Abstract

This is an essay submitted for step 2 of the project for course PHYS/CISC650 - Introduction to Quantum Computing and Quantum Information Theory taught by Prof Alexei Kananenka at University of Delaware in Fall 2025. I have explained a few quantum error correction algorithms in this paper. The content is sourced mainly from [4]. Figures are as attributed but were texified manually by me.

The current trends in quantum computing make use of photons [6], trapped ions [1], superconducting qubits and spins in semiconductors to act as qubits. A common shortcoming of all these technologies is scalability and fidelity. *Scalability* - Current quantum computing devices operate with on the order of a hundred qubits with approximately 20–30 gate operations and are not capable of error correction [3]. *Fidelity* - It is not possible to isolate these qubits from the environment easily. This results in several errors acting on the qubits such as X, Y and Z errors. The X error is a bit flip error which also occurs in classical information processing systems. Quantum information processors are different in that they also suffer from phase errors like Y and Z. The third problem is that it is difficult to construct an information processor with several hundreds of too many qubits which interact with each other coherently. The advantage of quantum computing is that information is stored not simply in the bits (qubits) but largely in the probability correlation representing the probability of each qubit occurring in a certain state. This makes quantum systems susceptible to decoherence. This means that if the entanglement breaks down, then the information is lost.

To combat these issues, Peter Shor proposed the first quantum error correction scheme in 1995 [5]. This method demonstrated how quantum information can be redundantly stored and error be suppressed. In this paper, I will describe some modern quantum error correction codes such as the stabiliser code.

1 Example - The Three Qubit Error Correction Code

In this section I first describe the two qubit error detection code which utilises projective measurements on the syndrome qubit. Then I describe the three qubit error correction code.

Classical error correction codes depend on the reliable duplication of the source information into multiple instances, and using more resources to represent the duplicated information. Quantum information suffers from lack of such reliability and the impossibility of doing so described in [7], as the no-cloning theorem raises challenges in error correction protocols. The smallest correction code which can correct all errors on a single qubit uses 5 qubits [2]. Any state ψ can not be cloned into another qubit simultaneously. the operation

$$\psi \xrightarrow{\text{two-qubit encoder}} \psi \otimes \psi$$

cannot be possibly done using any Unitary operator. Since all quantum gates are described by unitary transformations, this operation is not possible to be done with any quantum information processor at all.

If such an operator U existed, then

$$A|0\rangle = |00\rangle \qquad \text{and} \qquad A|1\rangle = |11\rangle .$$

So,

$$\begin{aligned} A(\alpha|0\rangle + \beta|1\rangle) &= \alpha^2|00\rangle + \alpha\beta(|01\rangle + |10\rangle) + \beta^2|11\rangle \\ &\neq \alpha|00\rangle + \beta|11\rangle = \alpha(A|0\rangle) + \beta(A|1\rangle). \end{aligned}$$

This shows that A cannot even be a linear operator, let alone a unitary. This theorem and proof are sourced from [7]. Even though this proof is written as if for two states, it forbids cloning for arbitrary states in any system.

The means to circumvent this error is to pass the state through a two qubit encoder

$$\alpha|0\rangle + \beta|1\rangle \xrightarrow{\text{two-qubit encoder}} \alpha|00\rangle + \beta|11\rangle = \alpha|0\rangle_L\beta|1\rangle_L.$$

An error X_1, respectively X_2, which flips the first, respectively second bit, sends the state ψ into the subspace $\mathcal{F}$ spanned by $|01\rangle$ and $|10\rangle$. Two qubit encoding thus renders both errors indistinguishable. It is thus capable of detecting errors but not of correcting them.

The code subspace $\mathcal{C}$ is the span of $|00\rangle$ and $|11\rangle$. Since $\mathcal{C}$ and $\mathcal{F}$ are mutually orthogonal subspaces it is possible to make a projective measurement on one of the qubits without destroying any of the information.

2 The projective measurement Z_1Z_2.

The Z_1Z_2 gate is described as follows

$$Z_1Z_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}..$$

If an error occurs, then $\psi \in \mathcal{F}$

$$|01\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Thus $Z_1Z_2|10\rangle$ is given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = - \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = -\psi.$$

Similarly, if

$$\psi = |01\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

then

$$Z_1 Z_2 \psi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} = -\psi.$$

Thus errors are detected by presence of a state in the characteristic subspace corresponding to -1. The fact that there are only two characteristic spaces, of values 1 and -1, can be restated as follows. All the results of a single error on one qubit is anticommutative with one of the projective measurements.

3 Three Qubit error correction code

Akin to the two qubit code, this algorithm also utilises projective measurements which act on the main state and extract two qubits in the syndrome. Similar the the two qubit encoder, this algorithm prepares the state in the first qubit $\psi = \alpha|0\rangle + \beta|1\rangle$ by duplicating it into the second and third qubits. The latter two are prepared in the state zero and encoded as:

$$\mathrm{CNOT}_{1,2}\,\mathrm{CNOT}_{1,3}\,\psi \otimes |0\rangle \otimes |0\rangle = \alpha|000\rangle + \beta|111\rangle.$$

Following the error, the state enters one of the four orthogonal subspaces $\mathcal{C} = \mathrm{span}\{|000\rangle, |111\rangle\}$, $\mathcal{F}_1 = \mathrm{span}\{|100\rangle, |011\rangle\}$, $\mathcal{F}_2 = \mathrm{span}\{|010\rangle, |101\rangle\}$, and $\mathcal{F}_3 = \mathrm{span}\{|001\rangle, |110\rangle\}$. Following the duplication encoding, ψ enters $\mathcal{C}$ and stays in this subspace $\mathcal{C}$. if no error occurs. In case X_1, X_2, respectively X_3 error occurs, it enters $\mathcal{C}_1$, $\mathcal{C}_2$, respectively $\mathcal{C}_3$.

$$Z_1 Z_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

As this matrix leaves

$$\begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\1\end{bmatrix}$$

unchanged, these are in the characteristic subspace of Z_1Z_2. The -1 subspace of Z_1Z_2 is spanned by

$$\begin{bmatrix}0\\0\\1\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\1\\0\\0\end{bmatrix}.$$

Thus the operation $C_{Z_1Z_2}$ sends the first syndrome qubit A_1 from state $|+\rangle$ to $|-\rangle$ if the first two qubits are in the -1 characteristic space and leaves it unchanged if it is in the $+1$ characteristic state.

So HA_1 measures 1 if ψ is in $\mathcal{F}_1$ or in $\mathcal{F}_2$, the ones where first two qubits are different. Similarly, HA_2 measures 1 if the second two qubits are different. If both happen, then the state is in $\mathcal{F}_2$. The other cases are

space	A_1A_2
$\mathcal{C}$	00
$\mathcal{F}_1$	10
$\mathcal{F}_2$	11
$\mathcal{F}_3$	01

.

The three qubit correction code only protects against single bit flips. It does not protect against multiple bit flips or Z errors.

4 Stabiliser Codes

The three qubit code is an example of a stabiliser code. In general, an k qubit code can be entangled with $m = n - k$ redundancy qubits and stabiliser measurements can be performed on the first k qubits. Then the result of each of the projective measurements P_i can be recorded in the A_i syndrome qubit. The group $\mathcal{G}_n$ of Pauli operators of weight upto n consists of elements such as $I \otimes X \otimes Z \otimes X$. This operator is an element of $\mathcal{G}_4$ and has weight 3 because it has 3 non identity measurements. An element of $\mathcal{G}_n$ is called a stabiliser if it leaves the codespace unchanged. The stabilisers $\{P_i\}$ are selected such that they are all

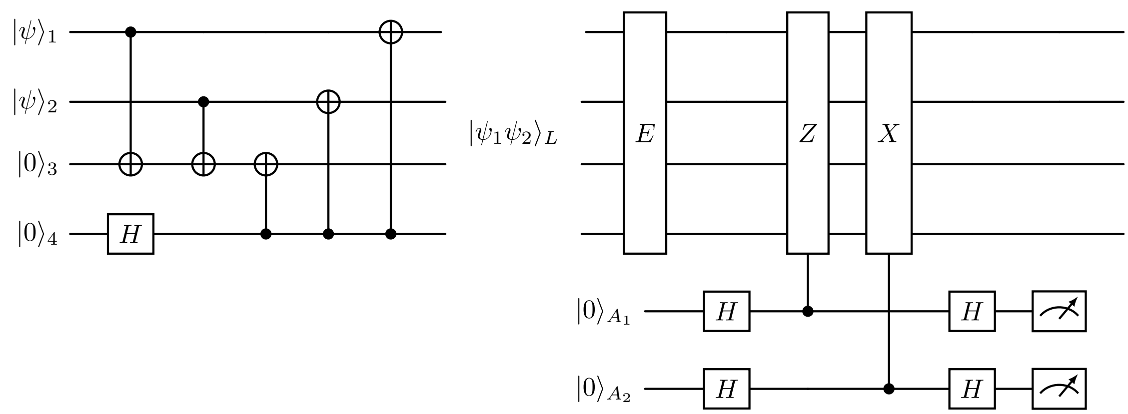

Figure 1: Circuit for the $[\![4,2,2]\!]$ algorithm. source: [4, figure 5]

elements of the Pauli group $\mathcal{G}_n$ and leave the codespace unchanged. Moreover, it is essential that the stabilisers all commute with each other. This ensures that the measurements can be performed in any order or even simultaneously. (n,k,d) notation

1. n is the number of bits per codeword.
2. k is the number of encoded bits.
3. d is the code distance

Here d is the number minimum size of an error which goes undetected.

5 The $[\![4,2,2]\!]$ detection code

The $[\![4,2,2]\!]$ detection code is the smallest error correction code which can detect X type as well as Z type errors. However, it cannot correct these errors though.

The encoder sends $|00\rangle$ to $\frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$, $|01\rangle$ to

$$\begin{aligned}
&\mathrm{CNOT}_{4,1}\,\mathrm{CNOT}_{4,2}\,\mathrm{CNOT}_{4,3}\,|011+\rangle \\
=\;&\mathrm{CNOT}_{4,1}\,\mathrm{CNOT}_{4,2}\,\frac{1}{\sqrt{2}}(|0110\rangle + |0101\rangle) \\
=\;&\mathrm{CNOT}_{4,1}\,\frac{1}{\sqrt{2}}(|0110\rangle + |0001\rangle) \\
=\;&\frac{1}{\sqrt{2}}(|0110\rangle + |1001\rangle)
\end{aligned}$$

Essentially, the first syndrome qubit stores the parity of the data qubits and the second one entangles the first three with their flips. So the codespace of this algorithm is

$$\mathcal{C} = \text{span}\left\{\begin{array}{l} |00\rangle_L = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle) \\ |01\rangle_L = \frac{1}{\sqrt{2}}(|0110\rangle + |0110\rangle) \\ |10\rangle_L = \frac{1}{\sqrt{2}}(|1010\rangle + |0101\rangle) \\ |11\rangle_L = \frac{1}{\sqrt{2}}(|1100\rangle + |1101\rangle) \end{array}\right\}.$$

This space is stabilised by $X_1X_2X_3X_4$ and $Z_1Z_2Z_3Z_4$. Also, the result of an X flip is to make the first syndrome qubit 1, of a Z error is to make the second syndrome qubit 1. Thus the extraction of a syndrome allows detection of some error but does not tell us which error occurred. This is also consistent with the equation

$$d = 2t + 1,$$

as the highest weight of logical operator in this algorithm is 2, $d = 2$. So $t = 0$, that means it can not correct errors.

The algorithm which can correct all errors one one logical qubit is the $[\![5, 1, 3]\!]$ algorithm. Some errors can propagate to correlated errors of higher weight on the data. This algorithm works by recording such errors on a flag which is switched from $|+\rangle$ to $|-\rangle$ if such a propagative error has occured.

6 The Shor $[\![9, 1, 3]\!]$ Code

The code works on one logical qubit and encodes it into 9 qubits. This is done by concatenating the codeword with itself three times. So the codeword $|-\rangle_{3p} = \frac{1}{2}(|000\rangle + |111\rangle)$ is transformed to

$$|-\rangle_{3p}\,|-\rangle_{3p}\,|-\rangle_{3p} = \frac{1}{\sqrt{8}}((|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)).$$

The code $|+\rangle_{3p}$ is transformed similarly, and the codespace is

$$\mathcal{C}_{[\![9,1,3]\!]} = \text{span}\left\{\begin{array}{l} |0\rangle_9 = \frac{1}{\sqrt{8}}\left((|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)\right) \\ |1\rangle_9 = \frac{1}{\sqrt{8}}\left((|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)\right) \end{array}\right\}.$$

The stabilisers of this space are

$$\mathcal{S}_{[\![9,1,3]\!]} = \begin{array}{l} \langle Z_1Z_2, Z_2Z_3, Z_4Z_5, Z_5Z_6, Z_7Z_8, Z_8Z_9, \\ X_1X_2X_3X_4X_5X_6, X_4X_5X_6X_7X_8X_9\rangle. \end{array}$$

Note that these are simply stabilisers of the individual components which are being concatenated together.

If the X_5 occurs, then Z_4Z_5 and Z_5Z_6 send the state $\psi = (|000\rangle - |111\rangle\,(|000\rangle - |111\rangle))(|000\rangle\,0\,|111\rangle)$ (for example) to the -1 characteristic space. All the other stabilisers send it to the +1 subspace. So the syndrome extracted is 001100000. This is because

$$X_5 |0\rangle_{3p} = X_5 \left(\frac{1}{\sqrt{8}} ((|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)) \right)$$
$$= \left(\frac{1}{\sqrt{8}} ((|000\rangle + |111\rangle)(|010\rangle + |101\rangle)(|000\rangle + |111\rangle)) \right).$$

The Z_4Z_5 operator acts on this state to make it

$$\frac{1}{\sqrt{8}} ((|000\rangle + |111\rangle)(-|010\rangle - |101\rangle)(|000\rangle + |111\rangle)).$$

upon entanglement with a syndrome qubit corresponding to Z_4Z_5, $\psi \otimes |+\rangle_4 = \frac{1}{\sqrt{2}}(\psi \otimes |0\rangle + \psi \otimes |1\rangle)$ transforms to $Z_4Z_5\psi \otimes |+\rangle_4 = \frac{1}{\sqrt{2}}(\psi \otimes |0\rangle - \psi \otimes |1\rangle) = \psi \otimes |-\rangle_4$. The syndrome $|+\rangle_4$ is passed through the Hadamard gate and extracted as a 0. Note that the key here is that Z_4Z_5 acts on it only if the syndrome qubit is in state 1. In this way the other syndrome bits can be extracted for all the states in the codespace. If the syndrome reads 001100000 then the error recovery is done by applying Z_4Z_5 and Z_5Z_6 on the state. In this way it can correct alll single qubit errors having distance $d = 3$.

7 Scalability

Some of the challenges with Stabiliser codes are finding the sets of stabilisers, and finding codes which work well with the chosen stabilisers. Hamming Codes, which are $[\![2^r - 1, 2^r - 1 - 2r, 3]\!]$ algorithms of the Stabliiser algorithms, can be scaled. g Another modern class of scalable quantum error correction algorithms is the Surface code algorithms. The Surface code algorithms belong to a class of codes called the topological codes. Surface codes mitigate the problem of finding stabilisers problem by patching together smaller repeated elements of smaller codes. Hence they can be scaled to larger codes easily. Moreover, these only require interaction of qubits with near neighbours. This is helpful because high fidelity interactions over long ranges are not easily possible between qubits in most quantum computing platforms. This makes surface codes the current standard in quantum error correction. However, there are challenges to surface codes. It has a poor encoding density. While the distance of the code can be increased, the cost of doing so is a vanishing code rate. It is also highly resource intensive.

8 Current and future trends

The future of quantum computing requires processors which can satisfy the following four requirements. The architecture should have a high noise threshold. It should enable implementation of universal sets of logic gates. The processing should be possible with few physical qubits per logical qubits. The need for more qubits arises from the role of redundancy in error correction. The times required to execute the error correction should also be low so as to be compatible with the main computation itself. These constraints are physical. Low density parity check codes are one such class of error correction codes which satsify certain upper bounds on the number of qubits involved in each parity check and the number of parity checks as well. These factors bring a great deal of attention to these algorithms in the field of quantum error correction.

References

[1] C. J. Ballance, T. P. Harty, N. M. Linke, M. A. Sepiol, and D. M. Lucas. High-fidelity quantum logic gates using trapped-ion hyperfine qubits. *Phys. Rev. Lett.*, 117:060504, Aug 2016.

[2] Rui Chao and Ben W. Reichardt. Quantum error correction with only two extra qubits. *Physical Review Letters*, 121(5), August 2018.

[3] A. K. Fedorov, N. Gisin, S. M. Beloussov, and A. I. Lvovsky. Quantum computing at the quantum advantage threshold: a down-to-business review. Jan 2022.

[4] Joschka Roffe. Quantum error correction: an introductory guide. *Contemporary Physics*, 60(3):226–245, July 2019.

[5] Peter W. Shor. Scheme for reducing decoherence in quantum computer memory. *Phys. Rev. A*, 52:R2493–R2496, Oct 1995.

[6] Xi-Lin Wang, Luo-Kan Chen, W. Li, H.-L. Huang, C. Liu, C. Chen, Y.-H. Luo, Z.-E. Su, D. Wu, Z.-D. Li, H. Lu, Y. Hu, X. Jiang, C.-Z. Peng, L. Li, N.-L. Liu, Yu-Ao Chen, Chao-Yang Lu, and Jian-Wei Pan. Experimental ten-photon entanglement. *Phys. Rev. Lett.*, 117:210502, Nov 2016.

[7] William K Wootters and Wojciech H Zurek. A single quantum cannot be cloned. *Nature*, 299(5886):802–803, 1982.