CASE STUDY ON PORTFOLIO OPTIMIZATION

Dr. Aric LaBarr
Institute for Advanced Analytics

PORTFOLIO OPTIMIZATION

Financial Portfolio

- A portfolio is a collection of assets where the investor chooses the investment amount of each investment in the portfolio.
- Portfolio performance is typically measured by total value of the portfolio at the end of a period of time.
- To determine how much to allocate in each part of a portfolio, two things must be considered – risk and return.

Risk vs. Return

- **Return** – percentage growth in the value of an asset
- **Risk** – variability / volatility associated with the returns on the stock

- We can look at historical data to estimate both risk and return.
  - Example: overall means and variance over a certain period of time.
- We can also **forecast** this risk with ARCH/GARCH models.

- **Goal** – maximize return while minimizing risk!

Portfolio Optimization Example

- Portfolio of Technology Stocks
  - Want to invest in 5 tech stocks.
    - Apple, Google, Microsoft, Paypal, Ebay
  - High return → High volatility
  - Return:

$$Return = p_1 r_1 + p_2 r_2 + p_3 r_3 + p_4 r_4 + p_5 r_5$$

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Average return of each stock

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Proportion of wealth in each stock

$$p_1 + p_2 + p_3 + p_4 + p_5 = 1$$

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  - Risk:

$$\sum_j \sum_k p_j * \sigma_{j,k} * p_k = p_1 \sigma_{1,1} p_1 + p_1 \sigma_{1,2} p_2 + \cdots + p_5 \sigma_{5,5} p_5$$

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Covariance between each stock combination

Portfolio Optimization Example

- Have two choices on what to do:
    1. Minimize risk for a given return (typical)
    2. Maximize return for a given risk

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- Have two choices on what to do:
    1. Minimize risk for a given return (typical)
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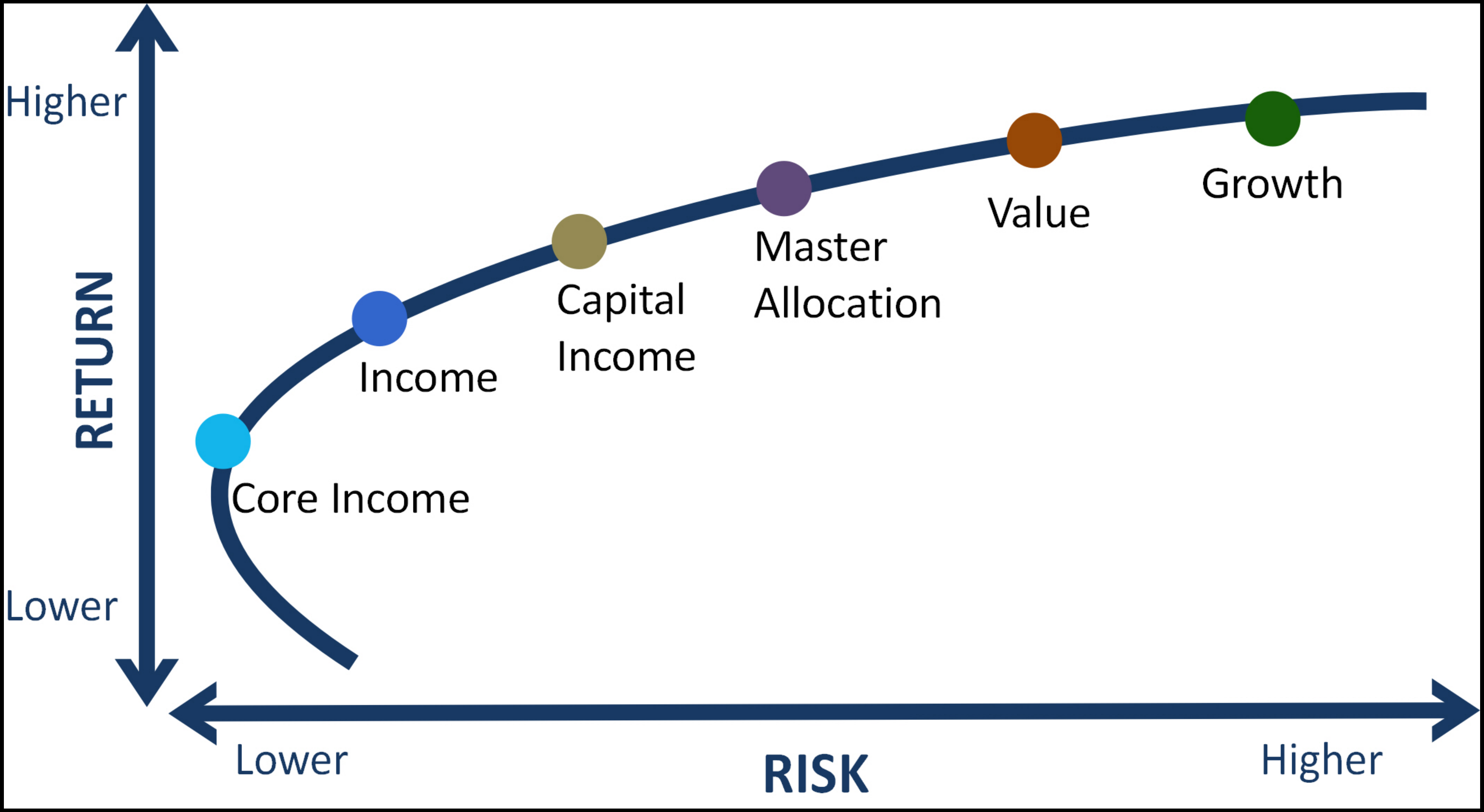
- There is a trade-off between risk and return!

Efficient Frontier



Estimating Risk

- **Risk** – variability / volatility associated with the returns on the stock

$$Risk = \sum_j \sum_k p_j * \sigma_{j,k} * p_k = p_1\sigma_{1,1}p_1 + p_1\sigma_{1,2}p_2 + \cdots + p_k\sigma_{k,k}p_k$$

Projected covariance between each stock combination

- 3 Different Estimates of Risk
  1. Historical Variance / Standard Deviation
  2. Capital Asset Pricing Model (CAPM)
  3. Generalized Autoregressive Conditional Heteroscedasticity (GARCH) Model

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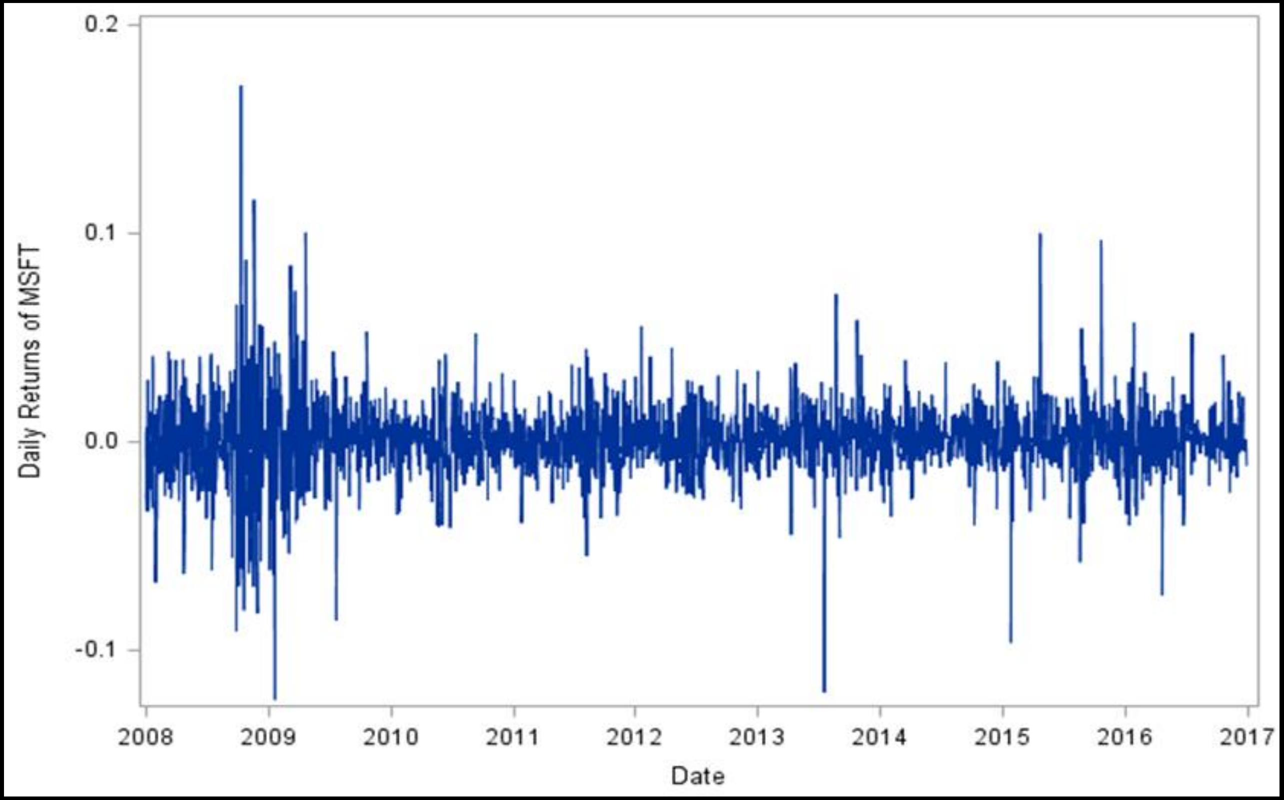
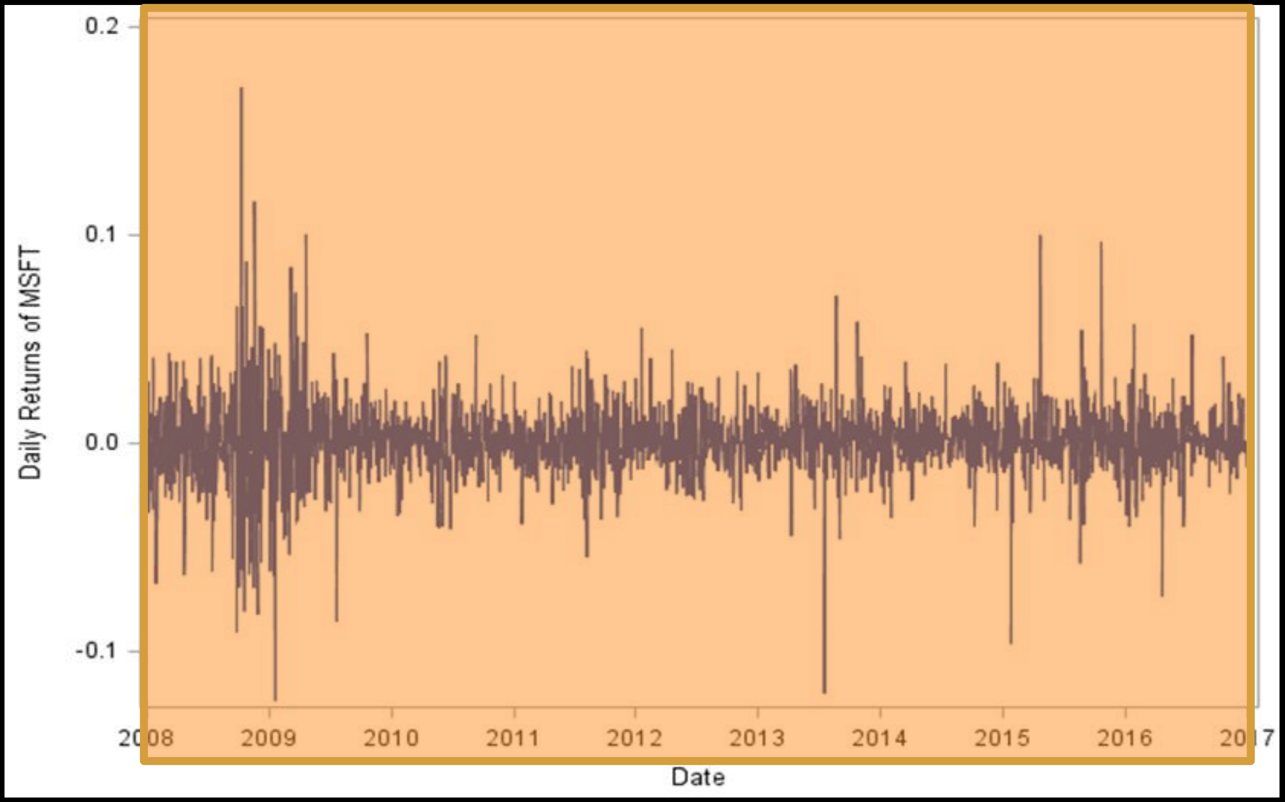
HISTORICAL VARIANCE APPROACH

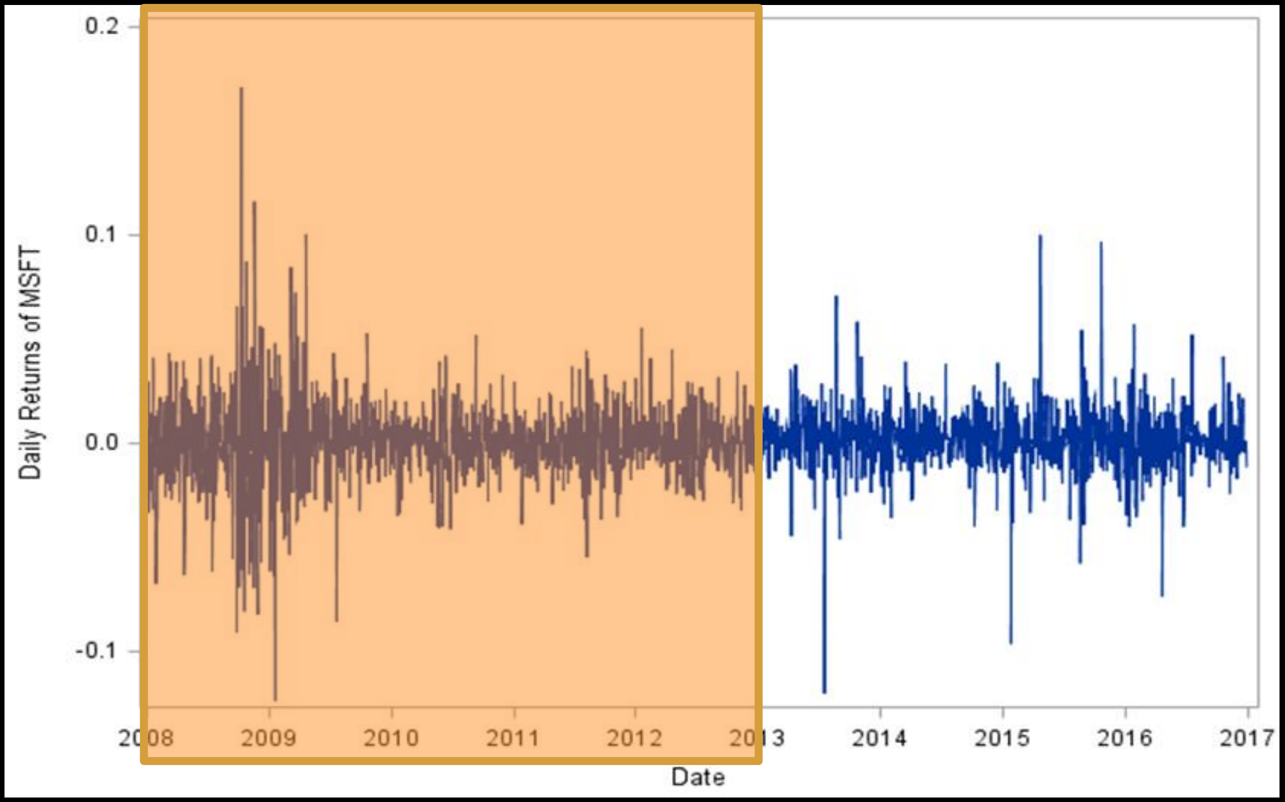
Constant Volatility?



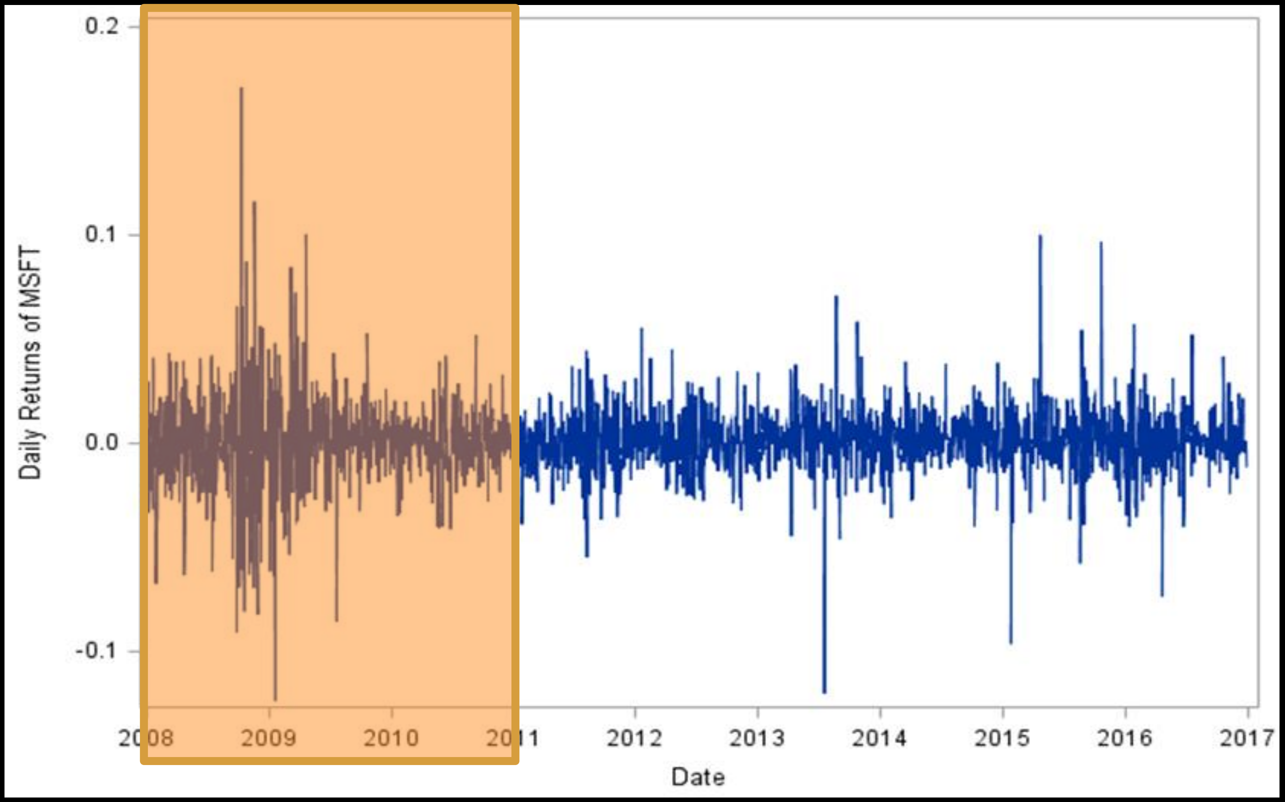
Use all Data?


0.2
0.1
0.0
-0.1
Daily Returns of MSFT
2008
2009
2010
2011
2012
2013
2014
2015
2016
2017
Date

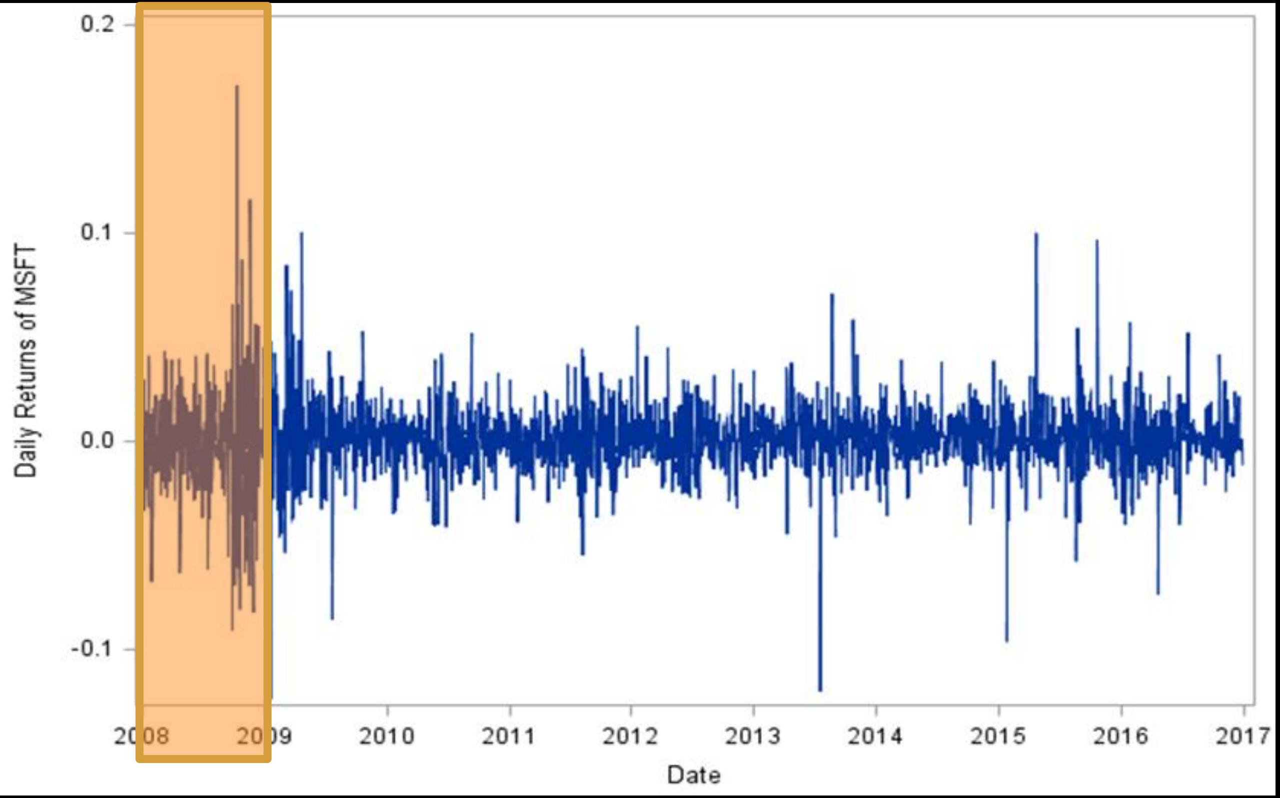
Rolling Window Calculation



Rolling Window Calculation



Rolling Window Calculation



Historical Variance Approach

- The historical variance approach uses a rolling window estimate of variance from previous *m* observations:

$$\sigma_t^2 = \frac{1}{m-1}\sum_{i=1}^{m}(r_{t-i} - \bar{r})^2$$

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Window of Time	Estimate of MSFT Volatility
2/1/2008 – 2/1/2019	0.000308
2/1/2014 – 2/1/2019	0.000215
2/1/2016 – 2/1/2019	0.000199
2/1/2018 – 2/1/2019	0.000339

Estimating Risk

- **Risk** – variability / volatility associated with the returns on the stock

$$Risk = \sum_j \sum_k p_j * \sigma_{j,k} * p_k = p_1\sigma_{1,1}p_1 + p_1\sigma_{1,2}p_2 + \cdots + p_k\sigma_{k,k}p_k$$

Projected covariance between each stock combination

Historical Variance Approach

- The historical variance approach uses a rolling window estimate of variance **as well as covariance** from previous *m* observations:

$$\sigma_t^2 = \frac{1}{m-1}\sum_{i=1}^{m}(r_{t-i} - \bar{r})^2$$

$$\sigma_{j,k} = \frac{1}{m-1}\sum_{i=1}^{m}\left(r_{j,t-i} - \bar{r}_j\right)\left(r_{k,t-i} - \bar{r}_k\right)$$

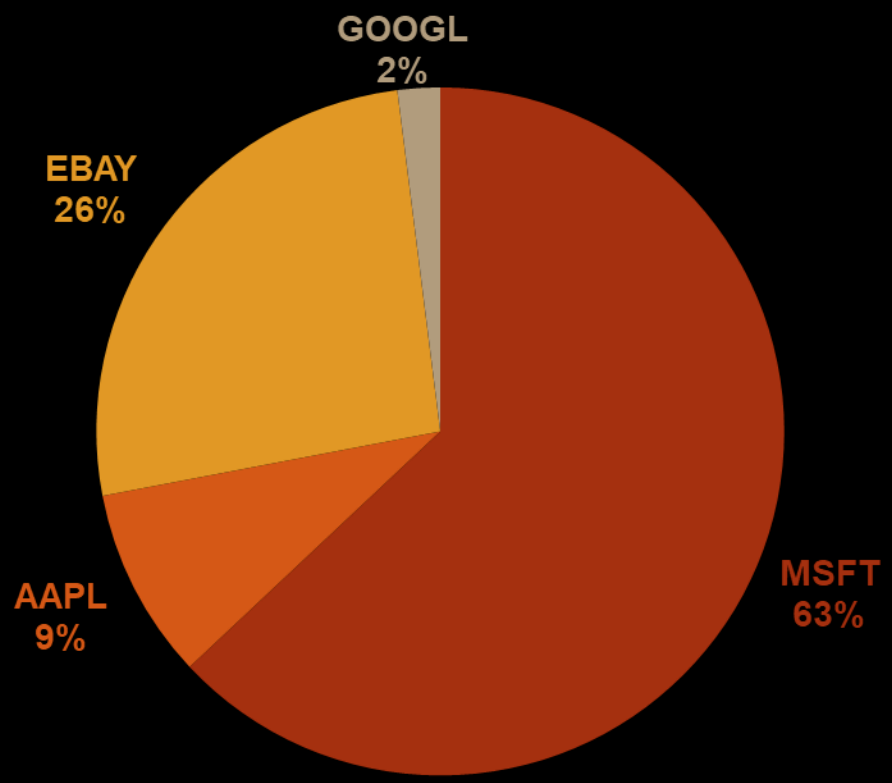
Historical Variance Approach

- The historical variance approach uses a rolling window estimate of variance **as well as covariance** from previous *m* observations:

Window of Time	Estimate of MSFT/AAPL Covariance
2/1/2008 – 2/1/2019	0.000168
2/1/2014 – 2/1/2019	0.000121
2/1/2016 – 2/1/2019	0.000124
2/1/2018 – 2/1/2019	0.000261

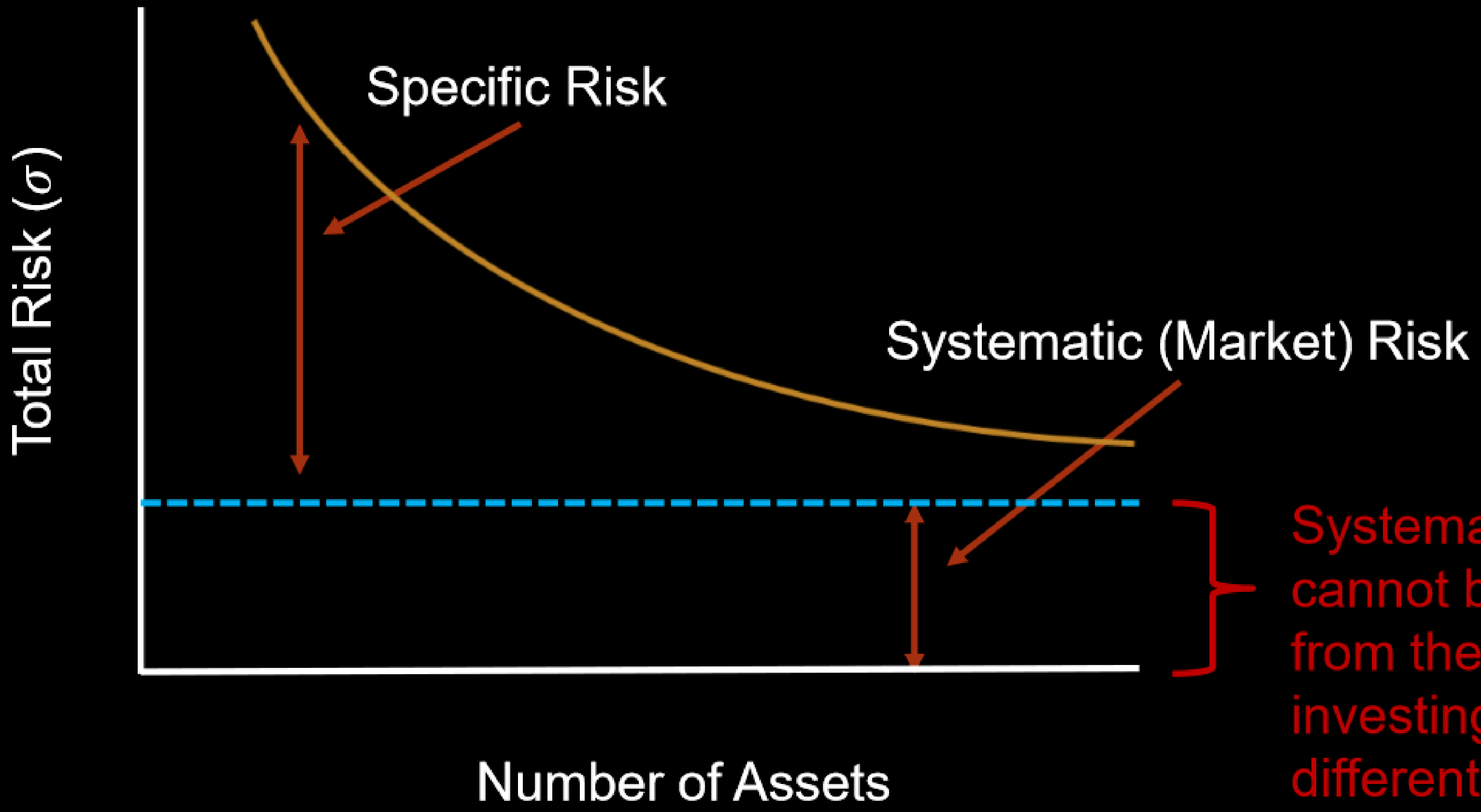
$$\sigma_{j,k} = \frac{1}{m-1}\sum_{i=1}^{m}(r_{j,t-i} - \bar{r}_j)(r_{k,t-i} - \bar{r}_k)$$

Optimal Portfolio – Historical



CAPM APPROACH

Systematic vs. Specific Risk



Systematic Risk cannot be eliminated from the portfolio by investing in more or different assets.

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- **Goal** – maximize return while minimizing risk!

$$Return = \sum_{i=1}^{m} p_i \hat{\alpha}_i + \bar{X}\left(\sum_{i=1}^{m} p_i \hat{\beta}_i\right)$$

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Excess return from assets

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Market return

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$$Risk = \sum_{i=1}^{m} w_i \sigma_{\varepsilon_i}^2 + \sigma_X^2 \left(\sum_{i=1}^{m} w_i \hat{\beta}_i\right)^2$$

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Asset risk

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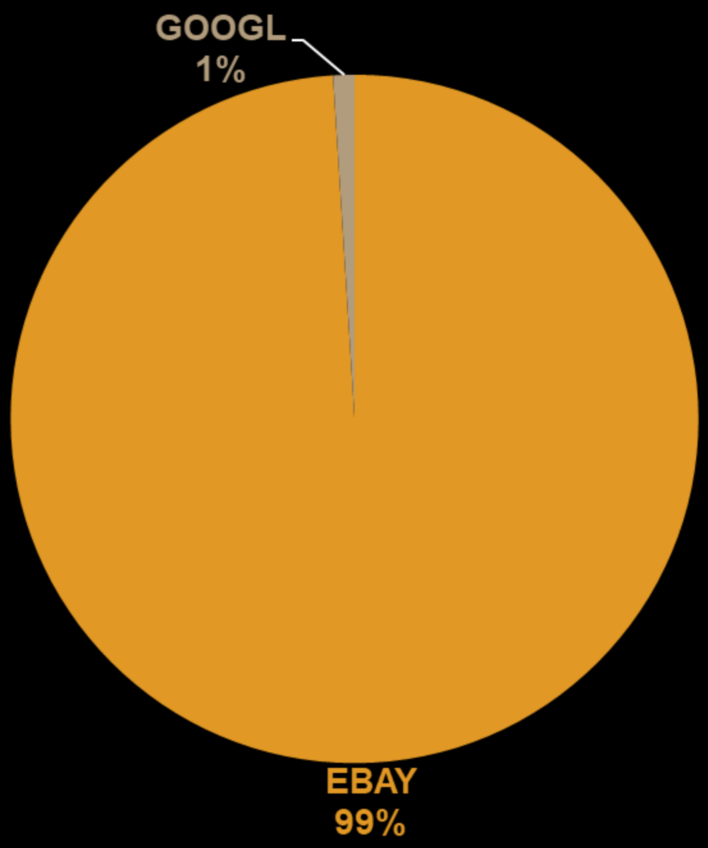
Market risk

Optimal Portfolio – CAPM



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GARCH APPROACH

Extensions to ARCH / GARCH Models

- **G**eneralized **A**uto**R**egressive **C**onditional **H**eteroscedasticity (**G**ARCH):

$$\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2 + \beta_1 \hat{\sigma}_{t-1}^2$$

- Original model assumes normality of returns! ☹
- GARCH-t model assumes t-distribution of returns ☺

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- GARCH-t model assumes t-distribution of returns ☺

- Original model assumes symmetry of returns (no leverage effect)! ☹
- QGARCH model assumes asymmetry (leverage effect) ☺

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- QGARCH-t combines them both! ☺ ☺

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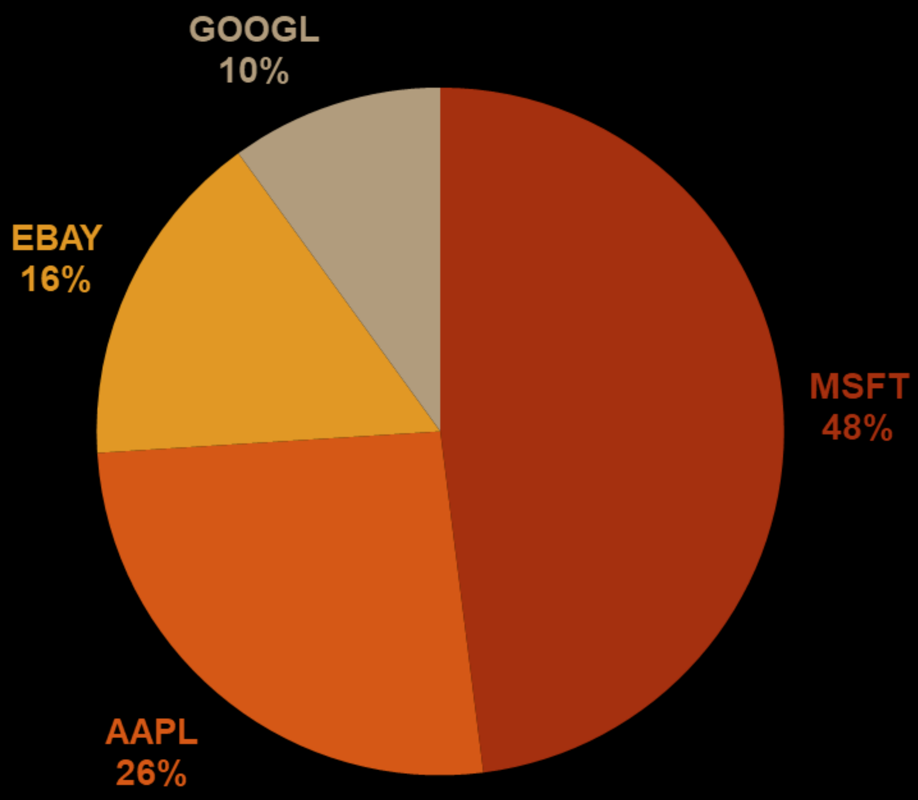
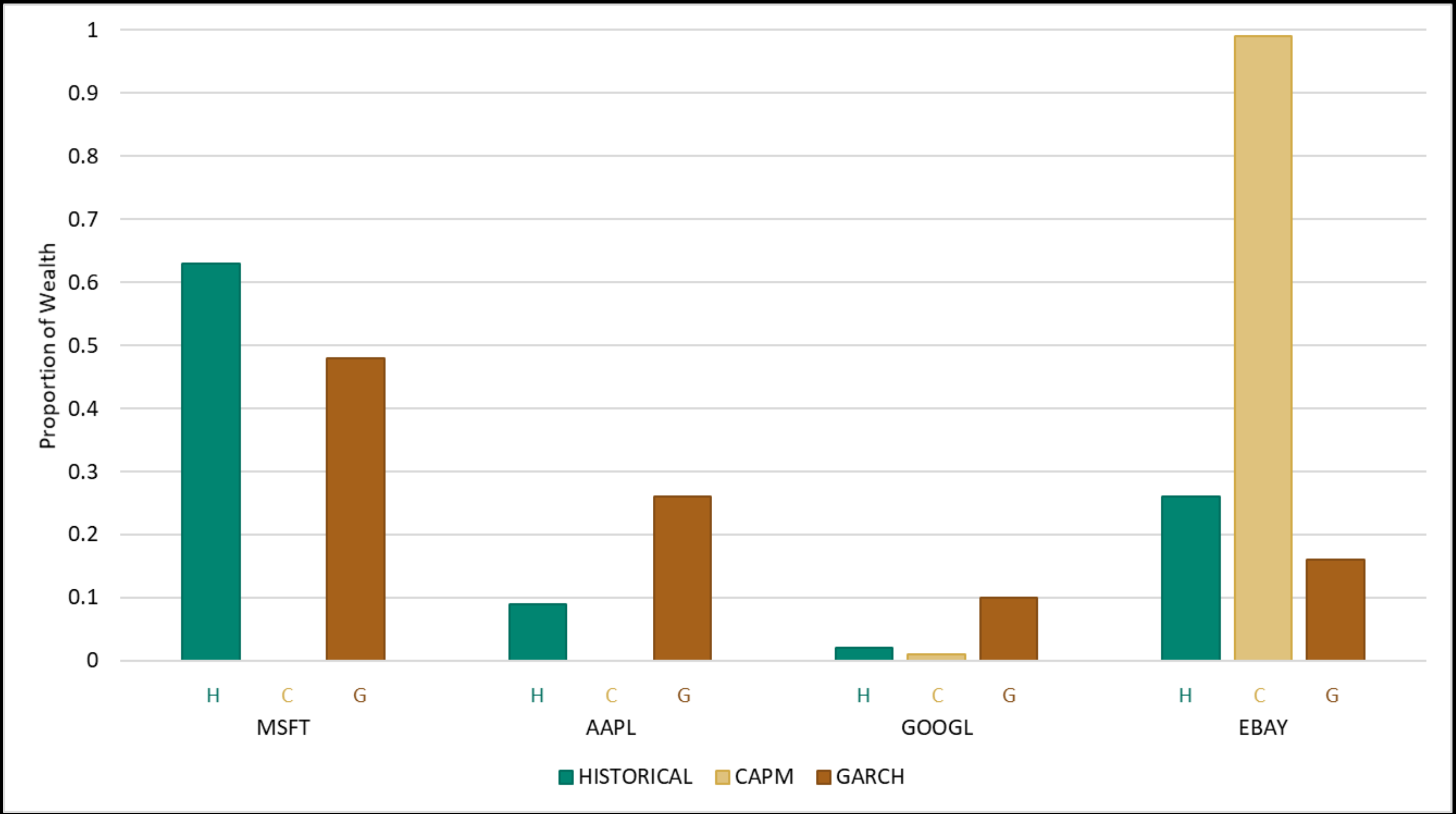
Forecasted variance and **historical** covariance

Optimal Portfolio – GARCH



COMPARISON / EVALUATION

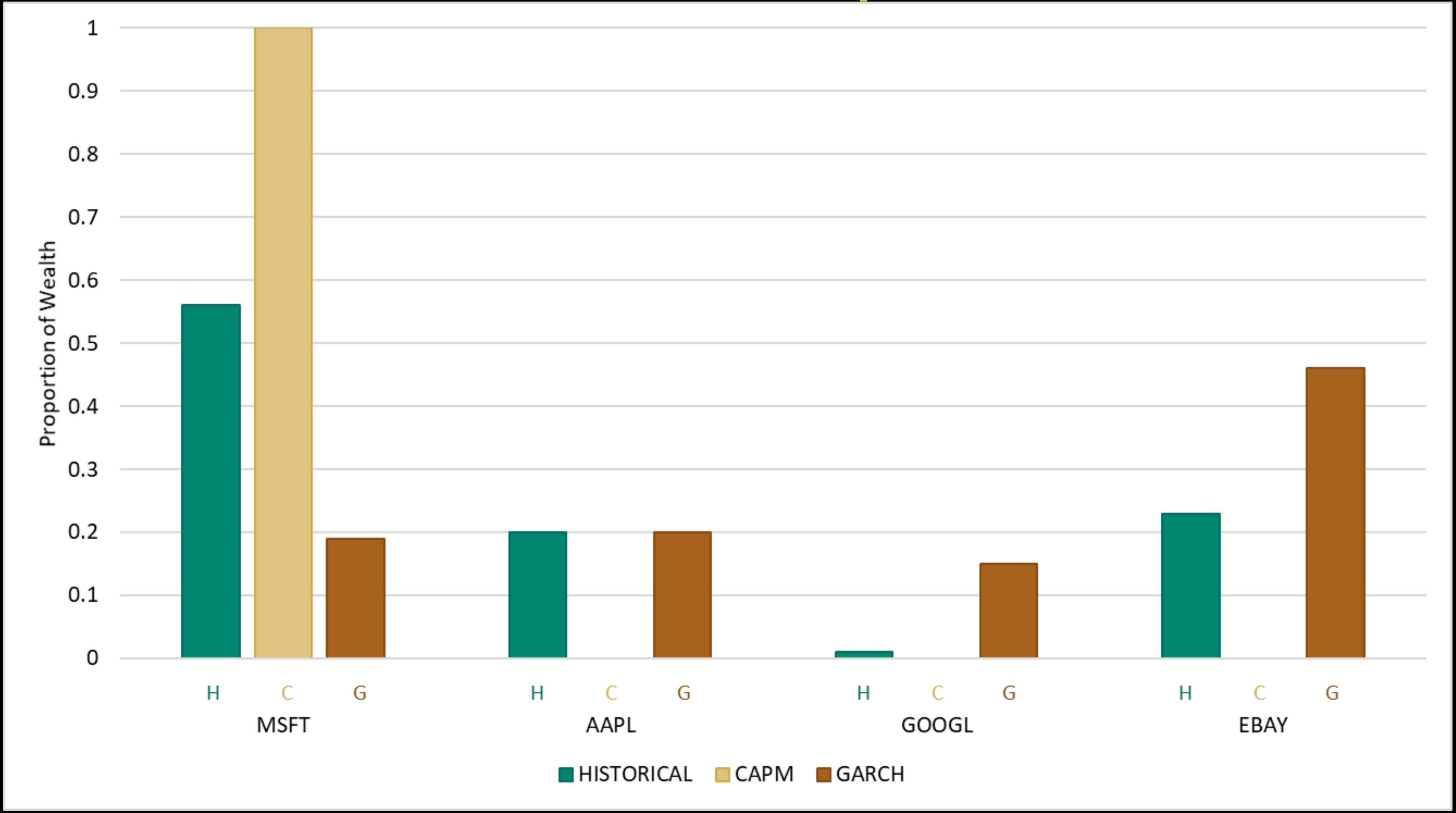
Allocation of Wealth Comparison



Return Per Strategy

Strategy	Return
Historical Variance	2.67%
CAPM	2.74%
GARCH	2.26%

Allocation of Wealth Comparison – SAS



Return Per Strategy – SAS

Strategy	Return
Historical Variance	2.66%
CAPM	2.81%
GARCH	2.06%

RISKLESS ASSET EXTENSION

Economics Extension

- What if we do **not** limit ourselves to have the following restriction?

$$p_1 + p_2 + p_3 + p_4 + p_5 = 1$$

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- What if we do **not** limit ourselves to have the following restriction?

$$p_1 + p_2 + p_3 + p_4 + p_5 = 1$$

- If the sum is less than 1 → invest the remainder in riskless asset.
- If the sum is greater than 1 → borrow money at riskless rate, invest, payback to achieve results.

Invest in Riskless Asset

- If the sum is less than 1 → invest the remainder in riskless asset.
  1. Invest \$0.88676 into stocks.
  2. Invest (1-0.88676) = \$0.11324 in riskless asset (assume 0.5% annual interest).
  3. After time period is up, earn return from portfolio (0.88676 + 0.014434) = \$0.901194 and return from riskless asset (0.11324 + 0.0005662) = \$0.11381
  4. \$0.901194 + \$0.11381 = \$1.015

Invest in Riskless Asset

- If the sum is greater than 1 → borrow money at riskless rate, invest, payback to achieve results.
  1. Borrow \$0.7767 at riskless rate (assume 0.5% annual interest).
  2. Invest (1+0.7767) = \$1.7767 in stocks.
  3. After time period is up, earn return from portfolio (1.7767 + 0.028883) = \$1.805583 and payback investment with interest (0.7767 + 0.0038835) = \$0.78058
  4. \$1.805583 - \$0.78058 = \$1.025