Nonlinear Optimization

Types of Optimization

There are 4 main types of optimization problems:

1. **Linear Programming** – objective function and constraints are linear.
2. **Integer Linear Programming** – objective function and constraints are linear but decision variables must be integers.
3. **Mixed Integer Linear Programming** – same as ILP with only some decision variables restricted to integers.
4. **Non-linear Programming** – objective function and constraints continuous but not all linear.

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Linear vs. Nonlinear

- Examples of linear relationships:

$y = ax + b$ $\qquad$ $z = ax + by$

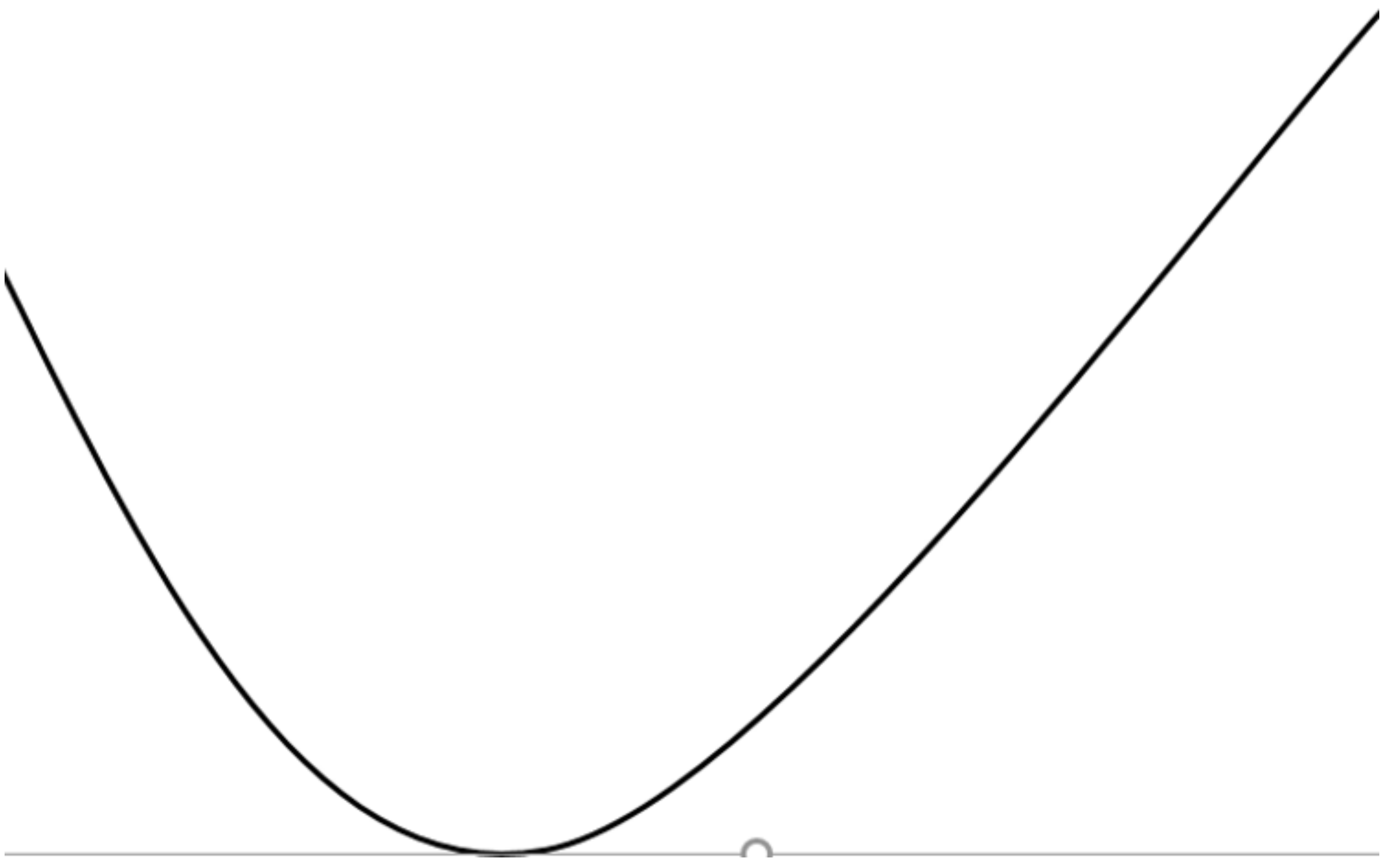
- Examples of nonlinear relationships:

$y = ax^b$ $\qquad$ $z = axy$

Algorithms

- Nonlinear optimization is a lot harder of a process than linear optimization (careful of local optimum)
- Many algorithms use gradients to solve the optimization.
  - Conjugate gradient method
  - Newton method with line search
  - Trust region
- Genetic Algorithms

An example of Gradient Descent (minimizing a function)



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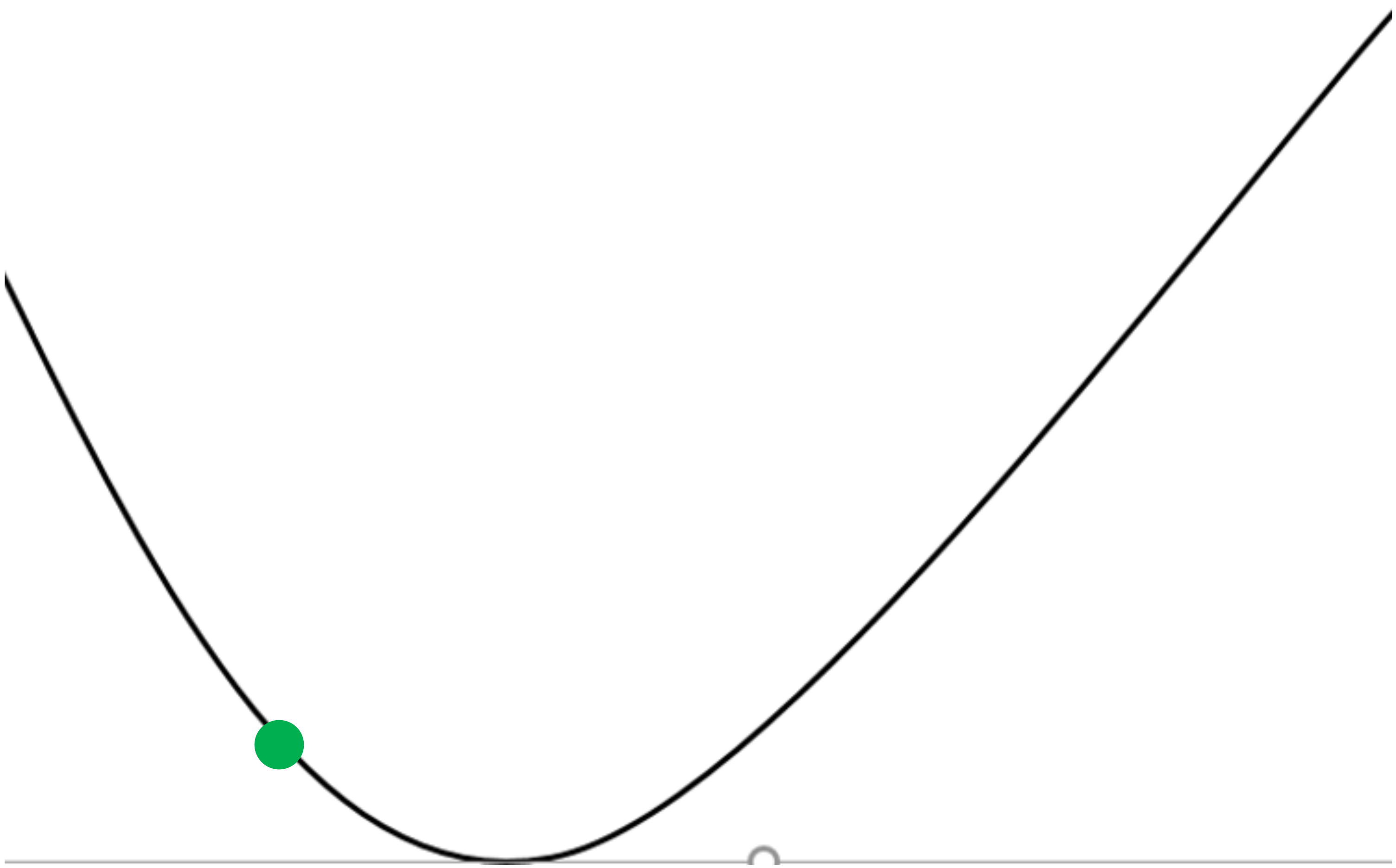

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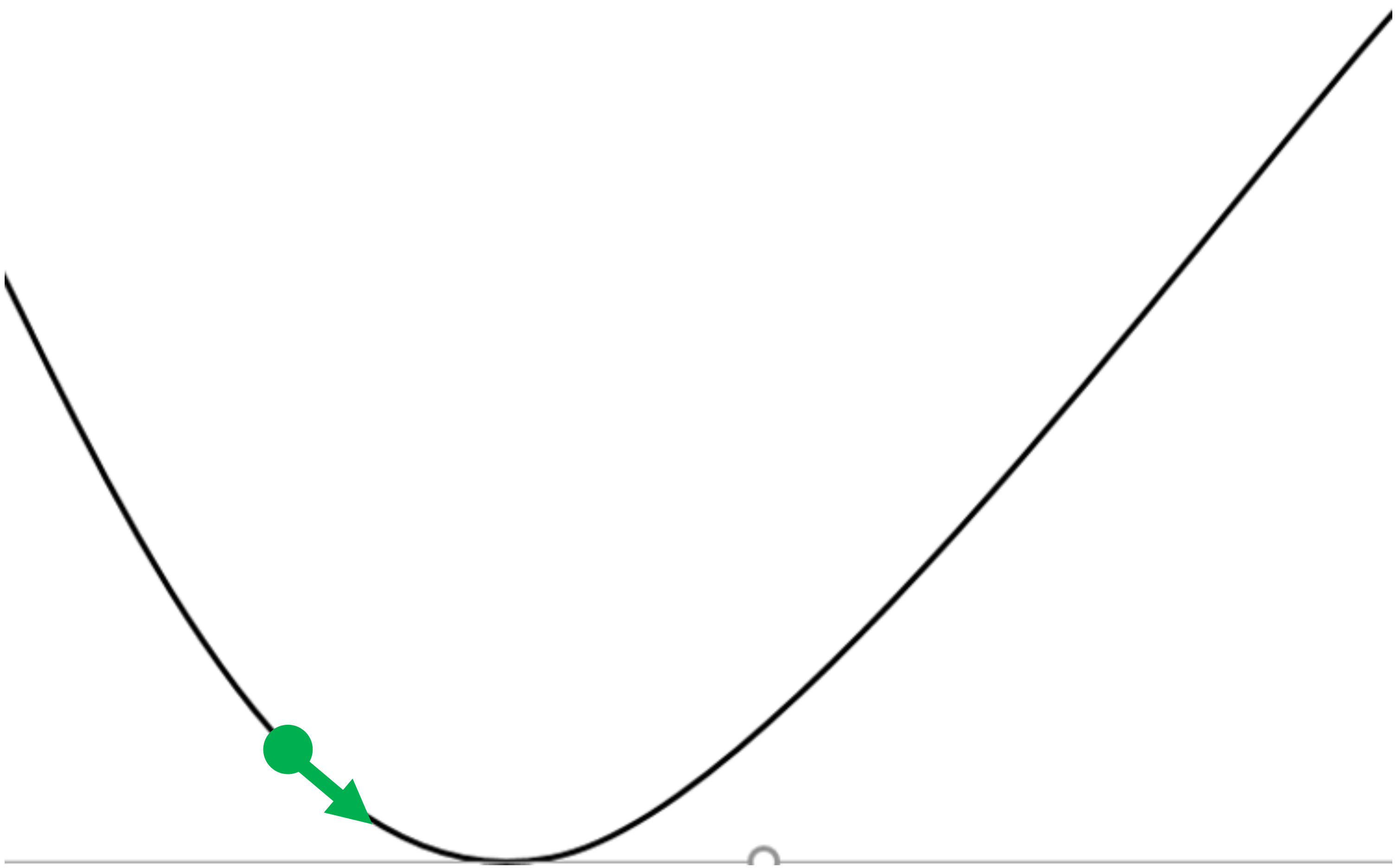

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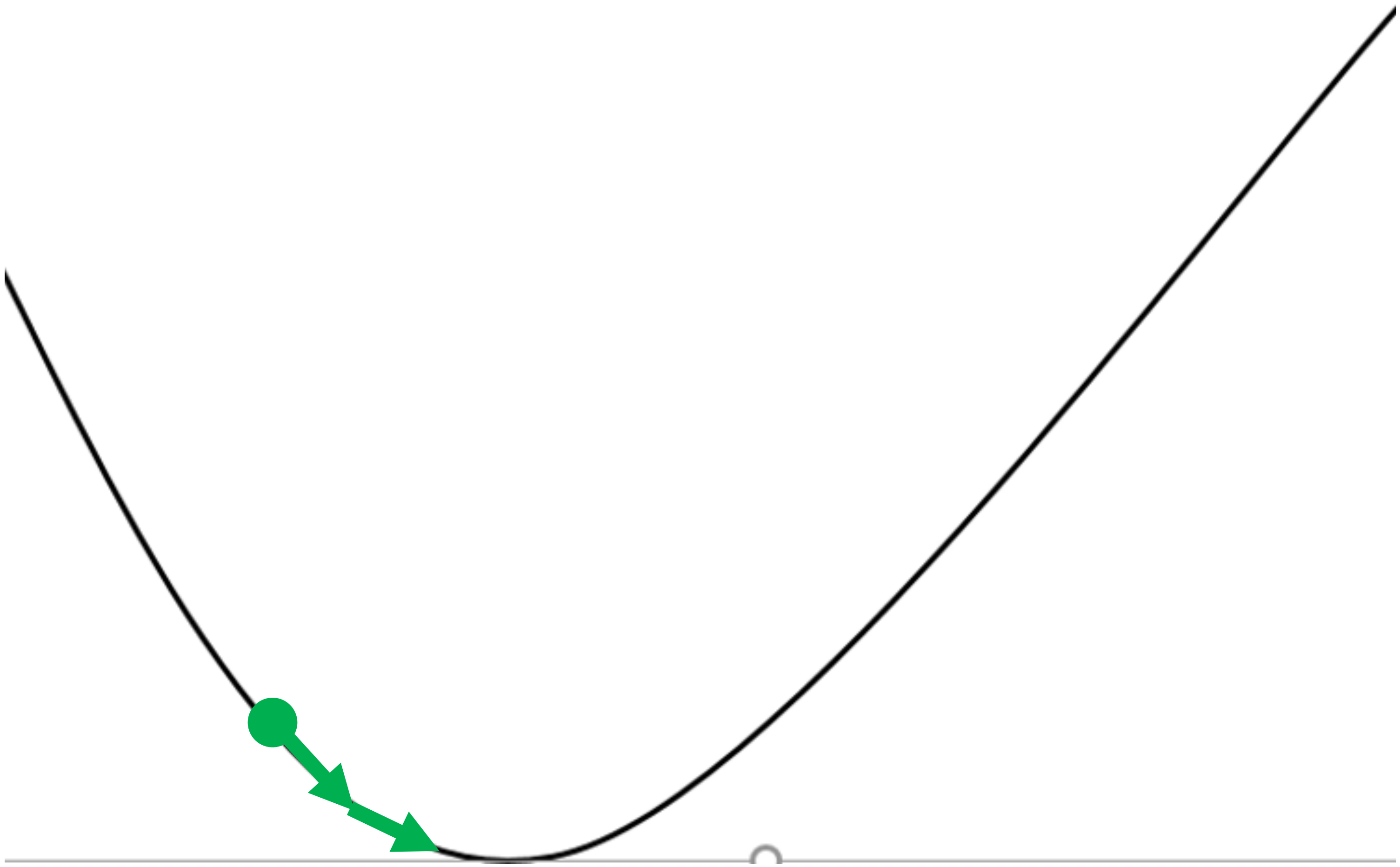

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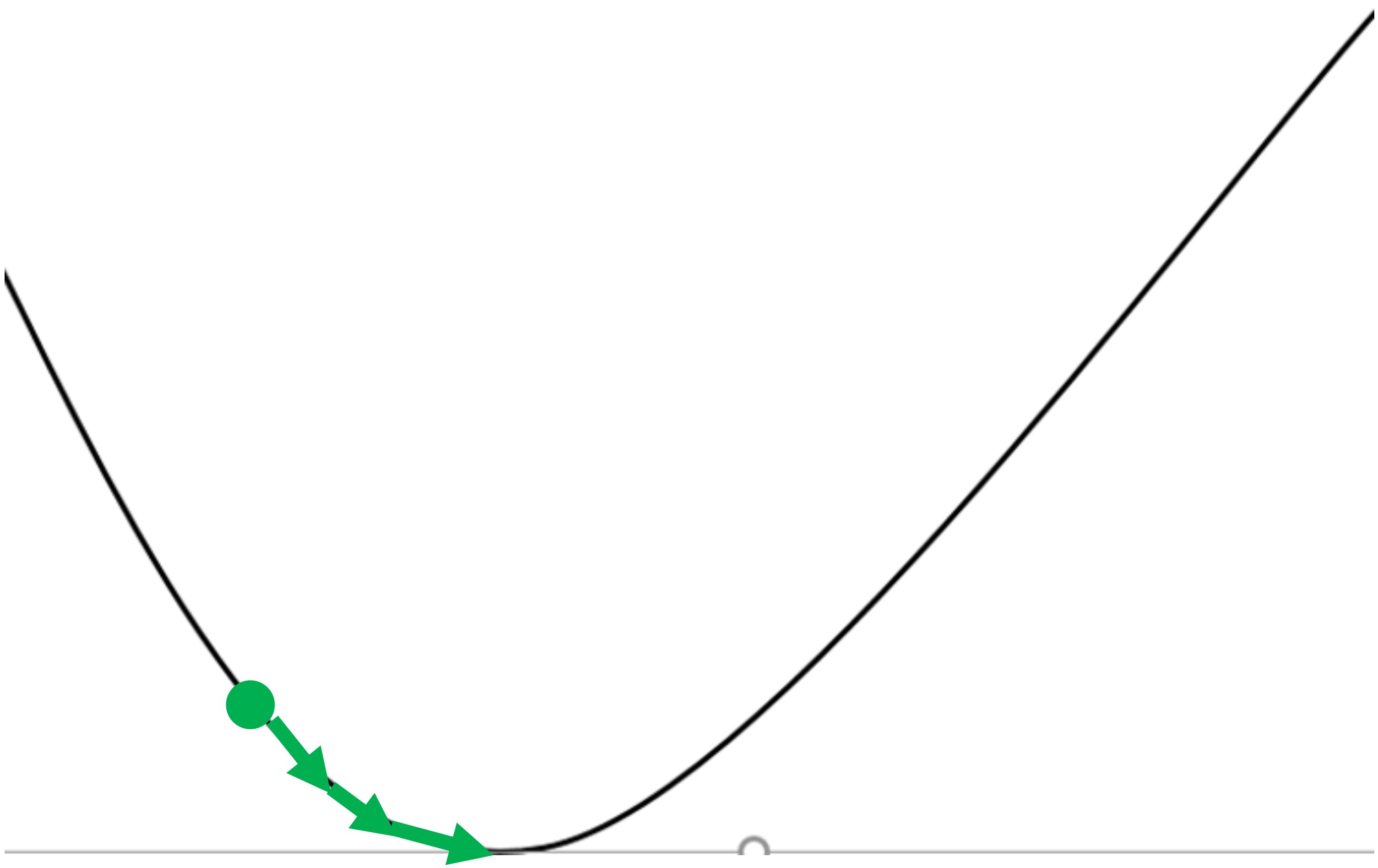

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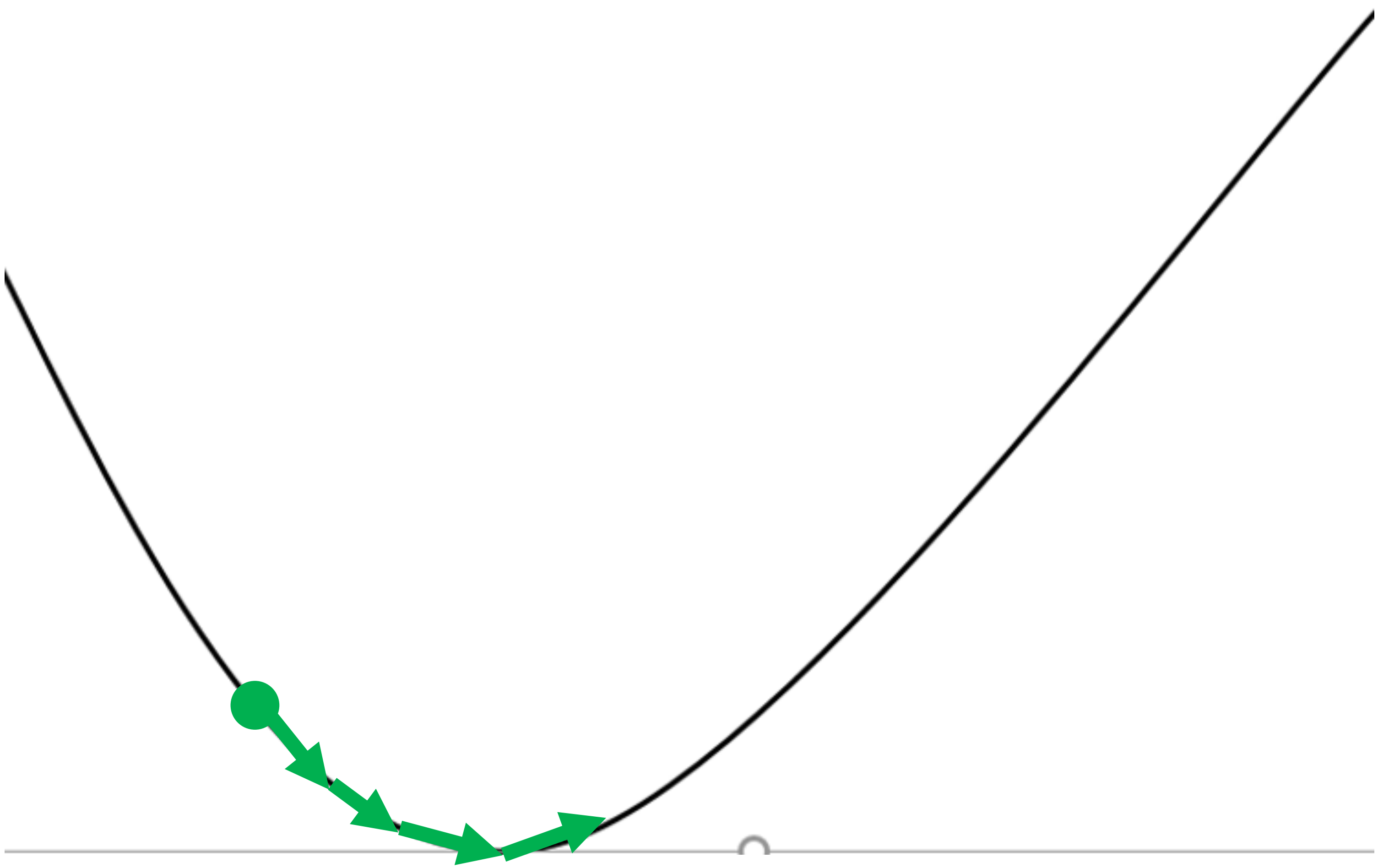

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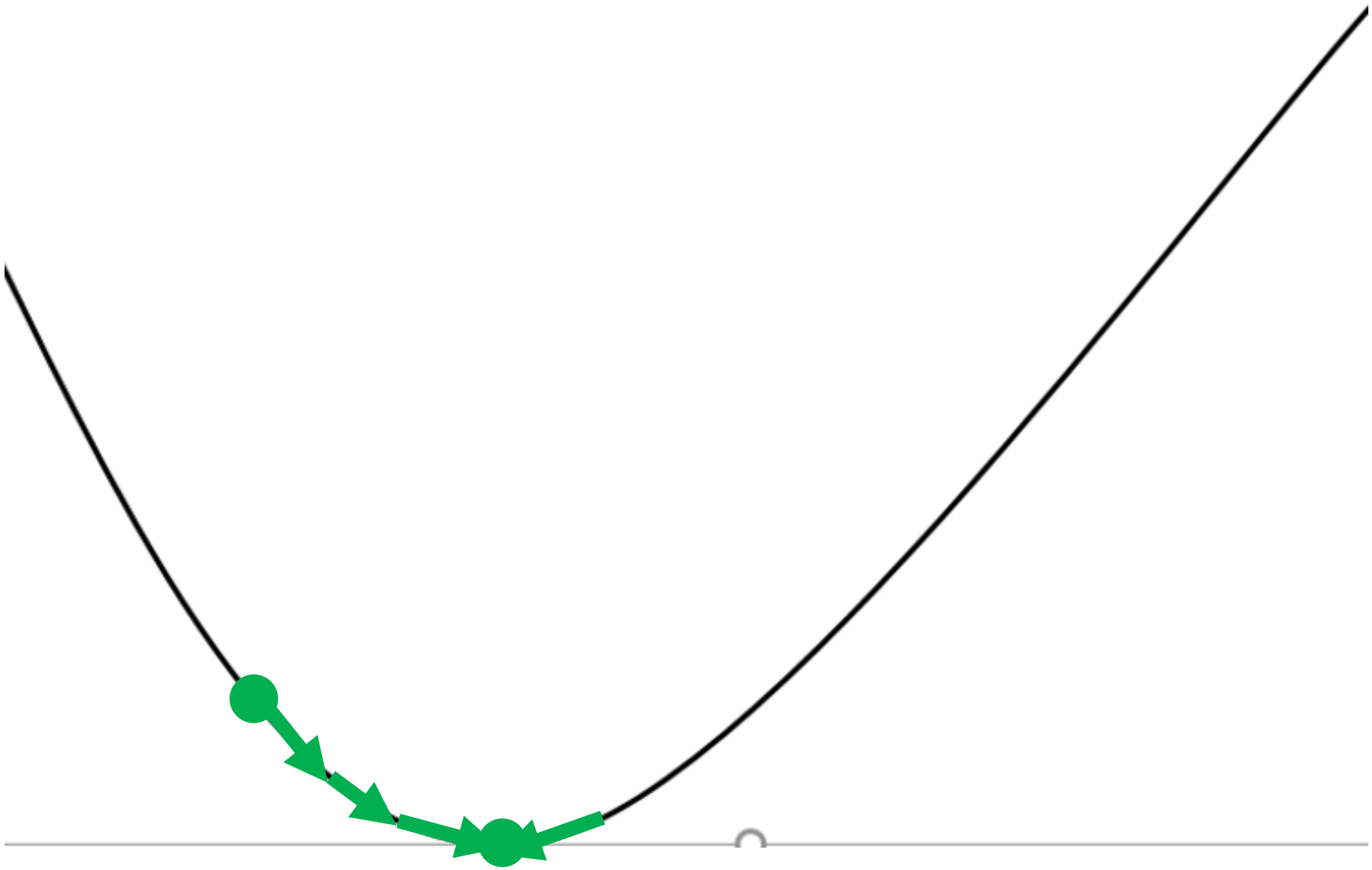
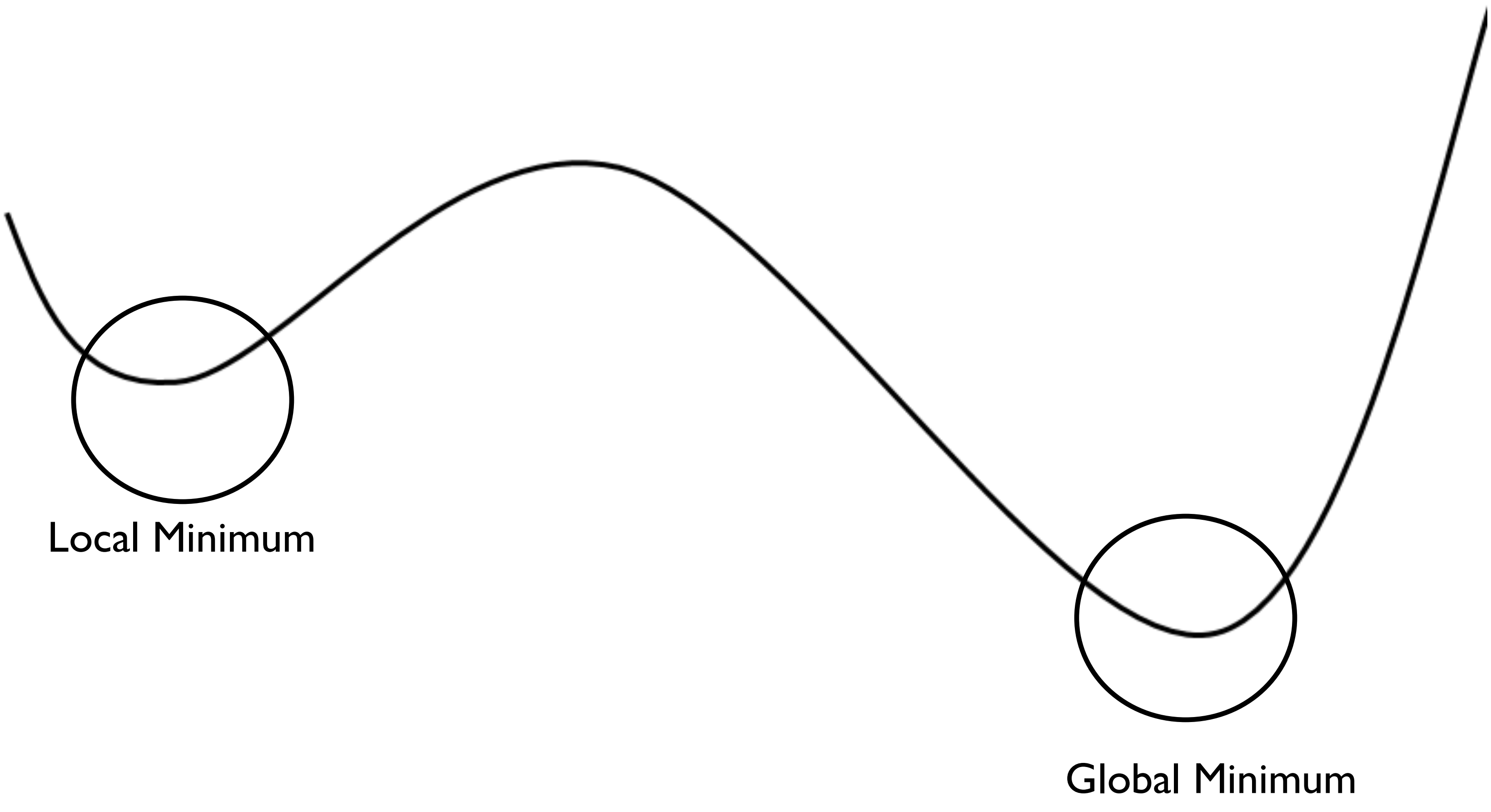


Potential issues



Multiple Answers

- Depending on where we start determines our answer!
- In cases with both local and global optima, there is no guarantee that a single run will produce the correct answer.
- The best method is to try many different starting points to get an idea of how good our answer actually is.

Portfolio Optimization

Financial Portfolio

- A ***portfolio*** is a collection of assets where the investor chooses the investment amount of each investment in the portfolio.
- Portfolio performance is typically measured by total value of the portfolio at the end of a period of time.
- To determine how much to allocate in each part of a portfolio, two things must be considered – risk and return.

Risk versus Return

- **Return** –growth in the value of an asset (can also be percentage growth)
- **Risk** – variability / volatility associated with the returns on the stock (can use standard deviation or variance)
- We can look at historical data to estimate both risk and return.
- Example: overall means and variance over a certain period of time (use historical data).

Optimizing a portfolio

- When optimizing a portfolio, we focus on risk and return, therefore, we could either:

1. Minimize risk for a given return (typical)
2. Maximize return for a given risk

Portfolio Optimization Example

- Advising Ms Womack
  - Ms Womack has some savings to invest and has 5 preferable stocks to invest in.
  - These five stocks are in 5 different industries over the past two years.
  - High return → High volatility
  - Return:

$$Return = p_1 r_1 + p_2 r_2 + p_3 r_3 + p_4 r_4 + p_5 r_5$$

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Average return of each stock

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$$Return = p_1 r_1 + p_2 r_2 + p_3 r_3 + p_4 r_4 + p_5 r_5$$

sum p > 1 must borrow money
sum p < 1 not investing all money
(invest in risk free rate)

Proportion of wealth in each stock

$$p_1 + p_2 + p_3 + p_4 + p_5 = 1$$

Risk of a portfolio

- Risk of a portfolio is the variation (volatility of the portfolio)….
- Need to go back to statistics, where we define Cov(Y_1 +Y_2)

$$Cov(aY_1 + bY_2) = a^2V(Y_1) + b^2V(Y_2) + 2abCov(Y_1, Y_2)$$

- We want to find the Cov(p_1r_1 +p_2r_2+p_3r_3 + p_4r_4 + p_5r_5)

where p_1,..p_5 are the proportion in each stock and $r_1 \ldots r_5$ are the returns for each stock

$$Cov(p_1r_1 + p_2r_2 + p_3r_3 + p_4r_4 + p_5r_5) = p_1^2V(r_1) + p_2^2V(r_2) + p_3^2V(r_3) + p_4^2V(r_4) + p_5^2V(r_5) + 2p_1p_2Cov(r_1, r_2) + 2p_1p_3Cov(r_1, r_3) + \ldots + +2p_4p_5Cov(r_4, r_5)$$

In Matrix form… (bet you thought you wouldn't see this again….)

$$\begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} & \sigma_{15} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} & \sigma_{25} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} & \sigma_{35} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} & \sigma_{45} \\ \sigma_{51} & \sigma_{52} & \sigma_{53} & \sigma_{54} & \sigma_{55} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{bmatrix}$$

Portfolio Optimization Example

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  - Return:
  - Risk:

$$\sum_{j}\sum_{k} p_j * \sigma_{j,k} * p_k = p_1\sigma_{1,1}p_1 + p_1\sigma_{1,2}p_2 + \cdots + p_5\sigma_{5,5}p_5$$

Covariance between each stock combination

Portfolio data

```
data portfolio;
input Computer Chemical Power Auto Electronics;
cards;
0.22816   -0.07205  0.0173    0.22266   0.08202
0.09134   0.02588   0.05646   0.01278   -0.03499
-0.01288  -0.04771  0.0228    0.00379   0.01662
-0.17196  0.06343   0         0.04101   -0.07496
0.16557   0.0367    0.0051    0.07576   -0.0081
•••
;
```

2 years of monthly stock returns from these 5 stocks (i.e. 24 observations)

Creating data sets in SAS

```
proc corr data=portfolio cov out=Corr;
var Computer Chemical Power Auto Electronics;
run;
data Cov;
set Corr;
where _TYPE_='COV';
run;
data Mean;
set Corr;
where _TYPE_='MEAN';
run;
```

Corr data set

	TYPE	_NAME_	Computer	Chemical	Power	Auto	Electronics
1	COV	Computer	0.0096204201	-0.000656562	0.0003096089	0.0026898786	0.0012240949
2	COV	Chemical	-0.000656562	0.0036308674	-0.000042252	-0.000575105	0.0003071201
3	COV	Power	0.0003096089	-0.000042252	0.0013255352	0.0005815396	-0.000146178
4	COV	Auto	0.0026898786	-0.000575105	0.0005815396	0.0068849184	0.0017932533
5	COV	Electronics	0.0012240949	0.0003071201	-0.000146178	0.0017932533	0.0024922448
6	MEAN		0.02087875	0.0120470833	0.00685375	0.0225908333	0.0133495833
7	STD		0.0980837402	0.0602566793	0.0364078999	0.0829754084	0.0499223878
8	N		24	24	24	24	24
9	CORR	Computer	1	-0.111089574	0.0867003429	0.3305112628	0.2499900535
10	CORR	Chemical	-0.111089574	1	-0.019259775	-0.115024983	0.1020957485
11	CORR	Power	0.0867003429	-0.019259775	1	0.1925015793	-0.080425249
12	CORR	Auto	0.3305112628	-0.115024983	0.1925015793	1	0.4329092655
13	CORR	Electronics	0.2499900535	0.1020957485	-0.080425249	0.4329092655	1

SAS Code

```
proc optmodel;
        set <str> Assets1, Assets2, Assets3;
        num Covariance{Assets1,Assets2};
        num Mean{Assets1};
        read data Cov into Assets1=[_NAME_];
        read data Cov into Assets2=[_NAME_] {i in Assets1} <Covariance[i,_NAME_]=col(i)>;
        read data Mean into Assets3=[_NAME_] {i in Assets1} <Mean[i]=col(i)>;
        var Proportion{Assets1}>=0 init 0;
        min Risk = sum{i in Assets1}(sum{j in Assets1}Proportion[i]*Covariance[i,j]*Proportion[j]);
        con Return: 0.015 <= sum{i in Assets1}Proportion[i]*Mean[i];
        con Sum: 1 = sum{i in Assets1}Proportion[i];
        solve;
print Covariance Mean;
print Proportion 'Sum ='(sum{i in Assets1}Proportion[i]);
quit;
```

The OPTMODEL Procedure

Problem Summary	
Objective Sense	Minimization
Objective Function	Risk
Objective Type	Quadratic
Number of Variables	5
Bounded Above	0
Bounded Below	5
Bounded Below and Above	0
Free	0
Fixed	0
Number of Constraints	2
Linear LE (<=)	1
Linear EQ (=)	1
Linear GE (>=)	0
Linear Range	0
Constraint Coefficients	10

Performance Information	
Execution Mode	Single-Machine
Number of Threads	4

The SAS System

The OPTMODEL Procedure

Solution Summary	
Solver	QP
Algorithm	Interior Point
Objective Function	Risk
Solution Status	Optimal
Objective Value	0.0012749728
Primal Infeasibility	0
Dual Infeasibility	9.75782E-19
Bound Infeasibility	0
Duality Gap	2.750367E-17
Complementarity	0
Iterations	5
Presolve Time	0.00
Solution Time	0.01

Covariance					
	Auto	Chemical	Computer	Electronics	Power
Auto	0.006884918	-0.000575105	0.002689879	0.001793253	0.000581540
Chemical	-0.000575105	0.003630867	-0.000656562	0.000307120	-0.000042252
Computer	0.002689879	-0.000656562	0.009620420	0.001224095	0.000309609
Electronics	0.001793253	0.000307120	0.001224095	0.002492245	-0.000146178
Power	0.000581540	-0.000042252	0.000309609	-0.000146178	0.001325535

[1]	Mean
Auto	0.0225908
Chemical	0.0120471
Computer	0.0208788
Electronics	0.0133496
Power	0.0068538

[1]	Proportion
Auto	0.22471
Chemical	0.30406
Computer	0.13235
Electronics	0.18083
Power	0.15805

Sum =	1

In R

```
setwd('C:/Users/Susan/Google Drive/Optimization')
port=read.table('portfolio_r.csv',sep=',',header=T)
mean.vec=apply(port,2,mean)
cov.vec=cov(port)
library(quadprog)
Dmat=cov.vec
dvec=rep(0,5)
Amat=t(matrix(c(1,1,1,1,1,mean.vec),nrow=2,byrow=T))
bvec=c(1,0.015)
meq=1
ln.model=solve.QP(Dmat,dvec,Amat,bvec,meq)
ln.names=c('Computer','Chemical','Power','Auto','Electroni
cs')

names(ln.model$solution)=ln.names
ln.model$solution
ln.model$value
```

Output

```
> ln.model$solution
   Computer    Chemical     Power      Auto   Electronics
  0.1323478   0.3040609 0.1580473 0.2247138   0.1808302
> ln.model$value
[1] 0.0006374864
```

Efficient Frontier

- Higher risk yields higher return. What is the best return we can achieve for a given level of risk (or what is the lowest risk for a given level of return)
- The efficient frontier is the set of optimal portfolios that offers the highest expected return for a defined level of risk or the lowest risk for a given level of expected return. Portfolios that lie below the efficient frontier are sub-optimal, because they do not provide enough return for the level of risk. (Investopedia)