KEY RISK MEASURES

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INTRODUCTION TO RISK

Key Risk Characteristics

- Risk is an uncertainty that affects a system in an unknown fashion and brings great fluctuation in value and outcome.
- **Risk is the outcome of uncertainty** – fluctuations can be measured in a probabilistic sense.
- Risk has a time horizon.
- Risk measurement has to be set against a benchmark.

Statistics of Risk

- Risk analysis is using some of the “typical” statistical measures.
  - ~~Mean~~
  - Variance
  - Skewness
  - Kurtosis – used for catastrophic, extreme tail events

Common Risk Measures

- There are some common measures that are used in risk analysis:
  1. Probability of Occurrence
  2. Standard Deviation / Variance / Coefficient of Variation
  3. Semi-standard Deviation
  4. Volatility
  5. Value at Risk (VaR)
  6. Expected Shortfall (ES)

Common Risk Measures

- **Probability of Occurrence**
  - Examples – Probability of failure of a project, probability of default, migration probabilities, transition matrices.

- **Standard Deviation, Variance, Coefficient of Variation**
  - Two-sided measures
  - Sufficient only under normality or maybe symmetry

Common Risk Measures

- **Semi-standard Deviation (Downside Risk)**
  - Measure of dispersion for the values falling below the mean.

$$\sigma_{semi} = \sqrt{\frac{1}{T}\sum_{t=1}^{T} \min(X_t - \bar{X}, 0)^2}$$

- **Volatility**
  - Standard deviation of an asset's logarithmic returns

$$\sigma_{volatility} = \sqrt{\frac{1}{T}\sum_{t=1}^{T} \ln\left(\frac{X_t}{X_{t-1}}\right)^2}$$

Common Risk Measures

- **Value at Risk – VaR**
  - The amount of capital reserves at risk given a particular holding period at a particular probability of loss
  - Example – 1 year 99.9% VaR

- **Expected Shortfall – ES**
  - The expected capital reserve given a particular holding period in the worst q% of the cases

VALUE AT RISK

History of VaR

- Developed in the early 1990's by JP Morgan
- The *"4:15pm" report*
- JP Morgan launched *RiskMetrics* ® (1994)
- VaR has been widely used since that time
- Currently, researchers are looking into more advanced "VaR-like" measures.

Definition

- The VaR calculation is aimed at making a statement of the following form:
  - We are 99% certain that we will not lose more than \$10,000 in the next 3 days.
  - \$10,000 is the 3-day 99% VaR

- VaR is the maximum amount at risk to be lost…

…**over a period of time**…

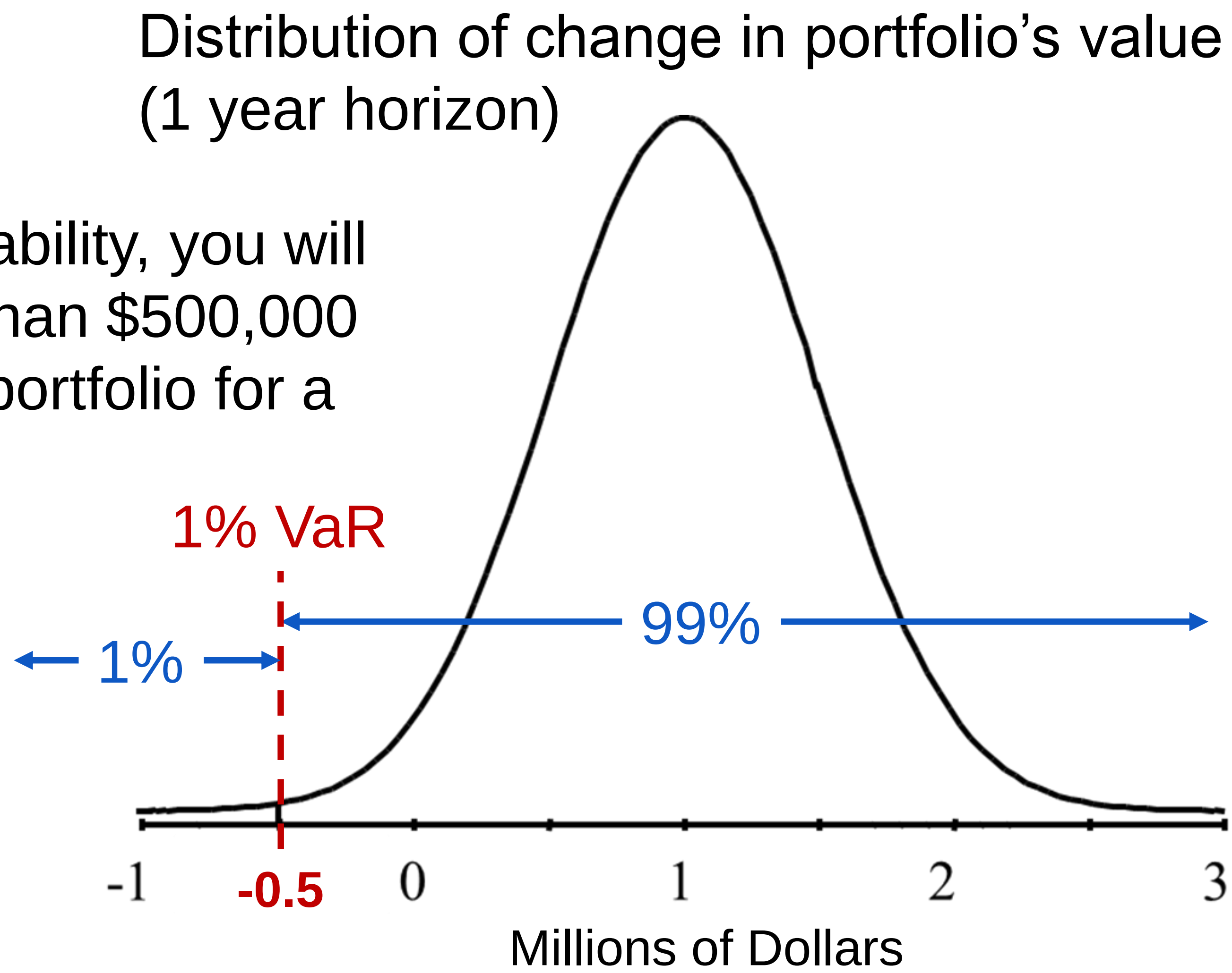
…**at a particular level of confidence**.

Focus on the Tail

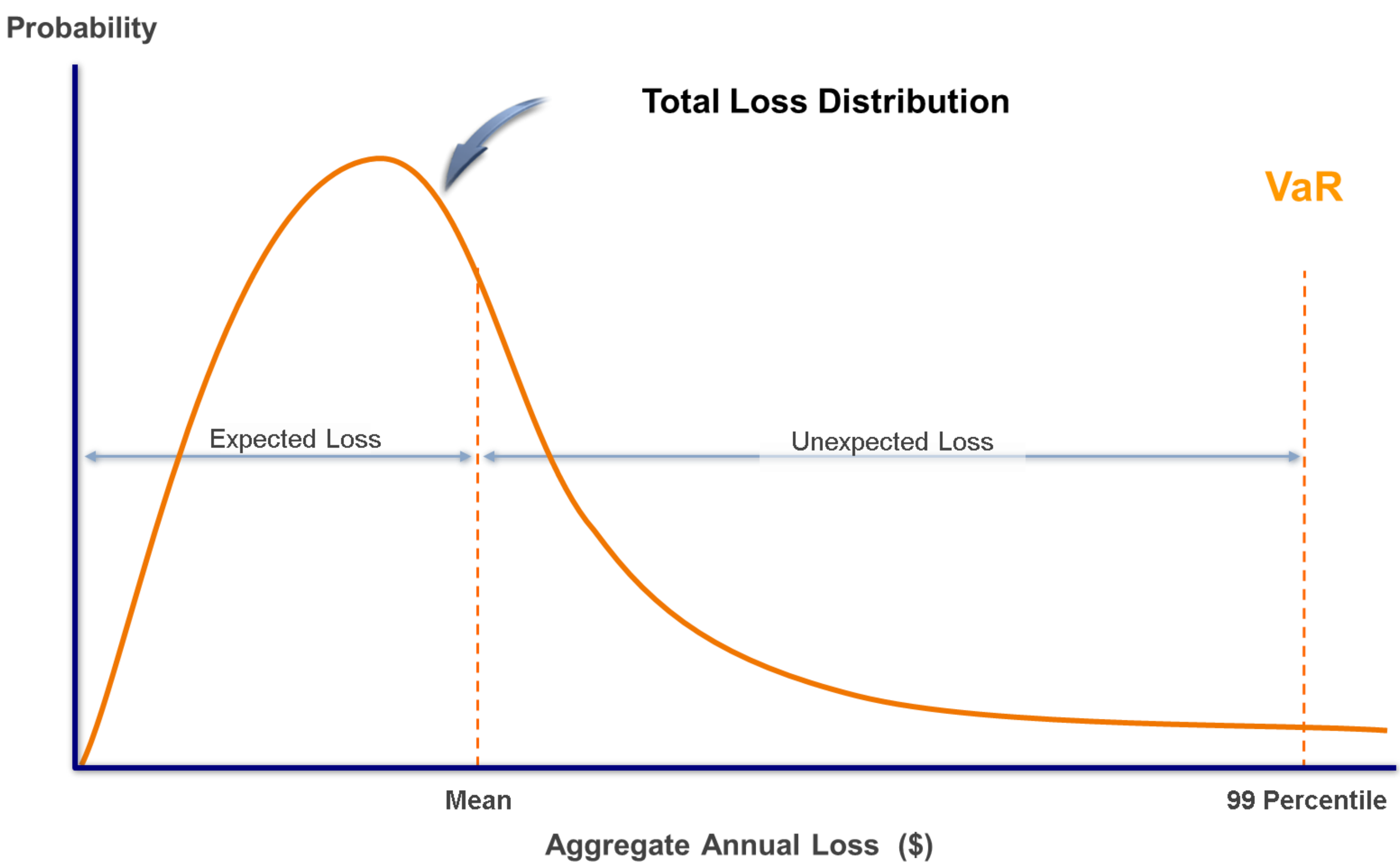
- The Value at Risk is associated with a percentile (quantile) of a distribution.
- Focused on the tail of the distribution.

Visualizing VaR – Left Tail

With 99% probability, you will not lose more than $500,000 by holding the portfolio for a year



Visualizing VaR – Right Tail

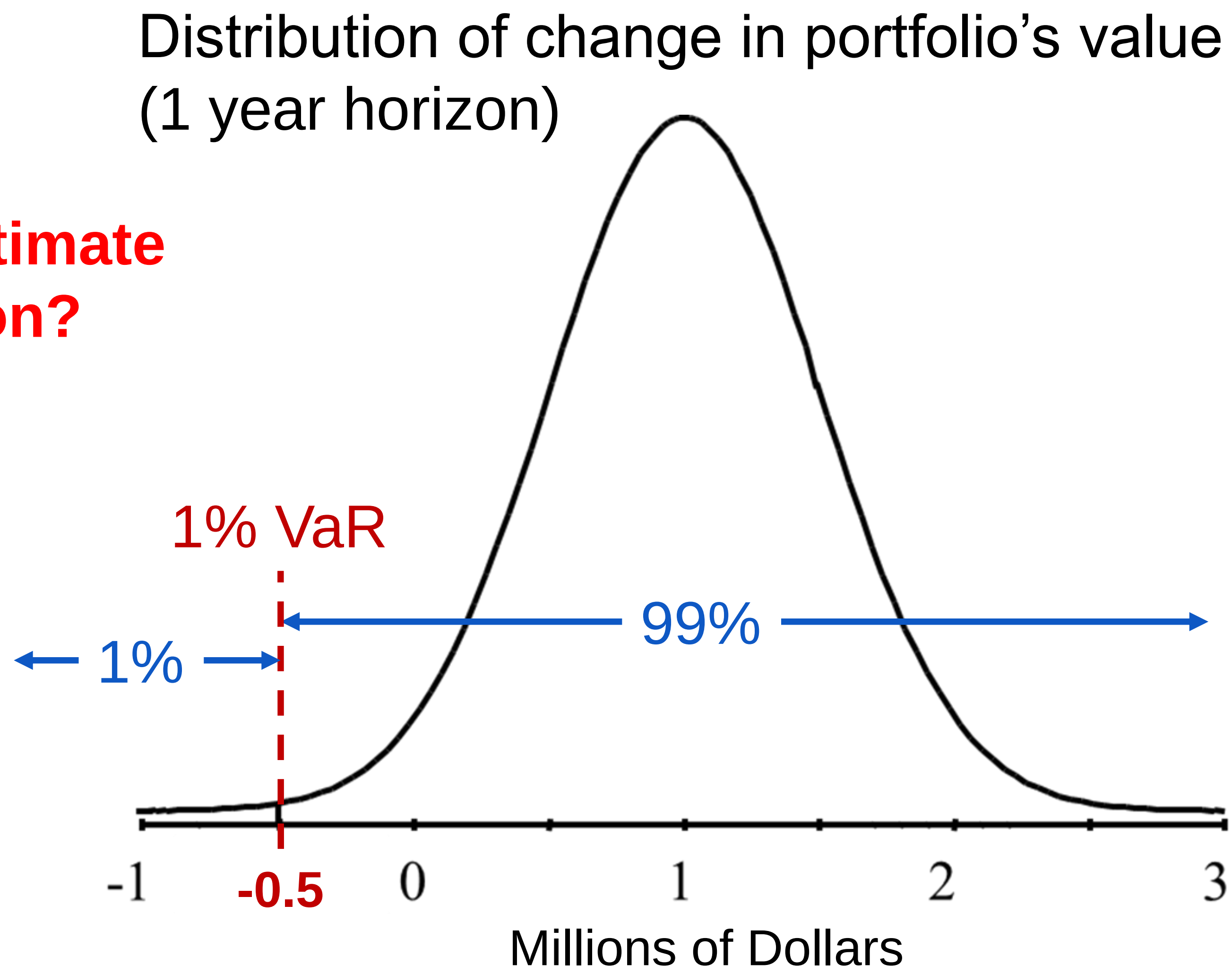


VaR Estimation

- Main Steps:
  1. Identify the variable of interest (asset value, portfolio value, credit losses, insurance claims, etc.)
  2. Identify the key risk factors that impact the variable of interest (assets prices, interest rates, duration, volatility, default probabilities, etc.)
  3. Perform deviations in the risk factors to calculate the impact in the variable of interest

Visualizing VaR – Left Tail

How do we estimate this distribution?



VaR Estimation

- How do we estimate this distribution?

- 3 Main Approaches
  1. Delta-Normal or Variance-Covariance Approach
  2. Historical Simulation (variety of approaches)
  3. Monte Carlo Simulation

EXPECTED SHORTFALL

Drawbacks of VaR – Magnitude

- VaR ignores the distribution of a portfolio's return beyond its VaR.
- Example:
  - The 99.9% VaR for an investment in stock A is $100K. The 99.9% VaR for an investment in stock B is $100K.
  - Are you indifferent between the two?

Drawbacks of VaR – Magnitude

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- Example:
  - The 99.9% VaR for an investment in stock A is $100K. The 99.9% VaR for an investment in stock B is $100K.
  - Are you indifferent between the two?

- Stock A: The loss can be up to $250K
- Stock B: The loss can be up to $950K
- **VaR ignores the magnitude of the worst returns.**

Drawbacks of VaR - Diversification

- Under non-normality, VaR may not capture diversification.
- VaR fails to satisfy the **subadditivity property**.

$$Risk(A + B) \leq Risk(A) + Risk(B)$$

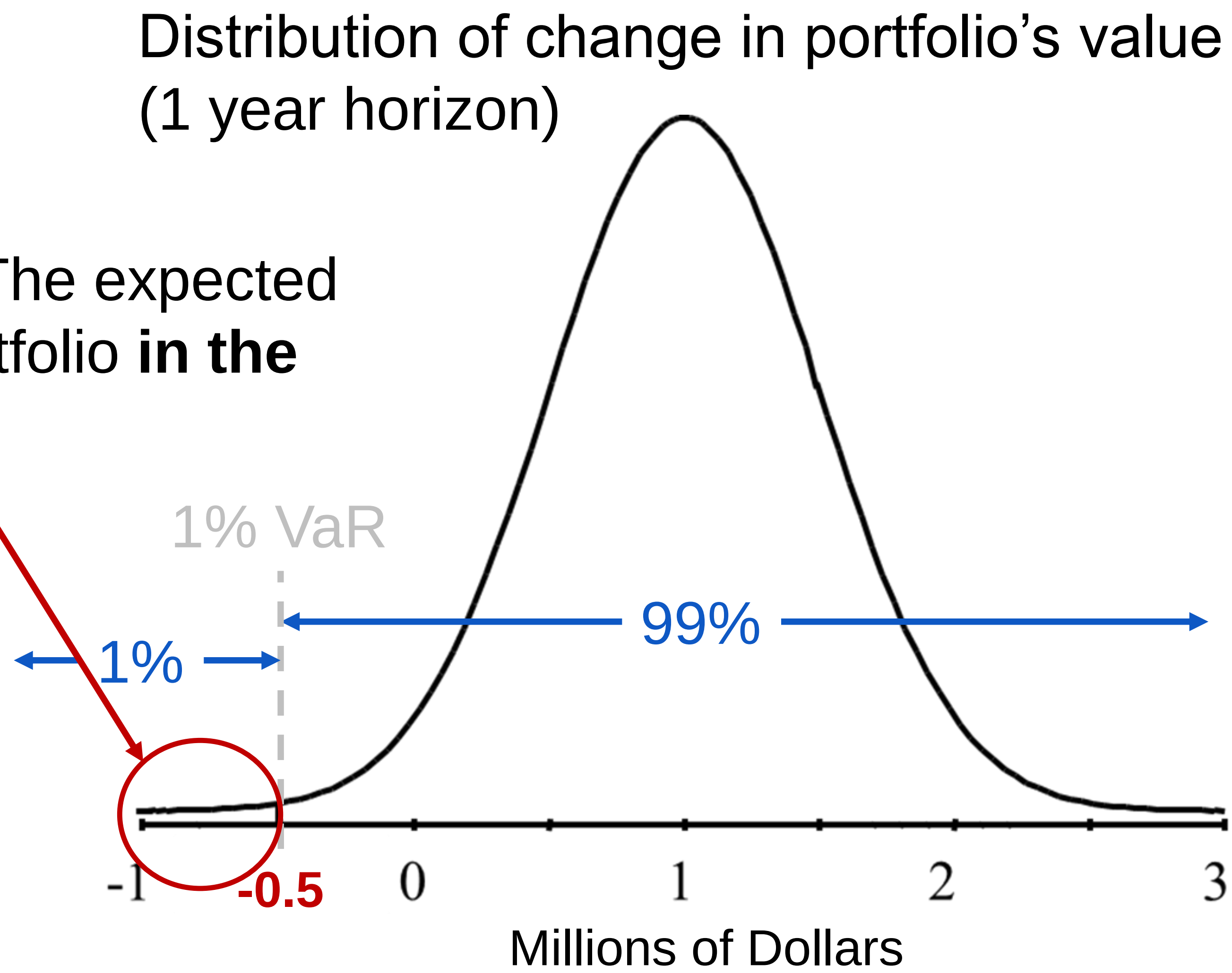
- The VaR of a portfolio with two securities may be larger than the sum of the VaR's of the securities in the portfolio.

VaR Alternative – CVaR

- The **Conditional Value at Risk** (**CVaR**) or **Expected Shortfall** (**ES**) is a measure that doesn't have the two drawbacks of the VaR.
- Given a confidence level and a time horizon, a portfolio's CVaR is the **expected loss** one suffers given that a "bad" event occurs.
- The CVaR is a conditional expectation.
- If my loss exceeds the VaR level, what should I expect it to be equal to?

Visualizing CVaR (ES) – Left Tail

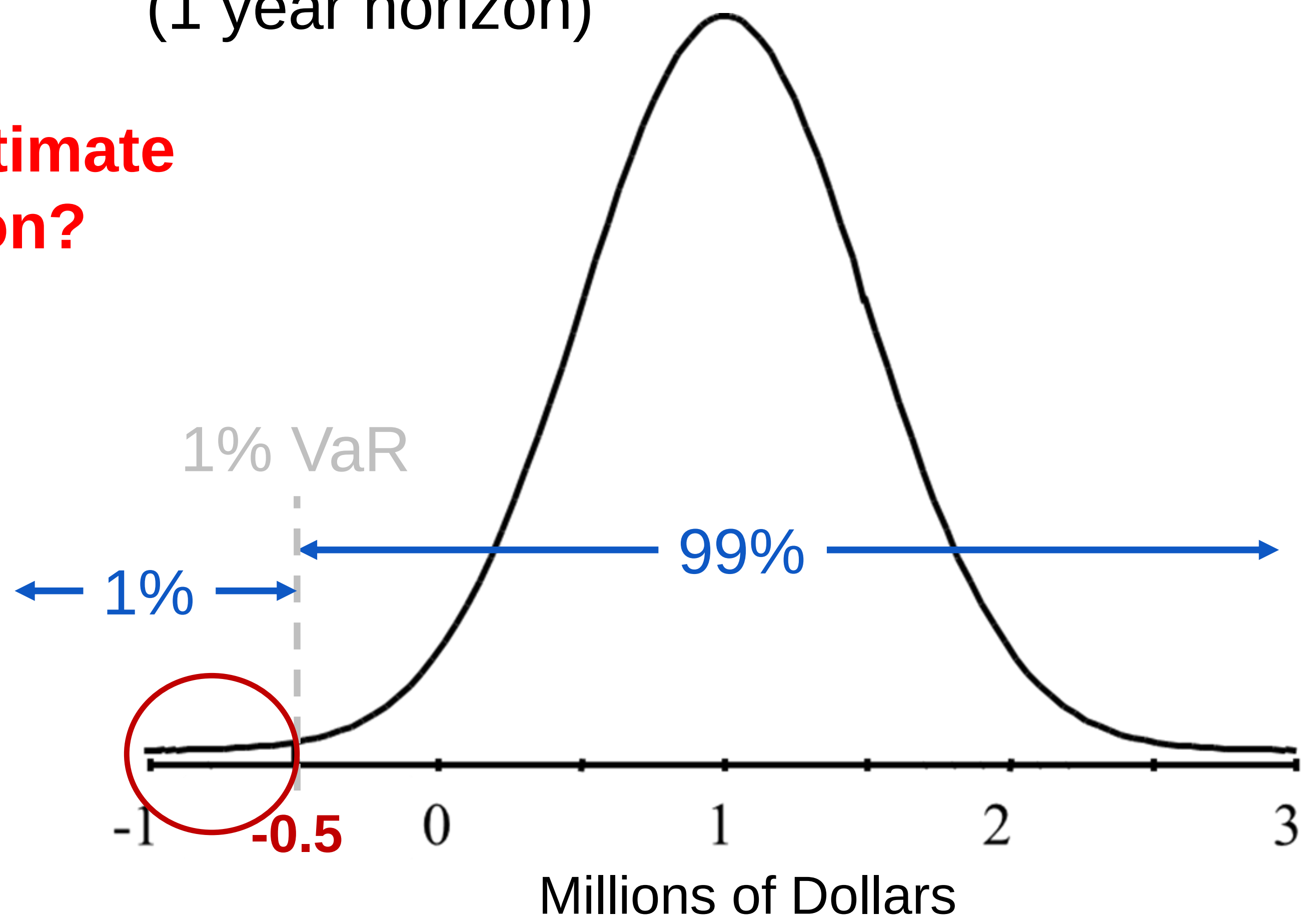
CVaR (or ES): The expected value of the portfolio **in the circled region.**



Visualizing CVaR (ES) – Left Tail

Distribution of change in portfolio's value (1 year horizon)

How do we estimate this distribution?



CVaR (ES) Estimation

- How do we estimate this distribution?

- 3 Main Approaches
  1. Delta-Normal or Variance-Covariance Approach
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CALCULATING RETURNS

Returns on Assets

- A lot of the calculations we will be making in this course will revolve around calculating the returns on assets.
- There are 2 main methods for calculating returns:
  1. Arithmetic Return
  2. Geometric Return

Basic Notation

- Here is the basic notation needed to calculate returns:
  - Return (r_t) – return at a period *t* (holding an asset from period *t-1* to period *t*)
  - Price (P_t) – price at a given time period *t*
  - Lag Price (P_{t-1}) – price a time period *t-1*
  - Dividend (D_t) – dividend payment at time period *t*

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  - Lag Price (P_{t-1}) – price a time period *t-1*
  - Dividend (D_t) – dividend payment at time period *t*
    - For small time periods we typically ignore dividend (set equal to 0)
    - Equivalently: P_t is the price of an asset where dividends are fully reinvested (and thus reflected in P_t itself)

Arithmetic Return

$$r_t = \frac{P_t + D_t - P_{t-1}}{P_{t-1}}$$

$$D_t = 0$$

$$r_t = \frac{P_t - P_{t-1}}{P_{t-1}}$$

- If $r_1 = 5\%$ and $r_2 = -5\%$, what is the total return of the two days?

Arithmetic Return

$$r_t = \frac{P_t + D_t - P_{t-1}}{P_{t-1}}$$

$D_t = 0$

$$r_t = \frac{P_t - P_{t-1}}{P_{t-1}}$$

- If $r_1 = 5\%$ and $r_2 = -5\%$, what is the total return of the two days? **NOT ZERO!**

Arithmetic Return

$$r_t = \frac{P_t + D_t - P_{t-1}}{P_{t-1}}$$

$$D_t = 0$$

$$r_t = \frac{P_t - P_{t-1}}{P_{t-1}}$$

- If $r_1 = 5\%$ and $r_2 = -5\%$, what is the total return of the two days?
- How do we get $r_{0,2}$ as a function of r_1 and r_2?

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- If $r_1 = 5\%$ and $r_2 = -5\%$, what is the total return of the two days?
- How do we get $r_{0,2}$ as a function of r_1 and r_2?

$$r_{0,2} = \frac{P_2 - P_0}{P_0} = \cdots = \frac{P_1}{P_0} r_2 + r_1 \neq r_2 + r_1$$

Arithmetic Return

- If $r_1 = 5\%$ and $r_2 = -5\%$, what is the total return of the two days?
- How do we get $r_{0,2}$ as a function of r_1 and r_2?

$$P_0 = 1 \qquad P_1 = 1.05$$

$$r_1 = 5\%$$

Arithmetic Return

- If $r_1 = 5\%$ and $r_2 = -5\%$, what is the total return of the two days?
- How do we get $r_{0,2}$ as a function of r_1 and r_2?

$$P_0 = 1 \qquad P_1 = 1.05 \qquad P_2 = 0.9975$$

$$r_1 = 5\% \qquad r_2 = -5\%$$

Arithmetic Return

- If $r_1 = 5\%$ and $r_2 = -5\%$, what is the total return of the two days?
- How do we get $r_{0,2}$ as a function of r_1 and r_2?

$$P_0 = 1 \qquad P_1 = 1.05 \qquad P_2 = 0.9975$$

$$r_1 = 5\% \qquad r_2 = -5\%$$

$$r_{0,2} = \frac{P_2 - P_0}{P_0} = \frac{0.9975 - 1}{1} = -0.0025 = -0.25\%$$

Geometric Return

$$R_t = \ln\left(\frac{P_t + D_t}{P_{t-1}}\right)$$

$$D_t = 0$$

$$R_t = \ln\left(\frac{P_t}{P_{t-1}}\right) = \ln(P_t) - \ln(P_{t-1})$$

- If $R_1 = 5\%$ and $R_2 = -5\%$, what is the total return of the two days?
- How do we get $R_{0,2}$ as a function of R_1 and R_2?

$$R_{0,2} = \ln\left(\frac{P_2}{P_0}\right) = \ln\left(\frac{P_2}{P_1} \times \frac{P_1}{P_0}\right) = \ln\left(\frac{P_2}{P_1}\right) + \ln\left(\frac{P_1}{P_0}\right) = R_2 + R_1$$

Geometric Return

- What if we took the same prices and measured returns geometrically instead?
- How do we get $R_{0,2}$ as a function of R_1 and R_2?

$$P_0 = 1 \qquad P_1 = 1.05$$

$$R_1 = 4.88\%$$

Geometric Return

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$$P_0 = 1 \qquad P_1 = 1.05 \qquad P_2 = 0.9975$$

$$R_1 = 4.88\% \qquad R_2 = -5.13\%$$

Geometric Return

- What if we took the same prices and measured returns geometrically instead?
- How do we get $R_{0,2}$ as a function of R_1 and R_2?

$$P_0 = 1 \qquad P_1 = 1.05 \qquad P_2 = 0.9975$$

$$R_1 = 4.88\% \qquad R_2 = -5.13\%$$

$$R_{0,2} = \ln\left(\frac{P_2}{P_0}\right) = -0.25\% = 4.88\% - 5.13\%$$

Mathematical Relation

- What is the difference between the two?

$$R_t = \ln\left(\frac{P_t}{P_{t-1}}\right) = \ln\left(\frac{P_t - P_{t-1}}{P_{t-1}} + 1\right) = \ln(1 + r_t)$$

$$= r_t - \frac{r_t^2}{2} + \frac{r_t^3}{3} - \cdots \approx r_t \text{ when } r_t \text{ small}$$

- For a typical **daily** return, the difference between R_t and r_t is very close to 0.

Mathematical Relation

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- For a typical **daily** return, the difference between R_t and r_t is very close to 0.

$$r_{0,2} = \frac{P_2 - P_0}{P_0} = \frac{0.9975 - 1}{1} = -0.0025 = \mathbf{-0.25\%}$$

$$R_{0,2} = \ln\left(\frac{P_2}{P_0}\right) = \mathbf{-0.250313\%}$$

Mathematical Relation

- What is the difference between the two?

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- For a typical **daily** return, the difference between R_t and r_t is very close to 0.

$$r_{0,2} = \frac{P_2 - P_0}{P_0} = \frac{0.9975 - 1}{1} = -0.0025 = \mathbf{-0.25\%}$$

$$R_{0,2} = \ln\left(\frac{P_2}{P_0}\right) = \mathbf{-0.250313\%}$$

VERY CLOSE!

Empirical Relation (Google Inc.)

Date	Close	Arithmetic Return	Geometric Return
10/4/2018	1168.19	-	-
10/5/2018	1157.35	-0.928%	-0.932%
10/6/2018	1148.97	-0.724%	-0.727%
10/7/2018	1138.82	-0.883%	-0.887%
10/8/2018	1081.22	-5.058%	-5.190%
10/9/2018	1079.32	-0.176%	-0.176%
10/10/2018	1110.08	2.850%	2.810%
10/11/2018	1100.92	-0.825%	-0.829%