Advanced Treatise on Number Systems and Quantitative Aptitude: Theory, Algorithms, and Applications

1. Introduction: The Mathematical Architecture of Quantitative Aptitude

The study of numbers is not merely a collection of arithmetic operations but a profound exploration of the structural properties that govern the quantitative universe. In the realm of quantitative aptitude, the number system serves as the bedrock upon which all logical reasoning, data interpretation, and advanced problem-solving techniques are constructed. To master aptitude problems, one must transcend the superficial understanding of numbers as mere counting tools and appreciate them as elements of complex algebraic fields, rings, and groups. This treatise aims to provide an exhaustive, expert-level analysis of number systems, covering classification, divisibility theory, modular arithmetic, factorial properties, and the nuanced behaviors of digits in exponential expansions.

The utility of this knowledge extends far beyond academic curiosity. In professional testing environments and computational applications, the ability to rapidly decompose integers into prime factors, determine remainders without long division, or predict the trailing digits of astronomical numbers is invaluable. We shall dissect these topics with rigorous attention to the underlying theorems—ranging from Euclid's fundamental algorithms to the elegant modularity of Fermat and Euler—while maintaining a focus on practical application through shortcuts and heuristic methods.

1.1 The Classification of the Real Number System

The hierarchy of numbers is a nested structure where each category refines the definition of the previous one, introducing new properties and capabilities.

1.1.1 The Structure of Real Numbers ($\mathbb{R}$)

At the apex of the standard quantitative framework lies the set of Real Numbers ($\mathbb{R}$), which encompasses every value that can be mapped to a continuous number line. This continuum is essential for calculus and physics but, in discrete aptitude contexts, serves primarily as the universe containing our specific sets of interest: Rational and Irrational numbers.

Rational Numbers ($\mathbb{Q}$): A rational number is defined by its ability to be expressed as a ratio $\frac{p}{q}$, where p and q are integers and $q \neq 0$. The decimal expansion of a rational number exhibits a specific behavior: it either terminates (e.g., $\frac{1}{8} = 0.125$) or repeats indefinitely (e.g., $\frac{4}{11} = 0.3636\ldots$). This property of repetition is not random; it is a direct consequence of the division algorithm in base-10, where the finite number of possible remainders guarantees that a sequence must eventually loop.

Irrational Numbers ($\mathbb{R} \setminus \mathbb{Q}$): In contrast, irrational numbers cannot be expressed as a simple fraction. Their decimal expansions are chaotic, non-terminating, and non-recurring. The most famous examples include the transcendental numbers like π and e, and algebraic irrationals like $\sqrt{2}$. In aptitude problems, the distinction is vital when dealing with surds and approximations. For instance, while $\frac{22}{7}$ is a rational approximation of π, it is functionally distinct in high-precision contexts.

1.1.2 Integers ($\mathbb{Z}$) and Their Subsets

The set of Integers ($\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$) introduces the concept of directionality (sign) to quantity. Within this set, we identify subsets that are frequently targeted in classification problems:

- **Natural Numbers ($\mathbb{N}$):** The counting numbers $\{1, 2, 3, \dots\}$.
- **Whole Numbers (W):** The union of Natural Numbers and the identity element zero $\{0, 1, 2, \dots\}$.

The inclusion of zero is non-trivial. Zero is the additive identity and plays a unique role in divisibility (it is divisible by every non-zero integer, but no integer is divisible by zero).

1.2 Prime and Composite Theory

The Fundamental Theorem of Arithmetic states that every integer greater than 1 is either a prime itself or can be represented as the product of prime numbers in a unique way, up to the order of factors. This theorem is the "atomic theory" of numbers, establishing primes as the building blocks of the integers.

1.2.1 Prime Numbers

A prime number has exactly two distinct divisors: 1 and itself. The sequence $2, 3, 5, 7, 11, 13, 17, 19, 23, \dots$ reveals several critical properties:

- **Uniqueness of 2:** It is the only even prime. All other primes are odd. This implies that the sum of two primes is even, *unless* one of them is 2. This fact is often exploited in parity-based logic puzzles.
- **Distribution:** There are 25 primes between 1 and 100. Memorizing these is often recommended for speed in competitive scenarios.
- **Form of Primes:** Every prime number greater than 3 can be expressed in the form $6k \pm 1$ for some integer k. While the converse is not always true (e.g., $25 = 6(4) + 1$ is not prime), this form is a powerful filter for identifying potential prime candidates.

1.2.2 Composite Numbers and Unity

Composite numbers have more than two factors. They can be decomposed into prime factors (e.g., $12 = 2^2 \times 3$).

- **The Number 1:** It occupies a unique status as neither prime nor composite. It is the multiplicative identity unit. Including 1 in prime factorization would violate the uniqueness condition of the Fundamental Theorem of Arithmetic.

1.2.3 Co-Primality

Two numbers a and b are co-prime (or relatively prime) if their Highest Common Factor (HCF) is 1. This relationship, denoted as $(a, b) = 1$, does not require a or b to be prime themselves. For example, 15 (3×5) and 8 (2^3) are co-prime. Co-primality is the prerequisite for many advanced theorems, including Euler's Totient Theorem and the divisibility rules for composite numbers.

2. The Theory of Divisibility

Divisibility rules are often presented as arbitrary lists of checks, but they are deeply rooted in the properties of modular arithmetic. Understanding the *why* behind these rules empowers one to derive rules for unusual numbers and handle complex divisibility queries. The decimal number system expresses a number N as a sum of powers of 10:

$$N = a_n 10^n + a_{n-1} 10^{n-1} + \cdots + a_1 10^1 + a_0 10^0$$

Divisibility rules essentially check $N \pmod d$ by exploiting the behavior of $10^k \pmod d$.

2.1 Standard Divisibility Rules (2 through 11)

Divisor	Rule	Mathematical Derivation
2	Last digit is even (0, 2, 4, 6, 8).	$N = 10k + a_0$. Since $10 \equiv 0 \pmod 2$, $N \equiv a_0 \pmod 2$. If a_0 is divisible by 2, N is divisible by 2.
3	Sum of digits is divisible by 3.	$10 \equiv 1 \pmod 3$, so $10^k \equiv 1^k \equiv 1 \pmod 3$. Thus, $N \equiv \sum a_i \pmod 3$. The remainder of N divided by 3 is the same as the sum of digits divided by 3.
4	Last two digits divisible by 4.	$100 = 4 \times 25 \equiv 0 \pmod 4$. All terms involving 10^2 or higher vanish. Divisibility depends only on $10a_1 + a_0$.
5	Last digit is 0 or 5.	$10 \equiv 0 \pmod 5$. $N \equiv a_0 \pmod 5$. N is divisible if a_0 is 0 or 5.
6	Divisible by 2 AND 3.	Since $6 = 2 \times 3$ and gcd(2,3)=1, divisibility by both factors is required and sufficient.
8	Last three digits divisible by 8.	$1000 = 8 \times 125 \equiv 0 \pmod 8$. Higher powers vanish. Only the number formed by the last three digits matters.
9	Sum of digits divisible by 9.	Similar to 3: $10 \equiv 1 \pmod 9$. $N \equiv \sum a_i \pmod 9$. This is often called "casting out nines".
10	Last digit is 0.	Requires divisibility by 2 and 5 simultaneously.
11	Difference of alternating digit sums is divisible by 11.	$10 \equiv -1 \pmod{11}$. Thus $10^k \equiv (-1)^k \pmod{11}$. $N \equiv a_0 - a_1 + a_2 - a_3 \dots \pmod{11}$. We sum digits at odd places and subtract the sum of digits at even places.

2.2 Advanced Divisibility: The Osculator Method

For prime numbers like 7, 13, 17, 19, 23, and 29, simple digit summation or truncation rules do not apply directly. Instead, we utilize a technique involving "seed numbers" or "osculators." This method relies on finding a multiple of the divisor that is close to a multiple of 10.

2.2.1 The General Theory of Seed Numbers

To find the divisibility rule for a prime P ending in 1, 3, 7, or 9:

1. Find a multiple of P that equals $10k+1$ or $10k-1$.
2. If the multiple is $10k-1$ (ends in 9), the **Seed Number** is k, and the operation is **Addition**.
3. If the multiple is $10k+1$ (ends in 1), the **Seed Number** is k, and the operation is **Subtraction**. The rule involves truncating the last digit of the number, multiplying it by the Seed Number, and applying the operation to the remaining part of the number. This process is recursive.

2.2.2 Specific Prime Rules

Divisibility by 7:

- Step 1: Find multiple. $7 \times \text{something} \approx 10k$. $7 \times 3 = 21$.
- Step 2: $21 = 2 \times 10 + 1$. This is of the form $10k+1$ with $k = 2$.
- Step 3: Operation is **Subtraction**. Seed is 2.
- **Rule:** Subtract 2 times the last digit from the remaining number.
- *Example:* Check 343.
  - $34 - (2 \times 3) = 34 - 6 = 28$.
  - 28 is divisible by 7. Thus, 343 is divisible by 7.

Divisibility by 13:

- Step 1: Find multiple. $13 \times 3 = 39$.
- Step 2: $39 = 4 \times 10 - 1$. This is of the form $10k-1$ with $k = 4$.
- Step 3: Operation is **Addition**. Seed is 4.
- **Rule:** Add 4 times the last digit to the remaining number.
- *Example:* Check 169.
  - $16 + (4 \times 9) = 16 + 36 = 52$.
  - $52 = 13 \times 4$. Thus, 169 is divisible by 13.

Divisibility by 17:

- Step 1: Find multiple. $17 \times 3 = 51$.
- Step 2: $51 = 5 \times 10 + 1$. This is $10k+1$ with $k = 5$.
- Step 3: Operation is **Subtraction**. Seed is 5.
- **Rule:** Subtract 5 times the last digit from the remaining number.

Divisibility by 19:

- Step 1: Find multiple. 19 itself is $2 \times 10 - 1$.
- Step 2: Form is $10k-1$ with $k = 2$.
- Step 3: Operation is **Addition**. Seed is 2.

- **Rule:** Add 2 times the last digit to the remaining number.

Divisibility by 23:

- Step 1: $23 \times 3 = 69 = 7 \times 10 - 1$.
- Step 2: Seed is 7. Operation is **Addition**.
- **Rule:** Add 7 times the last digit to the remaining number.
- *Example:* Check 529.
  - $52 + (7 \times 9) = 52 + 63 = 115$.
  - Check 115: $11 + (7 \times 5) = 11 + 35 = 46$.
  - 46 is 23×2. Confirmed.

2.3 Composite Divisibility Rules

For a composite number $N = a \times b$, a number is divisible by N if and only if it is divisible by both a and b, provided that a **and** b **are co-prime**.

- **12:** Check for 3 and 4. (HCF(3,4)=1).
- **24:** Check for 3 and 8. (HCF(3,8)=1).
- **72:** Check for 8 and 9.
- **99:** Check for 9 and 11.
- **88:** Check for 8 and 11.
- **Invalid Method:** For 24, checking 4 and 6 is invalid because HCF(4, 6) = 2. A number like 12 is divisible by both 4 and 6, but not by 24. The factors used must have no common factors other than 1.

3. The Architecture of Factors

The study of factors allows for deep insight into the composition of numbers. When we prime factorize a composite number N, we express it as:

$$N = p_1^a \times p_2^b \times p_3^c \dots$$

where $p_1, p_2, \dots$ are distinct prime numbers and a, b, c are positive integers. This "canonical form" is the starting point for calculating various properties.

3.1 Total Number of Factors

Every divisor of N is formed by taking a combination of the prime factors. For the prime p_1, we can choose to include it in the divisor with a power of $0, 1, 2, \dots, a$. This gives $(a + 1)$ choices. Similarly, there are $(b + 1)$ choices for p_2, and so on. The Fundamental Counting Principle tells us that the total number of distinct divisors is the product of these choices.

$$\text{Total Factors } (F) = (a + 1)(b + 1)(c + 1) \dots$$

- *Example:* $N = 360 = 2^3 \times 3^2 \times 5^1$.

- Factors = $(3+1)(2+1)(1+1) = 4 \times 3 \times 2 = 24$.

3.2 Sum of Factors

The sum of all divisors is the product of the geometric series sums associated with each prime factor. The sum of divisors $\sigma(N)$ is given by:

$$\sigma(N) = \left(\frac{p_1^{a+1} - 1}{p_1 - 1}\right)\left(\frac{p_2^{b+1} - 1}{p_2 - 1}\right)\dots$$

- *Derivation:* The sum is the expansion of the product $(p_1^0 + p_1^1 + \dots + p_1^a)(p_2^0 + \dots + p_2^b)\dots$.
- *Example:* $360 = 2^3 \times 3^2 \times 5^1$.
  - Term 1: $\frac{2^4-1}{2-1} = 15$.
  - Term 2: $\frac{3^3-1}{3-1} = 13$.
  - Term 3: $\frac{5^2-1}{5-1} = 6$.
  - Sum = $15 \times 13 \times 6 = 1170$.

3.3 Product of Factors

Often, numbers display a symmetry in their factors. If d is a divisor of N, then N/d is also a divisor. Thus, factors come in pairs that multiply to N.

$$\text{Product of Factors} = N^{\frac{\text{Total Factors}}{2}}$$

- *Example:* For 360, factors = 24.
  - Product = $360^{24/2} = 360^{12}$.
- *Perfect Squares:* If N is a perfect square, it has an odd number of factors. The "middle" factor is $\ N$. The formula still holds: $N^{F/2}$ will simply result in a fractional power that resolves to an integer because $\ N$ is an integer.
  - Example: $N = 9$. Factors: 1, 3, 9 (3 factors).
  - Product = $9^{3/2} = (\ 9)^3 = 3^3 = 27$. Correct ($1 \times 3 \times 9 = 27$).

3.4 Specialized Factor Properties

- **Odd Factors:** To find the number of odd factors, we simply ignore the even prime factor (2) in the calculation.
  - For $360 = 2^3 \times 3^2 \times 5^1$, consider only $3^2 \times 5^1$.
  - Odd Factors = $(2+1)(1+1) = 6$.
- **Even Factors:** Total Factors − Odd Factors.
  - For 360: $24 - 6 = 18$.

    - *Alternative:* Multiply the count of choices for the prime 2 excluding the 2^0 case (so a choices instead of $a+1$) by the standard choices for other primes: $a(b+1)(c+1) = 3(3)(2) = 18$.
- **Perfect Square Factors:** We need factors where every prime exponent is even.
    - For $360 = 2^3 \times 3^2 \times 5^1$.
    - Possible exponents for 2: 0, 2 (2 choices).
    - Possible exponents for 3: 0, 2 (2 choices).
    - Possible exponents for 5: 0 (1 choice).
    - Total perfect square factors = $2 \times 2 \times 1 = 4$ (They are 1, 4, 9, 36).

4. Multiples and Commonality: HCF and LCM

The study of Highest Common Factors (HCF) and Least Common Multiples (LCM) allows for the synchronization of periodic events and the simplification of complex fractions.

4.1 Highest Common Factor (HCF)

The HCF is the largest number that divides two or more numbers without leaving a remainder.

4.1.1 Methods of Calculation

1. **Prime Factorization:** Take the lowest power of each common prime.
2. **Long Division (Euclidean Algorithm):**
    - Divide the larger number by the smaller.
    - Take the remainder and divide the previous divisor by it.
    - Repeat until remainder is 0. The last divisor is the HCF.
3. **Difference Method (The Shortcut):** The HCF of a set of numbers cannot be larger than the difference between any two of the numbers.
    - Rule: HCF(a, b) divides $|a-b|$. Often, HCF is either the difference itself or a factor of the difference.
    - *Example:* HCF(41, 71, 91). Differences: $71-41=30, 91-71=20, 91-41=50$.
    - HCF must divide 30, 20, and 50. HCF(30, 20, 50) is 10.
    - Check if 10 divides 41, 71, 91. No. The HCF of 41, 71, 91 is actually 1 (since they are prime/coprime). *Correction:* The difference method helps narrow down candidates. If the numbers leave the *same* remainder, then the HCF of differences is the exact answer for the "largest number dividing them leaving same remainder".

4.1.2 Applications

- **Tiling Problems:** Finding the largest square tile to cover a floor of size $L \times W$. Side of tile = HCF(L, W).
- **Maximum Capacity:** Finding the largest container to measure exactly different volumes of liquids.

4.2 Least Common Multiple (LCM)

The LCM is the smallest number divisible by all numbers in a set.

4.2.1 Methods of Calculation

1. **Prime Factorization:** Take the highest power of every prime present in the numbers.
2. **Formula (Two Numbers Only):**

$$LCM(a, b) = \frac{a \times b}{HCF(a, b)}$$

   This relationship is crucial for checking consistency in problems.

4.2.2 Applications

- **Traffic Lights/Bells:** If lights change every x, y, z seconds, they change together at $LCM(x, y, z)$.
- **Circular Motion:** If runners complete a lap in t_1, t_2 seconds, they meet at the start at $LCM(t_1, t_2)$.

4.3 Fractions and Decimals

To find HCF/LCM of fractions:

- $\text{HCF of Fractions} = \frac{\text{HCF of Numerators}}{\text{LCM of Denominators}}$
- $\text{LCM of Fractions} = \frac{\text{LCM of Numerators}}{\text{HCF of Denominators}}$
- *Note:* Fractions must be in their simplest form before applying these formulas.

5. Modular Arithmetic and Remainder Theorems

Finding remainders for large exponents is a staple of quantitative aptitude. We move beyond basic division to theorems that describe the cyclical nature of remainders.

5.1 The Concept of Negative Remainders

Standard arithmetic uses positive remainders ($0 \leq r < d$). However, modular arithmetic allows for negative remainders, which often simplify calculations significantly.

- *Concept:* $N = d \times q + r$. If we choose a quotient such that $d \times (q + 1)$ exceeds N, the "deficiency" is the negative remainder.
- *Example:* $23 \div 5$. Standard: $23 = 4 \times 5 + 3$ (Rem +3). Negative: $23 = 5 \times 5 - 2$ (Rem -2).
- *Utility:* To find remainder of $23^{100} \div 5$.
  - Using +3: 3^{100} (Hard).
  - Using -2: $(-2)^{100} = 2^{100}$ (Still hard).
  - Better Example: $24^{100} \div 5$.
    - Standard: $24 \equiv 4 \pmod 5 \rightarrow 4^{100}$ (Hard).
    - Negative: $24 \equiv -1 \pmod 5 \rightarrow (-1)^{100} = 1$.

- Result: Remainder is 1. Much faster.

5.2 Fermat's Little Theorem

This theorem provides a powerful shortcut when the divisor is a prime number. **Statement:** If p is a prime number and a is an integer co-prime to p:

$$a^{p-1} \equiv 1 \pmod p$$

- *Example:* Find remainder of $2^{74} \div 73$.
  - Divisor 73 is prime. $a = 2$ is co-prime to 73.
  - Fermat states: $2^{72} \equiv 1 \pmod{73}$.
  - Break down $2^{74} = 2^{72} \times 2^2$.
  - $1 \times 4 = 4$. Remainder is 4.

5.3 Euler's Totient Theorem

A generalization of Fermat's Theorem for composite divisors. **Statement:** If N is any integer and $\gcd(a, N) = 1$:

$$a^{\phi(N)} \equiv 1 \pmod N$$

where $\phi(N)$ is Euler's Totient Function, representing the count of numbers less than N that are co-prime to N.

5.3.1 Calculating $\phi(N)$

For $N = p_1^a p_2^b \dots$:

$$\phi(N) = N\,(1 - \frac{1}{p_1})\,(1 - \frac{1}{p_2})\,\dots$$

- *Example:* Find remainder of $7^{50} \div 15$.
  - $N = 15 = 3 \times 5$.
  - $\phi(15) = 15(1 - 1/3)(1 - 1/5) = 15(2/3)(4/5) = 8$.
  - This means $7^8 \equiv 1 \pmod{15}$.
  - $50 = 8 \times 6 + 2$.
  - $7^{50} = (7^8)^6 \times 7^2 \equiv 1^6 \times 49 \pmod{15}$.
  - $49 \div 15$ leaves remainder 4.

5.4 Wilson's Theorem

This theorem handles factorials modulo a prime. **Statement:** If p is a prime:

$$(p-1)! \equiv -1 \pmod{p}$$

Which is equivalent to $(p-1)! \equiv (p-1) \pmod{p}$.

- *Example:* Remainder of $30! \div 31$.
  - $p = 31$. $(30)! \equiv -1 \pmod{31}$.
  - Remainder is 30.

5.5 Chinese Remainder Theorem (Simplified)

Used when the divisor is composite ($N = a \times b$) and a, b are co-prime.

- *Problem:* Find remainder of x divided by N.
- *Method:* Find remainder modulo a and modulo b, then find the number that satisfies both.
- *Example:* Remainder of $123 \div 10$.
  - $123 \equiv 3 \pmod{5}$.
  - $123 \equiv 1 \pmod{2}$.
  - Check numbers that are $1 \pmod{2}$ (odd numbers): 1, 3, 5, 7, 9...
  - Check which of these is $3 \pmod{5}$: 3.
  - So remainder is 3. (Trivial example, but effective for large numbers).

6. Digit Analysis: Unit Digits and Last Two Digits

Solving for specific digits in exponential expansions (x^y) is a pattern-recognition task based on the periodicity of number endings.

6.1 Unit Digit Cyclicity

The last digit of any number N^P depends only on the last digit of the base N. These digits repeat in cycles of 1, 2, or 4.

Base End Digit	Cyclicity	Pattern (Powers 1, 2, 3, 4...)
0, 1, 5, 6	1	$0 \to 0, 1 \to 1, 5 \to 5, 6 \to 6.$
4	2	Odd Power: 4, Even Power: 6.
9	2	Odd Power: 9, Even Power: 1.
2	4	2, 4, 8, 6.

Base End Digit	Cyclicity	Pattern (Powers 1, 2, 3, 4...)
3	4	3, 9, 7, 1.
7	4	7, 9, 3, 1.
8	4	8, 4, 2, 6.

6.1.1 Algorithm

To find unit digit of N^P:

1. Identify cyclicity (C) of the unit digit of N.
2. Divide exponent P by C to get remainder R.
3. If $R \neq 0$, unit digit is $(\text{last digit of } N)^R$.
4. If $R = 0$, unit digit is $(\text{last digit of } N)^C$.
5. *Example:* 2^{34}. Cycle is 4. $34 \div 4$ rem 2. Unit digit is $2^2 = 4$.

6.2 Last Two Digits Shortcuts

Finding the last two digits is equivalent to finding $N^P \pmod{100}$.

6.2.1 Numbers Ending in 1

If base ends in 1 $(\dots d1)$, the last two digits of $(\dots d1)^P$ are:

- **Unit Place:** Always 1.
- **Tens Place:** The unit digit of $(d \times \text{unit digit of } P)$.
- *Example:* 41^{137}.
  - Unit: 1.
  - Tens: $4(\text{tens of base}) \times 7(\text{unit of exp}) = 28$. Take 8.
  - Result: 81.

6.2.2 Numbers Ending in 3, 7, 9

Convert these to a form ending in 1.

- **Ends in 9:** $9^2 = 81$. Change base to N^2.
- **Ends in 3:** $3^4 = 81$. Change base to N^4.
- **Ends in 7:** $7^4 = 2401$ (ends 01). Change base to N^4.
- *Example:* 19^{26}.
  - $19^2 = 361$.

- $(19^2)^{13} = 361^{13}$.
- Apply ending-in-1 rule to 61^{13}.
- Unit: 1. Tens: $6 \times 3 = 18 \rightarrow 8$.
- Result: 81.

6.2.3 Numbers Ending in 2, 4, 6, 8 (Even Numbers)

Use the property that $2^{10} = 1024$.

- 24^{Odd} ends in 24.
- 24^{Even} ends in 76.
- 76^{Any} ends in 76.
- *Strategy:* Extract powers of 2 until you form 2^{10}.
- *Example:* 2^{54}.
  - $(2^{10})^5 \times 2^4 = (1024)^5 \times 16$.
  - Ends in $24^5 \times 16$.
  - $24^{\text{Odd}} \rightarrow 24$.
  - $24 \times 16 = 384$. Last two digits: 84.

6.2.4 Numbers Ending in 5

- If the tens digit of base is odd AND unit digit of exponent is odd, last two digits are **75**.
- Otherwise, they are **25**.
- *Example:* 135^{123}. Base tens (3) is Odd. Exp unit (3) is Odd. Result: 75.
- *Example:* 135^{122}. Exp unit (2) is Even. Result: 25.

7. Factorial Analysis

Factorials ($n!$) are central to permutations and combinations, but in number theory, we focus on their prime composition and trailing zeroes.

7.1 Legendre's Formula

This formula determines the highest power of a prime p that divides $n!$.

$$E_p(n!) = \sum_{k=1}^{\infty} \lfloor \frac{n}{p^k} \rfloor$$

- *Example:* Highest power of 3 in $50!$.
  - $\lfloor 50/3 \rfloor = 16$.
  - $\lfloor 50/9 \rfloor = 5$.

- $\lfloor 50/27 \rfloor = 1$.
- Total = $16 + 5 + 1 = 22$. So 3^{22} divides $50!$.

7.2 Trailing Zeroes

A trailing zero is produced by a factor of 10. Since $10 = 2 \times 5$, we count the pairs of 2s and 5s. In a factorial, 5s are always scarcer than 2s. Therefore, the number of zeroes equals the number of factors of 5.

- **Formula:** Zeroes in $n! = E_5(n!)$.
- *Example:* Zeroes in $1000!$.
  - $1000/5 = 200$.
  - $1000/25 = 40$.
  - $1000/125 = 8$.
  - $1000/625 = 1$.
  - Total = 249 zeroes.

8. Base Systems

While the decimal system (Base 10) is standard, aptitude problems often feature Base 2 (Binary), Base 8 (Octal), or Base 16 (Hexadecimal).

8.1 Base Conversions

- **Decimal to Base N:** Continuously divide the decimal number by N and record the remainders in reverse order.
- **Base N to Decimal:** Expand the number using powers of N.
  - $(234)_5 = 2(5^2) + 3(5^1) + 4(5^0) = 50 + 15 + 4 = 69_{10}$.

8.2 Arithmetic in Other Bases

- **Addition:** Add digits. If sum $\geq$ Base, carry over 1 and keep $(\text{sum} - \text{Base})$.
  - *Example in Base 5:* $3_5 + 4_5$. Sum is 7. In Base 5, $7 = 1 \times 5 + 2$. Write 2, carry 1. Result: 12_5.

9. Recurring Decimals to Fractions

Converting recurring decimals to fractions is a key skill for simplification.

9.1 Pure Recurring

If all digits after the decimal repeat:

- Numerator = Repeating digits.
- Denominator = Number of 9s equal to number of repeating digits.
- $0.\overline{abc} = \frac{abc}{999}$.

9.2 Mixed Recurring

If there are non-repeating digits after the decimal:

- **Formula:**

$$\frac{(\text{Total Number}) - (\text{Non-recurring part})}{\text{9s for recurring} \dots \text{0s for non-recurring}}$$

- *Example:* $0.12\overline{3}$.
  - Total number digits seen: 123.
  - Non-recurring: 12.
  - Recurring part is '3' (1 digit) $\longrightarrow$ one 9.
  - Non-recurring part is '12' (2 digits) $\longrightarrow$ two 0s.
  - Fraction: $\frac{123-12}{900} = \frac{111}{900}$.

10. Conclusion

The landscape of quantitative aptitude is vast, yet connected by the simple threads of number theory. From the basic classification of integers to the complex periodicity of Euler’s theorem, the tools provided in this treatise allow for the dismantling of even the most formidable numerical problems. Mastery comes not just from memorizing these formulas (like Legendre’s or the Last Two Digits shortcuts) but from understanding the logic of modular arithmetic and factorization that sustains them. This deep comprehension enables the solver to adapt techniques to novel problems, transforming "math" from a calculation task into a strategic exercise