Assignment 1

Assignment due: 11:59 pm, Thursday, Sep-12.

1. Please ensure that you submit the assignment to Canvas by the deadline. Late submissions will result in a penalty.

2. Please ensure that your submission is clear and easy to read. Remember to include your name and clearly state which question you are addressing.

3. Please organize your solutions in ascending order, combine them into a single file, and upload the file to Canvas.

4. It is encouraged to discuss assignment questions with your peers to enhance your learning experience. However, it is important to remember that the purpose of these discussions is to help you understand the material better. You are expected to write down the solutions independently and be able to explain your solutions on your own.

5. Your submission will be graded following a 3-2-1-0 grading scale.

- 3 marks means your answer is totally correct.
- 2 marks means your answer is mostly right but suffers from some small issue, possibly logical, possibly notational.
- 1 mark means your answer contains some important ideas but is missing other necessary elements or details.
- 0 marks means you did not achieve any of the above.

1. Let Ω be a G-set. Prove the following relation between two stabilizers: for each $g \in G$ and $\alpha \in \Omega$, $G_{\alpha \cdot g} = g^{-1} G_\alpha g$.

2. Let G act on Ω. Prove the orbits of the action partition Ω, namely,
   (a) Ω is a union of the orbits.
   (b) Any two different orbits are disjoint.

3. Let G act on the set of all subsets of G via conjugation. Namely, for a subset $X \subset G$ and $g \in G$, the action is defined by $X \cdot g = \{g^{-1} x g \mid x \in X\}$.
   (a) For a subset $X \subset G$, determine the stabilizer G_X.
   (b) Recall that the normalizer of X in G is $N_G(X) = \{g \in G \mid g^{-1} X g = X\}$ and the stabilizer of X in G is $C_G(X) = \{g \in G \mid g^{-1} x g = x \text{ for each } x \in X\}$. Prove that $C_G(X) \trianglelefteq N_G(X)$ and $N_G(X)/C_G(X)$ is isomorphic to a subgroup of $\mathrm{Sym}(X)$.

4. Show that a finite simple group G whose order is at least $r!$ cannot have a proper nontrivial subgroup of index r.

5. Let $N \trianglelefteq G$, and $y \in N$. Show that the conjugacy class of y in G is a union of conjugacy classes of the group N. Show further that there is a bijective correspondence between the conjugacy classes of N which comprise the conjugacy class of y in G and the cosets of $NC_G(y)$ in G.

6. Let H and K be finite subgroups of G. Let $g \in G$. Show that

$$|HgK| = |H||K|/|K \cap H^g|.$$

7. Let $\varphi : G \to H$ be a surjective homomorphism with G finite and let $g \in G$. Show that

$$|C_G(g)| \geq |C_H(\varphi(g))|.$$

8. Let G be a finite group. Let G act transitively on Ω. Define an action of G on the set $\Omega \times \Omega$ by putting $(\alpha, \beta) \cdot g = (\alpha \cdot g, \beta \cdot g)$. Let $\alpha \in \Omega$. Show that G has the same number of orbits on $\Omega \times \Omega$ that G_α has on Ω.

9. Let G be finite. Show that the probability that two elements of G chosen at random commute with each other is equal to $k/|G|$, where k is the number of conjugacy classes of G (the two elements are allowed to be equal: they are chosen randomly with replacement).

10. An action of G on Ω is *doubly transitive* if G is transitive on the set of those ordered pairs (α, β) with $\alpha \neq \beta$ (with the action as in Problem 8). Show that G is doubly transitive on Ω if

$$\frac{1}{|G|} \sum_{g \in G} \chi(g)^2 = 2,$$

where χ is the associated permutation character, Do not assume that G is transitive.