Computer methods in applied mechanics and engineering

Comput. Methods Appl. Mech. Engrg. 191 (2002) 1723–1751

www.elsevier.com/locate/cma

Mixed finite element formulations of strain-gradient elasticity problems

E. Amanatidou, N. Aravas *

Department of Mechanical and Industrial Engineering, University of Thessaly, Pedion Areos, Volos 38334, Greece

Received 8 May 2001; received in revised form 25 September 2001; accepted 28 September 2001

Abstract

Theories on intrinsic or material length scales find applications in the modeling of size-dependent phenomena. In elasticity, length scales enter the constitutive equations through the elastic strain energy function, which, in this case, depends not only on the strain tensor but also on gradients of the rotation and strain tensors. In the present paper, the strain-gradient elasticity theories developed by Mindlin and co-workers in the 1960s are treated in detail. In such theories, when the problem is formulated in terms of displacements, the governing partial differential equation is of fourth order. If traditional finite elements are used for the numerical solution of such problems, then C^1 displacement continuity is required. An alternative "mixed" finite element formulation is developed, in which both the displacement and the displacement gradients are used as independent unknowns and their relationship is enforced in an"integral-sense". A variational formulation is developed which can be used for both linear and non-linear strain-gradient elasticity theories. The resulting finite elements require only C^0 continuity and are simple to formulate. The proposed technique is applied to a number of problems and comparisons with available exact solutions are made.

Keywords: Strain-gradient elasticity; Finite elements; Mixed formulation

1. Introduction

Classical (local) continuum constitutive models possess no material/intrinsic length scale. The typical dimensions of length that appear in the corresponding boundary value problems are associated with the overall geometry of the domain under consideration. In spite of the fact that classical theories are quite sufficient for most applications, there is ample experimental evidence which indicates that, in certain applications, there is significant dependence on additional length/size parameters. An extensive summary of this experimental evidence is given in a recent review article by Fleck and Hutchinson [13]. In "gradient-type" plasticity theories, length scales are introduced through the coefficients of spatial gradients of one or

* Corresponding author. Tel.: +30-421-074002; fax: +30-421-074009.
E-mail address: aravas@mie.uth.gr (N. Aravas).

0045-7825/02/$ - see front matter
PII: S0045-7825(01)00353-X

more internal variables. In elasticity, length scales enter the constitutive equations through the elastic strain energy function, which, in this case, depends not only on the strain tensor but also on gradients of the rotation and strain tensors.

A first attempt to incorporate length scale effects in elasticity was made by Mindlin [25], Koiter [21,22] and Toupin [40], who considered "strain-gradient" elasticity theories, in which the elastic strain energy density is a function of strain as well as strain and rotation gradients. They solved also a number of problems and demonstrated the effects of the material length scales that enter the strain-gradient elasticity theories [22,25,26,28]. Several theoretical issues related to strain-gradient elasticity were addressed later by Germain [16–18]. Fleck, Hutchinson and their co-workers extended the Mindlin gradient-elasticity theory and developed a "deformation theory" of strain-gradient plasticity [12–14]. In the recent years, a variety of "non-local" or "gradient-type" theories have been used in order to introduce material length scales into constitutive models [1,7,24,30,41,44–46].

The solution of a number of boundary value problems is now available in the literature (e.g. [6,11,47,48]) including problems of fracture mechanics [2,15,20,39,42,43,49,50].

The finite element implementation of "strain-gradient" constitutive models has been the subject of several publications, especially in the recent years (e.g. [8–10,19,29,31–33,35–37,49]).

In the present paper, a general methodology for the finite element solution of strain-gradient elasticity problems is presented. In particular, the general family of strain-gradient models developed by Mindlin and co-workers is treated in detail. Mindlin [25] presented several alternative *equivalent* formulations of his theory by using different kinematic variables in the elastic strain energy density function W. Here, Mindlin's "Type I" and "Type III" equivalent formulations are analyzed in detail [25,27]. In the "Type I" formulation, W is written as a function of the strain tensor and the second spatial gradient of displacement; in "Type III", W is written in terms of the strain, the spatial gradient of rotation, and the fully symmetric part of the second spatial gradient of displacement (or of the gradient of strain). In such theories, when the problem is formulated in terms of displacements, the governing partial differential equations are of fourth order. Therefore, if traditional finite elements are used for the numerical solution of such problems, then C^1 displacement continuity is required. In this paper, a "mixed" finite element formulation is presented, in which the displacement and the displacement gradients are used as independent unknowns and their relationship is enforced in an "integral-sense". The developed "Type I" and "Type III" variational formulations can be used for both linear and non-linear elasticity theories such as the strain-gradient "deformation theory of plasticity" introduced recently by Fleck and Hutchinson [12–14]. The resulting finite elements require only C^0 continuity and are simple to formulate. Stress-like quantities which are "work-conjugate" to the aforementioned kinematic quantities are introduced and their relationship to the true stress $\boldsymbol{\sigma}$ and true couple stress $\boldsymbol{\mu}$ is discussed in detail. It is shown that the calculation of $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ is straightforward, when a "Type III" formulation is used; however, if the finite element solution is based on a "Type I" formulation, the calculation of $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ requires additional "post-processing" which reduces the accuracy of the calculated values of $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$. Several finite elements that can be used together with the aforementioned variational statements are studied. Numerical examples for a simple model problem, the problem of a plate with a hole, and a mode-III crack are presented and comparisons with available analytical solutions are made.

2. Notation and conventions

2.1. Tensor products

Standard notation is used throughout. Boldface symbols denote tensors the orders of which are indicated by the context. All tensor components are written with respect to a fixed Cartesian coordinate system with base vectors $\mathbf{e}_i$ $(i = 1, 2, 3)$, and the summation convention is used for repeated Latin indices, unless otherwise indicated. The prefixes tr and det indicate the trace and the determinant, respectively, a superscript T the transpose of a second-order tensor, and the subscripts S and A the symmetric and anti-symmetric parts of a second-order tensor. Let $(\mathbf{a}, \mathbf{b})$ be the vectors, and $(\mathbf{A}, \mathbf{B})$ the second-order tensors; the following products are used in the text $\mathbf{a} \cdot \mathbf{b} = a_i b_i$, $(\mathbf{ab})_{ij} = a_i b_j$, $(\mathbf{a} \cdot \mathbf{A})_i = a_k A_{ki}$, $(\mathbf{A} \cdot \mathbf{a})_i = A_{ik} a_k$, $(\mathbf{A} \cdot \mathbf{B})_{ij} = A_{ik} B_{kj}$, and $\mathbf{A} : \mathbf{B} = A_{ij} B_{ij}$. A comma followed by a subscript, say i, denotes partial differentiation with respect to the spatial coordinate x_i, i.e., $A_{,i} = \partial A / \partial x_i$. The following notation is also used for the symmetric and anti-symmetric parts of a second-order tensor:

$$A^{\mathrm{S}}_{ij} = A_{(ij)} \equiv \tfrac{1}{2}(A_{ij} + A_{ji}), \quad A^{\mathrm{A}}_{ij} = A_{[ij]} \equiv \tfrac{1}{2}(A_{ij} - A_{ji}). \tag{1}$$

2.2. "Normal" and "tangential" parts of tensors

Let S be the bounding surface of a body and $\mathbf{n}$ the outward unit normal to S. We consider a vector field $\mathbf{a}$ defined on S and define its "normal" $(\mathbf{a}^{\mathrm{n}})$ and "tangential" $(\mathbf{a}^{\mathrm{t}})$ parts as

$$\mathbf{a}^{\mathrm{n}} = (\mathbf{a} \cdot \mathbf{n})\mathbf{n} \quad \text{and} \quad \mathbf{a}^{\mathrm{t}} = \mathbf{a} - \mathbf{a}^{\mathrm{n}}. \tag{2}$$

Similarly, if $\mathbf{A}$ is a second-order tensor field $\mathbf{A}$ on S, we define its "normal" $(\mathbf{A}^{\mathrm{n}})$ and "tangential" $(\mathbf{A}^{\mathrm{t}})$ parts as

$$\mathbf{A}^{\mathrm{n}} = (\mathbf{A} \cdot \mathbf{n})\mathbf{n} = \mathbf{A} \cdot (\mathbf{n}\mathbf{n}) \quad \text{and} \quad \mathbf{A}^{\mathrm{t}} = \mathbf{A} - \mathbf{A}^{\mathrm{n}}. \tag{3}$$

If $\mathbf{A}$ and $\mathbf{B}$ are second-order tensors, it can be shown easily that

$$\mathbf{A}^{\mathrm{n}} : \mathbf{B}^{\mathrm{t}} = \mathbf{A}^{\mathrm{t}} : \mathbf{B}^{\mathrm{n}} = 0, \quad \mathbf{A} : \mathbf{B} = \mathbf{A}^{\mathrm{n}} : \mathbf{B}^{\mathrm{n}} + \mathbf{A}^{\mathrm{t}} : \mathbf{B}^{\mathrm{t}}, \quad \text{and} \quad \mathbf{A}^{\mathrm{t}} \cdot \mathbf{n} = \mathbf{0}. \tag{4}$$

On S we define also the normal derivative D of a field f as the magnitude of the normal part of $\nabla f = f_{,i}\mathbf{e}_i$, i.e.,

$$Df = (\nabla f) \cdot \mathbf{n} = f_{,i} n_i. \tag{5}$$

The "surface gradient" $\mathbf{D}f$ on S is defined as the tangential part of ∇f, i.e.,

$$\mathbf{D}f = \nabla f - (Df)\,\mathbf{n} \quad \text{or} \quad D_i f = f_{,i} - f_{,k} n_k n_i. \tag{6}$$

It should be noted that the tangential derivative of the unit vector $\mathbf{n}$ normal to the bounding surface S obeys the law

$$D_j n_i = D_i n_j. \tag{7}$$

If we consider the second-order tensor, say $\mathbf{A}$, defined by the gradient of a vector field $\mathbf{u}$ on S, i.e.,

$$\mathbf{A} = \mathbf{u}\nabla \quad \text{or} \quad A_{ij} = u_{i,j}, \tag{8}$$

then it can be shown easily that

$$\mathbf{A}^{\mathrm{n}} = (D\mathbf{u})\,\mathbf{n} \quad \text{and} \quad \mathbf{A}^{\mathrm{t}} = \mathbf{u}\mathbf{D} \tag{9}$$

or

$$A^{\mathrm{n}}_{ij} = (Du_i)\,n_j \quad \text{and} \quad \mathbf{A}^{\mathrm{t}}_{ij} = D_j u_i, \tag{10}$$

i.e., $\mathbf{A}^{\mathrm{n}}$ consists of the derivatives of the components u_i in the direction normal to S, whereas $\mathbf{A}^{\mathrm{t}}$ contains the derivatives of u_i on a plane tangent to S.

In Appendix A we show that for every second-order tensor $\mathbf{A}$,

$$A^{\mathrm{n}}_{ij} = n_k A^{\mathrm{t}}_{ki} n_j + 2A_{[ik]} n_k n_j + n_k A_{(kp)} n_p n_i n_j. \tag{11}$$

If we set now $A_{ij} = u_{i,j}$ in the last equation, we find that

$$(Du_i)\, n_j = n_k\, n_j\, D_i u_k + 2\, u_{[i,k]}\, n_k\, n_j + n_k\, u_{(k,p)}\, n_p\, n_i\, n_j. \tag{12}$$

3. A review of strain-gradient elasticity theories

3.1. Kinematic variables

Let $\mathbf{u}$ be the displacement field. The following quantities are defined:

$$\epsilon_{ij} = u_{(i,j)} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) = \text{strain}, \tag{13}$$

$$\Omega_{ij} = u_{[i,j]} = \tfrac{1}{2}(u_{i,j} - u_{j,i}) = -e_{ijk}\, \omega_k = \text{rotation tensor}, \tag{14}$$

$$\omega_i = \tfrac{1}{2}(\nabla \times \mathbf{u})_i = \tfrac{1}{2} e_{ijk}\, u_{k,j} = -\tfrac{1}{2} e_{ijk}\, \Omega_{jk} = \text{rotation vector (axial vector of } \boldsymbol{\Omega}), \tag{15}$$

$$\bar{\kappa}_{ij} = \omega_{j,i} = \text{rotation gradient}, \quad \bar{\kappa}_{ii} = 0, \tag{16}$$

$$\tilde{\kappa}_{ijk} = u_{k,ij} = \epsilon_{jk,i} + \epsilon_{ki,j} - \epsilon_{ij,k} = \tilde{\kappa}_{jik} = \text{second gradient of displacement}, \tag{17}$$

$$\hat{\kappa}_{ijk} = \tfrac{1}{2}(u_{j,ki} + u_{k,ji}) = \epsilon_{jk,i} = \hat{\kappa}_{ikj} = \text{strain gradient}, \tag{18}$$

$$\begin{aligned}\bar{\bar{\kappa}}_{ijk} &= \tfrac{1}{3}(u_{i,jk} + u_{j,ki} + u_{k,ij}) = \tfrac{1}{3}(\epsilon_{ij,k} + \epsilon_{jk,i} + \epsilon_{ki,j}) \\ &= \bar{\bar{\kappa}}_{jik} = \bar{\bar{\kappa}}_{ikj} = \bar{\bar{\kappa}}_{kji} = \text{symmetric part of } \tilde{\kappa}_{ijk} \text{ or } \hat{\kappa}_{ijk},\end{aligned} \tag{19}$$

where e_{ijk} is the alternating symbol.

The above quantities are related by the following expressions [27]:

$$\tilde{\kappa}_{ijk} = \hat{\kappa}_{ijk} + \hat{\kappa}_{jki} - \hat{\kappa}_{kij} = \bar{\bar{\kappa}}_{ijk} + \tfrac{2}{3}\bar{\kappa}_{ip}\, e_{pjk} + \tfrac{2}{3}\bar{\kappa}_{jp}\, e_{pik}, \tag{20}$$

$$\hat{\kappa}_{ijk} = \tfrac{1}{2}(\tilde{\kappa}_{ijk} + \tilde{\kappa}_{ikj}) = \bar{\bar{\kappa}}_{ijk} - \tfrac{1}{3}\bar{\kappa}_{jp}\, e_{kip} - \tfrac{1}{3}\bar{\kappa}_{kp}\, e_{jip}, \tag{21}$$

$$\bar{\kappa}_{ij} = \tfrac{1}{2}\tilde{\kappa}_{ipk}\, e_{jpk} = \hat{\kappa}_{pik}\, e_{jpk}, \tag{22}$$

$$\bar{\bar{\kappa}}_{ijk} = \tfrac{1}{3}(\tilde{\kappa}_{ijk} + \tilde{\kappa}_{jki} + \tilde{\kappa}_{kij}) = \tfrac{1}{3}(\hat{\kappa}_{ijk} + \hat{\kappa}_{jki} + \hat{\kappa}_{kij}). \tag{23}$$

3.2. Constitutive equations

The alternative forms of the strain-gradient elasticity theory given by Mindlin [25,27] are summarized in the following. The strain energy density W is written in three equivalent forms:

$$W = \tilde{W}(\boldsymbol{\epsilon}, \tilde{\boldsymbol{\kappa}}) = \hat{W}(\boldsymbol{\epsilon}, \hat{\boldsymbol{\kappa}}) = \bar{W}(\boldsymbol{\epsilon}, \bar{\boldsymbol{\kappa}}, \bar{\bar{\boldsymbol{\kappa}}}). \tag{24}$$

Mindlin refers to the description $W = \tilde{W}(\boldsymbol{\epsilon}, \tilde{\boldsymbol{\kappa}})$ as "Type I", to $W = \hat{W}(\boldsymbol{\epsilon}, \hat{\boldsymbol{\kappa}})$ as "Type II", and to $W = \bar{W}(\boldsymbol{\epsilon}, \bar{\boldsymbol{\kappa}}, \bar{\bar{\boldsymbol{\kappa}}})$ as "Type III". Mindlin and Eshel [27] present the most general form of $\tilde{W}$, $\hat{W}$ and $\bar{W}$ for an isotropic linear elastic material and derive the relationships among the material constants that appear in these three equivalent descriptions.

Using the above forms of the elastic strain energy density, one defines the following quantities:

$$\bar{\sigma}_{ij} = \frac{\partial \tilde{W}}{\partial \epsilon_{ij}} = \frac{\partial \hat{W}}{\partial \epsilon_{ij}} = \frac{\partial \bar{W}}{\partial \epsilon_{ij}} = \bar{\sigma}_{ji}, \tag{25}$$

$$\tilde{\mu}_{ijk} = \frac{\partial \tilde{W}}{\partial \tilde{\kappa}_{ijk}} = \tilde{\mu}_{jik}, \tag{26}$$

$$\hat{\mu}_{ijk} = \frac{\partial \hat{W}}{\partial \hat{\kappa}_{ijk}} = \hat{\mu}_{ikj}, \tag{27}$$

$$\bar{\mu}_{ij} = \frac{\partial \bar{W}}{\partial \bar{\kappa}_{ij}}, \quad \bar{\bar{\mu}}_{ijk} = \frac{\partial \bar{W}}{\partial \bar{\bar{\kappa}}_{ijk}} = \bar{\bar{\mu}}_{jik} = \bar{\bar{\mu}}_{ikj} = \bar{\bar{\mu}}_{kji}. \tag{28}$$

It is worthy of note that $\bar{\sigma}_{ij}$, $\tilde{\mu}_{ijk}$, $\hat{\mu}_{ijk}$, $\bar{\mu}_{ij}$, and $\bar{\bar{\mu}}_{ijk}$ are introduced as "conjugate" quantities to ϵ_{ij}, $\tilde{\kappa}_{ijk}$, $\hat{\kappa}_{ijk}$, $\bar{\kappa}_{ij}$, and $\bar{\bar{\kappa}}_{ijk}$, and their relationship to "true" stress or "true" couple stress is not obvious; this relationship is discussed in Section 3.4.

The quantities defined above are related by the following expressions [27]:

$$\tilde{\mu}_{ijk} = \tfrac{1}{2}(\hat{\mu}_{ijk} + \hat{\mu}_{jik}) = \bar{\bar{\mu}}_{ijk} + \tfrac{1}{4}\bar{\mu}_{ip}\, e_{pjk} + \tfrac{1}{4}\bar{\mu}_{jp}\, e_{pik}, \tag{29}$$

$$\hat{\mu}_{ijk} = \tilde{\mu}_{ijk} + \tilde{\mu}_{kij} - \tilde{\mu}_{jki} = \bar{\bar{\mu}}_{ijk} + \tfrac{1}{2}\bar{\mu}_{jp}\, e_{pik} + \tfrac{1}{2}\bar{\mu}_{kp}\, e_{pij}, \tag{30}$$

$$\bar{\mu}_{ij} = \tfrac{4}{3}\tilde{\mu}_{ikp}\, e_{jkp} = \tfrac{2}{3}\hat{\mu}_{kpi}\, e_{jkp}, \tag{31}$$

$$\bar{\bar{\mu}}_{ijk} = \tfrac{1}{3}(\tilde{\mu}_{ijk} + \tilde{\mu}_{jki} + \tilde{\mu}_{kij}) = \tfrac{1}{3}(\hat{\mu}_{ijk} + \hat{\mu}_{jki} + \hat{\mu}_{kij}). \tag{32}$$

It should be noted also that the so-called "micropolar theory of elasticity" [21,22,28,40] is a special case of the above theory. In fact, if $\bar{W}$ is independent of $\bar{\bar{\kappa}}_{ijk}$, i.e., $W = \bar{W}(\epsilon_{ij}, \bar{\kappa}_{ij})$, then $\bar{\bar{\mu}}_{ijk} \equiv 0$ and $\bar{\mu}_{ij}$ becomes the usual "couple-stress" tensor.

The variation of the internal work is given in a separate expression for each of the three equivalent forms of W:

$$\delta W^{\text{int}} = \int_V \delta W \, \mathrm{d}V = \int_V (\bar{\sigma}_{ij}\, \delta\epsilon_{ij} + \tilde{\mu}_{ijk}\, \delta\tilde{\kappa}_{ijk})\, \mathrm{d}V \tag{33}$$

$$= \int_V (\bar{\sigma}_{ij}\, \delta\epsilon_{ij} + \hat{\mu}_{ijk}\, \delta\hat{\kappa}_{ijk})\, \mathrm{d}V \tag{34}$$

$$= \int_V (\bar{\sigma}_{ij}\, \delta\epsilon_{ij} + \bar{\mu}_{ij}\, \delta\bar{\kappa}_{ij} + \bar{\bar{\mu}}_{ijk}\, \bar{\bar{\kappa}}_{ijk})\, \mathrm{d}V, \tag{35}$$

where V is the volume of the elastic body.

Let f_i be the "body force" per unit volume and Φ_{ij} the "body double force" per unit volume. Using Eqs. (33)–(35) and making use of Stokes' surface divergence theory, one is motivated to adopt the following form for the variation of work done by external forces for each form of W [25,27]:

$$\begin{aligned}\delta W^{\text{ext}} &= \int_V (f_i\, \delta u_i + \Phi_{ij}\, \delta u_{j,i})\, \mathrm{d}V + \int_S [\tilde{P}_i\, \delta u_i + \tilde{R}_i\, D(\delta u_i)]\, \mathrm{d}V + \sum_\alpha \oint_{C^\alpha} \tilde{E}_i\, \delta u_i\, \mathrm{d}s \\ &= \int_V (f_i\, \delta u_i + \Phi_{ij}\, \delta u_{j,i})\, \mathrm{d}V + \int_S [\hat{P}_i\, \delta u_i + \hat{R}_i\, D(\delta u_i)]\, \mathrm{d}V + \sum_\alpha \oint_{C^\alpha} \hat{E}_i\, \delta u_i\, \mathrm{d}s \\ &= \int_V (f_i\, \delta u_i + \Phi_{[ij]}\, \delta\Omega_{ji} + \Phi_{(ij)}\delta\epsilon_{ji})\, \mathrm{d}V + \int_S (\bar{P}_i\, \delta u_i + \bar{Q}_i^{\mathrm{t}}\, \delta\omega_i^{\mathrm{t}} + \bar{R}\, \delta\epsilon^{\mathrm{n}})\, \mathrm{d}V + \sum_\alpha \oint_{C^\alpha} \bar{E}_i\, \delta u_i\, \mathrm{d}s,\end{aligned} \tag{36}$$

where S is the bounding surface of the elastic body, $\epsilon^{\mathrm{n}} = n_i \epsilon_{ij} n_j$ the component of the strain tensor in the direction normal to S, and $(\tilde{\mathbf{P}}, \tilde{\mathbf{R}}, \tilde{\mathbf{E}})$, $(\hat{\mathbf{P}}, \hat{\mathbf{R}}, \hat{\mathbf{E}})$, $(\bar{\mathbf{P}}, \bar{\mathbf{Q}}^{\mathrm{t}}, \bar{R}, \bar{\mathbf{E}})$ are generalized external forces, which are defined precisely in the following. In the above equation, the line integrals over C^α are included when the outer surface S is piecewise smooth; in such a case, the surface S can be divided into a finite number of smooth surfaces S^α $(\alpha = 1, 2, \ldots)$ each bounded by an edge C^α, and integration is conducted along the arc length of each C^α.

We note that the body moment per unit volume is $M_i = e_{ijk} \Phi_{[jk]}$, so that the term $\Phi_{[ij]}\, \delta\Omega_{ji}$ in (36) can be written also as $\Phi_{[ij]}\, \delta\Omega_{ji} = M_i\, \delta\omega_i$, where $\delta\omega_i = -\frac{1}{2} e_{ijk}\, \delta\Omega_{jk}$.

The identity $\delta W^{\mathrm{int}} = \delta W^{\mathrm{ext}}$ leads to the following relations for the "external forces" [25,27]:

Type I

$$\tilde{P}_i = n_j(\bar{\sigma}_{ji} - \tilde{\mu}_{kji,k} - \Phi_{ji}) - [D_j - (D_p n_p) n_j](n_k \tilde{\mu}_{kji}), \tag{37}$$

$$\tilde{R}_i = n_k n_j \tilde{\mu}_{jki}, \tag{38}$$

$$\tilde{E}_i = [[\ell_j n_k \tilde{\mu}_{kji}]], \tag{39}$$

Type II

$$\hat{P}_i = n_j(\bar{\sigma}_{ji} - \hat{\mu}_{kji,k} - \Phi_{ji}) - [D_j - (D_p n_p) n_j](n_k \hat{\mu}_{kji}), \tag{40}$$

$$\hat{R}_i = n_k n_j \hat{\mu}_{jki}, \tag{41}$$

$$\hat{E}_i = [[\ell_j n_k \hat{\mu}_{kji}]], \tag{42}$$

Type III

$$\bar{P}_i = n_j(\bar{\sigma}_{ji} - \tfrac{1}{2}\bar{\mu}_{pk,p} e_{jik} - \bar{\bar{\mu}}_{kji,k} - \Phi_{ji}) - \tfrac{1}{2} n_j \bar{\mu}^{\mathrm{n}}_{,k} e_{ijk} - [D_j - (D_p n_p) n_j](n_k \bar{\bar{\mu}}_{kji} + n_i n_q n_p \bar{\bar{\mu}}_{pqj}), \tag{43}$$

$$\bar{Q}^{\mathrm{t}}_i = n_j \bar{\mu}^{\mathrm{t}}_{ji} + 2 n_q n_j n_k \bar{\bar{\mu}}_{kjp} e_{qpi}, \tag{44}$$

$$\bar{R} = n_i n_j n_k \bar{\bar{\mu}}_{ijk}, \tag{45}$$

$$\bar{E}_i = [[\tfrac{1}{2} s_i \bar{\mu}^{\mathrm{n}} + \ell_j n_k (\bar{\bar{\mu}}_{kji} + n_i n_p \bar{\bar{\mu}}_{pkj}]], \tag{46}$$

where $\bar{\mu}^{\mathrm{n}} = n_i \bar{\mu}_{ij} n_j$ on S.

In the above expressions, the double brackets [[ ]] indicate the jump in the value of the enclosed quantity across C^α, and $\boldsymbol{l} = \mathbf{s} \times \mathbf{n}$, where $\mathbf{s}$ is the unit vector tangent to C^α.

3.3. Boundary value problems

Mindlin and Eshel [27] have shown that the equilibrium equations and the appropriate boundary conditions for the three equivalent formulations are as follows:

Type I

$$(\bar{\sigma}_{ji} - \tilde{\mu}_{kji,k} - \Phi_{ji})_{,j} + f_i = 0 \quad \text{in } V. \tag{47}$$

At each point on the boundary S the following are specified: (i) u_i or $\tilde{P}_i$, and (ii) Du_i or $\tilde{R}_i$. If S is piecewise smooth, then u_i or $\tilde{E}_i$ is also specified at each point on the edges C^α.

Type II

$$(\hat{\sigma}_{ji} - \hat{\mu}_{kji,k} - \Phi_{ji})_{,j} + f_i = 0 \quad \text{in } V. \tag{48}$$

At each point on the boundary S the following are specified: (i) u_i or $\hat{P}_i$, and (ii) Du_i or $\hat{R}_i$. If S is piecewise smooth, then u_i or $\hat{E}_i$ is also specified at each point on the edges C^α.

Type III

$$(\bar{\sigma}_{ji} - \bar{\bar{\mu}}_{kji,k} - \tfrac{1}{2}\bar{\mu}_{pk,p}\, e_{jik} - \Phi_{ji})_{,j} + f_i = 0 \quad \text{in } V. \tag{49}$$

At each point on the boundary S the following are specified: (i) u_i or $\bar{P}_i$, (ii) ω_i^{t} or $\bar{Q}_i^{\mathrm{t}}$, and (iii) ϵ^{n} or $\bar{R}$. If S is piecewise smooth, then u_i or $\bar{E}_i$ is also specified at each point on the edges C^α.

The relationship between the Type I and Type III formulations has been discussed also by Smyshlyaev and Fleck [38], and Fleck and Hutchinson [13].

3.4. Relation to true stress, true couple-stress and true loads

In this section we discuss the relationship between $\bar{\sigma}_{ij}$, $\tilde{\mu}_{ijk}$, $\hat{\mu}_{ijk}$, $\bar{\mu}_{ij}$, $\bar{\bar{\mu}}_{ijk}$ defined in Section 3.2 and the "true" stresses. Also the relations between the "external loads", $\tilde{P}_i$, $\hat{P}_i$, $\bar{P}_i$, $\tilde{R}_i$, $\hat{R}_i$, $\tilde{R}$, $\bar{Q}_i$, $\tilde{E}_i$, $\hat{E}_i$, $\bar{E}_i$, and the "true" loads are discussed.

Let σ_{ij} be the usual true stress tensor and μ_{ij} the couple-stress tensor. On an infinitesimal area with unit normal vector $\mathbf{n}$, the traction vector $\mathbf{t}$ and the couple vector $\mathbf{m}$ are related to $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ by

$$t_i = n_j\,\sigma_{ji} \quad \text{and} \quad m_i = n_j\,\mu_{ji}. \tag{50}$$

If the body force per unit volume is f_i and the body moment per unit volume is $M_i = e_{ijk}\Phi_{[jk]}$, then the principles of linear and angular momentum lead to the well-known equations

$$\sigma_{ji,j} + f_i = 0 \quad \text{and} \quad \sigma_{[ij]} + \tfrac{1}{2}\mu_{pk,p}\, e_{ijk} + \Phi_{[ij]} = 0. \tag{51}$$

Mindlin and Eshel [27] adopt the following expression for the variation of the work done by external forces:

$$\delta W^{\mathrm{ext}} = \int_V (f_i\,\delta u_i + \Phi_{[ij]}\,\delta\Omega_{ji} + \Phi_{(ij)}\,\delta\epsilon_{ji})\,\mathrm{d}V + \int_S (t_i\,\delta u_i + m_i\,\delta\omega_i + n_i\,\bar{\bar{\mu}}_{ijk}\,\delta\epsilon_{jk})\,\mathrm{d}V + \sum_\alpha \oint_{C^\alpha} \bar{E}_i\,\delta u_i\,\mathrm{d}s. \tag{52}$$

It should be noted that the last two terms in each of the integrals of (52) can be written as

$$\Phi_{[ij]}\,\delta\Omega_{ji} + \Phi_{(ij)}\,\delta\epsilon_{ji} = \Phi_{ij}\,\delta u_{j,i} \quad \text{and} \quad m_i\,\delta\omega_i + n_i\,\bar{\bar{\mu}}_{ijk}\,\delta\epsilon_{jk} = T_{[ij]}\,\delta\Omega_{ji} + T_{(ij)}\,\delta\epsilon_{ji} = T_{ij}\,\delta u_{j,i}, \tag{53}$$

where $\boldsymbol{\Phi}$ is the "body double force" per unit volume, and $\mathbf{T}$ the "surface double force" per unit area (or "double traction") defined on S as

$$T_{ij}^{\mathrm{S}} \equiv T_{(ij)} = n_k\,\bar{\bar{\mu}}_{kij} \quad \text{and} \quad T_{ij}^{\mathrm{A}} \equiv T_{[ij]} = \tfrac{1}{2}e_{ijk}\,m_k = \tfrac{1}{2}e_{ijk}\,n_p\,\mu_{pk}. \tag{54}$$

Using the above expression for δW^{ext}, and setting it equal to

$$\delta W^{\mathrm{int}} = \int_V (\bar{\sigma}_{ij}\,\delta\epsilon_{ij} + \bar{\mu}_{ij}\,\delta\bar{\kappa}_{ij} + \bar{\bar{\mu}}_{ijk}\,\delta\bar{\bar{\kappa}}_{ijk})\,\mathrm{d}V, \tag{55}$$

Mindlin and Eshel [27] have shown that the true stress $\boldsymbol{\sigma}$ is related to $\bar{\boldsymbol{\sigma}}$, $\tilde{\boldsymbol{\mu}}$, $\hat{\boldsymbol{\mu}}$, $\bar{\boldsymbol{\mu}}$, and $\bar{\bar{\boldsymbol{\mu}}}$ by the expressions

$$\sigma_{(ij)} = \bar{\sigma}_{ij} - \tfrac{1}{3}(\tilde{\mu}_{ijk,k} + \tilde{\mu}_{jki,k} + \tilde{\mu}_{kij,k}) - \Phi_{(ij)} \tag{56}$$

$$= \bar{\sigma}_{ij} - \tfrac{1}{3}(\hat{\mu}_{ijk,k} + \hat{\mu}_{jki,k} + \hat{\mu}_{kij,k}) - \Phi_{(ij)} \tag{57}$$

$$= \bar{\sigma}_{ij} - \bar{\bar{\mu}}_{ijk,k} - \Phi_{(ij)}, \tag{58}$$

$$\sigma_{[ij]} = -\tfrac{4}{3}\tilde{\mu}_{k[ij],k} - \Phi_{[ij]} \tag{59}$$
$$= -\tfrac{2}{3}\hat{\mu}_{[ij]k,k} - \Phi_{[ij]} \tag{60}$$
$$= -\tfrac{1}{2}e_{ijk}\,\bar{\mu}_{pk,p} - \Phi_{[ij]}, \tag{61}$$

and the true couple stress $\boldsymbol{\mu}$ is

$$\mu_{ij} = \tfrac{4}{3}\tilde{\mu}_{ikp}\,e_{jkp} = \tfrac{2}{3}\hat{\mu}_{kpi}\,e_{jkp} = \bar{\mu}_{ij}. \tag{62}$$

In view of the last equation the double traction vector $\mathbf{T}$ can be written as

$$T_{ij} = n_k\,\bar{\bar{\mu}}_{kij} + \tfrac{1}{2}e_{ijk}\,n_p\,\bar{\mu}_{pk}. \tag{63}$$

Using the definition of $\bar{P}_i$, $\bar{Q}_i^{\mathrm{t}}$, $\bar{R}$, and $\bar{E}_i$ given in (43)–(46) and Eqs. (58)–(62) above, we can show easily that the "mathematical" loads $(\bar{P}_i, \bar{Q}_i^{\mathrm{t}}, \bar{R}, \bar{E}_i)$ are related to the true loads $(t_i, m_i, T_{(ij)})$ by the following expressions:

$$\bar{P}_i = t_i - \tfrac{1}{2}e_{ijk}\,D_k(m_p\,n_p\,n_j) + [(D_p n_p)\,n_j - D_j]\,(T_{(ji)} + n_k\,T_{(kj)}\,n_i), \tag{64}$$

$$\bar{Q}_i^{\mathrm{t}} = m_i^{\mathrm{t}} + 2\,e_{qpi}\,n_q\,n_j\,T_{(pj)}, \tag{65}$$

$$\bar{R} = n_i\,T_{(ij)}\,n_j, \tag{66}$$

$$\bar{E}_i = [[\tfrac{1}{2}s_i\,m_j\,n_j + \ell_j\,(T_{(ij)} + n_i\,T_{(jk)}\,n_k)]]. \tag{67}$$

Mindlin and Eshel [27] have shown also that

$$\tilde{P}_i = \hat{P}_i = \bar{P}_i + \tfrac{1}{2}n_j\,e_{jkp}\,D_k(\bar{Q}_p^{\mathrm{t}}\,n_i), \tag{68}$$

$$\tilde{R}_i = \hat{R}_i = \tfrac{1}{2}e_{ijk}\,\bar{Q}_j^{\mathrm{t}}\,n_k + \bar{R}\,n_i, \tag{69}$$

$$\tilde{E}_i = \hat{E}_i = \bar{E}_i - \tfrac{1}{2}s_k[[\bar{Q}_k^{\mathrm{t}}\,n_i]]. \tag{70}$$

Using direct notation, we can write the above relationships in the form

$$\bar{\mathbf{P}} = \mathbf{t} + \tfrac{1}{2}\mathbf{D} \times [(\mathbf{m}\cdot\mathbf{n})\,\mathbf{n}] + [(\mathbf{D}\cdot\mathbf{n})\,\mathbf{n} - \mathbf{D}]\cdot(\mathbf{T}^{\mathrm{S}} + \mathbf{n}\cdot\mathbf{T}^{\mathrm{S}}\,\mathbf{n}) \tag{71}$$

$$\bar{\mathbf{Q}}^{\mathrm{t}} = \mathbf{m}^{\mathrm{t}} + 2\,\mathbf{n} \times (\mathbf{n}\cdot\mathbf{T}^{\mathrm{S}}), \tag{72}$$

$$\bar{R} = \mathbf{n}\cdot\mathbf{T}^{\mathrm{S}}\cdot\mathbf{n}, \tag{73}$$

$$\bar{\mathbf{E}} = [[\tfrac{1}{2}(\mathbf{m}\cdot\mathbf{n})\,\mathbf{s} + \boldsymbol{l}\cdot\mathbf{T}^{\mathrm{S}}\cdot(\mathbf{I} + \mathbf{n}\,\mathbf{n})]], \tag{74}$$

and

$$\tilde{\mathbf{P}} = \hat{\mathbf{P}} = \bar{\mathbf{P}} + \tfrac{1}{2}\mathbf{n} \times \mathbf{D}(\mathbf{Q}^{\mathrm{t}}\,\mathbf{n}), \tag{75}$$

$$\tilde{\mathbf{R}} = \hat{\mathbf{R}} = \tfrac{1}{2}\bar{\mathbf{Q}}^{\mathrm{t}} \times \mathbf{n} + \bar{R}\,\mathbf{n}, \tag{76}$$

$$\tilde{\mathbf{E}} = \hat{\mathbf{E}} = \bar{\mathbf{E}} - \tfrac{1}{2}\mathbf{s}\cdot[[\bar{\mathbf{Q}}^{\mathrm{t}}\,\mathbf{n}]]. \tag{77}$$

The relationship between "true" and "mathematical" stresses and loads is discussed also by Germain in [16] and [18], where "higher-order" volumetric forces are also considered.

It should be noted that in physical problems one would, presumably, have information about the body force f_i and the body double force Φ_{ij} throughout the volume V, and the true traction t_i, the true moment per unit area m_i, and the double traction $T_{(ij)}$ on the surface S. Eqs. (64)–(70) indicate the way in which this information is to be employed in setting up the boundary "loads" in any of the three equivalent boundary value problems discussed in Section 3.3 within the framework of strain-gradient elasticity. It should be noted though that the true loads $(t_i, m_i, T_{(ij)})$ cannot be prescribed directly; instead, they enter the boundary conditions in certain combinations according to (64)–(70). Finally, after any of the three boundary value problems of Section 3.3 has been solved, the true stresses σ_{ij} and true couple stresses μ_{ij} are determined from Eqs. (56)–(62).

4. Variational formulation

A given boundary value problem in strain-gradient elasticity can be formulated in any of the three equivalent ways discussed in the previous section. Here we discuss the Type III formulation and emphasize the calculation of true stresses and true couple stresses. A brief discussion of the Type I formulation is also presented.

4.1. Type III formulation

The governing equations in V are:

$$\sigma_{ji,j} + f_i = 0, \tag{78}$$

$$\sigma_{ij} = \bar{\sigma}_{ij} + \bar{\sigma}_{ij}^{(2)}, \tag{79a}$$

$$\bar{\sigma}_{ij}^{(2)} = -\bar{\bar{\mu}}_{kij,k} - \tfrac{1}{2} e_{ijk}\, \bar{\mu}_{pk,p}, \tag{79b}$$

$$\epsilon_{ij} = u_{(i,j)}, \quad \omega_i = -\tfrac{1}{2} e_{ijk}\, u_{j,k} \quad \text{or} \quad u_{[i,j]} = -e_{ijk}\, \omega_k, \tag{80}$$

$$\bar{\kappa}_{ij} = \omega_{j,i}, \quad \bar{\bar{\kappa}}_{ijk} = \tfrac{1}{3}(\epsilon_{ij,k} + \epsilon_{jk,i} + \epsilon_{ki,j}), \tag{81}$$

$$\bar{\sigma}_{ij} = \frac{\partial \bar{W}}{\partial \epsilon_{ij}}, \quad \bar{\mu}_{ij} = \frac{\partial \bar{W}}{\partial \bar{\kappa}_{ij}}, \quad \bar{\bar{\mu}}_{ijk} = \frac{\partial \bar{W}}{\partial \bar{\bar{\kappa}}_{ijk}}. \tag{82}$$

The corresponding boundary conditions are

$$u_i = \bar{u}_i \quad \text{on } S_u, \tag{83}$$

$$n_j \sigma_{ji} - \tfrac{1}{2} n_j \bar{\mu}^{\rm n}_{,k} e_{ijk} + [(D_p n_p) n_j - D_j](n_k \bar{\bar{\mu}}_{kji} + n_i n_p n_k \bar{\bar{\mu}}_{kpj}) = \bar{P}_i \quad \text{on } S_P, \tag{84}$$

$$\omega_i^{\rm t} = \bar{\omega}_i^{\rm t} \quad \text{on } S_\omega, \tag{85}$$

$$n_j \bar{\mu}_{ji}^{\rm t} + 2 n_q n_j n_k \bar{\bar{\mu}}_{kjp} e_{qpi} = \bar{Q}_i^{\rm t} \quad \text{on } S_Q, \tag{86}$$

$$n_i n_j \epsilon_{ij} = \bar{\epsilon} \quad \text{on } S_\epsilon, \tag{87}$$

$$n_i n_j n_k \bar{\bar{\mu}}_{ijk} = \bar{R} \quad \text{on } S_R, \tag{88}$$

$$u_i = \bar{u}_i^\alpha \quad \text{on } C_u^\alpha, \tag{89}$$

$$[[\tfrac{1}{2}s_i\,\bar{\mu}^{\mathrm{n}} + \ell_j\,n_k\,(\bar{\bar{\mu}}_{kji} + n_i\,n_p\,\bar{\bar{\mu}}_{pjk})]] = \bar{E}_i^{\alpha} \quad \text{on } C_E^{\alpha}, \tag{90}$$

where $\bar{\mu}^{\mathrm{n}} = n_i\,n_j\,\bar{\mu}_{ij}$, $(\bar{\mathbf{u}}, \bar{\mathbf{P}}, \bar{\boldsymbol{\omega}}^{\mathrm{t}}, \bar{\mathbf{Q}}^{\mathrm{t}}, \bar{\epsilon}, \bar{R}, \bar{\mathbf{u}}^{\alpha}, \bar{\mathbf{E}}^{\alpha})$ are known functions, $S_u \cup S_P = S_\omega \cup S_Q = S_\epsilon \cup S_R = S$, $S_u \cap S_P = S_\omega \cap S_Q = S_\epsilon \cap S_R = \emptyset$, $C_u^{\alpha} \cup C_E^{\alpha} = C^{\alpha}$, and $C_u^{\alpha} \cap C_E^{\alpha} = \emptyset$.

We recall that, if we omit the terms involving $\bar{\bar{\kappa}}$ in $\bar{W}$, set $\bar{\bar{\boldsymbol{\mu}}} = \mathbf{0}$ and $S_\epsilon = S_R = \emptyset$, then we recover the standard boundary value problem of "micropolar elasticity" [21,22,28,40].

In the following, we present a variational formulation of the problem, in which $\mathbf{u}, \boldsymbol{\omega}, \boldsymbol{\epsilon}$, and $\bar{\boldsymbol{\sigma}}^{(2)}$ are viewed as the primary unknowns. In particular, the quantities $\mathbf{u}$, $\boldsymbol{\omega}$, and $\boldsymbol{\epsilon}$ are considered as independent variables subject to suitable side conditions. These side conditions are:

1. the kinematical equations $u_{i,j} = \epsilon_{ij} - e_{ijk}\,\omega_k$ in the entire body, and
2. the expression of the tangential part of the $\epsilon_{ij} - e_{ijk}\,\omega_k$ on the entire surface S in terms of the tangential derivatives $D_j u_i$ of the displacement, i.e.,

$$D_j u_i = (\epsilon_{ij} - e_{ijk}\,\omega_k)^{\mathrm{t}} = \epsilon_{ij} - e_{ijk}\,\omega_k - \epsilon_{ik}\,n_k\,n_j + e_{ipk}\,\omega_k\,n_p\,n_j \quad \text{on } S. \tag{91}$$

In the variational formulation, the following equations are stated in an integral form:

- the equilibrium equations (78) and the traction boundary conditions (84) and (90),
- the relationship between $\bar{\sigma}_{ij}^{(2)}$, $\bar{\mu}_{ij}$, and $\bar{\bar{\mu}}_{ijk}$ (79b) and the double force boundary conditions (86) and (88),
- the kinematical relation $u_{i,j} = \epsilon_{ij} - e_{ijk}\,\omega_k$ in V, and
- the condition $D_j u_i = (\epsilon_{ij} - e_{ijk}\,\omega_k)^{\mathrm{t}}$ on S.

The integral form of the aforementioned equations is

$$\begin{aligned}
&\int_V [(\bar{\sigma}_{ji} + \bar{\sigma}_{ji}^{(2)})_{,j} + f_i]\,u_i^*\,\mathrm{d}V \\
&\quad + \int_{S_P} \{\bar{P}_i - n_j(\bar{\sigma}_{ji} + \bar{\sigma}_{ji}^{(2)}) + \tfrac{1}{2}n_j\,\bar{\mu}_{,k}^{\mathrm{n}}\,e_{ijk} - [(D_p n_p)\,n_j - D_j](n_k\,\bar{\bar{\mu}}_{kji} + n_i\,n_p\,n_k\,\bar{\bar{\mu}}_{kpj})\}u_i^*\,\mathrm{d}S \\
&\quad + \sum_{\alpha} \oint_{C_E^{\alpha}} \{\bar{E}_i^{\alpha} - [[\tfrac{1}{2}s_i\,\bar{\mu}^{\mathrm{n}} + \ell_j\,n_k\,(\bar{\bar{\mu}}_{jki} + n_i\,n_p\,\bar{\bar{\mu}}_{pjk})]]\}\,u_i^*\,\mathrm{d}s = 0,
\end{aligned} \tag{92}$$

$$\int_V (e_{ijk}\,\bar{\sigma}_{[jk]}^{(2)} + \bar{\mu}_{ji,j})\,\omega_i^*\,\mathrm{d}V + \int_{S_Q} (\bar{Q}_i^{\mathrm{t}} - n_j\,\bar{\mu}_{ji}^{\mathrm{t}} - 2\,n_q\,n_j\,n_k\,\bar{\bar{\mu}}_{jkp}\,e_{qpi})\,\omega_i^{*\mathrm{t}}\,\mathrm{d}S = 0, \tag{93}$$

$$\int_V (\bar{\sigma}_{(ij)}^{(2)} + \bar{\bar{\mu}}_{ijk,k})\,\epsilon_{ij}^*\,\mathrm{d}V + \int_{S_R} (\bar{R} - n_i\,n_j\,n_k\,\bar{\bar{\mu}}_{ijk})\,n_p\,n_q\,\epsilon_{pq}^*\,\mathrm{d}S = 0, \tag{94}$$

$$\int_V [u_{i,j} - (\epsilon_{ij} + \Omega_{ij}(\boldsymbol{\omega}))]\,\bar{\sigma}_{ji}^{(2)*}\,\mathrm{d}V = 0, \tag{95}$$

$$\int_S [D_j u_i - (\epsilon_{ij} - e_{ijk}\,\omega_k - \epsilon_{ik}\,n_k\,n_j + e_{ipk}\,\omega_k\,n_p\,n_j)]\,X_{ij}^*\,\mathrm{d}S = 0, \tag{96}$$

where $\bar{\mu}^{\mathrm{n}} = \mathbf{n} \cdot \bar{\boldsymbol{\mu}} \cdot \mathbf{n}$, $\Omega_{ij}(\boldsymbol{\omega}) = -e_{ijk}\,\omega_k$,

$$\bar{\sigma}_{ij} = \frac{\partial \bar{W}}{\partial u_{(i,j)}}, \quad \bar{\mu}_{ij} = \frac{\partial \bar{W}}{\partial \bar{\kappa}_{ij}}, \quad \bar{\bar{\mu}}_{ijk} = \frac{\partial \bar{W}}{\partial \bar{\bar{\kappa}}_{ijk}}, \quad \bar{W} = \bar{W}(u_{(i,j)}, \bar{\kappa}_{ij}, \bar{\bar{\kappa}}_{ijk}), \tag{97}$$

$$\bar{\kappa}_{ij}(\boldsymbol{\omega}) = \omega_{j,i}, \quad \bar{\bar{\kappa}}_{ijk}(\boldsymbol{\epsilon}) = \tfrac{1}{3}(\epsilon_{ij,k} + \epsilon_{jk,i} + \epsilon_{ki,j}). \tag{98}$$

Eqs. (92)–(96) are required to hold for all $\mathbf{u}^* \in L^2$ such that $\mathbf{u}^* = \mathbf{0}$ on S_u and C_u^α, for all $\boldsymbol{\omega}^* \in L^2$ such that $\boldsymbol{\omega}^{*t} = \mathbf{0}$ on S_ω, for all $\boldsymbol{\epsilon}^* \in L^2$ such that $\mathbf{n} \cdot \boldsymbol{\epsilon}^* \cdot \mathbf{n} = 0$ on S_ϵ, and for all $\mathbf{X}^* \in L^2$, where L^2 is the space of square integrable functions.

In the following, we use Green's theorem in Eqs. (92)–(94), reformulate Eq. (96), and derive the weak form of the problem.

Equation (92)

In Appendix B we show that

$$\int_S \{[(D_p n_p) n_j - D_j](n_k \bar{\bar{\mu}}_{kji} + n_i n_p n_k \bar{\bar{\mu}}_{kpj})\} u_i^* \,\mathrm{d}S = \int_S (n_k \bar{\bar{\mu}}_{kji} + n_i n_p n_k \bar{\bar{\mu}}_{kpj}) D_j u_i^* \,\mathrm{d}S - \sum_\alpha \oint_{C^\alpha} [[\ell_j n_k (\bar{\bar{\mu}}_{kji} + n_i n_p \bar{\bar{\mu}}_{pjk})]] u_i^* \,\mathrm{d}s. \tag{99}$$

Using Eq. (12) we find that

$$D_j u_i^* = u_{i,j}^* - (D u_i^*) n_j = u_{i,j}^* - n_k n_j D_i u_k^* - 2 u_{[i,k]}^* n_k n_j - n_k u_{(k,p)}^* n_p n_i n_j. \tag{100}$$

Therefore,

$$(n_k \bar{\bar{\mu}}_{kji} + n_i n_p n_k \bar{\bar{\mu}}_{kpj}) D_j u_i^* = n_k \bar{\bar{\mu}}_{kji} (u_{(i,j)}^* - 2 n_j n_p u_{[i,p]}^* - n_i n_j n_p n_q u_{(p,q)}^*) \tag{101}$$

so that (99) can be written as

$$\int_S \{[(D_p n_p) n_j - D_j](n_k \bar{\bar{\mu}}_{kji} + n_i n_p n_k \bar{\bar{\mu}}_{kpj})\} u_i^* \,\mathrm{d}S = \int_S n_k \bar{\bar{\mu}}_{kji} (u_{(i,j)}^* - 2 n_j n_p u_{[i,p]}^* - n_i n_j n_p n_q u_{(p,q)}^*) \,\mathrm{d}S - \sum_\alpha \oint_{C^\alpha} [[\ell_j n_k (\bar{\bar{\mu}}_{kji} + n_i n_p \bar{\bar{\mu}}_{kpj})]] u_i^* \,\mathrm{d}s. \tag{102}$$

Also, according to Stokes' theorem

$$\int_S [\nabla \times (\bar{\mu}^{\mathrm{n}} \mathbf{u}^*)] \cdot \mathbf{n} \,\mathrm{d}S = \sum_\alpha \oint_{C_E^\alpha} [[\bar{\mu}^{\mathrm{n}}]] \mathbf{u}^* \cdot \mathbf{s} \,\mathrm{d}s \tag{103}$$

so that

$$\int_S \tfrac{1}{2} n_j \bar{\mu}_{,k}^{\mathrm{n}} e_{jki} u_i^* \,\mathrm{d}S = \sum_\alpha \oint_{C^\alpha} \tfrac{1}{2} [[\bar{\mu}^{\mathrm{n}}]] u_i^* s_i \,\mathrm{d}s - \int_S \tfrac{1}{2} \bar{\mu}^{\mathrm{n}} e_{ijk} u_{k,j}^* n_i \,\mathrm{d}S. \tag{104}$$

Taking into account (102), (104) and that $\mathbf{u}^* = \mathbf{0}$ on S_u and C_u^α, we can write (92) as

$$\int_V [(\bar{\sigma}_{ji} + \bar{\sigma}_{ji}^{(2)})_{,j} + f_i] u_i^* \,\mathrm{d}V + \int_{S_P} \bar{P}_i u_i^* \,\mathrm{d}S - \int_S n_j (\bar{\sigma}_{ji} + \bar{\sigma}_{ji}^{(2)}) u_i^* \,\mathrm{d}S - \int_S n_k \bar{\bar{\mu}}_{kji} (u_{(i,j)}^* - 2 n_j n_p u_{[i,p]}^* - n_i n_j n_p n_q u_{(p,q)}^*) \,\mathrm{d}S - \int_S \frac{1}{2} \bar{\mu}^{\mathrm{n}} e_{ijk} u_{k,j}^* n_i \,\mathrm{d}S + \sum_\alpha \oint_{C_E^\alpha} \bar{E}_i^\alpha u_i^* \,\mathrm{d}s = 0. \tag{105}$$

Finally, integrating by parts, we end up with the following equation of "virtual work"

$$\int_V [(\bar{\sigma}_{ji} + \bar{\sigma}_{ji}^{(2)}) u_{i,j}^* \,\mathrm{d}V + \int_S n_k \bar{\bar{\mu}}_{kji} (u_{(i,j)}^* - 2 n_j n_p u_{[i,p]}^* - n_i n_j n_p n_q u_{(p,q)}^*) \,\mathrm{d}S + \int_S \tfrac{1}{2} \bar{\mu}^{\mathrm{n}} e_{ijk} u_{k,j}^* n_i \,\mathrm{d}S = \int_V f_i u_i^* \,\mathrm{d}V + \int_{S_P} \bar{P}_i u_i^* \,\mathrm{d}S + \sum_\alpha \oint_{C_E^\alpha} \bar{E}_i^\alpha u_i^* \,\mathrm{d}s. \tag{106}$$

Equation (93)

Taking into account that $n_j\,\bar{\mu}^{\mathrm{t}}_{ji}\,\omega^{*\mathrm{t}}_i = n_j\,\bar{\mu}_{ji}\,\omega^{*\mathrm{t}}_i = n_j\,\bar{\mu}_{ji}\,\omega^*_i - n_j\,\bar{\mu}_{ji}\,n_i\,n_k\,\omega^*_k$, $\boldsymbol{\omega}^{*\mathrm{t}} = \mathbf{0}$ on S_ω, $n_q\,e_{pqi}\,\omega^{*\mathrm{t}}_i = n_q\,e_{pqi}\,\omega^*_i$ and integrating by parts, we can write (93) as

$$\int_V (-e_{ijk}\,\bar{\sigma}^{(2)}_{[jk]}\,\omega^*_i + \bar{\mu}_{ij}\,\bar{\kappa}^*_{ij})\,\mathrm{d}V - \int_S (2\,n_q\,n_j\,n_k\,\bar{\bar{\mu}}_{kjp}\,e_{pqi}\,\omega^*_i + n_j\,\bar{\mu}_{ji}\,n_i\,n_k\,\omega^*_k)\,\mathrm{d}S = \int_{S_Q} \bar{Q}^{\mathrm{t}}_i\,\omega^{*\mathrm{t}}_i\,\mathrm{d}S, \tag{107}$$

where $\bar{\kappa}^*_{ij} = \omega^*_{j,i}$.

Equation (94)

Taking into account that $n_p\,n_q\,\epsilon^*_{pq} = 0$ on S_ϵ, and integrating by parts, we can write (94) as

$$\int_V (-\bar{\sigma}^{(2)}_{(ij)}\,\epsilon^*_{ij} + \bar{\bar{\mu}}_{ijk}\,\bar{\bar{\kappa}}^*_{ijk})\,\mathrm{d}V + \int_S n_k\,\bar{\bar{\mu}}_{kij}(-\epsilon^*_{ij} + n_i\,n_j\,n_p\,n_q\,\epsilon^*_{pq})\,\mathrm{d}S = \int_{S_R} \bar{R}\,n_i\,n_j\,\epsilon^*_{ij}\,\mathrm{d}S, \tag{108}$$

where

$$\bar{\bar{\kappa}}^*_{ijk} = \tfrac{1}{3}(\epsilon^*_{ij,k} + \epsilon^*_{jk,i} + \epsilon^*_{ki,j}).$$

Equation (96)

Guided by (101) and (104) and in order to simplify the formulation, we set in (96)

$$X^*_{ij} = T^*_{(ij)} + n_i\,n_k\,T^*_{(kj)} + \tfrac{1}{2}e_{ijk}\,n_k\,\bar{\mu}^{\mathrm{n}*}, \tag{109}$$

without loss of generality (as X^*_{ij}, $T^*_{(ij)}$, and $\bar{\mu}^{\mathrm{n}*}$ are all arbitrary). Then (96) is equivalent to the following two equations:

$$\int_S [u_{(i,j)} - 2\,n_j\,n_k\,u_{[i,k]} - n_i\,n_j\,n_p\,n_q\,u_{(p,q)} - (\epsilon_{ij} + 2\,n_j\,n_k\,e_{ikp}\,\omega_p - n_i\,n_j\,n_p\,n_q\,\epsilon_{pq})]\,T^*_{(ij)}\,\mathrm{d}S = 0, \tag{110}$$

$$\int_S (\tfrac{1}{2}e_{ijk}\,u_{k,j}\,n_i - \omega_i\,n_i)\,\bar{\mu}^{\mathrm{n}*}\,\mathrm{d}S = 0, \tag{111}$$

which must hold for all $T^*_{(ij)} \in L^2$ and for all $\bar{\mu}^{\mathrm{n}*} \in L^2$.

Summarizing, we state the *weak form* of the problem. Find

1. $\mathbf{u}(\mathbf{x}) \in H^1$ satisfying $\mathbf{u}|_{S_u} = \bar{\mathbf{u}}$ and $\mathbf{u}|_{C^\alpha_u} = \bar{\mathbf{u}}^\alpha$, where H^k is the space of functions with square-integrable derivatives through order k,
2. $\boldsymbol{\omega}(\mathbf{x}) \in H^1$ satisfying $\boldsymbol{\omega}^{\mathrm{t}}|_{S_\omega} = \bar{\boldsymbol{\omega}}^{\mathrm{t}}$,
3. $\boldsymbol{\epsilon}(\mathbf{x}) = \boldsymbol{\epsilon}^{\mathrm{T}}(\mathbf{x}) \in H^1$ satisfying $\mathbf{n}\cdot\boldsymbol{\epsilon}|_{S_\epsilon}\cdot\mathbf{n} = \bar{\epsilon}$, and
4. $\bar{\boldsymbol{\sigma}}^{(2)}(\mathbf{x}) \in L^2$,

such that for all $\mathbf{u}^* \in H^1$ satisfying $\mathbf{u}^*|_{S_u} = \mathbf{u}^*|_{C^\alpha_u} = \mathbf{0}$, for all $\boldsymbol{\omega}^* \in H^1$ satisfying $\boldsymbol{\omega}^{*\mathrm{t}}|_{S_\omega} = \mathbf{0}$, for all $\boldsymbol{\epsilon}^* = \boldsymbol{\epsilon}^{*\mathrm{T}} \in H^1$ satisfying $\mathbf{n}\cdot\boldsymbol{\epsilon}^*|_{S_\epsilon}\cdot\mathbf{n} = 0$, for all $T^*_{(ij)} \in L^2$, and for all $\bar{\mu}^{\mathrm{n}*} \in L^2$:

$$\begin{aligned}\int_V (\bar{\sigma}_{ji} + \bar{\sigma}^{(2)}_{ji})\,u^*_{i,j}\,\mathrm{d}V &+ \int_S T_{(ij)}\,(u^*_{(i,j)} - 2\,n_j\,n_k\,u^*_{[i,k]} - n_i\,n_j\,n_p\,n_q\,u^*_{(p,q)})\,\mathrm{d}S + \int_S \tfrac{1}{2}\bar{\mu}^{\mathrm{n}}\,e_{ijk}\,u^*_{k,j}\,n_i\,\mathrm{d}S\\ &= \int_V f_i\,u^*_i\,\mathrm{d}V + \int_{S_P} \bar{P}_i\,u^*_i\,\mathrm{d}S + \sum_\alpha \oint_{C^\alpha_E} \bar{E}^\alpha_i\,u^*_i\,\mathrm{d}s,\end{aligned} \tag{112}$$

$$\int_V (-e_{ijk}\,\bar{\sigma}^{(2)}_{[jk]}\,\omega^*_i + \bar{\mu}_{ij}\,\bar{\kappa}^*_{ij})\,\mathrm{d}V - \int_S (2\,T_{(ij)}\,n_j\,n_k\,e_{ikp}\,\omega^*_p + \bar{\mu}^{\mathrm{n}}\,n_i\,\omega^*_i)\,\mathrm{d}S = \int_{S_Q} \bar{Q}^{\mathrm{t}}_i\,\omega^{*\mathrm{t}}_i\,\mathrm{d}S, \tag{113}$$

$$\int_V (-\bar{\sigma}^{(2)}_{(ij)}\,\epsilon^*_{ij} + \bar{\bar{\mu}}_{ijk}\,\bar{\bar{\kappa}}^*_{ijk})\,\mathrm{d}V + \int_S T_{(ij)}\,(-\epsilon^*_{ij} + n_i\,n_j\,n_p\,n_q\,\epsilon^*_{pq})\,\mathrm{d}S = \int_{S_R} \bar{R}\,n_i\,n_j\,\epsilon^*_{ij}\,\mathrm{d}S, \tag{114}$$

$$\int_V [u_{i,j} - (\epsilon_{ij} - e_{ijk}\,\omega_k)]\,\bar{\sigma}^{(2)*}_{ji}\,\mathrm{d}V = 0, \tag{115}$$

$$\int_S [u_{(i,j)} - 2\,n_j\,n_k\,u_{[i,k]} - n_i\,n_j\,n_p\,n_q\,u_{(p,q)} - (\epsilon_{ij} + 2\,n_j\,n_k\,e_{ikp}\,\omega_p - n_i\,n_j\,n_p\,n_q\,\epsilon_{pq})]\,T^*_{(ij)}\,\mathrm{d}S = 0, \tag{116}$$

$$\int_S (\tfrac{1}{2}e_{ijk}\,u_{k,j}\,n_i - \omega_i\,n_i)\,\bar{\mu}^{\mathrm{n}*}\,\mathrm{d}S = 0, \tag{117}$$

where $\bar{\mu}^{\mathrm{n}} = \mathbf{m}\cdot\mathbf{n}$ is the normal component of the couple vector $\mathbf{m} = \mathbf{n}\cdot\bar{\boldsymbol{\mu}}$ on S, $T_{(ij)} = n_k\,\bar{\bar{\mu}}_{kij}$ is the symmetric part of the double traction vector $\mathbf{T}$ on S,

$$\bar{\sigma}_{ij} = \frac{\partial\bar{W}}{\partial u_{(i,j)}}, \quad \bar{\mu}_{ij} = \frac{\partial\bar{W}}{\partial\bar{\kappa}_{ij}}, \quad \bar{\bar{\mu}}_{ijk} = \frac{\partial\bar{W}}{\partial\bar{\bar{\kappa}}_{ijk}}, \tag{118}$$

with $\bar{W} = \bar{W}(u_{(i,j)}, \bar{\kappa}_{ij}(\boldsymbol{\omega}), \bar{\bar{\kappa}}_{ijk}(\boldsymbol{\epsilon}))$, and

$$\bar{\kappa}_{ij}(\boldsymbol{\omega}) = \omega_{j,i}, \quad \bar{\bar{\kappa}}_{ijk}(\boldsymbol{\epsilon}) = \tfrac{1}{3}(\epsilon_{ij,k} + \epsilon_{jk,i} + \epsilon_{ki,j}). \tag{119}$$

The finite element formulation presented in Section 4.2 is based on the above weak form. The quantities $\mathbf{u}$, $\boldsymbol{\omega}$, $\boldsymbol{\epsilon}$, and $\bar{\boldsymbol{\sigma}}^{(2)}$ are the nodal unknowns and the true stress and true couple stress are calculated easily in a finite element solution by using the equations

$$\sigma_{ij} = \bar{\sigma}_{ij} + \bar{\sigma}^{(2)}_{ij} = \frac{\partial\bar{W}}{\partial u_{(i,j)}} + \bar{\sigma}^{(2)}_{ij} \quad \text{and} \quad \mu_{ij} = \frac{\partial\bar{W}}{\partial\bar{\kappa}_{ij}}, \tag{120}$$

where $\Phi_{ij} = 0$ is assumed.

It should be noted that in the special case of a two-dimensional problem we have that

$$u_1 = u_1(x_1, x_2), \quad u_2 = u_2(x_1, x_2) \quad \text{and} \quad \mathbf{n} = n_1\,\mathbf{e}_1 + n_2\,\mathbf{e}_2, \tag{121}$$

where all components refer to the Cartesian axes Ox_1–Ox_2 on the plane of the problem, and $\mathbf{e}_1$ and $\mathbf{e}_2$ are the corresponding base vectors. In that case

$$\tfrac{1}{2}e_{ijk}\,u_{k,j}\,\mathbf{e}_i = \tfrac{1}{2}(u_{2,1} - u_{1,2})\,\mathbf{e}_3 \quad \text{and} \quad \boldsymbol{\omega} = \omega_3\,\mathbf{e}_3, \tag{122}$$

where $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$ so that Eq. (117) is satisfied automatically.

Returning to the general three-dimensional case, we note that the solution to the problem can be given also by the stationarity condition $\delta\Pi = 0$ of the functional

$$\begin{aligned}\Pi(\mathbf{u}, \boldsymbol{\omega}, \boldsymbol{\epsilon}, \bar{\boldsymbol{\sigma}}^{(2)}) = {} & \int_V \bar{W}(u_{(i,j)}, \bar{\boldsymbol{\kappa}}(\boldsymbol{\omega}), \bar{\bar{\boldsymbol{\kappa}}}(\boldsymbol{\epsilon}))\,\mathrm{d}V + \int_V [u_{i,j} - (\epsilon_{ij} - e_{ijk}\,\omega_k)]\,\bar{\sigma}^{(2)}_{ji}\,\mathrm{d}V - \int_V f_i\,u_i\,\mathrm{d}V \\ & - \int_{S_P} \bar{P}_i\,u_i\,\mathrm{d}S - \int_{S_Q} \bar{Q}^{\mathrm{t}}_i\,\omega^{\mathrm{t}}_i\,\mathrm{d}S - \int_{S_R} \bar{R}\,n_i\,n_j\,\epsilon_{ij}\,\mathrm{d}S - \sum_\alpha \oint_{C^\alpha_E} \bar{E}^\alpha_i\,u_i\,\mathrm{d}s \\ & + \int_S [u_{(i,j)} - 2\,n_j\,n_k\,u_{[i,k]} - n_i\,n_j\,n_p\,n_q\,u_{(p,q)} - (\epsilon_{ij} + 2\,n_j\,n_k\,e_{ikp}\,\omega_p - n_i\,n_j\,n_p\,n_q\,\epsilon_{pq})]\,n_r\,\bar{\bar{\mu}}_{rij}\,\mathrm{d}S \\ & + \int_S (\tfrac{1}{2}e_{ijk}\,u_{k,j}\,n_i - \omega_i\,n_i)\,n_p\,\bar{\mu}_{pq}\,n_q\,\mathrm{d}S,\end{aligned} \tag{123}$$

where $\epsilon_{ji} = \epsilon_{ij}$, $\bar{\kappa}_{ij}(\boldsymbol{\omega}) = \omega_{j,i}$, $\bar{\bar{\kappa}}_{ijk}(\boldsymbol{\epsilon}) = \frac{1}{3}(\epsilon_{ij,k} + \epsilon_{jk,i} + \epsilon_{ki,j})$, $\bar{\mu}_{ij} = \partial\bar{W}/\partial\bar{\kappa}_{ij}$, $\bar{\bar{\mu}}_{ijk} = \partial\bar{W}/\partial\bar{\bar{\kappa}}_{ijk}$, $\delta\mathbf{u} = \mathbf{0}$ on S_u and C^α_u, $\delta\boldsymbol{\omega}^{\mathrm{t}} = \mathbf{0}$ on S_ω, and $\mathbf{n}\cdot\delta\boldsymbol{\epsilon}\cdot\mathbf{n} = 0$ on S_ϵ. The stationarity condition $\delta\Pi = 0$ implies the appropriate field equations and boundary conditions, $\bar{\sigma}^{(2)}_{ij} = -\bar{\bar{\mu}}_{kij,k} - \frac{1}{2}e_{ijk}\,\bar{\mu}_{pk,p}$ in V, and $D_j u_i = (\epsilon_{ij} - e_{ijk}\,\omega_k)^{\mathrm{t}}$ on S. It should be noted that the quantities $\bar{\sigma}_{ji}$, $n_r\bar{\bar{\mu}}_{rij}$ and $n_i\bar{\mu}_{ij}n_j$ in the above functional are Lagrange multipliers that enforce the corresponding constraints in V and on S.

In the two-dimensional case, the last term in the above functional vanishes identically and the integrals over C_E^α are replaced by

$$\sum_\alpha \bar{E}_i^\alpha u_i, \tag{124}$$

where the sum over α refers to any corners that may exist on the bounding curve of the two-dimensional body.

In the special case of micropolar elasticity ($\bar{\bar{\mu}}_{ijk}=0,\ S_\epsilon = S_R = \emptyset$), the above functional takes the form

$$\begin{aligned}\Pi(\mathbf{u},\boldsymbol{\omega},\bar{\boldsymbol{\sigma}}^{(2)}) = {} & \int_V \bar{W}(u_{(i,j)},\bar{\boldsymbol{\kappa}}(\boldsymbol{\omega}))\,\mathrm{d}V + \int_V (u_{[i,j]} + e_{ijk}\,\omega_k)\,\bar{\sigma}_{ji}^{(2)}\,\mathrm{d}V - \int_V f_i\,u_i\,\mathrm{d}V - \int_{S_P} \bar{P}_i\,u_i\,\mathrm{d}S \\ & - \int_{S_Q} \bar{Q}_i^{\mathrm{t}}\,\omega_i^{\mathrm{t}}\,\mathrm{d}S - \sum_\alpha \oint_{C_E^\alpha} \bar{E}_i^\alpha\,u_i\,\mathrm{d}s + \int_S \left(\tfrac{1}{2}e_{ijk}\,u_{k,j}\,n_i - \omega_i\,n_i\right)n_p\,\bar{\mu}_{pq}\,n_q\,\mathrm{d}S,\end{aligned} \tag{125}$$

where $\bar{\kappa}_{ij}(\boldsymbol{\omega}) = \omega_{j,i}$, $\bar{\sigma}_{ij}^{(2)} = -\bar{\sigma}_{ji}^{(2)}$, $\delta\mathbf{u} = \mathbf{0}$ on S_u and C_u^α, $\bar{\mu}_{ij} = \partial\bar{W}/\partial\bar{\kappa}_{ij}$, and $\delta\boldsymbol{\omega}^{\mathrm{t}} = \mathbf{0}$ on S_ω. The stationarity condition $\delta\Pi = 0$ implies the appropriate field equations and boundary conditions, $\bar{\sigma}_{ij}^{(2)} = -\frac{1}{2}e_{ijk}\,\bar{\mu}_{pk,p}$ in V, and $\frac{1}{2}e_{ijk}\,u_{k,j}\,n_i = \omega_i\,n_i$ on S.

In the two-dimensional case, the last term in the above functional vanishes identically, and the functional reduces to that used by Herrmann [19] for the solution of two-dimensional problems using the finite element method. It should be emphasized though that, in three-dimensional problems, the last term in (125) does not vanish identically and must be included in the formulation.

4.2. Type I formulation

In this section we discuss briefly the alternative Type I formulation. Following the steps of the previous section, we can show easily that the solution to the problem can be formulated alternatively by the stationarity condition $\delta\Pi = 0$ of the functional

$$\begin{aligned}\Pi(\mathbf{u},\boldsymbol{\alpha},\tilde{\boldsymbol{\sigma}}^{(2)}) = {} & \int_V \tilde{W}(u_{(i,j)},\tilde{\boldsymbol{\kappa}}(\boldsymbol{\alpha}))\,\mathrm{d}V + \int_V (u_{i,j} - \alpha_{ij})\,\tilde{\sigma}_{ji}^{(2)}\,\mathrm{d}V - \int_V f_i\,u_i\,\mathrm{d}V - \int_{\Gamma_P} \tilde{P}_i\,u_i\,\mathrm{d}S \\ & - \int_{\Gamma_R} \tilde{R}_i\,n_j\,\alpha_{ij}\,\mathrm{d}S - \sum_\alpha \oint_{C_E^\alpha} \tilde{E}_i^\alpha\,u_i\,\mathrm{d}s + \int_S (D_j u_i - \alpha_{ij}^{\mathrm{t}})\,n_k\,\tilde{\mu}_{kji}\,\mathrm{d}S,\end{aligned} \tag{126}$$

where $\tilde{\kappa}_{ijk}(\boldsymbol{\alpha}) = \alpha_{k(i,j)}$, $\tilde{\mu}_{ijk} = \partial\tilde{W}/\partial\tilde{\kappa}_{ijk}$, with $\delta\mathbf{u} = \mathbf{0}$ on $S - \Gamma_P$ and $C^\alpha - C_E^\alpha$, and $\mathbf{n}\cdot\delta\boldsymbol{\alpha} = \mathbf{0}$ on $S - \Gamma_R$. The stationarity condition $\delta\Pi = 0$ implies the appropriate field equations and boundary conditions, the relationship $\tilde{\sigma}_{ij}^{(2)} = -\tilde{\mu}_{kij,k}$ in V, and $D_j u_i = \alpha_{ij}^{\mathrm{t}}$ on S. In the above functional the quantities $\tilde{\sigma}_{ji}^{(2)}$ and $n_k\,\tilde{\mu}_{kji}$ are Lagrange multipliers that enforce the corresponding constraints in V and on S.

The above functional can form the basis for a finite element solution, in which the nodal unknowns are $\mathbf{u}$, $\boldsymbol{\alpha}$, and $\tilde{\boldsymbol{\sigma}}^{(2)}$. In fact, Shu and Barlow [36] presented recently a finite element formulation which corresponds to the above functional, if the last term in (126) that enforces the condition $D_j u_i = \alpha_{ij}^{\mathrm{t}}$ on the boundary S is omitted. Omission of this term results in non-symmetric stiffness matrices in the finite element solution. Our experience has shown that omission of the aforementioned boundary terms causes undesirable oscillations of the numerical solution on the boundary of the body.

It should be mentioned also that the numerical calculation of the true stresses and true couple stress in a finite element solution is not trivial, when a Type I formulation is used. In such a finite element solution, one can calculate easily the stress-like quantity

$$\tilde{\sigma}_{ij} = \bar{\sigma}_{ij} + \tilde{\sigma}^{(2)}_{ij} = \bar{\sigma}_{ij} - \tilde{\mu}_{kij,k}, \tag{127}$$

which is different from the true stress σ_{ij} in general. In fact, using Eqs. (56), (59) and (62) one can show that the true stress $\boldsymbol{\sigma}$ and the true couple stress $\boldsymbol{\mu}$ are given by the expressions (for $\Phi_{ij} = 0$)

$$\sigma_{ij} = \bar{\sigma}_{ij} + \tilde{\sigma}^{(2)}_{ij} - \tfrac{1}{3}\tilde{\sigma}^{(2)}_{ji} - \tfrac{1}{3}\tilde{\mu}_{ijk,k} \quad \text{and} \quad \mu_{ij} = \tfrac{4}{3}\tilde{\mu}_{ikp}\, e_{jkp}. \tag{128}$$

In a finite element solution of a Type I formulation the quantities $\bar{\sigma}_{ij}$, $\tilde{\sigma}^{(2)}_{ij}$, and $\tilde{\mu}_{ijk}$ are available readily. However, the calculation of the true stress σ_{ij} according to Eq. (128) requires the numerical evaluation of $\tilde{\mu}_{ijk,k}$, which reduces the accuracy of the calculated value of σ_{ij}.

We conclude this section with a statement of the weak form of the Type I formulation. Find

1. $\mathbf{u}(\mathbf{x}) \in H^1$ satisfying $\mathbf{u} = \tilde{\mathbf{u}}$ on S_u and $\mathbf{u} = \tilde{\mathbf{u}}^{\alpha}$ on C^{α}_u,
2. $\boldsymbol{\alpha}(\mathbf{x}) \in H^1$ satisfying $\boldsymbol{\alpha} \cdot \mathbf{n} = \tilde{\mathbf{d}}$ on $S - \Gamma_P$, and
3. $\tilde{\boldsymbol{\sigma}}^{(2)}(\mathbf{x}) \in L^2$,

such that for all $\mathbf{u}^* \in H^1$ satisfying $\mathbf{u}^* = \mathbf{0}$ on S_u and C^{α}_u, for all $\boldsymbol{\alpha}^* \in H^1$ satisfying $\boldsymbol{\alpha}^* \cdot \mathbf{n} = \mathbf{0}$ on $S - \Gamma_P$, and for all $\tilde{\mathbf{T}}^* \in L^2$:

$$\int_V (\tilde{\sigma}_{ji} + \tilde{\sigma}^{(2)}_{ji})\, u^*_{i,j}\, \mathrm{d}V + \int_S \tilde{T}_{ik}\, u^{*\mathrm{t}}_{i,k}\, \mathrm{d}S = \int_V f_i\, u^*_i\, \mathrm{d}V + \int_{\Gamma_P} \tilde{P}_i\, u^*_i\, \mathrm{d}S + \sum_{\alpha} \oint_{C^{\alpha}_E} \tilde{E}^{\alpha}_i\, u^*_i\, \mathrm{d}s, \tag{129}$$

$$\int_V (\tilde{\sigma}^{(2)}_{ij}\, \alpha^*_{ji} + \tilde{\mu}_{ijk}\, \tilde{\kappa}^*_{ijk})\, \mathrm{d}V - \int_S \tilde{T}_{ik}\, \alpha^{*\mathrm{t}}_{ik}\, \mathrm{d}S = \int_{\Gamma_R} \tilde{R}_i\, n_j\, \alpha^*_{ij}\, \mathrm{d}S, \tag{130}$$

$$\int_V (u_{i,j} - \alpha_{ij})\, \tilde{\sigma}^{(2)*}_{ji}\, \mathrm{d}V = 0, \tag{131}$$

$$\int_S (u^{\mathrm{t}}_{i,j} - \alpha^{\mathrm{t}}_{ij})\, \tilde{T}^*_{ij}\, \mathrm{d}S = 0, \tag{132}$$

where

$$\bar{\sigma}_{ij} = \frac{\partial \tilde{W}}{\partial u_{(i,j)}}, \quad \tilde{T}_{ij} = n_k\, \tilde{\mu}_{kji}, \quad \tilde{\mu}_{ijk} = \frac{\partial \tilde{W}}{\partial \tilde{\kappa}_{ijk}}, \tag{133}$$

with $\tilde{W} = \tilde{W}(u_{(i,j)}, \tilde{\kappa}_{ijk}(\boldsymbol{\alpha}))$, and

$$\tilde{\kappa}_{ijk}(\boldsymbol{\alpha}) = \tfrac{1}{2}(\alpha_{ki,j} + \alpha_{kj,i}), \quad \tilde{\kappa}^*_{ijk}(\boldsymbol{\alpha}^*) = \tfrac{1}{2}(\alpha^*_{ki,j} + \alpha^*_{kj,i}). \tag{134}$$

5. Finite element applications

In the examples that follow, we consider a material with an elastic strain energy density of the form

$$W = \tilde{W}(\boldsymbol{\epsilon}, \tilde{\boldsymbol{\kappa}}) = \hat{W}(\boldsymbol{\epsilon}, \hat{\boldsymbol{\kappa}}) = \bar{W}(\boldsymbol{\epsilon}, \bar{\boldsymbol{\kappa}}, \bar{\bar{\boldsymbol{\kappa}}}), \tag{135}$$

where

$$\tilde{W}(\boldsymbol{\epsilon}, \tilde{\boldsymbol{\kappa}}) = \tfrac{1}{2}\lambda\, \epsilon_{ii}\, \epsilon_{kk} + \mu\, \epsilon_{ij}\, \epsilon_{ij} + \tfrac{1}{2}\ell^2\, [\lambda\, \tilde{\kappa}_{ijj}\, \tilde{\kappa}_{ikk} + \mu\, (\tilde{\kappa}_{ijk}\, \tilde{\kappa}_{ijk} + \tilde{\kappa}_{ijk}\, \tilde{\kappa}_{kji})], \tag{136}$$

$$\hat{W}(\boldsymbol{\epsilon}, \hat{\boldsymbol{\kappa}}) = \tfrac{1}{2}\lambda\, \epsilon_{ii}\, \epsilon_{kk} + \mu\, \epsilon_{ij}\, \epsilon_{ij} + \tfrac{1}{2}\ell^2\, (\lambda\, \hat{\kappa}_{ijj}\, \hat{\kappa}_{ikk} + 2\, \mu\, \hat{\kappa}_{ijk}\, \hat{\kappa}_{ijk}), \tag{137}$$

$$\bar{W}(\boldsymbol{\epsilon},\bar{\boldsymbol{\kappa}},\bar{\bar{\boldsymbol{\kappa}}}) = \tfrac{1}{2}\lambda\,\epsilon_{ii}\,\epsilon_{kk} + \mu\,\epsilon_{ij}\,\epsilon_{ij} + \ell^2\left[\tfrac{2}{9}(\lambda+3\,\mu)\,\bar{\kappa}_{ij}\,\bar{\kappa}_{ij} - \tfrac{2}{9}\lambda\,\bar{\kappa}_{ij}\,\bar{\kappa}_{ji} + \tfrac{1}{2}\lambda\,\bar{\bar{\kappa}}_{iij}\,\bar{\bar{\kappa}}_{kkj} + \mu\,\bar{\bar{\kappa}}_{ijk}\,\bar{\bar{\kappa}}_{ijk} + \tfrac{2}{3}\lambda\,e_{ijk}\,\bar{\kappa}_{ij}\,\bar{\bar{\kappa}}_{kpp}\right], \tag{138}$$

so that

$$\bar{\sigma}_{ij} = \frac{\partial\tilde{W}}{\partial\epsilon_{ij}} = \frac{\partial\hat{W}}{\partial\epsilon_{ij}} = \frac{\partial\bar{W}}{\partial\epsilon_{ij}} = 2\,\mu\,\epsilon_{ij} + \lambda\,\epsilon_{kk}\,\delta_{ij}, \tag{139}$$

$$\tilde{\mu}_{ijk} = \frac{\partial\tilde{W}}{\partial\tilde{\kappa}_{ijk}} = \tfrac{1}{2}\ell^2\left[\lambda\,(\tilde{\kappa}_{ipp}\,\delta_{jk} + \tilde{\kappa}_{jpp}\,\delta_{ik}) + \mu\,(2\,\tilde{\kappa}_{ijk} + \tilde{\kappa}_{kji} + \tilde{\kappa}_{kij})\right], \tag{140}$$

$$\hat{\mu}_{ijk} = \frac{\partial\hat{W}}{\partial\hat{\kappa}_{ijk}} = \ell^2\,(2\,\mu\,\hat{\kappa}_{ijk} + \lambda\,\hat{\kappa}_{ipp}\,\delta_{jk}) = \ell^2\,(2\,\mu\,\epsilon_{jk,i} + \lambda\,\epsilon_{pp,i}\,\delta_{jk}), \tag{141}$$

$$\bar{\mu}_{ij} = \frac{\partial\bar{W}}{\partial\bar{\kappa}_{ij}} = \tfrac{2}{9}\ell^2\left[2\,(\lambda+3\,\mu)\,\bar{\kappa}_{ij} - 2\,\lambda\,\bar{\kappa}_{ji} + 3\,\lambda\,e_{ijk}\,\bar{\bar{\kappa}}_{kpp}\right], \tag{142}$$

$$\bar{\bar{\mu}}_{ijk} = \frac{\partial\bar{W}}{\partial\bar{\bar{\kappa}}_{ijk}} = \tfrac{1}{9}\ell^2\left[3\,\lambda\,(\bar{\bar{\kappa}}_{ppk}\,\delta_{ij} + \bar{\bar{\kappa}}_{ppi}\,\delta_{jk} + \bar{\bar{\kappa}}_{ppj}\,\delta_{ki}) + 18\,\mu\bar{\bar{\kappa}}_{ijk} + 2\,\lambda\,\bar{\kappa}_{pq}\,(\delta_{ij}\,e_{pqk} + \delta_{jk}\,e_{pqi} + \delta_{ki}\,e_{pqj})\right]. \tag{143}$$

In the above equations, μ and λ are the usual Lamé constants and ℓ is a material length scale.

It should be noted that, if we define $\hat{\sigma}^{(2)}_{ij} = -\hat{\mu}_{kij,k} = \hat{\sigma}^{(2)}_{ji}$ and $\hat{\sigma}_{ij} = \bar{\sigma}_{ij} - \hat{\mu}_{kij,k} = \hat{\sigma}_{ji}$, then it can be shown readily that, for the aforementioned constitutive model, the following equation holds

$$\hat{\sigma}_{ij} = \bar{\sigma}_{ij} - \ell^2\,\nabla^2\bar{\sigma}_{ij}, \quad \text{where } \bar{\sigma}_{ij} = 2\,\mu\,\epsilon_{ij} + \lambda\,\epsilon_{kk}\,\delta_{ij}. \tag{144}$$

For this particular constitutive model, Ru and Aifantis [34] have shown that, for a traction boundary value problem ($S_P = S,\ S_u = \emptyset$), the solution for the aforementioned quantity $\hat{\sigma}_{ij}$ of the strain-gradient elasticity problem equals the stress field of the corresponding classical elastic solution (i.e., with $\ell = 0$); however, the corresponding displacement fields are different in general.

For the gradient model presented above, several boundary value problems have been solved analytically (e.g., [11,15]). In the following, a number of boundary value problems are solved by using the variational formulations presented in Section 4 together with the finite element method and the obtained numerical solutions are compared to the corresponding analytical solutions.

5.1. Plane strain problems

5.1.1. Element development

We consider three different plane-strain elements and apply the "patch test" [51] in order to evaluate their performance. The finite element formulation is "mixed" [51] (as opposed to"irreducible") and independent interpolations for $\mathbf{u}$, $\boldsymbol{\omega}$, $\boldsymbol{\epsilon}$, and $\boldsymbol{\sigma}^{(2)}$ (or $\mathbf{u}$, $\boldsymbol{\alpha}$, and $\tilde{\boldsymbol{\sigma}}^{(2)}$) are used.

The following elements with isoparametric interpolations are studied first (see Fig. 1):

1. III5-28: is a 5-node element with 28 degrees of freedom. The quantities $(u_1, u_2, \omega_3, \epsilon_{11}, \epsilon_{22}, 2\epsilon_{12})$ are used as degrees of freedom at the four corner nodes; the fifth node is located at the centroid of the element and the nodal degrees of freedom are $(\bar{\sigma}^{(2)}_{11}, \bar{\sigma}^{(2)}_{22}, \bar{\sigma}^{(2)}_{(12)}, \bar{\sigma}^{(2)}_{[12]})$. A bi-linear interpolation for $(u_1, u_2, \omega_3, \epsilon_{11}, \epsilon_{22}, 2\epsilon_{12})$ is used in the isoparametric plane; the quantities $(\bar{\sigma}^{(2)}_{11}, \bar{\sigma}^{(2)}_{22}, \bar{\sigma}^{(2)}_{(12)}, \bar{\sigma}^{(2)}_{[12]})$ are constant over the element and the resulting global interpolation is piecewise constant in a finite element mesh.
2. III13-70: is a 13-node element with 70 degrees of freedom. The first nine nodes are located at the standard Lagrangian positions and the nodal degrees of freedom are $(u_1, u_2, \omega_3, \epsilon_{11}, \epsilon_{22}, 2\epsilon_{12})$. The four

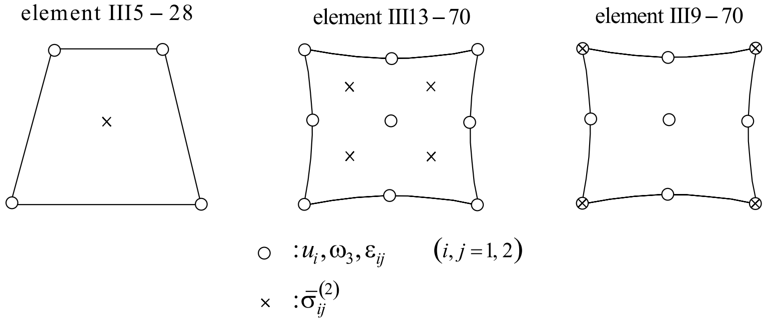

Fig. 1. Finite elements for a Type III formulation.

additional internal nodes are located at the corresponding 2×2 Gauss integration points and the nodal degrees of freedom are $(\bar{\sigma}_{11}^{(2)}, \bar{\sigma}_{22}^{(2)}, \bar{\sigma}_{(12)}^{(2)}, \bar{\sigma}_{[12]}^{(2)})$. A bi-quadratic interpolation for $(u_1, u_2, \omega_3, \epsilon_{11}, \epsilon_{22}, 2\epsilon_{12})$ and a bi-linear interpolation for $(\bar{\sigma}_{11}^{(2)}, \bar{\sigma}_{22}^{(2)}, \bar{\sigma}_{(12)}^{(2)}, \bar{\sigma}_{[12]}^{(2)})$ are used in the isoparametric plane. The resulting global interpolation for $(\bar{\sigma}_{11}^{(2)}, \bar{\sigma}_{22}^{(2)}, \bar{\sigma}_{(12)}^{(2)}, \bar{\sigma}_{[12]}^{(2)})$ is piecewise continuous in a finite element mesh.

3. III9-70: is a 9-node isoparametric element with 70 degrees of freedom and is used with the Type III formulation. The quantities $(u_1, u_2, \omega_3, \epsilon_{11}, \epsilon_{22}, 2\epsilon_{12})$ are used as degrees of freedom at all nodes; the quantities $(\bar{\sigma}_{11}^{(2)}, \bar{\sigma}_{22}^{(2)}, \bar{\sigma}_{(12)}^{(2)}, \bar{\sigma}_{[12]}^{(2)})$ are additional degrees of freedom at the corner nodes. A bi-quadratic Lagrangian interpolation for $(u_1, u_2, \omega_3, \epsilon_{11}, \epsilon_{22}, 2\epsilon_{12})$ and a bi-linear interpolation for $(\bar{\sigma}_{11}^{(2)}, \bar{\sigma}_{22}^{(2)}, \bar{\sigma}_{(12)}^{(2)}, \bar{\sigma}_{[12]}^{(2)})$ are used in the isoparametric plane. The resulting global interpolation for all nodal quantities is now continuous in a finite element mesh.

The general mathematical conditions for solvability and stability of mixed finite element discretizations are the so-called Babuška–Brezzi (BB) [3,4] conditions (see also [5]). In a recent article, Zienkiewicz and Taylor [52] claim that the satisfaction of the patch test in a mixed finite element formulation is equivalent to the satisfaction of the mathematical BB conditions.

The patch test is applied in the domain shown in Fig. 2. In a Type III formulation the following quantities are prescribed:

- $(u_1, u_2, \omega_3, \epsilon_{22})$ on AB and CD, and
- $(u_1, u_2, \omega_3, \epsilon_{11})$ on BC and AD.

The aforementioned boundary conditions are consistent with a displacement field of the form

$$u_i = a_i + b_i x_1 + c_i x_2 + d_i x_1 x_2, \quad i = 1, 2 \quad \text{for the element III5-28,} \tag{145}$$

and

$$u_i = A_i + B_i x_1 + C_i x_2 + D_i x_1 x_2 + E_i x_1^2 + F_i x_2^2 + G_i x_1 x_2^2 + H_i x_1^2 x_2 + K_i x_1^2 x_2^2,$$
$$i = 1, 2, \quad \text{for the elements III13-70 and III9-70.} \tag{146}$$

The coefficients $a_i, b_i, \ldots, H_i, K_i$ in the above equations are all constants. All three element types are capable of representing exactly the aforementioned fields.

The true stresses σ_{ij} corresponding to the above displacement fields are calculated easily by using the relationship $\sigma_{ij} = \bar{\sigma}_{ij} + \bar{\sigma}_{ij}^{(2)} = \bar{\sigma}_{ij} - \bar{\bar{\mu}}_{kij,k} - \frac{1}{2} e_{ijk} \bar{\mu}_{pk,p}$ together with the constitutive equation presented in the beginning of Section 5. The required body forces for equilibrium are $f_i = -\sigma_{ji,j}$, i.e., $f_1 = -d_2(\lambda + \mu)$ and $f_2 = -d_1(\lambda + \mu)$ for the displacement field (145), and

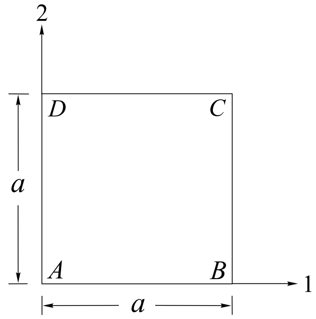

Fig. 2. The domain of interest.

$$f_1 = (4K_1\ell^2 - D_2 - 2E_1)\lambda + (12K_1\ell^2 - D_2 - 2F_1 - 4E_1)\mu - 2[H_2(\lambda+\mu) + G_1\mu]x_1 - 2[G_2(\lambda+\mu) + H_1(\lambda+2\mu)]x_2 - 4K_2(\lambda+\mu)x_1x_2 - 2K_1(\lambda+2\mu)x_2^2 - 2K_1\mu x_1^2, \tag{147}$$

$$f_2 = (4K_2\ell^2 - D_1 - 2F_2)\lambda + (12K_2\ell^2 - D_1 - 2E_2 - 4F_2)\mu - 2[G_2(\lambda+2\mu) + H_1(\lambda+\mu)]x_1 - 2[G_1(\lambda+\mu) + H_2\mu]x_2 - 4K_1(\lambda+\mu)x_1x_2 - 2K_2(\lambda+2\mu)x_1^2 - 2K_2\mu x_2^2, \tag{148}$$

for the displacement field (146).

Typical patch configurations are shown in Fig. 3. The discrete finite element equations are derived from (112)–(117) or, equivalently, from the functional (123). The element stiffness matrices are calculated numerically: for the element III5-28 a 2×2 Gauss rule is used for the area integrals and a 2-point Gauss integration is used for the line integrals over the boundary in (112)–(117). Similarly, a 3×3 Gauss rule is used for the area integrals and a 3-point Gauss integration for the line integrals over the boundary are used for the elements III13-70 and III9-70. The resulting global stiffness matrices are symmetric and have zeros on the diagonal locations corresponding to the degrees of freedom $\bar{\sigma}_{ij}^{(2)}$. The solution of the linear system of equations is obtained by using the LAPACK library [23] (subroutine DGBSV for banded matrices).

In the patch test, after all boundary conditions are enforced, the "count test" [52] is applied for the remaining degrees of freedom in the problem:

$$n_u + n_\omega + n_\epsilon \geqslant n_\sigma, \tag{149}$$

where $n_u,\ n_\omega,\ n_\epsilon$, and n_σ are the number of the remaining degrees of freedom corresponding to $\mathbf{u},\ \boldsymbol{\omega},\ \boldsymbol{\epsilon}$, and $\boldsymbol{\sigma}^{(2)}$, respectively. The above condition is necessary for solvability [5]; obviously, if (149) is not satisfied, the patch test has failed. If (149) is satisfied, the solvability of the finite element equations can be ascertained by computing the eigenvalues of the symmetric global stiffness matrix and ensuring that no zero eigenvalues are present.

It is a straightforward exercise to show that with an $N\times N$ patch of elements, the count test is passed always by all three elements considered.

The eigenvalues of the stiffness matrices are examined next and the results are summarized in Table 1.

The spurious zero eigenvalues appearing for the III5-28 and III13-70 elements indicate that they fail the patch test. The element III9-70, with the globally continuous $\boldsymbol{\sigma}^{(2)}$ field, has no spurious zero eigenvalues, reproduces the exact solution and passes the patch test (except for the special case of the 1×1 patch). [1]

[1] Single-element tests are known to be very stringent; e.g., in incompressible linear isotropic elasticity, the one-element test has failed by most continuous pressure approximations whose performance is known to be acceptable in many situations [51]. For this reason, more importance is attached to the assembly test.

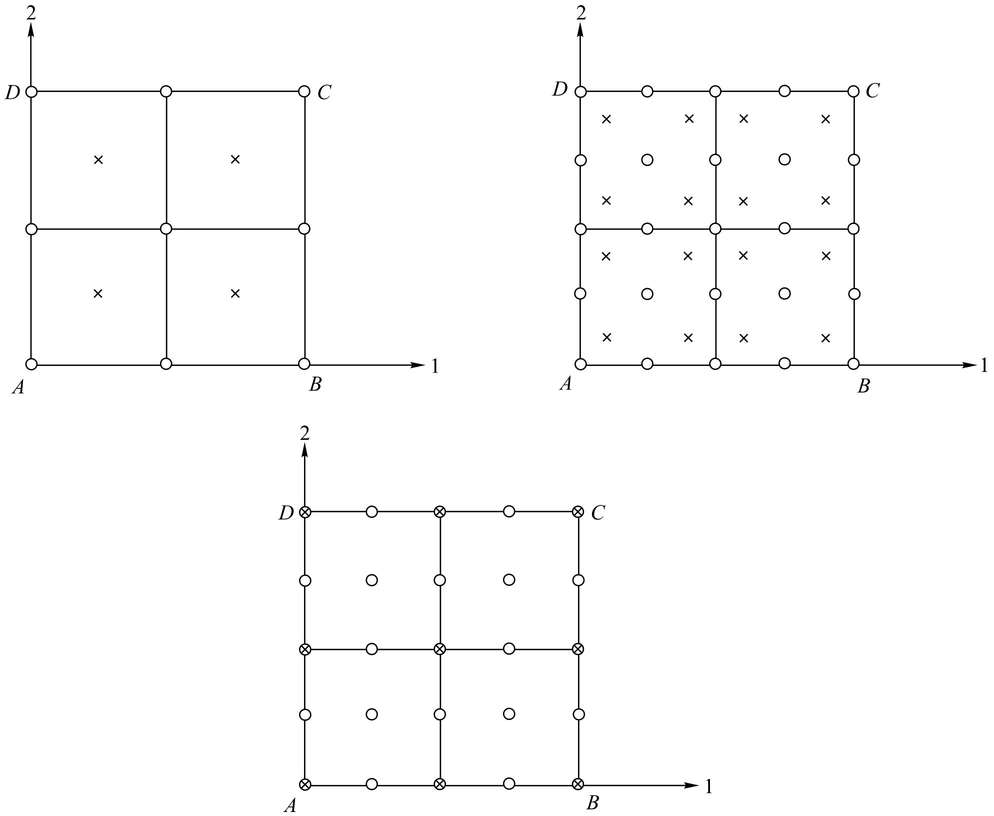

Fig. 3. Finite element meshes used in the patch test.

Table 1

Element type	$N \times N$ elements	Number of zero eigenvalues
III5-28	1×1	1
III5-28	2×2	4
III5-28	4×4	4
III13-70	1×1	5
III13-70	2×2	5
III13-70	4×4	5
III9-70	1×1	10
III9-70	2×2	0
III9-70	4×4	0

It should be mentioned also that if the terms involving integrals over the boundary S in (123) are omitted, then III9-70 fails to produce the exact solution.

Several elements that can be used in a Type-I formulation are examined next. The elements are shown in Fig. 4 and the corresponding nodal degrees of freedom are $\mathbf{u}$, $\boldsymbol{\alpha}$, and $\tilde{\boldsymbol{\sigma}}^{(2)}$. The domain shown in Fig. 1 is used again in the patch test. The following boundary conditions are prescribed in this case:

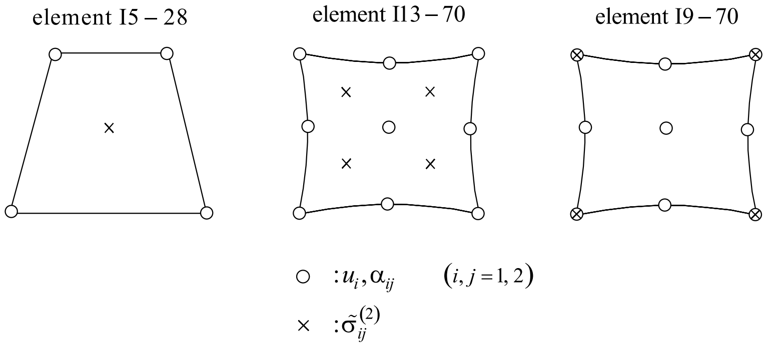

Fig. 4. Finite elements for a Type I formulation.

- $(u_1, u_2, \alpha_{12} = u_{1,2}, \alpha_{22} = u_{2,2})$ on AB and CD, and
- $(u_1, u_2, \alpha_{11} = u_{1,1}, \alpha_{21} = u_{2,1})$ on BC and AD.

The behaviors of these elements are similar to those of the corresponding elements in the Type III formulation, i.e., the element I9-70 passes the patch test, whereas the elements I5-28 and I13-70 fail in it.

5.1.2. An infinite plate with a hole

In this section we consider the problem of an infinite plate with a hole of radius a subjected to biaxial tension p at infinity under plane strain conditions as shown in Fig. 5. The constitutive model presented in the beginning of Section 5 is used to describe the mechanical response of the elastic material.

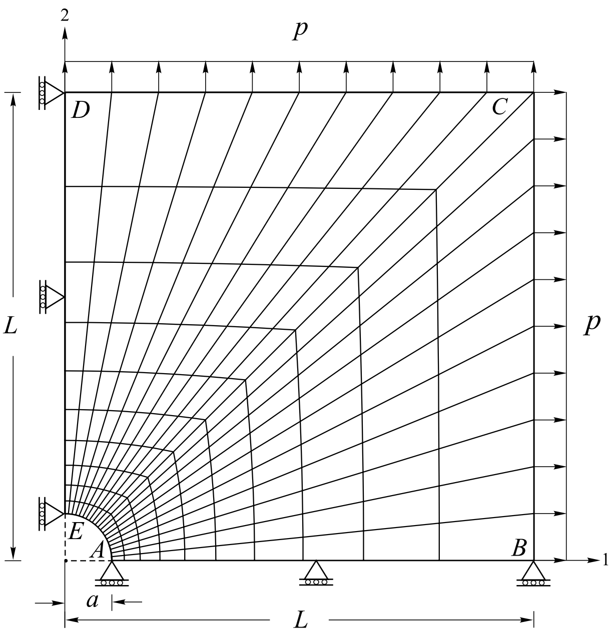

Fig. 5. Plate with a hole.

The problem is axially symmetric and the displacement field is of the form

$$u_r = u(r), \quad u_\theta = u_z = 0, \tag{150}$$

where (r, θ, z) are cylindrical coordinates centered at the center of the hole.

The only non-zero kinematical quantities in this case are

$$\epsilon_{rr} = u', \quad \epsilon_{\theta\theta} = \frac{u}{r}, \quad \bar{\bar{\kappa}}_{rrr} = u'', \quad \bar{\bar{\kappa}}_{r\theta\theta} = \bar{\bar{\kappa}}_{\theta r\theta} = \bar{\bar{\kappa}}_{\theta\theta r} = \frac{u'}{r} - \frac{u}{r^2}. \tag{151}$$

The corresponding non-zero conjugate quantities $\bar{\boldsymbol{\sigma}}$, $\bar{\boldsymbol{\mu}}$, and $\bar{\bar{\boldsymbol{\mu}}}$ are

$$\bar{\sigma}_{rr} = (2\mu + \lambda)\, u' + \lambda\, \frac{u}{r}, \quad \bar{\sigma}_{\theta\theta} = (2\mu + \lambda)\, \frac{u}{r} + \lambda u', \quad \bar{\sigma}_{zz} = \nu\,(\bar{\sigma}_{rr} + \bar{\sigma}_{\theta\theta}), \tag{152}$$

$$\bar{\mu}_{\theta z} = -\bar{\mu}_{z\theta} = \tfrac{2}{3}\ell^2\, \lambda \left(u'' + \frac{u'}{r} - \frac{u}{r^2} \right) \equiv \bar{\mu}, \tag{153}$$

and

$$\bar{\bar{\mu}}_{rrr} = \ell^2 \left[(\lambda + 2\mu)\, u'' + \lambda \left(\frac{u'}{r} - \frac{u}{r^2} \right) \right] \equiv \bar{\bar{\mu}}_1, \tag{154}$$

$$\bar{\bar{\mu}}_{r\theta\theta} = \bar{\bar{\mu}}_{\theta r\theta} = \bar{\bar{\mu}}_{\theta\theta r} = \tfrac{1}{3}\ell^2 \left[\lambda u'' + (\lambda + 6\mu) \left(\frac{u'}{r} - \frac{u}{r^2} \right) \right] \equiv \bar{\bar{\mu}}_2, \tag{155}$$

$$\bar{\bar{\mu}}_{rzz} = \bar{\bar{\mu}}_{zrz} = \bar{\bar{\mu}}_{zzr} = \tfrac{1}{3}\ell^2\, \lambda \left(u'' + \frac{u'}{r} - \frac{u}{r^2} \right) \equiv \bar{\bar{\mu}}_3. \tag{156}$$

The quantities $\bar{\sigma}^{(2)}_{ij} = -\bar{\bar{\mu}}_{kij,k} - \frac{1}{2} e_{ijk}\, \bar{\mu}_{pk,p}$ are

$$\bar{\sigma}^{(2)}_{rr} = -\tfrac{1}{3}\ell^2 \left[3\,(\lambda + 2\mu)\, u''' + 2\,(2\lambda + 3\mu)\, \frac{u''}{r} - (5\lambda + 12\mu) \left(\frac{u'}{r^2} - \frac{u}{r^3} \right) \right], \tag{157}$$

$$\bar{\sigma}^{(2)}_{\theta\theta} = -\tfrac{1}{3}\ell^2 \left[(\lambda u''' + 2\,(2\lambda + 3\mu)\, \frac{u''}{r} + (\lambda + 6\mu) \left(\frac{u'}{r^2} - \frac{u}{r^3} \right) \right], \tag{158}$$

$$\bar{\sigma}^{(2)}_{zz} = -\tfrac{1}{3}\ell^2\, \lambda \left(u''' + \frac{2u''}{r} - \frac{u'}{r^2} + \frac{u}{r^3} \right). \tag{159}$$

Finally, the true stresses σ_{ij} are the sum of $\bar{\sigma}_{ij}$ and $\bar{\sigma}^{(2)}_{ij}$, i.e., $\sigma_{ij} = \bar{\sigma}_{ij} + \bar{\sigma}^{(2)}_{ij}$.

On a surface with normal $\mathbf{n} = n_r\, \mathbf{e}_r + n_\theta\, \mathbf{e}_\theta$ the "true loads" $\mathbf{t}$, $\mathbf{m}$, and $\mathbf{T}^{\mathrm{S}}$ are given by

$$\mathbf{t} = \mathbf{n} \cdot \boldsymbol{\sigma} = n_r\, \sigma_{rr}\, \mathbf{e}_r + n_\theta\, \sigma_{\theta\theta}\, \mathbf{e}_\theta, \quad \mathbf{m} = \mathbf{n} \cdot \bar{\boldsymbol{\mu}} = \bar{\mu}\, n_r\, \mathbf{e}_z, \tag{160}$$

and

$$\mathbf{T}^{\mathrm{S}} = \mathbf{n} \cdot \bar{\bar{\boldsymbol{\mu}}} = n_r\, (\bar{\bar{\mu}}_1\, \mathbf{e}_r\, \mathbf{e}_r + \bar{\bar{\mu}}_2\, \mathbf{e}_\theta\, \mathbf{e}_\theta + \bar{\bar{\mu}}_3\, \mathbf{e}_z\, \mathbf{e}_z) + n_\theta\, \bar{\bar{\mu}}_2\, (\mathbf{e}_r\, \mathbf{e}_\theta + \mathbf{e}_\theta\, \mathbf{e}_r), \tag{161}$$

where $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ are the unit base vectors of the cylindrical system.

On a circle of radius $r = b$ with unit outward normal $\mathbf{n} = \mathbf{e}_r$, we have that $\mathbf{t} = \sigma_{rr}\, \mathbf{e}_r$, $\mathbf{m} = \bar{\mu}\, \mathbf{e}_z$, $\mathbf{T}^{\mathrm{S}} = \bar{\bar{\mu}}_1\, \mathbf{e}_r\, \mathbf{e}_r + \bar{\bar{\mu}}_2\, \mathbf{e}_\theta\, \mathbf{e}_\theta + \bar{\bar{\mu}}_3\, \mathbf{e}_z\, \mathbf{e}_z$, and

$$\nabla = \mathbf{e}_r\, \frac{\partial}{\partial r} + \mathbf{e}_\theta\, \frac{1}{r}\, \frac{\partial}{\partial \theta}, \quad D = \frac{\partial}{\partial r}, \quad \mathbf{D} = \mathbf{e}_\theta\, \frac{1}{r}\, \frac{\partial}{\partial \theta}, \quad \frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta, \quad \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r. \tag{162}$$

Therefore, using Eqs. (71)–(74), we find that the "loads" $\bar{\mathbf{P}}$, $\bar{\mathbf{Q}}^{\mathrm{t}}$, and $\bar{R}$ that appear in the boundary conditions are given by

$$\bar{\mathbf{P}} = \left(\sigma_{rr} + \frac{\bar{\bar{\mu}}_2}{b} \right) \mathbf{e}_r, \quad \bar{\mathbf{Q}}^{\mathrm{t}} = \bar{\mu} \mathbf{e}_z, \quad \bar{R} = \bar{\bar{\mu}}_1, \tag{163}$$

where $(\mathbf{e}_1, \mathbf{e}_2)$ are unit vectors along the coordinate axes (see Fig. 5).

On the surface of the hole ($r = a$, $\mathbf{n} = -\mathbf{e}_r$), the boundary condition $\bar{\mathbf{P}} = \mathbf{0}$ is used. This is equivalent to

$$\bar{\mathbf{P}} = -\left(\sigma_{rr} + \frac{\bar{\bar{\mu}}_2}{a} \right) \mathbf{e}_r = \mathbf{0} \quad \text{or} \quad \sigma_{rr} = -\frac{\bar{\bar{\mu}}_2}{a}, \tag{164}$$

which is different from the usual boundary condition $\sigma_{rr} = 0$.

The exact solution of this problem has been developed by Exadaktylos [11] and is of the form

$$u(r) = \frac{p}{2G} \left\{ (1 - 2\nu) r + \frac{a^2}{r} + \frac{\ell}{c} \left[\frac{a}{r} K_1\left(\frac{a}{\ell}\right) - (1 - 2\nu) K_1\left(\frac{r}{\ell}\right) \right] \right\}, \tag{165}$$

where

$$c = \frac{1 - 2\nu}{2} K_0\left(\frac{a}{\ell}\right) + \frac{1 - \nu}{2} \left(\frac{4\ell}{a} + \frac{a}{\ell} \right) K_1\left(\frac{a}{\ell}\right), \tag{166}$$

and $K_n(x)$ are the well-known "modified Bessel functions of the second kind".

The problem is solved numerically using the element III9-70. One-quarter of the plate is analyzed; the finite element mesh used in the calculations is shown in Fig. 5. A total of 200 elements and 6090 degrees of freedom are used in the calculations. The numerical calculations are carried out for $\nu = 0.3$ and $a = 3\ell$. The size L of the domain analyzed (see Fig. 5) is taken to be $L = 10a$. Since L is large compared to both a and ℓ, the solution of this problem is expected to be close to that of an infinite plate.

The following boundary conditions are prescribed (see Fig. 5):

- $u_2 = 0,\ \bar{P}_1 = 0,\ \omega_3 = 0,\ \bar{R} = 0$ on AB,
- $u_1 = 0,\ \bar{P}_2 = 0,\ \omega_3 = 0,\ \bar{R} = 0$ on DE,
- $\bar{P}_1 = p,\ \bar{P}_2 = 0,\ \bar{Q}_3^{\mathrm{t}} = 0,\ \bar{R} = 0$ on BC,
- $\bar{P}_2 = p,\ \bar{P}_1 = 0,\ \bar{Q}_3^{\mathrm{t}} = 0,\ \bar{R} = 0$ on CD,
- $\bar{\mathbf{P}} = \mathbf{0},\ \bar{Q}_3^{\mathrm{t}} = 0,\ \bar{R} = 0$ on AE,
- $\bar{\mathbf{E}} = \mathbf{0}$ at all corners.

It should be noted that far from the hole the fields are uniform and the prescribed boundary loads are the same as those of classical elasticity, i.e. ($\sigma_{11} = p,\ \sigma_{12} = 0$) on BC, and ($\sigma_{22} = p,\ \sigma_{21} = 0$) on CD. As mentioned earlier, on the surface AE of the hole the boundary condition $\bar{\mathbf{P}} = \mathbf{0}$ is equivalent to $\sigma_{rr} = -\bar{\bar{\mu}}_2/a$.

Fig. 6 shows the results of the finite element calculations together with the exact solution. Fig. 6(a) shows the calculated variation of u_r along the x_1-axis together with the exact solution given in Eq. (165). Fig. 6(b) shows the calculated variation of σ_{rr} and $\sigma_{\theta\theta}$ along the radial line $\theta = 2.25°$ (i.e., along the radial line passing through the centers of the row of elements close to the x_1-axis) and the corresponding exact solution. The numerical solution agrees very well with the exact solution.

We conclude this section by mentioning that the classical elasticity solution with $\ell = 0$ predicts a stress concentration $\sigma_{\theta\theta}|_{r=a} = 2p$ and a value $\sigma_{rr}|_{r=a} = 0$, whereas the present gradient elasticity solution with $a = 3\ell$ predicts $\sigma_{\theta\theta}|_{r=a} = 1.94p$ and $\sigma_{rr}|_{r=a} = 0.16p$.

5.2. The mode-III crack

In this section we consider the problem of a mode-III (antiplane shear) crack. In particular, we consider the near-tip region of a mode-III crack in a homogeneous elastic material and use a boundary layer formulation to study the near-tip fields. Fig. 7 shows a schematic representation of half of the

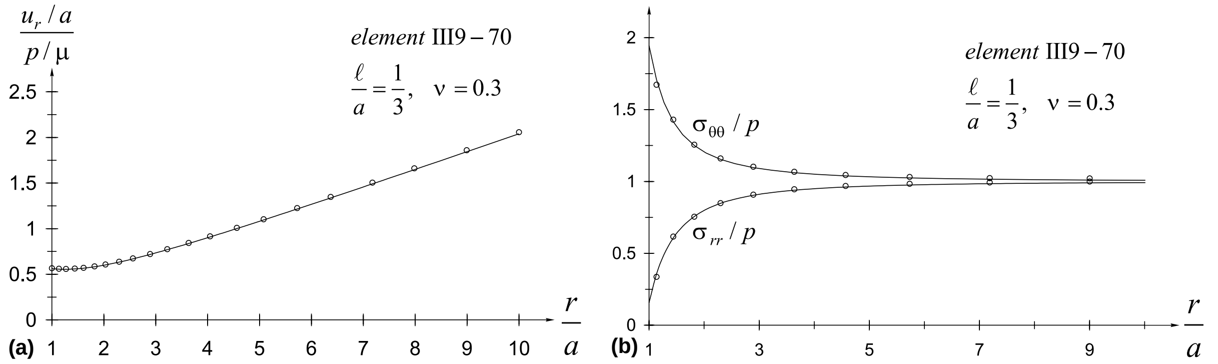

Fig. 6. Variation of u_r, σ_{rr}, and $\sigma_{\theta\theta}$ for the plate with a hole.

region near the crack tip, which is located at point O. The asymptotic mode-III linear elastic displacement field

$$u_1 = u_2 = 0, \quad u_3 = \frac{2K_{\mathrm{III}}}{\mu}\sqrt{\frac{r}{2\pi}}\sin\tfrac{1}{2}\theta, \tag{167}$$

is applied remotely on the boundary ABCD of the domain shown in Fig. 7. In the above equation u_3 is the out-of-plane displacement component, K_{III} is the usual mode-III stress intensity factor, and (r,θ) are crack tip polar coordinates. The constitutive model presented in the beginning of Section 5 is used to describe the mechanical response of the elastic material.

The displacement field in this problem is of the form

$$u_1 = u_2 = 0, \quad u_3 = u_3(x_1, x_2). \tag{168}$$

The only non-zero kinematical quantities in this case are $\alpha_{31} = u_{3,1}$, $\alpha_{32} = u_{3,2}$, $\tilde{\kappa}_{113}$, $\tilde{\kappa}_{223}$, and $\tilde{\kappa}_{123} = \tilde{\kappa}_{213}$. The corresponding non-zero conjugate quantities are $\bar{\sigma}_{13} = \bar{\sigma}_{31}$, $\bar{\sigma}_{23} = \bar{\sigma}_{32}$, $\tilde{\mu}_{113}$, $\tilde{\mu}_{223}$, $\tilde{\mu}_{131} = \tilde{\mu}_{311}$, $\tilde{\mu}_{232} = \tilde{\mu}_{322}$, $\tilde{\mu}_{123} = \tilde{\mu}_{213}$, $\tilde{\mu}_{312} = \tilde{\mu}_{132}$, and $\tilde{\mu}_{231} = \tilde{\mu}_{321}$. Also, the quantities $\tilde{\sigma}^{(2)}_{13}, \tilde{\sigma}^{(2)}_{23}, \tilde{\sigma}^{(2)}_{31}$, and $\tilde{\sigma}^{(2)}_{32}$ are non-zero in general.

The solution of this problem has been developed by Georgiadis [15], who used a Type II formulation and presented his results for the stress-like quantity $\hat{\sigma}_{ij}$; however, it can be shown easily that for this particular problem $\tilde{\sigma}_{13} = \hat{\sigma}_{13} = \hat{\sigma}_{31} = \sigma_{13}$ and $\tilde{\sigma}_{23} = \hat{\sigma}_{23} = \hat{\sigma}_{32} = \sigma_{23}$ (but $\tilde{\sigma}_{31} \neq \sigma_{31}$ and $\tilde{\sigma}_{32} \neq \sigma_{32}$). Georgiadis [15] has shown that the variation of σ_{23} ahead of the crack is of the form

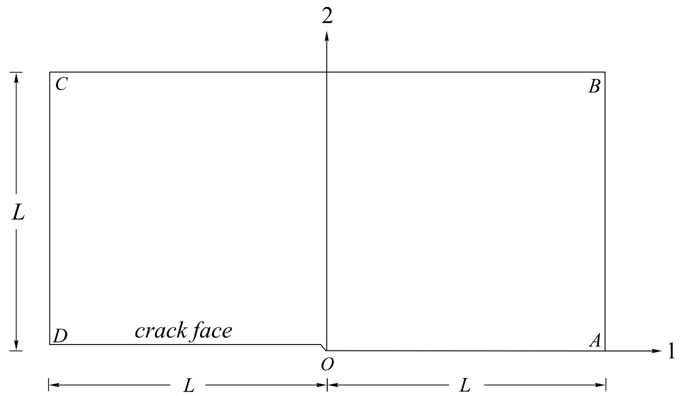

Fig. 7. Near-tip region of a mode-III crack.

$$\sigma_{23}(r,\theta=0)=\frac{\sqrt{3}}{2\pi}\frac{K_{\mathrm{III}}}{\sqrt{\ell}}f\left(\frac{r}{\ell}\right), \tag{169}$$

where

$$f(x)=\int_0^1\left[\frac{2(1-\omega)^2(1+\omega)^{3/2}\,\mathrm{e}^{-\omega x}}{3\,\omega N(\omega)}-\frac{(1-\omega)^{1/2}N(-1/\omega)}{\omega^{5/2}\,\mathrm{e}^{x/\omega}}\right]\mathrm{d}\omega, \tag{170}$$

and

$$N(x)=\exp\left\{\frac{1}{\pi}\int_0^1\tan^{-1}\left[\frac{(1-z^2)^{3/2}}{z^3}\right]\frac{\mathrm{d}z}{z+x}\right\}. \tag{171}$$

Georgiadis [15] has shown also that the asymptotic behavior of Eq. (169) as $r\to 0$ is

$$\sigma_{23}(r,\theta=0)=-\frac{\sqrt{3}}{4\sqrt{\pi}}\frac{K_{\mathrm{III}}\,\ell}{r^{3/2}}. \tag{172}$$

Also, the asymptotic form of the crack opening displacement is [15]

$$u_3(r,\theta=\pi)=\tfrac{4}{3}\sqrt{3\pi}\,\frac{K_{\mathrm{III}}}{\mu\ell}\,r^{3/2}\quad\text{as } r\to 0. \tag{173}$$

The problem is solved numerically using the element shown in Fig. 8. The element I9-35 is a 9-node isoparametric element with 35 degrees of freedom and is used with the Type I formulation. The quantities $(u_3,\alpha_{31},\alpha_{32})$ are used as degrees of freedom at all nodes; the quantities $(\tilde{\sigma}_{13}^{(2)},\tilde{\sigma}_{23}^{(2)})$ are additional degrees of freedom at the corner nodes. A bi-quadratic Lagrangian interpolation for $(u_3,\alpha_{31},\alpha_{32})$ and a bi-linear interpolation for $(\tilde{\sigma}_{13}^{(2)},\tilde{\sigma}_{23}^{(2)})$ are used in the isoparametric plane. The resulting global interpolation for all nodal quantities is continuous in a finite element mesh.

The following boundary conditions are prescribed (see Fig. 7):

- $u_3=0$ and $\tilde{R}_3=0$ on OA,
- u_3 and $\alpha_{31}=u_{3,1}$ on AB,
- u_3 and $\alpha_{32}=u_{3,2}$ on BC,
- u_3 and $\alpha_{31}=u_{3,1}$ on CD,
- $\tilde{P}_3=0$ and $\tilde{R}_3=0$ on OD,
- $\tilde{E}_3=0$ at all corners.

The finite element mesh used in the computations is shown in Fig. 9. The triangular elements at the crack tip are derived from the I9-35 element by coalescing all three nodes on one side of the original element; all nodes are "tied" together at the crack tip. In the finite element computations the domain size L (see Fig. 10)

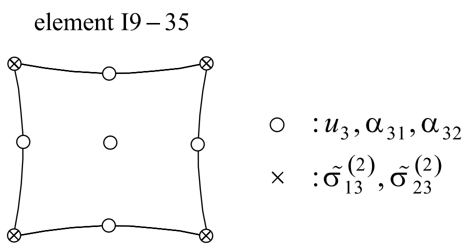

Fig. 8. Finite element used for the mode-III crack problem.

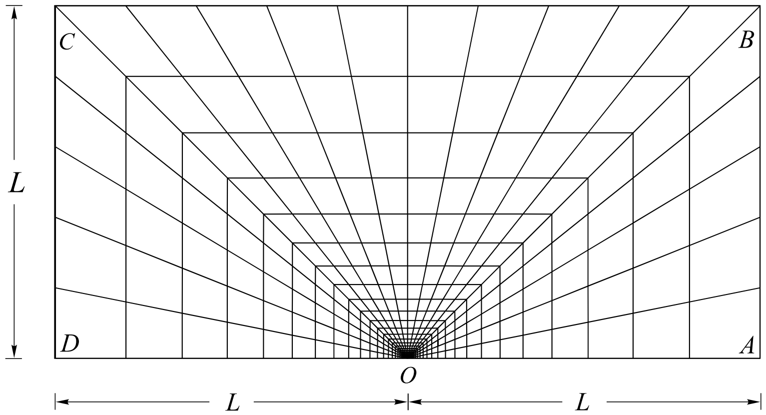

Fig. 9. Finite element mesh used for the mode-III crack problem.

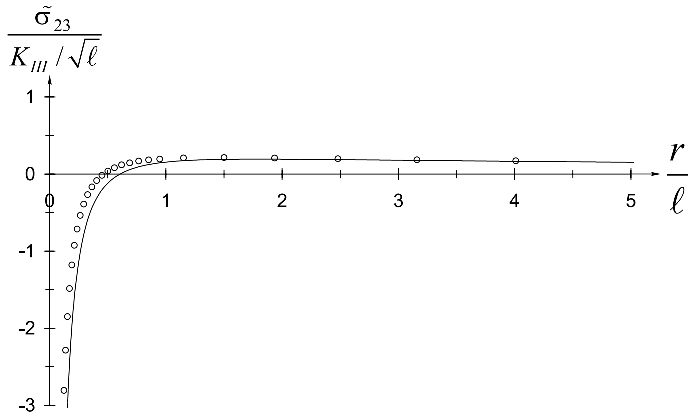

Fig. 10. Variation of σ_{23} ahead of the crack tip.

is $1000\,\ell$ and the radial size of the crack tip elements is $0.0004\,\ell$. A total of 1600 elements and 23 045 degrees of freedom are used in the calculations.

Fig. 10 shows the variation of $\tilde{\sigma}_{23}(r, \theta = 0)$; the solid line corresponds to Eq. (169) and the open circles indicate the results of the finite element solution on the radial line $\theta = 4.5°$. Fig. 11 shows the variation of $\tilde{\sigma}_{23}(r, \theta = 0)$ on a logarithmic scale. Curve II in Fig. 11 represents the near-tip asymptotic solution of Eq. (172), whereas curve I corresponds to the value of σ_{23} given by the traditional mode-III asymptotic solution of linear elastic fracture mechanics, i.e.,

$$\sigma_{23} = \frac{K_{\mathrm{III}}}{\sqrt{2\pi r}}. \tag{174}$$

As before, the results of the finite element solution are along the radial line $\theta = 4.5°$. The finite element solution agrees well with the asymptotic solution (172) near the crack tip and follows closely the traditional asymptotic solution (174) far away from the crack tip.

Fig. 12 shows the radial variation of the crack opening displacement $u_3(r, \theta = \pi)$ together with the asymptotic curve of Eq. (173).

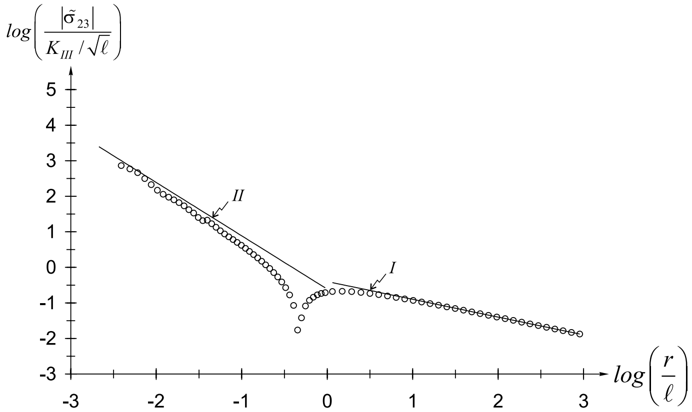

Fig. 11. Variation of σ_{23} ahead of the crack tip.

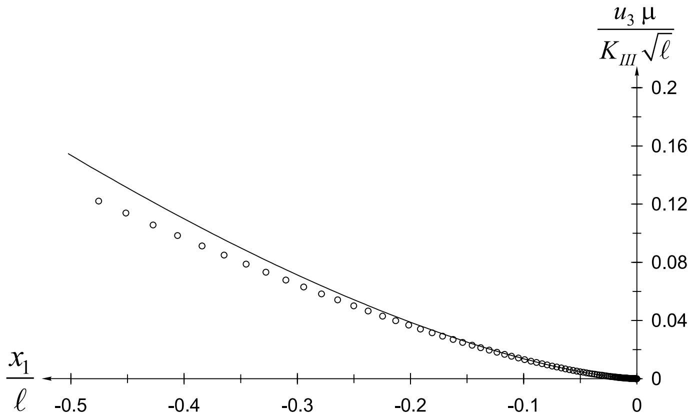

Fig. 12. Variation of the crack opening displacement.

6. Closure

The numerical examples presented in the previous section are all for the linear constitutive model presented in the beginning of Section 5. It should be emphasized though that the Type I and Type III variational formulations developed in Section 4 can be used for both linear and non-linear elasticity theories such as the strain-gradient "deformation theory of plasticity" introduced recently by Fleck and Hutchinson [12–14].

Acknowledgements

This work was carried out while the authors were supported by the GSRT of Greece under grant no. PENED 99 ED 642.

Appendix A

The normal part $A^{\rm n}_{ij}$ is defined as $A^{\rm n}_{ij} = A_{ik}\, n_k\, n_j$. Therefore, we can write

$$A^{\rm n}_{ij} = \left(A_{(ik)} + A_{[ik]}\right) n_k\, n_j = \tfrac{1}{2}\left(A_{ik} + A_{ki}\right) n_k\, n_j + A_{[ik]}\, n_k\, n_j \tag{A.1}$$

$$= \tfrac{1}{2} A^{\mathrm{n}}_{ij} + \tfrac{1}{2} A_{ki}\, n_k\, n_j + A_{[ik]}\, n_k\, n_j \tag{A.2}$$

or

$$A^{\mathrm{n}}_{ij} = A_{ki}\, n_k\, n_j + 2 A_{[ik]}\, n_k\, n_j. \tag{A.3}$$

Next, we set $A_{ki} = A^{\mathrm{n}}_{ki} + A^{\mathrm{t}}_{ki} = A_{kp}\, n_p\, n_i + A^{\mathrm{t}}_{ki}$ in the above equation to find

$$A^{\mathrm{n}}_{ij} = A_{kp}\, n_p\, n_i\, n_k\, n_j + A^{\mathrm{t}}_{ki}\, n_k\, n_j + 2 A_{[ik]}\, n_k\, n_j, \tag{A.4}$$

which is the desired result.

Appendix B

In the following we prove the identity:

$$\int_S \{[(D_p n_p)\, n_j - D_j](n_k\, \bar{\bar{\mu}}_{kji} + n_i\, n_p\, n_k\, \bar{\bar{\mu}}_{kpj})\}\, u^*_i \,\mathrm{d}S$$
$$= \int_S (n_k\, \bar{\bar{\mu}}_{kji} + n_i\, n_p\, n_k\, \bar{\bar{\mu}}_{kpj})\, D_j u^*_i \,\mathrm{d}S - \sum_\alpha \oint_{C^\alpha} [[\ell_j\, n_k\, (\bar{\bar{\mu}}_{kji} + n_i\, n_p\, \bar{\bar{\mu}}_{kpj})]]\, u^*_i \,\mathrm{d}s. \tag{B.1}$$

Using the definition of the tangential derivative D_k together with identical notation, we can show readily that

$$D_j q_j = (D_p n_p)\, n_j\, q_j + [\nabla \times (\mathbf{n} \times \mathbf{q})] \cdot \mathbf{n}. \tag{B.2}$$

Using the last equation and applying the Stokes' theorem we conclude that

$$\int_S D_j q_j \,\mathrm{d}S = \int_S (D_p n_p)\, n_j\, q_j \,\mathrm{d}S + \sum_\alpha \oint_{C^\alpha} [[\mathbf{n} \times \mathbf{q}]] \cdot \mathbf{s} \,\mathrm{d}s. \tag{B.3}$$

Taking into account that

$$[[\mathbf{n} \times \mathbf{q}]] \cdot \mathbf{s} = [[e_{ijk}\, n_j\, q_k]]\, s_i = [[(\mathbf{s} \times \mathbf{n})_k\, q_k]] = [[\ell_k\, q_k]], \tag{B.4}$$

we can write Eq. (B.3) as

$$\int_S D_j q_j \,\mathrm{d}S = \int_S (D_p n_p)\, n_j\, q_j \,\mathrm{d}S + \sum_\alpha \oint_{C^\alpha} [[\ell_j\, q_j]] \,\mathrm{d}s. \tag{B.5}$$

The last equation is sometimes referred to as the ‘‘surface divergence theorem’’.

If we set $q_j = \bar{\bar{m}}_{ij}\, u^*_i$, we can write the last equation as

$$\int_S D_j(\bar{\bar{m}}_{ij}\, u^*_i) \,\mathrm{d}S = \int_S (D_p n_p)\, n_j\, \bar{\bar{m}}_{ij}\, u^*_i \,\mathrm{d}S + \sum_\alpha \oint_{C^\alpha} [[\ell_j\, \bar{\bar{m}}_{ij}]]\, u^*_i \,\mathrm{d}s. \tag{B.6}$$

Since

$$D_j(\bar{\bar{m}}_{ij}\, u^*_i) = (D_j \bar{\bar{m}}_{ij})\, u^*_i + \bar{\bar{m}}_{ij}\, D_j u^*_i, \tag{B.7}$$

the last equation can be written as

$$\int_S \{[(D_p n_p)\, n_j - D_j] \bar{\bar{m}}_{ij}\}\, u^*_i \,\mathrm{d}S = \int_S \bar{\bar{m}}_{ij}\, D_j u^*_i \,\mathrm{d}S - \sum_\alpha \oint_{C^\alpha} [[\ell_j\, \bar{\bar{m}}_{ij}]]\, u^*_i \,\mathrm{d}s. \tag{B.8}$$

Finally, setting $\bar{\bar{m}}_{ij} = n_k\, \bar{\bar{\mu}}_{kji} + n_i\, n_p\, n_k\, \bar{\bar{\mu}}_{kpj}$ in (B.8), we obtain the desired equation (B.1).

References

[1] E.C. Aifantis, On the microstructural origin of certain inelastic models, J. Engrg. Matl. Tech. 106 (1984) 326–330.
[2] C. Atkinson, F.G. Leppington, The effect of couple stresses on the tip of a crack, Int. J. Solids Struct. 13 (1977) 1103–1122.
[3] I. Babuška, The finite element method with Lagrange multipliers, Numer. Math. 20 (1973) 179–192.
[4] F. Brezzi, On the existence, uniqueness, and approximations of saddle-point problems arising from Lagrange multipliers, RAIPO B-R2 (1974) 129–151.
[5] F. Brezzi, K.-J. Bathe, A discourse on the stability conditions for mixed finite element formulations, Comput. Methods Appl. Mech. Engrg. 82 (1990) 27–57.
[6] T.S. Cook, Y. Weitsman, Strain-gradient effects around inclusions and cavities, Int. J. Solids Struct. 2 (1966) 393–406.
[7] R. de Borst, Simulation of strain localisation: a reappraisal of the Cosserat continuum, Engrg. Comput. 8 (1991) 317–332.
[8] R. de Borst, H.B. Muhlhaus, Gradient dependent plasticity: formulation and algorithmic aspects, Int. J. Numer. Methods Engrg. 35 (1992) 521–540.
[9] R. de Borst, J. Pamin, Some novel developments in finite element procedures for gradient-dependent plasticity, Int. J. Numer. Methods Engrg. 39 (1996) 2477–2505.
[10] R. de Borst, L.J. Sluys, Localisation in a Cosserat continuum under static and dynamic loading conditions, Comput. Methods Appl. Mech. Engrg. 90 (1991) 805–827.
[11] G.E. Exadaktylos, On the problem of the circular hole in an elastic material with microstructure, private communication.
[12] N.A. Fleck, J.W. Hutchinson, A phenomenological theory for strain gradient effects in plasticity, J. Mech. Phys. Solids 41 (1993) 1825–1857.
[13] N.A. Fleck, J.W. Hutchinson, Strain gradient plasticity, Adv. Appl. Mech. 33 (1997) 295–361.
[14] N.A. Fleck, G.M. Muller, M.F. Ashby, J.W. Hutchinson, Strain gradient plasticity: theory and experiment, Acta Metall. Mater. 42 (1994) 475–487.
[15] H. Georgiadis, Exact analysis of crack-tip fields in gradient-elastic anti-plane shear deformation, in: J.T. Katsikadelis, D.E. Beskos, E.E. Gdoutos (Eds.), Recent Advances in Mechanics, Honorary Volume for Prof. A.N. Kounadis, National Technical University of Athens, 2000, pp. 263–272.
[16] P. Germain, Sur l'application de la méthode des puissances virtuelles en mécanique des milieux continus, C.R. Acad. Sci. Paris 274 (1972) 1051–1055.
[17] P. Germain, The method of virtual power in continuum mechanics. Part 2: Microstructure, SIAM J. Appl. Math. 25 (1973) 556–575.
[18] P. Germain, La méthode des puissances virtuelles en mécanique des milieux continus, J. Mécanique 12 (1973) 235–274.
[19] L.R. Herrmann, Mixed finite elements for couple-stress analysis, in: S.N. Atluri, R.H. Gallagher, O.C. Zienkiewicz (Eds.), Hybrid and Mixed Finite Element Methods, Wiley, New York, 1983, pp. 1–17.
[20] Y. Huang, L. Zhang, T.F. Guo, K.C. Hwang, Mixed mode near-tip fields for cracks in materials with strain gradient effects, J. Mech. Phys. Solids 45 (1997) 439–465.
[21] W.T. Koiter, Couple-stresses in the theory of elasticity. I, Proc. K. Ned. Akad. Wet. (B) 67 (1964) 17–29.
[22] W.T. Koiter, Couple-stresses in the theory of elasticity. II, Proc. K. Ned. Akad. Wet. (B) 67 (1964) 30–44.
[23] LAPACK Users' Guide, 3rd ed., 1999, The Society of Industrial and Applied Mathematics.
[24] J.B. Leblond, G. Perrin, J. Devaux, Bifurcation effects in ductile materials with damage localization, J. Appl. Mech. 61 (1994) 236–242.
[25] R.D. Mindlin, Microstructure in linear elasticity, Arch. Rational Mech. Anal. 10 (1964) 51–78.
[26] R.D. Mindlin, Second gradient of strain and surface tension in linear elasticity, Int. J. Solids Struct. 1 (1965) 417–438.
[27] R.D. Mindlin, N.N. Eshel, On first-gradient theories in linear elasticity, Int. J. Solids Struct. 4 (1968) 109–124.
[28] R.D. Mindlin, H.F. Tiersten, Effects of couple-stresses in linear elasticity, Arch. Rational Mech. Anal. 11 (1962) 415–448.
[29] P. Papanastasiou, I. Vardoulakis, Numerical analysis of progressive localization with application to borehole stability, Int. J. Numer. Anal. Methods Geomech. 16 (1992) 183–198.
[30] G. Pijaudier-Cabot, Z.P. Bazant, Nonlocal damage theory, J. Engrg. Mech. ASCE 113 (1987) 1512–1533.
[31] E. Providas, M.A. Katis, Finite element method in plane Cosserat elasticity, in: B.H.V. Topping (Ed.), Computational Techniques for Materials, Composites and Composite Structures, Civil-Comp Press, 2000, pp. 47–56.
[32] S. Ramaswamy, N. Aravas, Finite element implementation of gradient plasticity models. Part I: Gradient-dependent yield functions, Comput. Methods Appl. Mech. Engrg. 163 (1998) 11–32.
[33] S. Ramaswamy, N. Aravas, Finite element implementation of gradient plasticity models. Part II: Gradient-dependent evolution equations, Comput. Methods Appl. Mech. Engrg. 163 (1998) 33–53.
[34] C.Q. Ru, E.C. Aifantis, A simple approach to solve boundary-value problems in gradient elasticity, Acta Mech. 101 (1993) 59–68.
[35] J.Y. Shu, N.A. Fleck, The prediction of a size effect in microindentation, Int. J. Solids Struct. 35 (1999) 1363–1383.
[36] J.Y. Shu, C.Y. Barlow, Strain gradient effects on microscopic strain field in a metal matrix composite, Int. J. Plast. 16 (2000) 563–591.

[37] J.Y. Shu, W.E. King, N.A. Fleck, Finite elements for materials with strain gradient effects, Int. J. Numer. Methods Engrg. 44 (1999) 373–391.

[38] V.P. Smyshlyaev, N.A. Fleck, The role of strain gradients in the grain size effect for polycrystals, J. Mech. Phys. Solids 44 (1996) 465–496.

[39] E. Sternberg, R. Muki, The effect of couple-stresses on the stress concentration around a crack, Int. J. Solids Struct. 3 (1996) 69–95.

[40] R.A. Toupin, Elastic materials with couple-stresses, Arch. Rational Mech. Anal. 11 (1962) 385–414.

[41] V. Tvergaard, A. Needleman, Effects of nonlocal damage in porous plastic solids, Int. J. Solids Struct. 32 (1995) 1063–1077.

[42] D.J. Unger, E.C. Aifantis, Strain gradient elasticity theory for antiplane shear cracks. Part I: Oscillatory displacements, Theoret. Appl. Fracture Mech. 34 (2000) 243–252.

[43] D.J. Unger, E.C. Aifantis, Strain gradient elasticity theory for antiplane shear cracks. Part II: Monotonic displacements, Theoret. Appl. Fracture Mech. 34 (2000) 253–265.

[44] I. Vardoulakis, Shear-banding and liquefaction in granular materials on the basis of a Cosserat continuum theory, Ingenieur-Archiv 59 (1989) 106–113.

[45] I. Vardoulakis, E.C. Aifantis, Gradient dependent dilatancy and its implications in shear banding and liquefaction, Ingenieur-Archiv 59 (1989) 197–208.

[46] I. Vardoulakis, J. Sulem, Bifurcation Analysis in Geomechanics, Blackie/Chapman & Hall, London, 1995.

[47] Y. Weitsman, Couple-stress effects on stress concentration around a cylindrical inclusion in a field of uniaxial tension, J. Appl. Mech. (1965) 424–428.

[48] Y. Weitsman, Strain-gradient effects around cylindrical inclusions and cavities in a field of cylindrically symmetric tension, J. Appl. Mech. (1966) 57–67.

[49] Z.C. Xia, J.W. Hutchinson, Crack tip fields in strain gradient plasticity, Int. J. Fracture 92 (1998) 325–348.

[50] L. Zhang, Y. Huang, J.Y. Chen, K.C. Hwang, The mode III full-field solution in elastic materials with strain gradient effects, J. Mech. Phys. Solids 44 (1998) 1621–1648.

[51] O.C. Zienkiewicz, R.L. Taylor, The Finite Element Method, 5th ed., Butterworth/Heinemann, London, 2000.

[52] O.C. Zienkiewicz, R.L. Taylor, The finite element patch test revised. A computer test for convergence, validation and error estimates, Comput. Methods Appl. Mech. Engrg. 149 (1997) 223–254.