Assignment 2

Assignment due: 11:59 pm, Thursday, Sep-26.

1. Please ensure that you submit the assignment to Canvas by the deadline. Late submissions will result in a penalty.

2. Please ensure that your submission is clear and easy to read. Remember to include your name and clearly state which question you are addressing.

3. Please organize your solutions in ascending order, combine them into a single file, and upload the file to Canvas.

4. It is encouraged to discuss assignment questions with your peers to enhance your learning experience. However, it is important to remember that the purpose of these discussions is to help you understand the material better. You are expected to write down the solutions independently and be able to explain your solutions on your own.

5. Your submission will be graded following a 3-2-1-0 grading scale.

- 3 marks means your answer is totally correct.
- 2 marks means your answer is mostly right but suffers from some small issue, possibly logical, possibly notational.
- 1 mark means your answer contains some important ideas but is missing other necessary elements or details.
- 0 marks means you did not achieve any of the above.

1. Let X be a set with at least three elements. Show that a group G acts doubly transitively on X if and only if G_x acts transitively on $X \setminus \{x\}$ for every $x \in X$.

2. Show directly from definitions that a doubly transitive action is primitive.

3. Let $G = \langle (12345) \rangle \leqslant S_5$ act on $X = \{1, 2, 3, 4, 5\}$. Show that this action is primitive, but not doubly transitive.

4. Let $G \leqslant \mathrm{Sym}(\Omega)$ be a transitive group and let Γ and Δ be finite subsets of Ω. Suppose that $G_{(\Gamma)}$ and $G_{(\Delta)}$ act primitively on $\Omega \setminus \Gamma$ and $\Omega \setminus \Delta$, respectively, and $G = \langle G_{(\Gamma)}, G_{(\Delta)} \rangle$. Show that the action of G on Ω is primitive.

5. If Δ is a block for a group G and $\alpha \in \Delta$, show that Δ is a union of orbits for G_α. (This is often useful in looking for blocks.)

6. Let $S \in \mathrm{Syl}_p(G)$ and $N \trianglelefteq G$. Show that $S \cap N \in \mathrm{Syl}_p(N)$. In particular, if N is a p-group, then $N \leqslant S$.

7. Let G be finite and suppose $\varphi : G \to H$ is a surjective homomorphism.

   (a) If $P \in \mathrm{Syl}_p(G)$, show that $\varphi(P) \in \mathrm{Syl}_p(H)$.

(b) If $Q \in \mathrm{Syl}_p(H)$, show that $Q = \varphi(P)$ for some $P \in \mathrm{Syl}_p(G)$.

(c) Show that $n_p(H) \leq n_p(G)$.

8. Let G be finite and suppose $H \leqslant G$ satisfies the condition that $C_G(x) \leqslant H$ for all $x \in H \setminus \{1_G\}$. Show that $\gcd(|H|, |G : H|) = 1$. (hint: Choose $P \in \mathrm{Syl}_p(H)$ and show that $P \in \mathrm{Syl}_p(G)$).

9. Let $|G| = p^2q^2$ where $p > q$ are primes. If $|G| \neq 36$, show that G has a normal Sylow p-subgroup.

10. Let $P \in \mathrm{Syl}_p(G)$. Show that $N_G(N_G(P)) = N_G(P)$.