Engineering Sciences (SPI) Doctoral School

Analytic stochastic processes for signal processing

INTRODUCTION TO THE CONCEPT OF GRAVITATIONAL ALLOCATIONS

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1. Point process
2. Gravitational allocation

- Homogeneous Poisson point process PPP(λ)

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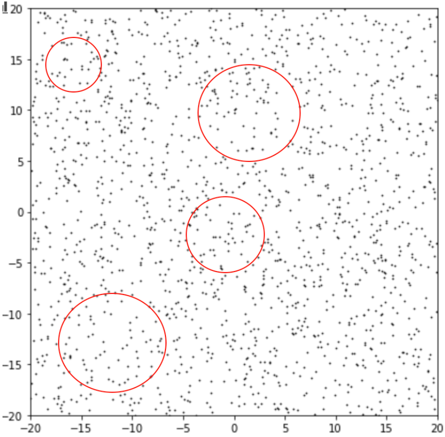

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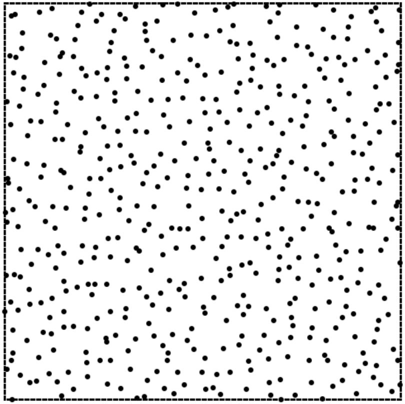

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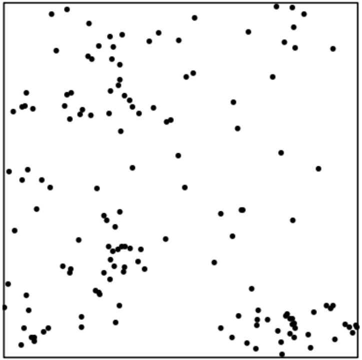

Transportation, Matching, Allocation:

Given two measures ν and μ on Λ

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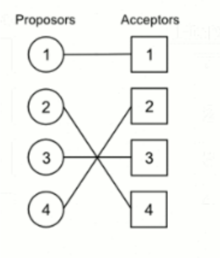

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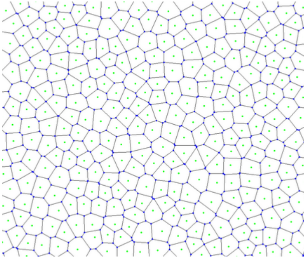

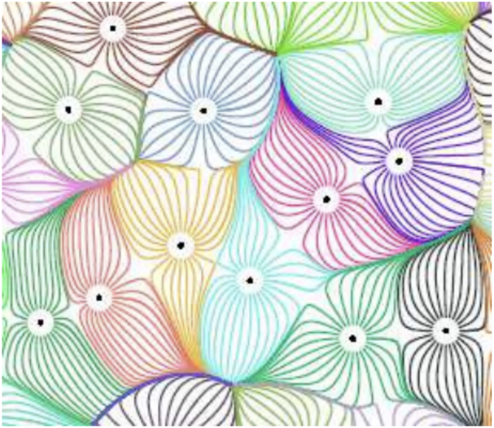

Gravitational allocation from Lebesgue to the Poisson point process:

- Let $\mathcal{Z}$ be a standard Poisson point process on $\mathbb{R}^d$
- Consider the vector function

$$F(x) := \sum_{z \in \mathcal{Z}, |z-x| \uparrow} \frac{z-x}{|z-x|^d},$$

to be the force of attraction acting on each point x of $\mathbb{R}^d \setminus \mathcal{Z}$.

- For any $x \in \mathbb{R}^d \setminus \mathcal{Z}$, consider the integral curve $Y_x(t)$, to be unique solution of

$$\frac{dY_x(t)}{dt} = F(Y_x(t)), \; Y_x(0) = x.$$

define for some maximal time $\tau_x \in]0 \,, \, \infty]$.
We call these curves the gravitational flow curves.

Proposition:
Assume $d \geq 3$. Almost surely, the above series defining the force function

$$F(x) := \sum_{z \in \mathcal{Z}, |z-x| \uparrow} \frac{z-x}{|z-x|^d},$$

converges simultaneously for all x for which it is defined, and defines a translation-invariant (in distribution) vector valued random function. The random function F is almost surely continuously differentiable where it is defined.

Alternative formulation of F:

$$F(x) := \sum_{z\in\mathcal{Z},|z-x|\uparrow} \frac{z-x}{|z-x|^d},$$

$$\sum_{z\in\mathcal{Z},\,|z-u|\uparrow} \frac{z-x}{|z-x|^d} - \sum_{z\in\mathcal{Z},\,|z-v|\uparrow} \frac{z-x}{|z-x|^d} = \kappa_d(u-v). \tag{1}$$

This equation lead to a new formulation of the force function F as follow:

$$F(x) := \sum_{z \in \mathcal{Z}, |z-x| \uparrow} \frac{z-x}{|z-x|^d},$$

$$F(x) := \sum_{z \in \mathcal{Z}, |z| \uparrow} \frac{z-x}{|z-x|^d} + \kappa_d x.$$

The force function was taken to be:

$$F(x) = \sum_{z \in \mathcal{Z}, |z| \uparrow} \frac{z - x}{|z - x|^d} + \kappa_d x,$$

while its the divergence is:

$$\mathrm{div}(F) = -d\kappa_d \sum_{z \in \mathcal{Z}} \delta_z + d\kappa_d,$$

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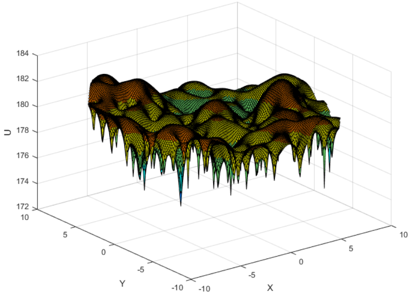

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- The gradient curves $Y_x(t)$ are the unique solution of the differential equation

$$\frac{dY_x(t)}{dt} = F(Y_x(t)) = -\nabla u(Y_x(t)), \qquad Y_x(0) = x.$$

THANK YOU.