Tailored Finite Point Method

UNDERGRADUATE THESIS

Submitted in partial fulfillment of the requirements of
BITS F421T Thesis

By

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May 2017

Certificate

This is to certify that the thesis entitled, "*Tailored Finite Point Method*" and submitted by Dhaval Milind Mohandas ID No. 2013A4TS856G in partial fulfillment of the requirements of BITS F421T Thesis embodies the work done by him under my supervision.

Supervisor
Dr. Dhanumjaya Palla
BITS-Pilani Goa Campus
Date:

Date:

Abstract

Tailored Finite Point Method

The tailored finite point method has been explored. Many a times while modelling multi- scale phenomena we require a highly refined mesh to capture all the phenomena present. Traditional methods like the finite difference and finite element method may not always be the best choice for problems involving such phenomena as they may be computationally expensive. The tailored finite point method overcomes this problem by requiring a less refined mesh. The tailored finite point method has been applied to the one dimensional wave equation, a parabolic equation and a two dimensional equation. Among these some problems are singularly perturbed. The plots for the problems solved have been obtained and the error analysis results are tabulated.

Acknowledgements

I would like to thank my institute, Birla Institute of Technology and Science, Pilani, K.K Birla Goa Campus for providing me this opportunity to pursue an undergraduate thesis. I would like to thank Dr. Dhanumjaya Palla, my guide for this thesis, for guiding me throughout the course of this thesis and for providing valuable inputs. Last but not the least I am aslo thankful to Dr. Shibu Clement, HOD of Mechanical Department.

Contents

Chapter 1

Introduction

The tailored Finite Point Method is a modification of the finite point method. This method is highly efficient in multi-scale and singularly perturbed phenomena, such as the high frequency waves propagation, the singular perturbation problems with boundary or interior layers. These problems arise in many fields such as:

- the elastic/electromagnetic wave propagation in heterogeneous media
- the seismic wave propagation in geophysics
- aerodynamics
- flows involving multiphase phenomena

1.1 Challenges in traditional numerical methods for boundary/interior layers

In conventional simulation methods for problems involving layers, to capture the boundary/interior layers or the small scale waves, one needs a very high resolution mesh to give good accuracy. For problems involving time dependency, we need a very small step size in time, especially when there is some stability criterion involved. Thus the computational costs are very expensive.
Boundary or interior layers are characterized by rapid transitions in the solution, and are therefore difficult to capture in numerically. Many times such layers are also characterized by spurious oscillations.

1.2 Difficulties for high frequency waves

When the wave frequencies are high, i.e. the wavelengths ϵ are short compared to the overall size of the problem domain, it will be ineffecient solve the problem using any direct method.. Using conventional finite difference or finite element schemes, $N \sim \epsilon^{-d}$ points are required to catpure all the details of the problem where d is the dimension of the problem domain.

1.3 Wave Equation

$$u_t + a(x)u_x = 0 \tag{1.1}$$

We now look at various finite point schemes developed for the numerical simulation of singular perturbation problems, high frequency wave propagation problems and interface problems.
We use the exact solution of the local approximate problem i.e the problem at one point on the mesh and to find global approximate solution to our problem in the entire domain.

1.4 Rectangular Cell Tailored Finite Point Scheme (RCTFPM)

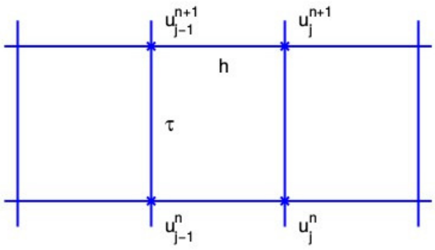


FIGURE 1.1

$$u_j^{n+1} = \alpha_{-1}u_{j-1}^{n+1} + \beta_{-1}u_{j-1}^n + \beta_0 u_j^n$$

We expect the scheme to hold exactly for all the wave-formed functions in space.

$$V = \{v(x,t)|v(x,t) = c_1 + c_2\ \exp(ik_j(at-x)) + c_3\ \exp(-ik_j(at-x)), \quad \forall c_1, c_2, c_3 \in \mathbb{C}\} \tag{1.2}$$

where k_j is the wave number in the j-th $[x_j, x_{j+1}]$ cell and 'i' is the imaginary unit. The complex-valued initial condition is denoted by $u_0(x)$. In the most general form it can be written as, $u_0(x) = A_0(x)\exp(iS_0(x))$, where $A_0(x)$ and $S_0(x)$ are real-valued functions.

We take the wave number as $k_j = S_0^{'}(x_j)$ in each cell.

Taking the wave functions of $(c_1, c_2, c_3) = (1,0,0), (0,1,0), (0,0,1)$ in gives

$$\begin{cases} 1 = \alpha_{-1} + \beta_{-1} + \beta_0 \\ \cos(k_j a\tau) = \alpha_{-1}\cos(k_j(a\tau+h)) + \beta_{-1} + \beta_0 \\ \sin(k_j a\tau) = \alpha_{-1}\sin(k_j(a\tau+h)) - \beta_{-1}\sin(k_j h) \end{cases}$$

By solving the above system we get,

$$\begin{aligned} \alpha_{-1} &= \frac{\sin(k_j(a\tau-h)/2)}{\sin(k_j(a\tau+h)/2)} \\ \beta_{-1} &= 1 \\ \beta_0 &= -\alpha_{-1} \end{aligned}$$

1.4.1 Stability Analysis

We perform the Von Neumann Stability Analysis to get the stability criterion. We susbtitute $u_j^n = e^{ij\xi h}$ and $u_j^{n+1} = Ge^{ij\xi h}$ where G is the growth factor. For the scheme to be stable we require

$$|G| \leq 1$$

Substituting in the scheme we get

$$\begin{aligned}
Ge^{ij\xi h} &= \alpha_{-1}Ge^{i(j-1)\xi h} + \beta_{-1}e^{i(j-1)\xi h} + \beta_0 e^{ij\xi h} \\
G &= \alpha_{-1}Ge^{-i\xi h} + e^{-i\xi h} - \alpha_{-1} \\
(1-\alpha_{-1}e^{-i\xi h})G &= e^{-i\xi h} - \alpha_{-1} \\
G &= \frac{e^{-i\xi h} - \alpha_{-1}}{1-\alpha_{-1}e^{-i\xi h}}
\end{aligned}$$

Taking absolute on both sides

$$|G| = \left|\frac{e^{-i\xi h} - \alpha_{-1}}{1-\alpha_{-1}e^{-i\xi h}}\right|$$

$$|G| = \frac{|e^{-i\xi h} - \alpha_{-1}|}{|1-\alpha_{-1}e^{-i\xi h}|}$$

$$|G| = \frac{|\cos(\xi h) + i\sin(\xi h) - \alpha_{-1}|}{|1-\alpha_{-1}(\cos(\xi h) - i\sin(\xi h))|}$$

$$|G| = \frac{|\cos(\xi h) - \alpha_{-1} + i\sin(\xi h)|}{|1-\alpha_{-1}\cos(\xi h) + i\alpha_{-1}\sin(\xi h)|}$$

$$|G| = \sqrt{\frac{(\cos(\xi h) - \alpha_{-1})^2 + (\sin(\xi h))^2}{(1-\alpha_{-1}\cos(\xi h))^2 + (\alpha_{-1}\sin(\xi h))^2}}$$

$$|G| = 1$$

Thus the scheme is unconditionally stable.

This scheme is claimed to be second order in space.

1.4.2 Example 1

Consider the wave equation along with the given initial condition

$$\begin{aligned}
u_t + u_x &= 0 \\
u(x,0) &= e^{-50x^2}e^{i\sin(x)}
\end{aligned} \tag{1.3}$$

The analytical solution is given by

$$u(x,t) = e^{-50(x-t)^2}e^{i\sin(x-t)} \tag{1.4}$$

The following plots and error estimates have been obtained on solving the above example using the tailored finite point method.

(h, τ)	$(1/2^6, 1/2^{10})$	$(1/2^7, 1/2^{10})$	$(1/2^8, 1/2^{10})$	$(1/2^9, 1/2^{10})$
Error	0.0293	0.0070	0.0016	0.0003
Order	-	2.05	2.08	2.32

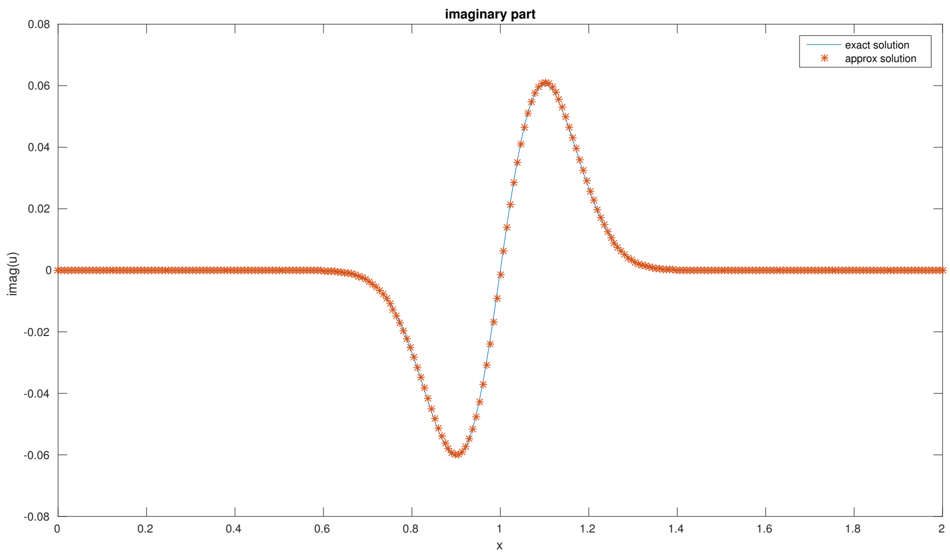


FIGURE 1.2

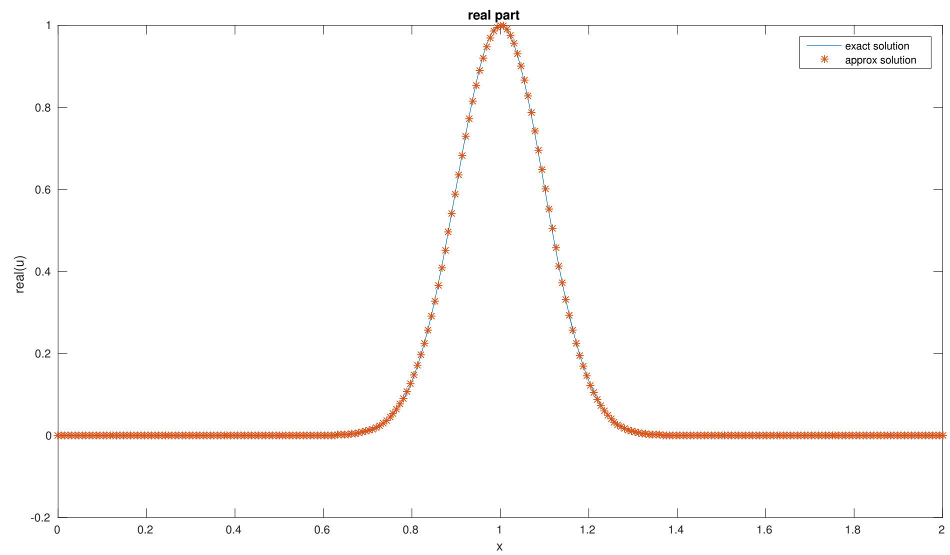


FIGURE 1.3

1.4.3 Example 2

$$u_t + a(x)u_x = 0 \tag{1.5}$$

The wave speed and initial condition are given as follows:

$$a(x) = \pi + x \tag{1.6}$$
$$u_0(x) = e^{-50x^2} e^{i\sin(x)} \tag{1.7}$$

The exact solution is given by

$$u(x,t) = e^{-50((\pi+x)e^{-t}-\pi)^2} e^{(i\sin(\pi+x)e^{-t}-\pi)} \tag{1.8}$$

The wave number k_j is taken to be $k_j = \cos(x_j)$ Here we see that the wave speed is not a constant. It is a linear function. The following approach is adopted when 'a' is not a constant: We approximate $a(x)$ as

$$a(x) \approx b_j + c_j x \qquad \text{where} \tag{1.9}$$
$$b_j = \frac{a(x_{j-1})x_j - a(x_j)x_{j-1}}{h} \tag{1.10}$$

Then at the n^{th} time step the solution can be locally approximated as

$$u(x_j, t^n) = A_0(y_j^n)e^{(iS_0(y_j^n)/\epsilon)} \tag{1.11}$$

where

$$y_j^n = ((b_j + c_j x_j)e^{-c_j t^n} - b_j)/c_j \tag{1.12}$$

We take the wave number as follows

$$k_j^n = S_0'(y_j^n)e^{-c_n t^n}/\epsilon \tag{1.13}$$

This example has been solved using the RCTFPM method. The following are the expressions for the coeffecients

$$\alpha_{-1}^n = \frac{\sin(k_j^n(a_j\tau - h)/2)}{\sin(k_j^n(a_j\tau + h)/2)} \tag{1.14}$$
$$\beta_{-1}^n = 1 \tag{1.15}$$
$$\beta_0^n = -\alpha_{-1}^n \tag{1.16}$$

The error estimates and plots obtained are given as follows:

(h, τ)	$(1/2^6,1/2^{10})$	$(1/2^7,1/2^{10})$	$(1/2^8,1/2^{10})$	$(1/2^9,1/2^{10})$
Error	0.0178	0.0058	0.0018	0.0007
Order	-	1.608	1.623	1.405

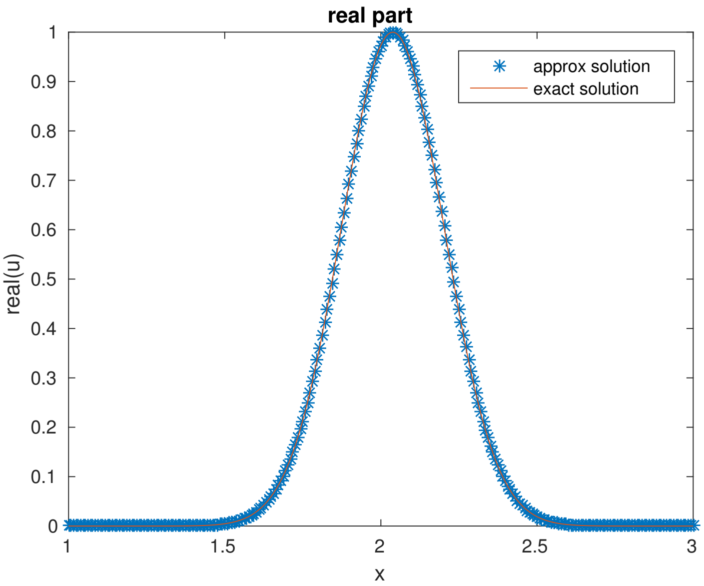


FIGURE 1.4

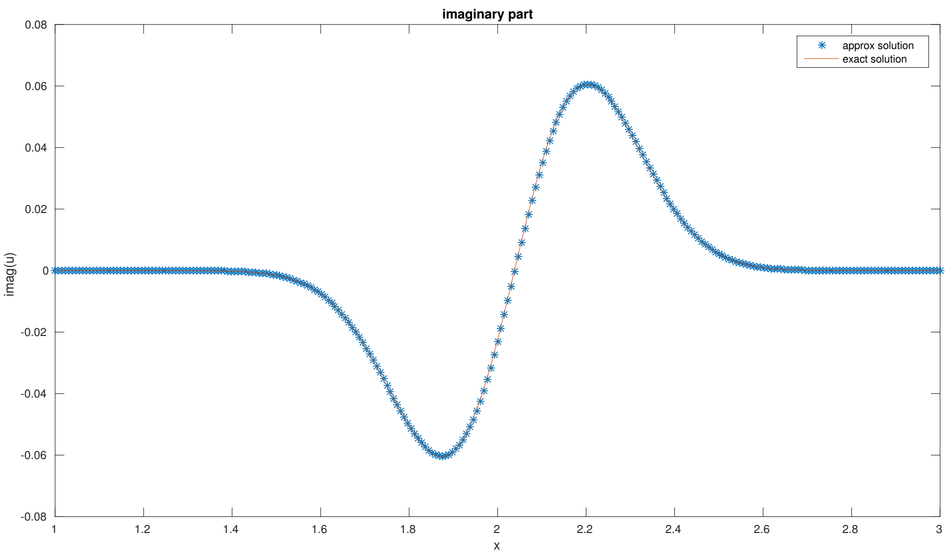


FIGURE 1.5

1.4.4 Example 3

$$u_t + a(x)u_x = 0 \tag{1.17}$$

The wave speed and initial condition are given as follows:

$$a(x) = 0.5 + x \tag{1.18}$$
$$u_0(x) = e^{-200x^2} e^{i\sin(x)/\epsilon} \tag{1.19}$$

The exact solution is given by

$$u(x,t) = e^{-50((\pi+x)e^{-t}-\pi)^2} e^{(i\sin(\pi+x)e^{-t}-\pi)} \tag{1.20}$$

The wave number k_j is taken to be $k_j = \cos(x_j)$

This example has been solved using the RCTFPM method. The error estimates and plots obtained are given as follows:

(h, τ)	$(1/2^7,1/2^6)$	$(1/2^8,1/2^6)$	$(1/2^9,1/2^6)$	$(1/2^{10},1/2^6)$
Error	0.0816	0.0381	0.0204	0.0105
Order	-	1.088	0.9102	0.9588

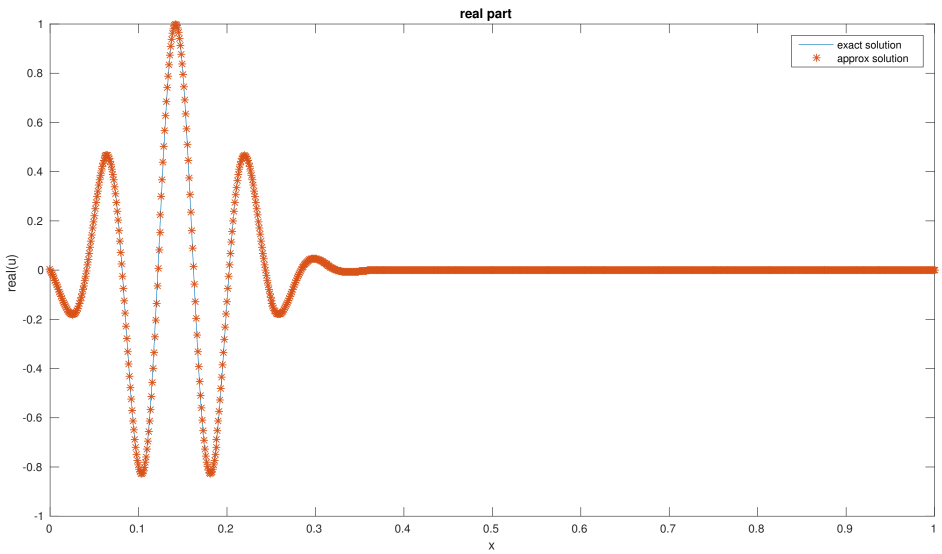


FIGURE 1.6

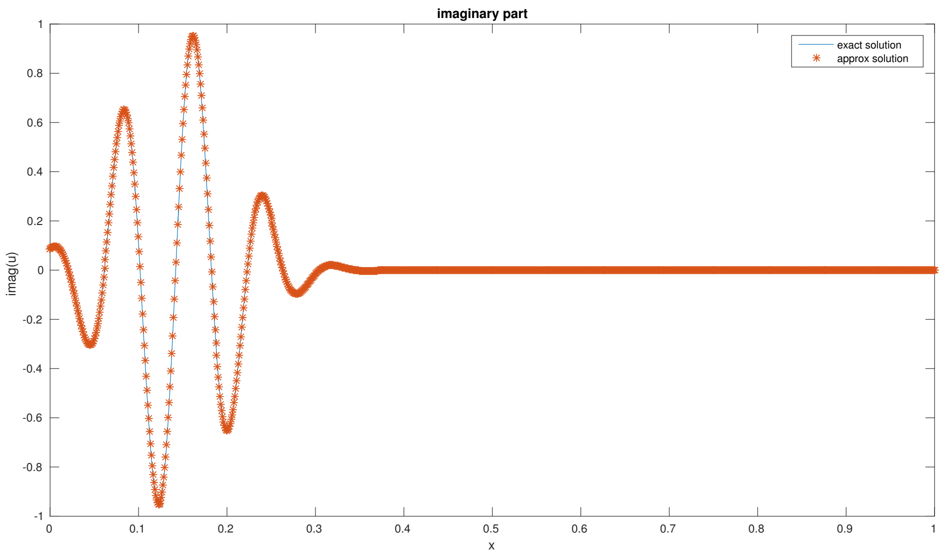

imaginary part
exact solution
approx solution
imag(u)
x
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

FIGURE 1.7

1.5 Centered tailored finite point method (CTFPM)

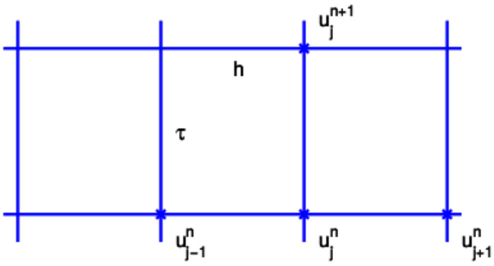


FIGURE 1.8

We derive an explicit scheme as follows:

$$u_j^{n+1} = \alpha_{-1} u_{j-1}^n + \alpha_0 u_j^n + \alpha_1 u_{j+1}^n \tag{1.21}$$

The coefficients are taken as $(c_1, c_2, c_3) = (1, 0, 0), (0, 1, 0), (0, 0, 1)$ which gives us

$$\begin{cases} 1 = \alpha_{-1} + \alpha_0 + \alpha_1 \\ \cos(k_j a\tau) = \alpha_{-1}\cos(k_j h) + \alpha_0 + \alpha_1 \cos(k_j h) \\ \sin(k_j a\tau) = \alpha_{-1}\sin(k_j h) - \alpha_1 \sin(k_j h) \end{cases}$$

Solving this system yields,

$$\begin{aligned} \alpha_{-1} =& \frac{\sin(k_j a\tau/2)\sin(k_j(a\tau + h)/2)}{\sin(k_j h/2)\sin(k_j h)} \\ \alpha_1 =& \frac{\sin(k_j a\tau/2)\sin(k_j(a\tau - h)/2)}{\sin(k_j h/2)\sin(k_j h)} \\ \alpha_0 =& 1 - \alpha 1 - \alpha_{-1} \end{aligned}$$

1.5.1 Stability Analysis

We perform the Von Neumann Stability Analysis to get the stability criterion. Similar to the RCTFPM method we get,

$$G = \alpha_{-1} e^{-i\xi h} + \alpha_0 + \alpha_1 e^{i\xi h}$$

After calculating, the condition $|G| \leq 1$ is the same as

$$\begin{aligned} 1 - \cos^2(k_j h) - \cos(k_j h) + \cos(k_j a\tau) + \cos(\xi h) - \cos(k_j h)\cos(k_j a\tau)\cos(\xi h) \geq 0 \\ (1 + \cos(\xi h))(1 - \cos(k_j h)\cos(k_j a\tau)) + (1 + \cos(k_j h))(\cos(k_j a\tau) - \cos(k_j h)) \geq 0 \end{aligned}$$

As $k_j a\tau, k_j h << 1$. Thus for the scheme to be stable we require $a\tau \leq h$.

1.5.2 Accuracy

We use the taylor series expansion to see the accuracy of the scheme. Consider the terms

$$\begin{aligned} u_{j-1}^n &\approx u(x-h,t) \approx u(x,t) - hu_x(x,t) + \frac{h^2}{2}u_{xx}(x,t) + o(h^3) \\ u_{j+1}^n &\approx u(x+h,t) \approx u(x,t) + hu_x(x,t) + \frac{h^2}{2}u_{xx}(x,t) + o(h^3) \\ u_j^n &\approx u(x,t) \end{aligned}$$

Substituting the above expressions in the L.H.S of (1.22). We get

$$\begin{aligned} &= \alpha_{-1}(u(x,t) - hu_x(x,t) + \frac{h^2}{2}u_{xx}(x,t)) + \alpha_0(u(x,t)) + \alpha_1(u(x,t) + hu_x(x,t) + \frac{h^2}{2}u_{xx}(x,t)) \\ &= (\alpha_{-1} + \alpha_0 + \alpha_1)u(x,t) + h(\alpha_1 - \alpha_{-1})u_x(x,t) + \frac{h^2}{2}(\alpha_1 + \alpha_{-1})u_{xx}(x,t) \\ &= u(x,t) + h(\alpha_1 - \alpha_{-1})u_x(x,t) + \frac{h^2}{2}(\alpha_1 + \alpha_{-1})u_{xx}(x,t) \end{aligned}$$

In th discrete form

$$= u_j^n + h((\alpha_1 - \alpha_{-1})\frac{(u_{j+1}^n - u_{j-1}^n)}{2h}) + \frac{h^2}{2}(\alpha_1 + \alpha_{-1})\frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{h^2}$$

From calculating we have

$$\begin{aligned} \alpha_1 - \alpha_{-1} &= -\frac{\sin(k_j a\tau)}{\sin(k_j h)} \\ \alpha_1 + \alpha_{-1} &= \frac{\sin^2((k_j a\tau)/2)}{\sin^2((k_j h)/2)} \end{aligned}$$

As $k_j a\tau, \quad k_j h << 1$. We have

$$\begin{aligned} \alpha_1 - \alpha_{-1} &= -\frac{a\tau}{h} \\ \alpha_1 + \alpha_{-1} &= \frac{(a\tau)^2}{h^2} \end{aligned}$$

Substituting above we get,

$$= u_j^n - \frac{(a\tau)}{2h}(u_{j+1}^n - u_{j-1}^n) + \frac{(a\tau)^2}{2h^2}(u_{j-1}^n - 2u_j^n + u_{j+1}^n)$$

Equating with R.H.S we get

$$u_j^{n+1} = u_j^n - \frac{a\tau}{2h}(u_{j+1}^n - u_{j-1}^n) + (a\tau)^2\left(\frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{2h^2}\right)$$

This is the Lax Wendroff scheme which is second order accurate.

This scheme is second order in space.

1.5.3 Example

Consider the wave equation along with the given initial condition

$$\begin{aligned} u_t + u_x &= 0 \\ u(x,0) &= e^{-50x^2} e^{i\sin(x)} \end{aligned} \tag{1.22}$$

The analytical solution is given by

$$u(x,t) = e^{-50(x-t)^2} e^{i\sin(x-t)} \tag{1.23}$$

The following plots and error estimates have been obtained on solving the above example using the tailored finite point method.

(h, τ)	($1/2^6$,$1/2^7$)	($1/2^7$,$1/2^8$)	($1/2^8$,$1/2^9$)	($1/2^9$,$1/2^{10}$)
Error	0.03996	0.01049	0.00265	6.65558e-04
Order	-	1.92	1.98	1.99

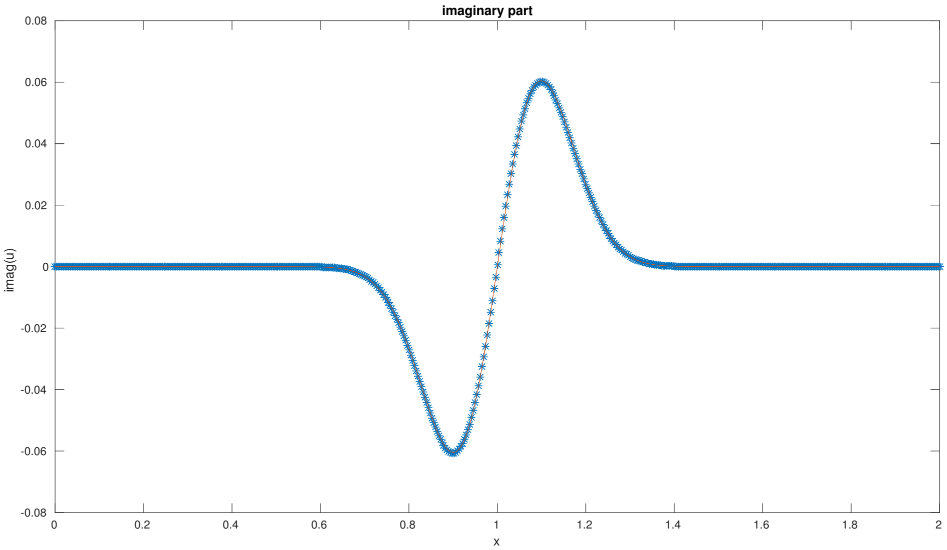


FIGURE 1.9

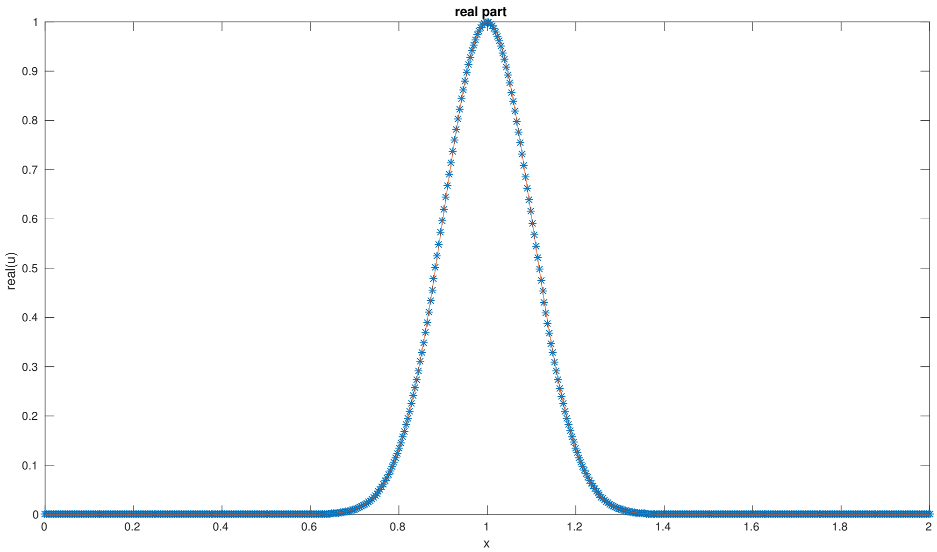


FIGURE 1.10

1.6 One Sided Tailored Finite Point Method (OSTFPM)

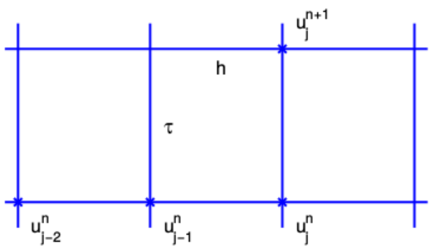


FIGURE 1.11

Many a times for boundary value problems the OSTFPM is an efficient method.The scheme is given as follows:

$$u_j^{n+1} = \alpha_{-2} u_{j-2}^n + \alpha_{-1} u_{j-1}^n + \alpha_0 u_j^n \tag{1.24}$$

The coefficients are taken as $(c_1, c_2, c_3) = (1, 0, 0), (0, 1, 0), (0, 0, 1)$ which gives us

$$\begin{cases} 1 = \alpha_{-2} + \alpha_{-1} + \alpha_0 \\ \cos(k_j a\tau) = \alpha_{-2}\cos(2k_j h) + \alpha_{-1}\cos(k_j h) + \alpha_0 \\ \sin(k_j a\tau) = \alpha_{-2}\sin(2k_j h) + \alpha_{-1}\sin(k_j h) \end{cases}$$

this gives us

$$\begin{aligned} \alpha_{-2} &= \frac{\sin(k_j a\tau/2)\sin(k_j(a\tau - h)/2)}{\sin(k_j h/2)\sin(k_j h)} \\ \alpha_{-1} &= \frac{\sin(k_j a\tau/2)\sin(k_j(a\tau - 2h)/2)}{\sin^2(k_j h/2)} \\ \alpha_0 &= 1 - \alpha_{-2} - \alpha_{-1} \end{aligned}$$

1.6.1 Stability Criterion

The stability criterion of the scheme is a better than CTFPM. For the scheme to be stable we require $\tau \leq 2h$.

This scheme is second order in space.

1.6.2 Accuracy

$$
\begin{aligned}
&u_{j-2}^n \approx u(x-2,t) \approx u(x,t) - 2hu_x(x,t) + 2h^2 u_{xx}(x,t) + o(h^3)\\
&u_{j-1}^n \approx u(x-1,t) \approx u(x,t) - hu_x + \frac{h^2}{2} u_{xx}(x,t) + o(h^3)\\
&u_j^n \approx u(x,t)
\end{aligned}
$$

Substituting these values in the scheme, we get

$$
\begin{aligned}
&= \alpha_{-2}(u - 2hu_x + 2h^2 u_{xx}) + \alpha_{-1}(u - hu_x + \frac{h^2}{2}) + \alpha_0 u_j^n\\
&= (\alpha_{-2} + \alpha_{-1} + \alpha_0)u - h(2\alpha_{-2} + \alpha_{-1})u_x + h^2(\frac{\alpha_{-1}}{2} + 2\alpha_{-2})u_{xx}\\
&= u - h(\alpha_{-2} + \alpha_{-1})u_x + h^2(2\alpha_{-2} + \alpha_{-1}/2)u_{xx}\\
&= u - a\tau u_x + \frac{(a\tau)^2}{2} u_{xx}
\end{aligned}
$$

In discrete form

$$
u_j^n - \frac{a\tau}{2h}(u_{j+1}^n - u_{j-1}^n) + \frac{(a\tau)^2}{2}\left(\frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{h^2}\right)
$$

Equating with R.H.S we get

$$
u_j^{n+1} = u_j^n - \frac{a\tau}{2h}(u_{j+1}^n - u_{j-1}^n) + \frac{(a\tau)^2}{2h^2}(u_{j-1}^n - 2u_j^n + u_{j+1}^n)
$$

This is the Lax Wendroff scheme which is second order accurate.

1.6.3 Example

Consider the wave equation along with the given initial condition

$$
\begin{aligned}
u_t + u_x &= 0\\
u(x,0) &= e^{-50x^2} e^{i\sin(x)}
\end{aligned}
\tag{1.25}
$$

The analytical solution is given by

$$u(x,t) = e^{-50(x-t)^2} e^{i\sin(x-t)} \tag{1.26}$$

The following plots and error estimates have been obtained on solving the above example using the tailored finite point method.

(h, τ)	($1/2^6$,$1/2^7$)	($1/2^7$,$1/2^8$)	($1/2^8$,$1/2^9$)	($1/2^9$,$1/2^{10}$)
Error	0.1203	0.0434	0.0153	0.0054
Order	-	1.47	1.50	1.49

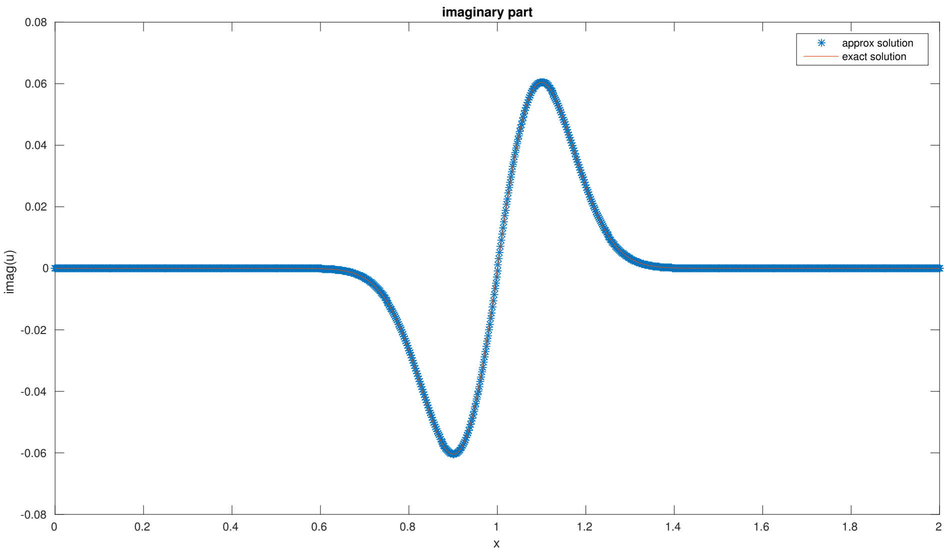


FIGURE 1.12

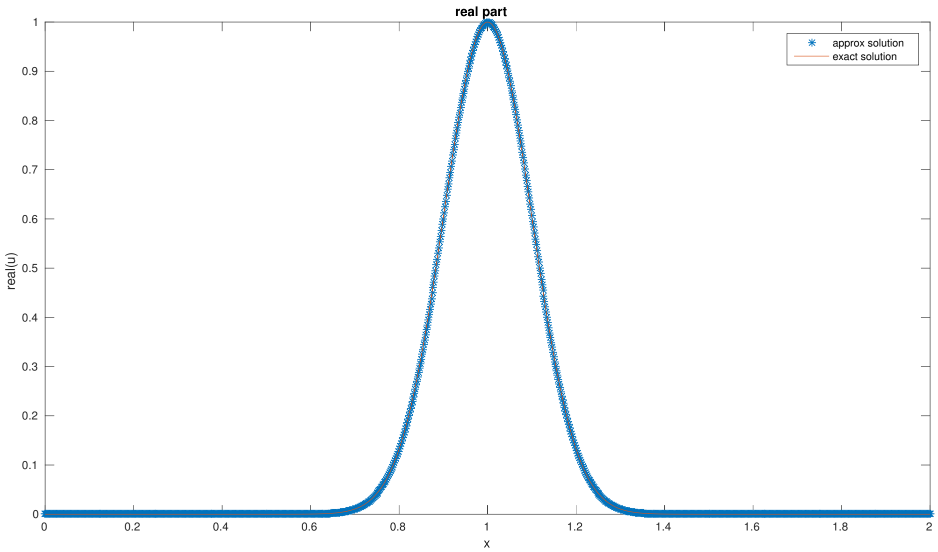

real part
approx solution
exact solution
real(u)
x
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

FIGURE 1.13

Chapter 2

Parabolic Problem

2.1 Parabolic Problem

We consider the convection-diffusion-reaction equation with the following conditions

$$u_t - \epsilon u_{xx} + bu_x + cu = f \text{ for } (x,t) \in \Omega = (0,1) \tag{2.1}$$

$$u(x,0) = \phi(x) \tag{2.2}$$

$$u(0,t) = \alpha(t), u(1,t) = \beta(t) \tag{2.3}$$

The coefficients on the grid points are approximated as follows:

$$b(x,t) \approx b_j^n, \quad c(x,t) \approx c_j^n, \quad f(x,t) \approx f_j^n \text{ for } x \in [x_{j-1}, x_{j+1}], \;\; t \in [t^n, t^{n+1}]$$

Thus the equation is now,

$$u_t - \epsilon u_{xx} + b_j^n + c_j^n = f_j^n \text{ for } x \in [x_{j-1}, x_{j+1}], \;\; t \in [t^n, t^{n+1}]$$

For $c_j^n > 0$, we let $v(x,t) = u(x,t) - f_j^n / c_j^n$. Then $v(x,t)$ satisfies

$$v_t - \epsilon v_{xx} + b_j^n + c_j^n v = 0 \text{ for } x \in [x_{j-1}, x_{j+1}], \;\; t \in [t^n, t^{n+1}]$$

Let

$$H_3 = \{w(x,t)| = w(x,t) = \beta_0 e^{-c_j^n t}, \beta_1 e^{\lambda_+ x}, \beta_2 e^{\lambda_- x} \text{ for } \beta_i \in R\}$$

with

$$\lambda_{\pm} = \frac{b_j^n}{2\epsilon} \pm \sqrt{\frac{(b_j^n)^2}{4\epsilon^2} + \frac{c_j^n}{\epsilon}}$$

We use the following scheme

$$v_j^{n+1} = \alpha_1 v_{j-1}^n + \alpha_2 v_j^n + \alpha_3 v_{j+1}^n$$

Substituting the basis functions in the above scheme we obtain

$$\alpha_1 + \alpha_2 + \alpha_3 = e^{-c_j^n \tau} \tag{2.4}$$

$$\alpha_1 e^{-\lambda_+ h} + \alpha_2 + \alpha_3 e^{\lambda_+ h} = 1 \tag{2.5}$$

$$\alpha_1 e^{-\lambda_- h} + \alpha_2 + \alpha_3 e^{\lambda_- h} = 1 \tag{2.6}$$

On solving the above system we get

$$\alpha_1 = \frac{e^{-\lambda_+ h}(1 - e^{-c_j^n}\tau)}{(1 - e^{-\lambda_+ h})(1 - e^{\lambda_- h})} \tag{2.7}$$

$$\alpha_3 = \frac{e^{\lambda_- h}(1 - e^{-c_j^n}\tau)}{(1 - e^{-\lambda_+ h})(1 - e^{\lambda_- h})} \tag{2.8}$$

$$\alpha_2 = e^{-c_j^n \tau} - \alpha_1 - \alpha_3 \tag{2.9}$$

On substituting the expressions in the scheme we have the following expression

$$u_j^{n+1} = \alpha_1 u_{j-1}^n + \alpha_2 u_j^n + \alpha_3 u_{j+1}^n + (1 - e^{-c_n^j \tau})\frac{f_j^n}{c_j^n}$$

2.1.1 Stability criterion

The stability criterion is given as

$$\alpha_1 + \alpha_3 \leq \frac{1 + e^{-c\tau}}{2}$$

This scheme is claimed to have second order convergence.

2.1.2 Example

Consider the equation along with the given initial conditions and boundary conditions

$$
\begin{aligned}
&u_t - \epsilon u_{xx} + 2u_x + u = f(x,t) \\
&f(x,t) = e^{2(x-1)/\epsilon}(2\cos(2t) + \sin(2t)) + e^{(x-1)}(cos(t) - \epsilon\sin(t) + 3\sin(t))
\end{aligned}
\tag{2.10}
$$

The analytical solution is given by

$$u(x,t) = e^{2(x-1)/\epsilon} sin(2t) + e^{(x-1)} sin(t) \tag{2.11}$$

The following plots and error estimates have been obtained on solving the above example using the tailored finite point method.

(h, τ)	(1/16,2 $* 10^{-5}$)	(1/32,2 $* 10^{-5}$)	(1/64,2 $* 10^{-5}$)	(1/256,2 $* 10^{-5}$)
Error	0.00253	0.00064	0.00015	0.00003
Order	-	1.97	2.013	2.09

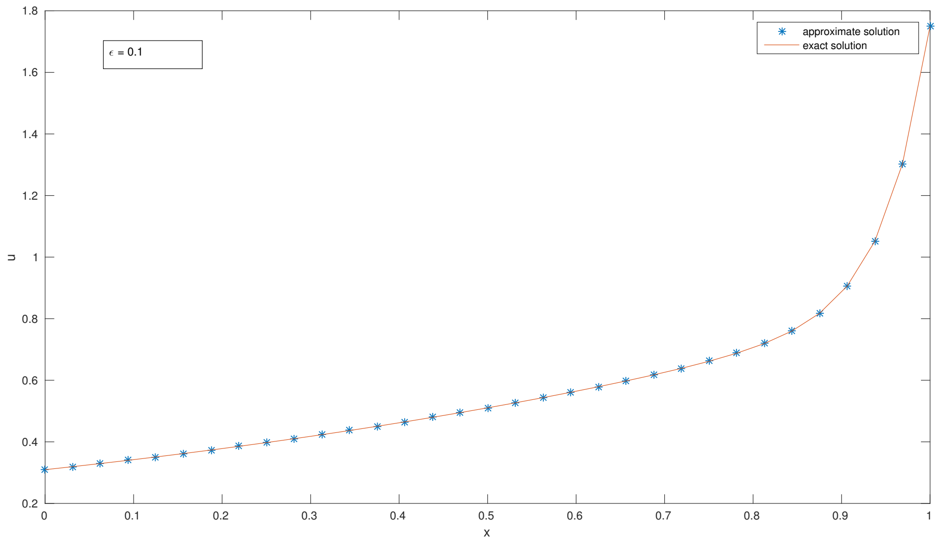

ϵ = 0.1
approximate solution
exact solution
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
u
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x

Figure 2.1

Chapter 3

TFPM in two dimensions

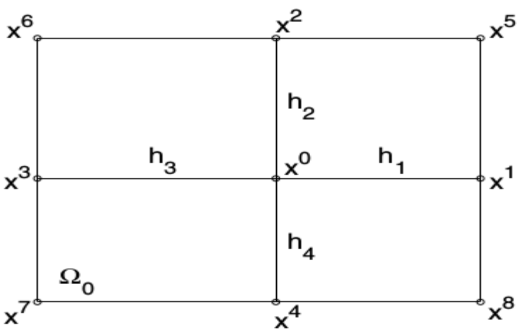
3.0.1 Convection-Diffusion-Reaction Problem

$$\mathbb{L}u \equiv -\epsilon^2 \Delta u + p(\mathbf{x})u_x + q(\mathbf{x})u_y + b(\mathbf{x})u = f(x), \forall \mathbf{x} = (x, y) \in \Omega \tag{3.1}$$

$$u|_{\partial\Omega} = 0 \tag{3.2}$$

For the sake of simplicity, we assume that $\Omega = [0, 1] \times [0, 1]$ and we have a uniform mesh, i.e $h = N^{-1}$ be the mesh size and

$$x_i = ih, y_j = jh, 0 \leq i, j \leq N \tag{3.3}$$



Then $\{P_{i,j} = (x_i, y_j), \quad 0 \leq i, j \leq N\}$ are the mesh points. We construct our TFP scheme for (3.1) on the cell Ω_0. We approximate the problem on the cell as follows

$$-\epsilon^2 \Delta + p_0(x)u_x + q_0 u_y + b_0 u = f_0 \tag{3.4}$$

with $p_0 = p(\mathbf{x}^0),\ q_0 = q(\mathbf{x}^0),\ b_0 = b(\mathbf{x}^0),\ f_0 = f(\mathbf{x}^0)$

We consider the case $b_0 > 0$

Let

$$u(x,y) = \frac{f_0}{b_0} + v(x,y)\exp\left(\frac{p_0 x + q_0 y}{2\epsilon^2}\right) \tag{3.5}$$

Then we have,

$$-\epsilon^2 \Delta v + d_0^2 = 0 \tag{3.6}$$

with $d_0^2 = b_0 + \frac{p_0^2+q_0^2}{4\epsilon^2}$
Let $\mu_0 = d_0/\epsilon$, and

$$H_4 = \left\{ v(x,y)|\ \ v = c_1 e^{-\mu_0 x} + c_2 e^{\mu_0 x} + c_3 e^{-\mu_0 y} + c_4 e^{-\mu_0 y}, \quad \forall c_i \in \mathbb{R} \right\}$$

Then we take the scheme as

$$\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 + \alpha_4 V_4 + \alpha_0 V_0 = 0 \tag{3.7}$$

with $V_j = v(\mathbf{x}^j)$. Thus we obtain

$$\alpha_1 e^{-\mu_0 h} + \alpha_2 + \alpha_3 e^{\mu_0 h} + \alpha_4 + \alpha_0 = 0 \tag{3.8}$$

$$\alpha_1 e^{\mu_0 h} + \alpha_2 + \alpha_3 e^{-\mu_0 h} + \alpha_4 + \alpha_0 = 0 \tag{3.9}$$

$$\alpha_1 + \alpha_2 e^{-\mu_0 h} + \alpha_3 + \alpha_4 e^{\mu_0 h} + \alpha_0 = 0 \tag{3.10}$$

$$\alpha_1 + \alpha_2 e^{\mu_0 h} + \alpha_3 + \alpha_4 e^{-\mu_0 h} + \alpha_0 = 0 \tag{3.11}$$

For any constant $\alpha_0 \in \mathbb{R}$ the system (1.10) to (1.13) has the unique solution

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{-\alpha_0}{e^{\mu_0 h} + e^{-\mu_0 h} + 2} \equiv \frac{-\alpha_0}{4\cosh^2\left(\frac{\mu_0 h}{2}\right)} \tag{3.12}$$

On taking

$$\alpha_0 = \frac{e^{\mu_0 h} + e^{-\mu_0 h} + 2}{e^{\mu_0 h} + e^{-\mu_0 h} - 2} \equiv \frac{\cosh^2\left(\frac{\mu_0 h}{2}\right)}{\sinh^2\left(\frac{\mu_0 h}{2}\right)} \tag{3.13}$$

we have

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = -\frac{1}{e^{\mu_0 h} + e^{-\mu_0 h} + 2} \equiv -\frac{1}{4sinh^2\left(\frac{\mu_0 h}{2}\right)} \tag{3.14}$$

We ultimately get the following scheme

$$\begin{aligned} U_0 - &\frac{e^{-\frac{p_0 h}{2\epsilon^2}} U_1 + e^{-\frac{q_0 h}{2\epsilon^2}} U_2 + e^{\frac{p_0 h}{2\epsilon^2}} U_3 + e^{\frac{q_0 h}{2\epsilon^2}} U_4}{4cosh^2\left(\frac{\mu_0 h}{2}\right)} \\ = &\frac{f_0}{b_0}\left(1 - \frac{e^{-\frac{p_0 h}{2\epsilon^2}} + e^{-\frac{q_0 h}{2\epsilon^2}} + e^{\frac{p_0 h}{2\epsilon^2}} + e^{\frac{q_0 h}{2\epsilon^2}}}{4cosh^2\left(\frac{\mu_0 h}{2}\right)}\right) \end{aligned} \tag{3.15}$$

with

$$U_j = u(\mathbf{x}^j) = \frac{f_0}{b_0} + V_j \exp\left(\frac{p_0 x + q_0 y}{2\epsilon^2}\right) \tag{3.16}$$

3.0.2 Example

Consider the following convection-diffusiion-reaction problem along with the given conditions

$$-\epsilon^2 \Delta u + p(\mathbf{x}) u_x + q(\mathbf{x}) + u_y + b(\mathbf{x}) u = f(\mathbf{x}), \qquad \forall x = (x, y) \in \Omega \tag{3.17}$$

$$u|_{\partial\Omega} = 0 \tag{3.18}$$

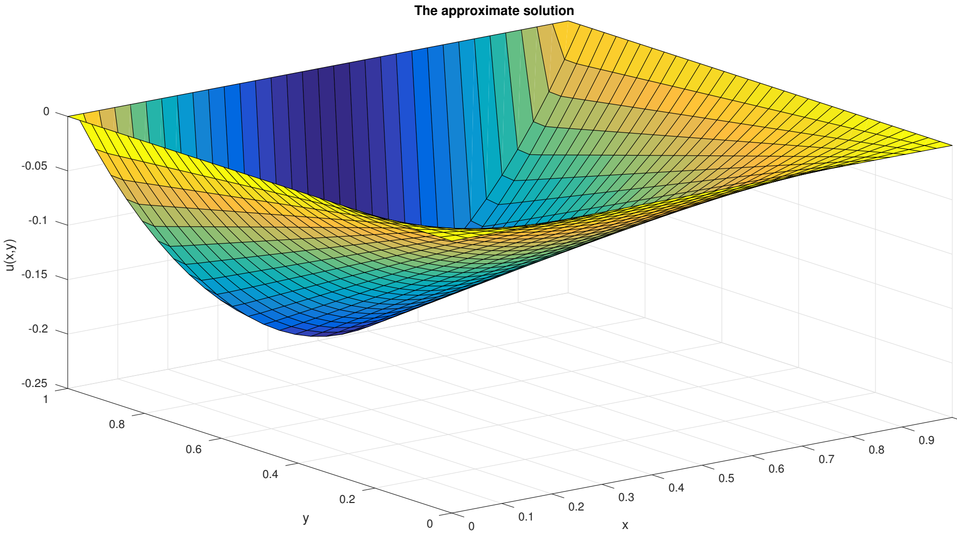
where

$$\begin{aligned} &\Omega = [0,1]^2, \;\; p(\mathbf{x}) = 1, \;\; q(\mathbf{x} = 0), \;\; b(\mathbf{x}) = 1, \\ &f(x,y) = [2\epsilon^2 + y(1-y)]\left[e^{\frac{x-1}{\epsilon^2}} + (x-1)e^{-\frac{1}{\epsilon^2} - x} - x\right] + y(1-y)\big(e^{-\frac{1}{\epsilon^2}} - 1\big) \end{aligned}$$

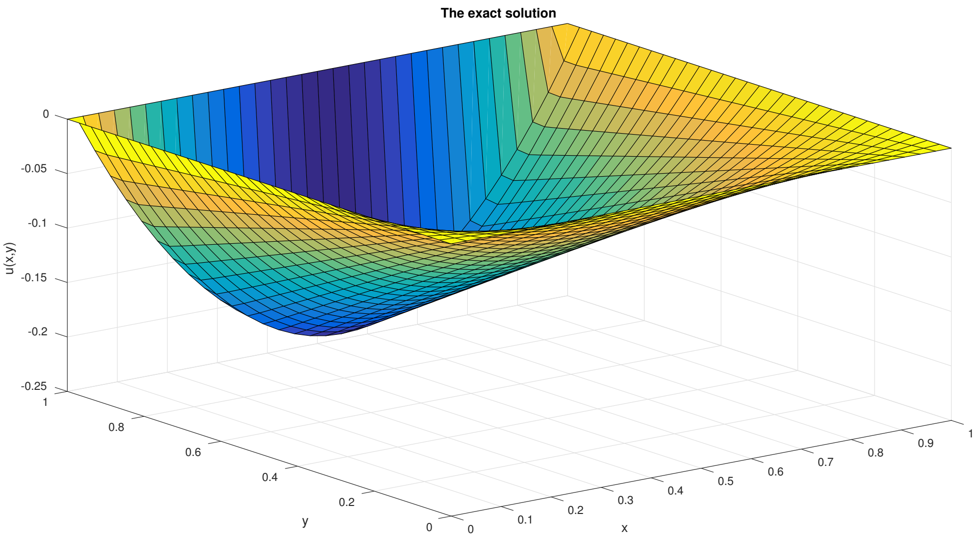
The exact solution of the problem is given by

$$u(x,y) = y(1-y)\left[e^{\frac{x-1}{\epsilon^2}} + (x-1)e^{-\frac{1}{\epsilon^2}} - x\right]$$

The following plots and error estimates have been obtained on solving the above example using the



tailored finite point method.



	mesh size h	1/16	1/32	1/64	1/128
$\epsilon = 0.1$	Error	0.0082	0.0032	9.4071e-04	2.2276e-04
	Order	-	1.3576	1.7662	2.0783
$\epsilon = 0.05$	Error	0.0061	0.0038	0.0021	8.400e-04
	Order	-	0.6828	0.8556	1.32
$\epsilon = 0.01$	Error	0.0048	0.0025	0.0013	6.7100e-04
	Order	-	0.9411	0.9434	0.9541

Chapter 4

PCTFPM and Shishkin Mesh Theory

The theory of some other concepts/methods which were encountered during the course of this thesis is described here.

4.1 Parallelogram Centred Tailored Finite Point Method (PCTFPM)

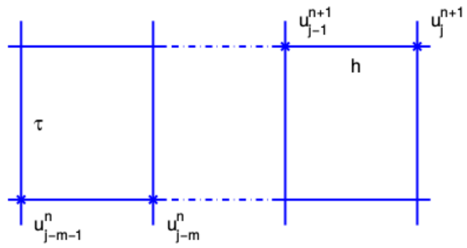


FIGURE 4.1

The parallelogram scheme is given as follows:

$$u_j^{n+1} = \alpha_{-1} u_{j-1}^{n+1} + \beta_{-1} u_{j-m-1}^{n} + \beta_0 u_{j-m}^{n}$$

As seen from the mesh and from the scheme, there is a diagonal like correspondance between the points of the next time step and the current time step. This can be useful whenever there is a discontinuity in the initial condition or an interior layer is being formed because of a discontinuous source term or a discontinuous coefficient.

By using the basis

$$V = \{v(x,t)|v(x,t) = c_1 + c_2 e^{(ik_j(at-x))} + c_3 e^{-(ik_j(at-x))}, \quad \forall c_1, c_2, c_3 \in \mathbb{C}\}$$

Taking the coeffecients (c_1, c_2, c_3) as (1,0,0), (0,1,0), (0,0,1) we get

$$\begin{aligned}
1 &= \alpha_{-1} + \beta_{-1} + \beta_0 \\
cos(k_j a\tau) &= \alpha_{-1}\cos(k_j(a\tau + h)) + \beta_{-1}\cos(k_j(m+1)h) + \beta_0 \cos(mh) \\
sin(k_j a\tau) &= \alpha_{-1}\sin(k_j(a\tau + h)) + \beta_{-1}\sin(k_j(m+1)h) + \beta_0 \sin(mh)
\end{aligned}$$

Thus we get

$$\begin{aligned}
\alpha_{-1} &= \frac{\sin(k_j(a\tau - (m+1)h)/2)}{\sin(k_j(a\tau - (m-1)h)/2)} \\
\beta_{-1} &= 1 \\
\beta_0 &= -\alpha_{-1}
\end{aligned}$$

PCTFPM is unconditionally stable and has second order convergence in space.

4.2 Shishkin Mesh

An approach to solving parabolic problem on a Shishkin mesh is described here.

The problem is as follows:

$$\begin{aligned}
&u_t - \epsilon u_{xx} + b u_x = 0 \text{ for } (x,t) \in \Omega = (0,1) \\
&u(x,0) = \phi(x) \\
&u(0,t) = \alpha(t), \quad u(1,t) = \beta(t)
\end{aligned}$$

4.2.1 Construction of the mesh and an Implicit Scheme

We consider the interval $[0,1]$. Let $\sigma = \max\{\frac{1}{2}, 1 - \frac{2\epsilon}{b} ln(N)\}$, where N is the number of intervals in the mesh. The mesh is divided into $[0,\sigma]$ and $[\sigma, 1]$ in $(N/2)$ equidistant parts.

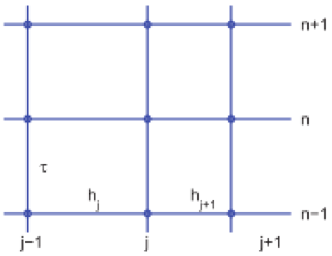


FIGURE 4.2

$$x_j = \begin{cases} jH & \text{when} j \leq N/2 \\ \sigma + (j - \frac{N}{2})h & \text{when} j > N/2 \end{cases}$$

where

$$H = \frac{2\sigma}{N}$$
$$h = \frac{2(1-\sigma)}{N}$$

$$u_j^n = a_1 u_{j-1}^{n+1} + a_2 u_j^{n+1} + a_3 u_{j+1}^{n+1}$$

The basis used is

$$\{1, x - bt, e^{\frac{bx}{\epsilon}}\}$$

Substituting these basis functions we get

$$a_1 + a_2 + a_3 = 1$$
$$a_1(-h_j) + a_3 h_{j+1} = b\tau$$
$$a_1 e^{-\frac{bh_j}{\epsilon}} + a_2 + a_3 e^{-\frac{bh_{j+1}}{\epsilon}} = 1$$

Solving the above system leads to

$$
\begin{aligned}
a_1 &= \frac{b\tau(e^{bh_{j+1}/\epsilon} - 1)}{h_{j+1}(1 - e^{-bh_j/\epsilon}) - h_j(e^{bh_{j+1}/\epsilon} - 1)} \\
a_3 &= \frac{b\tau(1 - e^{-bh_j/\epsilon})}{h_{j+1}(1 - e^{-bh_j/\epsilon}) - h_j(e^{bh_{j+1}/\epsilon} - 1)} \\
a_2 &= 1 - a_1 - a_3
\end{aligned}
$$

Thus the above scheme can be used to solve parabolic problems on a Shishkin mesh.

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