Quantum Error Correction: Stabiliser Codes

PHYS/CISC 650 – Step 4 Presentation

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Motivation

Quantum computing platforms:

- Photonic qubits [3]
- Trapped ions [1]
- Superconducting qubits, semiconductor spins

Challenges:

- **Scalability:** only 100–200 qubits, shallow circuits [2]
- **Fidelity:** vulnerable to X, Y, Z errors
- **Decoherence:** information stored in correlations $\rightarrow$ fragile

Why Error Correction?

Quantum states cannot be freely copied (No-Cloning Theorem).

$$
\begin{aligned}
A(\alpha|0\rangle + \beta|1\rangle) &= (\alpha\,|0\rangle + \beta\,|1\rangle)(\alpha\,|0\rangle + \beta\,|1\rangle) \\
&= \alpha^2\,|00\rangle + \alpha\beta\,|01\rangle + \alpha\beta\,|10\rangle + \beta^2\,|11\rangle \neq \alpha\,|00\rangle + \beta\,|11\rangle = \alpha A|0\rangle + \beta A|1\rangle
\end{aligned}
$$

Implications:

- Cannot use classical redundancy (bit duplication)
- Need encoding into larger entangled states
- Detect and correct errors without collapsing the state

Two-Qubit Detection Code

Encoding:

$$\alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|00\rangle + \beta|11\rangle$$

Detects bit-flip (X) errors, but cannot distinguish where they occurred.
Measurement:

$$Z_1 Z_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Outcome:

- $+1 \Rightarrow$ no error
- $-1 \Rightarrow$ some error occurred

Three-Qubit Bit-Flip Code

Encoding:

$$|\psi\rangle \otimes |00\rangle \xrightarrow{\text{CNOT}} \alpha|000\rangle + \beta|111\rangle$$

Four orthogonal subspaces:

$$\mathcal{C}, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$$

Syndromes:

Subspace	A_1A_2
$\mathcal{C}$	00
$\mathcal{F}_1$	10
$\mathcal{F}_2$	11
$\mathcal{F}_3$	01

Corrects any single X error.

Stabilizer Codes

Pauli group $\mathcal{G}_n$:

$$\{I, X, Y, Z\}^{\otimes n}$$

Stabilizer code:

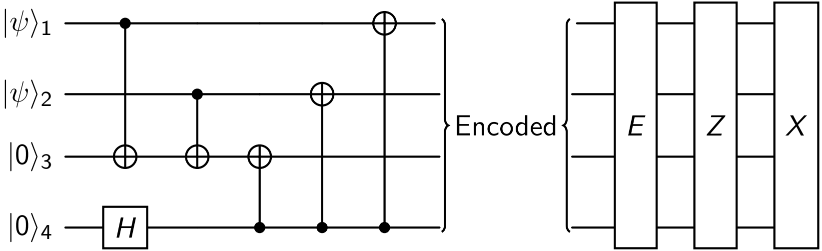
- Codespace $= +1$ eigenspace of all stabilizers
- Generators commute
- (n, k, d) notation

Distance d:

$$d = \text{minimum weight of undetectable error}$$

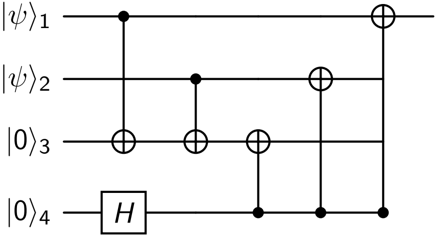
4, 2, 2 Detection Code

Smallest stabilizer code detecting both X and Z errors.



Syndrome bits detect error type but cannot correct them because $d = 2$.

Codespace



1. Given $\psi = |00\rangle$, we compute ψ_L.
2. First anciliary bit is the parity check. So $a_1 = |0\rangle$. $a_2 = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.
3. Entangle $|000\rangle$ with a_2 to get $\frac{1}{2}(|000\rangle\,(|0\rangle + |1\rangle)$.
4. Finally, whenever a_2 is $|1\rangle$, flip the first three qubits. $\psi_L = \frac{1}{2}(|0000\rangle + |1111\rangle)$.

Shor Code: $9, 1, 3$

First quantum error-correcting code (1995).
Encoding:

$$|0\rangle_9 = \frac{1}{\sqrt{8}}(|000\rangle + |111\rangle)^{\otimes 3}$$

Stabilizers include:

$$Z_1Z_2, Z_2Z_3, \ldots, \; X_1X_2X_3X_4X_5X_6, \ldots$$

Corrects **any** single-qubit error ($d = 3$).

Syndrome Example (Shor Code)

If error X_5 occurs:

$$Z_4Z_5, Z_5Z_6 \Rightarrow -1$$

All other stabilizers return $+1$.
Syndrome:

$$001100000$$

Recovery:

$$Z_4Z_5,\ Z_5Z_6$$

Scalability Challenges

Stabilizer code issues:

- Choosing stabilizers
- Constructing codes with good distance

Hamming codes:

$$2^r - 1,\ 2^r - 1 - 2r,\ 3$$

Surface codes:

- Topological, local interactions
- Repeated small patches $\rightarrow$ scalable
- High noise threshold
- But: low encoding density, many physical qubits needed

Other Correction Algorithms

Requirements for practical quantum processors:

- High noise threshold
- Universal logical gate set
- Low qubit overhead per logical qubit
- Fast error correction cycles

Low-Density Parity-Check (LDPC) codes:

- Sparse parity checks
- Lower resource overhead
- Growing research interest

Conclusion

- QEC is essential for scalable quantum computation
- Early codes (3-qubit, Shor) illustrate basic principles
- Stabilizer formalism unifies many codes
- Surface codes dominate current hardware implementations
- LDPC codes may significantly reduce qubit overhead in the future

Questions?

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