

# ESTIMATION AND CI FOR VALUE AT RISK & EXPECTED SHORTFALL

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# VALUE AT RISK ESTIMATION

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# VaR Estimation

- Main Steps:
  1. Identify the variable of interest (asset value, portfolio value, credit losses, insurance claims, etc.)
  2. Identify the key risk factors that impact the variable of interest (assets prices, interest rates, duration, volatility, default probabilities, etc.)
  3. Perform deviations in the risk factors to calculate the impact in the variable of interest

# VaR Estimation

- 3 Main Approaches
  1. Delta-Normal or Variance-Covariance Approach
  2. Historical Simulation (variety of approaches)
  3. Monte Carlo Simulation

# DELTA-NORMAL APPROACH

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# Delta – Normal

- Suppose that the value,  $V$ , of an asset is a function of a Normally distributed risk factor,  $RF$ .
- What if the relationship between the two is linear?

$$V = \beta_0 + \beta_1 RF$$

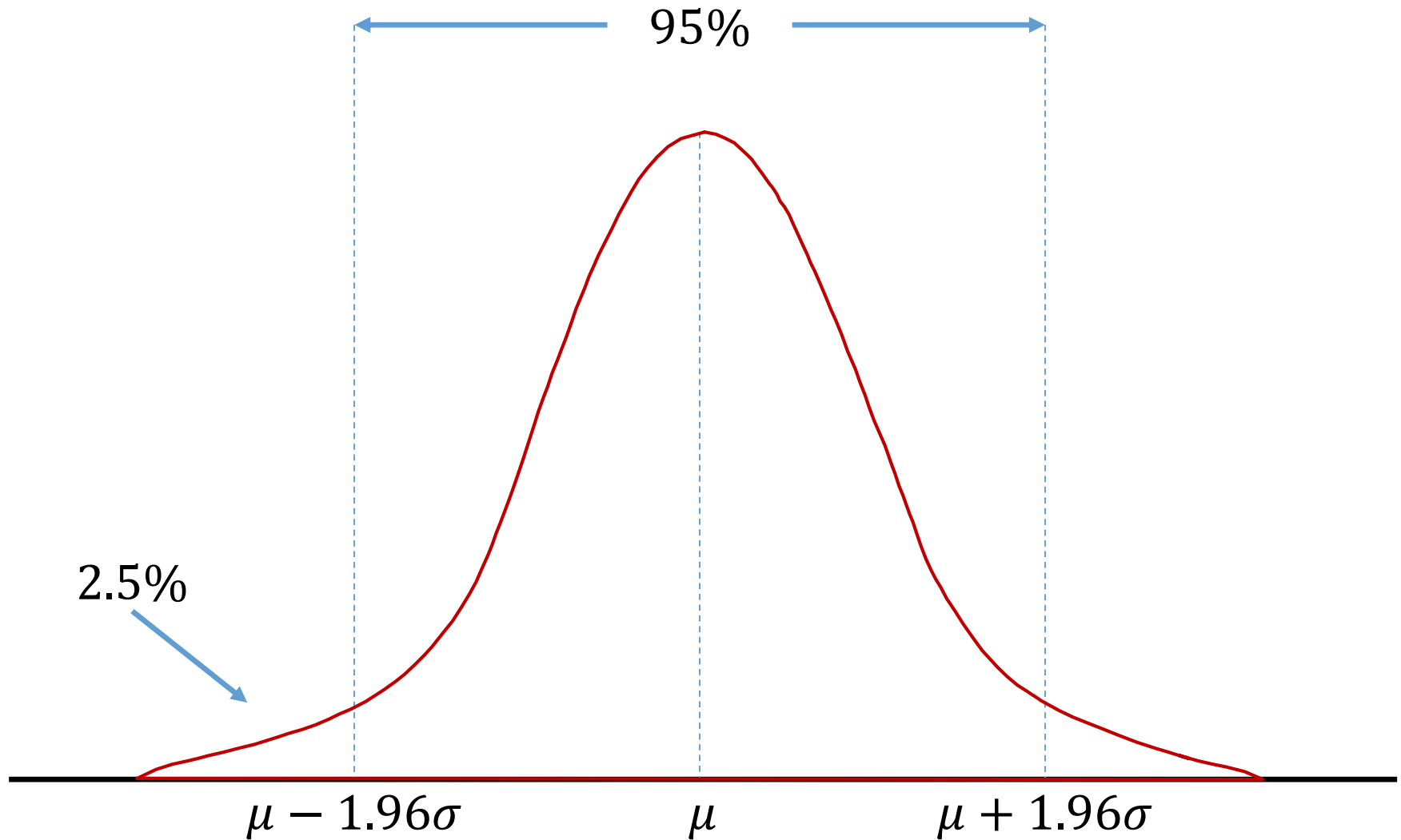
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- Easy to calculate VaR!
- If  $RF$  is Normally distributed, then  $V$  would be as well.
- What is the 2.5% VaR on any Normal distribution?

# Normal Distribution





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$$VaR_{2.5\%} = 1.96 * \sigma$$

- Just need to estimate  $\sigma$ !

# Delta – Normal

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- What if the relationship between the two is **non-linear**?

$$V = \beta_0 + \beta_1 RF^2$$

- How can we calculate the Value at Risk by taking advantage of the Normality assumption?

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$$V = \beta_0 + \beta_1 RF^2$$

- How can we calculate the Value at Risk by taking advantage of the Normality assumption?
- Finding the extreme of a Normally distributed value and squaring that **DOES NOT EQUAL** the extreme value for the squared risk factor.

# Delta – Normal

- First, value the instrument (asset) at the initial value of the risk factor,  $V(RF_0)$ .
- Next, calculate  $\Delta$ , the first derivative of  $V$  with respect to  $RF$ , evaluated at the initial value  $RF_0$ .
- Small changes of the risk factor result in small changes of the value – **approximate using the slope!**
- **This is a linear relationship!**
- Hence the name Delta – Normal; have the Normal assumption as well as taking small changes ( $\Delta$ ) as an approximation.

# Delta – Normal

- How to calculate the first derivative?
- If you know the formula relating the  $RF$  with the  $V$  then it is easy.
- The change in any value is a function of all of the derivatives of that function:

$$dV = \frac{\partial V}{\partial RF} \cdot dRF + \frac{1}{2} \cdot \frac{\partial^2 V}{\partial RF^2} \cdot dRF^2 + \dots$$

# Delta – Normal

- Taylor-Series expansion:
- The change in any value is a function of all of the derivatives of that function:

$$dV = \frac{\partial V}{\partial RF} \cdot dRF + \frac{1}{2} \cdot \frac{\partial^2 V}{\partial RF^2} \cdot dRF^2 + \dots$$

- Delta – Normal approach assumes that only the first derivative is actually important:

$$dV = \left. \frac{\partial V}{\partial RF} \right|_{RF_0} \cdot dRF \quad \Rightarrow \quad dV = \Delta_0 \cdot dRF$$

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- Change in value of the portfolio is a constant ( $\Delta_0$ ) times the change in the risk factor.

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- This is a **linear function!**
- What is the distribution of the change in  $RF$ ?

$$dRF = RF_t - RF_{t-1}$$



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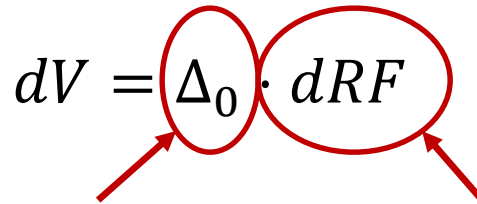
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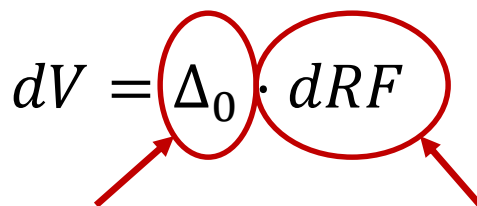
Constant

Normal distribution

- Therefore, the change in  $V$ ,  $dV$ , also follows a Normal distribution.

# Delta – Normal

- Difference of Normal distributions = Normal distribution!

$$dV = \Delta_0 \cdot dRF$$


Constant

Normal distribution

- Therefore, the change in  $V$ ,  $dV$ , also follows a Normal distribution.
- The worst loss for  $V$  is attained for an extreme value of  $RF$ .
- $RF$  is Normally distributed, so use the standard deviation of  $RF$  and an  $\alpha$  level to calculate the VaR of the instrument.

# Example of Delta – Normal

- Suppose that the variable of interest is a portfolio consisting of  $N$  units in a certain stock,  $S$ .
- The price of the stock at time  $t$  is denoted by  $P_t$ .
- Value of the portfolio:  $N \times P_t$
- Change in the portfolio value =  $N \times \Delta P_t$
- Assume that the price of the stock is a random walk:

$$P_t = P_{t-1} + \varepsilon_t$$

$$\varepsilon_t \sim N(0, \sigma)$$

- What is  $\Delta_0$  and  $dRF$ ?

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# Example of Delta – Normal

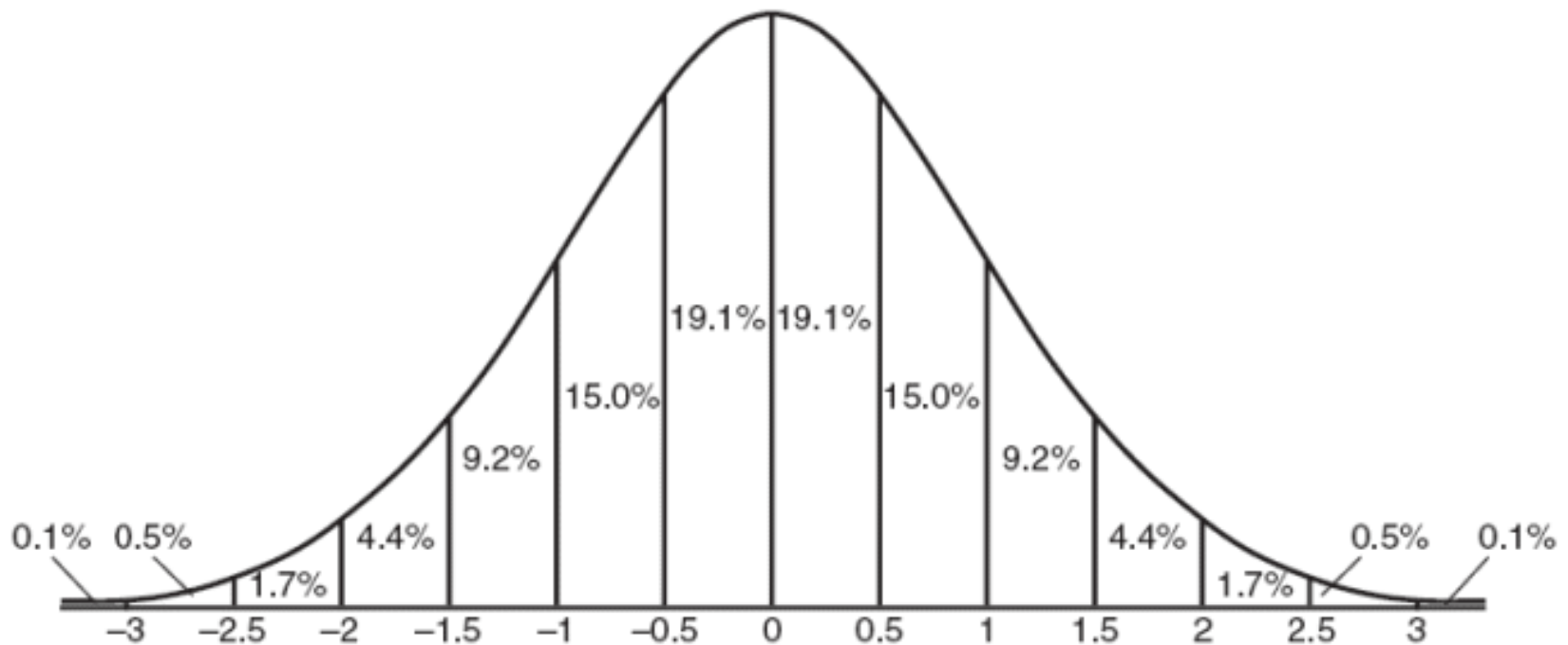
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- Assume that the price of the stock is a random walk:

$$P_t = P_{t-1} + \varepsilon_t \quad \varepsilon_t = P_t - P_{t-1}$$
$$\varepsilon_t \sim N(0, \sigma)$$

- What is  $\Delta_0$  and  $dRF$ ?

# Variance – Covariance

- The Delta – Normal approach is sometimes called the **variance-covariance approach** because you rely on the known variance relationships on the Normal distribution.



# Variance – Covariance

- Popular percentiles for Value at Risk calculations (left tail):
  - 0.1% (99.9% confidence level) –  $VaR = -3.09\sigma$
  - 0.5% (99.5% confidence level) –  $VaR = -2.58\sigma$
  - 1.0% (99% confidence level) –  $VaR = -2.33\sigma$
  - 5.0% (95% confidence level) –  $VaR = -1.64\sigma$



# Variance – Covariance

- The variance piece of “variance-covariance” is rather apparent from our last example, but what about the covariance piece?
- If all you have is a single portfolio or asset (or an independent one) then all you need is the variance.
- However, if you have multiple portfolio's with a dependence structure then you need the covariance as well.

# V – C: Single Position Portfolio

- Suppose you invested \$100,000 in Apple today (bought Apple stock).
- Daily standard deviation of Apple return = 1.28%.
- Daily mean of Apple return = 0%.
- Data gathered from 10/24/2016 – 10/17/2018.
- Assume the Normal distribution on Apple returns (like the previous example).
- What is the daily VaR of your position at 99% confidence?

# V – C: Single Position Portfolio

- What is the daily VaR of your position at 99% confidence?
- The percentile of the returns is -2.33 standard deviations below the mean of 0.

$$VaR = \$100,000 \times (-2.33) \times 0.0128 = -\$2,981.68$$

## V – C: Single Position Portfolio

- What is the daily VaR of your position at 99% confidence?
- The percentile of the returns is -2.33 standard deviations below the mean of 0.

$$VaR = \$100,000 \times (-2.33) \times 0.0128 = -\$2,981.68$$

- With 99% confidence, you expect not to lose more than \$2,981.68 by holding Apple stock for one day.

**OR**

- There is a 1% chance of losing at least \$2,981.68 by holding Apple stock for one day.

# V – C: Two Position Portfolio

- Suppose you invest \$300,000 as follows:
  - \$200,000 in MSFT and \$100,000 in Apple
- Daily Returns:
  - $\bar{x}_{MSFT} = 0\%, \sigma_{MSFT} = 1.23\%$
  - $\bar{x}_{APPLE} = 0\%, \sigma_{APPLE} = 1.28\%$
  - Correlation of returns = 0.577
  - Assume Normal distribution for both.
- What is the 99% VaR of the portfolio?

## V – C: Two Position Portfolio

- What is the 99% VaR of the portfolio?
- Need to find the variance of the portfolio's return!
- Portfolio's return =  $2/3(\text{MSFT Return}) + 1/3(\text{Apple Return})$
- Portfolio's variance:

$$\sigma_P^2 = \left(\frac{2}{3}\right)^2 \sigma_M^2 + \left(\frac{1}{3}\right)^2 \sigma_A^2 + 2 * \left(\frac{2}{3}\right) \left(\frac{1}{3}\right) \text{Cov}(\text{Apple}, \text{MSFT})$$

$$\sigma_P^2 = \left(\frac{2}{3}\right)^2 0.0123^2 + \left(\frac{1}{3}\right)^2 0.0128^2 + \frac{4}{9} * 0.577 * 0.0123 * 0.0128$$

$$\sigma_P^2 = 0.0001255957 \Rightarrow \sigma_P = 0.0112$$

## V – C: Two Position Portfolio

- What is the 99% VaR of the portfolio?
- Need to find the variance of the portfolio's return!
- Portfolio's return =  $2/3(\text{MSFT Return}) + 1/3(\text{Apple Return})$
- Portfolio's variance:

$$\sigma_P^2 = 0.0001255957 \Rightarrow \sigma_P = 0.0112$$

- The 99% VaR is given by:

$$VaR = \$300,000 \times (-2.33) \times 0.0111 = -\$7,833.66$$

# Using Normality

- Under the assumption of Normality, we can get the following relationship between 1-day and n-day VaR:

$$VaR_N = \sqrt{N} \times VaR_1$$

- The general relation between a and b periods VaR is:

$$VaR_a = \sqrt{\frac{a}{b}} \times VaR_b$$





# HISTORICAL SIMULATION

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# Historical Simulation Idea

- Non-parametric methodology (distribution free).
- Based solely on historical data.
- If history suggests that only 1% of the time Apple's daily returns were below -4%, what do you think the VaR at a 99% confidence level should be?

# Historical: Single Position Portfolio

- \$100,000 invested in Apple today.
- You have 500 observations on Apple's daily returns. You want to compute the daily VaR of your portfolio at the 99% confidence level.
- The 99% VaR will be a loss value that will not be exceeded 99% of the time **OR** the loss will be exceeded only 1% of the time.
- Find the 1% quantile of your data!

# Historical: Single Position Portfolio

- Find the 1% quantile of your data!
- Using the 500 observations on daily returns, calculate the portfolio's value ( $\$100,000 \times R_A$ ).
- Sort the 500 observations from worst to best.
- The 1% of 500 days is 5 – find a loss observation in our data set that is only **exceeded** 5 times.
- The 99% VaR will be the 6<sup>th</sup> observation of your sorted data set.

# Historical: Single Position Portfolio

- The 99% VaR will be the 6<sup>th</sup> observation of your sorted data set.

| Observation Number | Date       | Return |
|--------------------|------------|--------|
| 495                | 10/10/2018 | -4.74% |
| 322                | 2/2/2018   | -4.44% |
| 375                | 4/20/2018  | -4.18% |
| 158                | 6/9/2017   | -3.95% |
| 201                | 8/10/2017  | -3.63% |
| 142                | 5/17/2017  | -3.42% |

# Historical: Two Position Portfolio

- \$200,000 invested in MSFT & \$100,000 in Apple today.
- You have 500 observations on both returns.
- Calculate the portfolio's value using each one of the historical daily returns:

$$\$200,000 \times R_M + \$100,000 \times R_A$$

- Sort the 500 portfolio values from worst to best.
- The 99% VaR will be the 6<sup>th</sup> observation.

# Historical: Two Position Portfolio

- The 99% VaR will be the 6<sup>th</sup> observation of your sorted data set.

| Observation Number | Date       | Portfolio Value |
|--------------------|------------|-----------------|
| 495                | 10/10/2018 | -\$15,917.39    |
| 326                | 2/8/2018   | -\$13,329.83    |
| 358                | 3/27/2018  | -\$12,007.20    |
| 323                | 2/5/2018   | -\$10,941.70    |
| 322                | 2/2/2018   | -\$9,768.47     |
| 142                | 5/17/2017  | -\$9,055.16     |



# Historical Simulation Assumptions

- There are some key assumptions we are making with this approach:
  1. The past will repeat itself.
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# Historical Simulation Assumptions

- There are some key assumptions we are making with this approach:
  1. The past will repeat itself.
  2. The historical period covered is long enough to get a good representation of “tail” events.
- These have led to “alternative” historical simulation approaches...

# Stressed VaR (and ES)

- Instead of basing calculations on the movements in market variables over the last  $n$  days, we can base calculations on movements during a period in the past that would have been particularly bad for the current portfolio.
- This produces measures known as “stressed VaR” and “stressed ES.”

# Stressed VaR: Two Position Portfolio

- The 99% Stressed VaR will be the 6<sup>th</sup> observation of your sorted 500 observation data set from 3/15/2007 – 3/9/2009.

| Observation Number | Date       | Portfolio Value |
|--------------------|------------|-----------------|
| 390                | 9/29/2008  | -\$38,000.46    |
| 396                | 10/7/2008  | -\$23,561.21    |
| 434                | 12/1/2008  | -\$20,714.13    |
| 382                | 9/17/2008  | -\$20,245.49    |
| 417                | 11/5/2008  | -\$19,901.08    |
| 406                | 10/21/2008 | -\$18,639.23    |

# Weighted VaR (and ES)

- Let weights assigned to observations decline exponentially as we go back in time.
- Rank observations from worst to best.
- Starting at worst observation sum weights until the required quantile is reached.

# Weighted VaR: Two Position Portfolio

- The 99% weighted VaR will be the observation of your sorted data set when you cross the cumulative quantile.

| Observation Number | Date            | Portfolio Value     | Weight         | Cumulative Weight |
|--------------------|-----------------|---------------------|----------------|-------------------|
| 495                | 10/10/2018      | -\$15,917.39        | 0.00488        | 0.00488           |
| 326                | 2/8/2018        | -\$13,329.83        | 0.00209        | 0.00697           |
| 358                | 3/27/2018       | -\$12,007.20        | 0.00245        | 0.00942           |
| <b>323</b>         | <b>2/5/2018</b> | <b>-\$10,941.70</b> | <b>0.00206</b> | <b>0.01148</b>    |
| 322                | 2/2/2018        | -\$9,768.47         | 0.00205        | 0.01353           |
| 142                | 5/17/2017       | -\$9,055.16         | 0.00083        | 0.01436           |



# Monte Carlo Simulation

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# MC Simulation: Main Idea

- Estimate the VaR through the simulation of results of statistical / mathematical models.
- Simulate the value of the portfolio using some statistical / financial model that explains the behavior of the random variables of interest.
- If we have “enough” simulations then we have simulated the distribution of the portfolio’s value.
- Use the empirical distribution to find the VaR at any point you wish.

# MC Simulation

- Monte Carlo simulation is not easy to use, but able to handle the following details:
  - Non-normal models
  - Nonlinear models
  - Multidimensional problems
  - History changing

# MC: Single Position Portfolio

- You have a portfolio with 2500 Apple stocks.
- Current stock price is \$221.19.
- How can we use MC simulation to estimate the 1-day ahead 99% VaR of the portfolio's value?
- What is the variable of interest?

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Key is Apple's price, 1 day ahead.  
How does the price evolve from  
one day to the next?

# MC: Single Position Portfolio

- Key is Apple's price, 1 day ahead.
- How does the price evolve from one day to the next?
- Use the random walk model:

$$\ln(P_{t+1}) = \ln(P_t) + \varepsilon_{t+1}$$

- The error,  $\varepsilon_{t+1}$ , follows a  $N(0, \sigma)$ .
- Why am I using the natural log of prices?

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- The error,  $\varepsilon_{t+1}$ , follows a  $N(0, \sigma)$ .
- Why am I using the natural log of prices?

$$\ln(P_{t+1}) - \ln(P_t) = \ln\left(\frac{P_{t+1}}{P_t}\right) = R_{t+1} = \varepsilon_{t+1}$$

- Assuming returns follow a Normal distribution.



# MC: Single Position Portfolio

- Let's use 10,000 simulations.
- In each simulation:
  - Draw a value from Normal distribution to get  $R_{t+1}$ .
  - Use  $\ln(P_{t+1}) = R_{t+1} + \ln(P_t)$  to get estimate of  $\ln(P_{t+1})$ .
  - Estimate Apple's price tomorrow:  $P_{t+1} = e^{\ln(P_{t+1})}$ .
  - Get the portfolio's value:  $V = 2500 \times P_{t+1}$

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  - Get the portfolio's value:  $V = 2500 \times P_{t+1}$
- Now we have 10,000 simulated portfolio-change values.
- Create an empirical distribution of portfolio changes to look at quantiles and calculate VaR.
- 99% VaR would be 101<sup>st</sup> observation.

# MC: Two Position Portfolio

- 2500 stocks of Apple (\$221.19) and 1700 stocks of Microsoft (\$110.71).
- The model now has correlation between the two that we have to account for in the simulation.
- Assume each still have returns that follow Normal distributions.
- Repeat last example with correlation structure now added.

# MC: Two Position Portfolio

- Let's use 10,000 simulations.
- In each simulation:
  - Draw a value from bivariate Normal distribution with covariance matrix to get  $R_{t+1}$  for each Apple and Microsoft.

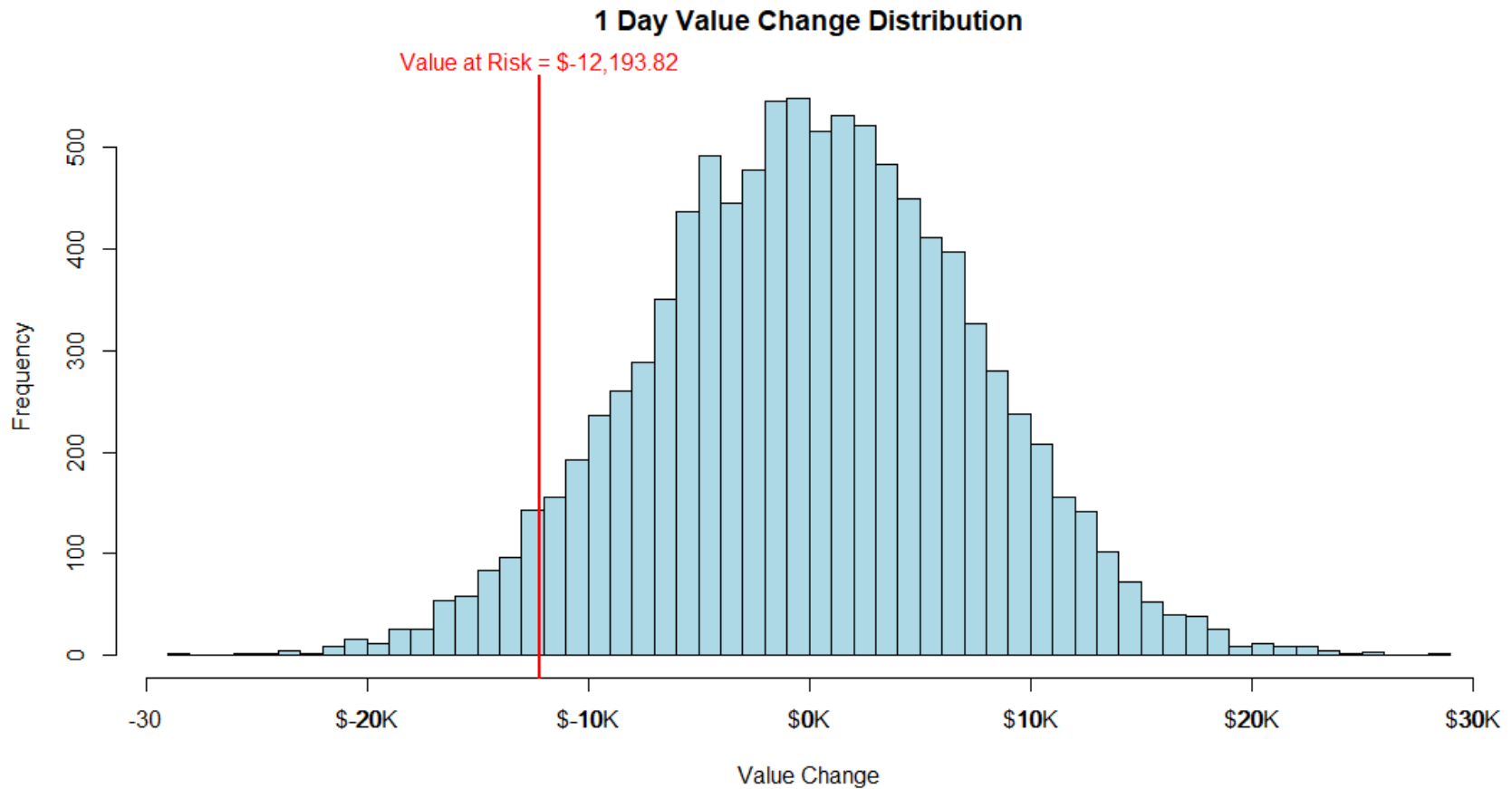
**OR**

- Draw a value from a Normal distribution for each Apple and Microsoft and add correlation structure after.
- Use  $\ln(P_{t+1}) = R_{t+1} + \ln(P_t)$  to get estimate of  $\ln(P_{t+1})$  for each.
- Estimate both prices for tomorrow:  $P_{t+1} = e^{\ln(P_{t+1})}$ .
- Portfolio's value:  $V = 2500 \times P_{A,t+1} + 1700 \times P_{M,t+1}$

# MC: Two Position Portfolio

- Now we have 10,000 simulated portfolio-change values.
- Create an empirical distribution of portfolio changes to look at quantiles and calculate VaR.
- 99% VaR would be 101<sup>st</sup> observation.

# MC: Two Position Portfolio



# Extensions

- Due to the Normality assumption in the last example we could have also used the Delta – Normal (Variance-covariance) approach.
- However, this illustrates the Monte Carlo principle and shows how easily it can be extended.
- You can use a mixture of distributions and variance structures that change over time with the Monte Carlo simulation approach.

# Monte Carlo Simulation Assumptions

- You are assuming a couple of key things with this approach:
  1. The model used is an accurate representation of the reality.
  2. The number of draws is enough to capture the tail behavior.





# Comparison of Three Approaches

|                    | Delta – Normal /<br>Variance – Cov.  | Historical<br>Simulation      | Monte Carlo<br>Simulation                  |
|--------------------|--------------------------------------|-------------------------------|--|
| <b>Attractions</b> | Intuitive                            | Intuitive and easy to explain | Extremely powerful and flexible            |
|                    | Easy formula for VaR                 | Non-parametric                | Handles non-linearity, non-normality, etc. |
|                    | Ideal for linear and Normal factors  | Easy to implement             | Ideal for complex problems                 |
| <b>Limitations</b> | Normality assumption                 | Problems obtaining data       | Hard to explain                            |
|                    | Linearity assumption                 | Complete dependence on past   | Computer-time intensive                    |
|                    | Covariance might not be well behaved | Length of estimation          | Considerable investment                    |



# CONFIDENCE INTERVAL ESTIMATION

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# Confidence Intervals for VaR

- Under the Normality assumption:

$$VaR = q_{\alpha} \times \sigma$$

$$SE(VaR) = SE(\hat{\sigma})$$

$$CI(\hat{\sigma}) = \left( \sqrt{\frac{(n-1)\hat{\sigma}^2}{\chi_{\frac{\alpha}{2}, n-1}^2}}, \sqrt{\frac{(n-1)\hat{\sigma}^2}{\chi_{1-\frac{\alpha}{2}, n-1}^2}} \right)$$

- Under Non-Normal distributions:

$$SE(VaR) = \sqrt{\frac{C(1-C)}{T \times f(q)^2}}$$

# Bootstrapping

- Steps of Bootstrapping:
  1. Resample from the simulated data using their empirical distribution; or rerun the simulation several times.
  2. In each new sample (from either approach in step 1) calculate the VaR.
  3. Repeat steps 1 and 2 many times to get several VaR estimates; use these estimates to get the expected VaR and its confidence interval.



# CONDITIONAL VALUE AT RISK (EXPECTED SHORTFALL) ESTIMATION

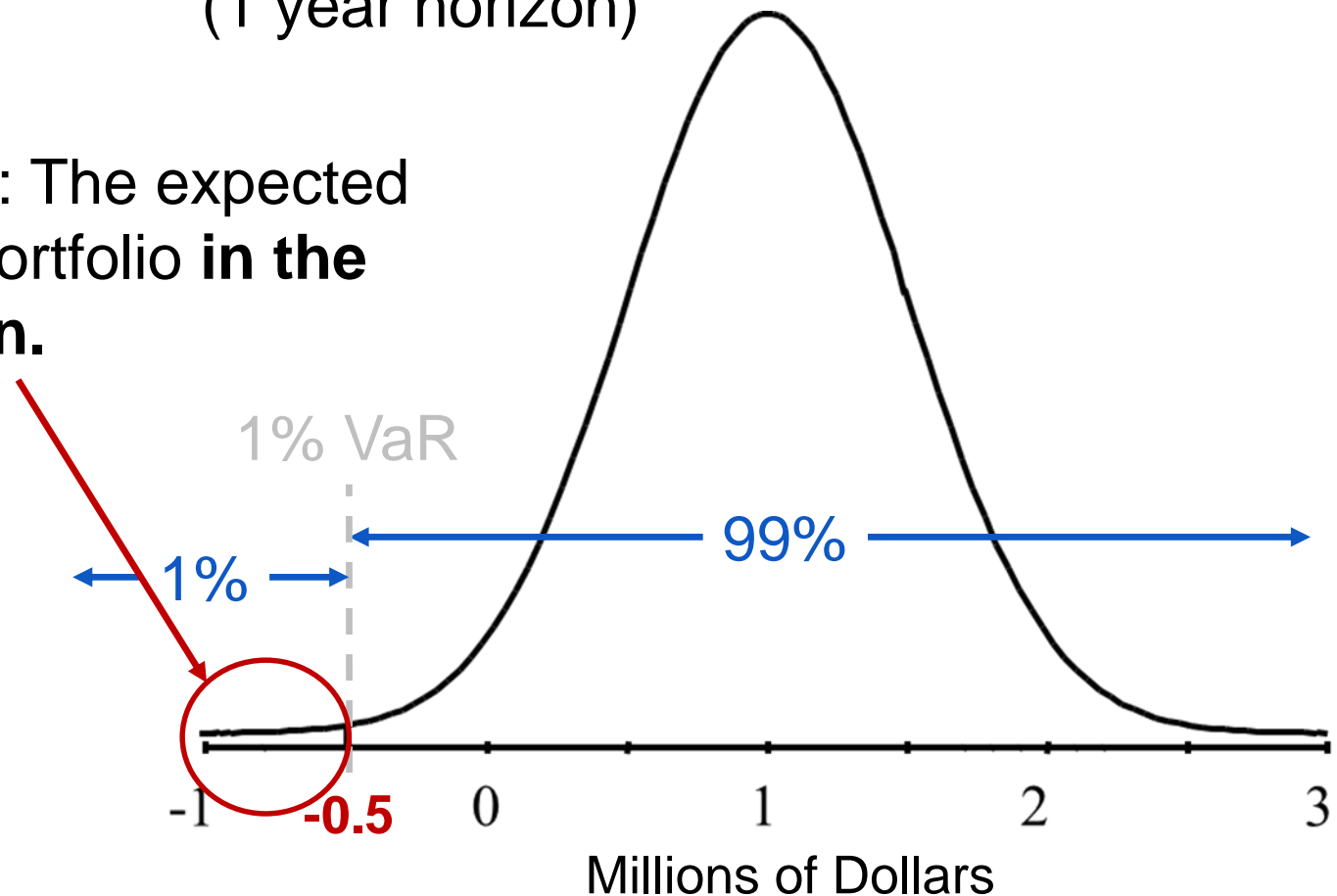
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# Visualizing CVaR (ES) – Left Tail

Distribution of change in portfolio's value  
(1 year horizon)

CVaR (or ES): The expected value of the portfolio **in the circled region**.



# CVaR (ES) Estimation

- 3 Main Approaches
  1. Delta-Normal or Variance-Covariance Approach
  2. Historical Simulation
  3. Monte Carlo Simulation

# CVaR: Variance – Covariance

- In the case of the variance-covariance approach, the CVaR can be calculated as follows:

$$CVaR = \mu - \sigma \times \frac{e\left(\frac{-q_{\alpha}^2}{2}\right)}{\alpha\sqrt{2\pi}}$$

- $\sigma$ : standard deviation
- $\alpha$ : percentile we are working on (e.g. 1%)
- $q_{\alpha}$ : tail  $100\alpha$  percentile of the standard Normal distribution (e.g. -2.33)

# CVaR: Historical Simulation

- Suppose you have 500 observations for the daily return on Apple and Microsoft.
- In order to find CVaR at the 99% confidence level, you need to do the following:
  - Sort the data from worst to best
  - Calculate the VaR (6<sup>th</sup> value in this example)
  - The CVaR is the **average** of the values that are worst than the VaR (the average of the first 5 values in our example)

# Historical: Two Position Portfolio

- The 99% CVaR will be the average of the first 5 observations of your sorted data set.

| Observation Number | Date       | Portfolio Value |
|--------------------|------------|-----------------|
| 495                | 10/10/2018 | -\$15,917.39    |
| 326                | 2/8/2018   | -\$13,329.83    |
| 358                | 3/27/2018  | -\$12,007.20    |
| 323                | 2/5/2018   | -\$10,941.70    |
| 322                | 2/2/2018   | -\$9,768.47     |
| 142                | 5/17/2017  | -\$9,055.16     |

$$\text{CVaR} = -\$12,392.92$$

# CVaR: MC Simulation

- Follow the steps described earlier to create the 10,000 simulated, sorted, portfolio values for the VaR calculation.
- Take the average of all the values that are worst than the VaR.
- Example – average of first 100 observations is 99% CVaR.

# CVaR: MC Simulation

