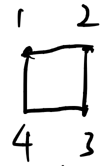
Lecture 6

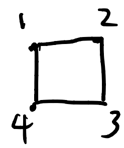
Objective:

Book reference:

Example (imprimitive action) 

$G = \langle (1234) \rangle \cong C_4$ acts imprimitively on $\{1,2,3,4\}$, with montrivial blocks $\{1,3\}$, $\{2,4\}$

Definition (primitive action) A transtive action G on Ω is primitive if Ω has no montrivial blocks w.r.t. the action.

Example (primitive action) 

$G = \langle (1234), (24) \rangle \cong D_4$ acts primitively on $\{1,2,3,4\}$.

Why are blocks important?

Def (system of blocks) Suppose G acts transitively on Ω and that Δ is a block for G. Define $\Sigma = \{\Delta \cdot x \mid x \in G\}$. Then each element in Σ is a block for the action and the sets in Σ form a partition of Ω.

Examine the relation between blocks and subgroups.

Def (pointwise stabilizer and setwise stablizer) Let G act on Ω and $\Delta \subseteq \Omega$. The pointwise stablizer of Δ in G is $G_{(\Delta)} = \{x \in G \mid \delta \cdot x = \delta \text{ for all } \delta \in \Delta\}$ and the setwise stabilizer of Δ in G is $G_{\{\Delta\}} = \{x \in G \mid \Delta \cdot x = \Delta\}$.

Clearly, $G_{(\Delta)} \trianglelefteq G_{\{\Delta\}}$. For $\alpha \in \Omega$, $G_{(\{\alpha\})} = G_{\{\{\alpha\}\}} = G_\alpha$. Pointwise and setwise stabilizers are generalizations of point stabilizer.

Lemma (transitivity on block) If G acts transitively on Ω, and Δ is a block for G then $G_{\{\Delta\}}$ acts transitively on Δ

Pf. For $x, y \in \Delta$, there exists $g \in G$ s.t. $y = x \cdot g$. Hence $(\Delta \cdot g) \cap \Delta \neq \phi$. Since Δ is a block, then $\Delta \cdot g = \Delta$. Therefore, $g \in G_{\{\Delta\}}$. Thus $G_{\{\Delta\}}$ acts transitively on Δ.

Thm (relation between blocks and subgroups) Let G act transitively on a set Ω, and let $\alpha \in \Omega$. Let $\mathcal{B}$ be the set of all blocks Δ for G with $\alpha \in \Delta$, and let S denote the set of all subgroups H of G with $G_\alpha \leq H$. Then there is a bijection Ψ of $\mathcal{B}$ onto S given by $\Psi(\Delta) = G_{\{\Delta\}}$, whose inverse mapping Φ is given by $\Phi(H) = \alpha \cdot H$. The mapping is order-preserving in the sense if $\Delta, \Gamma \in \mathcal{B}$ then $\Delta \subseteq \Gamma \Leftrightarrow \Psi(\Delta) \leq \Psi(\Gamma)$.

Pf. We first show that Ψ maps $\mathcal{B}$ into S. Let $\Delta \in \mathcal{B}$. Then $x \in G_\alpha$ implies $\alpha \in \Delta \cap \Delta \cdot x$ and so $\Delta = \Delta \cdot x$ as Δ is a block. This shows $x \in G_{\{\Delta\}}$ and $G_\alpha \leq G_{\{\Delta\}}$ for all $\Delta \in \mathcal{B}$. Thus, Ψ maps $\mathcal{B}$ into S.

We next show that Φ maps S into $\mathcal{B}$. Let $H \leq G$ with $G_\alpha \leq H$. Set $\Delta = \alpha \cdot H$ and $x \in G$. Clearly $\Delta \cdot x = \Delta$ if $x \in H$ and we claim $\Delta \cdot x \cap \Delta = \phi$ otherwise. If $\Delta \cdot x \cap \Delta \neq \phi$, let $\beta \in \Delta \cdot x \cap \Delta$. There exists $h_1, h_2 \in H$, s.t. $\beta = \alpha \cdot h_1 = (\alpha \cdot h_2) \cdot x$. Therefore, $h_2 x h_1^{-1} \in G_\alpha \leq H$ which implies $x \in H$. Hence, $\Delta \cdot x = \Delta$ iff $x \in H$ and $G_{\{\Delta\}} = H$. Therefore, $\Delta = \alpha \cdot H$ is a block containing α and $\Delta \in \mathcal{B}$. Thus, Φ maps S into $\mathcal{B}$.

By Lemma (transitivity on block), $\Phi \circ \Psi(\Delta) = \Phi(\Psi(\Delta)) = \Phi(G_{\{\Delta\}}) = \alpha \cdot G_{\{\Delta\}} = \Delta$. Hence, $\Phi \circ \Psi$ is an identity mapping on $\mathcal{B}$.

Let $H \in S$, $\Psi \circ \Phi(H) = \Psi(\Phi(H)) = \Psi(\alpha \cdot H) = G_{\{\alpha \cdot H\}} = H$. Hence, $\Psi \circ \Phi$ is an identity mapping on S.

It remains to show that Ψ is order-preserving. For $\Delta, \Gamma \in \mathcal{B}$, if $G_{\{\Delta\}} \leq G_{\{\Gamma\}}$, then $\Delta = \alpha \cdot G_{\{\Delta\}}$ and $\Gamma = \alpha \cdot G_{\{\Gamma\}}$ satisfy $\Delta \subseteq \Gamma$. If $\Delta \subseteq \Gamma$, for any $x \in G_{\{\Delta\}}$, $\Delta \cdot x = \Delta$ implies $\Gamma \cdot x \cap \Gamma \neq \phi$. As Γ is a block, $\Gamma \cdot x = \Gamma$ and $x \in G_{\{\Gamma\}}$. Thus, $G_{\{\Delta\}} \leq G_{\{\Gamma\}}$

Cor (characterization of primitive action) Let G act transitively on Ω with $|\Omega|>1$. Then G is primitive $\Leftrightarrow$ for each $\alpha \in \Omega$, G_α is a maximal subgroup of G.

Pf. By Thm (relation between blocks and subgroups), G_α is a maximal subgroup of G implies the bijection $\Psi: \mathcal{B} \mapsto \mathcal{S}$ has only two images G_α and G, where $\Psi^{-1}(G_\alpha) = \{\alpha\}$ and $\Psi^{-1}(G) = \Omega$. Hence, all the blocks containing α are the trivial $\{\alpha\}$ and Ω. Thus, G has no nontrivial blocks and is primitive.

Conversely, if G is primitive, then all the blocks contain any $\alpha \in \Omega$ are $\{\alpha\}$ and Ω. Hence, $\mathcal{B} = \{\{\alpha\}, \Omega\}$ and $\mathcal{S} = \Psi(\mathcal{B}) = \{G_\alpha, G\}$. Hence, all subgroups of G containing G_α are G_α and G. Then G_α is a maximal subgroup of G

Cor (regular and primitive group action) Let G act on Ω regularly and primitively. Then $|G| = |\Omega|$ is a prime.

Pf: Since the action is regular, then $G_\alpha = \{1_G\}$ for all $\alpha \in \Omega$.
Since the action is primitive, for each $\alpha \in \Omega$, $G_\alpha = \{1_G\}$ is a maximal subgroup of G. Hence, $|G| = |\Omega|$ must be a prime.