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Lecture - 08 **16 Sep 2024, Mon**

Theorem 1 (Eueler-Hierholzer Theorem)**.** *A graph G is Eulerian, i.e., has a trail containing each edge exactly once, if and only if it is connected and each vertex has even degree.*

Proof. Assume that G is connected and each of its vertices has even degree. We want to prove that it is Eulerian. We will use induction on the number $|E|$ of edges in G.

Let $m = 1$. The only connected graphs having exactly one edge are $G_1 = (\{v_1\}, \{\{v_1, v_1\}\})$, the graph with one vertex and a loop on it, and $G_2 = (\{v_1, v_2\}, \{\{v_1, v_2\}\})$ the graph with two vertices and exactly one edge between them. The latter has vertices with odd degrees, so only consider G_1. G_1 is Eulerian because the loop is a trail.

Let us assume that the statement is true for $|E| = 1, \dots, k$ where $k \in \mathbb{N}$ is randomly chosen. We now take a connected graph G with $k+1$ edges such that every vertex has even degree. By the preceeding lemma, G contains a cycle C because every vertex has degree ≥ 2. Let G' be the graph obtained by removing this cycle form G.

Let $\tilde{G}$ be a connected component of G'. Then $\tilde{G}$ is trivially connected, and for all vertices v in $V(\tilde{G})$, then the degree of a vertex v in the induced subgraph G' is

$$\deg_{G'}(v) = \begin{cases} \deg_G(v) & \text{if } v \notin C \\ \deg_G(v) & \text{if } v \in C \end{cases}.$$

Thus, each vertex of $\tilde{G}$ is even and it satisfies the induction hypothesis. It is thus Eulerian.

We proceed to construct an Eulerian walk on G. Let $P = v_0, e_1, v_1, \dots, e_s, v_s$ be an Eulerian walk on a collection $\mathcal{C}$ containing the cycle C and some or none of the connected components of G'. If there is a vertex v which is not in P, then let $\tilde{G}$ be the connected component containing v. There exists some $l \in \mathbb{N}$ such that $v_l \in \tilde{G}$ is contained in P. This is because we assumed G is connected and removed C from it. So, v must have a path connecting it to C, and it can only contain vertices from $\tilde{G}$ and from C as any vertex in this path is either in P or in the same connected component as v which is $\tilde{G}$. As C is non empty, $\tilde{G}$ has fewer than k edges and has an Eulerian trail by the induction hypothesis. Let this trail be $v_l = x_0, e_1, x_1, e_2, x_2, \dots, e_r, x_r = v_l$ We then find that $v_0, e_1, v_1, \dots, e_l, v_l = x_0, e_1, x_1, e_2, x_2, \dots, e_r, x_r = v_l, e_{l+1}, v_{l+1}, \dots, e_s = v_s = v_0$ is an Eulerian trail on $\mathcal{C} \cup \left\{\tilde{G}\right\}$.

By induction, we have an Eulerian trail on G.

$\square$

Definition 1 (Bipartite graphs)**.** *A graph $G = (V, E)$ is said to be bipartite if $V = A \cup B$ with $A \cap B = \phi$, such that both A and B are independent sets. In this case, we call A and B the* partitite *sets of G.*

We have the following characterisation of bipartite graphs.

Theorem 2 (König)**.** *A graph G is bipartite if and only if it has no odd cycles.*

Lemma 1. *A graph is bipartite if and only if all its connected components are bipartite.*

Proof. Given this lemma, we can prove the theorem by assuming that G is connected. $\square$

Lecture - 09 **18 Sep 24, Wed**

Proof. Consider the permutation σ in S_n. Since the fixed points of σ do not appear in the cyclic decomposition of σ, we induct on m, the number of points in $[n]$ which are not fixed by σ. In other words, $m = |\{a \in [n] : \sigma(a) \neq a\}|$. If $m = 2$, then there are exactly two points a_1, a_2 which are not fixed by σ. Then $\sigma = (a_1, a_2)$.

Induction hypothesis: Suppose every permutation σ in S_n with at most $k-1$ non fixed points is a product of disjoint cycles. We proceed to prove that σ in S_n is a product of disjoint cycles if it has m non-fixed points. Pick a point a_0 in $[n]$ which is not a fixed point of σ. We then examine the elements

$$a_1 = \sigma(a_0), a_2 = \sigma^2(a_0), \ldots, a_m = \sigma^m(a).$$

As these are all non fixed points of σ, at least two of them are the same because these are $m+1$ in number and there are only m non fixed points of σ. Suppose $\sigma^j(a_0) = a_j = a_l = \sigma^l(a_0)$. If $j = 0$ and $l = m$, then

$$a_0, \sigma(a_0), \ldots, \sigma^{m-1}(a_0)$$

is an m-cycle. Otherwise $\tau = (a_j, a_{j+1}, \ldots, a_{l-1})$ is a cycle with $l-1-j$ non fixed points. For $s = \sigma(a_i)$, where $i = j, j+1, \ldots l-1$, we have

$$\tau^{-1} \circ \sigma(s) = \tau^{-1}(\sigma^{i+1}(a_0)) = \sigma^i(a_0)$$

so that $\tau^{-1} \circ \sigma$ leaves $a_j, a_{j+1}, \ldots, a_{l-1}$ fixed. It also fixes all the points which are fixed by σ. Thus, it has $m - l + j < m$ non fixed points. It can thus be expressed as a product of disjoint cycles. τ is itself a cycle. □

Alternative proof. Pick a_1 which is not a fixed point of σ. If $\sigma^j(a) \neq a$ for $j = 1, \ldots, m-1$, then $\sigma = (a, \sigma(a), \sigma^2(a), \ldots, \sigma^{m-1}(a))$ is a cycle of length m. If, otherwise, $\sigma^j(a) = a$ for some positive integer $j \leq m-1$, we have $\sigma = \tau \circ \mu$ where $\tau(a) = \sigma(a)$ and $\tau^l(a) = \sigma^l(a)$ is a permutation which leaves all points other than those in $T = \{a, \sigma(a), \ldots, \sigma^{j-1}(a)\}$ fixed and μ is the restriction of σ to $[n] \setminus T$. That is,

$$\mu(s) = \begin{cases} s, & \text{if } s \in T \\ \sigma(s), & \text{otherwise} \end{cases}$$

We know that τ fixes all points not in S. We want to show that μ fixes all points in S. Note that we can express $a = \sigma^j(a)$. If $\mu(b) = \sigma^i(a)$ for some element $\sigma^i(a)$ in T, then $\mu(b) = \sigma(b)$, meaning that $b = \sigma^{i-1}(a)$ is in T. However, this is not possible because μ is chosen to fix all points in T. We need to prove that μ is a permutation. Given t, r in $[n]$, we have $\mu(t) = t$ and $\mu(r) = r$ if r, t are in T $\mu(t) = \sigma(t) \neq \sigma(r) = \mu(r)$ if both r ant t are not in T, and $\mu(t) = \sigma(t) \neq r = \mu(r)$ if r is in T but t is not in T because $\sigma(t)$ is in not in T. This shows that μ is injective. Let s be in $[n]$. If $s = \sigma^i(a)$ for some positive integer $i \leq l$, then s is in T and $s = \mu(s)$, otherwise $s = \sigma(s') = \mu(s')$ for some s' in $[n] \setminus T$ because σ is surjective. This shows that μ is surjective. Finally, we have that $T = \left\{a, \sigma(a), \ldots, \sigma^{j-1}(a)\right\}$ has cardinality j which is less than m and so it leaves more than $n-m$ points fixed.

$\square$

Theorem 4 (Decomposition into Transpositions)**.** *Any permutation σ in S_n can be written as a product of transpositions which may not be disjoint.*

$$\begin{pmatrix} (a_{1,1}a_{1,2}\ldots a_{1,n_1}) \\ (a_{2,1}a_{2,2}\ldots a_{2,n_2}) \\ \vdots \\ (a_{k,1}a_{k,2}\ldots a_{k,n_k}) \end{pmatrix} = \begin{matrix} (a_{1,1}a_{1,2})(a_{1,1}a_{1,3})(a_{1,1}a_{1,4})\cdots(a_{1,1}a_{1,n_1}) \\ (a_{2,1}a_{2,2})(a_{2,1}a_{2,3})(a_{2,1}a_{2,4})\cdots(a_{2,1}a_{2,n_2}) \\ \vdots \\ (a_{k,1}a_{k,2})(a_{k,1}a_{k,3})(a_{k,1}a_{k,4})\cdots(a_{k,1}a_{k,n_2}) \end{matrix}$$

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Definition 4 (Order of a permutation)**.** *The order of a permutation σ in S_n, denoted by* $\operatorname{ord}(\sigma)$ *is the smallest natural number k such that $\sigma^k = \iota$.*

Remark 1. *For all σ in S_n is at most $n!$ All of $\sigma, \sigma^2, \ldots, \sigma^{n!}$ are distinct and cover all of S_n so applying the Pigeonhole Principle, we know that at least one of these should be ι the identity permutation. This means $\sigma \leq n!$.*

Remark 2. *The order of the identity permutation ι denoted by ι is 1.*

Corollary 1 (Order of product of disjoint cycles)**.** *Let $\sigma = \tau_1\tau_2\cdots\tau_k$ where $\tau_1, \tau_2, \ldots \tau_k$ are disjoint cycles in S_n. Then* $\operatorname{ord}(\sigma) = \operatorname{LCM}(k_1, k_2, \ldots, k_k)$*, where k_i is the length of τ_i for each $i = 1, 2, \ldots k$.*

1 Decomposition into transpositions

Consider the polynomial Δ_n in n variables $x_1, x_2, \ldots, x_n$ defined by

$$\Delta_n = \Pi_{1 \leq i \leq j \leq n}(x_i - x_j)$$

For any $\sigma \in S_n$, define the polynomial

$$\Delta_n(\sigma) = \Pi_{1 \leq i < j \leq n}(x_{\sigma(i)} - s_{\sigma(j)}).$$

Note that $\Delta_n = \Delta(\epsilon)$. Also, each factor in the expression of $\Delta_n(\sigma)$ coincides with a factor of Δ but possibly introduces a $-$ sign.

Definition 5 (Signature of a permutation)**.** *The signature of a permutation σ in S_n is denoted by* $\operatorname{sgn}(\sigma)$ *and defined as*

$$\operatorname{sgn}(\sigma) = \begin{cases} 1 & \textit{if } \Delta_n(\sigma) = \Delta_n \\ -1 & \textit{if } \Delta_n(\sigma) = -\Delta_n \end{cases}.$$

Theorem 5 (asd)**.** *For any natural number $n \geq 2$,*

1. *If σ is a transposition,* $\operatorname{sgn}(\sigma) = -1$.

2. *If σ, τ are in S_n, then* $\operatorname{sgn}(\sigma \circ \tau) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)$

Proof. Let $\sigma = (k\ l)$ is a transposition, then

$$\Delta_n(\sigma) = \Pi_{1 \leq i < j \leq n}\left(x_{\sigma(i)} - x_{\sigma(j)}\right).$$

I $i < j$ and $\{i, j\} \cap \{k, l\} = \phi$, then $x_{\sigma(i)} - x_{\sigma(j)} = x_i - x_j$. We now consider the case that $\{i, j\} \cap \{k, l\} \neq \phi$. If $i < k$, then $x_i - x_l$ becomes $x_i - x_k$ and $x_i - x_k$ becomes $x_i - x_l$. If $j > l$, $x_l - x_j$ becomes $x_k - x_j$ and $x_k - x_j$ becomes $x_l - x_j$. If $k < i < l$, $x_k - x_i$ becomes $x_l - x_i = -(x_i - x_l)$ and $x_i - x_l$ becomes $x_i - x_k = -(x_k - x_i)$. Each of these changes does not affect $\Delta_n(\sigma)$. The only case which does, is if $i = k$ and $j = l$, then $x_l - x_k$ becomes $x_k - x_l$.

Let σ, τ be in S_n. Suppose $\Delta_n(\tau)$ has exactly r factors of the form $x_j - x_i$ where $j > i$, so that $\operatorname{ord}(\sigma) = (-1)^r$. □

Definition 6 (Even and odd permutations)**.** *A permutation σ is said ot be even if, respectively. The set of all even permutations in S_n is a subgroup of S_n, i.e., it is closed under product. It is called the alternating group of degree n and denoted by*

$$A_n = \{\sigma \in S_n : \operatorname{sgn}(\sigma) = 1\}.$$

Remark 3. *While we can write a given $\sigma \in S_n$ as a product of transpositions in many different ways, what does not change is whether there are an odd or even number of transpositions.*

Exercise For a cycle, write it as a product

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2 Counting beyond permutations

Recall that $\binom{n}{k}$, read "n choose k," and represents the number of ways in which a subset of k objects may be chosen from a set of n objects. It is also written as ${}^nC_k$.

Definition 7 ($\binom{n}{k}$)**.** *For $n \in \mathbb{N}$, the* binomial coefficient $\binom{n}{k}$ *is the number of subsets of $[n]$ with k elements.*

Proposition 1. *For n in $\mathbb{N}$, we have $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.*

More egnerally, if n is in $\mathbb{N}$, the number of partitions of $[n]$ with sizes $k_1, \ldots, k_r$ is denoted by $\begin{pmatrix} & n & \\ k_1\ k_2 & \cdots & k_r \end{pmatrix}$, and called a *multinomial coefficient.*

Theorem 6. *For n in $\mathbb{N}$ and $k_1, k_2, \ldots, k_r$ in $\mathbb{N}$ such that $k_1 + k_2 + \cdots + k_r = n$, we have*

$$\begin{pmatrix} & n & \\ k_1\ k_2 & \cdots & k_r \end{pmatrix} = \frac{n!}{k_1!k_2!\cdots k_r!}.$$

Theorem 7 (Binomial Theorem)**.** *If n is in $\mathbb{N}$ and x, y are elements in $\mathbb{R}$ (or in any ring such that $xy = yx$), then*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Combinatorial Consequences
Settings $x = 1, y = 1$ here, we get

$$\sum_{j=0}^{n} \binom{n}{j} = (1+1)^n = 2^n.$$

Is there a bijective proof of this identity?

3 Recurrence Relations

Example 1 (Fibonacci Numbers)**.** *The Fibonacci numbers are a sequence $\{F_n\}_n$ of natural numbers defined by*

$$F_1 = 1 \quad F_2 = 1 F_n = F_{n-2} + F_{n-1},$$

for all natural numbers n.

This is a difference equation and may be viewd as an analogue of a differential equation in a *discrete time domain.* So, a solution for F_n for a general n in $\mathbb{N}$ without using recurrence relations may be though of as a solution to the difference equation. The solution is $F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}^n \right) - \left(\frac{1-\sqrt{5}}{2}^n \right) \right]$.

4 Power Series

A *Formal Power Series* is an expression of the form $\sum_{n \in \mathbb{Z}^+} a_n t^n$, where a_n is in $\mathbb{R}$ for each n in $\mathbb{Z}^+$.

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Definition 8 (Power series)**.** *A* formal power series *over a field $\mathbb{K}$ in a variable z is an expression of the form*

$$\sum_{n=0}^{\infty} a_n z^n,$$

where a_n is in $\mathbb{K}$ for each positive integer n.

The set of all formal power series over $\mathbb{K}$ in a variable t is denoted by $\mathbb{K}[[t]]$. A power series is distinct from a power series in that the latter is defined contigent upon convergence.

Definition 9 (Polynomial)**.** *A polynomial in a variable t over a field $\mathbb{K}$ is an expression*

$$f(t) = \sum j = 0^n a_j t^j$$

where n is a fixed nonnegative integer and a_n is in $\mathbb{K}$ for each integer $j \geq 0$ and $j \leq n$.

The degree of a polynomial $f(t) = \sum_{j=0}^{n} a_n t^j$ is denoted as $\deg(f(t)) = \max\{m \, : \, a_m \neq 0\}$. The set of polynomials in t with coefficients in $\mathbb{K}$ is denoted by $\mathbb{K}[t]$.

Given a polynomial $f(t) = \sum_{j=0}^{n} a_n t^n$ over $\mathbb{K}$ in t, we can talk about the associated polynomial function, which is independent of the variable t. Call it $\overline{f}$. It is defined as

$$\overline{f}(t) = \sum_{j=0}^{n} a_n t^n, \qquad \text{t } \in \mathbb{K}.$$

Defining an associated power series for a gievn formal power series needs more care as it may not converge in $\mathbb{K}$.

Operations on $\mathbb{K}[[t]]$ include

1. $(\sum_{n=0}^{\infty} a_n t^n) + (\sum_{n=0}^{\infty} b_n t^n) = \sum_{n=0}^{\infty} (a_n + b_n) t^n$.
2. $(\sum_{n=0}^{\infty} a_n t^n) \cdot (\sum_{n=0}^{\infty} b_n t^n) = \sum_{n=0}^{\infty} c_n t^n$, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

Example 2.

$$\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$$

for $|t| < 1$.

Example 3 (Exponential power series)**.** *For all t in $\mathbb{C}$, we have*

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$$

Example 4 (Derivative of a power series)**.** *Given a power series*

$$f(t) = \sum_{n=0}^{\infty} a_n t^n, \quad \textit{for all} \quad t \in \mathbb{C},$$

the derivative f' is well defined with

$$f'(t) = \sum_{n=0}^{\infty} n a_n t^{n-1}.$$

Last time, we talked about recurrence relations, generating functions and formal power series. For the Fibonacci sequence given by

$$F_0 = 0, \qquad F_1 = 1, \qquad F_n = F_{n-1} + F_{n-2}, \qquad \forall \text{ integers } n \geq 2,$$

we set $f(t) = \sum_{n \in \mathbb{N}} F_n t^n$ and call it the *Generating Function* of this sequence. Solving by substitution for $n = 1, 2$, we get

$$\begin{aligned} f(t) &= \sum_{n=1}^{\infty} F_n t^n \\ &= F_0 + tF_1 \sum_{n=3}^{\infty} F_{n-2} t^n + F_{n-2} t^n \\ &= t + (t + t^2) \sum_{n=1}^{\infty} F_n t^n \\ &= t + (t + t^2) f(t) \end{aligned}$$

The recurrence relation we get for it is

$$f(t) = \frac{t}{1 - t - t^2}.$$

Since $1 - t - t^2$ has roots

$$\alpha = \frac{1 + \sqrt{1+4}}{2} = \frac{1 + \sqrt{5}}{2}$$

and

$$\beta = \frac{1 - \sqrt{1+4}}{2} = \frac{1 - \sqrt{5}}{2},$$

we have

$$\begin{aligned} (1 - \alpha t)(1 - \beta t) &= 1 - (\alpha + \beta) t + \alpha \beta t^2 \\ &= 1 - t - t^2 \end{aligned}$$

implying that

$$\begin{aligned} f(t) &= \frac{t}{(1 - \alpha t)(1 - \beta t)} \\ &= \frac{1}{\sqrt{5}} \left[\frac{(\alpha - \beta) t}{(1 - \alpha t)(1 - \beta t)} \right] \\ &= \frac{1}{\sqrt{5}} \left[\frac{1}{1 - \alpha t} - \frac{1}{1 - \beta t} \right]. \end{aligned}$$

Recall that $\frac{1}{1-s} = \sum_{n=1}^{\infty} s^n$ for any real number s. This allows us to write

$$\sum_{n=1}^{\infty} F_n t^n = \frac{1}{\sqrt{5}} \left[\sum_{n=1}^{\infty} (\alpha t)^n - \sum_{n=1}^{\infty} (\beta t)^n \right]$$
$$= \sum_{n=1}^{\infty} \frac{1}{\sqrt{5}} \left(\alpha^n - \beta^n \right) t^n.$$

Therefore, $F_n = \frac{1}{\sqrt{5}} \left(\alpha^n - \beta^n \right)$ from comparing the coefficients.

Since $|\beta| < 1$, we get that $\beta^n \to 0$ as $n \to \infty$. So $|F_n - \frac{\alpha^n}{\sqrt{5}}| \to 0$ as $n \to \infty$. In other words $F_n \sim \frac{1}{\sqrt{5}} \alpha^n$ grows exponentially as $|\alpha| > 1$.

5 Notation for approximations

Definition 10 (O notation)**.** *If f, g are functions from $\mathbb{R}_+$ to itself, we say that $f(t) = O(g(t))$ if there exists finite real numbers c, T such that $f(t) \leq cg(t)$ whenever $t > T$.*

Definition 11 (O notation for sequences)**.** *Let $\{a_n : n \in \mathbb{N}\}$ be a sequence, and $f\colon \mathbb{N} \to \mathbb{R}$ be a function. We say that $a_n = O(f(n))$ if there exists c in $\mathbb{R}$ and $\mathbb{N}$ such that $a_n \leq cf(n)$ for all n in $\mathbb{N}$ whenever $n \geq N$.*

For example, $2^n = O(n!)$.

Definition 12 (Little 'o' notation)**.** *Let $f\colon \mathbb{R}_+ \to \mathbb{R}_+$. We say that $f(t) = 0(g(t))$ if $\lim_{t\to\infty} \frac{f(t)}{g(t)} = 0$, that is $f(t)$ is growinf slower than $g(t)$.*

Example 5. *We say that $a_n \sim f(n)$ if $\lim_{n\to\infty} \frac{a_n}{f(n)} = 1$.*

Note that $a_n = f(n) + o(f(n)) \Rightarrow a_n \sim f(n)$, if $\lim_{n\to\infty} f_n = \infty$.

Theorem 8 (Stirling's Approximation)**.** *For any positive integer n,*

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e} \right)^n.$$

In other words

$$\ln(n!) = n \ln \left(\frac{n}{e} \right) - n + \frac{1}{2} \ln(n) + \frac{1}{2} \ln(2\pi) + o(1).$$

Remark 4. *Note that*

$$n = o(n \log(n))$$
$$\ln(n) = o(n)$$
$$\frac{1}{2} \ln(2\pi) = o(\ln(n))$$
$$o(1) = \left(\frac{1}{2} \ln(2\pi) \right)$$

This means

$$\ln(n!) = n\ln(n) + o(n\ln(n)) = (n\ln n)(1 + o(1))$$

so that $\ln(n!) \sim n\ln n$.

Sketch of proof. For any integer $n > 0$, we have

$$\ln(n!) = \sum_{i=1}^{n} \ln i \sim \int_{1}^{n} \ln x \mathrm{d}x.$$

So, $\square$

Lecture - 14 **30 Sep 24, Mon**

6 General Linear Recurrence Relations

Definition 13 (Linear Recurrence Relation)**.** *A* recurrence relation, *or* difference equation *is some relation constraining the elements of a sequence* $\{y_n\}_n$.

A *linear recurrence relation* on $\{y_n\}$ is an equation of the form

$$y_n = a_1 y_{n-1} + a_2 y_{n-2} + \cdots + a_k y_{n-k}.$$

Remark 5. *In order to find a recurrence relation involving k terms, uniquely, we need initial conditions in the form of k linearly independent equations, or equivalently, the first k terms.*

Assuming that the y_n is expressed by x^n, for some each n in $\mathbb{N}$, where x is some value in $\mathbb{K}$, write the characteristic equation

$$x^k = a_1 x^{k-1} + a_2 x^{k-2} + \cdots + a_{k-1} x + a_k.$$

Suppose the equation has l distinct solutions $\alpha_1, \alpha_2, \ldots, \alpha_l$, then any linear combination of these solutions: $y^n = b_1\alpha_1^n + b_2\alpha_2^n + \cdots b_l\alpha_l^n$ is also a solution to the recurrence relation. If the solutions $\alpha_1, \alpha_2, \ldots, \alpha_l$ are not distinct, and α_i is a solution of multiplicity r where r is some positive integer, then $\alpha_i^n, n\alpha_i^n, \ldots, n^{r-1}\alpha_i^n$ are all solutions of the recurrence relation.

7 Catalan Numbers

Definition 14 (Curve)**.** *A* curve *in a space X is a continuous function $f\colon [0,1] \to X$.*

The points $f(0)$, respectively $f(1)$, as in the previous definition are known as the starting point, respectively ending point, of f.

Definition 15 (Simple Curve)**.** *A* simple curve *is a continuous function $f\colon [0,1] \to X$. which is injective on $(0,1)$.*

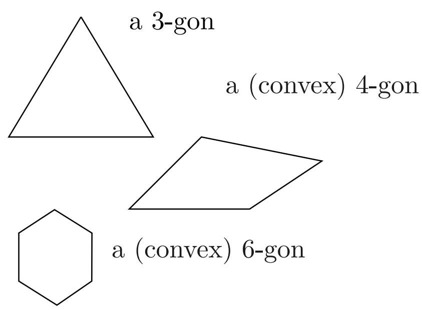
Definition 16 (Closed curve)**.** *A* closed curve *is a curve starting and ending at the same point.*

Definition 17 (Jordan Curve)**.** *A Jordan curve is a simple closed curve in the plane* $\mathbb{R}$*.*

Theorem 9 (Jordan Curve Theorem)**.** *If c is a Jordan curve in $\mathbb{R}^2$ then $\mathbb{R}^2 \setminus$* Image(c) *consists of exactly 2 connected components, one bounded, called the 'interior of c', and one unbounded, called the 'exterior of c.' The boundary of both of these connected components is* image(c).

Definition 18 (Polygon)**.** *A polygon is a Jordan curve which is piecewise linear.*

Definition 19 (Convex polygon)**.** *A convex polygon is a polygon whose interior is a convex set in* $\mathbb{R}^2$.



$C_2 = 2$, $C_3 = 5$, $C_4 = 14$, $C_5 = 42$, $C_6 = 132$, $C_7 = 429$, $C_8 = 1430$.

Lecture - 15 **02 Oct 24, Wed**

Definition 20 (Catalan numbers)**.** *For any positive integer n, the n^{th} Catalan number denoted by C_n is the number of ways of triangulating a convex n sided polygon.*

A triangulation of a polygon is a way of expressing the polygon as a union of non self intersecting triangles.

The recurrence relation for Catalan numbers C_n is

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}.$$

Proof. Consider any triangulation of an $n+1$ sided polygon and fix one edge of it. $\square$

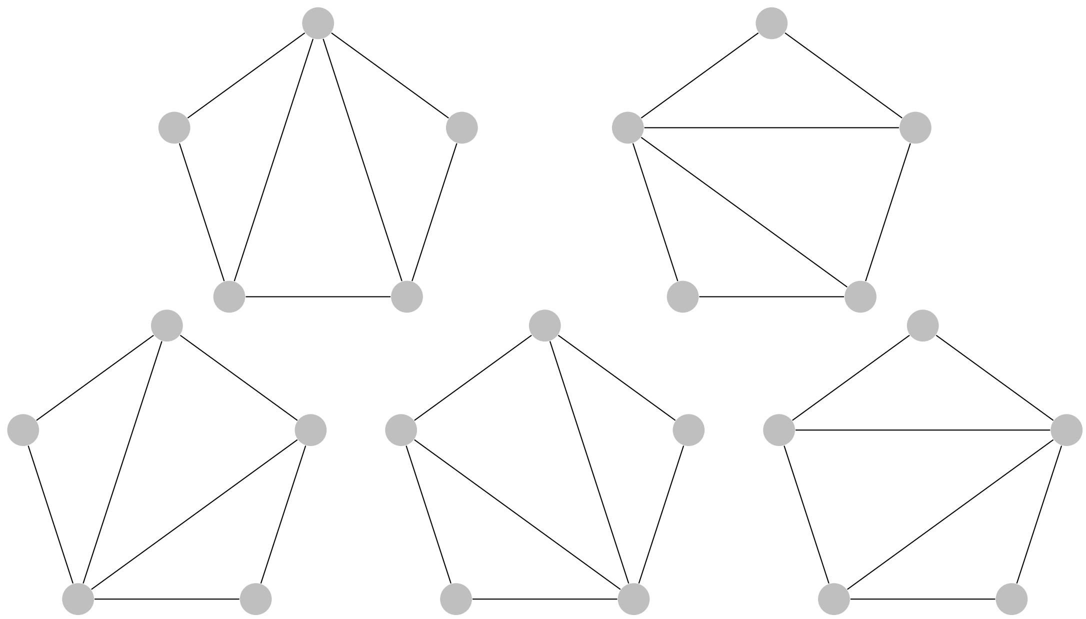


Figure 1: Triangulations of a pentagon

This is a nonlinear recurrence relation. To solve it, set

$$f(t) = \sum_{n=1}^{\infty} C_n t^n$$

to be the generating function of the Catalan numbers. Note

$$\begin{aligned}
\sum_{n=1}^{\infty} C_{n+1} t^n &= \sum_{n=1}^{\infty} C_n t^{n-1} \\
&= \frac{1}{t} \left[\sum_{n=0}^{\infty} C_n t^n - C_0 \right] \\
&= \frac{1}{t} \left[F(t) - 1 \right].
\end{aligned}$$

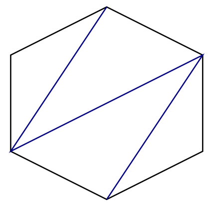


Figure 2: Triangulation of a hexagon

Also,

$$\begin{aligned}
\sum_{n=1}^{\infty} C_{n+1} t^n &= \sum_{k=0}^{\infty} i\left(C_k C_{n-k}\right) t^n \\
&= \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n} C_k t^k C_{n-k} t^{n-k} \right) \\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} C_k t^k C_{n-k} t^{n-k} \chi_{[1,n]}(k) \\
&= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} C_k t^k C_{n-k} t^{n-k} \chi_{[1,n]}(k) \\
&= \sum_{k=1}^{\infty} C_k t^k \left[\sum_{n=1}^{\infty} C_{n-k} t^{n-k} \chi_{[1,n](k)} \right] \\
&= \sum_{k=1}^{\infty} C_k t^k \left[\sum_{n=k}^{\infty} C_{n-k} t^{n-k} \right] \\
&= \sum_{k=1}^{\infty} C_k t^k \left[\sum_{n=0}^{\infty} C_n t^n \right]
\end{aligned}$$

Multiplying the entire equation by t, we get

$$\begin{aligned}
F(t) - 1 &= \sum_{n=2}^{\infty} C_n t^n \\
&= t \sum_{n=1}^{\infty} C_{n+1} t^n \\
&= t \sum_{k=1}^{\infty} C_n t^n \sum_{n=1}^{\infty} C_n t^n \\
&= tF(t)^2
\end{aligned}$$

This means $tF(t)^2 - F(t) + 1 = 0$. This is a quadratic equation for $F(t)$. Solving it gives us

$$F(t) = \frac{1 \pm \sqrt{1-4t}}{2t}.$$

As we have

$$\sqrt{1-4t} = \sum_{n=1}^{\infty} \binom{1/2}{n} (-4t)^n = 1 - 2t - 2t^2 - 4t^3 - 10t^4 - \cdots,$$

where

$$\binom{1/2}{n} = \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-n+1)}{n!},$$

the first solution yields

$$F(t) = \frac{1-\sqrt{1-4t}}{2t} = \frac{1}{t} - 1 - t - 2t^2 - 5t^3 - \cdots$$

Due to the $1/t$ term, this is not a power series. So we must have

$$\begin{aligned}
F(t) &= \frac{1-\sqrt{1-4t}}{2t} \\
&= \frac{1 - 1\sum_{n=1}^{\infty}\binom{1/2}{n}(-4t)^2}{2t} \\
&= \sum_{n=1}^{\infty} -\frac{\binom{1/2}{n+1}(-4t)^{n+1}}{2} t^n.
\end{aligned}$$

By comparing the terms of the power series, we get that

$$C_n = -\frac{\binom{1/2}{n+1}(-4)^{n+1}}{2}$$

Definition 21 (Double Factorial)**.** *For any odd integer n, we define the* double factorial $n!!$ *of* $n = 2k-1$ *by* $1 \times 3 \times 5 \times \cdots \times 2k-1$.

For $n = 2k-1$, we have
$$n!! = \frac{(2k)!}{k!2^k}.$$

Lemma 2. *For any positive integer n, we have*
$$\binom{1/2}{n+1} = \frac{(-1)^n}{(n+1)2^{2n+1}}\binom{2n}{n}.$$

Proof. Given a positive integer n, we have
$$\begin{aligned}\binom{1/2}{n+1} &= \frac{1/2(1/2-1)\cdots(1/2-n)}{(n+1)!}\\ &= \frac{(1-2)(1-4)\cdots(1-2n)}{2^{n+1}(n+1)!}\\ &= \frac{(-1)^n}{2^{n+1}(n+1)!}(2n-1)!!\\ &= \frac{(-1)^n}{2^{n+1}(n+1)!}\frac{(2n)!}{2^n(n!}\\ &= \frac{(-1)^n}{(n+1)2^{2n+1}}\frac{(2n)!}{(n!)^2}\end{aligned}$$

□

Using this expression for $\binom{1/2}{n}$, we can rephrase the expression for Catalan numbers in the following way. C_n

8 How does C_n behave as n gets large?

From Stirling's Formula we know that $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. We can deduce
$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2} \sim \frac{\sqrt{2\pi 2n}\left(\frac{2n}{e}^{2n}\right)}{2\pi n(\frac{n}{e}^{2n}}.$$

which implies that $\binom{2n}{n} \sim \frac{1}{\sqrt{\pi n}}2^{2n}$. Therefore
$$C_n = \frac{1}{(n+1)}\binom{2n}{n} \sim \frac{4^n}{(n+1)\sqrt{\pi n}} \sim \frac{4^n}{\pi n^{3/2}}.$$

In other words, $\log_2 C_n = 2n - \frac{3}{2}\log_2 n - \frac{1}{2}\pi + o(1)$.

9 Aside: Another use for Cataln Numbers

For any positive integer n, C_n also represents the number of ways of breaking down the sum of $n+1$ numbers into pairwise operations.

For instance, when $n = 2$, $(x_1 + x_2) + x_3 = x_1 + (x_2 + x_3)$, and when $n = 3$,

$$(x_1 + x_2) + (x_3 + x_4) = (x_1 + (x_2 + x_3)) + x_4$$
$$= ((x_1 + x_2) + x_3) + x_4 = x_1 + (x_2 + (x_3 + x_4))$$
$$= x_1 + (x_2 + x_3) + x_4.$$

10 Bell Numbers

Definition 22. *The n^{th}* Bell number *B_n is trhe number of partitions of a set of n elements.*

Recall that a partition of a set A is a collection of nonempty subsets $\{A_1, \ldots, A_k\}$ such that $A_i \cap A_j = \phi$ if $i \neq j$ and $\bigcup_{i\in[k]} A_i = A$.

Example 6. *$B_1 = 1$ because the only partition of $[1]$ is $\{\{1\}\}$.*

$B_2 = 2$ because the only partitions of $[2]$ are $\{\{1,2\}\}$ and $\{\{1\},\{2\}\}$.

$B_3 = 5$ because $\{\{1\},\{2,3\}\}$, $\{\{1\},\{2\},\{3\}\}$, $\{\{1,2,3\}\}$, $\{\{1,2\},\{3\}\}$ and $\{\{1,3\},\{2\}\}$ are the only partitions of $[3]$.

Lemma 3. *For n in $\mathbb{N}$,*

$$B_n = \sum_{k=1}^{n} \binom{n-1}{k-1} B_{n-k}.$$

Proof. In order for the formula to hold for $n = 2$, we need that $2 = B_2 = \sum_{k=1}^{2} \binom{1}{k} B_k = \binom{1}{0} B_0 + \binom{1}{1} B_1 = B_0 + B_1$. This means we need to set $B_0 = 1$. Suppose $\{A_1, A_2, \ldots, A_k\}$ is a partition of $[n]$. Then there is exactly one block, say A_j which contains n. Then, $A_j = \{n\} \cup Y$ for some $Y \subset [n-1]$. If $|A_j| = m$, then $|Y| = m - 1$; so Y could be one of $\binom{n-1}{m-1}$ subsets of $[n-1]$ The remaining elements in $[n] \setminus A_j$ could be partitioned in B_{n-m} ways. Therefore,

$$B_n = \sum_{m=1}^{n} \binom{n-1}{m-1} B_{n-m}.$$

□

This recurrence relaiton which we have obtained for B_n is not a linear recurrence relation because the number of terms is not fixed but grows with n.

Definition 23 (Exponential generating function)**.** *Given a sequence $\{a_n\}_n$, the* exponential generating function *of $\{a_n\}_{n\in\mathbb{N}}$ is*

$$G(t) = \sum_{n1}^{\infty} \frac{a_n}{n!} t^n$$

Consider the epxonential generating function of $\{B_n\}$ We have

$$G(t) = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n$$

Differentiating it with respect to t, we get

$$\begin{aligned}
G'(t) &= \sum_{n=1}^{\infty} \frac{b_n}{n!} n t^{n-1} \\
&= \sum_{n=1}^{\infty} \frac{B_n}{(n-1)!} t^{n-1} \\
&= \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} \sum_{k=1}^{n} \binom{n-1}{k-1} B_{n-k} \\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{t^{k-1}}{(k-1)!} \frac{B_{n-k} t^{n-k}}{(n-k)!} \\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} \frac{B_{n-k} t^{n-k}}{(n-k)!} \chi_{[1,n]}(k) \\
&= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{t^{k-1}}{(k-1)!} \frac{B_{n-k} t^{n-k}}{(n-k)!} \chi_{[1,n]}(k) \\
&= \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} \sum_{n=k}^{\infty} \frac{t^{n-k}}{(n-k)!} B_{n-k}
\end{aligned}$$

Replacing n with $n+k$, we get

$$\begin{aligned}
&= \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{n=0}^{\infty} \frac{t^n}{n!} B_n \\
&= \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} \right) \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} B_n \right) \\
&= e^t G(t).
\end{aligned}$$

Therefore, $G'(t) = e^t G(t)$ which means

$$[\log G(t)]' = \frac{G'(t)}{G(t)} = e^t.$$

So,

$$\log G(t) = e^t + k,$$

where k is some constant in $\mathbb{R}$. This means $G(t) = Ae^{e^t}$, where A is some positive number. We had that $G(0) = B(0) = 1$. So, $Ae^1 = 1$ and $A = \frac{1}{e}$. This means

$$G(t) = e^{e^t - 1}.$$

Lecture - 17 **07 Oct 24, Mon**

11 The Inclusion-Exclusion Principle

The simplest possible manifestation of the inclusion exclusion Principle is

$$|A \cap B| = |A| + |B| - |A \cap B|$$

for any two finite substes A and B of a set X.

Theorem 10. *Let A be a finite set, and $A_1, A_2, \dots A_k \subseteq A$. Then*

$$|A \setminus \bigcup_{j \in [k]} A_j| = \sum_{J \subseteq [k]} (-1)^{|J|} |A_J|$$

where $A_J = \bigcap_{j \in J} A_j$, and $A_\phi = A$.

Remark 6. *We can think of $A_1, A_2, \dots, A_k$ as a bunch of bad sets which we want to exclude from A.*

My Proof. Let $S_a = \{i \in [n] : a \in A_i\}$ for each a in A. Let $B = \bigcup_{i \in [n]} A_i$. We may write

$$\begin{aligned} |B| &= \sum_{a \in B} 1 \\ &= \sum_{a \in B} 1 - (1-1)^{|S_a|} \end{aligned}$$

(because $|S_a| \neq 0$ for a in B.)

$$\begin{aligned} &= \sum_{a \in B} \sum_{j=1}^{|S_a|} (-1)^j \binom{|S_a|}{j} \\ &= \sum_{a \in B} \sum_{j=1}^{|S_a|} (-1)^j \sum_{\substack{J \subseteq [S_a] \\ |J| = j}} 1 \end{aligned}$$

(the binomial coefficient is simply the number of ways of choosing j sized subsets from S_a.)

$$\begin{aligned} &= \sum_{a \in B} \sum_{j=1}^{|S_a|} \sum_{\substack{J \subseteq [S_a] \\ |J| = j}} (-1)^{|J|} \\ &= \sum_{a \in B} \sum_{J \subseteq [S_a]} (-1)^{|J|} \\ &= \sum_{a \in b} \sum_{J \subseteq [n]} (-1)^{|J|} \prod_{i \in J} \chi_i(a) \end{aligned}$$

(because the product $\prod_{i \in J} \chi_i(a)$ is 1 if a is in $\bigcap_{i \in J} A_i$ and 0 otherwise)

$$= \sum_{J \subseteq [n]} (-1)^{|J|} \sum_{a \in B} \chi_1(a) \cdots \chi_n(a)$$

(because finite double summations can be interchanged.)

$$= \sum_{J \subseteq [n]} (-1)^{|J|} |\bigcap_{j \in J} A_j|.$$

□

Lecture - 18 **09 Oct 24, Wed**

Definition 24 (Derangements)**.** *If n in $\mathbb{N}$ is a positive integer and $\sigma \colon [n] \to [n]$ is a permutation of $[n]$, i.e., $\sigma \in S_n$, such that $\sigma(i) \neq i$, for any iin $[n]$, then σ is said to be a derangement of $[n]$.*

Theorem 11. *If d_n is the number of derangements of $[n]$, then*

$$d_n = \sum_{j=0}^{n} (-1)^j \binom{n}{j} (n-j)! = \sum_{j=0}^{n} (-1)^j \frac{n!}{j!}$$

Remark 7. $\sum_{j=0}^{n} \frac{(-1)^j}{j!}$ *is an approximation for $\frac{1}{e}$.*

Proof. Let $A = S_n$ and A_i be the set of all permutations σ in S_n such that $\sigma)i) = i$. So, according to the Inclusion Exclusion Principle, the number of permutations which do not fix any elements at all, d_n, is

$$\begin{aligned} d_n &= |A \setminus \bigcup_{i \in [n]} A_i| \\ &= \sum_{J \subseteq [n]} (-1)^{|J|} (|A_J|) \\ &= \sum_{J \subseteq [n]} (-1)^{|J|} (n - |J|)! \\ &= \sum_{j=0}^{n} (-1)^j \binom{n}{j} (n-j)! \end{aligned}$$

□

This theorem has a nice probabilistic interpretation. Suppose we have a large number n of individuals in a room who put their umbrellas down at the door when they come in. If each person randomly picks up an umbrella while leaving, then the probability that no person gets their umbrella back is $\frac{d_n}{n!}$ which tends to $\frac{1}{e}$ as $n \to \infty$.

Theorem 12 (Counting Surjective maps)**.** *The number of surjective maps from* $[n]$ *to* $[k]$ *is*

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^n.$$

Proof. Let $A_i = \{f\colon [n]\to[k] \mid f(j)\neq i\,\forall\, j\in[n]\}$. So, $|A_i| = (k-|A_j|)^n$. By Inclusion Exclusion Principle, we get that the cardinality of S, the set of all surjective functions from $[n]$ to $[k]$.

$$\begin{aligned}|S| &= \sum_{J\subseteq[k]}(-1)^{|J|}|A_J|\\ &= \sum_{j=0}^{n}(-1)^j\sum_{\substack{J\subseteq[k]\\|J|=j}}(k-j)^n\\ &= \sum_{j=0}^{n}(-1)^j\binom{k}{j}(k-j)^n.\end{aligned}$$

□

12 Stirling numbers of the first kind

Recall that we hada factorization of permutations as a product of disjoint cucles (unique upto obvious invariances). The unsigned Stirling number of the first kind are

$$\begin{bmatrix}n\\k\end{bmatrix}$$

which is the number of permutations of $[n]$ which can be written as a product of k disjoint cycles, over positive integers n,k such that $k\leq n$. The *Signed Stirling numbers of the first kind are*

$$s(n,k) = (-1)^{n-k}\begin{bmatrix}n\\k\end{bmatrix},$$

where $k\leq n$ are positive integers.

Note $s(n,n)=1$ because only the identity permuatation on $[n]$ can be written as a product of n disjoint cycles. Observe that

$$\sum_{j=1}^{n}\begin{bmatrix}n\\k\end{bmatrix} = |S_n| = n!.$$

Theorem 13. *If* $n\geq k$ *are positive integers, then*

$$s(n+1,k) = -n\times s(n,k) + s(n,k-1).$$

Proof. Consider

$$S_{n,k} = \{\sigma \in S_n \ : \ \sigma \text{ can be written as a product of } k \text{ disjoint cycles } \},$$

and write $S_{n+1,k} = A \cup B$, where $A = \{\sigma \in S_{n+1,k} \ : \ \sigma(n+1) = n+1\}$. We may write $|A| = \begin{bmatrix} n \\ k-1 \end{bmatrix} = (-1)^{n-k+1}s(n, k-1)$. Given an element σ in $S_{n+1,k} \backslash A$, we observe that σ has k cycles in its decomposition. So, we can write $l = \sigma(n+1)$. Then $\sigma \circ (n+1 \ l)$ is in $S_{n,k}$. Conversely, if $\sigma \in S_{n,k}$ then for any l in $[n]$, $\sigma \circ (l \, n+1)$ is in $S_{n+1,k}$. To make this observation more ludic, we can write any element of $|S_{n+1,k} \setminus A|$ as $\sigma = \sigma_1 \sigma_2 \cdots \sigma_k$, where σ_i is a cycle for each i in $[k]$ and σ_i, σ_j are disjoint for disjoint i, j in $k]$. Let $l = \sigma^{-1}(n+1)$ and i be the unique element in $[k]$ such that $\sigma_i(l) = n+1$. This uniqueness follows from the disjointness of the cycles. So, we may write

$$\begin{aligned} \sigma &= \sigma_1 \cdots \sigma_{i-1}(a_1 \, a_2 \, \ldots l \, n+1)\sigma_{i+1} \cdots \sigma_n \\ &= \sigma_1 \cdots \sigma_{i-1}(a_1 \, a_2 \, \ldots l)(l \, n+1)\sigma_{i+1} \cdots \sigma_n \\ &= \sigma_1 \cdots \sigma_{i-1}(a_1 \, a_2 \, \ldots l)\sigma_{i+1} \cdots \sigma_n(l \, n+1). \end{aligned}$$

Here, $\sigma_1 \cdots \sigma_{i-1}(a_1 \, a_2 \, \ldots l)\sigma_{i+1} \cdots \sigma_n$ is in $S_{n,k}$. The number of elements in $S_{n+1,k}$ is $-s(n,k)$ and each element in $S(n,k)$ corresponds to n distinct elements in $S(n+1,k)$ arising from n distinct choices of the image $\sigma^{-1}(n+1) = l$. $\square$

Lecture - 19 **11 Oct 24, Fri**

Definition 25 (The falling factorial)**.** *For any positive integer n, we wdefine the* falling factorial *to be*

$$(t)_n = t(t-1)\cdots(t-n+1).$$

Definition 26 (The rising factorial)**.** *For any positive integer n, we define the* rising factoria *to be*

$$(t)^n = t(t+1)\cdots(t+n-1).$$

Remark 8.

$$\binom{\alpha}{n} = \frac{(\alpha)_n}{n!}$$

Remark 9. $t^{(n)} = (-1)^n(-t)_n$

Theorem 14. *For any t in $\mathbb{R}$ and nay n in $\mathbb{N}$, we have*

$$(t)_n = \sum_{k \in [n]} s(n,k)t^k.$$

Remark 10. *For fixed n in $\mathbb{N}$, the generating function of $\{s_{n,k} \ : \ k \in \mathbb{N}, k \leq n\}$ is*

$$(t)_n = \sum_{k \in [n]} s(n,k)t^k.$$

Proof. For $n = 1$, $s(1,1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Suppose the formula is true for some n in $\mathbb{N}$. Then,

$$\begin{aligned} (t)_{n+1} &= (t-n)(t)_n \\ &= (t-n)\left(\sum_{k\in[n]} s(n,k)t^k\right) \\ &= \sum_{k\in[n]} s(n,k)t^{k+1} - n\sum_{k\in[n]} s(n,k)t^k \\ &= \sum_{k=0}^{n-1} s(n,k-1)t^k - \sum_{k=0}^{n} ns(n,k)t^k \\ &= \sum_{k\in[n]} s(n+1,k)t^k. \end{aligned}$$

□

13 Stirling Numbers of the Second Kind

Definition 27. *The Stirling number of the second kind are denoted as*

$$\left\{ n \atop k \right\} = S(n,k)$$

for positive integers n, k and they represent the number of partitions of $[n]$ into k nonempty blocks.

Remark 11. *As the n^{th} Bell number B_n is the number of partitions is the toal number of partitions of $[n]$ into nonempty blocks, we have*

$$B_n = \sum_{k=1}^{n} \left\{ b \atop k \right\}.$$

Any partition of $[n]$ into k nonempty blocks is related to the surjective functions from $[n]$ to $[k]$. Given a surjective function $f\colon [n] \to [k]$, the sets $\{f^{-1}(1), \ldots, f^{-1}(k)\}$ form a partition of $[n]$ into k non empty blocks.

Conversely, given a partition $A_1, A_2, \ldots, A_k$ of $[n]$ into k nonempty blocks and σ in S_k, $f\colon [n] \to [k]$ defined by $f(i) = \sigma(j)$ such that i is in A_j is a surjective function. Thus, we have proved the following theorem

Theorem 15. *If $n \geq k$, then*

$$\begin{aligned} k! \begin{Bmatrix} n \\ k \end{Bmatrix} &= |\{f : [n] \to [k] \ : \ f \text{ is surjective } \}| \\ &= \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n \end{aligned}$$

Remark 12. *We have the recurrence relation for positive integers $k \leq n$,*

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} + k \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}.$$

Theorem 16. *for any t in $\mathbb{R}$, and any n in $\mathbb{N}$, we have*

$$t^n = \sum_{k \in [n]} \begin{Bmatrix} n \\ k \end{Bmatrix} (t)_k.$$

Proof. May be proven using induction. □

Remark 13. *Note that $(t)_n$*

Definition 28. *A vector space V over a field $\mathbb{F}$, is a set A equipped with two operations $+ \colon A \times A \to V$ usually called 'addition' and $\cdot : \mathbb{F} \times A \to A$ called the 'scalar multiplication' such that for any α, β in $\mathbb{F}$ and x, y and z in A,*

1. $\alpha(\beta x) = (\alpha\beta)x$,
2. $(\alpha + \beta)x = \alpha x + \beta x$,
3. $\alpha(x + y) = \alpha x + \alpha y$,
4. $(\alpha + \beta)x = \alpha x + \beta x$,
5. $1x = x$,
6. $x + y = y + x$,
7. $(x + y) + z = x + (y + z)$,

where 1 and 0 are the multiplicative and additive identity of $\mathbb{F}$ and there exists $\mathbf{0}$ in A such that $0x = \mathbf{0}$ for all x in A.

Hereafter, for a vector space V, the underlying set A will also be represented by V.

The vector space of polynomials over an unkown t with coefficients in $\mathbb{F}$, is a vector space over $\mathbb{F}$.

Let $\mathcal{P}_n$ be the (sub)space of all polynomials in $\mathbb{F}[t]$ of degree at most n. Then the dimension of $\mathcal{P}_n$ is $n + 1$ because $\{1, t, t^2, \ldots, t^n\}$ forms a basis of it.

Note that $\{(t)_0, (t)_1, \ldots, (t)_n\}$ and $\{t^{(0)}, t^{(1)}, \ldots, t^{(n)}\}$ form two separate bases of $\mathcal{P}_n$. The change of basis formulae for these bases are given by the relation between $(t)_i$ and $t^{(i)}$.

If $A = \{s(i,j)\}$ and $B = \{S(i,j)\}$ are $n \times n$ matrices, then $A = B^{-1}$.

The convention is to write

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{Bmatrix} n \\ 0 \end{Bmatrix} = \begin{cases} 0 & \text{if } n \in \mathbb{N} \\ 1 & \text{if } n = 0 \end{cases}.$$

k

Lecture - 20 **14 Oct 2024, Mon**

Note that $\{1 = t^0, t, t^2, \ldots, t^n\} = \mathcal{B}$, $\{(t)_0, (t)_1, (t)_2, \ldots, (t)_n\} = \mathcal{C}$, and $\{t^{(0)}, t^{(1)}, t^{(2)}, \ldots t^{(n)}\}$-$= \mathcal{D}$ form three separate bases of $\mathcal{P}_n$.

So, the formula

$$\begin{aligned} t^n &= \sum_{k \in [n]} \begin{Bmatrix} n \\ k \end{Bmatrix} t^k \\ &= \end{aligned}$$

Theorem 17. *If $\{f_n\}$ and $\{g_n\}$ are sequences related by*

$$g_n = \sum_{k=a}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} f_k, \qquad \forall n \geq a,$$

then we have the inversion formula

$$f_n = \sum_{k=a}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^{n-k} g_k, \qquad \forall n \geq a.$$

Remark 14. *If $a = 0$, or 1, this follows immediately from the fact that $B = A^{-1}$. Note that if $a = 0$, for $n = 0$, we have $f_0 = g_0$, and for $n = 1$, would imply*

$$g_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} f_1, \quad f_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} g_1.$$

An application to calculus

It is easier to compute the differences between functions rather than to compute the derivatives. In this case, we choose to approximate the interval by a grid. A real analytic function is a smooth i.e., infinitely differentiable function for which the infinite Taylor Series converges about any point. For a real analytic function, $f\colon \mathbb{R} \to \mathbb{R}$, we have

$$\begin{aligned} f(x+y) - f(x) &= yf'(x) + \frac{f^{(2)}}{2!} y^2 + \frac{f^{(3)}}{3!} y^3 + \cdots \\ &= \sum_{k=1}^{\infty} \frac{y^k}{k!} f^{(k)}(x). \end{aligned}$$

In particular, if Δ represents the finite difference operator, given by

$$(\Delta f)(x) = f(x+1) - f(x),$$

then

$$(\Delta f)(x) = \sum_{k=1}^{\infty} \frac{f^(k)(x)}{k!}.$$

More generally, we can talk about the higher order finite difference operators such as

$$\begin{aligned}
(\Delta^2 f)(f) &= (\Delta(\Delta f))(x) \\
&= \Delta\left[f(x+1) - f(x)\right] \\
&= \left[f(x+2) - f(x+1)\right] - \left[f(x+1) - f(x)\right] \\
&= f(x+1) - 2f(x+1) + f(x).
\end{aligned}$$

Similarly, we define $(\Delta^k f)(x) = (\Delta(\Delta^{k-1} f))(x)$.

Theorem 18. *If f is real anlaytic, then*

$$\frac{1}{k!}(\Delta^k f)(x) = \sum_{n=k}^{\infty} \frac{S(n,k)}{n!} f^{(n)}(x).$$

Consequently,

$$\frac{1}{k!} f^{(k)}(x) = \sum_{n=k}^{\infty}$$

FILL THIS

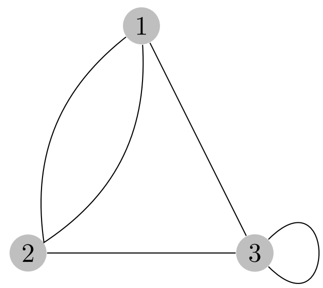
14 Graphs

Definition 29 (Adjacency Matrix)**.** *Given a graph $G = (V, E)$, we define its* adjacency matrix A, *as a matrix with rows and columns indexed by V, such that $A(x, y)$ is the number of edges from x to y.*

Example 7. *For the graph We have the adjacency matrix* $\begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

Theorem 19. *Let $G = (V, E)$ be a graph without loops and with adjacency matrix A. For x, y in V, and any l in $\mathbb{N}$, $A^l(x, y)$ is the number of (x, y) walks of length l in G.*

Here, $A^l(x, y)$ represents the $(x, y)^{\text{th}}$ entry of the matrix A^l.



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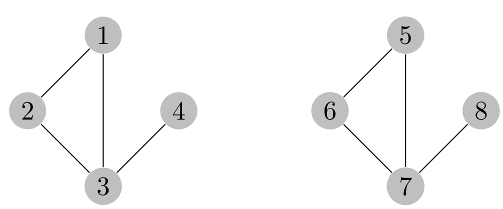
Proof. We prove this by induction on l. For $l = 1$, the theorem holds true because the only walks of length 1, from a vertex x to a vertex y are the edges from x to y. So the number of walks of length 1 from x to y is exactly the number of edges from x to y. Now suppose the theorem holds for $i = 1, \ldots, k-1$. Suppose x, y in V are arbitrarily given. We will prove that the number of walks from x to y of length k is $A^k(x, y)$. Let $x = v_0, e_1, v_1, \ldots, e_{k-1}, v_{k-1}, e_k, v_k = y$ denote a walk w from x to y of length k. So, each walk of length l from x to y is a concatenation of a some w' in $W_{k-1}(x, v_{k-1})$ and some edge from v_{k-1} to y where v_{k-1} is some vertex in V and $W_{k-1}(x, v)$ denotes the number of walks of length $k-1$ from a vertex x to a vertex v in V. Conversely, if w' is in $W_{k-1}(x, v)$ and e is an edge from v to y for some vertex v in V, then the concatenation $w' * e$ is a walk of length k from x to y. Using the induction hypothesis, we have $|W_{k-1}(x, v)| = A^{k-1}(x, v)$ for all x and v in V. So, we have the cardinality of $W_k(x, y) = \sum_{v \in V} |W_{k-1}(x, v)| A(v, y) = (A^{k-1}A)(x, y) = A^k(x, y)$. □

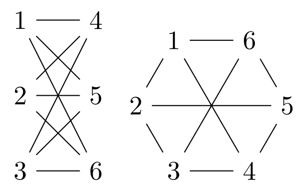
Corollary 2. *Let $G = (V, E)$ be a simple graph with $|V| = n$, adjacency matrix A, and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the characteristic values of A, then the number of closed walks of length l in G is $\lambda_l^1 + \lambda_2^2 + \cdots \lambda_n^l$.*



The two graphs above have different labels but the same structure.

Definition 30 (Isomorphism)**.** *An isomorphism between graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a bijection $\sigma\colon V_1 \to V_2$. such that $\{i, j\}$ is in E_1 if and only if $\{\sigma(i), \sigma(j)\}$ is in E_2.*

Remark 15. *The permutation σ relabels the vertices of G_1 but does not change its structure.*



Theorem 20. *Let G_1, G_2 are graphs on a vertex set V, then $G_1 \equiv G_2$ if and only if there exists a permutation matrix P such that $A_1 = P^{-1}A_2P$ where A_1, respectively A_2 are the adjacency matrices of G_1, respectively G_2.*

A permutation matrix is any matrix which has exactly one nonzero entry in each row or column and consists of only 0s and 1. So, there is a bijective correspondence between permutations σ in S_n and permutation matrices of size $n \times n$ given by $\sigma \leftrightarrow P_\sigma$ where

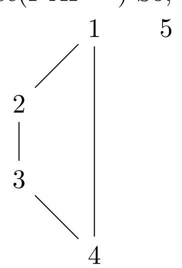
$$P_\sigma(i,j) = \begin{cases} 1 & \text{if } i = \sigma(j) \\ 0 & \text{otherwise} \end{cases}.$$

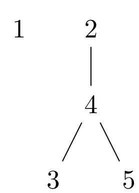
Also, $P_\sigma P_\tau = P_{\sigma\tau}$ and $P_\sigma^{-1} = P_{\sigma^{-1}}$.

Lecture - 22 **18 Oct 24, Fri**

Definition 31 (Spectrum)**.** *The spectrum* $\mathrm{Spec}(G)$ *of a graph G is the set of characteristic values of its adjacency matrix including multiplicity.*

The $\ker(A - \lambda I) = P^{-1}\ker(PAP^{-1} - \lambda I)P$ which implies that the $\mathrm{Spec}(A) = \mathrm{Spec}(PAP^{-1})$ So, G_1 is the same as G_2 for similar graphs G_1 and G_2.





Let K_n be the complete graph on n vertices. Then its adjacency matrix $A(K_n)$ has 1 everywhere except 0 on every position in the diagonal. In other words $A(K_n) = J_n - I_n$, where I_n is the $n \times n$ identity matrix and $J_n = \mathbf{1}\mathbf{1}^T$, where $\mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$. Since J_n is a rank 1 matrix, it has 0 as a characteristc value with multiplicity $n-1$ and $J_n\mathbf{1} = (\mathbf{1}\mathbf{1}^T)\mathbf{1} = \mathbf{1}(\mathbf{1}^T\mathbf{1}) = n\mathbf{1}$ implying that $\mathbf{1}$ is a characteristic vector of J_n with characteristic value n and multiplicity 1. So, the characteristic vlaues of $A(K_n)$ are 0 with multiplicity 1 and -1 with multiplicity $n-1$.

The complete bipartite graph on r, s vertices, $K_{r,s}$ has adjacency matrix

$$A(K_{r,s}) = \begin{bmatrix} 0_{r,r} 1_{r,s} \\ 1_{s,r} 0_{s,s} \end{bmatrix}$$

$A(K_{r,s})$ has characteristic values $\lambda_1 > \lambda_2$. As trace$(A(K_{r,s})) = \lambda_1 + \lambda_2$. Also, trace$((A(K_{r,s}))^2) = \lambda_1^2 + \lambda_2^2$ is the number of closed walks of length 2 in $K_{r,s}$ which is $|E(K_{r,s})|^2$ because the only closed walks of length 2 are those that walk along an edge and then back along it.

SO WHAT??? COMPLETE THIS!!!

Given the spectrum of a graph, we would like to know whether something may be said about the graph. For bipartite graphs, the answer is true.

Theorem 21 (Characterisation of spectrum of bipartite graphs)**.** *If G is a bipartite graph having a characteristic value λ with multiplicity n, then $-\lambda$ is also a characteristic value with multiplicity n.*

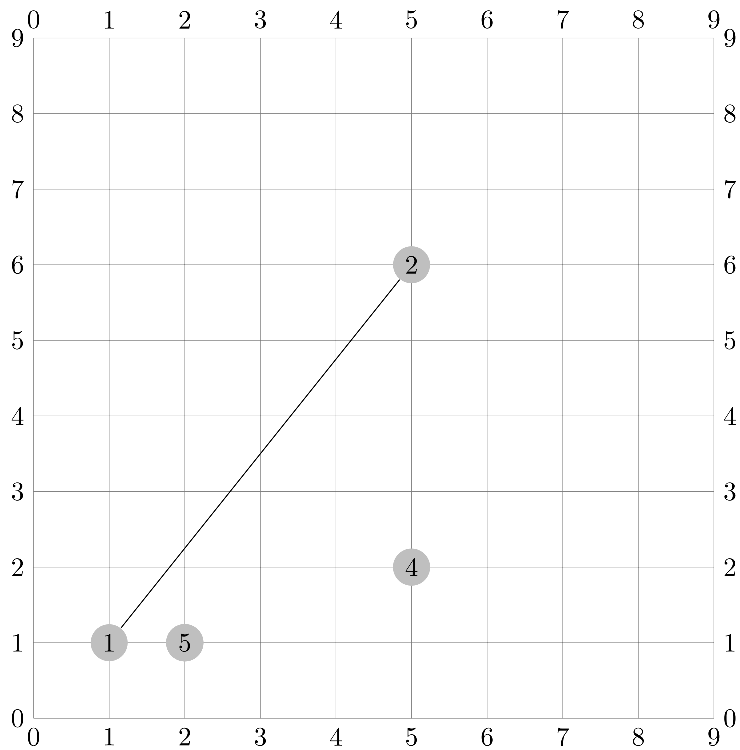
Proof. For any bipartite graph G, the adjacency matrix $A(G)$ is of the form

$$\begin{bmatrix} 0_{r,r} & B_{r,s} \\ B^T_{s,r} & 0_{s,s} \end{bmatrix}.$$

If $\lambda > 0$ is a characteristic value of $A(G)$, then there exists $w = \begin{bmatrix} u_{r,1} \\ v_{s,1} \end{bmatrix}$ such that

$$\begin{bmatrix} 0_{r,r} & B_{r,s} \\ B^T_{s,r} & 0_{s,s} \end{bmatrix} \begin{bmatrix} u_{r,1} \\ v_{s,1} \end{bmatrix} = \begin{bmatrix} Bv \\ B^T u \end{bmatrix} = \lambda \begin{bmatrix} u_{r,1} \\ v_{s,1} \end{bmatrix}.$$

CORRECT THIS!! In other words, $Bv = \lambda u$ and $B^T v = \lambda u$. If the multiplicity of λ is m, there are m linearly independent characteristic vectors for λ, we deduce there are also m linearly independent vectors obtained by negating one $\square$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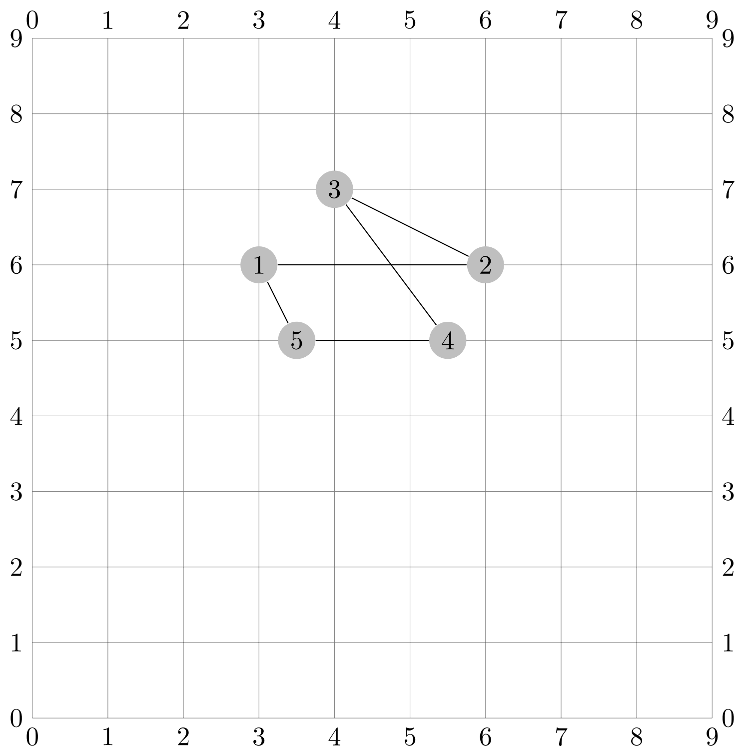
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