Assignment 3

Assignment due: 11:59 pm, Thursday, Oct-10.

1. Please ensure that you submit the assignment to Canvas by the deadline. Late submissions will result in a penalty.

2. Please ensure that your submission is clear and easy to read. Remember to include your name and clearly state which question you are addressing.

3. Please organize your solutions in ascending order, combine them into a single file, and upload the file to Canvas.

4. It is encouraged to discuss assignment questions with your peers to enhance your learning experience. However, it is important to remember that the purpose of these discussions is to help you understand the material better. You are expected to write down the solutions independently and be able to explain your solutions on your own.

5. Your submission will be graded following a 3-2-1-0 grading scale.

   - 3 marks means your answer is totally correct.
   - 2 marks means your answer is mostly right but suffers from some small issue, possibly logical, possibly notational.
   - 1 mark means your answer contains some important ideas but is missing other necessary elements or details.
   - 0 marks means you did not achieve any of the above.

1. Prove a group of order 90 is not simple.

2. If G is simple of order 60, show that G is isomorphic to a subgroup of S_5.

3. Let $|G| = p(p+1)$, where p is prime. Show that G has either a normal subgroup of order p or one of order $p+1$ (hint: If $n_p(G) > 1$, let $x \in G$ with $o(x) \neq 1, p$. Show that $|C_G(x)| = p+1$.)

4. Let H and K be two groups, and $\varphi : K \to \mathrm{Aut}(H)$ be a group homomorphism. Verify that the semidirect product $H \rtimes_\varphi K$ is indeed a group (verify the set is closed under operation, the existence of identity and inverse, as well as the associativity).

5. Let H and K be two groups, and $\varphi : K \to \mathrm{Aut}(H)$ be a group homomorphism. A semidirect product $H \rtimes_\varphi K$ is unchanged up to isomorphism if the action $\varphi : K \to \mathrm{Aut}(H)$ is composed with an automorphism of K. Namely, for automorphism $f : K \to K$, $H \rtimes_{\varphi \circ f} K \cong H \rtimes_\varphi K$.

6. Let $f : H_1 \to H_2$ and $f' : K_1 \to K_2$ be group isomorphisms. For each homomorphism $\varphi : K_1 \to \mathrm{Aut}(H_1)$, there is a corresponding homomorphism $\varphi' : K_2 \to \mathrm{Aut}(H_2)$ such that $H_1 \rtimes_\varphi K_1 \cong H_2 \rtimes_{\varphi'} K_2$.

7. Let $G = \mathrm{GL}(n, q)$. Prove that $Z(G) = \{\lambda I_n \mid \lambda \in \mathbb{F}_q^*\}$, where I_n is the $n \times n$ identity matrix.

8. (a) Show that $\mathrm{GL}(2, 2) \cong S_3$.

   (b) Construct an injective homomorphism $\varphi : \mathrm{GL}(1, 4) \to \mathrm{GL}(2, 2)$ that corresponds to the inclusion of A_3 in S_3.

   (c) Construct an explicit injective homomorphism from $\mathrm{GL}(n, 4)$ to $\mathrm{GL}(2n, 2)$ for each $n \geq 1$.

9. Let p be a prime and U be the subgroup of $\mathrm{GL}(3, p)$ consisting of all matrices of the form

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$$

   (a) Compute the center of U in $\mathrm{GL}(3, p)$.

   (b) Compute the normalizer of U in $\mathrm{GL}(3, p)$.

   (c) Compute the number of Sylow p-subgroups in $\mathrm{GL}(3, p)$.

10. Let β be a nontrivial automorphism of the field F. Use β to construct an outer automorphism of $\mathrm{GL}(n, F)$.