Name: ____________________ student ID: ____________________

1.

[2] (a) Prove that $R = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$ is an integral domain.

[1] (b) Prove that $R = \{\frac{1}{2}(a + b\sqrt{2}) \mid a, b \in \mathbb{Z}\}$ is not an integral domain.

[4] (c) Using the fact that $\alpha = \frac{1+\sqrt{-19}}{2}$ is a root of the quadratic polynomial $x^2 - x + 5$, prove that $R = \{a + b\alpha : a, b \in \mathbb{Z}\}$ is an integral domain.

Note: please check (but no need to prove) if the respective set R is a ring.

2.

[1] (a) If R is a commutative ring, define the circle operation $a \circ b$ by $a \circ b = a + b - ab$. Prove that the circle operation is associative and that $0 \circ a = a$ for all $a \in R$.

[4] (b) Prove that a commutative ring R is a field if and only if $\{r \in R \mid r \neq 1\}$ is an abelian group under the circle operation.

3.

[4] (a) Prove directly from the **definitions** of maximal ideals and prime ideals that every maximal ideal of a commutative ring R with identity is a prime ideal.

[3] (b) Assume R is commutative and for each $a \in R$ there is an integer $n_a > 1$ (depending on a) such that $a^{n_a} = a$. Prove that every prime ideal of R is a maximal ideal.

4. A proper ideal Q of the commutative ring R is called *primary* if whenever $ab \in Q$ and $a \notin Q$ then $b^n \in Q$ for some positive integer n. (Note that the symmetry between a and b in this definition implies that if Q is a primary ideal and $ab \in Q$ with neither a nor b in Q, then a positive power of a and a positive power of b both lie in Q.) Establish the following facts about primary ideals.

[4] (a) The primary ideals of $\mathbb{Z}$ are 0 and (p^n), where p is a prime and n is a positive integer.

[1] (b) Every prime ideal of R is a primary ideal.

[4] (c) An ideal Q of R is primary if and only if every zero divisor in R/Q is a nilpotent element of R/Q.

5. Prove the following are equivalent.

[4] (a) Every nonconstant polynomial in $\mathbb{C}[x]$ has a zero in $\mathbb{C}$.

[2] (b) Let $f(x) \in \mathbb{C}[x]$ such that each zero of f is a zero of $g(x) \in \mathbb{C}[x]$. Then $g(x)^r \in (f(x))$ for some $r \in \mathbb{Z}_+$.