CONVEXITY OF SETS

CONVEX SETS

- A set Ω is convex if given any two points $x, y \in \Omega$, and any $\theta \in [0, 1]$
- we have $\theta x + (1 - \theta)y \in \Omega$.

- Simple rearrangements provide alternative definitions for line segments,
- and hence convexity:

$$\{\theta x + (1 - \theta)y : 0 \leq \theta \leq 1\} = \{\theta_1 x + \theta_2 y : \theta_i \geq 0, \theta_1 + \theta_2 = 1\}$$

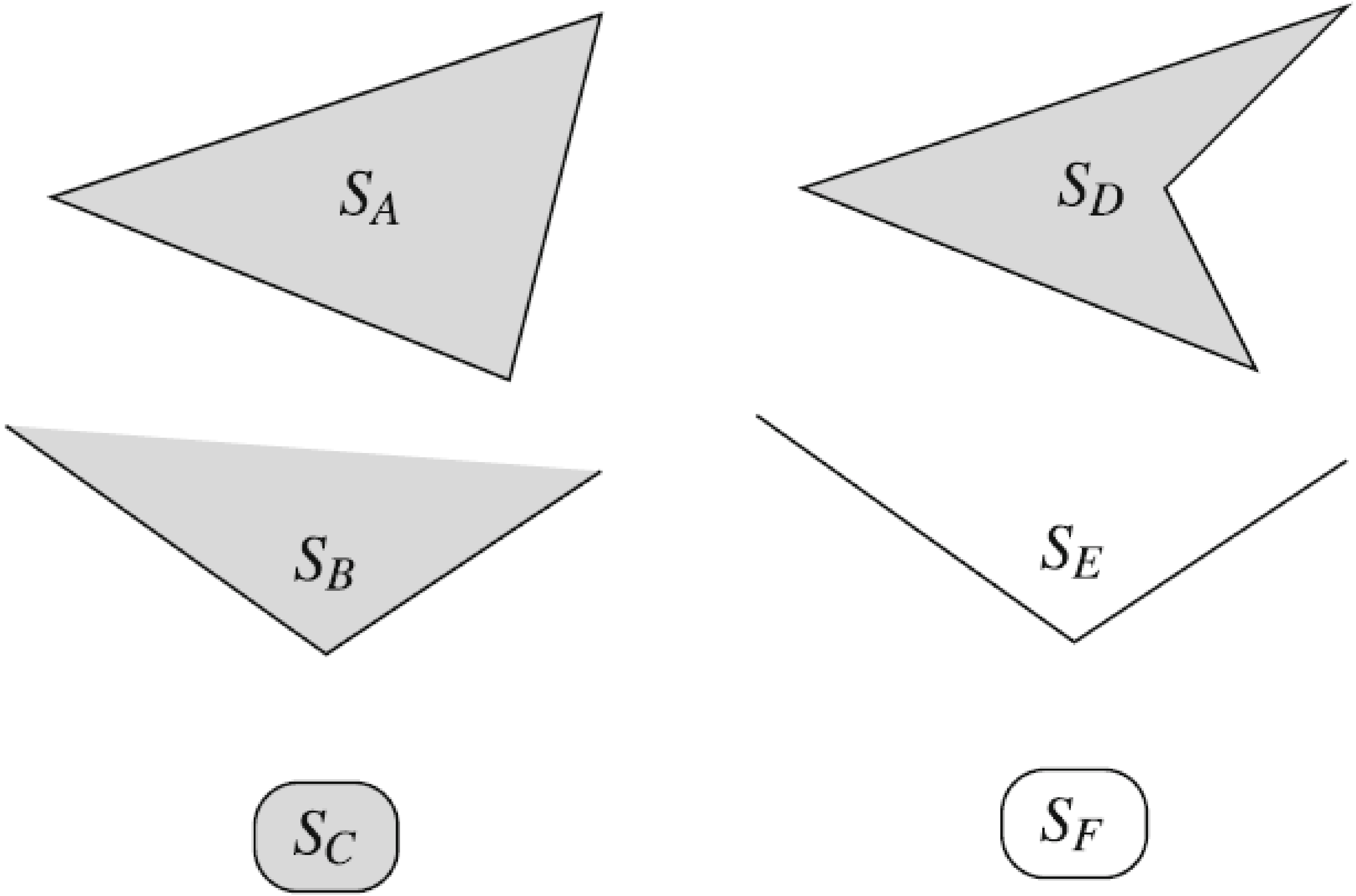
$$= \{y + \theta(x - y) : 0 \leq \theta \leq 1\}.$$

Example of Convex Sets

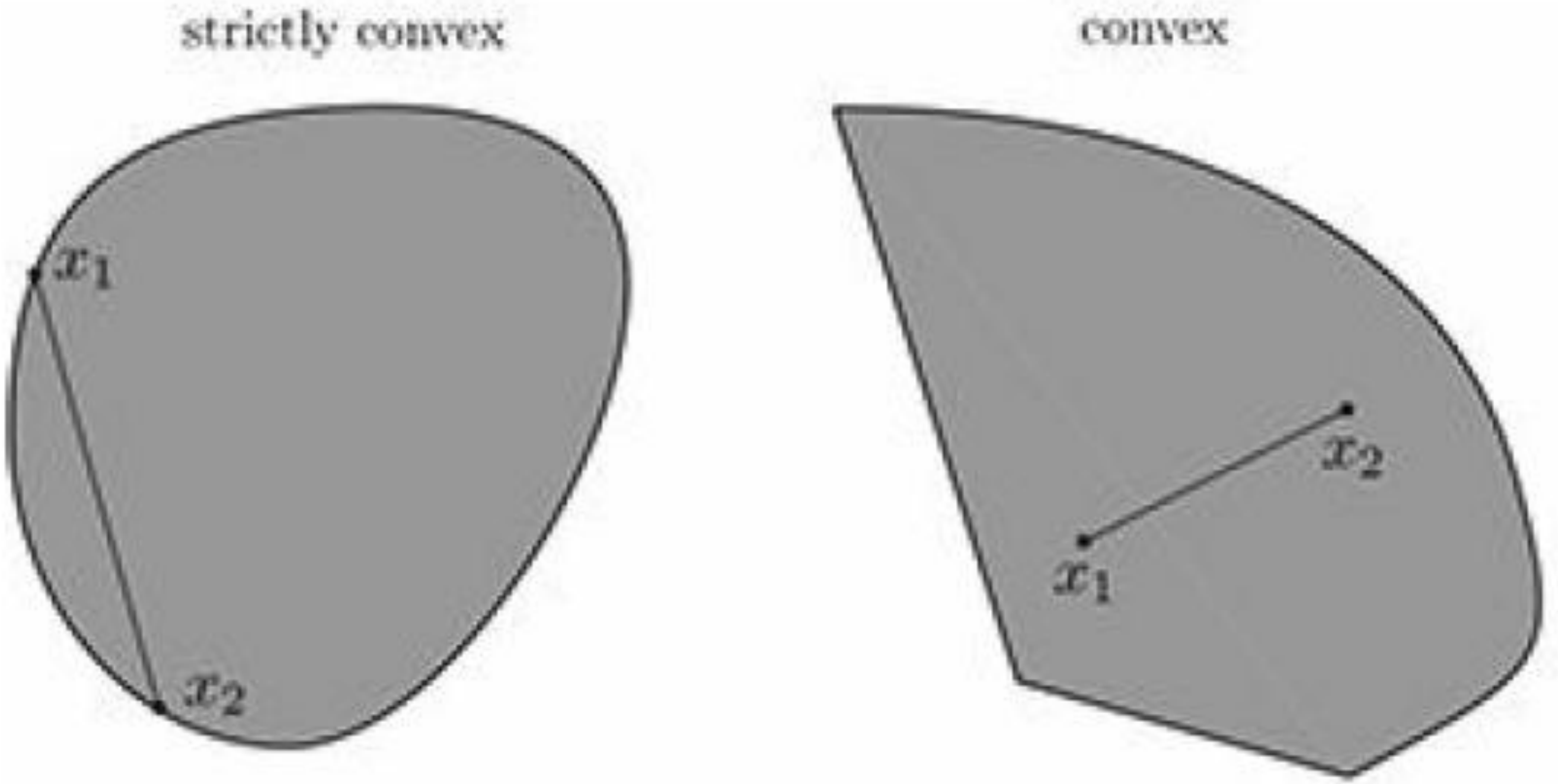
- Open ball: $B_\epsilon(x) = \{y: \|x - y\| < \epsilon\}$

- Closed ball: $\overline{B_\epsilon(x)} = \{y: \|x - y\| \le \epsilon\}$

EXAMPLE



EXAMPLE



Properties of Convex Sets

- Intersection of convex sets: $C_1 \cap C_2$

- Sum of convex sets: $C_1 + C_2$

- $\{Ax: x \in C \subset \mathbb{R}^n, A \in \mathbb{R}^{m\times n}\}$

Convex combination

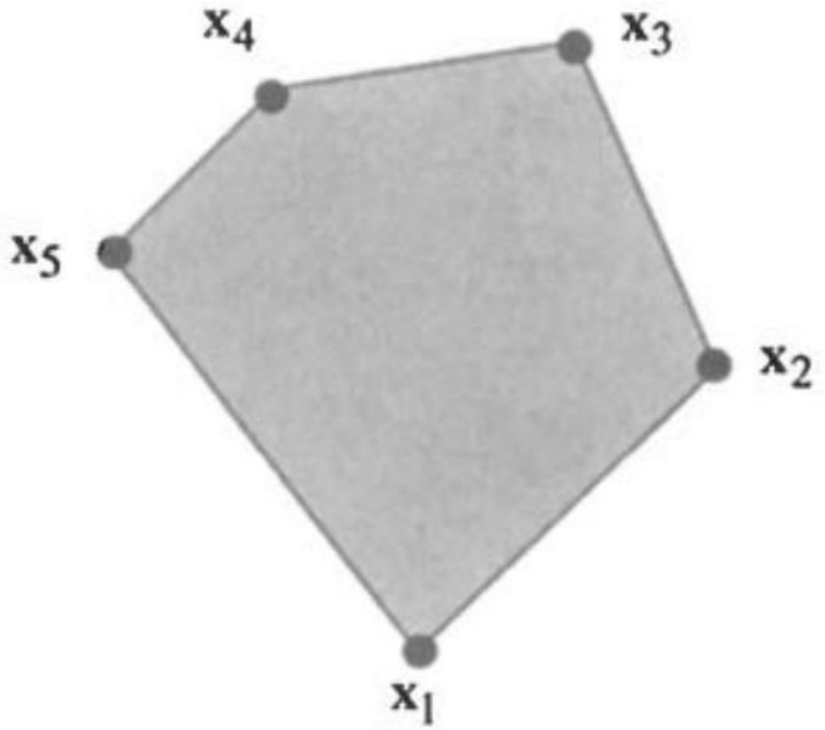
- A point $z \in \mathbb{R}^n$ is a convex combination of the points in $\{x_1, x_2, \ldots, x_m\} \subseteq \mathbb{R}^n$
- if there exists scalars $\theta_1, \ldots, \theta_m$, such that

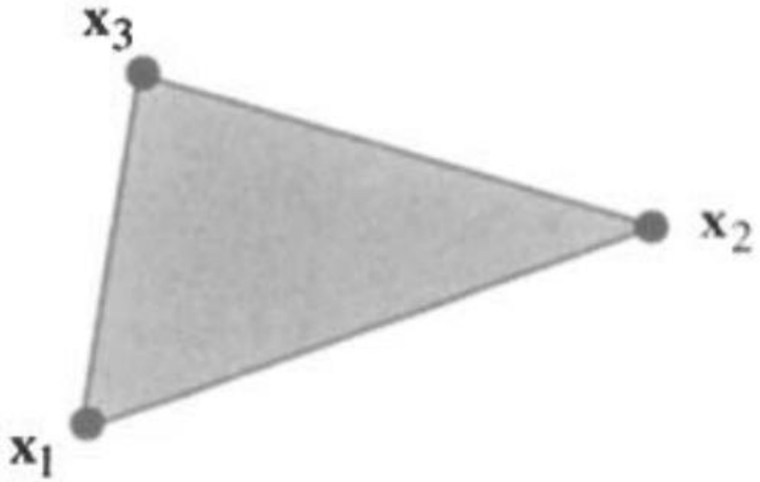
$$z = \sum_{i=1}^{m} \theta_i x_i, \sum_{i=1}^{m} \theta_i = 1 \text{ and } \theta_i \geq 0 \text{ for } i = 1, \ldots, m.$$

- Let $\Omega \subset \mathbb{R}^n$,

$$\operatorname{conv}(\Omega) = \left\{ y : y = \sum_{i=1}^{m} \theta_i x_i, \quad x_i \in \Omega, \quad \sum_{i=1}^{m} \theta_i = 1 \text{ and } \theta_i \geq 0 \text{ for } i = 1, \ldots, m \right\}.$$

EXAMPLE ON CONVEX HULL





- Let S be an arbitrary set in $\mathbb{R}^n$. If $x \in$ conv(S), then $x \in \text{conv}(x_1, x_2, \dots, x_{n+1})$.

Convex Polytopes

➢A convex polytope is a geometric object in $\mathbb{R}^n$ that has flat sides and

➢described as the intersection of a finite number of half-spaces, or

➢as the solution set to a matrix inequality

$$C = \{x: Ax \leq b, A \in \mathbb{R}^{m\times n}, x \in \mathbb{R}^n, b \in \mathbb{R}^m\}$$

➢or as the convex hull of a finite number of points.

SIMPLICES OR SIMPLEXES

- A simplex in $\mathbb{R}^n$ is a bounded convex polytope with nonempty interior
- and exactly $n + 1$ vertices.

- Applications in optimization
  - For the class of pattern search methods, convergence relies on the convex hull of the poll directions having a nonempty interior.
  - In model-based methods, good models rely on a well-poised sample set.

SIMPLEX TESTS

- Let $Y = \{y_0, y_1, \ldots, y_n\}$ be a set of $n + 1$ points in $\mathbb{R}^n$.

- Then the following are equivalent:

1. $\mathrm{Conv}(Y)$ is a simplex.
2. The set $\{(y_1 - y_0), (y_2 - y_0), \ldots, (y_n - y_0)\}$ is linearly independent.
3. The matrix $L = [(y_1 - y_0)\ (y_2 - y_0) \ldots (y_n - y_0)]$ is invertible.
4. The matrix $L = [(y_1 - y_0)\ (y_2 - y_0) \ldots (y_n - y_0)]$ satisfies $\det(L) \neq 0$.

- If conv(Y) is not a simplex, then we sometimes say Y forms a degenerate simplex.
- Degenerate or almost degenerate sets cause challenges in optimization because they do not effectively span the entire space.

Volume of a Simplex

- Suppose $Y = \{y_0, y_1, \ldots, y_n\}$ forms a simplex in $\mathbb{R}^n$ and
- $L = [(y_1 - y_0)\ (y_2 - y_0) \ldots (y_n - y_0)]$.

- Then volume of conv(Y) is given by

$$\mathrm{vol}\big(\mathrm{conv}(Y)\big) = \frac{|\det(L)|}{n!}$$

- For ease of writing, it is common to use $\mathrm{vol}(Y) = \mathrm{vol}(\mathrm{conv}(Y))$.

DIAMETER OF A SIMPLEX

- Suppose $Y = \{y_0, y_1, \ldots, y_n\}$ forms a simplex in $\mathbb{R}^n$.

- Then diameter of conv(Y) is given by

$$\mathrm{diam}\big(\mathrm{conv}(Y)\big) = \max\big\{\|x_i - x_j\| : x_i \in Y, x_j \in Y\big\}$$

- For ease of writing, it is common to use $\mathrm{diam}(Y) = \mathrm{diam}(\mathrm{conv}(Y))$.

Normalized Volume of a Simplex

π

- Suppose $Y = \{y_0, y_1, \ldots, y_n\}$ forms a simplex in $\mathbb{R}^n$.

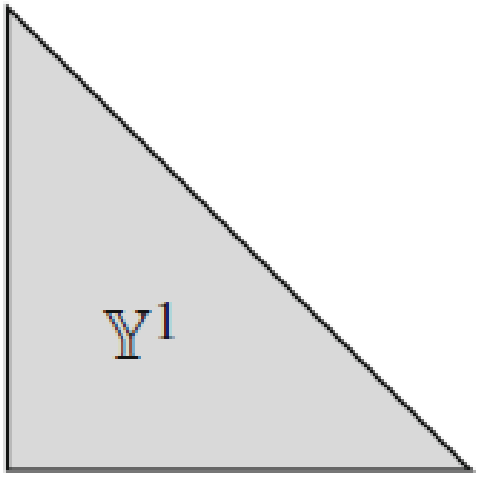
- Then normalised volume of conv(Y) is given by

$$\mathrm{von}\big(\mathrm{conv}(Y)\big) = \frac{\mathrm{vol}(Y)}{\big(\mathrm{diam}(Y)\big)^n} = \mathrm{vol}\left(\frac{Y}{\mathrm{diam}(Y)}\right)$$

- For ease of writing, it is common to use $\mathrm{von}(Y) = \mathrm{von}(\mathrm{conv}(Y))$.

EXAMPLES ON VOL(Y), VON(Y)

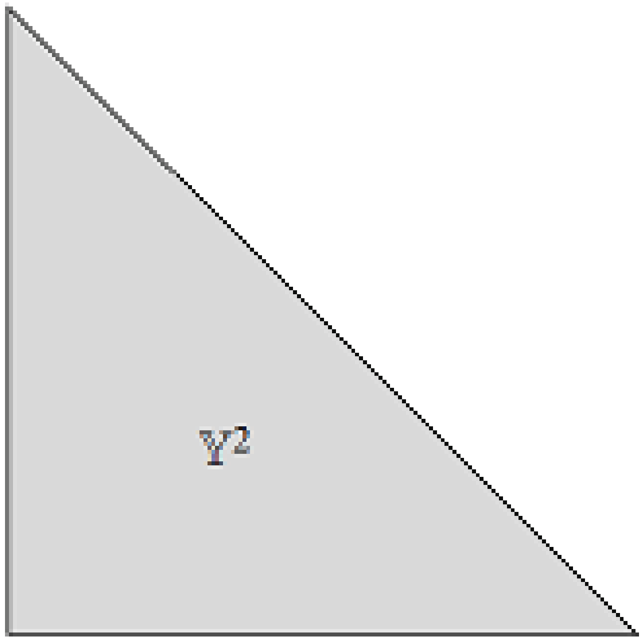
$$\mathbb{Y}^1 = \{[0,0]^\top, [1,0]^\top, [0,1]^\top\}.$$



$$\mathrm{vol}(\mathbb{Y}^1) = \frac{1}{2}, \quad \mathrm{diam}(\mathbb{Y}^1) = \sqrt{2}, \quad \mathrm{von}(\mathbb{Y}^1) = \frac{1}{4}.$$

EXAMPLES ON VOL(Y), VON(Y)

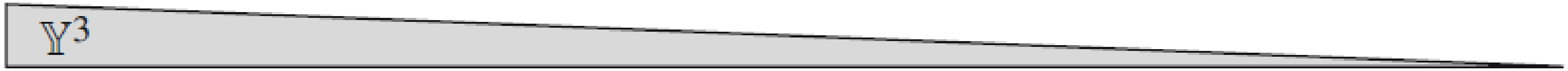
$$\mathbb{Y}^2 = \{[0,0]^\top, [2,0]^\top, [0,2]^\top\}.$$



$$\mathrm{vol}(\mathbb{Y}^2) = 2, \quad \mathrm{diam}(\mathbb{Y}^2) = 2\sqrt{2}, \quad \mathrm{von}(\mathbb{Y}^2) = \frac{1}{4}.$$

EXAMPLES ON VOL(Y), VON(Y)

$$\mathbb{Y}^4 = \{[0,0]^\top, [1,0]^\top, [1/2,1]^\top\}.$$



$$\mathrm{vol}(\mathbb{Y}^4) = \frac{1}{2}, \quad \mathrm{diam}(\mathbb{Y}^4) = \sqrt{1.25}, \quad \mathrm{von}(\mathbb{Y}^4) = \frac{2}{5}.$$

Approximate Diameter of a Simplex

- Suppose $Y = \{y_0, y_1, \ldots, y_n\}$ forms a simplex in $\mathbb{R}^n$.

- Then approximate diameter of conv(Y) is given by

$$\overline{\text{diam}}\big(\text{conv}(Y)\big) = \max\{\|x_i - x_0\| : x_i \in Y, x_0 \in Y\}$$

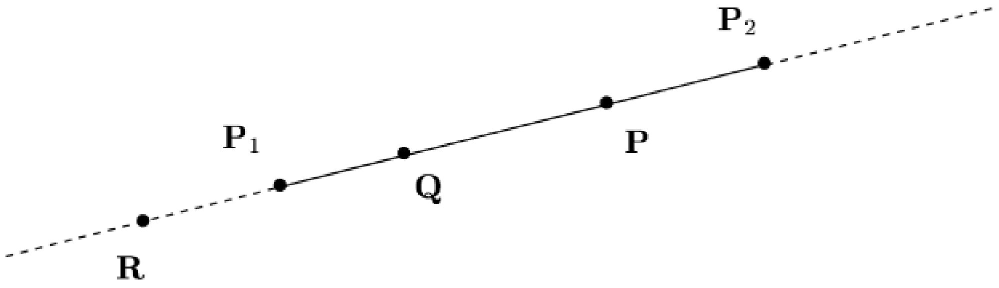
- For ease of writing, it is common to use $\overline{\text{diam}}(Y) = \overline{\text{diam}}(\text{conv}(Y))$.

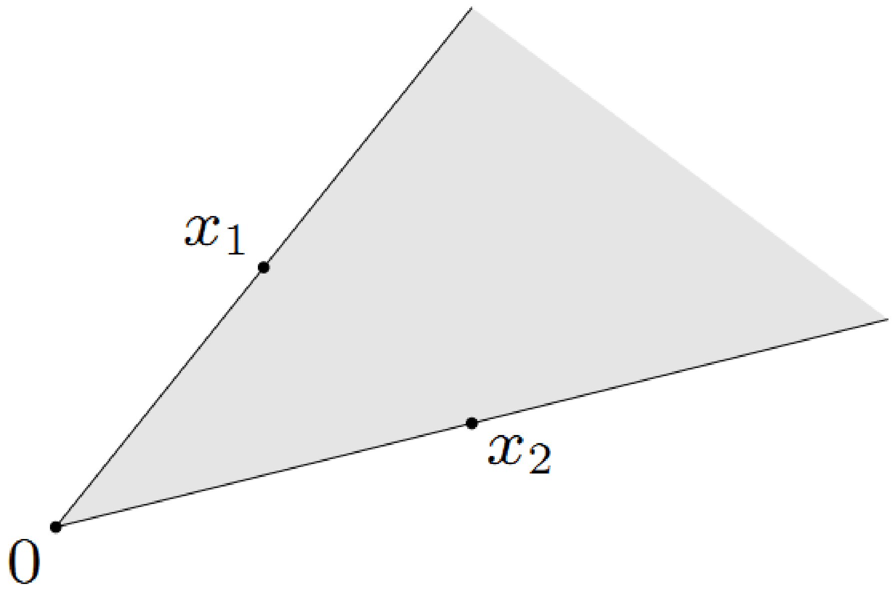
- Using triangle inequality:

$$\overline{\text{diam}}(Y) \leq \text{diam}\ (Y) \leq 2\overline{\text{diam}}(Y)$$

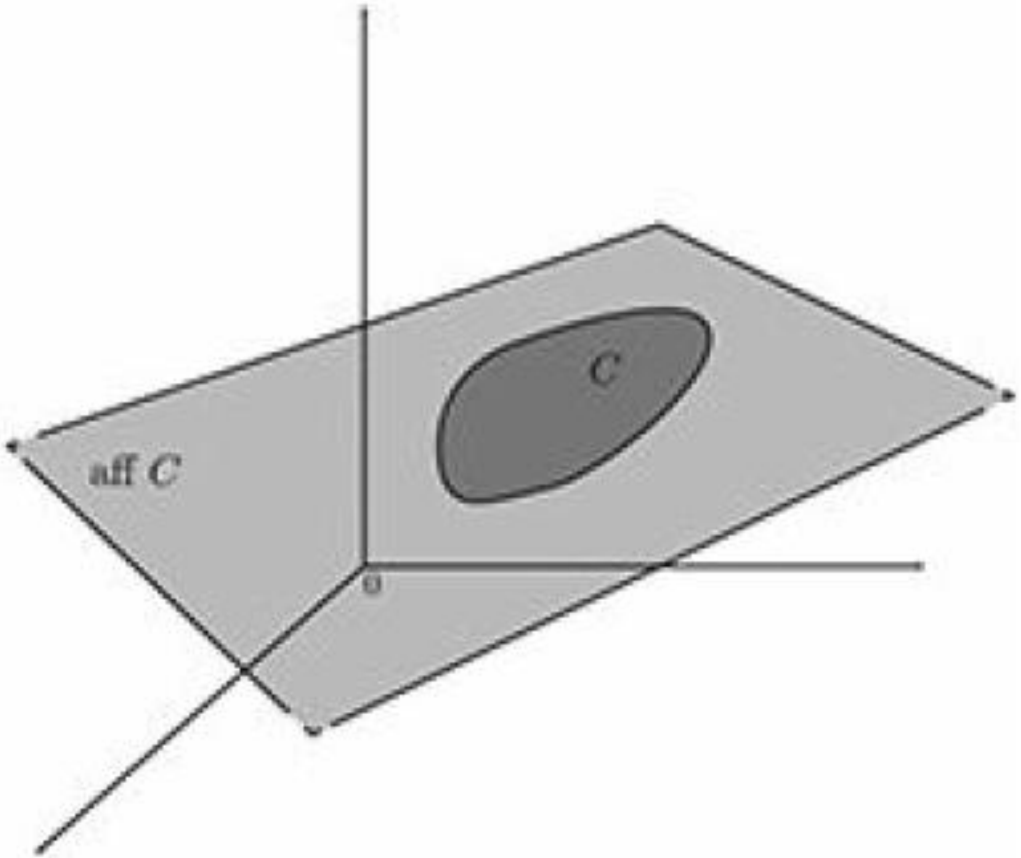
Different Combinations

- Convex combination
- Affine combination
- Conic combination:
- Linear combinations





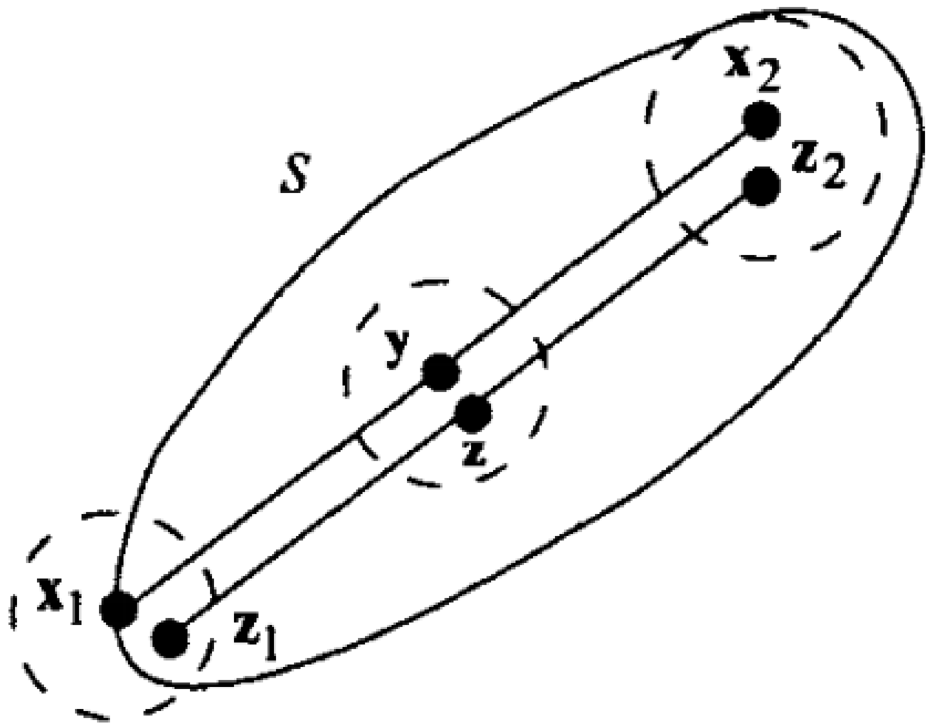
AFFINE HULL



SOME TOPOLOGICAL RESULTS ON CONVEX SETS

SOME TOPOLOGICAL RESULTS ON CONVEX SETS

- Let S be a convex set in $\mathbb{R}^n$ with a nonempty interior. Let $x_1 \in cl(S)$ and $x_2 \in int(S)$.
- Then $\lambda x_1 + (1-\lambda)x_2 \in int(S)$ for each $\lambda \in (0,1)$.
- Visualization:



WEIERSTRASS'S THEOREM

Existence of a minimizing solution for an optimization problem:

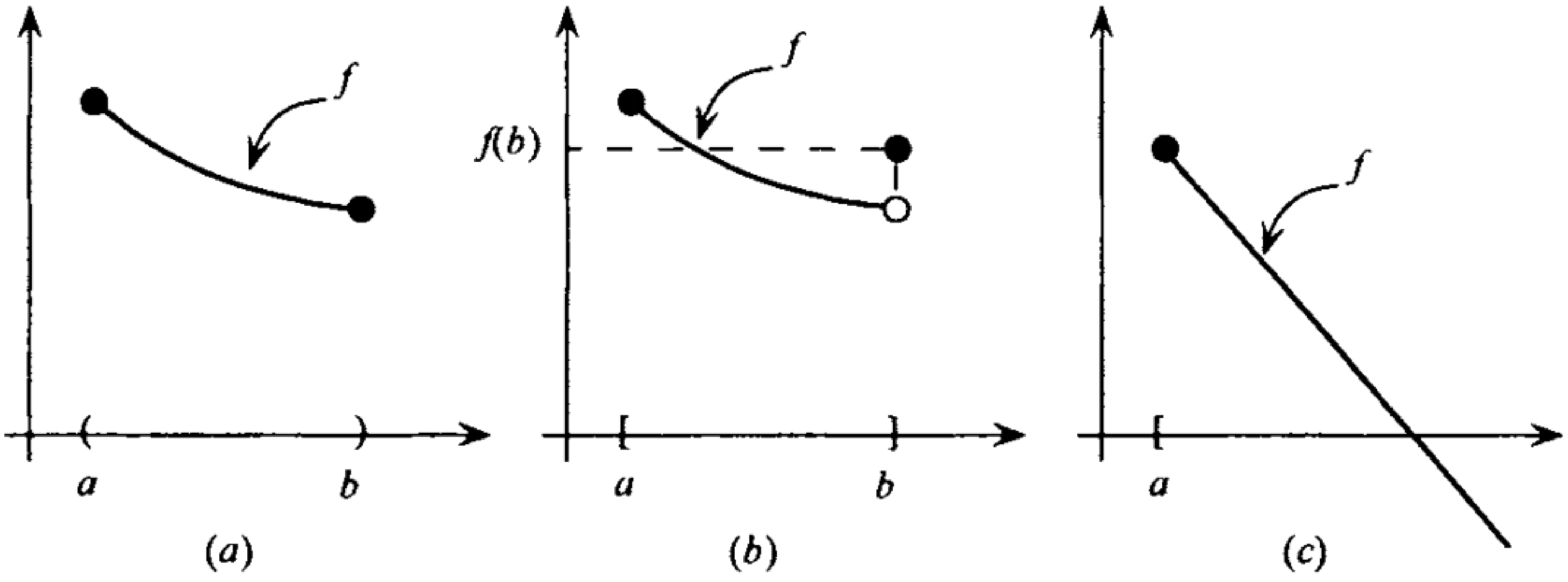
- $\bar{x}$ is a minimizing solution for the problem $\mathbf{min}\{\mathbf{f}(\mathbf{x}):\ \mathbf{x} \in \mathbf{S}\}$, provided that $\bar{x} \in S$ and $\mathbf{f}(\bar{\mathbf{x}}) \leq \mathbf{f}(\mathbf{x})$ for all $x \in S$.

- $\boldsymbol{\alpha} = \mathbf{inf}\{\mathbf{f}(\mathbf{x}):\ \mathbf{x} \in \mathbf{S}\}$, if α is the greatest lower bound of f on S.

- Let S be a nonempty, compact set, and let $f: S \to \mathbb{R}$ be continuous on S.

- Then the problem $\mathbf{min}\{\mathbf{f}(\mathbf{x}):\ \mathbf{x} \in \mathbf{S}\}$ attains its minimum; that is, there exists a minimizing solution to this problem.

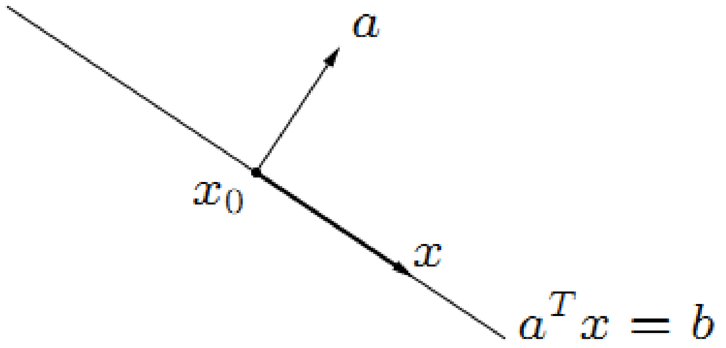
NONEXISTENCE OF A MINIMIZING SOLUTION



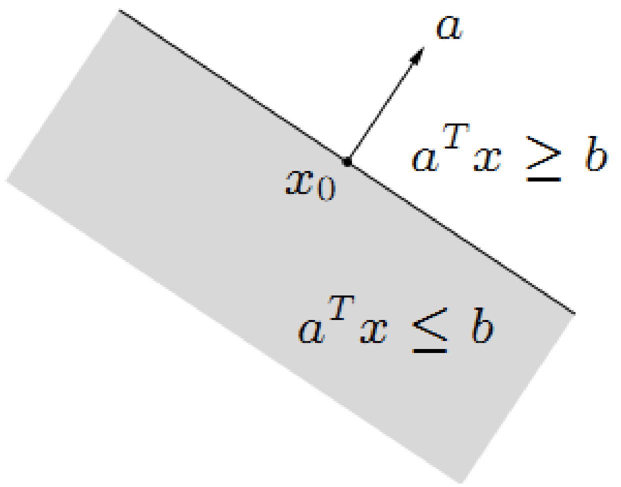
SEPARATION OF CONVEX SETS

HYPERPLANE AND HALF SPACES

hyperplane: set of the form $\{x \mid a^T x = b\}$ $(a \neq 0)$

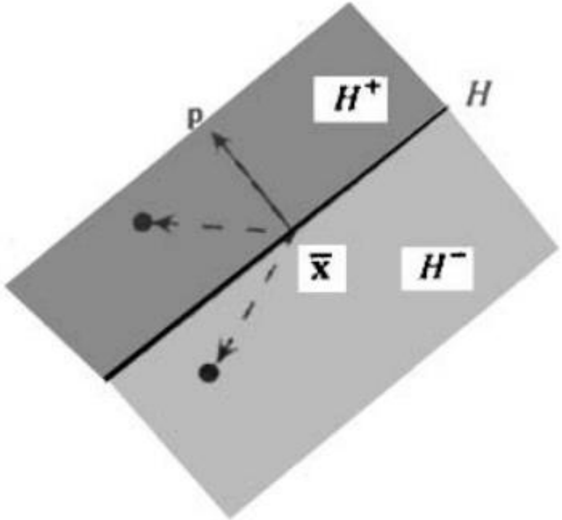


halfspace: set of the form $\{x \mid a^T x \leq b\}$ $(a \neq 0)$



Hyperplane and Half Spaces

Hyperplane and corresponding half-spaces:



- The hyperplane can be written in reference to a normal vector p of the form:

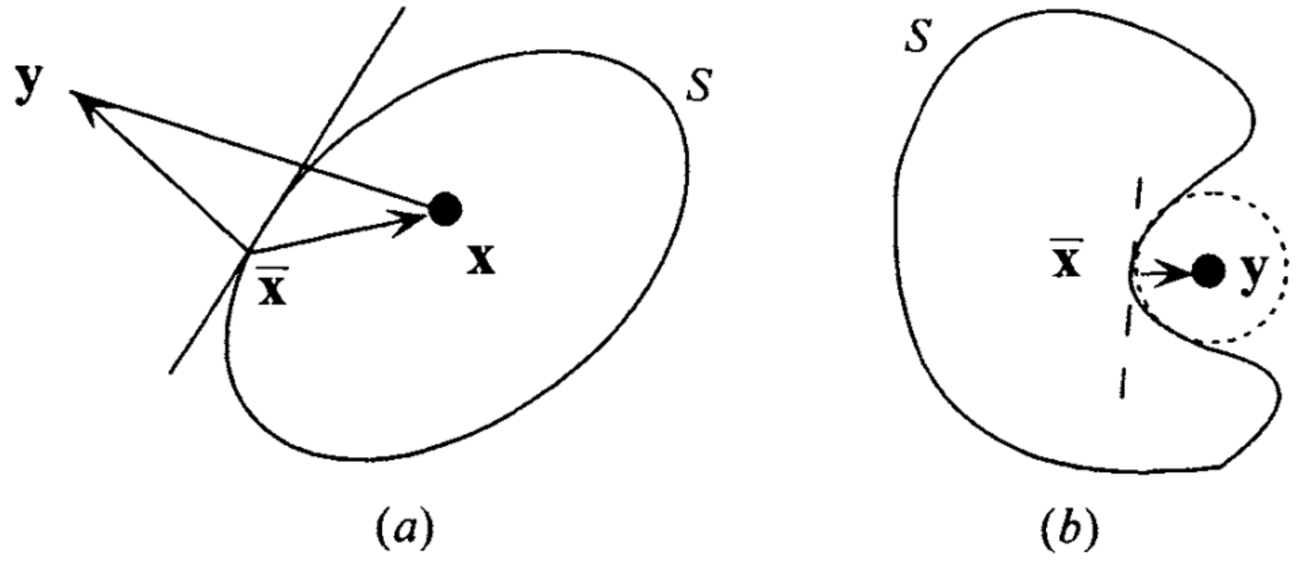
$$\mathrm{H} = \{x : p'x = \alpha\}$$

- Alternatively, the hyperplane can be written in reference to a point $\bar{x}$ in H

$$\mathrm{H} = \{x : p'(x - \bar{x}) = 0\}.$$

Closest-Point Theorem

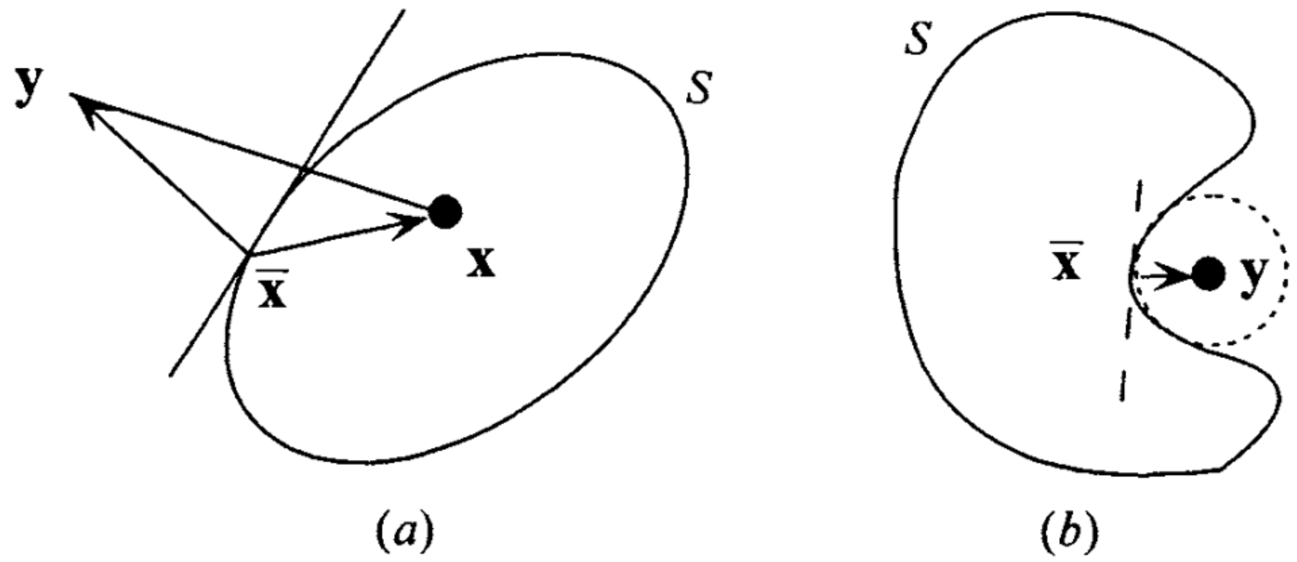
- Let S be a nonempty closed convex set in R^n and $y \notin S$.
- Then, there exists a unique point $\bar{x} \in S$ with minimum distance from y.
- Furthermore, $\bar{x}$ is the minimizing point if and only if $(y - \bar{x})'(x - \bar{x}) \leq 0$ for each $x \in S$.



SEPARATION OF A CONVEX SET AND A POINT

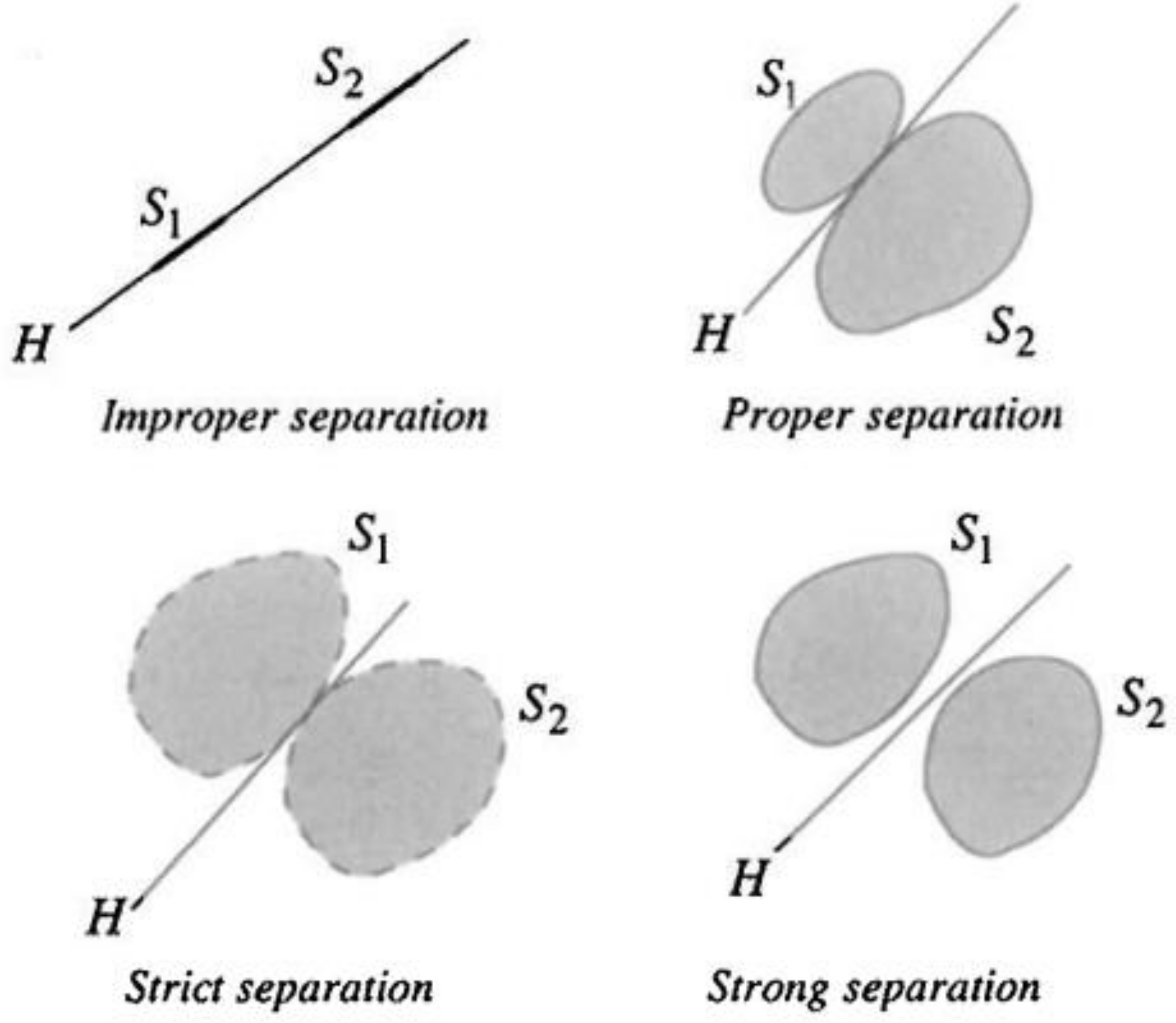
- Let S be a nonempty closed convex set in R^n and y ∉ S.

- Then, there exists a nonzero vector p and a scalar a such that $p'y > \alpha$ and $p'x \leq \alpha$ for each $x \in S$.



HYPERPLANES AND SEPARATION OF TWO SETS

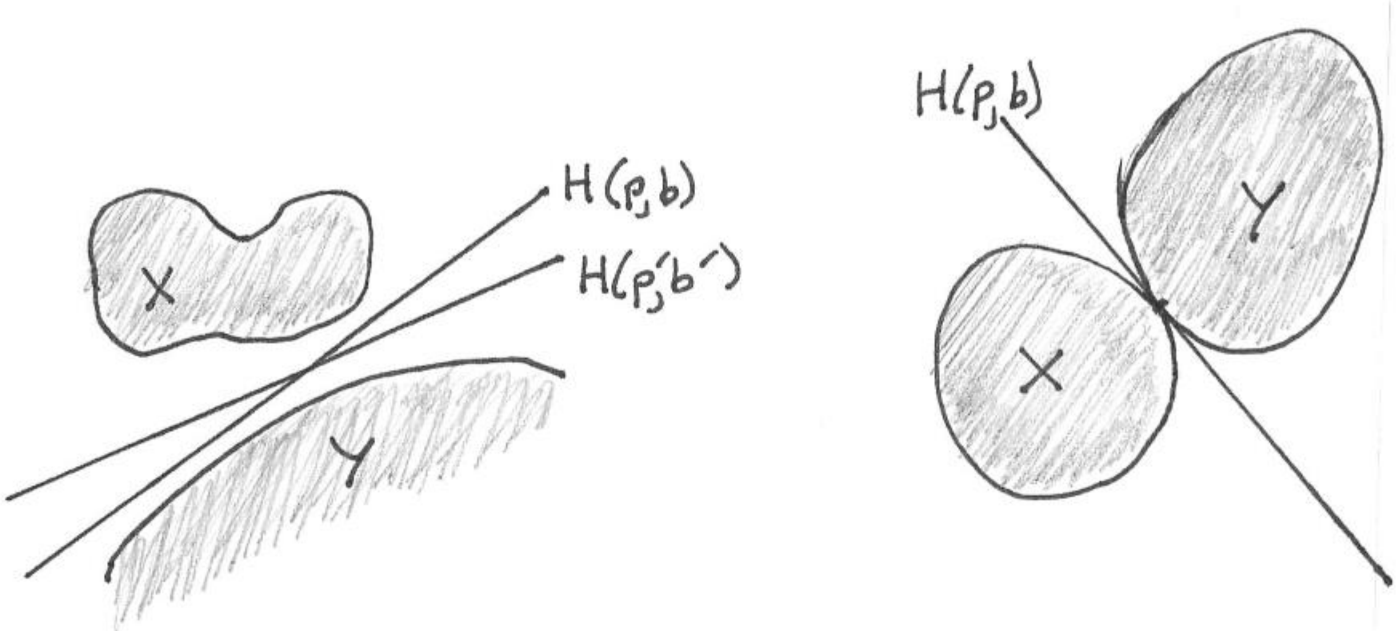
- Various types of separation:



Hyperplanes and separation of two sets

- The hyperplane $\mathrm{H}(\mathrm{p}, \mathrm{b})$ which separates sets X and Y in $\mathbb{R}^n$

- if for all $\mathrm{x} \in \mathrm{X}$ and $\mathrm{y} \in \mathrm{Y}$, we have $\mathrm{p}'\mathrm{x} \leq \mathrm{b} \leq \mathrm{p}'\mathrm{y}$.



MORE ON SEPARATION OF A CONVEX SET AND A POINT

- Let S be a nonempty set, and let $y \notin cl(conv(S))$. Then there exists a strongly separating hyperplane for S and y.

- There exists a hyperplane that strictly separates S and y.

- There exists a vector p such that $p'y > \sup\{p'x : x \in S\}$.

- There exists a vector p such that $p'y < \inf\{p'x : x \in S\}$

FARKAS'S THEOREM

FARKAS'S THEOREM

For given A, b, exactly one of the following statements is true:

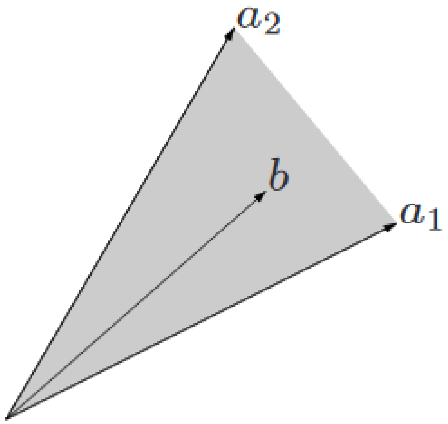
- there exists an x with $Ax = b, x \geq 0$,
- there exists a y with $A^T y \geq 0,\ b^T y < 0$

Geometric Interpretation of Farkas' Theorem

assume A is $m \times n$ with columns a_i

first alternative

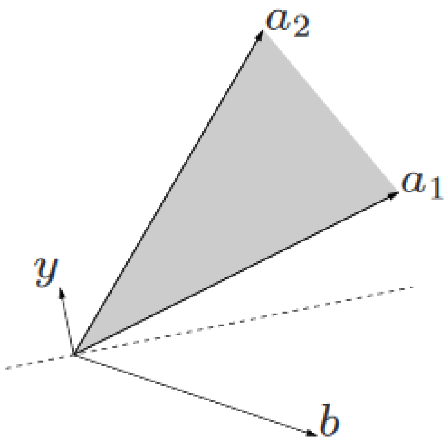
$$b = \sum_{i=1}^{n} x_i a_i, \qquad x_i \geq 0, \quad i = 1, \ldots, n$$



b is in the cone generated by the columns of A

second alternative

$$y^T a_i \geq 0, \quad i = 1, \ldots, m, \qquad y^T b < 0$$



the hyperplane $y^T z = 0$ separates b from $a_1, \ldots, a_m$

FARKAS'S THEOREM

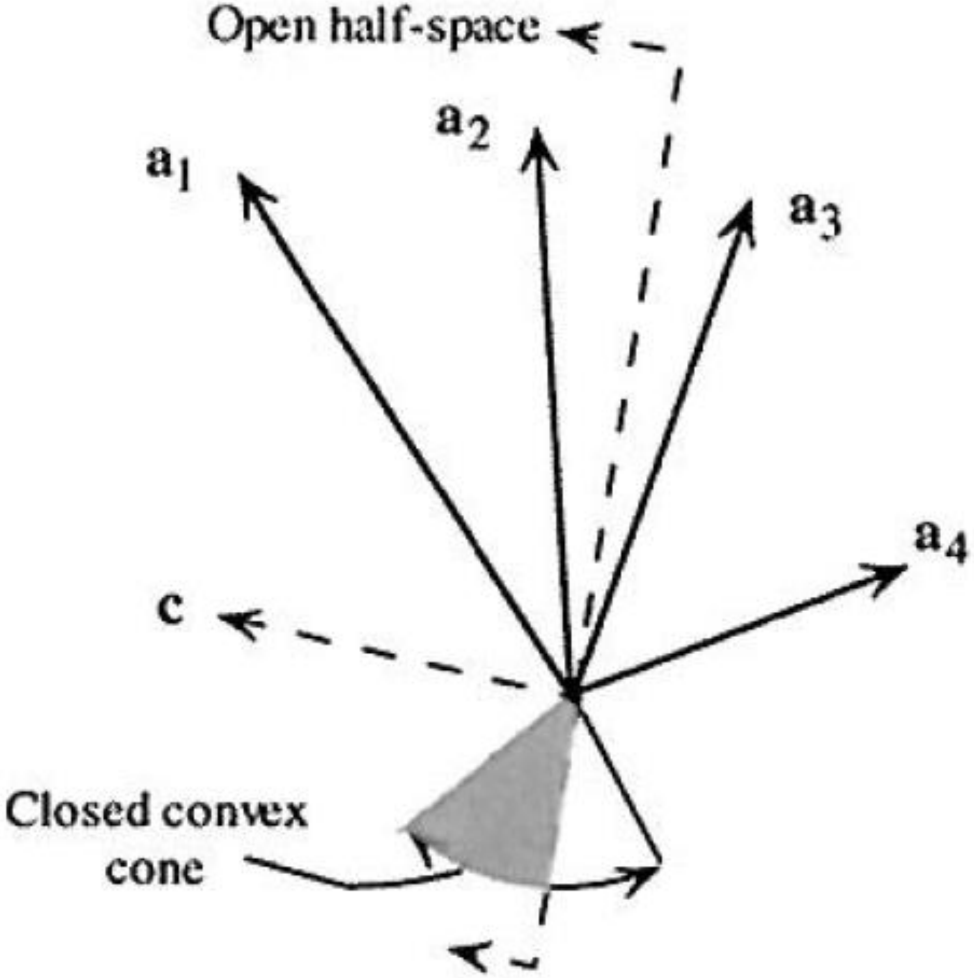
Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{c}$ be an n-vector. Then exactly one of the following two systems has a solution:

System 1: $\mathbf{Ax} \le \mathbf{0}$ and $\mathbf{c}^t\mathbf{x} > 0$ for some $\mathbf{x} \in R^n$.

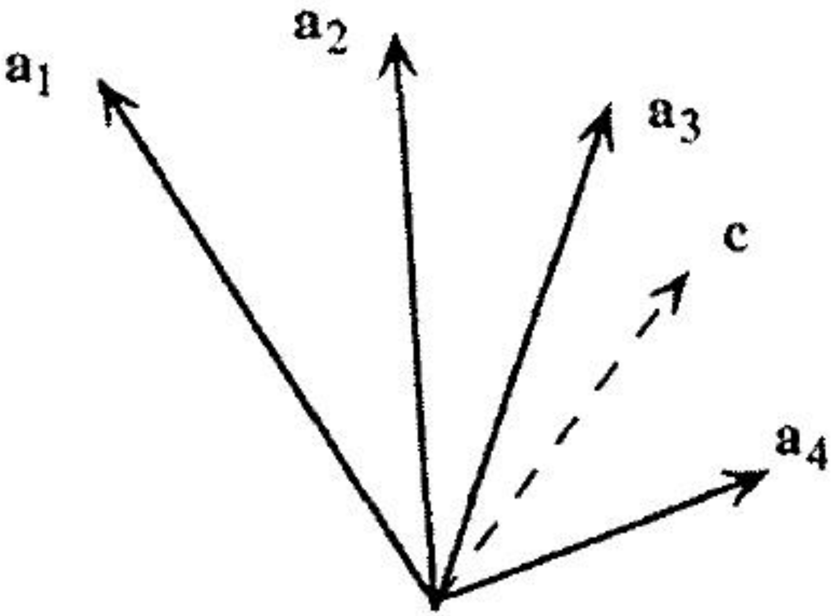
System 2: $\mathbf{A}^t\mathbf{y} = \mathbf{c}$ and $\mathbf{y} \ge \mathbf{0}$ for some $\mathbf{y} \in R^m$.

Suppose that System 2 has a solution; that is, there exists $\mathbf{y} \ge \mathbf{0}$ such that $\mathbf{A}^t\mathbf{y} = \mathbf{c}$. Let $\mathbf{x}$ be such that $\mathbf{Ax} \le \mathbf{0}$. Then $\mathbf{c}^t\mathbf{x} = \mathbf{y}^t\mathbf{Ax} \le 0$. Hence, System 1 has no solution. Now suppose that System 2 has no solution. Form the set $S = \{\mathbf{x} : \mathbf{x} = \mathbf{A}^t\mathbf{y}, \mathbf{y} \ge \mathbf{0}\}$. Note that S is a closed convex set and that $\mathbf{c} \notin S$. By Theorem

FARKAS'S THEOREM



System 1 has a solution



System 2 has a solution

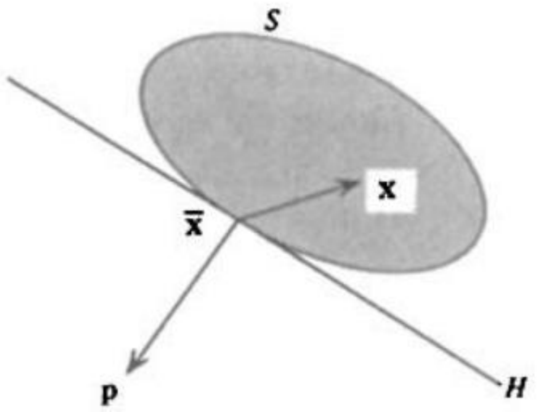
SUPPORTING HYPERPLANE

SUPPORTING HYPERPLANE

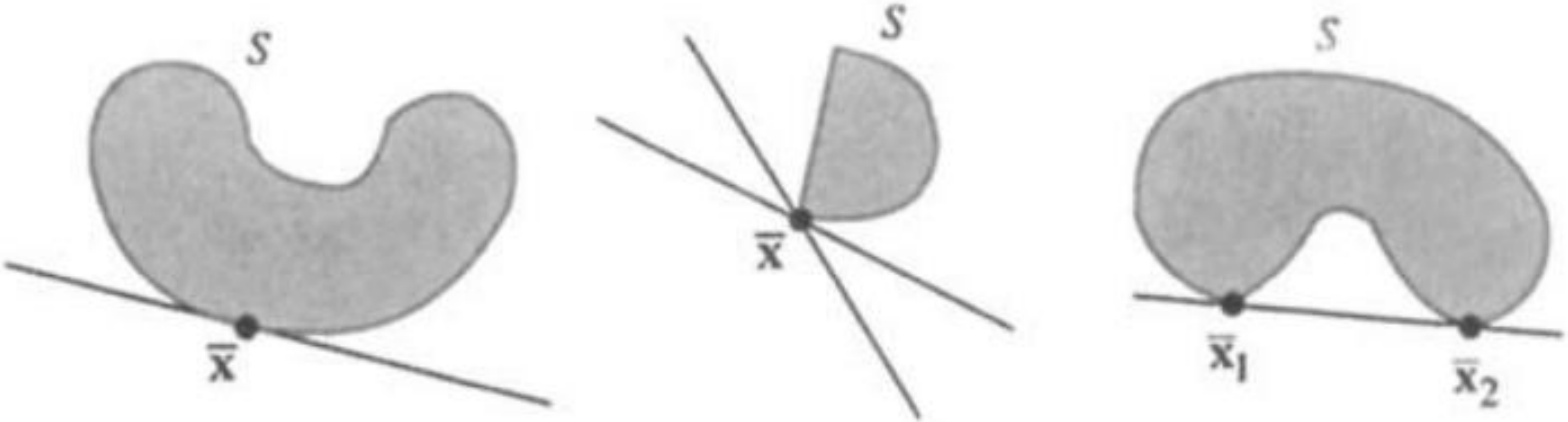
- Let S be a nonempty set in $\mathbb{R}^n$, and let $\bar{x} \in \partial S$.

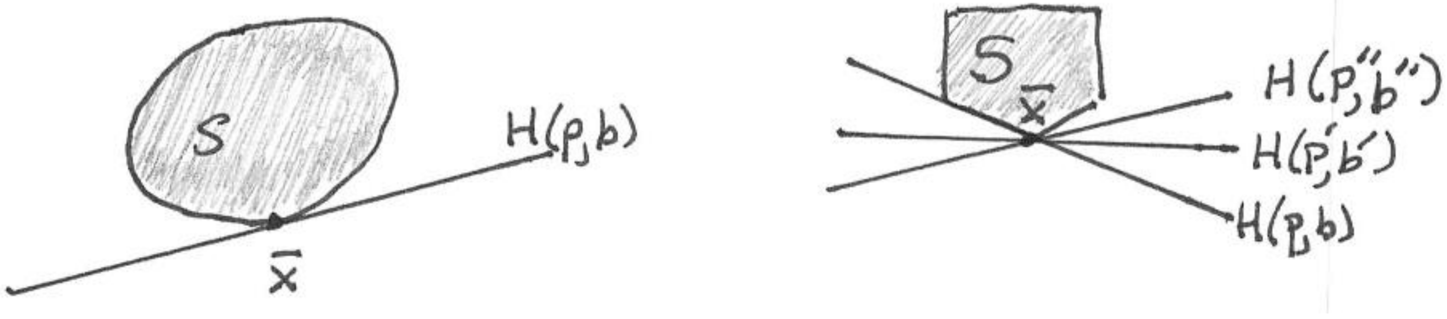
- A hyperplane $H = \{x : p'(x - \bar{x}) = 0\}$ is called a supporting hyperplane of S at $\bar{x}$ if either $S \subseteq H^+$, that is, $p'(x - \bar{x}) \leq 0$ for each $x \in S$,
- or else, $S \subseteq H^-$, that is, $p'(x - \bar{x}) \leq 0$ for each $x \in S$.

- If, in addition, $S \notin H$, H is called a proper supporting hyperplane of S at $\bar{x}$.



EXAMPLE ON SUPPORTING HYPERPLANE

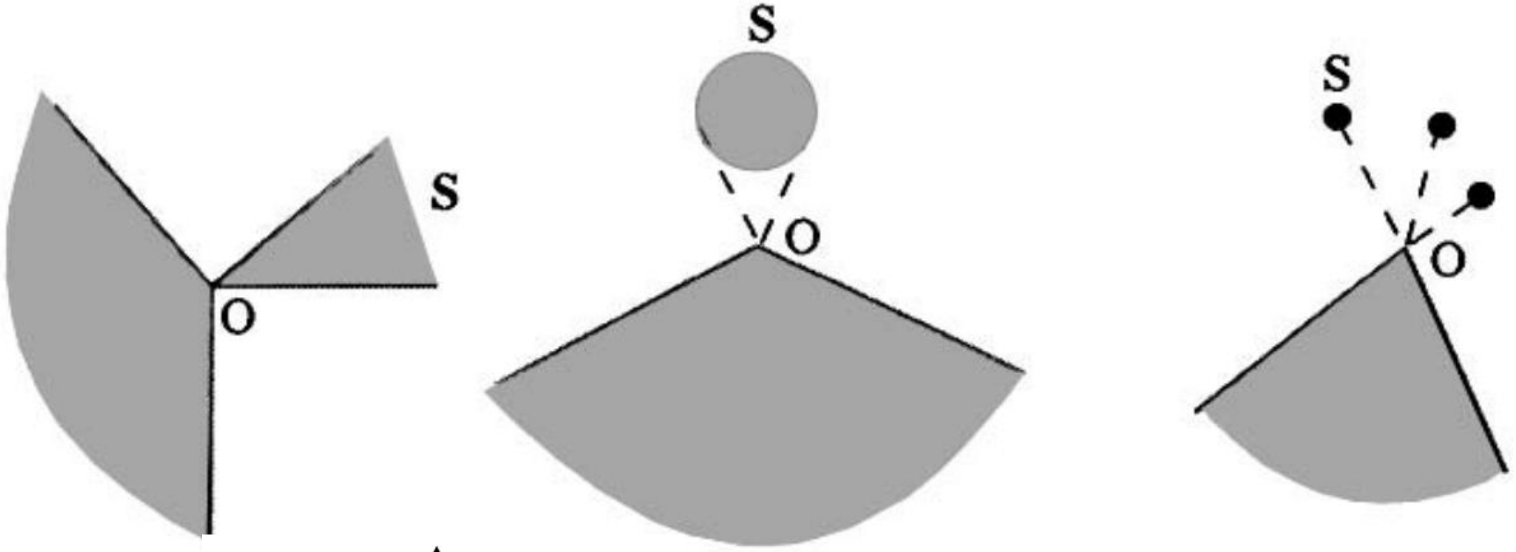


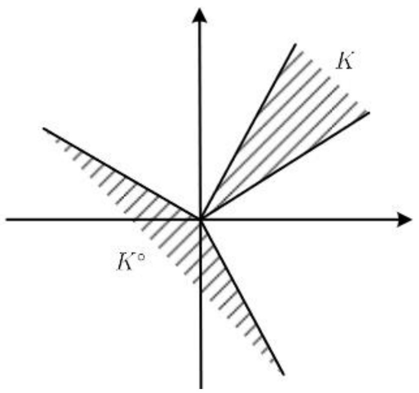


CONES AND POLARITY

CONES AND POLARITY

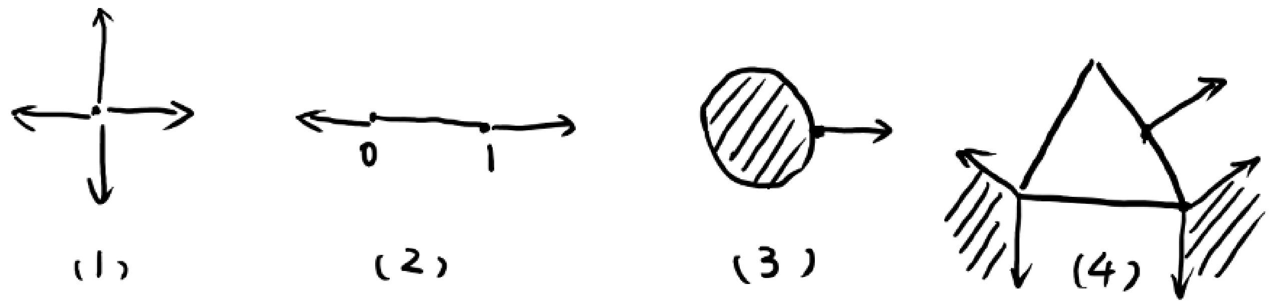
- Polar cone: Let S be a nonempty set in $\mathbb{R}^n$. Then the polar cone of S is given by $\{p: p'x \leq 0 \text{ for all } x \in S\}$.



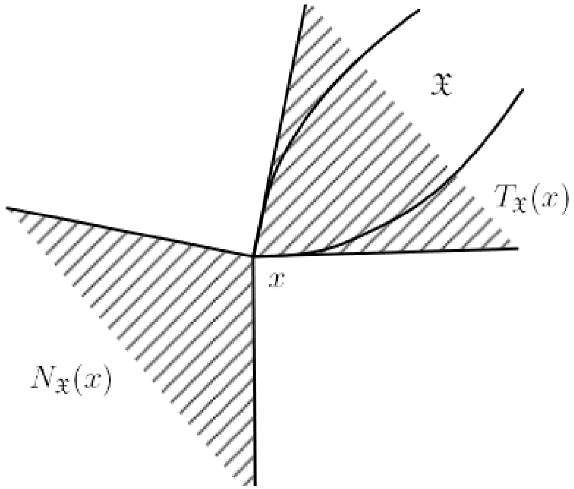


TANGENT CONE AND NORMAL CONES

- Normal cone: Let S be a nonempty set in $\mathbb{R}^n$. Then the normal cone of S at $x \in$ S is given by {p: $p'(y - x) \leq 0$ for all y∈S}.



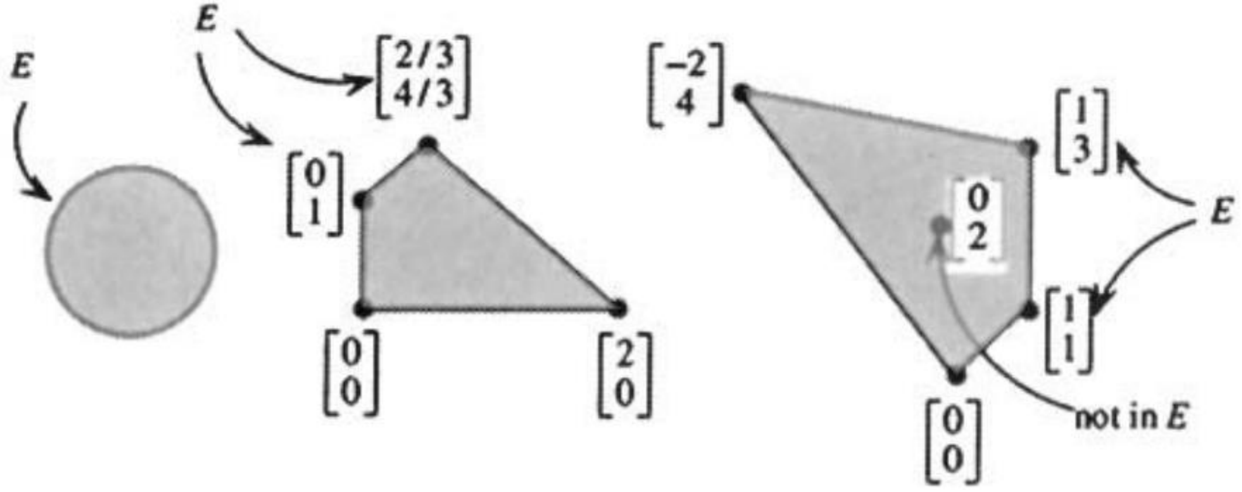
- Tangent cone: Let S be a nonempty set in $\mathbb{R}^n$. Then the feasible direction cone of S at $x \in$ S is given by {d: $\exists\, \epsilon > 0\ s.t.\ x + \epsilon d \in$ S}.



EXTREME POINTS AND DIRECTION

Extreme point:

- Let S be a nonempty convex set in $\mathbb{R}^n$. A vector x ∈S is called an extreme point of S
- if $x = \lambda x_1 + (1-\lambda)$with x_1, x_2 ∈S, and $\lambda \in (0,1)$ implies that $x = x_1 = x_2$.



Direction:

- Let S be a nonempty, closed convex set in $\mathbb{R}^n$.
- A nonzero vector d in $\mathbb{R}^n$ is called a direction, or a recession direction, of S
- if for each x ∈S, x + $\lambda d \in$ S for all $\lambda \geq 0$.

DIRECTION

Let C be a nonempty closed convex set.

- The recession cone R_C is a closed convex cone.
- A vector y belongs to R_C if and only if there exists a vector x ∈ C such that x + α y ∈ C for all α ≥ 0.
- R_C contains a nonzero direction if and only if C is unbounded.
- The recession cones of C and ri(C) are equal.

EXTREME DIRECTION

- Let S be a nonempty convex set in $\mathbb{R}^n$. A direction d is called an extreme direction of S
- if d$= \lambda d_1 + (1-\lambda)d_2$ with d_1, d_2, are two different directions and $\lambda \in (0,1)$ implies that d$= d_1 = d_2$.