FACTOR MODELS

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Introduction

- What is a factor model?
  - A model that describes the relationship between the expected return (r_t) of a portfolio and a set of market risk factors.
- Some of examples of risk factors:
  - Market Indices
  - Interest Rates
  - Exchange Rates
  - Inflation
  - Principal Components

Introduction

- What is a factor model?
  - A model that describes the relationship between the expected return (r_t) of a portfolio and a set of market risk factors.
- Single factor models: Expected return is a function of only one factor (simple linear regression)
- Multiple factor models: Expected return is a function of multiple factors (multiple linear regression)

Estimation Techniques

- The relationship between market factors and portfolio return is usually estimated through OLS.
- What if the relationship is time varying?
  - Exponentially Weighted Moving Average (EWMA)
  - Generalized Autoregressive Conditional Heteroscedasticity (GARCH)

?

CAPITAL ASSET PRICING MODEL (CAPM)

Background

- Factor Models are based on the Capital Asset Pricing Model (CAPM) equation:

$$E(R_i) - R_f = \beta_i\left(E(R_M) - R_f\right)$$

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Expected return of asset (or portfolio) of interest

Risk free rate of return (e.g. 3-month T-bill rate)

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Expected return of **market portfolio** (e.g. return of market index typically used)

Risk free rate of return (e.g. 3-month T-bill rate)

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Excess return of asset (or portfolio) of interest

Excess return of market

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$$E(R_i) - R_f = \beta_i\left(E(R_M) - R_f\right)$$

Relationship between individual asset and market

$\beta_i > 1$: Asset is riskier than the market

$\beta_i < 1$: Market is riskier than the asset

$\beta_i = 1$: Asset and market have same risk

Single Factor Models

- CAPM models with a single factor have the following representation:

$$R_{i,t} = \alpha_i + \beta_i X_t + \varepsilon_{i,t}$$

$$\varepsilon_{i,t} \sim iid\left(0, \sigma_i^2\right)$$

- X_t is the return on the market index during period *t*
- α_i is the assets' excess return, relative to the index
- σ_i is the "specific risk" of the asset
- $\beta_i \sigma_x$ is the "systematic risk" (market-risk) of the asset

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Obsession of asset managers!

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TYPES OF RISK

Usage

- There are two main uses for factor models:
  1. Relate the performance of an asset (or portfolio of them) to the state of the economy.
  2. Allows to identify the systematic and specific risk of an asset (or portfolio of them).

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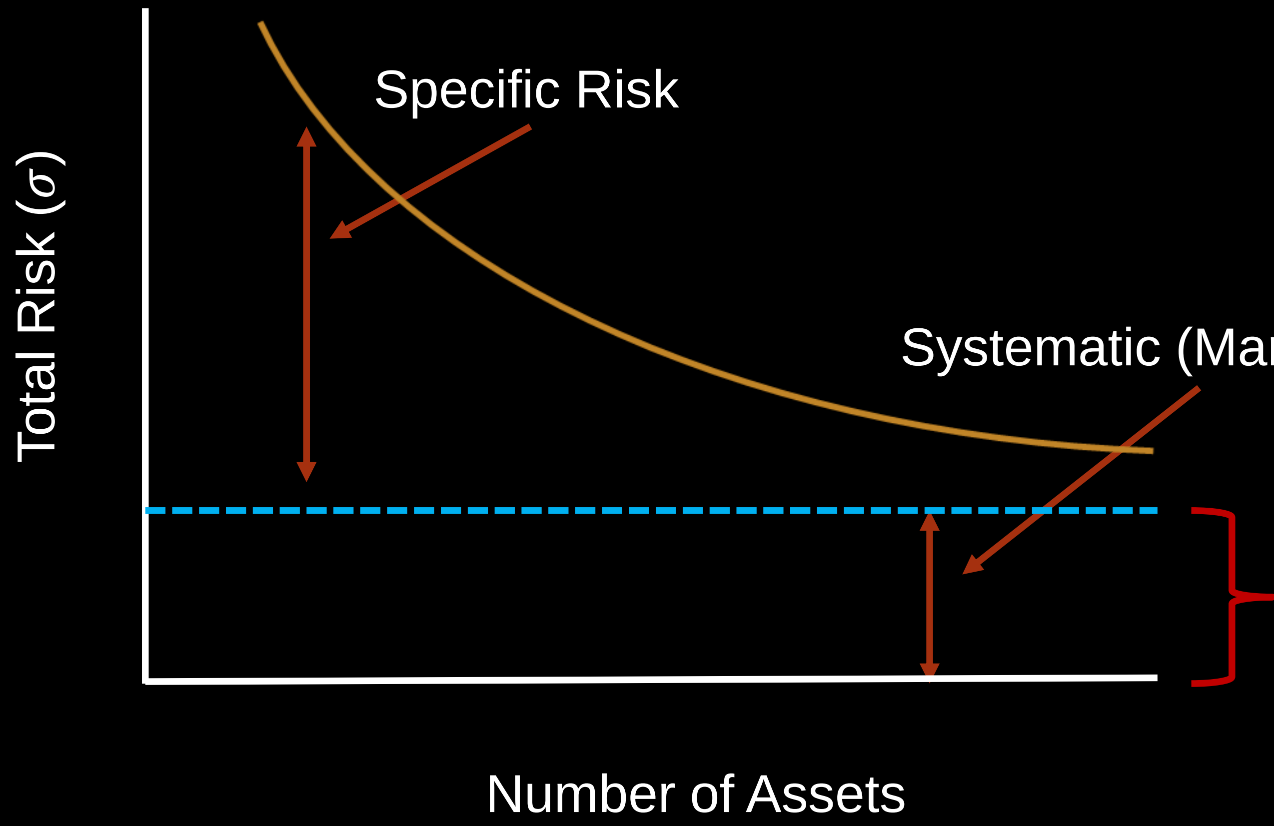
  1. Relate the performance of an asset (or portfolio of them) to the state of the economy.
     - Stress testing and scenario testing
     - Example – how would the portfolio behave if the whole market drops 10%?
  2. Allows to identify the systematic and specific risk of an asset (or portfolio of them).

Usage

- There are two main uses for factor models:
  1. Relate the performance of an asset (or portfolio of them) to the state of the economy.
  2. Allows to identify the systematic and specific risk of an asset (or portfolio of them).
     - **Systematic Risk** (un-diversifiable risk) – risk that **cannot** be reduced to 0, even if we hold a very broad and diversified portfolio (e.g. inherent risk of the market).
     - **Specific Risk** (idiosyncratic risk or residual risk) – risk that **can** be reduced to almost 0 by holding a very broad and diversified portfolio (e.g. specific risk of asset).

Systematic vs. Specific Risk



Systematic Risk cannot be eliminated from the portfolio by investing in more or different assets.

Systematic vs. Specific Risk

- Think about these two risks in terms of statistical concepts.
- Suppose you run an OLS regression of the portfolio return on a set of market factors (e.g. Dow Jones Index, US Inflation, 1-year T-bill rate, etc.).
  - Systematic risk – risk (standard deviation) of the market factors and the portfolio sensitivities to each risk factor (beta coefficients).
  - Specific risk – variance of the residuals.

Single Factor Model – Example

```
# Load Stock Data & Calculate Returns#
tickers = c("AAPL", "MSFT","EBAY","GOOGL", "^DJI")

getSymbols(tickers)

stocks_w <- cbind(last(to.weekly(AAPL)[,4], "5 years"),
                  last(to.weekly(MSFT)[,4], "5 years"),
                  last(to.weekly(EBAY)[,4], "5 years"),
                  last(to.weekly(GOOGL)[,4], "5 years"),
                  last(to.weekly(DJI)[,4], "5 years"))

stocks_w$msft_r <- periodReturn(stocks_w$MSFT.Close, period = "weekly")
stocks_w$aapl_r <- periodReturn(stocks_w$AAPL.Close, period = "weekly")
stocks_w$ebay_r <- periodReturn(stocks_w$EBAY.Close, period = "weekly")
stocks_w$googl_r <- periodReturn(stocks_w$GOOGL.Close, period = "weekly")
stocks_w$dji_r <- periodReturn(stocks_w$DJI.Close, period = "weekly")
```

Single Factor Model – Example

```
> # Single Factor Models #
> results_aapl <- lm(stocks_w$aapl_r ~ stocks_w$dji_r)
> summary(results_aapl)

Call:
lm(formula = stocks_w$aapl_r ~ stocks_w$dji_r)

Residuals:
      Min        1Q    Median        3Q       Max
-0.101887 -0.016169 -0.001555  0.012237  0.132723

Coefficients:
               Estimate Std. Error t value Pr(>|t|)
(Intercept)    0.001816   0.001829   0.993    0.322
stocks_w$dji_r 1.010162   0.100345  10.067   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.0295 on 260 degrees of freedom
Multiple R-squared:  0.2805,    Adjusted R-squared:  0.2777
F-statistic: 101.3 on 1 and 260 DF,  p-value: < 2.2e-16

> anova(results_aapl)
Analysis of Variance Table

Response: stocks_w$aapl_r
                Df   Sum Sq  Mean Sq F value    Pr(>F)
stocks_w$dji_r   1 0.088201 0.088201  101.34 < 2.2e-16 ***
Residuals      260 0.226285 0.000870
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>
```

Single Factor Model – Example

```
results <- lapply(6:9, function(x) lm(stocks_w[,x] ~ stocks_w[,10]))
names(results) <- c("MSFT", "AAPL", "EBAY", "GOOGL")

coefs <- sapply(results, coef, FUN.VALUE = matrix(nrow = 2, ncol = 4))

alphas <- coefs[1,]
betas <- coefs[2,]

sys_risk <- betas * sd(stocks_w$dji_r)

anova_res <- lapply(results, anova)

spec_risk <- NULL
for(i in names(results)){
  spec_risk[i] <- sqrt(anova_res[[i]][2,3])
}

CAPM_results <- rbind(alphas, betas, sys_risk, spec_risk)
```

Single Factor Model – Example

```
> CAPM_results
                 MSFT        AAPL        EBAY       GOOGL
alphas    0.002695375 0.001815797 0.000246736 0.001391715
betas     1.071611234 1.010162361 0.828031279 1.059924497
sys_risk  0.019501238 0.018382988 0.015068557 0.019288562
spec_risk 0.021852010 0.029501312 0.032293018 0.027541345
> |
```

Single Factor Model – Example

```
proc means data=stocks_w mean var;
        var DJI_r;
        output out=DJIX var=Var mean=Mean;
run;

proc reg data=stocks_w outest=Coef;
        MSFT: model MSFT_r = DJI_r;
        AAPL: model AAPL_r = DJI_r;
        GOOGL: model GOOGL_r = DJI_r;
        EBAY: model EBAY_r = DJI_r;
run;
quit;
```

Single Factor Model – Example

The MEANS Procedure

Analysis Variable : dji_r	
Mean	Variance
0.0014957	0.000331169

Single Factor Model – Example

VIEWTABLE: Work.Coef (Parameter Estimates and Statistics)

	Label of model	Type of statistics	Dependent variable	Root mean squared error	Intercept	dji_r
1	MSFT	PARMS	msft_r	0.0218520102	0.0026953748	1.0716112338
2	AAPL	PARMS	aapl_r	0.0295013123	0.001815797	1.0101623606
3	GOOGL	PARMS	googl_r	0.0275413453	0.0013917154	1.0599244966
4	EBAY	PARMS	ebay_r	0.0322930179	0.000246736	0.8280312793

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ASSET VS. RISK MANAGERS

Asset Manager's Objective

- The goal of an asset manager is the reduce the specific risk of their portfolio by diversifying their investments.
  - Passive Managers: Usually aim for **portfolio alpha** of 0 and **portfolio beta** of 1 while reducing the portfolio's specific risk as much as possible.
  - Active Managers: Usually "accept" **portfolio beta's** that are somewhat greater than 1 for an incremental return above the index (i.e. **alpha > 0**).

Asset Manager's View

- Suppose an asset manager has a portfolio with *M* stocks:

Observation	Stock	Portfolio Weight
1	IBM	5%
2	AAPL	7%
3	MSFT	7%
…	…	…
M	WMT	4%

- How can we use factor models to get the portfolio's α and β?

Asset Manager's View

- Each one of the *M* assets is represented by a single factor model:

$$R_{i,t} = \alpha_i + \beta_i X_t + \varepsilon_{i,t}$$

- The asset manager calculates all portfolio measures as follows:
  - $\alpha = \sum_{i=1}^{M} w_i \alpha_i$
  - $\beta = \sum_{i=1}^{M} w_i \beta_i$
  - $\varepsilon_t = \sum_{i=1}^{M} w_i \varepsilon_{i,t}$
  - Systematic Risk: $\left(\sum_{i=1}^{M} \widehat{w}_i \hat{\beta}_i\right) \hat{\sigma}_X$
  - Specific Risk: $\sqrt{\sum_{i=1}^{M} \widehat{w}_i^2 \hat{\sigma}_{\varepsilon_i}}$

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$$\left.\begin{array}{l} \bullet\ \alpha = \sum_{i=1}^{M} w_i \alpha_i \\ \bullet\ \beta = \sum_{i=1}^{M} w_i \beta_i \\ \bullet\ \varepsilon_t = \sum_{i=1}^{M} w_i \varepsilon_{i,t} \end{array}\right\} \text{Portfolio's Return}$$

  - Systematic Risk: $\left(\sum_{i=1}^{M} \widehat{w}_i \hat{\beta}_i\right) \hat{\sigma}_X$
  - Specific Risk: $\sqrt{\sum_{i=1}^{M} \widehat{w}_i^2 \hat{\sigma}_{\varepsilon_i}}$

Asset Manager’s View

- Each one of the *M* assets is represented by a single factor model:

$$R_{i,t} = \alpha_i + \beta_i X_t + \varepsilon_{i,t}$$

- There a couple of important assumptions:
    1. Assuming true parameters are constant (e.g. OLS is appropriate estimation of each factor)
    2. Using a simple weighted average of alpha’s and beta’s **assumes independence** of the specific risk of the assets

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How do we identify the optimal weights of the portfolio?

Asset Manager's Portfolio Approach

1. Estimate a single factor model for each asset in your portfolio:

$$R_{i,t} = \alpha_i + \beta_i X_t + \varepsilon_{i,t}$$

2. Define the portfolio's expected return:

$$E(R_{port}) = \sum_{i=1}^{m} w_i \hat{\alpha}_i + \bar{X}\left(\sum_{i=1}^{m} w_i \hat{\beta}_i\right)$$

3. Define the portfolio's risk (systemic + specific):

$$\sigma_{port}^2 = \sum_{i=1}^{m} w_i \sigma_{\varepsilon_i}^2 + \sigma_X^2 \left(\sum_{i=1}^{m} w_i \hat{\beta}_i\right)^2$$

Asset Manager's Portfolio Approach

4. Find the w_i's that solve the following problem:
   - Minimize:

$$\sigma_{port}^2 = \sum_{i=1}^{m} w_i \sigma_{\varepsilon_i}^2 + \sigma_X^2 \left(\sum_{i=1}^{m} w_i \hat{\beta}_i \right)^2$$

   - Constraints:

$$Return = \sum_{i=1}^{m} w_i \hat{\alpha}_i + \bar{X} \left(\sum_{i=1}^{m} w_i \hat{\beta}_i \right)$$

$$\sum_i w_i = 1 \ , \qquad w_i \geq 0$$

Asset Manager Portfolio – Example

```
# Optimize the Portfolio #
f <- function(x) x[1]*alphas[1] + x[2]*alphas[2] + x[3]*alphas[3] + x[4]*alphas[4] +
                 var(stocks_w$dji_r)*(x[1]*betas[1] + x[2]*betas[2] + x[3]*betas[3] + x[4]*betas[4])^2

theta <- c(0.2,0.2,0.2,0.2)

ui <- rbind(c(1,0,0,0),
            c(0,1,0,0),
            c(0,0,1,0),
            c(0,0,0,1),
            c(-1,-1,-1,-1),
            c(alphas + mean(stocks_w$dji_r)*betas))
ci <- c(0,
        0,
        0,
        0,
        -1,
        0.002)

port_opt <- constrOptim(theta = theta, f = f, ui = ui, ci = ci, grad = NULL)

port_weights <- port_opt$par
names(port_weights) <- names(results)
port_weights
```

Asset Manager Portfolio – Example

```
>
> round(port_weights*100,2)
 MSFT   AAPL   EBAY GOOGL
 2.66   0.51 67.97 28.85
>
```

Asset Manager Portfolio – Example

```
proc optmodel;

  /* Declare Sets and Parameters */
  set <str> Assets1;
  num Alpha{Assets1};
  num Beta{Assets1};
  num Sigma{Assets1};
  num MeanX;
  num VarX;

  /* Read in SAS Data Sets */
  read data Coef into Assets1=[_DEPVAR_] Alpha=col("Intercept");
  read data Coef into Assets1=[_DEPVAR_] Beta=col("DJI_r");
  read data Coef into Assets1=[_DEPVAR_] Sigma=col("_RMSE_");
  read data DJIX into MeanX=col("Mean");
  read data DJIX into VarX=col("Var");
```

Asset Manager Portfolio – Example

```
/* Declare Variables */
  var Proportion{Assets1} >= 0 init 0.2;

  /* Declare Objective Function */
  min Risk = sum{i in Assets1}Proportion[i]*Sigma[i]**2
             + VarX*(sum{i in Assets1}Proportion[i]*Beta[i])**2;

  /* Declare Constraints */
  con Return: 0.002 <= sum{i in Assets1}Proportion[i]*Alpha[i]
                   + MeanX*(sum{i in Assets1}Proportion[i]*Beta[i]);
  con Sum: 1 = sum{i in Assets1}Proportion[i];
```

Asset Manager Portfolio – Example

```
/* Call the Solver */
 solve;

 /* Print Solutions */
 print Alpha Beta Sigma MeanX VarX;
 print Proportion;* 'Sum='(sum{i in Assets1}Proportion[i]);

 /* Output Results */
 create data Weight_C from [Assets1] Proportion;

quit;
```

Asset Manager's Data

- Type of data that is commonly used:
  - Weekly or monthly returns
  - Three to five years of returns

Risk Manager's View

- Instead of modeling each asset individually with single factor models, use the w_i's of a portfolio to construct a weighted returns history of the whole portfolio before estimating:

$$Y_t = \alpha + \beta X_t + \varepsilon_t$$

- Systematic Risk: $\hat{\beta}\hat{\sigma}_X$

- Specific Risk: $\hat{\sigma}_\varepsilon$

Risk Manager’s Data

- Type of data that is commonly used:
  - Daily returns
  - One to two years of returns (or even less)

Comparison

Asset Manager	Risk Manager
Estimate M different models (Calculate weighted sum of α's and β's)	Weighted summation of returns (Calculate α and β from single model)
Weekly or monthly data	Daily data
3-5 Years of returns	1-2 Years of returns
OLS estimation	GARCH / EWMA