Constraint Satisfaction Problems

Chapter 6, Sections 1–5

Outline

- ♢ CSP examples
- ♢ Backtracking search for CSPs
- ♢ Problem structure and problem decomposition
- ♢ Local search for CSPs

Constraint satisfaction problems (CSPs)

Standard search problem:

– the state is a "black box"—any old data structure
that supports goal test, eval, successor

CSP is a more specific search problem:

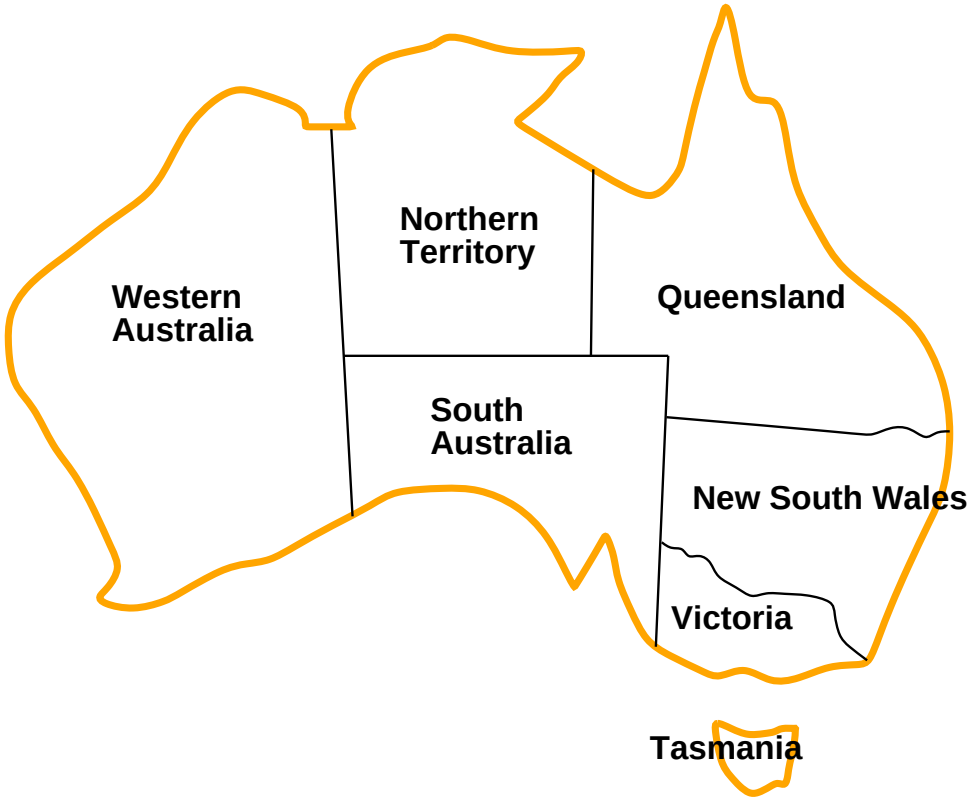
– the state is defined by variables X_i with values from domain D_i

– the goal test is a set of constraints specifying
allowable combinations of values for subsets of variables

Simple example of a **formal representation language**

Allows useful **general-purpose** algorithms with more power
than standard search algorithms

Example: Map-Coloring



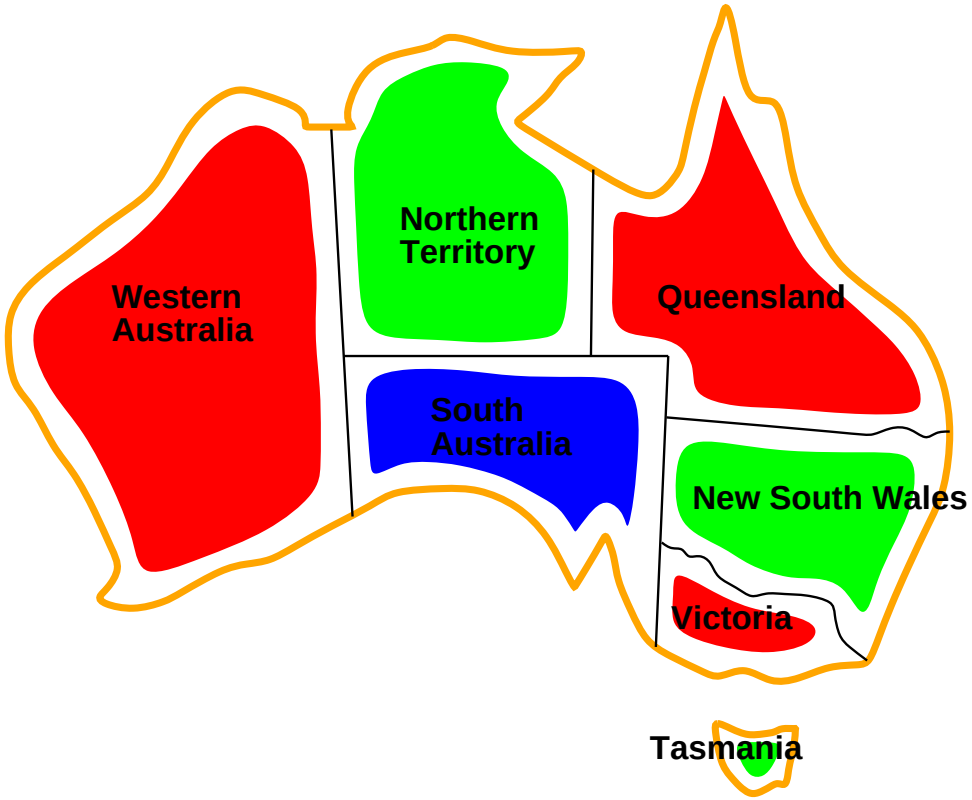
Variables WA, NT, Q, NSW, V, SA, T

Domains $D_i = \{red, green, blue\}$

Constraints: adjacent regions must have different colors

e.g., $WA \neq NT, WA \neq SA, NT \neq SA, NT \neq Q, SA \neq Q, \ldots$

Example: Map-Coloring contd.

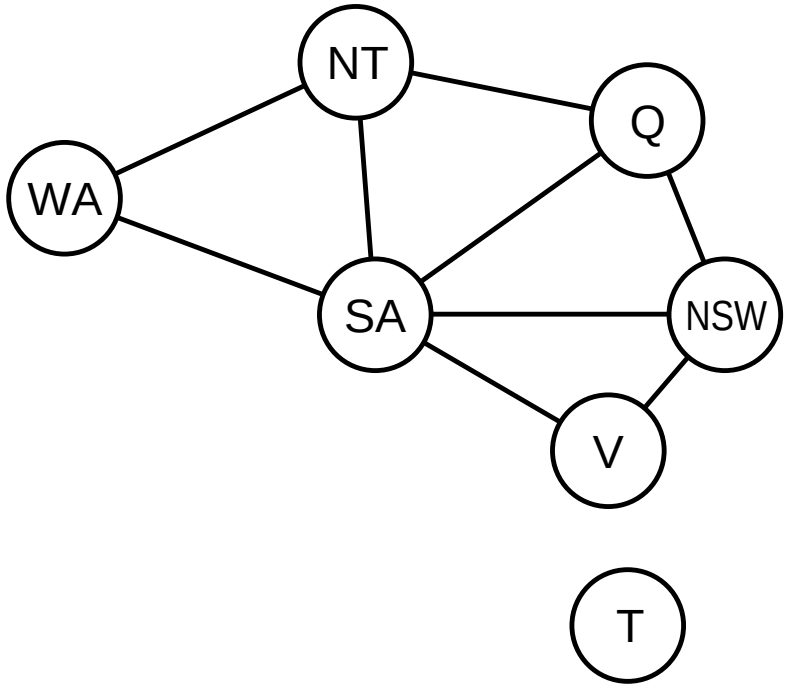


Solutions are assignments satisfying all constraints, e.g.,
$\{WA = red, NT = green, Q = red, NSW = green,$
$V = red, SA = blue, T = green\}$

Constraint graph

Binary CSP: each constraint relates at most two variables

Constraint graph: nodes are variables, arcs show constraints



General-purpose CSP algorithms use the graph structure
to speed up search. E.g., Tasmania is an independent subproblem!

Varieties of CSPs

Discrete variables

- ♢ finite domains; size d $\Rightarrow$ $O(d^n)$ complete assignments
  - e.g., Boolean CSPs, incl. Boolean satisfiability (NP-complete)
- ♢ infinite domains (integers, strings, etc.)
  - e.g., job scheduling, variables are start/end days for each job
  - need a constraint language, e.g., $StartJob_1 + 5 \leq StartJob_3$
  - linear constraints solvable, nonlinear undecidable

Continuous variables

- ♢ e.g., start/end times for Hubble Telescope observations
- ♢ linear constraints solvable in polynomial time by LP methods

Varieties of constraints

Unary constraints involve a single variable,
e.g., $SA \neq green$

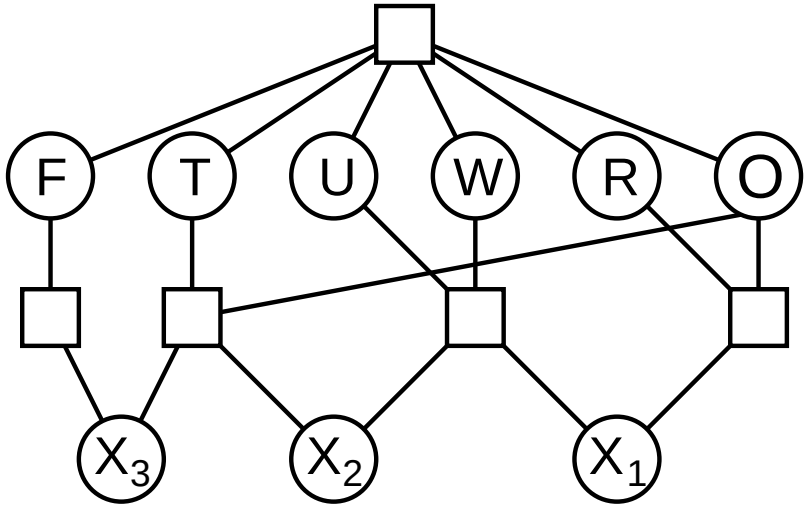
Binary constraints involve pairs of variables,
e.g., $SA \neq WA$

Higher-order constraints involve 3 or more variables,
e.g., cryptarithmetic puzzles

Preferences (soft constraints), e.g., red is better than $green$
often representable by a cost for each variable assignment
$\rightarrow$ constrained optimization problems

Example: Cryptarithmetic puzzle

$$\begin{array}{r} \ \mathsf{T\ W\ O} \\ +\ \mathsf{T\ W\ O} \\ \hline \mathsf{F\ O\ U\ R} \end{array}$$



Variables: $F\ T\ U\ W\ R\ O\ X_1\ X_2\ X_3$

Domains: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Constraints

$alldiff(F, T, U, W, R, O)$

$O + O = R + 10 \cdot X_1$, etc.

Real-world CSPs

Assignment problems
- e.g., who teaches what class

Timetabling problems
- e.g., which class is offered when and where?

Hardware configuration

Spreadsheets

Transportation scheduling

Factory scheduling

Floorplanning

Standard search formulation (incremental)

Let's start with the straightforward, dumb approach, then fix it

States are defined by the values assigned so far:

♢ Initial state: the empty assignment, $\{\}$
♢ Successor function: assign a value to an unassigned variable
that does not conflict with current assignment.
$\Rightarrow$ fail if there are no legal assignments
♢ Goal test: the current assignment is complete

1) This is the same for all CSPs! 
2) Every solution appears at depth n with n variables
$\Rightarrow$ use depth-first search
3) The path is irrelevant, so we can also use a complete-state formulation
4) $b = (n - \ell)d$ at depth ℓ, where d is the domain size
$\Rightarrow$ hence there are $n!\, d^n$ leaves!!!! 

Backtracking search

Variable assignments are commutative, i.e.,

[first $WA = red$ then $NT = green$] is the same as
[first $NT = green$ then $WA = red$]

We only need to consider assignments to a single variable at each node

$\Rightarrow$ $b = d$, so there are d^n leaves (instead of $n!d^n$)

Depth-first search for CSPs with single-variable assignments
is called backtracking search

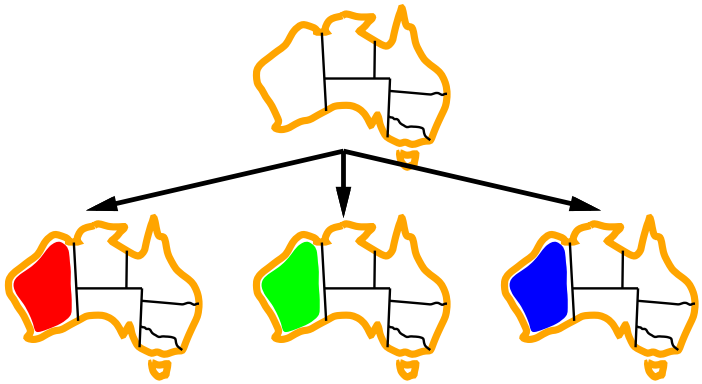
– backtracking search is the basic uninformed algorithm for CSPs
– can solve n-queens for $n \approx 25$

Backtracking search

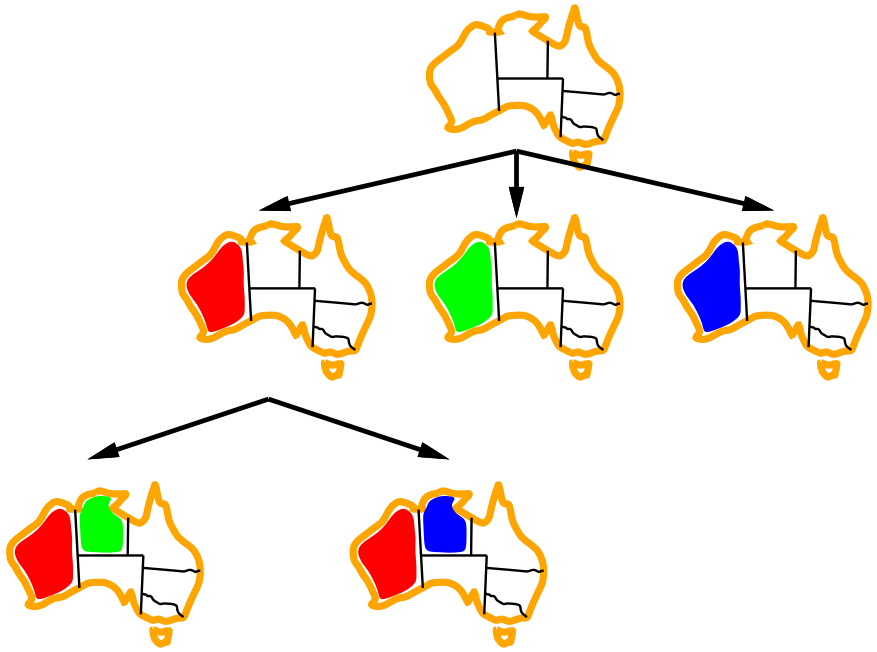
```
function BACKTRACKING-SEARCH(csp) returns solution/failure
    return RECURSIVE-BACKTRACKING({ }, csp)

function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
    if assignment is complete then return assignment
    var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
    for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
        if value is consistent with assignment given CONSTRAINTS[csp] then
            add {var = value} to assignment
            result ← RECURSIVE-BACKTRACKING(assignment, csp)
            if result ≠ failure then return result
            remove {var = value} from assignment
    return failure
```

Backtracking example

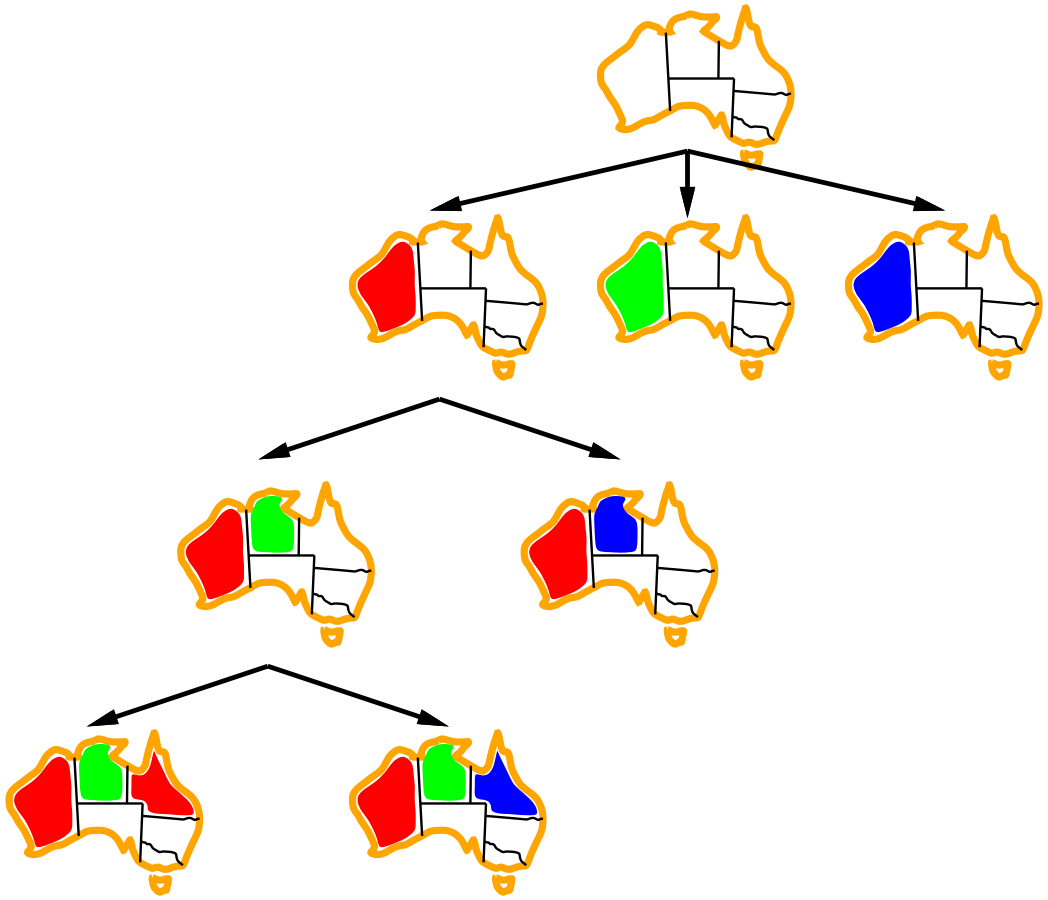
Backtracking example



Backtracking example



Backtracking example



Improving backtracking efficiency

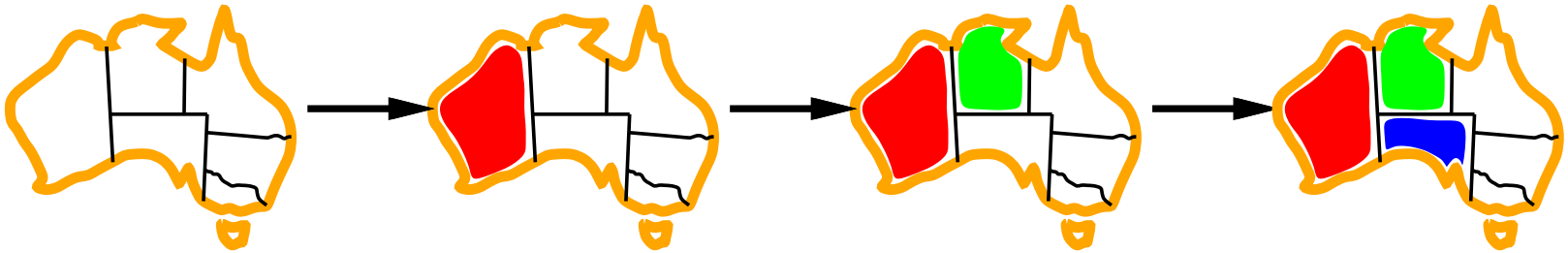
General-purpose methods can give huge gains in speed:

1. Which variable should be assigned next?
2. In what order should its values be tried?
3. Can we detect inevitable failure early?
4. Can we take advantage of problem structure?

Minimum remaining values

Minimum remaining values (MRV):

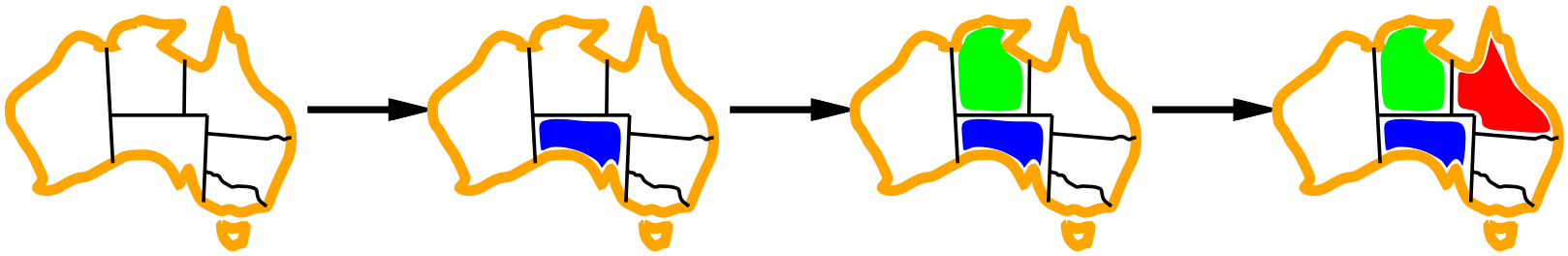
– choose the variable with the fewest legal values



Degree heuristic

If there are several MRV variables, we can use the degree heuristic:

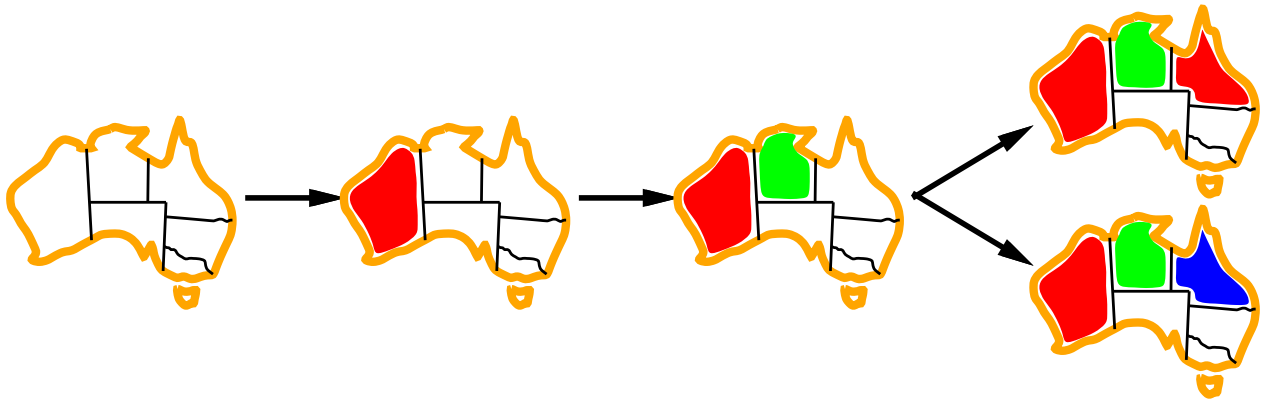
– choose the variable with the most constraints on remaining variables



Least constraining value

When we have selected a variable using MRV and degree heuristic,
we choose the least constraining value:

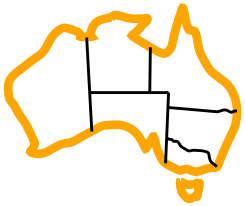
– the one that rules out the fewest values in the remaining variables



Combining these heuristics makes n-queens feasible for $n \approx 1000$

Forward checking

Idea: Keep track of remaining legal values for unassigned variables
Terminate search when any variable has no legal values



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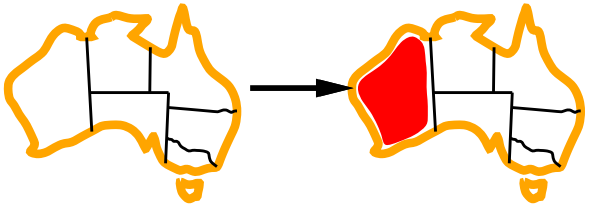

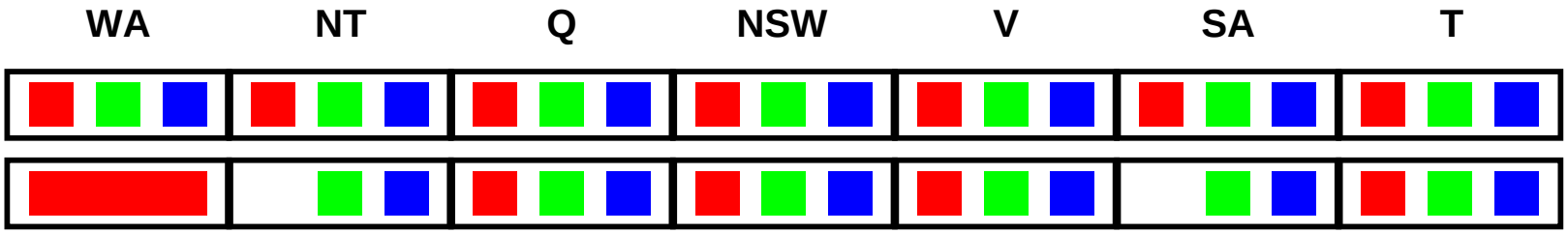

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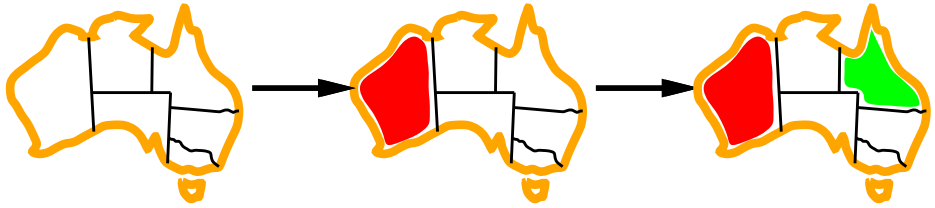

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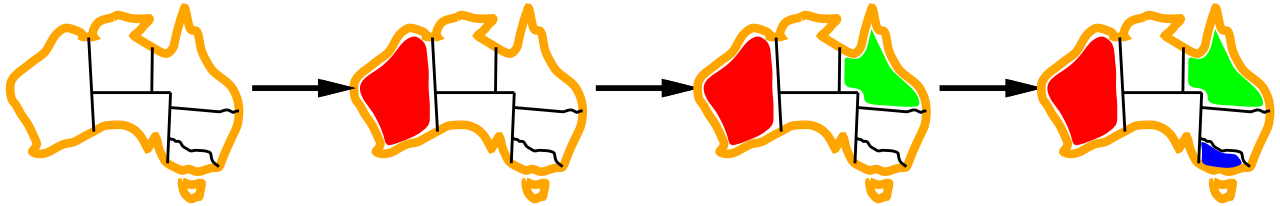
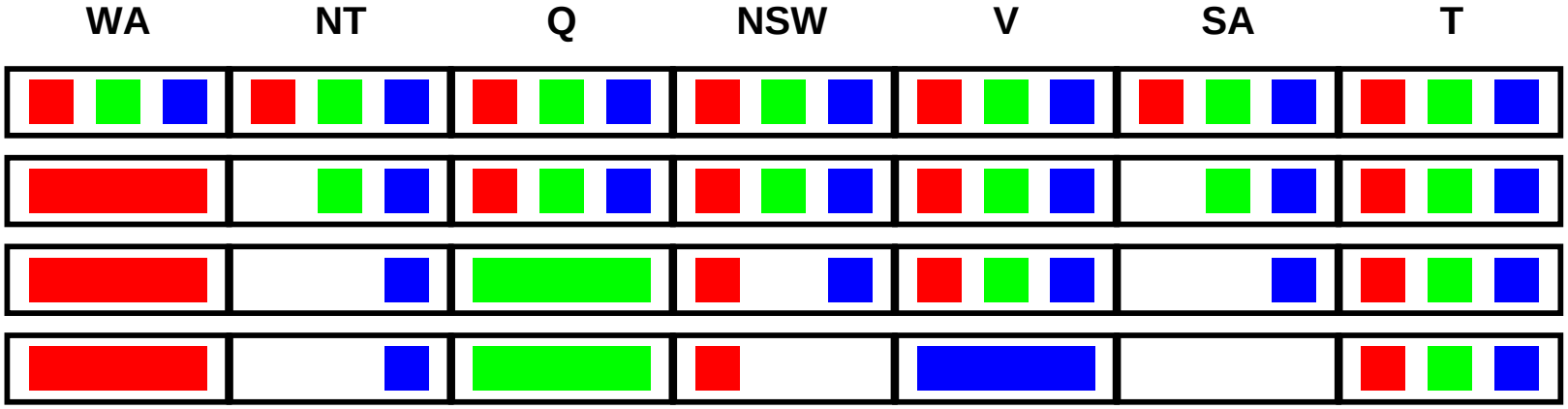
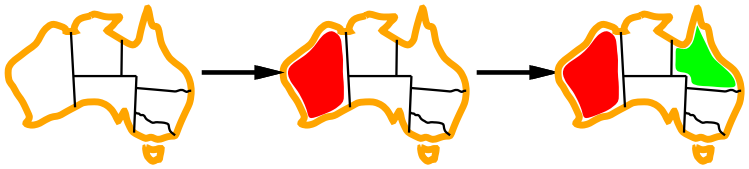




Constraint propagation

Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:



NT and SA cannot both be blue!

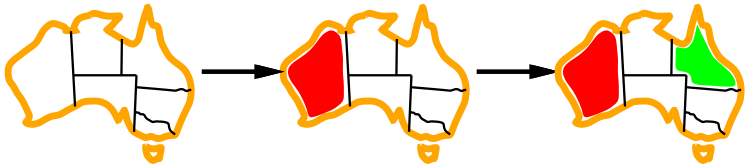
Constraint propagation repeatedly enforces constraints locally

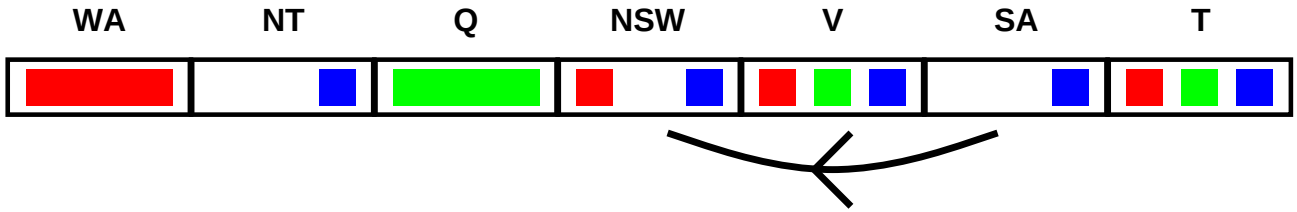
Arc consistency

Simplest form of propagation makes each arc consistent

$X \rightarrow Y$ is consistent iff

– for **every** value x of X there is **some** allowed y





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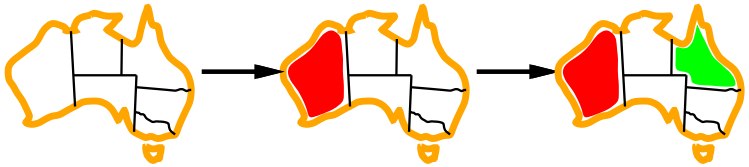

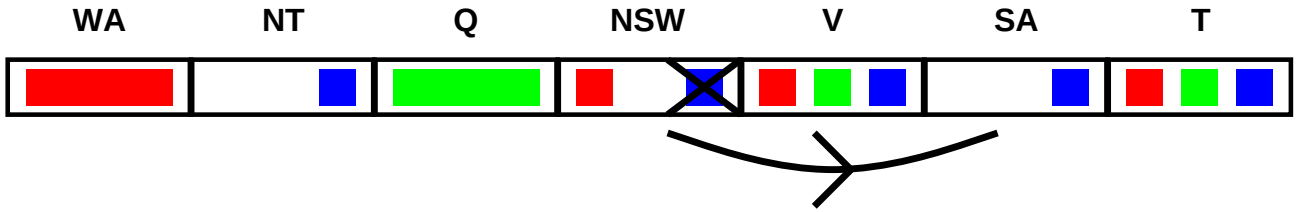

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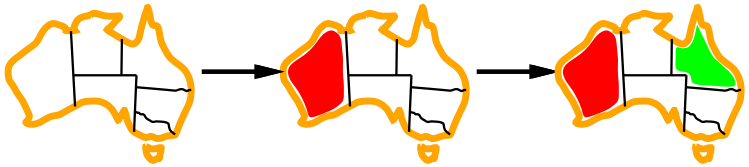

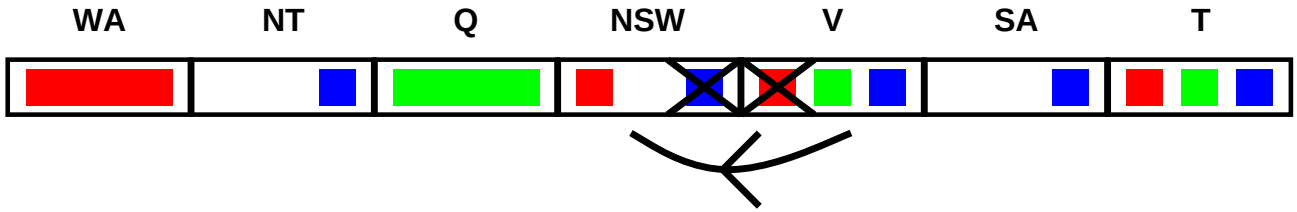

If X loses a value, neighbors of X need to be rechecked

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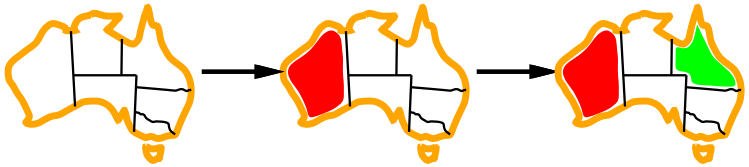
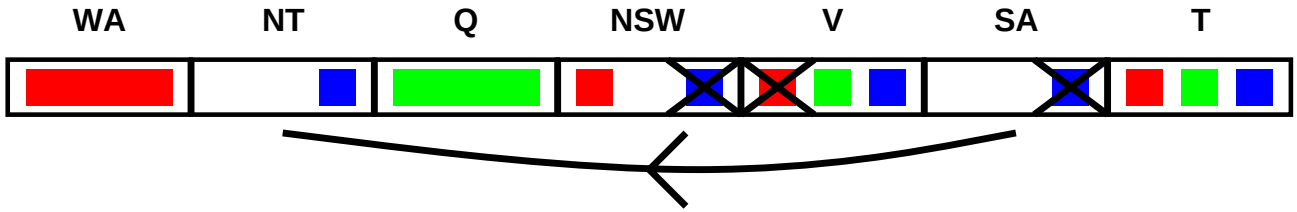




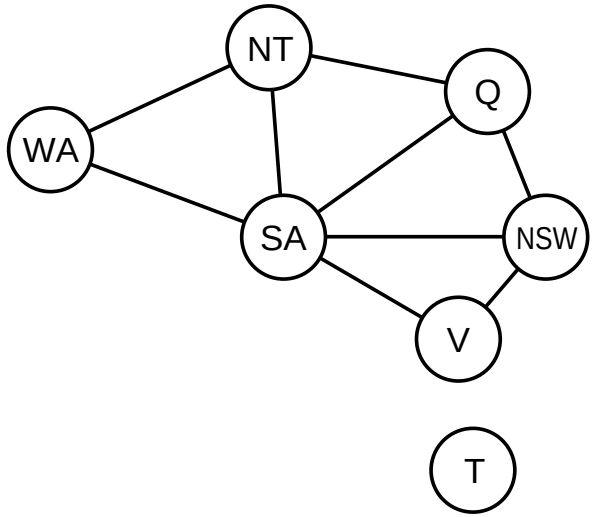
If X loses a value, neighbors of X need to be rechecked

Arc consistency detects failure earlier than forward checking

Can be run as a preprocessor or after each assignment

Problem structure

Tasmania and mainland are independent subproblems — identifiable as connected components of the constraint graph



Suppose that each subproblem has c variables out of n total

The worst-case solution cost is $n/c \cdot d^c$, which is **linear** in n

E.g., $n = 80$, $d = 2$, $c = 20$:

2^{80} = 4 billion years at 10 million nodes/sec

if we divide it into 4 equal-size subproblems:

$4 \cdot 2^{20}$ = 0.4 seconds at 10 million nodes/sec

Iterative algorithms for CSPs

Hill-climbing, simulated annealing typically work with "complete" states, i.e., all variables assigned

To apply to CSPs:
- we allow states with unsatisfied constraints
- the operators **reassign** variable values

The min-conflicts algorithm:
Variable selection:
- randomly select any conflicted variable

Value selection by the **min-conflicts** heuristic:
- choose the value that violates the fewest constraints
- i.e., hillclimb with $h(n) =$ total number of violated constraints

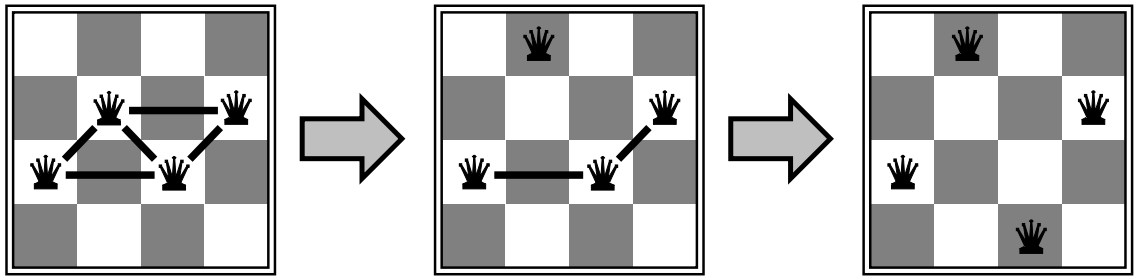
Example: 4-Queens

States: 4 queens in 4 columns ($4^4 = 256$ states)

Operators: move queen in column

Goal test: no attacks

Evaluation: $h(n)$ = number of attacks

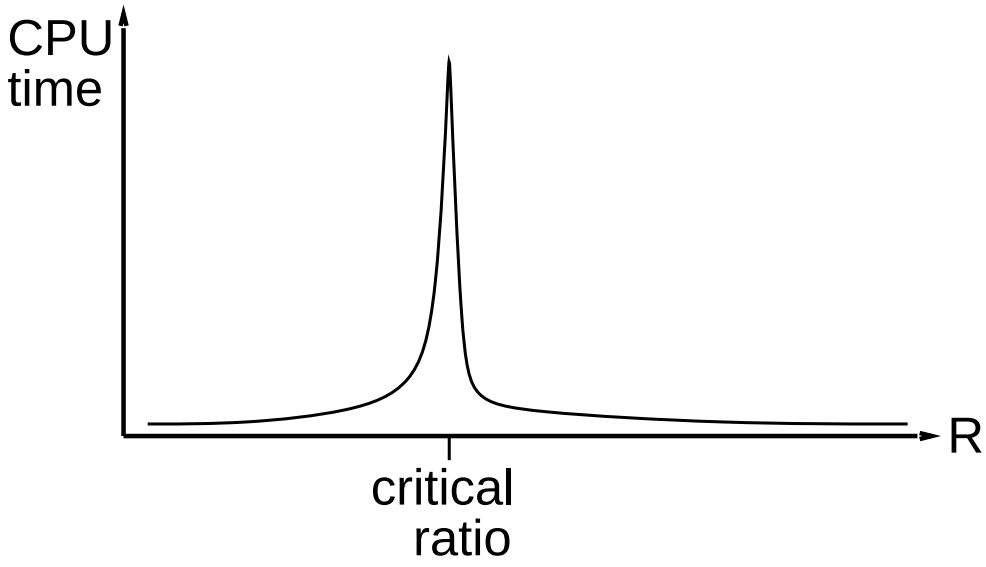


Performance of min-conflicts

Given a random initial state, we can solve n-queens in almost constant time for arbitrary n with high probability (e.g., $n = 10{,}000{,}000$)

The same appears to be true for any randomly-generated CSP
except in a narrow range of the ratio

$$R = \frac{\text{number of constraints}}{\text{number of variables}}$$



Summary

CSPs are a special kind of problem:
- states are defined by values of a fixed set of variables
- goal test is defined by constraints on variable values

Backtracking = depth-first search with one variable assigned per node

Variable ordering and value selection heuristics help significantly

Forward checking prevents assignments that guarantee later failure

Constraint propagation (e.g., arc consistency) does additional work to constrain values and detect inconsistencies

Iterative min-conflicts is usually effective in practice