Journal of International Economics 26 (1989) 309–325. North-Holland

INTERSECTORAL CAPITAL REALLOCATION UNDER PRICE UNCERTAINTY

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Received December 1987, revised version received May 1988

Consider a small open two-sector economy with costly capital mobility. The world price follows Brownian motion. The optimal reallocation policy is found using the theory of option pricing. The total value of each capital unit is its expected discounted marginal product in the current sector plus the option value of moving it to the other sector. A marginal unit is moved when its total value in the new sector exceeds that in the old sector, plus the adjustment cost. This yields 'hysteresis' – a zone of inaction and caution before reorienting capital in response to relative price shifts.

1. Introduction

The economies of many commodity-producing countries, especially LDCs, have experienced great strains over the last decade as the world prices of their exports have fluctuated. When commodity prices boomed, so did investment in those sectors. Subsequent price declines have meant a costly retrenchment. The experience of Nigeria with oil, or of Zambia with copper, are vivid examples of this phenomenon. With hindsight, we can say that the countries should have been more cautious in orienting their economies toward the booming sectors, given the possibility of future reversals. However, it is necessary to develop more precise criteria, both to judge if excesses were committed in the past, and to provide a better guide to future investment decisions.

The theory of intersectoral capital reallocation with adjustment costs is well developed; see the excellent exposition in Mussa (1984). However, that literature does not help us here. It deals with the response to a once-and-for-all unexpected shift of a given magnitude in the world price, whereas the problem posed here involves ongoing uncertainty. In this paper I shall construct a simple model of this situation, indicate the general method of its solution, and carry out that solution for an instructive special case. What emerges is a clearer formulation of the concept of caution in responding to

*I am grateful to Avishay Braverman and Andrew Caplin for valuable discussions, to Alan Blinder, Gene Grossman, and a referee for useful comments on an earlier draft, and to the National Science Foundation (Grant no. SES-8509536) and the International Finance Section, Princeton University, for financial support.

price fluctuations. There is a band of prices around the normal level within which no shifts of capital across sectors should be undertaken. When price changes are large enough to cross the bands, just enough capital should be shifted to bring the economy back to the edge of the band. A similar band of inaction exists in the case of a once-and-for-all price shift and linear adjustment costs; see Kemp and Wan (1974). But ongoing uncertainty makes the band wider. In fact I shall show that for some plausible functional forms and parameter values, the effect of uncertainty is much larger than that of linear adjustment costs alone.

A similar problem of labor allocation – search unemployment as demand fluctuates – was analyzed by Lucas and Prescott (1974). They used a general Markov process in discrete time, and obtained mostly qualitative results. I shall assume a stochastic process that is simple enough to be tractable, and more conducive to the specification of realistic parameters for numerical calculations, namely Brownian motion in continuous time.

The technique follows the analysis of entry decisions involving sunk costs for an exhaustible resource firm [Brennan and Schwartz (1985)], a manufacturing firm [Dixit (1989a)] and an export industry [Dixit (1989b)]. The idea comes from the theory of option pricing in financial economics.[1] An idle firm can be regarded as an asset that is an option on becoming an active firm by incurring the exercise price, namely the sunk cost of the investment. Likewise, an active firm is an option on becoming idle by paying any abandonment cost. Solving these two linked option pricing problems yields the values of the two assets, and the rules for exercising the options, namely the output prices that trigger entry and exit.

Here I consider a two-sector small open economy. Then a unit capital that is located in one sector is an asset that pays a dividend – its current marginal product – and is an option on another asset – a unit of capital in the other sector – with an exercise price equal to the adjustment cost.

The total stock of capital is fixed, and units of it are moved from one sector to the other. This is an admittedly unrealistic story, but one that is firmly established in the theory of adjustment dynamics in international trade. Therefore I adopt it to make it easier for readers to grasp the new issue and its relation to the familiar models. I shall later discuss the difference made by adopting a more realistic approach, where reallocation is achieved by letting capital depreciate in the sector that is to contract, and directing all gross investment to the expanding sector.

2. The model

The two sectors are labelled X and Y. Good Y is the numeraire, and the

[1] My exposition is reasonably self-contained, but readers who need more background in option pricing can consult Merton (1973).

world price of X is denoted by P. The stochastic dynamics of P over time follows a Brownian motion:

$$dP/P = \mu\, dt + \sigma\, dz, \tag{1}$$

where dz is the increment of the standard Wiener process, uncorrelated across time, and at any one instant satisfying

$$E(dz) = 0, \qquad E(dz^2) = dt,$$

where E denotes the expectations operator. Let P_0 be the initial value of P at $t=0$, consider the random P_t at a later date t. By the standard theory of Brownian motion, we know that $\ln P_t$ is normally distributed with mean $[\ln P_0 + (\mu - (1/2)\sigma^2)t]$ and variance $(\sigma^2 t)$. Then, from standard properties of the lognormal distribution, we have $E(P_t | P_0) = \exp(\mu t)$. Thus, μ is the trend rate of growth of the price; this may be negative for some commodities for Prebisch-like reasons, and positive for others for Hotelling-like reasons. Brownian motion has the merit of being a simple and yet rich process to describe commodity price fluctuations, but I shall suggest some alternatives for future work.

The economy's total capital is K. I shall assume this to consist of a large number of integer units. Of this, M units are currently in the X-sector and the rest in the Y-sector. The capital can be reallocated in integer units. This does not reflect any fundamental economic indivisibility, merely mathematical convenience. We can imagine each unit to be small enough that the process is close to a continuum limit, but the rigorous treatment of an actual continuum model is mathematically quite difficult.[2]

Production at an instant is modelled by means of a revenue function $R(P, M)$. Factors other than capital are constant and kept in the background. The simplest case, which I shall use in the solution, is one where capital stands for all the factors which can move even at a cost across sectors – all other factors are completely sector-specific. Then we have:

$$R(P, M) = PF(M) + G(K - M), \tag{2}$$

where F and G are the production functions in the two sectors. If there are other costlessly mobile factors like labor, $R(P, M)$ will be a strictly convex function of P.

Next I specify the adjustment costs. To move a unit of capital from the Y-sector to the X-sector, so as to increase the units in the latter from $(M-1)$ to M, costs $h(P, M)$. The opposite transfer costs $l(P, M)$. Once again, I shall

[2]See Bentolila and Bertola (1987).

use a simple special case in the basic solution. Suppose the cost of adjustment arises solely because the capital that is being moved must spend a certain amount of time in preparation, and cannot contribute to output in its initial sector during that time. Then for the production technology of (2), we can write:

$$h(P,M)=h_0[G(K-(M-1))-G(K-M)] \tag{3}$$

and

$$l(P,M)=l_0P[F(M)-F(M-1)]. \tag{4}$$

Here h_0 is the amount of time capital in the Y-sector must sit idle in the process of being moved to the X-sector, and the other way around for l_0. These can be corrected for the effect of discounting during the periods of idleness, but for realistic values that is a very small change. I shall vary these parameters to determine their effect on the solution. Note that as M increases, the contribution of a further unit to the X-sector output falls, while the opportunity cost, namely the output forgone in the Y-sector, rises. Therefore a higher P is necessary to justify such a further shift.

Think of the economy as being planned to maximize the expected discounted present value of national product net of adjustment costs. A competitive equilibrium with rational expectations can replicate the outcome, as in Lucas and Prescott (1974), but if there are market failures, decentralized implementation will need appropriate taxes or subsidies, as discussed by Mussa (1984). Start the process at $t=0$ with given initial values (P,M). Let $V(P,M)$ be the asset value of the capital when the optimal policies are followed – the Bellman value function of dynamic programming. We can determine $V(P,M)$ using the methods of option pricing.[3]

First consider an interval of time, or a range of prices, where M is kept unchanged. The capital assets yield a dividend flow $R(P,M)$, as well as a capital gain because V changes over time as P changes. By Itô's Lemma, the expected capital gain is:

$$\mathrm{E}(\mathrm{d}V)=V_P P\mu\,\mathrm{d}t+(1/2)V_{PP}\sigma^2P^2\,\mathrm{d}t.$$

The total expected rate of return is $[R+\mathrm{E}(\mathrm{d}V)/\mathrm{d}t]$. If the asset is to be willingly held, this total return must equal the required return ρV, where ρ is the appropriate social discount rate. I shall assume social risk-neutrality and a constant ρ, because this better highlights the role of option values in the

[3]A more rigorous treatment of a similar problem is in Dixit (1989a). The mathematics of optimal switching of Brownian motion is in Krylov (1980). Note also that my choice of the objective function amounts to assuming that the X good has a small share in the country's consumption, which is true for the most resource dependent countries.

adjustment decision. However, more sophisticated treatments based on techniques of financial economics are not difficult; see Dixit (1989a).

The equilibrium condition for the asset to be willingly held becomes the differential equation:

$$(1/2)\sigma^2 P^2 V_{PP}(P,M) + \mu P V_P(P,M) - \rho V(P,M) = -R(P,M). \tag{5}$$

This equation is linear in V and its derivatives. Therefore its general solution can be expressed as a sum of the complementary function (the general solution of the associated homogeneous equation) and a particular integral of (5). Suitable boundary conditions then determine the actual solution.

An obvious candidate for the particular integral is:

$$Q(P,M) = \mathrm{E}\int_0^\infty R(P_t, M)\exp(-\rho t)\,\mathrm{d}t. \tag{6}$$

This is just the expected discounted present value that would be obtained if the capital allocation were kept unchanged for ever. For complicated functions $R(P,M)$ this integral may not have a closed-form solution. I concentrate on the special case of the linear revenue function (2), where we have:

$$Q(P,M) = PF(M)/(\rho-\mu) + G(K-M)/\rho. \tag{7}$$

The expected discounted present value interpretation is now evident. Since the price of the X-sector output is expected to grow at the rate μ, its current value is capitalized at the effective discount rate $(\rho-\mu)$. Convergence requires $\rho > \mu$.

To find the complementary function, try a solution of the form P^ξ. Substitution yields:

$$(1/2)\sigma^2\xi(\xi-1) + \mu\xi - \rho = 0$$

or

$$\phi(\xi) \equiv \xi^2 - (1-m)\xi - r = 0,$$

where I have defined

$$m \equiv 2\mu/\sigma^2, \qquad r \equiv 2\rho/\sigma^2.$$

The convergence condition is now $r > m$. Therefore

$$\phi(0) = -r < 0, \qquad \phi(1) = -(r-m) < 0.$$

Since $\phi''(\xi) \equiv 2 > 0$, one root must be > 1 (call it β), and the other must be < 0 (call it $-\alpha$). Written out explicitly,

$$\beta = \{(1-m) + [(1-m)^2 + 4r]^{1/2}\}/2,$$

$$-\alpha = \{(1-m) - [(1-m)^2 + 4r]^{1/2}\}/2.$$

The general solution of (5) is therefore

$$V(P,M) = A(M)P^{-\alpha} + B(M)P^{\beta} + Q(P,M), \tag{8}$$

where $A(M)$ and $B(M)$ are arbitrary functions. The third term on the right-hand side is the value of preserving the initial capital allocation forever. The first two terms must then be the values of the option of changing this allocation. In fact, $A(M)P^{-\alpha}$ is the option value of being able to move some capital out of the X-sector, and $B(M)P^{\beta}$ is the option value of being able to bring some additional capital into the X-sector. This interpretation will be developed more fully in the light of the results below.

We can also formulate some boundary conditions for the endpoints $M = 0$ and $M = K$. Since $V(P,0)$ must remain finite as $P \to 0$, we must have $A(0) = 0$. This accords with the above interpretations: when there is no capital in the X-sector, the option value of moving it out is zero. On the other hand, consider $M = K$ and $P \to \infty$. Now all the capital is already on the X-sector, and since P is almost sure to stay very high for a long time into the future, we will not want to move any capital to the Y-sector for this time. Therefore $V(P,K)$ will be well approximated by the value with an invariant allocation, namely $Q(P,K)$. Then, since $\beta > 1$, the solution cannot contain a term in P^{β}, so $B(K)$ must be 0. This again accords with the option pricing interpretation: with all the capital in the X-sector, the option of adding more is worthless.

The solution can be displayed even more transparently for the case of the revenue function (2). Define:

$$f(M) = F(M) - F(M-1), \tag{9}$$

$$g(M) = G(K-(M-1)) - G(K-M), \tag{10}$$

$$a(M) = A(M) - A(M-1), \tag{11}$$

$$b(M) = B(M-1) - B(M). \tag{12}$$

Suppose units of capital are somehow ordered from 1 to K, those starting from 1 being given priority for X-sector deployment, and those starting from

K for Y-sector employment. Then $f(M)$ is the marginal product of the Mth unit of capital in the X-sector, and $g(M)$ is the marginal product of this same unit in the Y-sector. Similarly, $a(M)P^{-\alpha}$ is the option value of being able to shift this unit from the X-sector to the Y-sector, and $b(M)P^{\beta}$ the option value of the reverse shift. Then we can write (8) as:

$$V(P,M)=\sum_{j=1}^{M}\{a(j)P^{-\alpha}+Pf(j)/(\rho-\mu)\}$$

$$+\sum_{j=M+1}^{K}\{b(j)P^{\beta}+g(j)/\rho\}. \tag{13}$$

The total asset value has been expressed as the sum of the values of the individual units of capital, in each case comprising the marginal product in the sector of current deployment and the value of the option to switch to the other sector.

The discussion thus far has been confined to the intervals where it is assumed to be optimal to keep M unchanged for the present, even though the value of the option of changing it some time in the future is allowed for. The next step is to determine the prices at which a change becomes optimal. Let $P_H(M)$ be the price at which a switch from $(M-1)$ to M becomes optimal.[4] The theory of option pricing gives us two conditions that characterize $P_H(M)$: the value-matching condition,

$$V(P_H(M),M-1)=V(P_H(M),M)-h(P_H(M),M), \tag{14}$$

and the high-order contact or smooth pasting condition,

$$V_P(P_H(M),M-1)=V_P(P_H(M),M)-h_P(P_H(M),M). \tag{15}$$

The value-matching condition is evident; it equates the value of the option to the value of the asset being acquired *minus* the exercise price. If this failed, arbitrage profits would be possible. The high-order contact condition is trickier; if it failed, moving $P_H(M)$ in the direction of the steeper slope would raise the value of the option. See the discussion in Merton (1973, footnote 60), Brennan and Schwartz (1985, p. 143) and Majd and Pindyck (1987, footnote 11). For a heuristic derivation, see Dixit (1988).

[4] I should also consider transitions from $(M-2)$, etc. to M and vice versa. However, since the revenue function is concave in M, and the adjustment costs are convex, the prices that trigger single transitions are reached before those that trigger multiple transitions. Therefore the multiple transitions do not take place, except perhaps at $t=0$ when the initial capital allocation may be very inappropriate. I have omitted the rigorous treatment of this minor qualification for ease of exposition.

Similar conditions apply to the switch from M units to $(M-1)$ in the X-sector. The critical price $P_L(M)$ is characterized by

$$V(P_L(M), M) = V(P_L(M), M-1) - l(P_L(M), M) \tag{16}$$

and

$$V_P(P_L(M), M) = V_P(P_L(M), M-1) - l_P(P_L(M), M). \tag{17}$$

Using the general solution (8) and the definitions (11) and (12), we can write:

$$V(P, M) - V(P, M-1) = a(M)P^{-\alpha} - b(M)P^{\beta} + [Q(P, M) - Q(P, M-1)],$$

and similarly

$$V_P(P, M) - V_P(P, M-1) = -\alpha a(M)P^{-\alpha-1} - \beta b(M)P^{\beta-1}$$
$$+ [Q_P(P, M) - Q_P(P, M-1)].$$

Then (14)–(17) are four equations that determine the four unknowns $a(M)$, $b(M)$, $P_H(M)$ and $P_L(M)$, completing the solution.

For the case of the revenue function (2), the equations are more explicitly written as:

$$a(M)P_H(M)^{-\alpha} + P_H(M)f(M)/(\rho-\mu) - b(M)P_H(M)^{\beta} - g(M)/\rho$$
$$= h_0 g(M), \tag{18}$$

$$-\alpha a(M)P_H(M)^{-\alpha-1} + f(M)/(\rho-\mu) - \beta b(M)P_H(M)^{\beta-1} = 0, \tag{19}$$

$$a(M)P_L(M)^{-\alpha} + P_L(M)f(M)/(\rho-\mu) - b(M)P_L(M)^{\beta} - g(M)/\rho$$
$$= -l_0 P_L(M)f(M), \tag{20}$$

$$-\alpha a(M)P_L(M)^{-\alpha-1} + f(M)/(\rho-\mu) - \beta b(M)P_L(M)^{\beta-1} = -l_0 f(M), \tag{21}$$

In section 3 I shall obtain some illustrative numerical solutions to this system. Here I establish an important general property of the solution. For this purpose, fix M and define:

$$\Phi(P) = a(M)P^{-\alpha} + Pf(M)/(\rho-\mu) - b(M)P^{\beta} - g(M)/\rho.$$

This function is convex for small values of P and concave for large ones. The

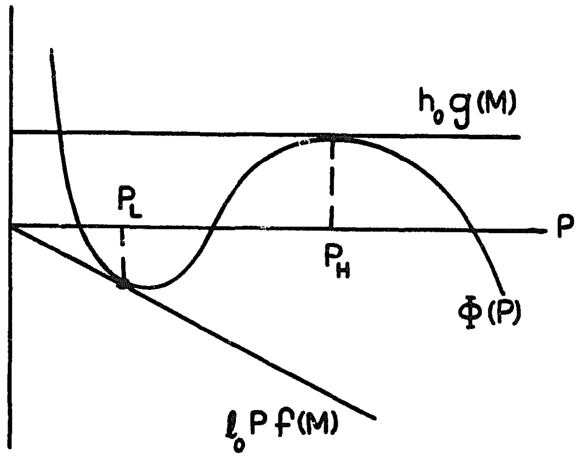

Fig. 1. Determination of P_L and P_H.

general shape is as shown in fig. 1. Using this function, eqs. (18)–(21) can be written:

$$\Phi(P_H(M)) = h_0 g(M), \qquad \Phi_P(P_H(M)) = 0,$$

$$\Phi(P_L(M)) = -l_0 P_L(M) f(M), \qquad \Phi_P(P_L(M)) = -l_0 f(M).$$

Then the solution can be visualized using fig. 1. Adjust $a(M)$ and $b(M)$ until the curve $\Phi(P)$ touches the lines $h_0 g(M)$ and $-l_0 P f(M)$. The abscissae of the points of tangency define $P_H(M)$ and $P_L(M)$, respectively. Note that $\Phi''(P)$ is negative at $P_H(M)$ and positive at $P_L(M)$.

Since Φ is formed by subtracting a solution of (5) corresponding to $(M-1)$ from a solution of the same equation for M, it satisfies the differential equation:

$$\begin{aligned}(1/2)\sigma^2 P^2 \Phi''(P) + \mu P \Phi'(P) - \rho \Phi(P) &= -[R(P,M) - R(P,M-1)] \\ &= -[Pf(M) - g(M)].\end{aligned}$$

Therefore

$$\begin{aligned}0 &< (1/2)\sigma^2 P_L(M)^2 \Phi''(P_L(M)) \\ &= \rho\Phi(P_L(M)) - \mu\Phi'(P_L(M)) - [P_L(M) f(M) - g(M)] \\ &= \rho[-l_0 P_L(M) f(M)] - \mu P_L(M)[-l_0 f(M)] - P_L(M) f(M) + g(M)\end{aligned}$$

or

$$g(M) > P_L(M) f(M) + (\rho - \mu) l_0 P_L(M) f(M). \tag{22}$$

Similarly, we find:

$$P_H(M)f(M) > g(M) + \rho h_0 g(M). \tag{23}$$

The two equations embody the idea that capital reallocation should proceed with caution when we have adjustment costs and price uncertainty. For example, (23) says that at the price that triggers the movement of an additional unit of capital into the X-sector, its marginal product there *exceeds* its marginal product in the Y-sector plus the interest on the adjustment cost. This extra gap arises because there is a value of waiting to see if future price movements are unfavorable, and thereby avoiding another costly move of the capital back to the Y-sector. At the price $P_H(M)$, the cost of the current allocative inefficiency just balances the value of waiting. Similarly, (22) shows the extra gap that is stipulated for a switch of capital out of the X-sector. Here the adjustment cost is the output forgone in the X-sector, for which the appropriate effective interest rate is $(\rho-\mu)$, as we saw earlier. The numerical calculations of the next section will give us a better quantitative idea of the magnitude of the gap representing the cautionary effect.

I should emphasize that the caution does not arise because of any risk aversion. In fact I deliberately assumed risk-neutrality to avoid confusion at this point. If anything, the caution and the waiting play just the opposite role: they convexify the payoff by letting us act if the future outturn is favorable and avoid acting if it is unfavorable.

To obtain further understanding of the issue, begin with neither adjustment costs nor uncertainty, and define:

$$P_0(M) = g(M)/f(M), \tag{24}$$

the price that equates the marginal products in the two sectors at the allocation M. As soon as the actual price differs from this, there is a benefit to reallocation, and M will be changed to the point where (24) holds at the current price. Now introduce adjustment costs. These are often assumed to be quadratic in the rate of adjustment. Then the cost of a small movement is of the second order, and at least some reallocation will be undertaken as soon as $P \neq P_0(M)$. In my model, and I would argue in reality too, a first-order adjustment entails a first-order cost. Then a slight gap between P and $P_0(M)$ will not call for any change in M. To be precise, the prices $P_+(M)$ and $P_-(M)$ that triggers respectively an increase and a decrease in M will be those that satisfy (23) and (22) with equality, that is,

$$P_+(M) = (1+\rho h_0)g(M)/f(M) \tag{25}$$

and

$$P_-(M) = [g(M)/f(M)]/[1 + (\rho - \mu)l_0]. \tag{26}$$

In other words, a band within which no reallocation is optimal arises even without uncertainty. This was the result of Kemp and Wan (1974) and one model in Mussa (1984, section IV B).

What uncertainty does is to widen the band of inactivity. Comparing (22)–(26), we see that

$$P_H(M) > P_+(M) > P_0(M) > P_-(M) > P_L(M). \tag{27}$$

At the price $P_+(M)$, it would be just profitable to move the unit M to the X-sector if we knew that the price would remain at $P_+(M)$ forever. When the price may go up or down in the future, there is a value of waiting to see how things develop. If the price falls, we will have avoided a move that is not sufficiently beneficial. In the language of option pricing, $P_+(M)$ is the price at which the option of moving unit M to the X-sector is only just in the money. However, it is not optimal to exercise the option at this point. It has to go deeper in the money, and $P_H(M)$ is the optimal price at which to exercise it. Similarly for movements in the other direction.

Incidentally, if the price process has mean-reversion, that will only widen the bands still further. This is because a favorable price is more likely to be temporary, and therefore the current benefit will have to be even higher to justify capital reallocation.

In the numerical work below, I shall examine the width of the bands without and with uncertainty. It turns out that the effect of even moderate amounts of uncertainty is much greater than that of quite sizeable adjustment costs alone.

3. Numerical results

In this section I obtain some numerical solutions for the above model. A configuration of parameters to constitute a central case is established, and some variations around it are considered. The choice of parameters is governed by concerns of simplicity and rounding as much as by reality. However, the exercise should give us an initial idea of the magnitude of the effects, and suggest directions for more detailed and realistic research.

All the calculations below use the special technology of (2), with Cobb–Douglas production functions:

$$F(M) = F_0 M^{\gamma} \qquad G(K-M) = G_0(K-M)^{\delta}. \tag{28}$$

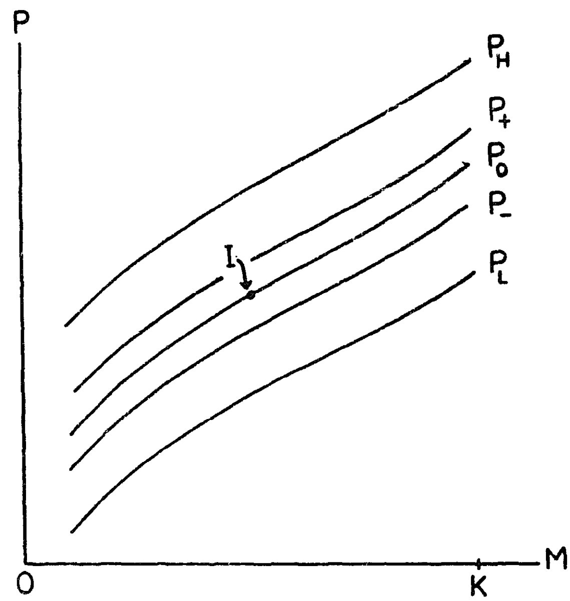

Fig. 2. Price bands governing capital reallocation.

The constants F_0 and G_0 are chosen to make the marginal products equal at the midpoint $M = K/2$. Then we can think of $P = 1$ as the normal price, and equal amounts of capital in the two sectors as the normal allocation. I set $\gamma = \delta = 0.5$ in the central case, and consider variations around that.

The other parameters in the central case are as follows: (1) $h_0 = l_0 = 1$, that is, a year's output is lost when moving capital. Most industrial activities would need a somewhat longer period. (2) $\sigma = 0.1$, or $\sigma^2 = 0.01$. The variance of the price is 1 percent per year; the standard deviation, which increases as the square root of time, is 10 percent for one year and 20 percent over four years. This is quite a modest amount of uncertainty by the standard of commodity price fluctuations. For example, Brennan and Schwartz (1985) use $\sigma^2 = 0.08$ for copper. (3) $K = 100$ (this is just a choice of units for capital, and enables us to think of the capital allocation in terms of percentages). (4) $\rho = 2.5$ percent per year (if anything a high real interest rate). (5) $\mu = 0$. Variations around all these values will be considered later.

The results can be summarized in terms of five critical prices in relation to the allocation. Fig. 2 shows the schematic picture.[5] The price functions $P_H(M)$, $P_L(M)$, $P_+(M)$, $P_-(M)$ and $P_0(M)$ defined above are drawn. For better visual appeal, the prices are shown as smooth functions of M rather than step functions. The curves are rising because of the concavity of the benefit, and the convexity of the cost, of successive adjustments, as was explained earlier.

Recall the interpretation of the curves. With neither adjustment costs nor

[5]The qualitative picture is valid in a more general model; see fig. 3 in Lucas and Prescott (1974).

Table 1

Central case.

M	10	20	30	40	50	60	70	80	90
P_H	0.42	0.64	0.84	1.04	1.28	1.57	1.95	2.54	3.77
P_+	0.33	0.50	0.66	0.83	1.01	1.24	1.55	2.01	3.00
P_0	0.32	0.49	0.65	0.81	0.99	1.21	1.51	1.97	2.92
P_-	0.32	0.49	0.64	0.80	0.97	1.18	1.47	1.90	2.79
P_L	0.25	0.38	0.50	0.63	0.77	0.94	1.17	1.53	2.26

uncertainty, the capital allocation would always be kept along the curve P_0. Now suppose we have adjustment costs but only a once-and-for-all price shift, and suppose the economy starts at the initial point I on this equilibrium curve. If the price moves to a new value lying within the band formed by the curves P_+ and P_-, the difference in marginal products is not sufficient to justify incurring the adjustment costs, and no reallocation is undertaken. If the price rises and crosses the P_+ curve, just enough capital is moved to the X-sector from the Y-sector to take the economy to the P_+ curve. Similarly for price declines and the P_- curve. Finally, let there be ongoing uncertainty as well as adjustment costs. Now the wider band formed by the curves P_H and P_L plays a similar role. For price fluctuations within the band, no capital reallocation is optimal. If the band is crossed, just enough capital is moved to bring the economy back to the nearer edge of the band.

For most of the cases considered below, the P_+ and P_- curves are so close to the P_0 curve as to be indistinguishable when sketched on a scale small enough to accommodate the P_H and P_L curves. In other words, the effect of the uncertainty is generally far more significant than that of adjustment costs alone. To bring this home without forgetting the P_+ and P_- curves altogether, I have provided the actual tables of the numbers.

Table 1 pertains to the central case. Note that P_+ and P_- are within 3 percent of P_0.[6] The band formed by P_H and P_L is much wider. For example, starting at $M=50$ and $P=1$, a rise to $P=1.28$ is needed before any capital is redeployed to the X-sector. If P doubles, 71 units of capital will be used in this sector. In the absence of uncertainty the number would have been 82. The response to a doubling of P is an interesting feature, and I shall examine it in all the variations to follow.

Each subsequent calculation changes just one of the parameters of the central case. Table 2 shows the effect of reducing l_0 to zero. Thus, moving capital from the X-sector to the Y-sector is made costless. This leaves the P_+

[6]Of course P_0 should equal 1.00, not 0.99, when $M=50$. The slight discrepancy is due to the fact that I am evaluating marginal products over integer units of capital, whereas F_0 and G_0 are chosen to ensure exact equality of the derivatives.

Table 2

$l_0=0$.

M	10	20	30	40	50	60	70	80	90
P_H	0.40	0.61	0.80	1.00	1.23	1.51	1.88	2.45	3.62
P_+	0.33	0.50	0.66	0.83	1.01	1.24	1.55	2.01	3.00
P_0	0.32	0.49	0.65	0.81	0.99	1.21	1.51	1.97	2.92
P_-	0.32	0.49	0.65	0.81	0.99	1.21	1.51	1.97	2.92
P_L	0.27	0.41	0.54	0.67	0.82	1.01	1.25	1.64	2.43

Table 3

$\sigma=0.25$.

M	10	20	30	40	50	60	70	80	90
P_H	0.51	0.78	1.02	1.28	1.57	1.92	2.39	3.12	4.63
P_+	0.33	0.50	0.66	0.83	1.01	1.24	1.55	2.01	3.00
P_0	0.32	0.49	0.65	0.81	0.99	1.21	1.51	1.97	2.92
P_-	0.32	0.49	0.64	0.80	0.97	1.18	1.47	1.90	2.79
P_L	0.20	0.31	0.41	0.51	0.62	0.76	0.95	1.24	1.84

and P_0 curves unaffected, and moves the P_- curve up to coincide with P_0. More interestingly, the P_H curve falls and the P_L curve rises. The latter makes immediate sense, but the former needs some thought. The cost of moving capital into the X-sector is not directly affected, and yet the trigger price curve for such movements is lowered. The point is that with uncertainty, a future price drop may require a move back to the Y-sector. This is now less costly, so the initial move to the X-sector is more willingly made. In fact the two curves move by an almost equal amount even though the parameter shift is not symmetric. This will not remain true for larger changes in the adjustment costs. For example, if l_0 goes to ∞, the $P_L(M)$ curve will go to zero, but the $P_H(M)$ curve will stay finite. Bentolila and Bertola (1987) discuss this asymmetry in more detail.

With $l_0=0$, a doubling of P leads to $M=72$, as compared to 71 in the central case. In this sense, the narrowing of the band is quite small.

Table 3 shows the effect of a higher σ. I use the value 0.25, which is still less than the actual uncertainty in many commodity prices. As expected, the band formed by P_H and P_L widens quite substantially. Starting with $P=1$ and $M=50$, a 56 percent price rise fails to trigger any capital redeployment, and a doubling of the price leads to $M=62$ as against 71 in the central case.

If the price rises to a much higher level, a large capital reallocation is justified even for a large σ. For example, with $\sigma=0.25$ and $P=4$, almost 87 percent of the capital gets shifted to the X-sector. Thus, an increase as large as that in oil prices in the early 1970s may optimally induce large structural changes in the economy. A subsequent unfavorable realization of the

Table 4

$\sigma = 0.01$.

M	10	20	30	40	50	60	70	80	90
P_H	0.34	0.52	0.69	0.86	1.05	1.29	1.61	2.10	3.12
P_+	0.33	0.50	0.66	0.83	1.01	1.24	1.55	2.01	3.00
P_0	0.32	0.49	0.65	0.81	0.99	1.21	1.51	1.97	2.92
P_-	0.32	0.49	0.64	0.80	0.97	1.18	1.47	1.90	2.79
P_L	0.30	0.46	0.60	0.76	0.93	1.14	1.41	1.85	2.71

Table 5

$F(M) \sim M^{0.75}$.

M	10	20	30	40	50	60	70	80	90
P_H	0.63	0.80	0.95	1.11	1.28	1.49	1.80	2.26	3.26
P_+	0.50	0.64	0.76	0.88	1.02	1.19	1.43	1.80	2.58
P_0	0.49	0.62	0.74	0.86	0.99	1.16	1.39	1.75	2.52
P_-	0.49	0.61	0.73	0.84	0.97	1.13	1.35	1.70	2.41
P_L	0.37	0.48	0.57	0.66	0.77	0.90	1.08	1.36	1.96

stochastic price path may make us regret that decision. However, the ex post regret does not make the ex ante decision irrational. So long as the parameters of the price process do not change, we should act as we did before if the price rises to its old height once again.

Table 4 takes σ down to the negligible level of 0.01. Even then, the gap between the two outermost curves, P_H and P_L, and the two interim curves, P_+ and P_-, is about as large as the gap between the latter and P_0. In other words, even this minor uncertainty has an effect as large as that of the adjustment cost of a year's output loss.

In table 5, the production function in the X-sector is altered by raising the power γ to 0.75, at the same time changing the constant F_0 to keep the marginal product at the midpoint unchanged. Thus, the marginal product in the X-sector is reduced for $M < K/2$ and raised for $M > K/2$. The effect is to twist all the curves clockwise around the midpoint $M = K/2$. The criteria for shifting capital into and out of the X-sector both become less stringent. However, in quantitative terms the effect is quite small.

In table 6, the price trend μ is raised to 2 percent per year. This lowers the P_H and P_L curves: capital is reallocated to the X-sector more readily, and is moved out of this sector more reluctantly. Again the effect is not very substantial; the price shifts are all less than 5 percent.

Finally, suppose capital reallocation is costless within the limits of depreciation in the contracting sector: one just directs all gross investment to the expanding sector. Does this solve the problem of responding to price fluctuations? In the central case and near the normal point, the elasticity of

Table 6

$\mu=0.02.$

M	10	20	30	40	50	60	70	80	90
P_H	0.39	0.60	0.78	0.98	1.20	1.47	1.83	2.38	3.53
P_+	0.33	0.50	0.66	0.83	1.01	1.24	1.55	2.01	3.00
P_0	0.32	0.49	0.65	0.81	0.99	1.21	1.51	1.97	2.92
P_-	0.32	0.49	0.65	0.81	0.99	1.21	1.50	1.96	2.89
P_L	0.23	0.36	0.47	0.58	0.72	0.88	1.09	1.42	2.11

the $P_0(M)$ curve is equal to 1. A price fluctuation of 0.1 in one year (one standard deviation) will entail a fall in the Y-sector capital from 50 units to 45 to restore efficiency. This is only just within the bounds of likely depreciation. Nearly a third of the time we will see wider price fluctuations, and the need for costly reallocation. For larger values of σ, or for the kinds of fluctuations in oil and copper prices we saw in the 1970s, depreciation does even less, and the kind of theory developed in this paper has a more important role.

4. Possible extensions

Several extensions of the model seem worth future attention to deepen our understanding of the issue. Most simply, more general production functions in the two sectors could be used. This will enable us to examine the role of the elasticity of substitution between capital and the specific inputs in each sector. We could also allow other factors that can move across sectors costlessly. This will make the revenue function nonlinear in P, and the particular integral Q in (6) will in general have to be evaluated numerically.

Other types of adjustment costs could be studied. Suppose labor L is costlessly mobile across sectors, and it takes h_0 units of labor services to move capital from the Y-sector to the X, and l_0 the other way round. Then, writing the revenue function as $R(P,M,L)$, we have the adjustment costs:

$$h(P,M,L)=R(P,M,L)-R(P,M,L-h_0)$$

and

$$l(P,M,L)=R(P,M,L)-R(P,M,L-l_0)$$

Finally, other stochastic processes could be considered. Poisson jumps are an appealing and tractable alternative; Dixit (1987a) discusses some general principles of their modelling in investment decisions. We could have a mean-reverting Brownian motion, or even a process in which the trend μ evolves stochastically, itself as a Brownian motion. Merton (1971) examines consumption and portfolio choices in such an environment and develops the necessary mathematics. For most of these extensions, more and much harder numerical solutions will be required. We could try to model an endogenous mechanism of learning about μ and σ, although that would be a more difficult modelling task.

5. Concluding remarks

Capital allocation is not the only decision sensitive to ongoing price uncertainty. The intertemporal consumption choice is at least as important, and probably just as easily mishandled, in many commodity-producing countries. The work of Merton (1971) and others is already a useful guide in this matter. An additional problem may also be relevant: a downward adjustment of consumption is especially costly because some infrastructure for the extra consumption is installed and then wasted, or because the people become 'addicted' to a higher standard of living. The technique developed in this paper seem useful for examining this situation, and deriving the optimal degree of caution that should be exercised when a commodity price boom makes a higher current consumption level feasible.

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