2 Historical review of research through 1960

This is an illustration of the ephemeral nature of utility curves.
C. Jackson Grayson, 1960, 309

The theory of choice under uncertainty is marked by two seminal works separated by more than two centuries, with remarkably little in the interim. These two contributions are D. Bernoulli's "Exposition of a New Theory on the Measurement of Risk" (1738) and John Von Neumann and Oskar Morgenstern's *Theory of Games and Economic Behavior* (1943 [1953]). A metaphor of "bookends" can be used to describe seminal works that bracket a large intervening literature. Because these two works stand almost alone, a more appropriate metaphor might be that they are "virtual bookends" across 200 years.

In his "mean utility [moral expectation]" Bernoulli introduced many of the critical elements of what is now called Expected Utility Theory. Although formal expositions of theories of risk attitudes came later, the core of his argument is a parallel assertion about an individual's nonlinear "utility" for money: specifically that individuals have diminishing marginal utility of income. Further, Bernoulli asserted a "moral expectation" that is a log function of income. The motivating factor for Bernoulli was the infamous St. Petersburg Paradox. This was a gamble that pays 2^n ducats with probability 2^{-n} for every $n = 1, 2, \ldots \infty$, and so has an expected value $1 + 1 + 1 + \ldots = \infty$. Apparently, neither Bernoulli nor his contemporaries could imagine trading anything greater than a modest amount of ducats for this gamble. Bernoulli "solved" the St. Petersburg Paradox by asserting that what mattered was not the mathematical expectation of the monetary returns of the St. Petersburg gamble, but rather the mathematical expectation of the utility of each of the outcomes, which implied only a modest monetary value of the gamble for most individuals.

Even as Bernoulli's logarithmic model became the foundation for the more general concave structure of the "utility function" that later writers used to axiomatize choice under uncertainty and to characterize risk attitudes, Bernoulli recognized situations in which a utility-of-income function would not be concave. However, he saw these counterexamples only as “exceedingly rare exceptions” (25).

Over two hundred years later, Von Neumann and Morgenstern developed an axiomatized structure of expected utility over “lotteries” that referenced Bernoulli’s logarithm function as a special case. This concept of expected utility was central to their exposition of a theory of games. They went on to describe a process by which such a utility function could be estimated from data on an individual’s choices between a series of pairs of “prospects” with certain versus risky outcomes.

In hindsight, it is astonishing that Bernoulli’s idea of expected utility received little, largely negative, attention between 1738 and 1943. On the other hand, the publication of *The Theory of Games and Economic Behavior* ignited a veritable explosion of interest in expected utility and theories of attitudes towards risk. By the mid-1970s, after the publication of Pratt’s “Risk Aversion in the Small and the Large” (1964) and the series of papers in the *Journal of Economic Theory* by permutations of Diamond, Rothschild, and Stiglitz (1970, 1971, 1974) neo-cardinalist theories of expected utility and risk preferences took the driver’s seat.

Of course, taking a commanding place in economic orthodoxy does not necessarily mean that the theory has also created a useful empirical structure for explaining or predicting “Economic Behavior” (as the second part of the Von Neumann and Morgenstern title suggests). That is the point of our book. However, before turning to that larger theme, it may be useful to contrast why Bernoulli’s foundational exposition received so little attention for more than two hundred years, and why Von Neumann–Morgenstern had such a revolutionizing impact on this aspect of economics.

To begin with, it seems commonplace to think of Bernoulli as modeling *some* concave utility function. In fact, he explicitly suggested only a *specific – logarithmic* – concave function. Even with this more general (concave) interpretation, there were (and remain) empirical problems with this model in terms of such things as the existence of gambling and the form of insurance contracts. In the case of gambling, Bernoulli argues (29) that “indeed this is Nature’s admonition to avoid the dice altogether.”

Schlee (1992) argues that until the arrival of the early marginal-utility theorists such as Jevons in the late nineteenth century, “few economists appear to have noticed” Bernoulli’s theory. Jevons (1871 [1931], 160)[1] cited Bernoulli’s work and asserted, “It is almost self-evident that the utility of money decreases as a person’s total wealth increases.” But Jevons argued that along the way to diagnosing a reasonable answer to “many important questions” (presumably including the St. Petersburg Paradox), “*having no means of ascertaining numerically the variation in utility*, [emphasis added] Bernoulli had to make assumptions of an arbitrary kind” (again, we presume Jevons means Bernoulli’s explicit adoption of the logarithmic form from the class of concave utility functions).

But the enthusiasm for Bernoulli, in spite of this rediscovery, was attenuated by several factors. First, there was extensive discussion of how to deal with the apparent inconsistency between declining marginal utility of income and the widespread phenomenon of gambling. According to Blaug (1968), Marshall also accepted the general idea of a diminishing marginal utility of income and begged the question of building an economic theory of choice under uncertainty by simply attributing gaming at less than fair odds directly to the "love of gambling." Specifically, in a footnote on page 843 of the eighth edition of his *Principles* (1890 [1920])[2] Marshall wrote:

> The argument that fair gambling is an economic blunder is generally based on Bernoulli's or some other definite hypothesis. But it requires no further assumption than that, firstly, the pleasures of gambling may be neglected; and secondly $\varphi''(x)$ is negative, where $\varphi(x)$ is the pleasure derived from wealth equal to x.

Furthermore, in a footnote on page 842, Marshall rederives Bernoulli's logarithmic utility function (in today's familiar notation) of:

$$U_i(y) = K \log (y / \alpha)$$

(where y is actual income and α is a subsistence level of income). *Contra* Bernoulli, Marshall suggests "Of course, both K and α vary with the temperament, the health, the habits, and the social surroundings of each individual." Marshall's treatment of Bernoulli is pretty much limited to this and a discussion of the implications for progressive income taxation. Schlee, in a different and extended discussion of Bernoulli, Jevons, and Marshall, argues that Marshall implicitly accepted the idea of some kind of expected utility calculations in valuations of durable goods, although he (Marshall) claimed that "this result belongs to Hedonics, and not properly to Economics."[3]

Second, even as Bernoulli was being cited by Jevons and Marshall, the movement to ordinal utilities soon gained traction, and indifference curves were described by utility functions defined only up to an increasing monotonic transformation. Changes of marginal utility per se were undefined in the new ordinal paradigm.

Moreover, there was Menger's demonstration (1934) that Bernoulli did not actually solve *the* St. Petersburg Paradox, he merely solved *one member of the family* of such paradoxes. Even with Bernoulli's logarithmic function, there exist other St. Petersburg gambles with appropriately increasing payments that would lead to the same conundrum that Bernoulli sought to resolve. A full solution required even more restrictive assumptions on the utility function (boundedness would be sufficient).[4]

Finally, Bernoulli's "solution" of a logarithmic utility function was not the only one put forth to the St. Petersburg Paradox. As Bernoulli himself noted, Cramer offered one approach from an entirely different direction, suggesting that the probabilities on the highest (outlying) payoffs were (either objectively or subjectively) treated as zero: "Let us suppose …. better yet, that I can never win more than that amount, no matter how long it takes before the coin falls with the cross upward."[5] In 1977 Shapley would reiterate a similar objection, in what he called the missing (and weak) link of the paradox: "One assumes that the subject believes the offer to be genuine, i.e., believes *that he will actually be paid, no matter how much he may win*" (440) (emphasis in the original). Shapley goes on to describe this assumption as "empirically absurd" (442).[6] One can argue that there was, in fact, no obvious empirical evidence to favor Bernoulli's solution over Cramer's (Shapley's). In fact, in terms of ad-hoc introspection, the conjecture that no gambler could afford to lose the infinitely large payoffs in the right tail of the St. Petersburg Paradox hardly stretches credulity. The idea that individuals treat very small probabilities as if they were zero lives on in modern risk analysis and behavioral economics.

With these limitations of Bernoulli's proposal, how can we explain its extraordinary revival after the appearance of the second virtual bookend, Von Neumann and Morgenstern's *Theory of Games and Economics Behavior*? What can be thought of as a "neo-cardinalist" revolution was launched in the mid-twentieth century with scant reference to the ordinalist foundation of neoclassical economics built over the preceding decades.[7] Within a quarter of a century, this revolution became, in at least some realms of economics, the new orthodoxy. One obvious argument is that the Von Neumann–Morgenstern (hereafter VNM) axiomatization was not tied to any specific functional form that was required to hold constant across all individuals. In fact, it didn't require even the most minimal assumption of concavity of the derived expected utility function. (A more detailed mathematical discussion of the EUT axioms and several related issues is included in the Appendix to this chapter).

As Blaug suggests (336), this led Friedman and Savage (1948) and also Markowitz (1952) to play a game of "my utility function has more inflection points than your utility function" in terms of purported empirical explanatory power, albeit without any indication of magnitudes on the x-axis of their free-hand drawings.

A Friedman–Savage individual (see Figure 2.1) had concave portions of his utility of income at relatively low and high levels of the income scale, explaining "both the tendency to buy insurance and the tendency to buy lottery tickets" (Blaug, 336). But it also had the uncomfortable implication that the very "poor" and the very "rich" will not accept fair bets. Friedman and Savage (288–289) seem to conceive of the convex section in the middle of the utility function as being rather large, because they motivate it in terms of a worker being able to change social classes.

Markowitz proposed that the utility function had three instead of two inflection points: a convex section followed by a concave section to the right of the origin, and a concave section followed by a convex section to the left of the origin (see Figure 2.2). In addition, the "origin" on one of his utility graphs had a specific meaning for Markowitz, namely "the 'customary' level

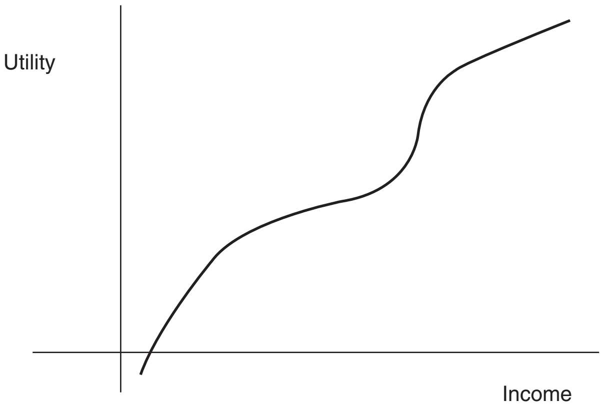

Figure 2.1 A Friedman–Savage Bernoulli function (schematic redrawn by authors based upon figure 3 in Friedman and Savage [1948, 297]).

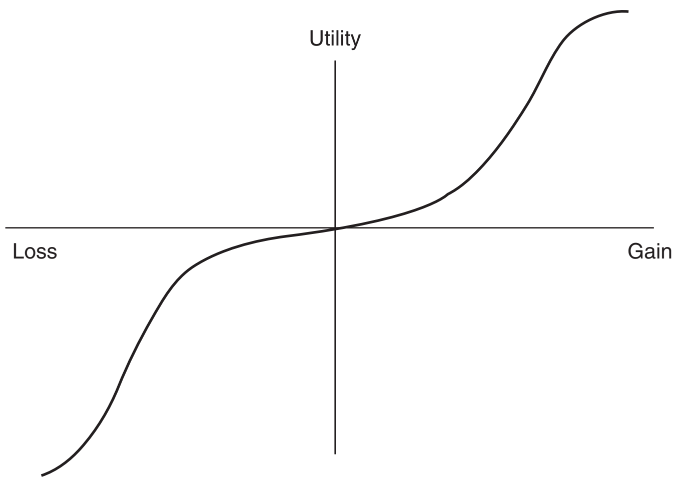

Figure 2.2 A Markowitz Bernoulli function (schematic redrawn by authors based upon figure 3 in Markowitz [1952, 152]).

of wealth" for the individual (155). In a Markowitz utility function (quoting Blaug)

> small increments in income yield increasing marginal utility, but large gains in income yield diminishing marginal utility; this accounts for people's reluctance to accept large but their eagerness to accept small "fair bets." On the other hand, small decrements in income yield increasing marginal disutility, while large losses in income yield diminishing marginal disutility; hence the eagerness to hedge against small losses but a devil-may-care attitude to very large losses.[8]

We will return to a discussion of the Friedman–Savage and Markowitz utility functions in several of the later chapters.

While Friedman and Savage and also Markowitz rejected the "shape" of the utility function of wealth proposed by Bernoulli, a standard convention that we will adopt is to refer to a utility function over some version of aggregate final states (including, but not limited to money, wealth, income, an aggregate measure of overall consumption, and so forth) that is also consistent with an axiomatic system of expected utility to be a "Bernoulli function." What we now know as the orthodox treatment of an individual Bernoulli utility function was cemented in place by the quick succession of Pratt (1964), Rothschild and Stiglitz (1970, 1971), and Diamond and Stiglitz (1974). We note here two issues: (1) this high orthodoxy embraces the idea of at least a quasi-cardinal concept of utility because key parameters of the utility function, such as various measures of "risk aversion," are robust with respect to some, but not all, monotonic transformations (see the Appendix to this chapter for more specifics), and (2) the equivalence relationships between risk aversion and underlying probability distributions highlight concepts of risk that were tied to the dispersion properties of the probability distribution of outcomes ("mean preserving" increases; see the discussion in the Appendix to this chapter). Likewise, Markowitz, through a different reasoning, ends up defining risk as dispersion. An interesting, but perhaps unintentional commentary on this state of affairs can be gleaned from Bernstein's popular book on risk, *Against the Gods* (1996). The dramatic flow of the book builds through the development of the concepts of probability, the St. Petersburg Paradox, normal distributions, and, eventually, Markowitz's portfolio theory of mean and variance of returns. But the narrative of nonmathematicians who have to deal with risk on a daily basis is intertwined in Bernstein's chapters, and it tells a story quite different from "risk as dispersion." Specifically, in actual practice (as well as in at least the regulation, insurance, and credit aspects of economics) risk equaled harm: "house breaking, highway robbery, death by gin-drinking, the death of horses" (90), fire (91), the loss of ships, and losses from hurricanes, earthquakes, and floods (92). On page 261, he discusses the investment philosophy of a former manufacturing executive turned trust manager: "In the simplest version of this approach, risk is just the chance of losing money." In the extensive literature on insurance (e.g., Williams [1966]) this idea of "pure risk" was clearly distinguished from "speculative risk" in which uncertainty pertains to the possibility of losses as well as gains. This distinction, well-established until the 1960s, between pure risk and speculative risk has been ignored, if not lost altogether, in many parts of the recent economics literature distracted by the algebraic convenience of dealing with the combination of second moments and quadratic utility functions.

Meanwhile, back in the academy, it is important to note that the neo-cardinalists did not displace the ordinalists from their commanding position in neoclassical economics. Instead, an interesting, inconsistent, but ultimately convenient live-and-let-live territorial division developed between the two. Ordinalism controls the center of orthodoxy in theories of consumer choice under certainty, while neo-cardinalism holds sway with regards to choice under risk or under uncertainty. Never mind that much consumer choice is made under uncertainty. After the mathematization of economics in which logical consistency is the commanding value, it is amazing that graduate seminars just skip over the inconsistency. The consensus view is well stated by Blaug (328):

> These two types of utility scales [fully ordinal versus cardinal but unique up to a linear transformation] differ strikingly in one respect. Scales that are monotone transformations of each other vary together in the same direction: this is the only property they have in common. But scales that are linear transformations of each other assert something much stronger: when the interval differences of one scale increase or decrease successively, the interval differences of the other scale increase or decrease successively to the same extent. … Measurability up to a linear transformation involves knowledge not only of the signs of the first differences of the utility scales but also of the signs of the second differences: the first differences tell us about the *direction* of preference; the second differences tell us about the *intensity* of preference [emphasis in the original].

Of course the measures of "risk aversion" (the Arrow–Pratt measures being the widely acknowledged gold standard) depend crucially upon the *second* derivative of the expected utility function. A specific mathematical treatment is included in the Appendix to this chapter.

In the neoclassical model of consumption, the broader concept of decreasing marginal utility[9] had been subsumed by the ordinal condition of strictly convex indifference curves (implying a declining marginal rate of substitution for any two goods). This condition is not enough to imply a concave utility function over money. Indeed, in certain cases the marginal utility of income (in effect, the Lagrangean multiplier on the budget constraint in the standard neoclassical set-up) can be found to be constant with respect to income. But the more pressing question for neoclassical economics was not to debate Bernoulli's concerns but to describe those conditions in which the marginal utility of income could be constant with regards to *all* prices and income (Samuelson [1947]).

To appreciate the depth of this coexistence-by-mutual-nonrecognition-of-inconsistencies, consider the standard texts of graduate microeconomics. Samuelson is straightforward: "Nothing at all is gained by the selection of individual cardinal measures of utility" (173). Another thoroughly typical example is the chapter entitled "Utility Maximization" (95) in Varian's graduate text (1992): "Utility function is often a very convenient way to describe preferences, but it should not be given any psychological interpretation. *The only relevant feature of a utility function is its ordinal character* [emphasis added]." Yet Varian's treatment of choice under uncertainty is equally orthodox in the VNM paradigm, concluding with the derivation that "an expected utility function is unique up to an affine transformation" which is, of course, a less restrictive assumption than a full ordinalism property of uniqueness up to any monotonic transformation. Friedman and Savage (1952, 464) have asserted that Bernoulli functions "are not derivable from riskless choices." Furthermore, there are many other areas in which, *contra* Samuelson, progress in microeconomic modeling has been enabled only by discarding some of the restrictions of pure ordinalism in favor of an additive separable utility function of the form $U_i(x_i, y_i) = x_i + v_i(y_i)$. Examples include mechanism design for public goods provision and the theory of bidding in auctions. And, of course, a large part of the theory of the firm assumes that firms maximize expected profits, which is to say that they have a VNM utility function that is linear in payoffs.

From Bernoulli through Markowitz and subsequently, the empirical content of these assertions about the shape of typical utility functions, and hence about an individual's neo-cardinalist measure of risk aversion (how these vary from individual to individual, and whether they are stable across individuals) was largely based upon appeals to assertions in the aggregate or what might charitably be called "stylized facts" (or, less charitably, "urban legends"). This is somewhat surprising because this is not what is at the heart of VNM. Indeed, a second explanation for the power of VNM to change the intellectual playing field regarding expected utility was that they presented not simply a toy model for armchair theorists but also a *process* for mapping out the utility function of any given individual in a given place and under a given set of ceteris-paribus circumstances.

If we are to argue that some of the influence of Von Neumann and Morgenstern derives from their assertion of empirical content of individual utility maps constructed from their method, it could not take long before someone would undertake to make such maps (rather than simply appeal to broad assertions about the behavior of people shopping or gambling as seen from looking out the windows of a faculty office). Indeed, the task was taken up in earnest almost immediately, in the 1950s.

Pioneering decision theorists Duncan Luce and Howard Raiffa, in their book *Games and Decisions* (1957) have a section on "Experimental Determinations of Utility." The paper they list as having the earliest publication date is the report of the laboratory experiments of Mosteller and Nogee (1951), henceforth MN. MN elicited data from payoff-motivated choice experiments over sample "poker" hands to construct Bernoulli/VNM utility functions. One interesting finding of MN (386) was a demographic difference. Harvard student subjects tended to have utility functions that were "conservative" (i.e., concave), whereas National Guard subjects tended to have utility functions that were "extravagant" (i.e., convex). Although the topic is beyond the scope of this chapter, it is worth noting for further reference Edwards's 1953 paper which was motivated by MN but which continued on into the realm of determination of the subjects' subjective probabilities rather than simply their utility over outcomes.

Also, at about the same time that *Games and Decisions* was in production, Raiffa was working with a graduate student, C. Jackson Grayson, whose dissertation, published in 1960 as *Decisions Under Uncertainty: Drilling Decisions by Oil and Gas Operators* is an underappreciated classic in the realm of using similar techniques to estimate individual VNM utility functions in the field (in this case, among independent Oklahoma oil and gas company owners and their employees). Grayson used linked hypothetical questions over different lotteries derived essentially as VNM had suggested. He obtained between 8 and 30 estimated utility points for each of 11 different people. Today, one would undoubtedly use a statistical process to create a "best fit" among the data. Grayson apparently used a free-hand drawing to approximate a "best fit."

In the introduction to this part of his book, Grayson says that he believes that this experiment in extracting utility functions met with "mixed success" (297). Nevertheless, there are some notable regularities and narratives that come from this early effort.

First, the curves estimated from choices made by various individuals showed widely different patterns of concavity, convexity, and inflection points. Perhaps surprisingly, given the results of later claims of an opposite characterization (such as those of Kahneman and Tversky [1979]), many of the individuals can roughly be described as risk-taking in gains and risk-averse in losses. Second, for one individual, Grayson had a chance to investigate the stability of the revealed utility function across time. After the passage of more than three months in-between, the individual appears to have generated a more "conservative" (Grayson's term for apparently more concave) utility function. Grayson (309) argues, albeit on the basis of this single subject, "This is an illustration of the ephemeral nature of utility curves." A final fascinating feature of Grayson's analysis of the usefulness of his labors was the frequency with which the individuals reported that they had difficulty making these (hypothetical) decisions as purely abstract constructs disassociated from the reality of their oil and gas exploration business: "There is no such thing as certainty in drilling wells." "Who gets the intangibles?" (This is a question about the obvious but often overlooked issue in these elicitation processes as to whether gains are to be considered as before-tax or after-tax income). "Is it an isolated area?" "Is it gas or oil?" (These are questions that relate to associated contracts and constraints). "Am I the operator?" (This is an important question not only about entrepreneurial control but also about already embedded risk sharing). We will return to the importance of these questions from Grayson's subjects in Chapter 6.

This chapter takes us up to the early 1960s. In the next chapter, we discuss the first wave of new theoretical work on utility and risk beginning in the period of the mid 1960s–1970s. This theoretical work in turn motivated a new effort at empirical mapping. Instead of the "mapping of utility functions" approach discussed here, empirical researchers turned to the elicitation of parametric representations of utility functions. The question we will pose next is: "To what extent did these elicitations yield dependable estimates of a person's propensity to choose under uncertainty, or to bear risk?"

Appendix. Mathematical details

We collect standard mathematical formulations of key concepts introduced in Chapter 2. Later chapters will draw on them, but readers uninterested in mathematical formalities should feel free to skip this Appendix.

We begin with some definitions. A *lottery* $L = (M, P)$ is a finite list of monetary outcomes $M = \{m_1, m_2, ..., m_k\} \subset \Re$ together with a corresponding list of probabilities $P = \{p_1, p_2, ..., p_k\}$, where $p_i \geq 0$ and $\sum_{i=1}^{k} p_i = 1$. The symbol $\Re$ denotes the real numbers $(-\infty, \infty)$. The *space of all lotteries* is denoted $\mathcal{L}$.

The *expected value* of lottery $L = (M, P)$ is $E_L m = \sum_{i=1}^{k} p_i m_i$.

A utility function over monetary outcomes (henceforth called a *Bernoulli function*) is a strictly increasing function $u : \Re \to \Re$. A *utility function over lotteries* is a function $U: \mathcal{L} \to \Re$.

Given a Bernoulli function u, the *expected utility* of lottery $L = (M, P)$ is $E_L u = \sum_{i=1}^{k} p_i u(m_i)$.

Preferences $\gtrsim$ over any set refer to a complete and transitive binary relation. A utility function U *represents* preferences $\gtrsim$ if $x \gtrsim y \Leftrightarrow U(x) \geq U(y)$ for all x, y in that set, where the symbol "$\Leftrightarrow$" means "if and only if." In particular, preferences $\gtrsim$ over L are represented by $U : \mathcal{L} \to \Re$ if

$$L \gtrsim L' \Leftrightarrow U(L) \geq U(L') \text{ for all } L, L' \in L.$$

Preferences $\gtrsim$ over $\mathcal{L}$ have the *expected utility property* if they can be represented by a utility function U that is the expected value of some Bernoulli function u. That is, there is some Bernoulli function u, such that for all $L, L' \in L$, we have

$$L = (M, P) \gtrsim L' = (M', P') \Leftrightarrow$$

$$U(L) \equiv E_L u = \sum_{i=1}^{k} p_i u(m_i) \geq U(L') \equiv E_{L'} u = \sum_{i=1}^{k} p_i{}' u(m_i{}'). \tag{2A.1}$$

It might seem that preferences with the expected utility property are quite special, and indeed they are. For example, their indifference surfaces are parallel and flat. Thus preferences over lotteries with only $k = 3$ monetary outcomes have indifference curves on L (represented as the probability simplex, here a triangle) that are all straight lines with the same slope.

The Expected Utility Theorem (EUT) is therefore surprising. It states that preferences over lotteries that satisfy a seemingly mild set of conditions will automatically satisfy the expected utility property, and thus be representable via a Bernoulli function.

Over the decades since the original results of Von Neumann and Morgenstern, many different sets of conditions have been shown to be sufficient. Here we mention the set used in a leading textbook, Mas-Colell, Whinston, and Green (1995). It consists of four axioms that an individual's preferences $\gtrsim$ over $\mathcal{L}$ should satisfy:

1. Rationality: Preferences $\gtrsim$ are complete and transitive on $\mathcal{L}$.
2. Continuity: The precise mathematical expressions are rather indirect (they state that certain subsets of real numbers are closed sets), but they capture the intuitive idea that U doesn't take jumps on the space of lotteries. This axiom rules out lexicographic preferences.
3. Reduction of Compound Lotteries: Compound lotteries have outcomes that are themselves lotteries in $\mathcal{L}$. By taking the expected value, one obtains the *reduced* lottery, a simple lottery in $\mathcal{L}$. The axiom states that the person is indifferent between any compound lottery and the corresponding reduced lottery.
4. Independence: Let $L, L', L'' \in \mathcal{L}$ and $\alpha \in (0,1)$. Suppose that $L \gtrsim L'$. Then $\alpha L + (1-\alpha)L'' \gtrsim \alpha L' + (1-\alpha)L''$. "In other words, if we mix two lotteries with a third one, then the preference ordering of the resulting two mixtures does not depend upon (is independent of) the particular third lottery used" (Mas-Colell, Whinston, and Green, 171).

Theorem 1 (EUT). *Let preferences $\gtrsim$ over $\mathcal{L}$ satisfy axioms 1–4 above. Then $\gtrsim$ has the expected utility property, i.e., there is a Bernoulli function u such that (2A.1) holds.*

As Mas-Colell et al. point out, all four axioms seem innocuous. Someone who cares only about the ultimate monetary payoffs and whose calculations are not affected by indirect ways of stating the probabilities will satisfy the third axiom. For example, such a person would be indifferent between the compound lottery "get 0 with probability 0.5 and with probability 0.5 play the lottery that pays 10 with independent probability 0.5 and 0 otherwise," and the reduced lottery "get 0 with probability 0.75 and 10 with probability 0.25." The fourth axiom enforces a degree of consistency by requiring that preference rankings over lotteries are not changed by nesting each of those lotteries within a generic compound lottery. The first two axioms are even less controversial or problematic.[10]

For a proof of the EUT, see Mas-Colell, Whinston, and Green and the references cited therein. Here is a sketch of how the function u can be constructed for given preferences. Denote by m_+ and m_- the maximum and minimum monetary outcomes in the lottery. Set $u(m_+) = 1$ and $u(m_-) = 0$. Consider any other monetary outcome m, and the set of lotteries $\{([m_+, m_-], [p, (1-p)]) : p \in [0,1]\}$. For $p = 1$ the lottery is preferred to m and for $p = 0$ the outcome m is preferred to the lottery. Using the continuity axiom, one can show that for some intermediate p^* the person is indifferent between m and that lottery. Set $u(m) = p^*$. Then use the other axioms to verify that the Bernoulli function so constructed indeed represents the given preferences.

Notice that this construction of the Bernoulli function suggests an empirical procedure: vary the probabilities on best and worst outcomes to try to find a person's point of indifference. This type of $u(m)$ elicitation is essentially what Mosteller and Nogee (1951) did in the laboratory and what Grayson (1960) did in the field using employees of independent Oklahoma oil companies. In both cases the authors estimated a finite number of $u(m)$ points from each individual's answers and then interpolated for intermediate values.

Given a twice continuously differentiable Bernoulli function u, the **coefficient of absolute risk aversion** at monetary outcome (or payoff) $m \in \Re$ is

$$A(m) = \frac{-u''(m)}{u'(m)} \tag{2A.2}$$

and the **coefficient of relative risk aversion** at $m > 0$ is

$$R(m) = \frac{-u''(m)}{u'(m)} m. \tag{2A.3}$$

A Bernoulli function is unique only up to a positive affine transformation. It is straightforward to check that if (2A.1) holds for some u, then (2A.1) still holds when u is replaced by $v(m) = cu(m) + b$ for any $c > 0$ and any $b \in \Re$. Note also that the functions $A(m)$ and $R(m)$ are the same for v as they are for u.

On the other hand, consider a positive and monotonic but nonlinear transformation, for example $w(m) = u(m)^2$, with $u(m) = m^r$ for $0 < r$. Using u we obtain $A(m) = (1-r)/m$ and $R(m) = (1-r)$. The risk aversion coefficients are not preserved with $w(m) = m^{2r}$; for that Bernoulli function we get $A(m) = (1-2r)/m$ and $R(m) = (1-2r)$.

Two families of Bernoulli functions often appear in the literature. From the previous example, it is easy to see that the *constant relative risk aversion* (CRRA) family $u(m;r) = \frac{m^{1-r}}{1-r}$ has constant relative risk aversion $R(m) = r$ for any $r \neq 1$. Bernoulli's original function $u(m;1) = \ln m$ has constant relative risk aversion $R(m) = 1$, and readers familiar with L'Hospital's rule can verify that it fits snugly in the CRRA family. The *constant absolute risk aversion* (CARA) family is $u(m;a) = -e^{-am}$ for $a > 0$; it is straightforward to check that it indeed has constant absolute risk aversion $A(m) = a$. In view of the paragraph before the previous one, readers who prefer to keep utility positive on

$[0,\infty)$ can replace the previous Bernoulli function u by the equivalent function $v = 1 + u$.

By Taylor's Theorem, any smooth (five times continuously differentiable) Bernoulli function u can be expanded at any point z in its domain as the following polynomial plus remainder:

$$u(z+h) = u(z) + u'(z)h + \frac{1}{2}u''(z)h^2 + \frac{1}{6}u'''(z)h^3 + \frac{1}{24}u''''(z)h^4 + R^5(z,h), \tag{2A.4}$$

where $R^5(z,h) = \dfrac{1}{120}u'''''(y)h^5$ for some point y between z and $z+h$.

Recall that any lottery $L = (M,P)$ has expected value $\bar{m} = E_L m$ as defined earlier, and second moment (or variance) $\sigma_L^2 = Var[L] = E(m-\bar{m})^2 = \sum_{i=1}^{k} p_i (m_i - \bar{m})^2$. The third moment (or unnormalized skewness) is $Sk[L] = E(m-\bar{m})^3$, and the fourth moment (or unnormalized kurtosis) is $Kur[L] = E(m-\bar{m})^4$. Of course, fifth and higher moments can also be defined, but they have no common names.

In equation (2A.4), set $z = \bar{m}$ and $m = z+h$, and take the expected value of both sides. The linear term disappears because $Eh = E(m-\bar{m}) = \bar{m} - \bar{m} = 0$. Hence we obtain

$$Eu(m) = u(\bar{m}) + \frac{1}{2}u''(\bar{m})Var[L] + \frac{1}{6}u'''(\bar{m})Sk[L] + \frac{1}{24}u''''(\bar{m})Kur[L] + E_L R^5. \tag{2A.5}$$

That is, expected utility of the lottery is equal to the utility of the mean outcome plus other terms that depend on the derivatives of u and the corresponding moments of L. The leading such term is proportional to the variance of the lottery and to the second derivative of u evaluated at the mean of the lottery. Of course, that second derivative is negative for a strictly concave function u, and as just noted, the Arrow–Pratt coefficient of absolute risk aversion $A(m)$ simply changes its sign and normalizes by the first derivative (to make the coefficient the same for all equivalent u's). That is, variance reduces expected utility to the extent that u is concave, as measured by $A(m)$.

As we will see in Chapter 5, some authors postulate signs for the third and fourth derivatives and interpret them in terms such as prudence and temperance.

Notes

1 Accessed on May 31, 2013 at http://socserv.mcmaster.ca/econ/ugcm/3ll3/jevons/TheoryPoliticalEconomy.pdf.

2 Accessed on May 31, 2013 at http://files.libertyfund.org/files/1676/Marshall_0197_Bk.pdf.

3 This is a quote from Marshall as stated on page 736 of Schlee.
4 Samuelson (1977) devotes a section of his article to extending Menger-like "Super-St. Petersburg" analysis.
5 The quote is in D. Bernoulli's article but is in fact from Cramer in a letter to Bernoulli's cousin.
6 Samuelson (1977, 27, emphasis in the original) says that he also agrees with those who question the infinite wealth assumption, "The *infinite* game cannot *feasibly* be played by both parties (or either party) whatever their utilities and desires. So the paradox aborts even before it dies."
7 The term "neo-cardinalist" is used to distinguish the approach from the previous "cardinal" utility theories in which the cardinal scale in "utils" was integral. A "neo-cardinalist" utility function is invariant to some transformations, but not as many as those of the ordinalist orthodoxy. See the Appendix to this chapter for more for specific details.
8 For a complete description of the hypothetical choices Markowitz offered to his "middle income acquaintances" see page 153 of his 1952 article.
9 The roots are in Bentham but diminishing marginal utility reached full development from Jevons; see Blaug's (1968) excellent treatment in his chapter 8.
10 On the other hand, the EUT's conclusion is quite strong, and not consistent with some actual choice data. As noted in later chapters, one reaction is to accommodate some of the anomalous data by weakening the axioms, usually the third or fourth.

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