is very much localized. The spectral characteristics of the input signal can only be used to determine the location of the mask. Hence, thresholding (or masking in the transform domain) is done in a conservative manner. Our experiments have revealed that to model (or estimate) any channel that is an LTI system, soft thresholding the coefficients can be done by choosing a Gaussian window of length 50 which is appropriate. True channel characteristics and the estimated channel using cross-spectral estimation are shown in Fig. 4.15. The chirp transform of the received signal at $\beta = 1$ at 0dB *SNR* and the masked transform are shown in Fig. 4.16. Then, the received signal is synthesized from this modified transform to obtain the denoised signal. The synthesized signal from the modified chirp transform is now used to estimate the channel by cross-spectral estimation. The estimated channel after denoising and the received signal after denoising together with received signal under no noise conditions, are shown in Fig. 4.17. To obtain the mean *SNR* curves, we have repeated the experiment over 100 iterations by varying the channel and the noise. The mean SNR curves for different original *SNR*, and the *SNR* after denoising, are shown in Fig. 4.18.

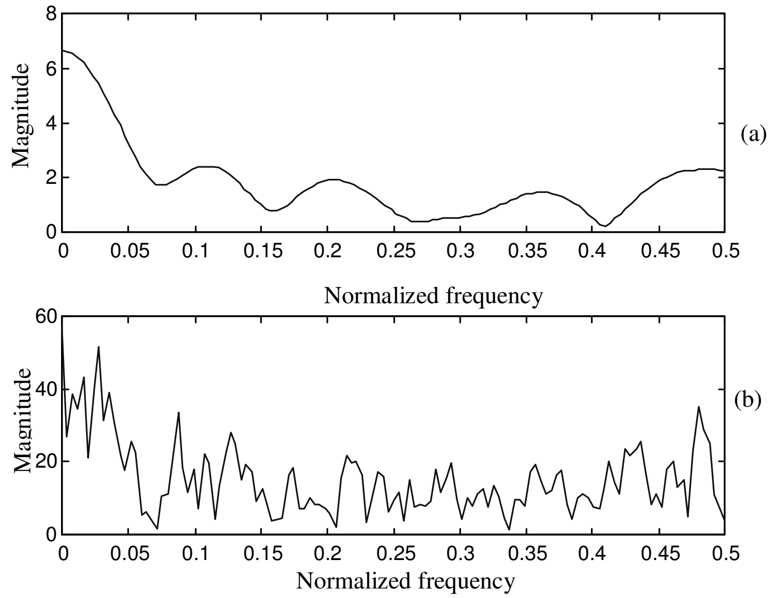

Fig. 4.15. (a) True channel characteristics and (b) The estimated channel using cross-spectral estimation method

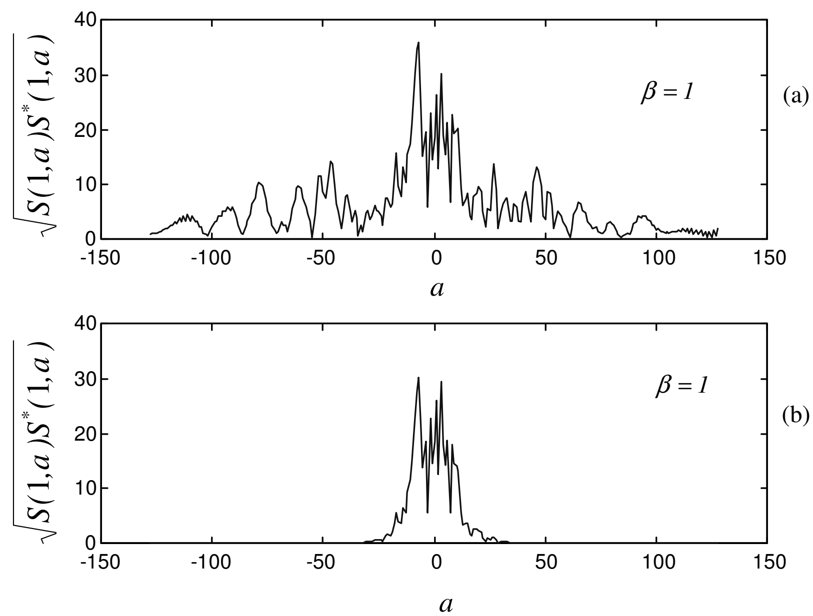

Fig. 4.16. (a) Chirp transform of the received signal at $\beta = 1$ and (b) The masked transform

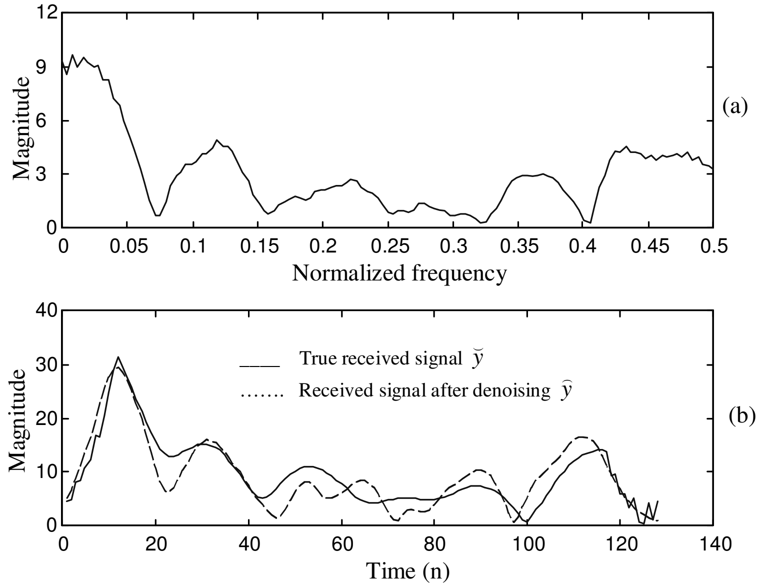

Fig. 4.17. (a) Estimated channel after denoising done using chirp transform and (b) True received signal $\hat{y}$ and the received signal after denoising $\breve{y}$

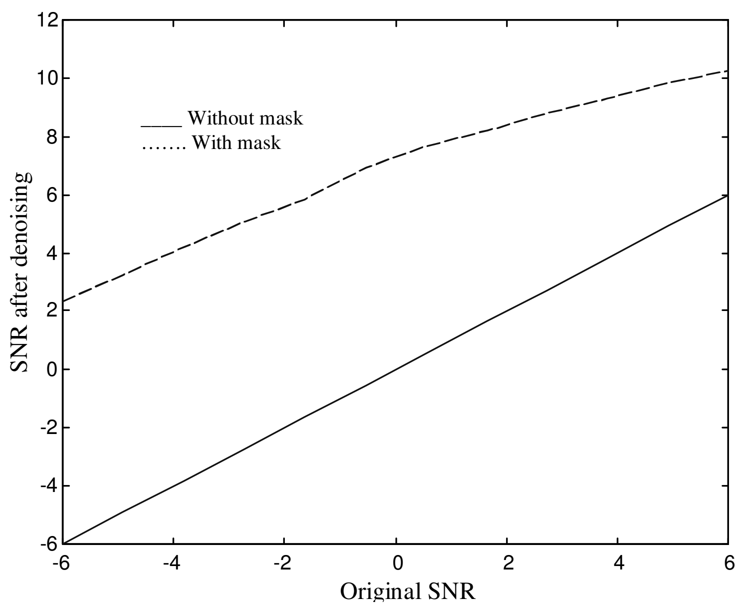

Fig. 4.18. Mean SNR curves with mask (denoising) and without mask (conventional cross-spectral estimation using chirp signal)

The *SNR*s in different cases can be computed as:

$$SNR_{Afer\ denoisnig} = 10\log_{10}\frac{\sum_n |\breve{y}(n)|^2}{\sum_n |\breve{y}(n) - \widehat{y}(n)|^2} \text{ and}$$

$$SNR_{Original} = 10\log_{10}\frac{\sum_n |\breve{y}(n)|^2}{\sum_n |\breve{y}(n) - y(n)|^2}. \tag{4.30}$$

It can be observed from Fig. 4.18 that there is a declinement in the *SNR* improvement with increasing original *SNR*. This has been accounted for the fact that soft thresholding allows the noise components that sit in the region corresponding to the transmitted input signal and that the noise suppression capability of the chirp transform lies in rejecting all components which lie *outside* the region of the input signal. In the method proposed by Xia, *iterative time-variant filtering* that progressively improves the *SNR* has been used. However, in the chirp transform domain such a repeated analysis followed by synthesis aiming at reducing the noise did not result in a significant improvement. The reason being:

The noise components that interfere with the signal components cannot be masked in the chirp transform domain and it can be done if and only if the true mask (the mask that exactly separates the noise components from the signal) is known.

As a logical extension of the above algorithm, the role of the signal is now replaced by interference and we may use chirp transform as an exciser for chirp-like interference in spread spectrum communication. Akansu and Bultan (Bultan *et al*, 1998) have shown that when the interference is strongly correlated with the basic atoms, chirps in the present case, interference excision is possible and demonstrated that it outperforms the Fourier-based excision techniques.

4.6. CONCLUSIONS

In this Chapter, we have developed chirp transform with chirps as the expansion set. The orthogonality of the expansion set, reconstruction of the signal using synthesis equation and discretization issues are considered. Some simple properties like shifting in time, frequency and variation of the envelope are discussed. Different signals, e.g., impulse, sinusoid and chirps are analyzed in the transform domain to enable us to investigate nonstationary signals. With the help of a multicomponent test signal, iterative analysis-synthesis based estimation technique that has the origin in adaptive signal decomposition is investigated. To estimate the frequency content of the residue, optimized window parameters for STFT are obtained using a much simplified concentration measure. Later, the problem of system identification has been addressed, where we demonstrate chirp transform as a denoising tool and observe significant improvement over conventional system identification.