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Faculty of Computer Science and Business Information Systems
5172080: Fundamentals of Mobile Robotics

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Tutorial 1 – Solutions

Topic: Linear Algebra

Q 1 – Vectors and Matrices

1.a

Calculate the values of the expressions

$$\boldsymbol{a}+\boldsymbol{b}, \quad \boldsymbol{b}+\boldsymbol{a}, \quad k\boldsymbol{a}+k\boldsymbol{b}, \quad k(\boldsymbol{a}+\boldsymbol{b}), \quad (k\boldsymbol{a})\cdot\boldsymbol{b}, \quad ||\boldsymbol{a}||$$

if $\boldsymbol{a} = \begin{bmatrix}1 & 2\end{bmatrix}^T$, $\boldsymbol{b} = \begin{bmatrix}2 & 3\end{bmatrix}^T$, *and* $k = 3$.

$\boldsymbol{a}+\boldsymbol{b}$ and $\boldsymbol{b}+\boldsymbol{a}$ are equal, because

$$\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}1+2\\2+3\end{bmatrix} = \begin{bmatrix}3\\5\end{bmatrix} = \begin{bmatrix}2+1\\3+2\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix} + \begin{bmatrix}1\\2\end{bmatrix}.$$

Equality also holds for $k\boldsymbol{a}+k\boldsymbol{b}$ and $k(\boldsymbol{a}+\boldsymbol{b})$, as

$$3\begin{bmatrix}1\\2\end{bmatrix} + 3\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}3\cdot 1\\3\cdot 2\end{bmatrix} + \begin{bmatrix}3\cdot 2\\3\cdot 3\end{bmatrix} = \begin{bmatrix}3\\6\end{bmatrix} + \begin{bmatrix}6\\9\end{bmatrix} = \begin{bmatrix}9\\15\end{bmatrix} = \begin{bmatrix}3\cdot 3\\3\cdot 5\end{bmatrix} = 3\begin{bmatrix}3\\5\end{bmatrix} = 3\left(\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}2\\3\end{bmatrix}\right).$$

The value of $(k\boldsymbol{a})\cdot\boldsymbol{b}$ can be found using the definition of the dot product, i.e.

$$\left(3\begin{bmatrix}1\\2\end{bmatrix}\right)\cdot\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}3\\6\end{bmatrix}\cdot\begin{bmatrix}2\\3\end{bmatrix} = 3\cdot 2 + 6\cdot 3 = 6 + 18 = 24.$$

$||\boldsymbol{a}||$ is the length of vector $\boldsymbol{a}$. The length can be found with its definition, i.e.

$$\sqrt{1^2+2^2} = \sqrt{1+4} = \sqrt{5} = 2.2361.$$

1.b

Determine if the vectors $\begin{bmatrix}1 & 2\end{bmatrix}^T$ *and* $\begin{bmatrix}-2 & -4\end{bmatrix}^T$ *are linearly dependent.*

The two vectors are indeed linearly dependent, as $k = -2$ can be found so that

$$\begin{bmatrix}-2\\-4\end{bmatrix} = -2\begin{bmatrix}1\\2\end{bmatrix}.$$

1.c

Calculate the values of the expressions

$$\boldsymbol{Aa}, \quad \boldsymbol{A^T A}, \quad \boldsymbol{AA^T}, \quad \boldsymbol{A^2}$$

if $\boldsymbol{A} = \begin{bmatrix}1 & -4\\-2 & 5\\3 & -6\end{bmatrix}$, *and* $\boldsymbol{a} = \begin{bmatrix}1\\2\end{bmatrix}$.

$\boldsymbol{Aa}$ is a matrix vector product. It can be calculated by multiplying all row vectors of the matrix $\boldsymbol{A}$ by the vector $\boldsymbol{a}$, i.e.

$$\begin{bmatrix}1 & -4\\-2 & 5\\3 & -6\end{bmatrix} \cdot \begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}1 & -4\end{bmatrix} \cdot \begin{bmatrix}1 & 2\end{bmatrix}^T\\ \begin{bmatrix}-2 & 5\end{bmatrix} \cdot \begin{bmatrix}1 & 2\end{bmatrix}^T\\ \begin{bmatrix}3 & -6\end{bmatrix} \cdot \begin{bmatrix}1 & 2\end{bmatrix}^T\end{bmatrix} = \begin{bmatrix}1 \cdot 1 - 4 \cdot 2\\ -2 \cdot 1 + 5 \cdot 2\\ 3 \cdot 1 - 6 \cdot 2\end{bmatrix} = \begin{bmatrix}1 - 8\\-2 + 10\\3 - 12\end{bmatrix} = \begin{bmatrix}-7\\8\\-9\end{bmatrix}.$$

$\boldsymbol{A^T A}$ is a matrix matrix product. It can be found by multiplying all row vectors of the matrix $\boldsymbol{A^T}$ by each column vector of the matrix $\boldsymbol{A}$, i.e.

$$\begin{bmatrix}1 & -4\\-2 & 5\\3 & -6\end{bmatrix}^T \cdot \begin{bmatrix}1 & -4\\-2 & 5\\3 & -6\end{bmatrix} = \begin{bmatrix}1 & -2 & 3\\-4 & 5 & -6\end{bmatrix} \cdot \begin{bmatrix}1 & -4\\-2 & 5\\3 & -6\end{bmatrix} =$$

$$\begin{bmatrix}\begin{bmatrix}1 & -2 & 3\end{bmatrix} \cdot \begin{bmatrix}1 & -2 & 3\end{bmatrix}^T & \begin{bmatrix}1 & -2 & 3\end{bmatrix} \cdot \begin{bmatrix}-4 & 5 & -6\end{bmatrix}^T\\ \begin{bmatrix}-4 & 5 & -6\end{bmatrix} \cdot \begin{bmatrix}1 & -2 & 3\end{bmatrix}^T & \begin{bmatrix}-4 & 5 & -6\end{bmatrix} \cdot \begin{bmatrix}-4 & 5 & -6\end{bmatrix}^T\end{bmatrix} =$$

$$\begin{bmatrix}1 + 4 + 9 & -4 - 10 - 18\\-4 - 10 - 18 & 16 + 25 + 36\end{bmatrix} = \begin{bmatrix}14 & -32\\-32 & 77\end{bmatrix}.$$

$\boldsymbol{AA^T}$ is also a matrix matrix product. Note that $\boldsymbol{A^TA} \neq \boldsymbol{AA^T}$ shows that matrix multiplication is not commutative.

$$\begin{bmatrix} 1 & -4 \\ -2 & 5 \\ 3 & -6 \end{bmatrix} \cdot \begin{bmatrix} 1 & -4 \\ -2 & 5 \\ 3 & -6 \end{bmatrix}^T = \begin{bmatrix} 1 & -4 \\ -2 & 5 \\ 3 & -6 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & 3 \\ -4 & 5 & -6 \end{bmatrix} =$$

$$\begin{bmatrix} \begin{bmatrix}1 & -4\end{bmatrix} \cdot \begin{bmatrix}1 & -4\end{bmatrix}^T & \begin{bmatrix}1 & -4\end{bmatrix} \cdot \begin{bmatrix}-2 & 5\end{bmatrix}^T & \begin{bmatrix}1 & -4\end{bmatrix} \cdot \begin{bmatrix}3 & -6\end{bmatrix}^T \\ \begin{bmatrix}-2 & 5\end{bmatrix} \cdot \begin{bmatrix}1 & -4\end{bmatrix}^T & \begin{bmatrix}-2 & 5\end{bmatrix} \cdot \begin{bmatrix}-2 & 5\end{bmatrix}^T & \begin{bmatrix}-2 & 5\end{bmatrix} \cdot \begin{bmatrix}3 & -6\end{bmatrix}^T \\ \begin{bmatrix}3 & -6\end{bmatrix} \cdot \begin{bmatrix}1 & -4\end{bmatrix}^T & \begin{bmatrix}3 & -6\end{bmatrix} \cdot \begin{bmatrix}-2 & 5\end{bmatrix}^T & \begin{bmatrix}3 & -6\end{bmatrix} \cdot \begin{bmatrix}3 & -6\end{bmatrix}^T \end{bmatrix} =$$

$$\begin{bmatrix} 1+16 & -2-20 & 3+24 \\ -2-20 & 4+25 & -6-30 \\ 3+24 & -6-30 & 9+36 \end{bmatrix} = \begin{bmatrix} 17 & -22 & 27 \\ -22 & 29 & -36 \\ 27 & -36 & 45 \end{bmatrix}.$$

$\boldsymbol{A^2} = \boldsymbol{AA}$ is undefined because the columns of the left matrix do not match the rows of the right matrix, as can be seen from

$$\begin{bmatrix} 1 & -4 \\ -2 & 5 \\ 3 & -6 \end{bmatrix} \cdot \begin{bmatrix} 1 & -4 \\ -2 & 5 \\ 3 & -6 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix}1 & -4\end{bmatrix} \cdot \begin{bmatrix}1 & -2 & 3\end{bmatrix}^T & \begin{bmatrix}1 & -4\end{bmatrix} \cdot \begin{bmatrix}-4 & 5 & -6\end{bmatrix}^T \\ \begin{bmatrix}-2 & 5\end{bmatrix} \cdot \begin{bmatrix}1 & -2 & 3\end{bmatrix}^T & \begin{bmatrix}-2 & 5\end{bmatrix} \cdot \begin{bmatrix}-4 & 5 & -6\end{bmatrix}^T \\ \begin{bmatrix}3 & -6\end{bmatrix} \cdot \begin{bmatrix}1 & -2 & 3\end{bmatrix}^T & \begin{bmatrix}3 & -6\end{bmatrix} \cdot \begin{bmatrix}-4 & 5 & -6\end{bmatrix}^T \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} \text{undefined} & \text{undefined} \\ \text{undefined} & \text{undefined} \\ \text{undefined} & \text{undefined} \end{bmatrix} \rightarrow \text{undefined}.$$

1.d

Calculate the determinants of the matrices

$$\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Determinants of 2×2 matrices can be easily found using the equation on slide 23 of the "Linear Algebra Primer" lecture, i.e.

$$det \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} = \begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} = 4 \cdot 2 - 1 \cdot 3 = 5,$$
$$det \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 1 \cdot 4 - 2 \cdot 2 = 0.$$

Similarly, the determinant of a 3×3 matrix can be calculated with Sarrus rule, i.e.

$$det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 1 \cdot 2 \cdot 3 + 0 \cdot 0 \cdot 0 + 0 \cdot 0 \cdot 0 - 1 \cdot 0 \cdot 0 - 0 \cdot 0 \cdot 3 - 0 \cdot 2 \cdot 1 = 6.$$

Q 2 – 2D Homogenous Transformations

Considering a robot moving on a plane, its pose w.r.t. a global coordinate frame is commonly written as $\boldsymbol{x} = (x, y, \theta)^T$, *where* (x, y) *denotes its position in the xy-plane and* θ *its orientation. The homogeneous transformation matrix that represents a pose* $\boldsymbol{x} = (x, y, \theta)^T$ *w.r.t. to the origin* $(0, 0, 0)^T$ *of the global coordinate system is given by*

$$^0\boldsymbol{T}_x = \begin{bmatrix} \boldsymbol{R}(\theta) & \boldsymbol{t} \\ 0 & 1 \end{bmatrix}, \boldsymbol{R}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, \boldsymbol{t} = \begin{bmatrix} x \\ y \end{bmatrix}$$

2.a

While being at pose $\boldsymbol{x_1} = (x_1, y_1, \theta_1)^T$, *the robot senses a landmark* l *at position* (l_x, l_y) *w.r.t. to its local frame. Use the matrix* $^g\boldsymbol{T}_{x_1}$ *to calculate the coordinates of* l *w.r.t. the global frame.*

$$\text{Let } ^g\boldsymbol{T}_{x_1} = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & x_1 \\ \sin(\theta_1) & \cos(\theta_1) & y_1 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } ^{x_1}\boldsymbol{l} = \begin{bmatrix} l_x \\ l_y \\ 1 \end{bmatrix} \text{ then}$$

$^g\boldsymbol{T}_{x_1}$ is the matrix expression in homogeneous form of pose $\boldsymbol{x_1}$ w.r.t. the global reference frame, while $^{x_1}\boldsymbol{l}$ is the vector expression in homogeneous form of the landmark w.r.t. the robot reference frame $\boldsymbol{x_1}$.

The question asks to compute the landmark coordinate w.r.t. the global frame, i.e. $^g\boldsymbol{l}$:

$$^g\boldsymbol{l} = {}^g\boldsymbol{T}_{x_1} \cdot {}^{x_1}\boldsymbol{l} \tag{1}$$

2.b

Now imagine that you are given the landmark's coordinates w.r.t. to the global frame. How can you calculate the coordinates that the robot will sense in his local frame?

We are given $^g\boldsymbol{l}$ and $^g\boldsymbol{T}_{x_1}$ and we want to compute $^{x_1}\boldsymbol{l}$. We can solve this either by taking (1) and solving with respect to $^{x_1}\boldsymbol{l}$ by multiplying to the left and right hand side by $(^g\boldsymbol{T}_{x_1})^{-1}$. Or we follow the same logic as the previous sub-question, i.e.

$$^{x_1}\boldsymbol{l} = {}^{x_1}\boldsymbol{T}_g \cdot {}^g\boldsymbol{l} = ({}^g\boldsymbol{T}_{x_1})^{-1} \cdot {}^g\boldsymbol{l}$$

2.c

The robot moves to a new pose $\boldsymbol{x_2} = (x_2, y_2, \theta_2)^T$ w.r.t. the global frame. Find the transformation matrix ${}^{x_1}\boldsymbol{T}_{x_2}$ that represents the new pose w.r.t. to $\boldsymbol{x_1}$. Hint: Write ${}^{x_1}\boldsymbol{T}_{x_2}$ as a product of homogeneous transformation matrices.

$$\text{Let } {}^{g}\boldsymbol{T}_{x_2} = \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) & x_2 \\ \sin(\theta_2) & \cos(\theta_2) & y_2 \\ 0 & 0 & 1 \end{bmatrix}$$

${}^{g}\boldsymbol{T}_{x_2}$ is the matrix expression in homogeneous form of pose $\boldsymbol{x_2}$ w.r.t. the global reference frame. This time we need to compute the homogeneous matrix form of the pose $\boldsymbol{x_2}$ expressed w.r.t. the reference frame of $\boldsymbol{x_1}$, i.e. ${}^{x_1}\boldsymbol{T}_{x_2}$. Again, we follow the rules of transformation concatenation and we find:

$${}^{x_1}\boldsymbol{T}_{x_2} = {}^{x_1}\boldsymbol{T}_{g} \cdot {}^{g}\boldsymbol{T}_{x_2} = \left({}^{g}\boldsymbol{T}_{x_1}\right)^{-1} \cdot {}^{g}\boldsymbol{T}_{x_2}$$