ARCH & GARCH MODELS

Dr. Aric LaBarr

Institute for Advanced Analytics

History

- Until 1980's: Econometrics focused almost solely on modeling the mean of the series (actual values of the target variable)
- Mid-1980's to now: Increased focus on volatility, what influences volatility and volatility's effect on the mean values.

- "One of the funny things about the stock market is that every time one person buys, another sells, and both think they are astute."

  \- William Feather

Unconditional vs. Conditional Variance

- Unconditional variance is the same standard variance calculation that we have done in the past:

$$\sigma^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

$$\sigma^2 = E\big(x - E(x)\big)^2$$

Unconditional vs. Conditional Variance

- Unconditional variance is the same standard variance calculation that we have done in the past:

$$\sigma^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

$$\sigma^2 = E\big(x - E(x)\big)^2$$

- Conditional variance is the measure of our uncertainty about a variable given a set of information (or data).
  - Heteroscedasticity – variance depends on external factors

$$\sigma_{cond}^2 = E(x - E(x|\boldsymbol{I})\,)^2$$

Heteroscedasticity

- Variance depends on external factors.
- Cross-sectional data:

$$Var(\varepsilon_i|\boldsymbol{x}_i) = \sigma_i^2$$

  - Heteroscedasticity is a nuisance we try to avoid or correct.

- Time series data:

$$Var(\varepsilon_t|\boldsymbol{I}_t) = \sigma_t^2$$

  - Heteroscedasticity is of interest, especially in finance, and we desire to model it!

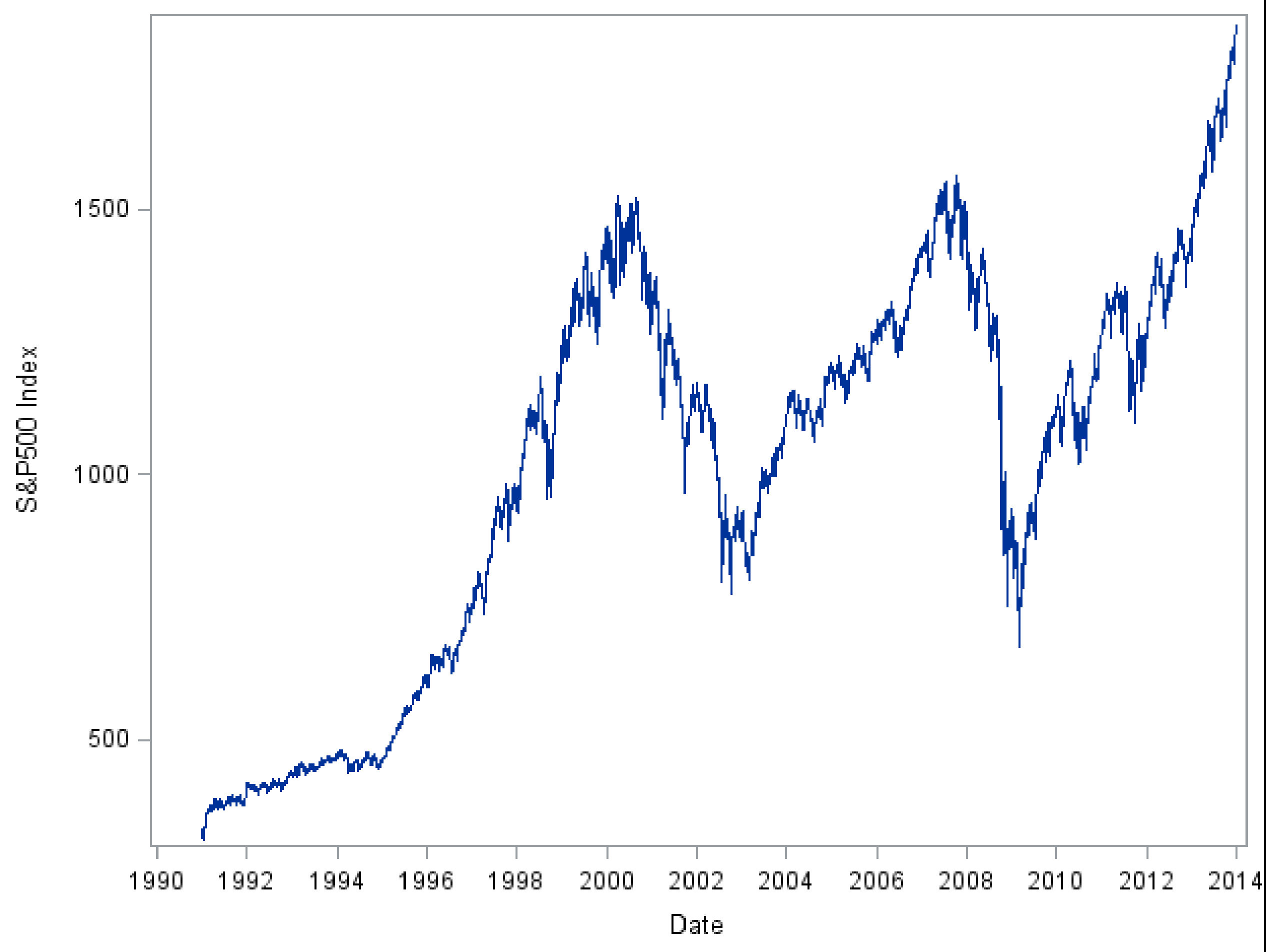
WHY DO WE MODEL VOLATILITY?

Facts About Financial Time Series

- Non-stationarity of prices.
- Mean-reversion of the returns of the series.

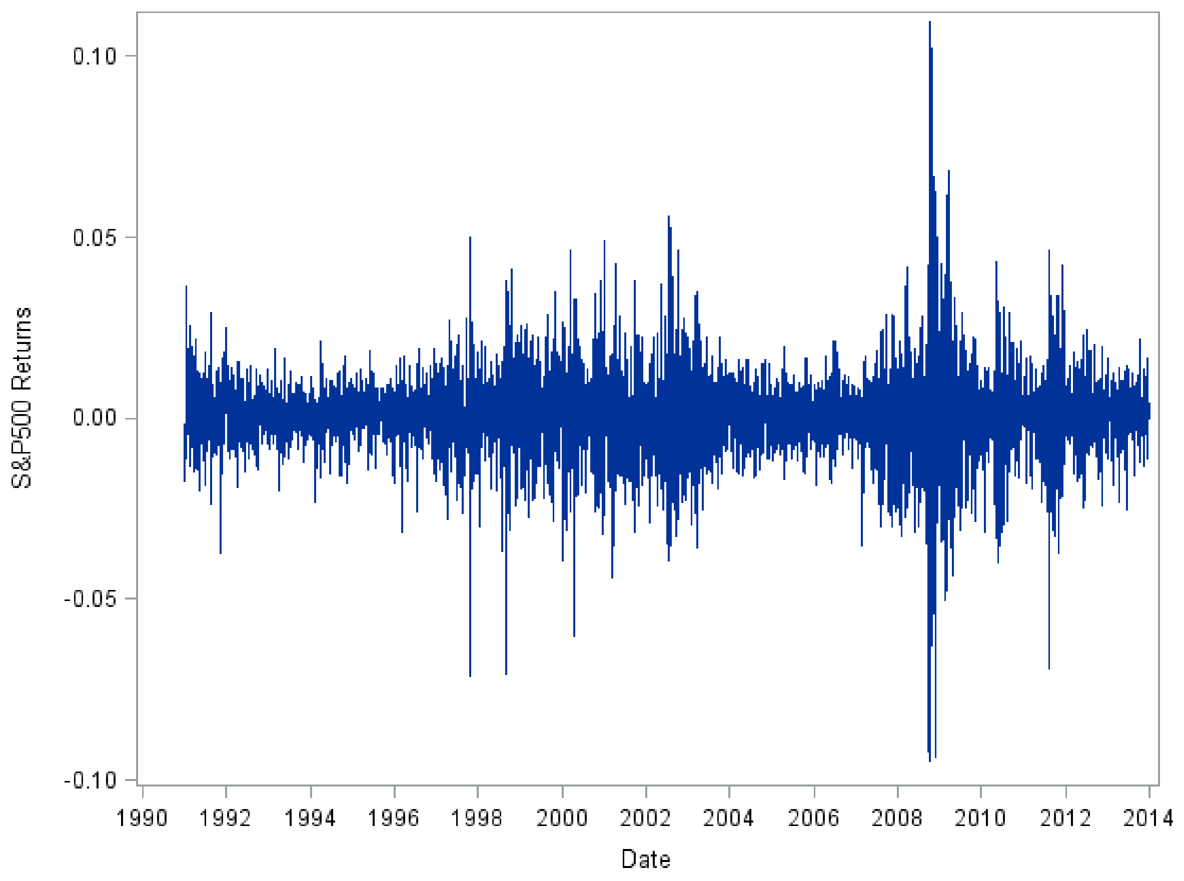
Facts About Financial Time Series

- Non-stationarity of prices.

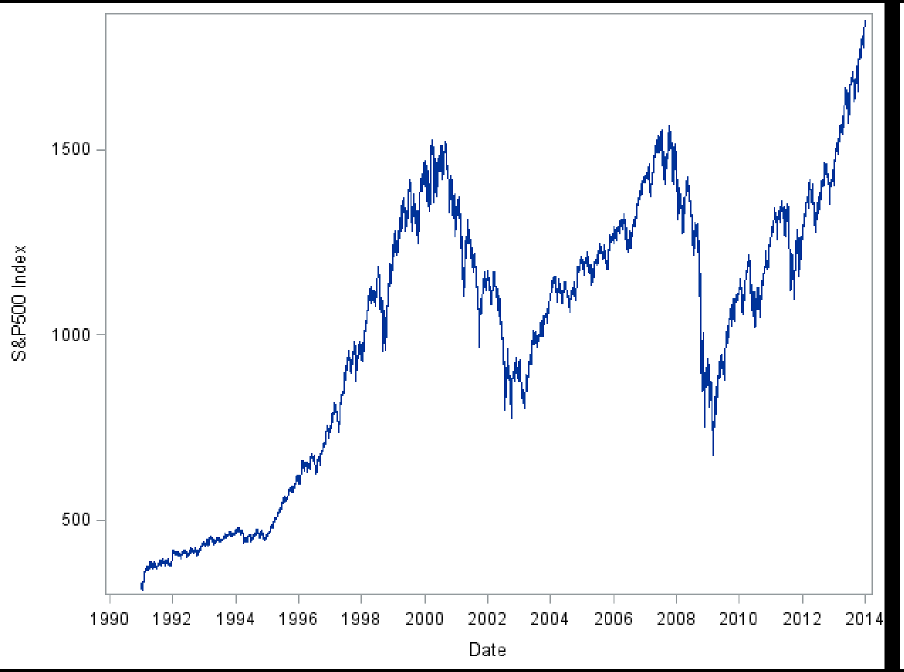


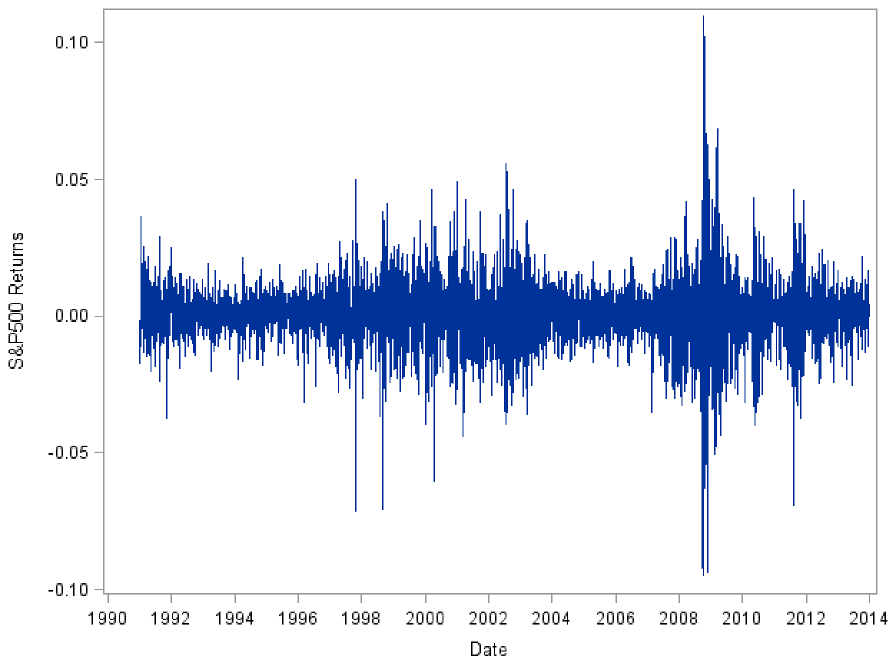
Facts About Financial Time Series

- Non-stationarity of prices.
- Mean-reversion of the returns of the series.



Facts About Financial Time Series





Facts About Financial Time Series

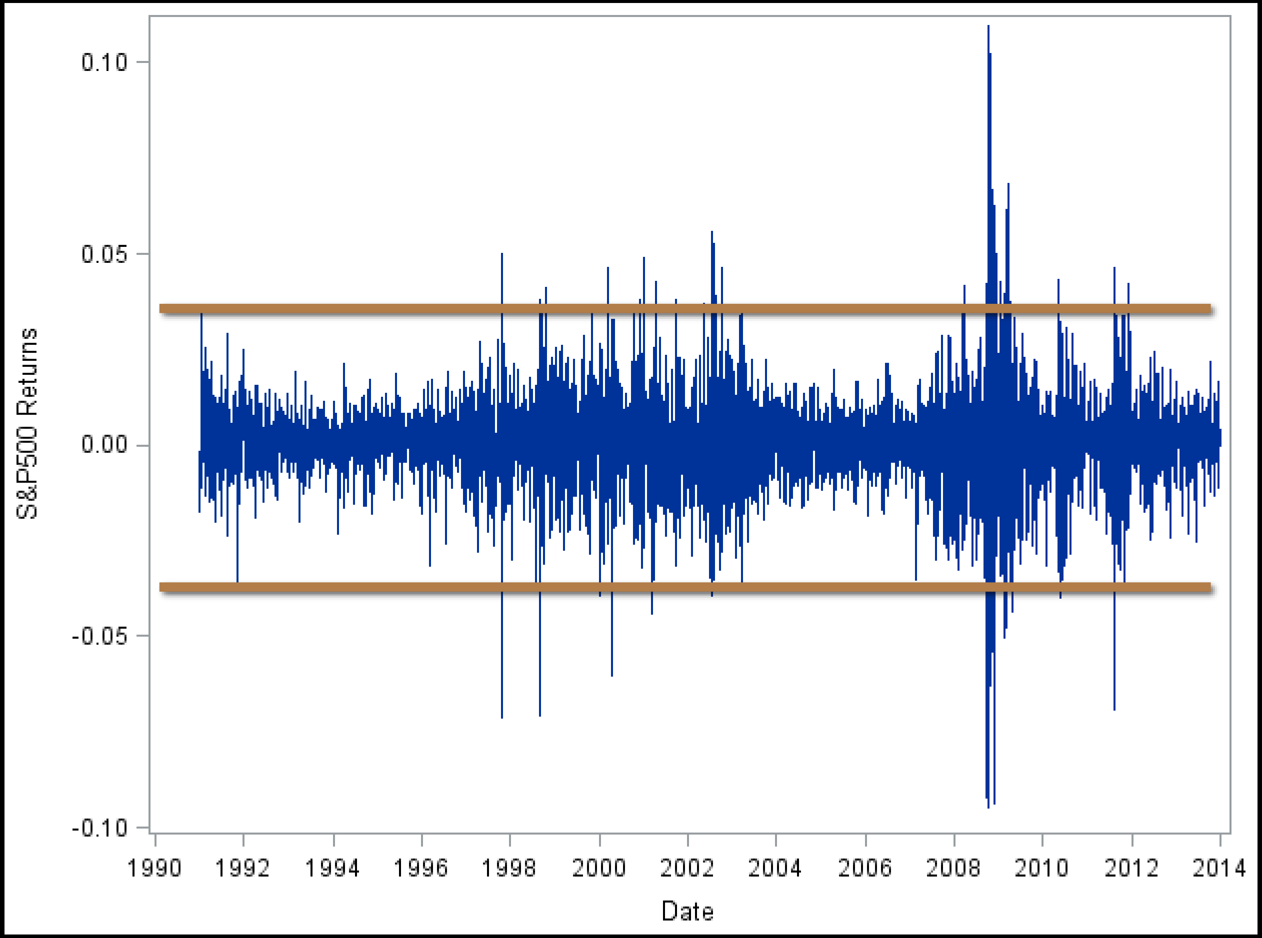
- Non-stationarity of prices.
- Mean-reversion of the returns of the series.

- THIS MAKES IT HARD TO GET INFORMATION FROM FORECASTING MARKET!!!

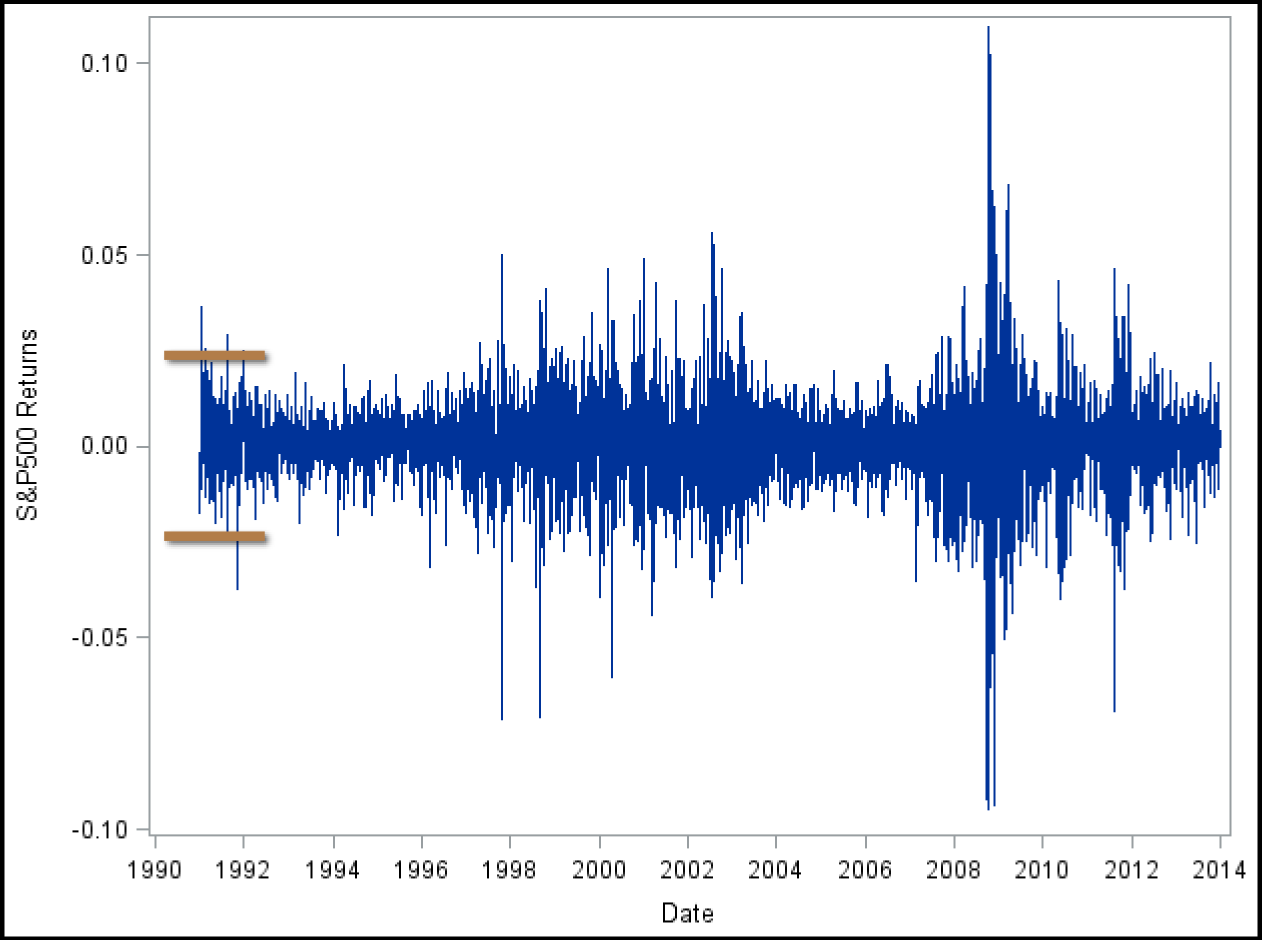
Facts About Financial Time Series

- **Thick tails** – more outliers than what the Normal distribution would suggest.
- **Volatility clustering** – large changes tend to be followed by large changes.
- **Leverage effects** – tendency for changes in stock prices to be negatively correlated with changes in volatility.
- **Non-trading period effects** – information accumulates at a different rate when market is closed as compared to when it is open.
- **Co-movements in volatility** – volatility is positively correlated across assets in a market and even across markets.

Constant Volatility?



Constant Volatility?



Applications

- Estimating the Value at Risk
- Optimizing Allocations of Assets
- Hedging Risk
- Pricing Multiple Assets in an Option

HOW TO MODEL VOLATILITY?

Simple Approaches

How Can We “Model” Variance

- Need to lay the foundation of how one can model variance over time.
- What is a reasonable model for Y, the actual price?

$$Y_t = \beta_0 + Y_{t-1} + \varepsilon_t$$

How Can We "Model" Variance

- Need to lay the foundation of how one can model variance over time.
- What is a reasonable model for Y, the actual price?

$$Y_t = \beta_0 + Y_{t-1} + \varepsilon_t$$

- How do we get to returns?

$$Y_t - Y_{t-1} = \beta_0 + \varepsilon_t$$

How Can We “Model” Variance

- Need to lay the foundation of how one can model variance over time.
- What is a reasonable model for Y, the actual price?

$$Y_t = \beta_0 + Y_{t-1} + \varepsilon_t$$

- How do we get to returns?

$$Y_t - Y_{t-1} = \beta_0 + \varepsilon_t$$

$$\frac{Y_t - Y_{t-1}}{Y_{t-1}} = \frac{\beta_0}{Y_{t-1}} + \frac{\varepsilon_t}{Y_{t-1}}$$

How Can We “Model” Variance

- Need to lay the foundation of how one can model variance over time.
- What is a reasonable model for Y, the actual price?

$$Y_t = \beta_0 + Y_{t-1} + \varepsilon_t$$

- How do we get to returns?

$$Y_t - Y_{t-1} = \beta_0 + \varepsilon_t$$

$$\frac{Y_t - Y_{t-1}}{Y_{t-1}} = \beta_0^* + \varepsilon_t^*$$

Still a Constant

Still Normal

How Can We “Model” Variance

- Need to lay the foundation of how one can model variance over time.
- What is a reasonable model for Y, the actual price?

$$Y_t = \beta_0 + Y_{t-1} + \varepsilon_t$$

- How do we get to returns?

$$Y_t - Y_{t-1} = \beta_0 + \varepsilon_t$$

$$r_t = \beta_0^* + \varepsilon_t^*$$

How Can We "Model" Variance

- Need to lay the foundation of how one can model variance over time.
- What is a reasonable model for Y, the actual price?

$$Y_t = \beta_0 + Y_{t-1} + \varepsilon_t$$

- How do we get to returns?

$$Y_t - Y_{t-1} = \beta_0 + \varepsilon_t$$

$$r_t = \beta_0^* + \varepsilon_t^*$$

Intercept only model, $\beta_0 = \bar{Y} \approx 0$

How Can We "Model" Variance

- Need to lay the foundation of how one can model variance over time.
- What is a reasonable model for Y, the actual price?

$$Y_t = \beta_0 + Y_{t-1} + \varepsilon_t$$

- How do we get to returns?

$$Y_t - Y_{t-1} = \beta_0 + \varepsilon_t$$

$$r_t = \varepsilon_t^*$$

$$r_t \sim N\left(0, \sigma^2\right)$$

How Can We "Model" Variance

- Need to lay the foundation of how one can model variance over time.
- What is a reasonable model for Y, the actual price?

$$Y_t = \beta_0 + Y_{t-1} + \varepsilon_t$$

- How do we get to returns?

$$Y_t - Y_{t-1} = \beta_0 + \varepsilon_t$$

$$r_t = \varepsilon_t^*$$

MODEL THIS!

$$r_t \sim N(0, \sigma_t^2)$$

Or…

- What IF we COULD model price?
- How would this change our model?
- All you need to do is model the RESIDUALS!

$$Y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \varepsilon_t$$

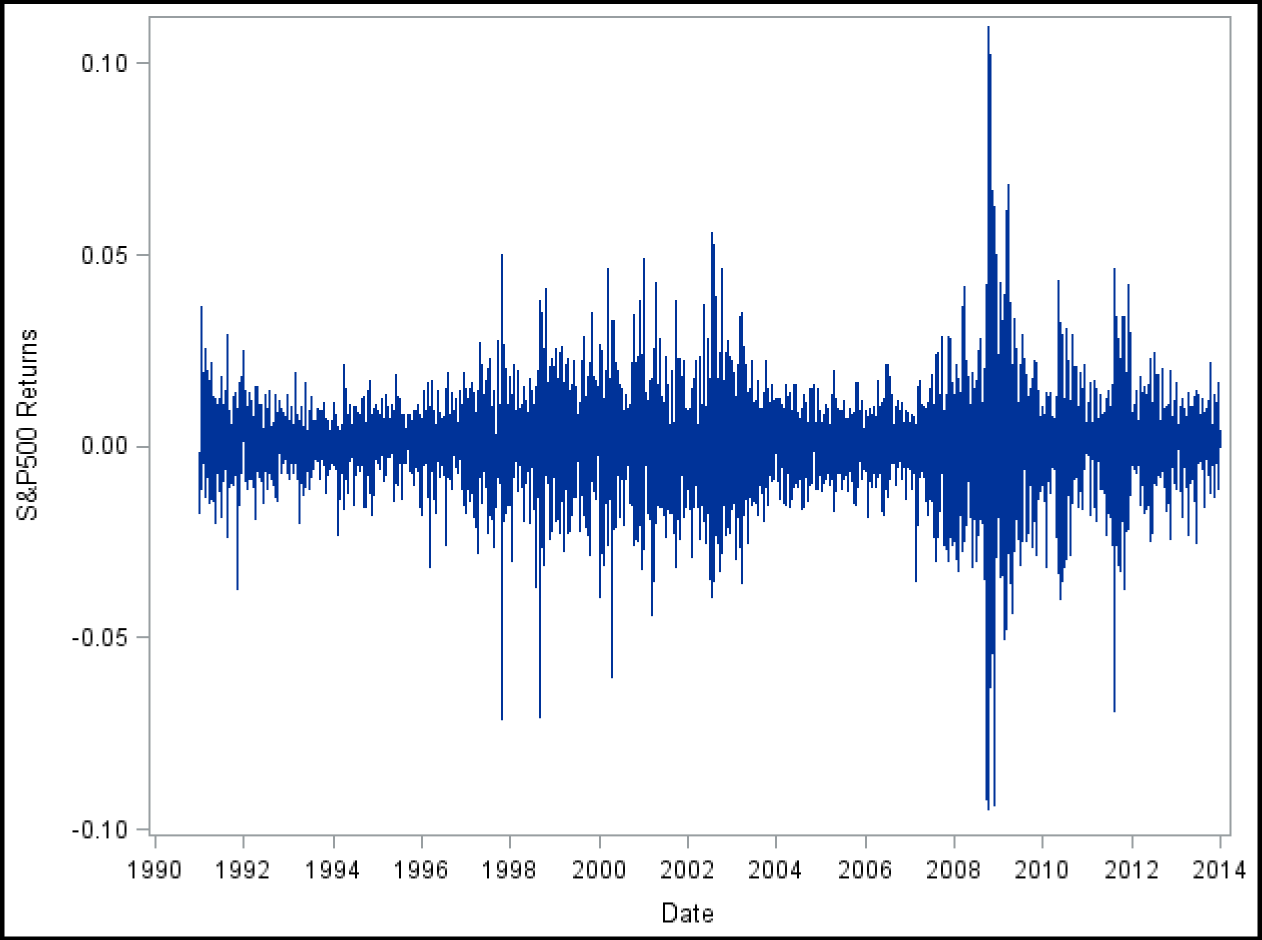
$$Y_t - \left(\hat{\beta}_0 + \hat{\beta}_1 x_{1,t} + \cdots + \hat{\beta}_k x_{k,t}\right) = \varepsilon_t$$

$$\hat{\varepsilon}_t = \varepsilon_t$$

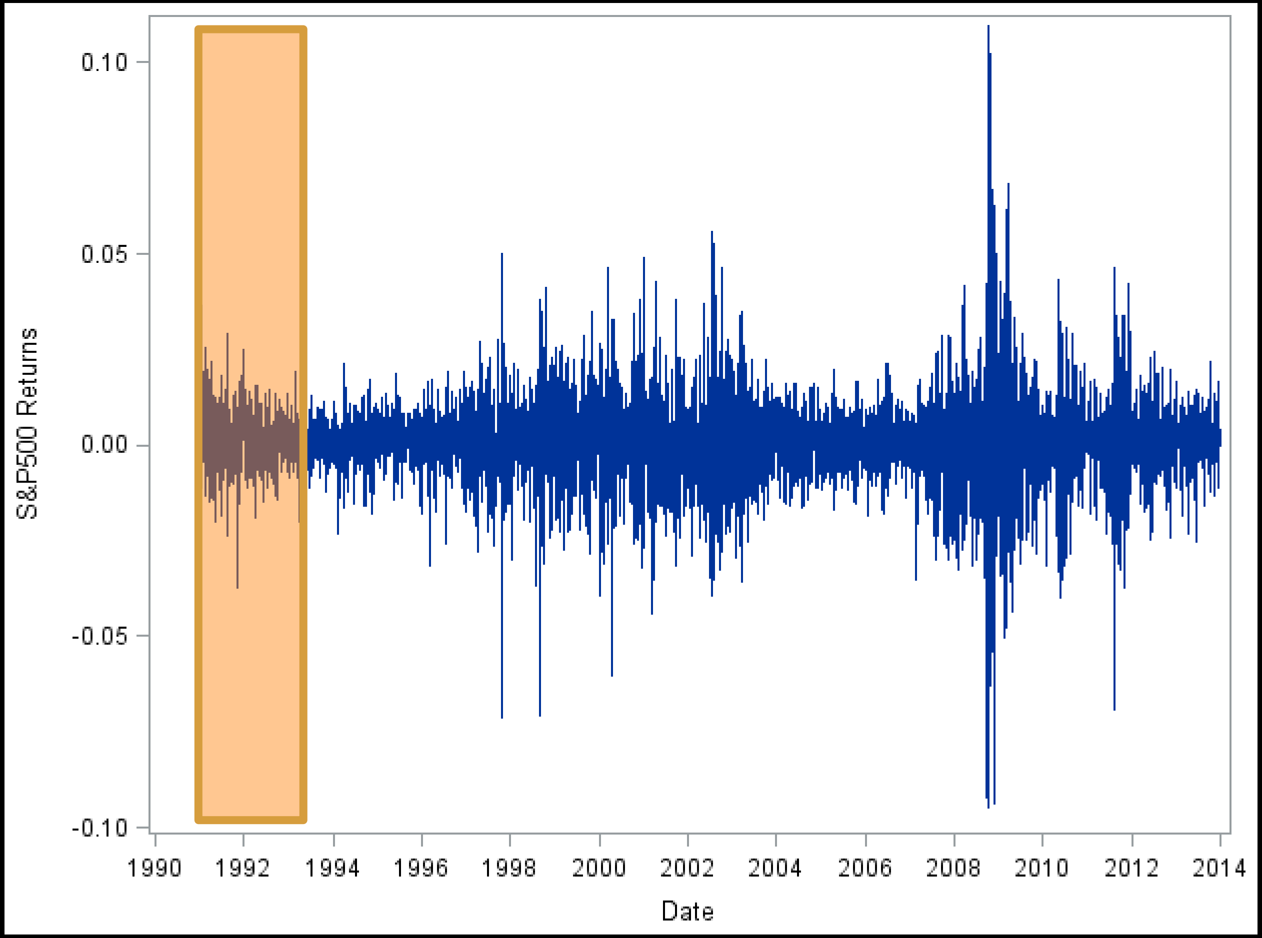
MODEL THIS!

$$\hat{\varepsilon}_t \sim N\left(0, \sigma_t^2\right)$$

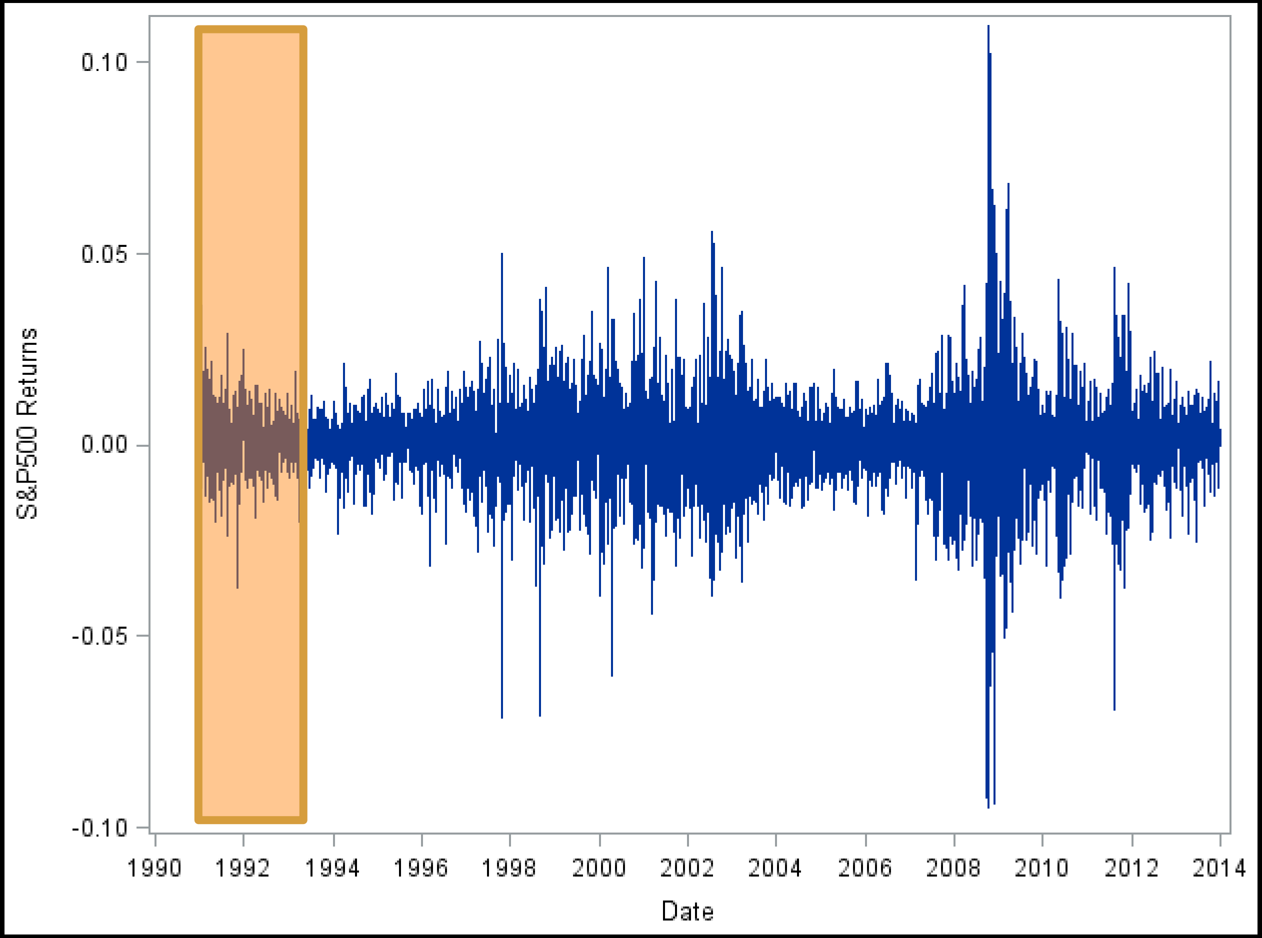
Rolling Window Calculation



Rolling Window Calculation



Rolling Window Calculation



Weighting Time Periods

- Why not weight more recent observations heavier than previous ones?
- Exponential Smoothing Models

$$\sigma_{t+1}^2 = \omega r_t^2 + (1 - \omega)\hat{\sigma}_t^2$$

Weighting Time Periods

- Why not weight more recent observations heavier than previous ones?
- Exponential Smoothing Models

$$\sigma_{t+1}^2 = \omega r_t^2 + (1-\omega)\hat{\sigma}_t^2$$

Volatility Tomorrow

"Actual" Today

Estimated Today

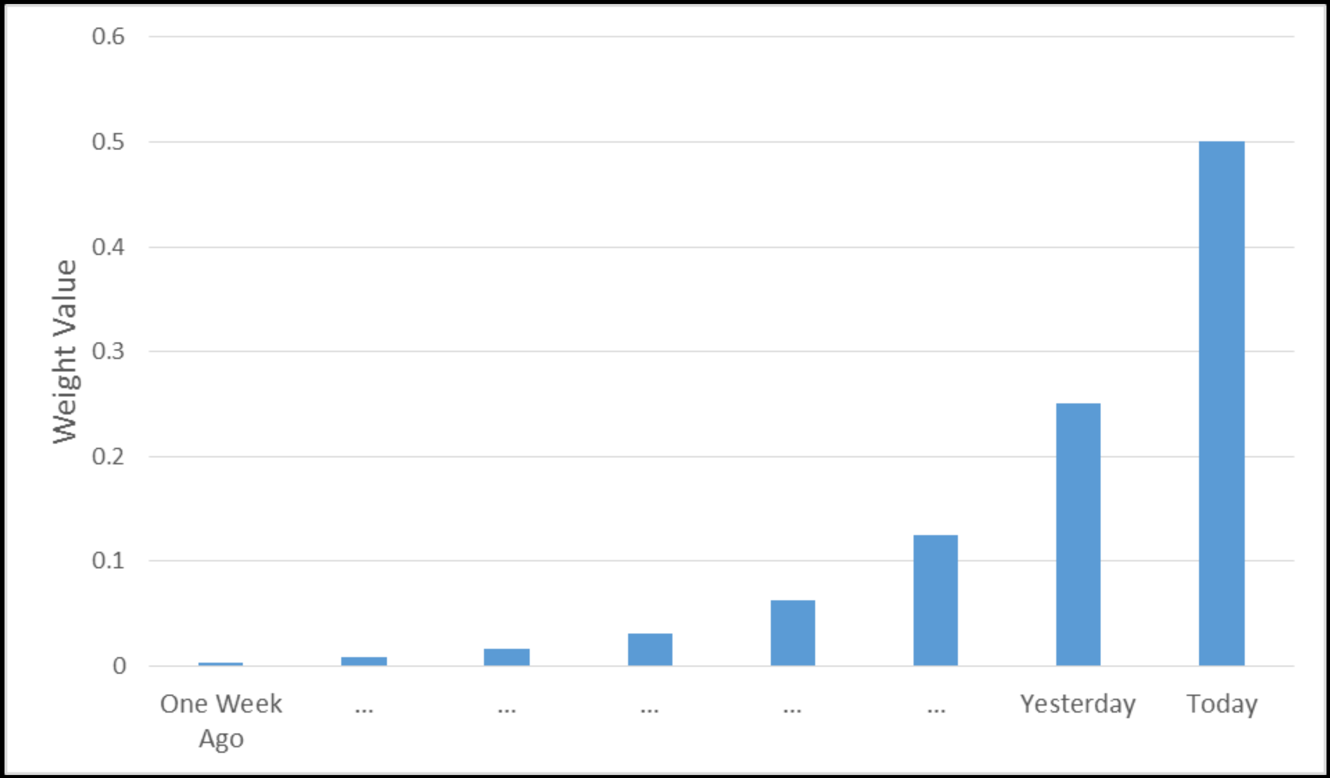
Weighting Time Periods

- Why not weight more recent observations heavier than previous ones?
- Exponential Smoothing Models

$$\sigma_{t+1}^2 = \omega r_t^2 + (1 - \omega)\hat{\sigma}_t^2$$

Weight

Weighting Time Periods – $\omega = 0.5$



HOW TO MODEL VOLATILITY?

Not as Simple Approaches

Time Series Framework

- Weighted average of the volatility
  - Higher weights on the recent past
  - Small but non-zero weights on the distant past

- Choose weights with “ARIMA-like” approaches.

AutoRegressive Conditional Heteroscedasticity (ARCH) Models

- Autoregressive time series approach to modeling volatility.
- Trying to account for time dependency and persistence.

$$\sigma_{t+1}^2 = \alpha_0 + \alpha_1 r_t^2$$

AutoRegressive Conditional Heteroscedasticity (ARCH) Models

- Autoregressive time series approach to modeling volatility.
- Trying to account for time dependency and persistence.

$$\sigma_{t+1}^2 = \alpha_0 + \alpha_1 r_t^2$$

Volatility Tomorrow

"Actual" Today

AutoRegressive Conditional Heteroscedasticity (ARCH) Models

- Autoregressive time series approach to modeling volatility.
- Trying to account for time dependency and persistence.

$$\sigma_{t+1}^2 = \alpha_0 + \alpha_1 r_t^2 + \alpha_2 r_{t-1}^2 + \cdots + \alpha_q r_{t-q+1}^2$$

"Actual" Today

"Actual" Yesterday

"Actual" (q-1) Days Ago

AutoRegressive Conditional Heteroscedasticity (ARCH) Models

- Autoregressive time series approach to modeling volatility.
- Trying to account for time dependency and persistence.

$$\sigma_{t+1}^2 = \alpha_0 + \alpha_1 r_t^2 + \alpha_2 r_{t-1}^2 + \cdots + \alpha_q r_{t-q+1}^2$$

"Actual" Today

"Actual" Yesterday

"Actual" (q-1) Days Ago

- Variances need to be positive, $\alpha_0 > 0,\ \alpha_i \geq 0$
- Model is a stationary one: $\sum_{i=1}^{q} \alpha_i < 1$

AutoRegressive Conditional Heteroscedasticity (ARCH) Models

- Autoregressive time series approach to modeling volatility.
- Trying to account for time dependency and persistence.

$$\sigma_{t+1}^2 = \alpha_0 + \alpha_1 r_t^2 + \alpha_2 r_{t-1}^2 + \cdots + \alpha_q r_{t-q+1}^2$$

Real World Data
Needs LARGE q!

Generalized ARCH (GARCH) Models

- Generalize the ARCH model
  - Similar to autoregressive (AR) model extends to the autoregressive moving average model (ARMA).

$$\sigma_{t+1}^2 = \alpha_0 + \alpha_1 r_t^2 + \beta_1 \hat{\sigma}_t^2$$

Generalized ARCH (GARCH) Models

- Generalize the ARCH model
  - Similar to autoregressive (AR) model extends to the autoregressive moving average model (ARMA).

$$\sigma_{t+1}^2 = \alpha_0 + \alpha_1 r_t^2 + \beta_1 \hat{\sigma}_t^2$$

Volatility Tomorrow

“Actual” Today

Forecasted Today

GARCH(1,1) Model: Restrictions

- Given that σ_t^2 is a variance, it needs to be positive:
  - $\alpha_0 > 0$
  - $\alpha_1 > 0,\ \beta_1 > 0$
- Stationary model:
  - $0 < \alpha_1 + \beta_1 < 1$

Alternative Representation

- A GARCH(1,1) model is an ARMA(1,1) model on the squared returns (or residuals)!

$$\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2 + \beta_1 \hat{\sigma}_{t-1}^2$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2 + \beta_1 \hat{\sigma}_{t-1}^2 + \beta_1 r_{t-1}^2 - \beta_1 r_{t-1}^2$$

$$\sigma_t^2 = \alpha_0 + (\alpha_1 + \beta_1) r_{t-1}^2 - \beta_1 \left(r_{t-1}^2 - \hat{\sigma}_{t-1}^2 \right)$$

$$\sigma_t^2 + r_t^2 = \alpha_0 + (\alpha_1 + \beta_1) r_{t-1}^2 - \beta_1 \left(r_{t-1}^2 - \hat{\sigma}_{t-1}^2 \right) + r_t^2$$

$$r_t^2 = \alpha_0 + (\alpha_1 + \beta_1) r_{t-1}^2 - \beta_1 \left(r_{t-1}^2 - \hat{\sigma}_{t-1}^2 \right) + (r_t^2 - \hat{\sigma}_t^2)$$

Alternative Representation

- A GARCH(1,1) model is an ARMA(1,1) model on the squared returns (or residuals)!

$$\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2 + \beta_1 \hat{\sigma}_{t-1}^2$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2 + \beta_1 \hat{\sigma}_{t-1}^2 + \beta_1 r_{t-1}^2 - \beta_1 r_{t-1}^2$$

$$\sigma_t^2 = \alpha_0 + (\alpha_1 + \beta_1) r_{t-1}^2 - \beta_1 \left(r_{t-1}^2 - \hat{\sigma}_{t-1}^2 \right)$$

$$\sigma_t^2 + r_t^2 = \alpha_0 + (\alpha_1 + \beta_1) r_{t-1}^2 - \beta_1 \left(r_{t-1}^2 - \hat{\sigma}_{t-1}^2 \right) + r_t^2$$

$$r_t^2 = \alpha_0 + (\alpha_1 + \beta_1) r_{t-1}^2 - \beta_1 \left(r_{t-1}^2 - \hat{\sigma}_{t-1}^2 \right) + \left(r_t^2 - \hat{\sigma}_t^2 \right)$$

Alternative Representation

- A GARCH(1,1) model is an ARMA(1,1) model on the squared returns (or residuals)!

$$\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2 + \beta_1 \hat{\sigma}_{t-1}^2$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2 + \beta_1 \hat{\sigma}_{t-1}^2 + \beta_1 r_{t-1}^2 - \beta_1 r_{t-1}^2$$

$$\sigma_t^2 = \alpha_0 + (\alpha_1 + \beta_1) r_{t-1}^2 - \beta_1 \left(r_{t-1}^2 - \hat{\sigma}_{t-1}^2 \right)$$

$$\sigma_t^2 + r_t^2 = \alpha_0 + (\alpha_1 + \beta_1) r_{t-1}^2 - \beta_1 \left(r_{t-1}^2 - \hat{\sigma}_{t-1}^2 \right) + r_t^2$$

$$r_t^2 = \alpha_0 + \delta_1 r_{t-1}^2 - \beta_1 u_{t-1} + u_t$$

Generalized ARCH (GARCH) Models

- Generalize the ARCH model
  - Similar to autoregressive (AR) model extends to the autoregressive moving average model (ARMA).

$$\sigma_{t+1}^2 = \alpha_0 + \alpha_1 r_t^2 + \beta_1 \hat{\sigma}_t^2$$

Generalized ARCH (GARCH) Models

- Generalize the ARCH model
  - Similar to autoregressive (AR) model extends to the autoregressive moving average model (ARMA).

$$\sigma_{t+1}^2 = \alpha_0 + \sum_{i=0}^{q} \alpha_i r_{t-i}^2 + \sum_{j=0}^{p} \beta_j \hat{\sigma}_{t-j}^2$$

Generalized ARCH (GARCH) Models

- Generalize the ARCH model
  - Similar to autoregressive (AR) model extends to the autoregressive moving average model (ARMA).

$$\sigma_{t+1}^2 = \alpha_0 + \alpha_1 r_t^2 + \beta_1 \hat{\sigma}_t^2$$

Real World Data "Typically" Only
Needs One of Each

Interpretations

- If we need to “force” the constraints about α_0, α_1, and β_1 its possible that our model is not appropriate and some other GARCH-type model should be used.

- The parameter α_i measures the reaction of conditional volatility to market shocks.
  - Large values (above 0.1) imply volatility is very sensitive to market events.
- The parameter β_i measures the persistence in conditional volatility.
  - Large values (above 0.9) imply volatility takes a long time to die out following a crisis in the market.

Interpretations

- The $(\alpha_i + \beta_i)$ determines the rate of convergence of the conditional volatility to the long term average level.
  - Large values (above 0.99) imply the terms structure of the volatility forecasts from the GARCH model is relatively flat.

- The constant α_0 together with $(\alpha_i + \beta_i)$ determines the level of the long term average volatility (the unconditional variance in the GARCH model).
  - The larger the value, the higher the long term volatility in the market.

TESTING FOR ARCH EFFECTS

Testing for ARCH Effects

- Just like in time series where we test for autocorrelations, we can also test for different ARCH effects, which are similar to autocorrelations across the squared residuals.
- There are two common tests for ARCH effects:
  - Lagrange Multiplier (LM) test
  - Portmanteau Q test

Testing for ARCH Effects

- Just like in time series where we test for autocorrelations, we can also test for different ARCH effects, which are similar to autocorrelations across the squared residuals.
- There are two common tests for ARCH effects:
  - Lagrange Multiplier (LM) test
  - Portmanteau Q test

- Null hypothesis for both tests is the same:

$$H_0: \alpha_1 = \alpha_2 = \cdots = \alpha_q = 0$$

Test for ARCH Effects – SAS

```
proc autoreg data=Stocks all plots(unpack);
   model msft_r = / archtest;
run;
```

Test for ARCH Effects – SAS

Tests for ARCH Disturbances Based on OLS Residuals

Order	Q	Pr > Q	LM	Pr > LM
1	74.7445	<.0001	73.9627	<.0001
2	130.8748	<.0001	112.1915	<.0001
3	162.6548	<.0001	125.6564	<.0001
4	210.0613	<.0001	149.3698	<.0001
5	284.1277	<.0001	187.2482	<.0001
6	329.7092	<.0001	199.1051	<.0001
7	392.3084	<.0001	219.8552	<.0001
8	418.4616	<.0001	221.6982	<.0001
9	452.1992	<.0001	226.4152	<.0001
10	550.9404	<.0001	267.5159	<.0001
11	636.4519	<.0001	290.2650	<.0001
12	665.1030	<.0001	290.3338	<.0001

Test for ARCH Effects – R

```
arch.test(arima(stocks$msft_r[-1], order = c(0,0,0)), output = TRUE)
```

```
ARCH heteroscedasticity test for residuals
alternative: heteroscedastic

Portmanteau-Q test:
     order  PQ p.value
[1,]     4 208       0
[2,]     8 413       0
[3,]    12 657       0
[4,]    16 716       0
[5,]    20 829       0
[6,]    24 911       0
Lagrange-Multiplier test:
     order   LM p.value
[1,]     4 5001       0
[2,]     8 1942       0
[3,]    12 1097       0
[4,]    16  810       0
[5,]    20  616       0
[6,]    24  508       0
>
```

GARCH Model – SAS

```
proc autoreg data=Stocks OUTEST=param_estimates;
   model msft_r = / noint garch=(p=1, q=1)
                    method=ml;
   output out=garch_n ht=predicted_var;
run;
```

GARCH Model – SAS

The AUTOREG Procedure

Ordinary Least Squares Estimates			
SSE	0.89945877	DFE	3018
MSE	0.0002980	Root MSE	0.01726
SBC	-15936.352	AIC	-15936.352
MAE	0.01161738	AICC	-15936.352
MAPE	101.145797	HQC	-15936.352
Durbin-Watson	2.1353	Total R-Square	0.0000
NOTE: No intercept term is used. R-squares are redefined.			

Algorithm converged.

GARCH Model – SAS

The AUTOREG Procedure

GARCH Estimates			
SSE	0.89945877	Observations	3018
MSE	0.0002980	Uncond Var	0.0002958
Log Likelihood	8295.74749	Total R-Square	0.0000
SBC	-16567.458	AIC	-16585.495
MAE	0.01161983	AICC	-16585.487
MAPE	100	HQC	-16579.009
		Normality Test	5832.2647
		Pr > ChiSq	<.0001
NOTE: No intercept term is used. R-squares are redefined.			

Parameter Estimates

Variable	DF	Estimate	Standard Error	t Value	Approx Pr > |t|
ARCH0	1	4.523E-6	5.11E-7	8.85	<.0001
ARCH1	1	0.0495	0.003517	14.08	<.0001
GARCH1	1	0.9352	0.004417	211.75	<.0001

GARCH Model – R

```
# Estimate Different GARCH Models #
GARCH.N <- garchFit(formula= ~ garch(1,1), data=stocks$msft_r[-1],
                    cond.dist="norm", include.mean = FALSE)
summary(GARCH.N)
```

GARCH Model – R

```
# Estimate Different GARCH Models #
GARCH.N <- garchFit(formula= ~ garch(1,1), data=stocks$msft_r[-1],
                    cond.dist="norm", include.mean = FALSE)

summary(GARCH.N)
```

GARCH Model – R

```
Call:
 garchFit(formula = ~garch(1, 1), data = stocks$msft_r[-1], con
orm",
    include.mean = FALSE)

Mean and Variance Equation:
 data ~ garch(1, 1)
<environment: 0x000001abd5ebbee0>
 [data = stocks$msft_r[-1]]

Conditional Distribution:
 norm

Coefficient(s):
     omega      alpha1       beta1
4.5081e-06  4.9738e-02  9.3519e-01

Std. Errors:
 based on Hessian

Error Analysis:
        Estimate  Std. Error  t value Pr(>|t|)
omega  4.508e-06   1.874e-06    2.405 0.016163 *
alpha1 4.974e-02   1.287e-02    3.863 0.000112 ***
beta1  9.352e-01   1.850e-02   50.562  < 2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Log Likelihood:
 8295.746    normalized:  2.748756
```

GARCH Model – R

```
Call:
 garchFit(formula = ~garch(1, 1), data = stocks$msft_r[-1], con
orm",
    include.mean = FALSE)

Mean and Variance Equation:
 data ~ garch(1, 1)
<environment: 0x000001abd5ebbee0>
 [data = stocks$msft_r[-1]]

Conditional Distribution:
 norm

Coefficient(s):
     omega      alpha1       beta1
4.5081e-06  4.9738e-02  9.3519e-01

Std. Errors:
 based on Hessian

Error Analysis:
        Estimate  Std. Error  t value Pr(>|t|)
omega  4.508e-06   1.874e-06    2.405 0.016163 *
alpha1 4.974e-02   1.287e-02    3.863 0.000112 ***
beta1  9.352e-01   1.850e-02   50.562  < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Log Likelihood:
 8295.746    normalized:  2.748756
```

GARCH Model – R

```
Standardised Residuals Tests:
                                   Statistic p-Value
 Jarque-Bera Test   R    Chi^2  5867.137  0
 Shapiro-Wilk Test  R    W      0.9430056 0
 Ljung-Box Test     R    Q(10)  21.29929  0.0191006
 Ljung-Box Test     R    Q(15)  26.13155  0.03666253
 Ljung-Box Test     R    Q(20)  37.35677  0.01060077
 Ljung-Box Test     R^2  Q(10)  2.133788  0.9952081
 Ljung-Box Test     R^2  Q(15)  4.94395   0.9925904
 Ljung-Box Test     R^2  Q(20)  7.459505  0.9948826
 LM Arch Test       R    TR^2   2.373413  0.9985811

Information Criterion Statistics:
      AIC       BIC       SIC      HQIC
-5.495524 -5.489548 -5.495526 -5.493375

>
```

EXTENSIONS TO ARCH/GARCH MODELING

Extensions to GARCH Framework

- What if the distribution is not Normal?
- What if the underlying distribution is not symmetric?
- What if the variance actually affected the value of the return directly?

Extensions to GARCH Framework

- What if the distribution is not Normal?
  - Test Normality with Jarque-Berra (J-B) test.
  - Null hypothesis is that the standardized residuals follow the Normal distribution.
  - Test follows χ_2^2.
- What if the underlying distribution is not symmetric?
- What if the variance actually affected the value of the return directly?

J-B Test of Normality – SAS

```
proc autoreg data=Stocks all plots(unpack);
   model msft_r = / normal;
run;
```

J-B Test of Normality – SAS

Miscellaneous Statistics			
Statistic	Value	Prob	Label
Normal Test	10717.2460	<.0001	Pr > ChiSq

J-B Test of Normality – R

```
jb.test(stocks$msft_r[-1])
```

```
              series 1
test stat 10717.25
p-value        0.00
>
```

Extensions to GARCH Framework

- What if the distribution is not Normal?
  - Bollerslev (1986) developed the t-GARCH model that has an underlying t-distribution instead of a Normal distribution.
- What if the underlying distribution is not symmetric?
- What if the variance actually affected the value of the return directly?

t-GARCH – SAS

```
proc autoreg data=Stocks OUTEST=param_estimates;
   model msft_r = / noint garch=(p=1, q=1)
                    dist=t method=ml;
   output out=garch_t ht=predicted_var;
run;
```

t-GARCH – SAS

Parameter Estimates						
Variable	**DF**	**Estimate**	**Standard Error**	**t Value**	**Approx Pr > |t|**	**Variable Label**
ARCH0	1	3.7094E-6	1.1E-6	3.37	0.0007	
ARCH1	1	0.0625	0.009735	6.42	<.0001	
GARCH1	1	0.9279	0.0103	90.43	<.0001	
TDFI	1	0.2391	0.0177	13.48	<.0001	Inverse of t DF

t-GARCH – R

```
GARCH.t <- garchFit(formula= ~ garch(1,1), data=stocks$msft_r[-1],
                    cond.dist="std", include.mean = FALSE)
summary(GARCH.t)
```

t-GARCH – R

```
GARCH.t <- garchFit(formula= ~ garch(1,1), data=stocks$msft_r[-1],
                    cond.dist="std", include.mean = FALSE)
summary(GARCH.t)
```

```
Error Analysis:
        Estimate  Std. Error  t value Pr(>|t|)
omega  3.688e-06   1.342e-06    2.748  0.00599 **
alpha1 6.287e-02   1.219e-02    5.156 2.52e-07 ***
beta1  9.280e-01   1.381e-02   67.181  < 2e-16 ***
shape  4.164e+00   3.240e-01   12.851  < 2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Log Likelihood:
 8527.507    normalized:  2.825549
```

Normal GARCH or t-GARCH

- Formal test for Normality.
- Compare using Information criteria.
- Formal test for d.f. of t-distribution.
  - Likelihood Ratio (LR) test to see if inverse of d.f. ≈ 0

Extensions to GARCH Framework

- What if the distribution is not Normal?
- What if the underlying distribution is not symmetric?
  - Nelson (1991) developed the EGARCH model to account for the **leverage effect** in certain data sets.
  - The leverage effect is when variance increases more when a return is negative compared to when a return is positive.
- What if the variance actually affected the value of the return directly?

EGARCH – SAS

```
proc autoreg data=Stocks OUTEST=param_estimates;
   model msft_r = / noint garch=(p=1, q=1,
                              type=exp) method=ml;
   output out=egarch ht=predicted_var;
run;
```

EGARCH – SAS

Parameter Estimates					
Variable	DF	Estimate	Standard Error	t Value	Approx Pr > |t|
EARCH0	1	-0.1260	0.0144	-8.75	<.0001
EARCH1	1	0.1003	0.006410	15.65	<.0001
EGARCH1	1	0.9837	0.001767	556.78	<.0001
THETA	1	-0.3457	0.0630	-5.49	<.0001

EGARCH – R

```
EGARCH <- ugarchfit(data = stocks$msft_r[-1],
                    spec = ugarchspec(variance.model = list(model = "eGARCH",
                                                            garchOrder = c(1,1))))
EGARCH
```

EGARCH – R

```
*---------------------------------*
*          GARCH Model Fit        *
*---------------------------------*

Conditional Variance Dynamics
-----------------------------------
GARCH Model      : eGARCH(1,1)
Mean Model       : ARFIMA(1,0,1)
Distribution     : norm

Optimal Parameters
------------------------------------
        Estimate  Std. Error   t value Pr(>|t|)
mu      0.000425    0.000214    1.9824 0.047434
ar1     0.782657    0.043066   18.1735 0.000000
ma1    -0.822819    0.039414  -20.8765 0.000000
omega  -0.138585    0.002286  -60.6263 0.000000
alpha1 -0.025584    0.006879   -3.7191 0.000200
beta1   0.982215    0.000229 4289.9173 0.000000
gamma1  0.108548    0.004829   22.4778 0.000000

Robust Standard Errors:
        Estimate  Std. Error   t value Pr(>|t|)
mu      0.000425    0.000229    1.8539 0.063748
ar1     0.782657    0.023923   32.7162 0.000000
ma1    -0.822819    0.019510  -42.1739 0.000000
omega  -0.138585    0.005463  -25.3699 0.000000
alpha1 -0.025584    0.015675   -1.6322 0.102634
beta1   0.982215    0.000544 1806.8671 0.000000
gamma1  0.108548    0.009359   11.5984 0.000000

LogLikelihood : 8317.322
```

EGARCH – R

```
*---------------------------------*
*          GARCH Model Fit        *
*---------------------------------*

Conditional Variance Dynamics
-----------------------------------
GARCH Model       : eGARCH(1,1)
Mean Model        : ARFIMA(1,0,1)
Distribution      : norm

Optimal Parameters
------------------------------------
        Estimate  Std. Error   t value Pr(>|t|)
mu      0.000425    0.000214    1.9824 0.047434
ar1     0.782657    0.043066   18.1735 0.000000
ma1    -0.822819    0.039414  -20.8765 0.000000
omega  -0.138585    0.002286  -60.6263 0.000000
alpha1 -0.025584    0.006879   -3.7191 0.000200
beta1   0.982215    0.000229 4289.9173 0.000000
gamma1  0.108548    0.004829   22.4778 0.000000

Robust Standard Errors:
        Estimate  Std. Error   t value Pr(>|t|)
mu      0.000425    0.000229    1.8539 0.063748
ar1     0.782657    0.023923   32.7162 0.000000
ma1    -0.822819    0.019510  -42.1739 0.000000
omega  -0.138585    0.005463  -25.3699 0.000000
alpha1 -0.025584    0.015675   -1.6322 0.102634
beta1   0.982215    0.000544 1806.8671 0.000000
gamma1  0.108548    0.009359   11.5984 0.000000

LogLikelihood : 8317.322
```

QGARCH Model

- An alternative model to capture the asymmetries and leverage effects.
- Introduces λ into model.

$$\sigma_t^2 = \alpha_0 + \alpha_1(\varepsilon_t - \lambda)^2 + \beta_1 \sigma_{t-1}^2$$

QGARCH – SAS

```
proc autoreg data=Stocks OUTEST=param_estimates;
   model msft_r = / noint garch=(p=1, q=1,
                    type=QGARCH) method=ml;
   output out=qgarch ht=predicted_var;

   model msft_r = / noint garch=(p=1, q=1,
                    type=QGARCH) dist=t method=ml;
   output out=qgarch_t ht=predicted_var;

run;
```

QGARCH – SAS

Parameter Estimates						
Variable	DF	Estimate	Standard Error	t Value	Approx Pr > |t|	Variable Label
QARCHA0	1	0	0	.	.	
QARCHA1	1	0.0662	0.009916	6.67	<.0001	
QARCHB1	1	0.009745	0.001019	9.57	<.0001	
QGARCH1	1	0.9170	0.0100	91.42	<.0001	
TDFI	1	0.2359	0.0168	14.07	<.0001	Inverse of t DF

Skewed GARCH – R

```
Skew.GARCH.N <- garchFit(formula= ~ garch(1,1), data=stocks$msft_r[-1],
                         cond.dist="snorm",include.mean = FALSE)
summary(Skew.GARCH.N)

Skew.GARCH.t <- garchFit(formula= ~ garch(1,1), data=stocks$msft_r[-1],
                         cond.dist="sstd", include.mean = FALSE)
summary(Skew.GARCH.t)
```

Skewed GARCH – R

```
Error Analysis:
        Estimate  Std. Error  t value Pr(>|t|)    
omega  3.720e-06   1.352e-06    2.751  0.00594 ** 
alpha1 6.305e-02   1.225e-02    5.146 2.66e-07 ***
beta1  9.278e-01   1.387e-02   66.878  < 2e-16 ***
skew   9.906e-01   2.122e-02   46.691  < 2e-16 ***
shape  4.159e+00   3.238e-01   12.846  < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Log Likelihood:
 8527.605    normalized:  2.825581
```

Extensions to GARCH Framework

- What if the distribution is not Normal?
- What if the underlying distribution is not symmetric?
- What if the variance actually affected the value of the return directly?
    - Referred to as GARCH-M models, where M stands for mean.

GARCH-M – SAS

```
proc autoreg data=Stocks OUTEST=param_estimates;
   model msft_r = / noint garch=(p=1, q=1,
                    mean=linear) method=ml;
   output out=garch_m ht=predicted_var;
run;
```

GARCH-M – R

```
GARCH.M <- ugarchfit(data = stocks$msft_r[-1],
                     spec = ugarchspec(variance.model=list(model="sGARCH",
                                                           garchOrder=c(1,1)),
                                       mean.model=list(armaOrder=c(0,0))))
GARCH.M
```

Exponentially Weighted Moving Average

- The Exponentially Weighted Moving Average (EWMA) model can be considered a "special" GARCH(1,1) model where:
  - $\alpha_0 = 0$
  - $(\alpha_1 + \beta_1) = 1$
  - RiskMetrics database (J.P. Morgan, 1994) has been using this methodology to forecast volatility.
  - RiskMetrics sets $\beta_1 = 0.94$ since this is the value that seems to produce the best out of sample forecasts.

Exponentially Weighted Moving Average

- The Exponentially Weighted Moving Average (EWMA) model can be considered a “special” GARCH(1,1) model where:
  - $\alpha_0 = 0$
  - $(\alpha_1 + \beta_1) = 1$ Nonstationary (called IGARCH)
  - RiskMetrics database (J.P. Morgan, 1994) has been using this methodology to forecast volatility.
  - RiskMetrics sets $\beta_1 = 0.94$ since this is the value that seems to produce the best out of sample forecasts.

EWMA – SAS

```
proc autoreg data=Stocks OUTEST=param_estimates;
   model msft_r = / noint garch=(p=1, q=1,
                    type=integ, noint)
                    method=ml;
   output out=ewma ht=predicted_var;
run;
```

EWMA – SAS

Parameter Estimates					
Variable	DF	Estimate	Standard Error	t Value	Approx Pr > |t|
ARCH1	1	1.0537E-8	0	Infty	<.0001
GARCH1	1	1.0000	0	Infty	<.0001

EWMA – R

```
EWMA <- ugarchfit(data = stocks$msft_r[-1],
                  spec = ugarchspec(variance.model=list(model="iGARCH",
                                                        garchOrder=c(1,1),
                                                        variance.targeting=0),
                                    mean.model=list(armaOrder=c(0,0))))
EWMA
```

EWMA – R

```
*---------------------------------*
*          GARCH Model Fit        *
*---------------------------------*

Conditional Variance Dynamics
-----------------------------------
GARCH Model     : iGARCH(1,1)
Mean Model      : ARFIMA(0,0,0)
Distribution    : norm

Optimal Parameters
------------------------------------
        Estimate  Std. Error  t value Pr(>|t|)
mu      0.000591    0.000269   2.1961 0.028081
alpha1  0.029232    0.002559  11.4226 0.000000
beta1   0.970768          NA       NA       NA
omega   0.000000          NA       NA       NA

Robust Standard Errors:
        Estimate  Std. Error  t value Pr(>|t|)
mu      0.000591    0.000339   1.7428 0.081369
alpha1  0.029232    0.005749   5.0843 0.000000
beta1   0.970768          NA       NA       NA
omega   0.000000          NA       NA       NA

LogLikelihood : 8263.003
```

Many, Many, Many GARCH Models

- AARCH
- ADCC-GARCH
- AGARCH
- ANN-ARCH
- ANST-GARCH
- APARCH
- ARCH-M
- ARCH-SM
- ATGARCH
- Aug-GARCH
- AVGARCH
- B-GARCH
- BEKK-GARCH
- CCC-GARCH
- Censored-GARCH
- CGARCH
- COGARCH
- CorrARCH
- DAGARCH
- DCC-GARCH
- Diag MGARCH
- DTARCH
- DVEC-GARCH
- EGARCH
- EVT-GARCH
- F-ARCH
- FDCC-GARCH
- FGARCH
- FIAPARCH
- FIEGARCH
- FIGARCH
- FIREGARCH
- Flex-GARCH
- GAARCH
- GARCH-Delta
- GARCH Diffusion
- GARCH-EAR
- GARCH-Gamma
- GARCH-M
- GARCHS
- GARCHSK
- GARCH-t
- GARCH-X
- GARCHX
- GARJI
- GDCC-GARCH
- GED-GARCH
- GJR-GARCH
- GO-GARCH
- GQARCH
- GQTARCH
- HARCH
- HGARCH
- HYGARCH
- IGARCH
- LARCH
- Latent GARCH
- Level GARCH
- LGARCH
- LMGARCH
- Log-GARCH
- MAR-ARCH
- MARCH
- Matrix EGARCH
- MGARCH
- Mixture GARCH
- MS-GARCH
- MV-GARCH
- NAGARCH
- NGARCH
- NL-GARCH
- NM-GARCH
- OGARCH
- PARCH
- PC-GARCH
- PGARCH
- PNP-GARCH
- QARCH
- QTARCH
- REGARCH
- RGARCH
- Robust GARCH
- Root GARCH
- RS-GARCH
- Robust DCC-GARCH
- SGARCH
- S-GARCH
- Sign-GARCH
- SPARCH
- Spline-GARCH
- SQR-GARCH
- STARCH
- Stdev-ARCH
- STGARCH
- Structural GARCH
- Strong GARCH
- SWARCH
- TGARCH
- t-GARCH
- Tobit-GARCH
- TS-GARCH
- UGARCH
- VCC-GARCH
- VGARCH
- VSGARCH
- Weak GARCH
- ZARCH

Many, Many, Many GARCH Models

- AARCH
- ADCC-GARCH
- AGARCH
- ANN-ARCH
- ANST-GARCH
- APARCH
- ARCH-M
- ARCH-SM
- ATGARCH
- Aug-GARCH
- AVGARCH
- B-GARCH
- BEKK-GARCH
- CCC-GARCH
- Censored-GARCH
- CGARCH
- COGARCH
- CorrARCH
- DAGARCH
- DCC-GARCH
- Diag MGARCH
- DTARCH
- DVEC-GARCH
- EGARCH
- EVT-GARCH
- F-ARCH
- FDCC-GARCH
- FGARCH
- FIAPARCH
- FIEGARCH
- FIGARCH
- FIREGARCH
- Flex-GARCH
- GAARCH
- GARCH-Delta
- GARCH Diffusion
- GARCH-EAR
- GARCH-Gamma
- GARCH-M
- GARCHS
- GARCHSK
- GARCH-t
- GARCH-X
- GARCHX
- GARJI
- GDCC-GARCH
- GED-GARCH
- GJR-GARCH
- GO-GARCH
- GQARCH
- GQTARCH
- HARCH
- HGARCH
- HYGARCH
- IGARCH
- LARCH
- Latent GARCH
- Level GARCH
- LGARCH
- LMGARCH
- Log-GARCH
- MAR-ARCH
- MARCH
- Matrix EGARCH
- MGARCH
- Mixture GARCH
- MS-GARCH
- MV-GARCH
- NAGARCH
- NGARCH
- NL-GARCH
- NM-GARCH
- OGARCH
- PARCH
- PC-GARCH
- PGARCH
- PNP-GARCH
- QARCH
- QTARCH
- REGARCH
- RGARCH
- Robust GARCH
- Root GARCH
- RS-GARCH
- Robust DCC-GARCH
- SGARCH
- S-GARCH
- Sign-GARCH
- SPARCH
- Spline-GARCH
- SQR-GARCH
- STARCH
- Stdev-ARCH
- STGARCH
- Structural GARCH
- Strong GARCH
- SWARCH
- TGARCH
- t-GARCH
- Tobit-GARCH
- TS-GARCH
- UGARCH
- VCC-GARCH
- VGARCH
- VSGARCH
- Weak GARCH
- ZARCH

Every day, self-proclaimed stock market "experts" tell us why the market just went up or down, as if they really knew. So where were they yesterday?!?

- Anonymous