Lecture 8.

Objective: Sylow counting theorem, apply Sylow's theorems to decide the simplicity of groups with prescribed order.

Book reference: Isaacs 5A, 5B

Lemma (p-subgroup of normalizer) Let G be finite and let $S \in Syl_p(G)$. Suppose P is a p-subgroup of $N_G(S)$. Then $P \leq S$.

Notation: $n_p(G) = |Syl_p(G)|$. $n_p(G) = |G : N_G(P)|$, $n_p(G) \mid |G:P|$.

Theorem (Sylow counting theorem) Let G be a finite group and write $n_p(G) = |Syl_p(G)|$. Then $n_p(G) \equiv 1 \pmod p$. In fact, $n_p(G) \equiv 1 \pmod{p^e}$ if $p^e \leq |S : S \cap T|$ for all $S, T \in Syl_p(G)$ with $S \neq T$.

Pf: Let $P \in Syl_p(G)$ and let P act on $Syl_p(G)$ via conjugation.

$\{P\}$ is one orbit of this action. It suffices to show that all the other orbits have size divisible by p^e.

Let $S \in Syl_p(G)$ with $S \neq P$. Let O_S be the orbit containing S, then $|O_S| = |P : N_P(S)|$.

Since $N_P(S)$ is a p-subgroup of $N_G(S)$, then by Lemma (p-subgroup of normalizer), $N_P(S) \leq S$. Thus, $N_P(S) \leq P \cap S$.

On the other hand, $P \cap S \leq N_P(S)$. Then $N_P(S) = P \cap S$. $|O_S| = |P : P \cap S| \geq p^e$.

Since $|P : P \cap S|$ is a power of P, then $p^e \mid |O_S|$. Consequently, $n_p(G) \equiv 1 \pmod{p^e}$.

Theorem (order pq group not simple) Let $|G| = pq$, where $p > q$ are primes. Then G has a normal Sylow p-subgroup. Also if G is nonabelian, then $q \mid p-1$ and G has exactly p Sylow q-subgroups.

Pf. By Cor (number of Sylow p-groups), $n_p(G) \in \{1, q\}$.

By the Sylow counting theorem, $n_p(G) \equiv 1 \pmod p$. Since $p > q$, $n_p(G) = 1$. Then by Cor (normal sylow p-groups), G has a unique normal Sylow p-subgroup P.

By Cor:(number of Sylow p-groups), $n_q(G) \in \{1, p\}$.

Since G/P is a group of order q, it is abelian and $G' \subseteq P$.

If $n_q(G)=1$, then by Cor (normal Sylow p-groups), G has a unique normal Sylow q-subgroup Q.

Since G/Q is a group of order q, it is abelian and $G' \subseteq Q$. Then $G' \subseteq P \cap Q = \{1_G\}$ and G is abelian. Hence, if G is nonabelian, $n_q(G)=p$.

By the Sylow counting theorem, $n_q(G) \equiv 1 \pmod q$. Then $q \mid p-1$

Theorem (order p^2q group not simple) Let $|G|=p^2q$, where p and q are primes. Then G has a normal Sylow p-group or a normal Sylow q-group.

pf. By Cor (number of Sylow p-groups), $n_q(G) \in \{1, p, p^2\}$ and $n_p(G) \in \{1, q\}$. By Sylow counting theorem, $n_p(G) \equiv 1 \pmod p$ and $n_q(G) \equiv 1 \pmod q$.

If $n_q(G)=1$, then by Cor (normal Sylow p-groups), the unique Sylow q-subgroup is normal in G, done.

If $n_q(G)=p$, then by Sylow counting theorem, $p \equiv 1 \pmod q$. Hence $p-1 \geq q$ and $q \not\equiv 1 \pmod p$, which forces $n_p(G)=1$. By Cor (normal Sylow p-groups), the unique Sylow p-subgroup is normal in G, done.

If $n_q(G)=p^2$. If $Q_1, Q_2 \in Syl_q(G)$, then $Q_1 \cap Q_2 = \{1\}$. Hence, the p^2 Sylow q-subgroups cover in total $p^2(q-1)$ elements of order q. Write $X = G \setminus \bigcup_{Q \in Syl_q(G)} Q \setminus \{1_G\}$. Then $|X| = p^2q - p^2(q-1) = p^2$, and every element of G with order unequal to q lies in X.

Let $S \in Syl_p(G)$, then $S = X$ is unique and normal in G.

Remark (Burnside) If $|G|=p^aq^b$, where p and q are primes, then G cannot be simple unless it has prime order.

Lemma (Sylow p-subgroup of simple group) Suppose $|G|=p^a m$, where $a>0$, $m>1$ and $p\nmid m$. If G is simple, then $n=n_p(G)$ satisfies $|G| \mid n!$

Pf By Cor (number of Sylow p-groups), $n=[G:N_G(P)]$. Since G is simple, then $N_G(P)<G$ and $n>1$. By Theorem (searching for normal subgroup), there exists a subgroup $N\leq N_G(P)$ s.t $N\trianglelefteq G$ and $[G:N] \mid n!$ Since G is simple and $N<G$, then $N=\{1_G\}$ and $|G| \mid n!$

Example If $|G|=2376=2^3\cdot3^3\cdot11$, then G is not simple

Pf. Assume G is simple, we prove by contradiction. Note that $n_{11}(G) \mid 2^3\cdot3^3$ and $n_{11}(G)\equiv 1 \pmod{11}$, we have $n_{11}(G)=12$.

Strategy: find a subgroup $H\leq G$ s.t. $n=|G:H|$ and $1<n<11$.

By Theorem (searching for normal subgroups), there exists $K\leq H$ s.t $K\trianglelefteq G$ and $[G:K] \mid n!$

Since $N<G$ and G is simple, then $N=\{1_G\}$ and $|G| \mid n!$, which is impossible.

Let $S\in Syl_{11}(G)$ and $N=N_G(S)$. Then $|G:N|=12$. Then $|N|=2\cdot3^2\cdot11$

Let $C=C_G(S)$. By the N/C theorem, N/C is isomorphic to a subgroup of $Aut(S)$.

Since $S\cong\mathbb{Z}_{11}$, then $|Aut(S)|=10$, $|N:C| \mid 10$, $(|N:C|,3)=1$ and $3^2 \mid |C|$.

Let $P\in Syl_3(C)$, then $|P|=3^2$. Let $H=N_G(P)$. Since G is simple, then $H<G$ and $|G:H|>1$.

We want to show $|G:H|<11$.

Since $P\leq C=C_G(S)$, $S\leq C_G(P)\leq N_G(P)=H$, hence $11 \mid |H|$.

By the Sylow development theorem, there exists $Q\in Syl_3(G)$, s.t. $P<Q$. Note that $|Q:P|=3$.

Then $P\trianglelefteq Q$. Hence $Q\leq N_G(P)=H$. Therefore $3^3 \mid |H|$.

Thus $3^3\cdot11 \mid |H|$ and $1<|G:H|\leq 2^3$, done.