CHAPTER 4

CHIRP TRANSFORM

4.1. INTRODUCTION

Analysis of multicomponent signals is a potential task in non-parametric modeling. STFT with optimized windows, kernel design for reduced interference, etc., are some of the approaches concerned with improving resolution and suppressing cross terms. The recent focus of signal processing is on extending the analysis domain to shear, scale, translation, etc., which has led to the evolution of chirplet transform containing all the above spaces with possible extension to even more dimensions (Mann *et al*, 1995). The principle involved in wavelets 'scaling', together with chirping and shifting the wavelet, causes the analyzing grid in t-f plane sheared, scaled and shifted. Hence, it can be considered as a generalization of STFT and wavelet analysis; and therefore can give a good estimate of the true spectrum for a large vocabulary. However, many a time, the signal need not be analyzed in all subspaces, but a few may well represent the signal's inner structure, as nonlinear frequency modulation can be approximated by piecewise linear FM, the case pertinent to the problem in spectral estimation of nonstationary signals. We restrict the analysis to a two-dimensional space comprising of chirping and shifting only, called the chirp transform. The tilings can be made arbitrary by allowing the nonlinearity of the frequency modulation to vary. However, from the computational and data storage point of view, this is cumbersome, uneconomical and unyielding (Baraniuk *et al*, 1996b). Different time-frequency tilings matched to the signal's characteristics give a better representation of the signal. In this Chapter, to understand the concept of arbitrary t-f tilings and their usefulness not only as adaptive signal representation schemes but also as signal processing tools, we consider chirps as an expansion set, where the chirp is a linear frequency modulated signal. The chirps with chirp rate and translation in time being the parameters are used to carry out the analysis in the transform domain. We present an analysis methodology in the transform domain for various signals after we discuss the orthogonality of the expansion set. Based on the interpretation and properties of the transform, we discuss its application in spectral estimation and system identification. In the spectral estimation problem, we present two methods based on WVD and STFT with optimized windows. Recently, nonstationary signals have been used in system identification, because of the wideband spectrum and time localization (Shalvi *et al*, 1996). This can be considered as denoising plus conventional system identification problem. We can make use of the localization of the chirp transform to a specific type of signal whose inner structure best matches with the expansion set in denoising experiments.

4.2. CHIRP TRANSFORM

The origin of chirp transform is not new, as it can be found dating back to 1820's (Lawrence, 1965) in optics known as Fresnal transformation, given by

$$S(a) = \int s(t)\, e^{-j(t+a)^2}\, dt \tag{4.1}$$

the basis of which can be found in chirplet transform with amplitude modulated chirp subjected to dilations. The analysis equation of the chirp transform is given by introducing another parameter, β, the chirp rate, as:

$$S(\beta,a) = \int s(t)\, e^{-j\beta(t+a)^2/2} dt\,, \tag{4.2}$$

where '*a*' is a time shift of the basis function that gets reflected as a frequency shift in the frequency domain and $S(\beta,a)$ is the subspace of chirplet transform. It can be expressed as the inner product of $s(t)$ and the expansion set (the reason for calling expansion set instead of basis will be explained shortly) as:

$$S(\beta,a) = \langle s(t)\,, e^{j\beta(t+a)^2/2} \rangle. \tag{4.3}$$

The synthesis equation is given by

$$\hat{s}(t) = \int\int S(\beta,a)\, e^{j\beta(t+a)^2/2}\; d\beta\;\; da\,. \tag{4.4}$$

The integral given in Eqn. (4.2) is in the form of Liebnitz's integral and can be solved by assuming slowly varying envelope for s*(t)* (Papoulis, 1984). However, the magnitude of the expansion coefficients can be obtained by using Moyal's law without imposing constraints. This helps us in studying the behavior of the transform analytically for various signals. Moyal's law can also be used to study the overlapping (in other words, time-frequency overlap for nonstationary signals and orthogonality in stationary cases) of two signals in the time-frequency plane, which is defined as:

$$\left| \int s_1(t)\, s_2^*(t)\, dt \right|^2 = \iint W_1(t,f)\, W_2(t,f)\, dt\, df\,, \tag{4.5}$$

where W_1 and W_2 correspond to the WVD of the s_1 and s_2, respectively. Before we present the applications and analysis of various signals, we take a brief look at the properties of the expansion set. In the following derivation, we prove that the signal can be reconstructed from the synthesis equation.

Let $\hat{s}$ be the signal reconstructed from the inverse transformation. Then

$$\hat{s}(t) = \int\int S(\beta, a)\, e^{j\beta(t+a)^2/2}\, da\, d\beta,$$

$$= \int\int\int s(\tau)\, e^{-j\beta(\tau+a)^2/2}\, e^{j\beta(t+a)^2/2}\, d\tau\, da\, d\beta. \tag{4.6}$$

After simplifying the above equation we get,

$$= \int\int\int s(\tau)\, e^{-j\beta(\tau^2-t^2)/2}\, e^{-j\beta a(\tau-t)}\, d\tau\, da\, d\beta \quad . \tag{4.7}$$

Integrating the above equation with respect to a gives us

$$= \int\int s(\tau)\, e^{-j\beta(\tau^2-t^2)/2}\, \partial(\beta(\tau-t))\, d\tau\, d\beta. \tag{4.8}$$

We consider the above integral in the following cases:

1. $(\tau - t) \neq 0$ *and* $\beta \neq 0$: The integral is clearly equal to zero since $\partial(\beta(\tau - t)) = 0$ in the region of interest.
2. $(\tau - t) = 0$ *and* $\forall \beta$: The integral becomes *s(t)*.
3. $\forall(\tau - t)$ *and* $\beta = 0$: When $\beta = 0$, the integral becomes $\int s(\tau)\, d\tau$.
4. $(\tau - t) = 0$ *and* $\beta = 0$: It also results in the same value as it is with the previous case, i.e., $\int s(\tau)\, d\tau$.

The final value of the integral would be the sum of the above different cases. It is to be noted that we have computed the expression twice at $(\tau - t) = 0$ *and* $\beta = 0$, firstly in $(\tau - t) = 0$ *and* $\forall\beta$, and next explicitly at $\forall(\tau - t)$ *and* $\beta = 0$. Hence, the final value of the integral would be:

$$\widehat{s}(t) \; = \; 0 + s(t) + \int \mathrm{s}(\tau)\, d\tau - \int \mathrm{s}(\tau)\, d\tau \; = \; s(t). \tag{4.9}$$

Further investigation is required on the existence and uniqueness of the inverse transformation and more mathematical treatment is necessary, even though we have made an attempt to discuss its uniqueness in the following pages. Now we try to prove that the expansion set is not orthogonal because they are not linearly dependent.

In general, we shall say that two functions *x(t)* and *y(t)* are orthogonal if their inner product is zero:

$$\langle x, y \rangle \; = \; 0 . \tag{4.10}$$

To investigate the orthogonality, we consider two atoms, s_1 and s_2 from the expansion set with different parameters (β_1, a_1) and (β_2, a_2), respectively.

Then,

$$\langle s_1, s_2 \rangle \; = \; \int e^{j\beta_1 (t + a_1)^2 / 2} \; e^{-j\beta_2 (t + a_2)^2 / 2} \, dt . \tag{4.11}$$

Since our objective is to find out the inner product magnitude, without loss of generality, we use Moyal's law to evaluate the absolute squared of the inner product of the two atoms. By noting that the WVD of any atom, which is of the form described above, is given by:

$$WVD_{s(t)}(t,\omega) = \partial(\omega - \beta t - \beta a) \qquad (4.12)$$

and recalling Eqn. (4.5), we obtain

$$\left|\langle s_1, s_2 \rangle\right|^2 = \iint \partial(\omega - \beta_1 t - \beta_1 a_1)\ \partial(\omega + \beta_2 t + \beta_2 a_2)\ dt\, d\omega. \qquad (4.13)$$

As mentioned earlier, the atoms which result in zero inner product are said to be orthogonal. This interpretation looks simpler when the atoms are stationary. In the present case, since the spectrum is varying in a time-dependent fashion we need to look into the joint t-f plane. We infer from Eqn. (4.13) that it goes conditionally to zero and the instances when the integral becomes non-zero are:

$$t(\beta_1 - \beta_2) + (\beta_1 a_1 - \beta_2 a_2) = 0. \qquad (4.14)$$

The above equation implies that the instant at which the instantaneous spectra are crossing (since the instantaneous spectra can overlap at only one instant given by the above condition), the inner product is non-zero. Hence, we say that when two atoms are time-frequency disjoint they are orthogonal, i.e., their instantaneous spectra do not cross (or overlap)[†]. The case when $\beta_1 = \beta_2$ in the above equation is of particular importance, since the expansion set is orthogonal in this subspace. For most of the analysis purpose, we exploit the fact that when the transform domain containing the chirp rate that

[†] When the concept of instantaneous frequency fails as is the case for multicomponent signals or nonanalytic signals, we shall look into the WVD.

corresponds to the signal under analysis gives the coefficient of maximum amplitude and we can identify components in the transform domain by looking either for local maxima or global maxima. In general, the expansion set is not orthogonal since the atoms are not always linearly dependent. When two functions are linearly dependent, then they cannot be orthogonal. To check whether the atoms are linearly independent or not we may compute the wronskain of the expansion set, where the wronskain of N functions (vectors) is defined as (Leon, 1994):

$$W = \begin{vmatrix} f_1 & f_2 & \cdots & f_N \\ f_1' & f_2' & \cdots & f_N' \\ \vdots & & & \vdots \\ f_1^{N-1} & f_2^{N-1} & \cdots & f_N^{N-1} \end{vmatrix}, \tag{4.15}$$

where $f^{\,n}$ corresponds to the n^{th} derivative of the function f. Substituting the atoms in place of f to compute the wronskain (for the sake of analysis we consider only two atoms, s_1 and s_2), we get

$$W = 2j\, e^{j\beta_1(t+a_1)^2/2}\, e^{j\beta_2(t+a_2)^2/2}\,(\,t(\beta_1-\beta_2)+(\beta_1 a_1-\beta_2 a_2)). \tag{4.16}$$

The result is same as that of Eqn. (4.14) and hence the expansion set is linearly dependent. Fortunately, they are orthogonal in the subspace containing all atoms with $\beta_i = \beta_j\ \forall\, i = j$.

4.3. CHOICE OF PARAMETERS

No discretization procedure exists for the above transformation. Hence, a close analysis on the t-f plane is required on how to choose β. We represent β and a $\in \Re$ in such a way that they do not range from $-\infty$ to $+\infty$ but belong to $\Re$ which suffices to approximate the signal reasonably. Now we will have a closer look at the phase term of the expansion set:

$$j\varphi(t) = j\beta(t+a)^2/2 = j\beta t^2/2 + j\beta ta + j\beta a^2/2. \tag{4.17}$$

The first term on the right hand side of the above equation causes chirping, the middle term causes a frequency shift (i.e., modulation in time domain) and the last term is just a phase constant. The instantaneous frequency of the expansion set is given as: $\varphi'(t) = \beta t + \beta a$, which is in the form of $y = mx + c$. Based on this rule, different locations covered by varying β and a are depicted in Fig. 4.1. The angle corresponding to the slope of the above expression is represented as '*chirp rate in radians*' for easy visualization of the chirp rate parameter in the t-f plane. In our simulations and analysis, we have varied β from –4 to 4 ; and a from -127 to 127 where the signal length is 128 samples. At $\beta = 0$, the transform coefficients represent the DC term in the Fourier transform. It is for this reason that, at $\beta = 0$ the parameter 'a' does not cause any frequency shift since the middle term on the right hand side of Eqn. (4.17) is zero. Hence, for slowly varying signals, i.e., $\beta \approx 0$, shifting in time does not alter the spectrum.. To accommodate such slowly varying signals and model sinusoids, a/β has to be used, and this ensures the desired frequency shift. A slowly varying signal (approximately a sinusoid) generated using $\beta = 0.001\, and\; a = 2000$ and $\beta = 0.001\, and\; a = 4000$ are shown in Fig. 4.2. We can observe from the figure that the periodicity is almost doubled in the latter case. In a similar manner, we can sweep the entire frequency plane by choosing the appropriate values for a. Close analysis is required on the t-f plane to select the expansion set and to select the dictionary appropriately.

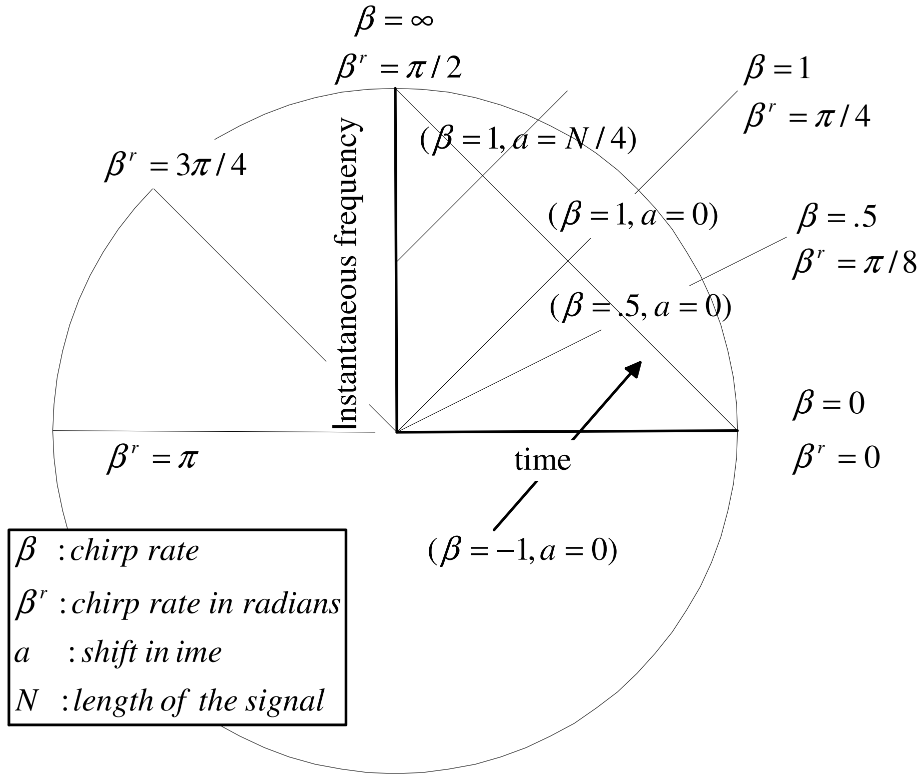

Fig. 4.1. Time-frequency locations and the energy concentration of the atoms in the expansion set and equivalent representation as a rotation parameter

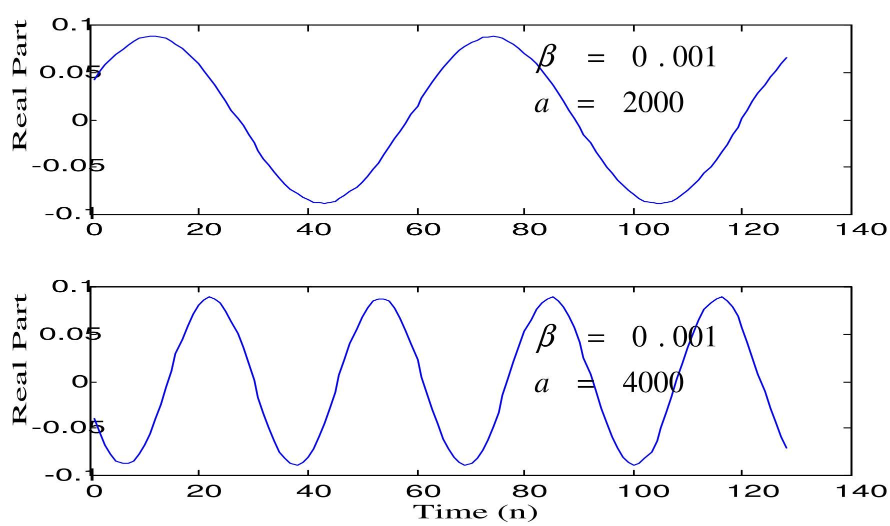

Fig. 4.2. Modeling of slowly varying signals

4.4. ANALYSIS IN THE TRANSFORM DOMAIN

We are interested in the transform coefficients where the expansion set would be orthogonal. To analytically solve the integrals and observe the synthesis criteria, we have chosen Gaussians as they have closed form expressions and easy to solve.

Let the signal be

$$s(t) = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} e^{-\frac{\alpha t^2}{2}} e^{j\beta_1 t^2/2}, \tag{4.18}$$

then, $S(\beta_1, a) = \int s(t) e^{-j\beta_1 (t+a)^2/2} dt$ (4.19)

can be given by

$$S(\beta_1, a) = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} \int e^{-\alpha t^2/2} e^{-j\beta_1 at} e^{-j\beta_1 a^2/2} dt .$$

Rewriting the above equation, we get

$$S(\beta_1, a) = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} e^{-j\beta_1 a^2/2} \int e^{-\frac{\alpha}{2}(t + j\frac{\beta_1 at}{\alpha})^2} e^{-j\frac{\beta_1^2 a^2}{2\alpha}} dt$$

$$= \left(\frac{4\pi}{\alpha}\right)^{\frac{1}{4}} e^{-j\beta_1 a^2/2} e^{-\frac{\beta_1^2 a^2}{2\alpha}} . \tag{4.20}$$

When we analyze the above signal using the chirp transform at a chirp rate equal to the signal, we obtain the following information:

The magnitude is controlled by the window parameter (i.e., amplitude modulation) in such a way that its spread is inversely proportional to the spread of the window. As $\alpha \to 0$, the magnitude response becomes an impulse, since the signal and expansion set are orthogonal. The phase response is insensitive to the amplitude variation, except that the phase is varying with the same chirp rate as does the signal and the magnitude response is controlled by the envelope of the signal. The magnitude response of the chirp signal for different windows is shown in Fig. 4.3. Now we move the window position time centered at $t_{0,}$ i.e.,

$$s(t) = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} e^{-\frac{\alpha(t-t_0)^2}{2}} e^{j\beta_1 t^2/2}$$

and the chirp transform of the signal at β_1 can be computed as is done in Eqn. (4.20) to get

$$S(\beta_1, a) = \left(\frac{4\pi}{\alpha}\right)^{\frac{1}{4}} e^{-\frac{\beta_1^2 a^2}{2\alpha}} e^{-j\beta_1(a+t_0)^2/2} e^{-j\beta_1^2 a^2} . \tag{4.21}$$

The above result says that shifting the window (not the phase term of the signal) causes the phase response to shift by the same amount and the magnitude response is the same as the earlier one but the direction is opposite. We now shift the time centering of the phase term of the signal by t_0, keeping the window time centered at the origin. Then,

$$s(t) = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} e^{-\frac{\alpha t^2}{2}} e^{j\beta(t-t_0)^2/2}$$

$$S(\beta_1, a) = \left(\frac{4\pi}{\alpha}\right)^{\frac{1}{4}} e^{-\frac{\beta_{12}^2 (a-t_0)^2}{2\alpha}} e^{-j\beta_1(a+t_0)^2/2} e^{-j\beta_1 t_0^2} . \tag{4.22}$$

The anticipated results are shift in magnitude response as well as in phase response without the change in numerical values. We observe shift in, both magnitude response and phase response, because of the time centerings of the window and phase term of the signal are not the same.

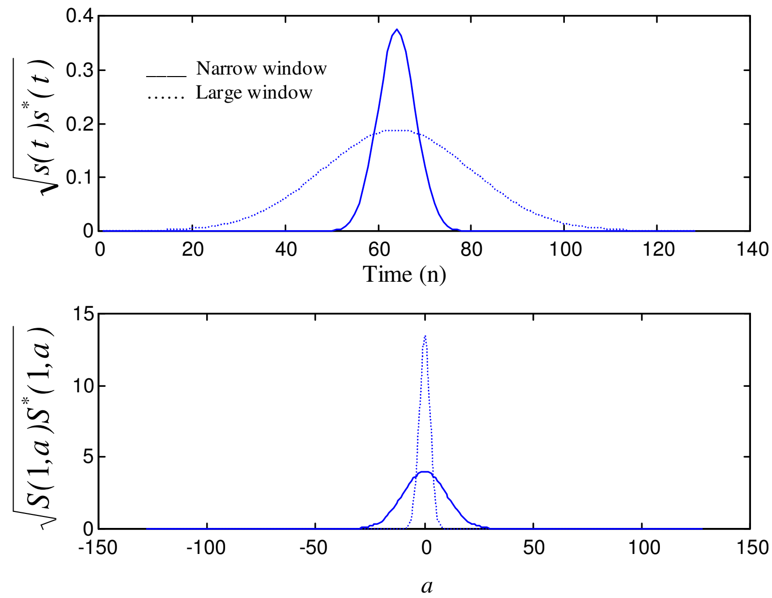

Fig. 4.3. (a) Envelope of the signal with different windows and (b) Magnitude response variation of the chirp transform at $\beta = 1$

We make the following conclusions after the observations:

- A shift in phase term of the signal is responsible for the shift in the magnitude response but not in the phase response of the transform.
- The location of the window determines the location of the phase response, which has the same instantaneous law as the signal does (i.e., $\beta(\,.\,)^2$) and it does not alter the magnitude response.

The results are similar to the properties of Fourier transform, since we are analyzing in the orthogonal subspace. We strengthen this viewpoint by observing the magnitude response for various signals of theoretical importance. An impulse, a rectangular gated sinusoid and a Guassian windowed chirp signals have been analyzed using the chirp transform. To study the interdependency of the coefficients, we concentrate at the chirp rates corresponding to the test signals. In Fig. 4.4(a), the chirp transform of an impulse at $\beta = 0.01, 1\ and\ 20$, respectively, is shown. For a rectangular windowed sinusoid, the chirp transform at the same chirp rate has a magnitude response similar to the Fourier transform of a rectangular pulse as shown in Fig. 4.4(b). It is to be observed that the transform has maximum amplitude in the region corresponding to $\beta = 0.01$ where the signal is orthogonal. Similarly, a Gaussian enveloped linear FM component analyzed using the chirp transform has a magnitude response again a Gaussian along the orhtogonal axis i.e., $\beta = 1$, depicted in Fig. 4.4(c). This Fourier transform interpretation of the individual components in the subspace of the chirp transform corresponding to their chirp rates helps us in identifying the individual components in a multicomponent scenario.

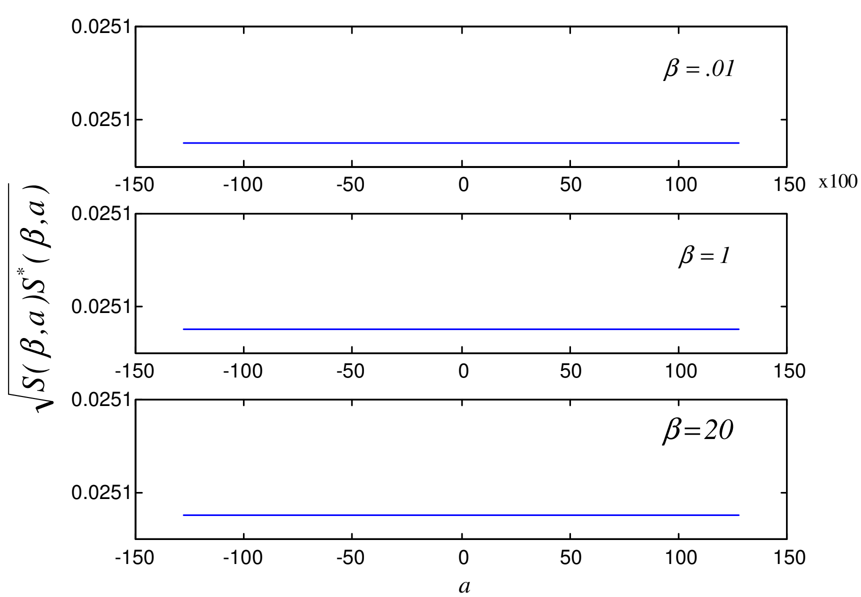

Fig. 4.4. (a) Chirp transform of an impulse

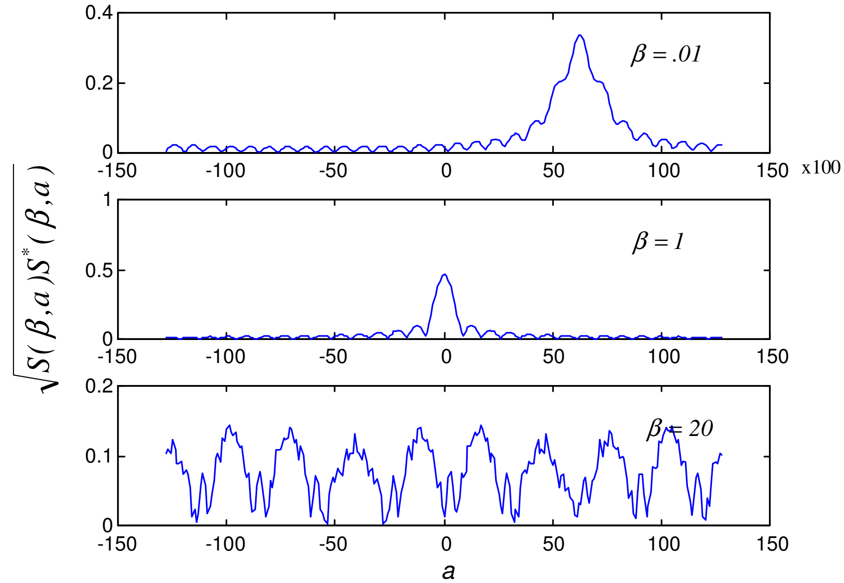

Fig. 4.4. (b) Chirp transform of a rectangular gated chirp

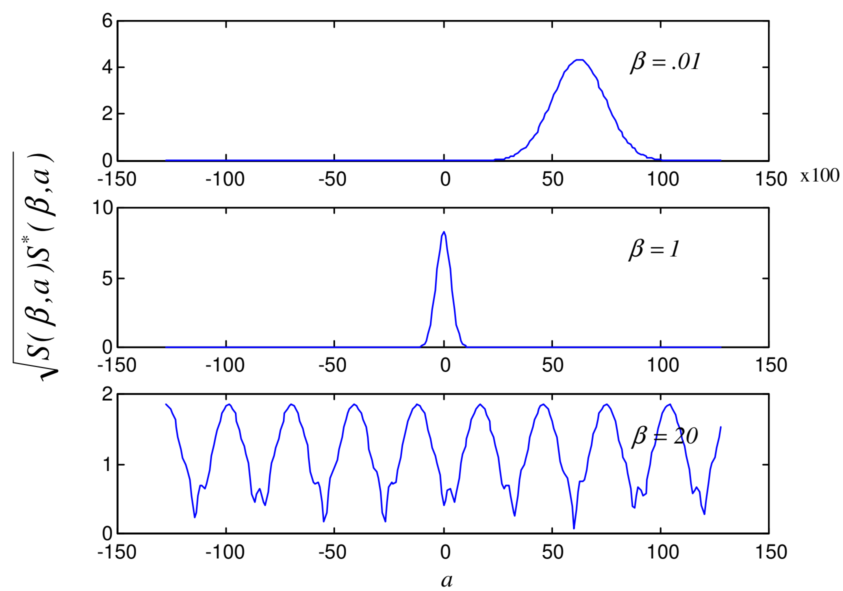

Fig. 4.4. (c) Chirp transform of a Gaussain windowed chirp