Lecture 9

Objective: we introduce semidirect product of groups.

Book reference: Dummit & Foote, sections 5.4, 5.5

Proposition Let H and K be subgroups of the group G. The number of distinct ways of each element in HK in the form hk, for some $h \in H$ and $k \in K$, is $|H \cap K|$. In particular, if $H \cap K = \{1_G\}$, then each element of HK can be written uniquely as a product hk, for some $h \in H$ and $k \in K$.

Direct product

When H and K are embedded into $H \times K$ in the standard way, namely,

$H \mapsto H \times K$ by $h \to (h,1)$ and $K \mapsto H \times K$ by $k \to (1,k)$, the three properties hold:

① they generate $H \times K$: $(h,k) = (h,1)(1,k)$

② they intersect trivially: $(h,1) = (1,k) \Rightarrow h=1, k=1$

③ they commute elementwise: $(h,1)(1,k) = (1,k)(h,1)$

On the other hand, how can we recognize a group is a direct product of other groups?

We have the following recognition theorem.

Theorem (direct product recognition theorem)

Let G be a group with subgroups H and K where

① $G = HK$

② $H \cap K = \{1_G\}$

③ $hk = kh$ for all $h \in H$ and $k \in K$

Then the map $H \times K \to G$ by $(h,k) \mapsto hk$ is an isomorphism.

The properties ① and ③ imply $H \trianglelefteq G$ and $K \trianglelefteq G$

Example: Let I be an m-subset of $\{1, 2, \dots, n\}$ and let G be the setwise stabilizer of I in S_n, i.e., $G = \{\sigma \in S_n \mid \sigma(i) \in I \text{ for each } i \in I\}$.

Let $J = \{1, 2, \dots, n\} \setminus I$. Note G is also setwise stabilizer of J.

Let H be a pointwise stabilizer of I and K be a pointwise stabilizer of J, i.e.

$H = \{\sigma \in G \mid \sigma(i) = i \text{ for each } i \in I\}$

$K = \{\sigma \in G \mid \sigma(j) = j \text{ for each } j \in J\}$.

Then $H \triangleleft G$, $K \triangleleft G$, and $H \cap K = \{1\}$. Thus $HK \cong H \times K$.

Each $\sigma \in G$ satabilize I and J. hence $\sigma = \sigma_I \sigma_J$, where $\sigma_I \in H$ and $\sigma_J \in K$.

Hence $G = HK \cong H \times K$.

Note that $H \cong S_{n-m}$ and $K \cong S_m$. Then $G \cong S_{n-m} \times S_m$.

Example In $G = S_n$, $n \geq 3$, let $H = A_n$ and $K = \langle (1,2) \rangle = \{(1), (1,2)\}$. We have $G = HK = H \cup H(1,2)$

Also $H \cap K = \{(1)\}$. However, $G \not\cong H \times K \cong A_n \times Z_2$ by noting $Z(G) = \{(1)\}$, $|Z(A_n \times Z_2)|$

$= |Z(A_n)|\,|Z(Z_2)| = 2$. Note also $K \ntriangleleft G$.

Semidirect products

Let H and K be subgroups of G. Suppose $H \triangleleft G$, then $HK \leq G$.

Consider the operations in group HK:

$(hk)(h'k') = hkh'k^{-1}kk' = (h\underline{kh'k^{-1}})(kk') \in HK$

$(hk)^{-1} = k^{-1}h^{-1} = k^{-1}h^{-1}kk^{-1} = (\underline{k^{-1}h^{-1}k})k^{-1} \in HK$

K acts on H via conjugation. Since $H \triangleleft G$, this action leads to an automorphism of H.

Consider two groups H and K, not initially inside a common group.

Suppose there is a group homomorphism $\varphi: K \to \mathrm{Aut}(H)$, $\varphi: k \to \varphi_k$

Since φ is a homomorphism, we have $\varphi_{k_1} \circ \varphi_{k_2} = \varphi_{k_1 k_2}$, $\varphi_1 = id_H$, $\varphi_k^{-1} = \varphi_{k^{-1}}$.

Def (semidirect product) For two groups H and K. Let $\varphi: K \to \mathrm{Aut}(H)$ be a group homomorphism, or equivalently, an action of K on H via φ.

The corresponding semidirect product $H \rtimes_\varphi K$ is a set $\{(h,k) \mid h \in H, k \in K\}$ with operation $(h,k)(h',k') = (h\,\varphi_k(h'), kk')$ for all $h, h' \in H$ and $k, k' \in K$.

Notation $\rtimes_\varphi$: the triangle $\triangleleft$ points to the normal subgroup.

To show $H \rtimes_\varphi K$ is a group, need to verify

① operation is closed

② $(1_H, 1_K)$ is the identity

③ the inverse of (h,k) is $(\varphi_{k^{-1}}(h^{-1}), k^{-1})$.

④ associativity

Proof of ③:

$(h,k)(\varphi_{k^{-1}}(h^{-1}), k^{-1}) = (h\,\varphi_k(\varphi_{k^{-1}}(h^{-1})), kk^{-1}) = (hh^{-1}, kk^{-1}) = (1_H, 1_K)$

$(\varphi_{k^{-1}}(h^{-1}), k^{-1})(h,k) = (\varphi_{k^{-1}}(h^{-1})\,\varphi_{k^{-1}}(h), k^{-1}k) = (\varphi_{k^{-1}}(h^{-1}h), k^{-1}k) = (1_H, 1_K)$

$(h,k)^{-1} = (\varphi_{k^{-1}}(h^{-1}), k^{-1})$

Example. $\{\pm 1\}$ is a multiplicative group acting as additive automorphism on $\mathbb{Z}$.

$\varphi: \{\pm 1\} \mapsto \mathrm{Aut}(\mathbb{Z})$, such that $\varphi_1(n) = n$ and $\varphi_{-1}(n) = -n$ for each $n \in \mathbb{Z}$.

We have a semidirect product $\mathbb{Z} \rtimes_\varphi \{\pm 1\}$ where $(a, \varepsilon)(a', \varepsilon')$ $= (a + \varphi_\varepsilon(a'), \varepsilon\varepsilon') = (a + \varepsilon a', \varepsilon\varepsilon')$.

Moreover, for the group homomorphism $\theta: \mathbb{Z} \mapsto \mathrm{Aut}(\mathbb{Z})$ given by $\theta_n(m) = (-1)^n m$ for $n, m \in \mathbb{Z}$, we have a semidirect product $\mathbb{Z} \rtimes_\theta \mathbb{Z}$ where $(m,n)(m',n') = (m + \varphi_n(m'), n+n')$ $= (m + (-1)^n m', n+n')$.

how H and K fit in $H \rtimes_\varphi K$.

Theorem (constructing semidirect product) Inside $H \rtimes_\varphi K$, we have

$H \cong H \times 1 = \{(h,1) : h \in H\}$ by $h \mapsto (h,1)$, $K \cong 1 \times K = \{(1,k) : k \in K\}$ by $k \mapsto (1,k)$.

For each $h \in H$ and $k \in K$, $(h,k) = (h,1)(1,k) = (1,k)(\varphi_k^{-1}(h),1)$, hence $H \rtimes_\varphi K$ is generated by $H \times 1$ and $1 \times K$. $H \times 1$ is a normal subgroup of $H \rtimes_\varphi K$ with conjugation $(1,k)(h,1)(1,k)^{-1} = (\varphi_k(h),1)$.

In particular, every $(h,1)$ computes with every $(1,k)$ iff $\varphi : K \to Aut(H)$ is the trivial action where φ_k is the identity mapping for each $k \in K$.

Pf: $\phi_1 : H \mapsto H \rtimes_\varphi K$, $\phi_1(h) = (h,1)$.

Note that $\phi_1(h)\phi_1(h') = (h, 1_K)(h', 1_K) = (h\varphi_1(h'), 1_K) = (hh', 1_K) = \phi_1(hh')$.

Hence, H is isomorphic to a subgroup $H \times \{1_K\}$ of $H \rtimes_\varphi K$ via ϕ_1.

Moreover, for each $(h_0, 1_K) \in H \times \{1_K\}$ and $(h,k) \in H \rtimes_\varphi K$,

$$(h,k)^{-1}(h_0, 1_K)(h,k) = (\varphi_{k^{-1}}(h^{-1}), k^{-1})(h_0, 1_K)(h,k) = (\varphi_{k^{-1}}(h^{-1})\varphi_{k^{-1}}(h_0), k^{-1})(h,k)$$
$$= (\varphi_{k^{-1}}(h^{-1}h_0)\varphi_{k^{-1}}(h), k^{-1}k) = (\varphi_{k^{-1}}(h^{-1}h_0h), 1_K) \in H \times \{1_K\}.$$

Hence $H \times \{1_K\} \trianglelefteq H \rtimes_\varphi K$. Set $h = 1_H$, we have $(1,k)^{-1}(h_0, 1_K)(1,k) = (\varphi_{k^{-1}}(h_0), 1_K)$, i.e. $(1,k)(h_0, 1_K)(1,k^{-1}) = (\varphi_k(h_0), 1_K)$.

$\phi_2 : K \longmapsto H \rtimes_\varphi K$, $\phi_2(k) = (1,k)$

Note that $\phi_2(k)\phi_2(k') = (1_H, k)(1_H, k') = (1_H \varphi_k(1_H), kk') = (1_H, kk') = \phi_2(kk')$.

Hence, K is isomorphic to a subgroup $\{1_H\} \times K$ of $H \rtimes_\varphi K$ via ϕ_2.

For any $h \in H$ and $k \in K$, $(h,1)(1,k) = (h\varphi_1(1), k) = (h,k)$, $(1,k)(h,1) = (\varphi_k(h), k)$.

$(h,1)$ and $(1,k)$ being commutative iff $\varphi_k(h) = h$ for all $h \in H$, $k \in K$, namely $\varphi : K \mapsto Aut(H)$ is the trivial action where φ_k is the identity mapping for each $k \in K$.