Let the likelihood function be

$$L(\boldsymbol{\theta};\boldsymbol{y}) = c\exp(-f(\boldsymbol{\theta};\boldsymbol{y})) \tag{1}$$

for some arbitrary bi-variate function f and for some constant c. It might be useful if we have a normal likelihood function in which case it is possible to compute posterior either analytically (eg. if the prior is also normal)or by resorting to numerical integration techniques [1](eg. Gauss-Hermite quadrature). It can also be used to calculate moments and cumulants in an efficient way[2].

In general, we can approximate the function $f(\boldsymbol{\theta})$ (in the neighborhood of $\boldsymbol{\tau}$) using a multi-variate Taylor series. In the present case, we will be considering bi-variate case. We consider only terms up to order two since the log-likelihood of normal density is quadratic,

$$\begin{aligned} f(\boldsymbol{\theta}) \approx a_{00} + a_{10}(\theta_1-\tau_1) + a_{01}(\theta_2-\tau_2) + a_{11}(\theta_1-\tau_1)(\theta_2-\tau_2) \\ + a_{20}(\theta_1-\tau_1)^2 + a_{02}(\theta_2-\tau_2)^2 \end{aligned} \tag{2}$$

where

$$a_{ij} = \frac{1}{i!\,j!}\left[\left(\frac{\partial}{\partial\theta_1}\right)^i\left(\frac{\partial}{\partial\theta_2}\right)^j f\right](\boldsymbol{\tau}) \tag{3}$$

Then the likelihood is approximated as:

$$L(\boldsymbol{\theta};\boldsymbol{y}) \sim N\left(\mu_L, \Sigma_L\right) \tag{4a}$$

$$\mu_L = -\frac{1}{2}\begin{bmatrix} (a_{11}+a_{10}-a_{11}\tau_2-2a_{20}\tau_1)/a_{20} \\ (a_{11}+a_{01}-a_{11}\tau_1-2a_{02}\tau_2)/a_{02} \end{bmatrix} \tag{4b}$$

$$\Sigma_L = \frac{-2a_{20}a_{02}}{4a_{20}a_{02}-a_{11}^2}\begin{bmatrix} 1/a_{02} & -2a_{11} \\ -2a_{11} & 1/a_{20} \end{bmatrix} \tag{4c}$$

We can expand the Taylor series around the *mode* or *ML* estimate. Having approximated the likelihood as a bivariate normal, below we will provide a closed form expression for calculating the posterior density for a given normal prior. If the prior $\pi(\boldsymbol{\theta})$is of the form

$$\pi(\boldsymbol{\theta}) \sim N(\mu_\pi, \Sigma_\pi) \tag{5}$$

then the posterior $p(\boldsymbol{\theta})$will be of the form

$$p(\boldsymbol{\theta}) \sim N(\mu_p, \Sigma_p) \tag{6a}$$

$$\Sigma_p = (\Sigma_L^{-1} + \Sigma_\pi^{-1})^{-1} \tag{6b}$$

$$\mu_p = \Sigma_p\left(\Sigma_L^{-1}\mu_L + \Sigma_\pi^{-1}\mu_\pi\right) \tag{6c}$$

References

[1] J. C. Naylor and A. F. M. Smith, "Applications of a method for the efficient computation of posterior distributions," *Applied Statistics*, vol. 31, 1982.

[2] Kostas Traintafyllopoulos, "Moments and cumulants of the multivariate real and complex gaussian distributions," 2002, www.stats.bris.ac.uk/research/stats/pub/ResRept/2002.html.