Lecture 7.

Objective: Sylow existance, development, conjugacy theorem

Book reference: Isaacs section 5A

Lagrange's theorem: $H \le G \Rightarrow |H| \mid |G|$.

the converse of Lagrange's theorem: let $m \mid |G|$, is there an order m subgroup of G?

finite abelian group ✓ consequence of the characterization of finite abelian group.

fintte nonabelian group × A_4 does not have an order 6 subgroup

Sylow's theorems provide a partial converse of Lagrange's theorem.

Lemma (binomial coefficients modulo prime) Let $n = p^a m$, where p is a prime.

Then $\binom{n}{p^a} \equiv m \pmod{p}$

Pf: $(x+1)^{p^a} \equiv x^{p^a}+1 \pmod{p}$, thus $(x+1)^{p^a m} = (x^{p^a}+1)^m \pmod{p}$

Compare the coefficients of x^{p^a}: $\binom{n}{p^a} \equiv \binom{m}{1} \equiv m \pmod{p}$

Thm (Sylow existence theorem) Let G be a finite group of order $p^a m$, where p is prime and $p \nmid m$. Then G has a subgroup H of order p^a.

Pf: Let Ω be the collection of all subsets $X \subseteq G$ with $|X| = p^a$.

Then $|\Omega| = \binom{|G|}{p^a} \equiv m \not\equiv 0 \pmod{p}$

Let G act on Ω via right multiplication. Since $p \nmid |\Omega|$, there exists an orbit $\mathcal{O}$ such that $p \nmid |\mathcal{O}|$. Let $X_0 \in \mathcal{O}$ and $H = G_{X_0}$. By orbit-stabilizer,

$|\mathcal{O}| = \frac{|G|}{|H|}$, which implies $p^a \mid |H|$.

For each $x \in X_0$ and $h \in H$, $xh \in X_0$. Thus, $xH \subseteq X_0$. Then $|H| = |xH| \le |X_0| = p^a$

Consequently, $|H| = p^a$.

Def (Sylow p-subgroup) Let G be a finite group and p a prime. A Sylow p-subgroup of G is a subgroup $P \leq G$ such that $|P| = p^a$ is in full power of p dividing $|G|$. The set of all Sylow p-subgroups of G is denoted $Syl_p(G)$.

Sylow existence theorem indicates that $Syl_p(G) \neq \phi$ for each prime $p \mid |G|$. For $p \nmid |G|$, we define $Syl_p(G) = \{\{1_G\}\}$. Thus, $Syl_p(G)$ is not empty for each prime p.

Cor. (Cauchy) Let G be finite with $p \mid |G|$, where p is prime. Then G has an element of order p.

Pf. Let $P \in Syl_p(G)$. Since $p \mid |G|$, then $P > 1$. Choose $x \in P \setminus \{1\}$, where $o(x) = p^e$. Hence, $x^{p^{e-1}} \in G$ and $o(x^{p^{e-1}}) = p$.

By the definition of Sylow p-subgroup, each $P \in Syl_p(G)$ is a maximal p-subgroup of G.

The relation between Sylow p-subgroups and other p-subgroups?

Thm (Sylow development theorem) Let G be finite and let $P \leq G$ be a p-subgroup. Then there exists $S \in Syl_p(G)$ with $P \leq S$.

The relation between distinct Sylow p-subgroups?

Thm (Sylow conjugacy theorem) Let G be a finite group. Then the set $Syl_p(G)$ is a single conjugacy class of subgroups of G.

Thm (conjugacy of Sylow p-groups) Let G be finite and suppose $P \leq G$ is a p-subgroup and $S \in Syl_p(G)$. Then $P \leq S^x$ for some $x \in G$.

Pf. Let $\Omega = \{Sx \mid x \in G\}$ and let P act on Ω via the right regular action. Then $|\Omega| = |G:S| = \frac{|G|}{|S|}$ and $p \nmid |\Omega|$. Since P is a p-group, the orbits of P acting on Ω must have sizes being power of p. Therefore, there exists $Sx \in \Omega$, which is fixed by P. Hence, for each $y \in P$, $Sxy = Sx$. Then $y \in x^{-1}Sx = S^x$, which implies $P \leq S^x$.

Proof of Sylow development theorem: Since $S^x \in Syl_p(G)$ if $S \in Syl_p(G)$, the proof is immediate.

Proof of Sylow conjugacy theorem: Let $P, S \in Syl_p(G)$. Then the result follows immediately.

Cor (number of Sylow p-groups) Let G be finite and let $P \in Syl_p(G)$. Then $|Syl_p(G)| = |G : N_G(P)|$.

In particular, $|Syl_p(G)| \mid |G : P|$.

Cor (normal Sylow p-groups) Let $S \in Syl_p(G)$. The following are equivalent:

(1) $S \trianglelefteq G$
(2) S is the unique Sylow p-subgroup of G
(3) Every p-subgroup of G is contained in S.
(4) S char G.

pf. (1) $\Rightarrow$ (2): $|Syl_p(G)| = |G : N_G(S)| = |G : G| = 1$.

(2) $\Rightarrow$ (3): use the Sylow development theorem.

(3) $\Rightarrow$ (4): Let $\sigma \in Aut(G)$. Then $\sigma(S)$ is a p-subgroup. By the Sylow development theorem, $\sigma(S) \leq S$.
Note that $|\sigma(S)| = |S|$. We have $\sigma(S) = S$. Thus, S char G

(4) $\Rightarrow$ (1) Clear.

Lemma (p-subgroup of normalizer) Let G be finite and let $S \in Syl_p(G)$. Suppose P is a p-subgroup of $N_G(S)$. Then $P \leq S$.

Pf: Since $P \leq N_G(S)$, then $PS \leq G$. Note that $|PS| = \frac{|P||S|}{|P \cap S|}$. PS is a p-group with $S \leq PS \leq G$. Since S is maximal p-subgroup of G, $PS = S$ and $P \leq S$.

Alternative proof: Since $S \in Syl_p(G)$ and $S \leq N_G(S)$, $S \in Syl_p(N_G(S))$. Since $S \trianglelefteq N_G(S)$, by Cor (normal Sylow p-groups) (3), $P \leq S$.