A Mean Field Game Analysis of Distributed MAC in Ultra-Dense Multichannel Wireless Networks

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ABSTRACT

This paper analyzes the performance of distributed Medium Access Control (MAC) protocols in ultra-dense multichannel wireless networks, where N frequency bands (or channels) are shared by $M = mN$ devices, and devices make decisions to probe and then transmit over available frequency bands. While such a system can be formulated as an M-player Baysian game, it is often infeasible to compute the Nash equilibria of a large-scale system due to *the curse of dimensionality*. In this paper, we exploit the Mean Field Game (MFG) approach and analyzes the system in the large population regime (N tends to ∞ and m is a constant). We consider a distributed and low complexity MAC protocol where each device probes d/k channels by following an exponential clock which ticks with rate k when it has a message to transmit, and optimizes the probing strategy to balance throughput and probing cost. We present a comprehensive analysis from the MFG perspective, including the existence and uniqueness of the Mean Field Nash Equilibrium (MFNE), convergence to the MFNE, and the price of anarchy with respect to the global optimal solution. Our analysis shows that the price of anarchy is at most one half, but is close to zero when the traffic load or the probing cost is low. Our numerical results confirm our analysis and show that the MFNE is a good approximation of the M-player system. Besides showing the efficiency of the considered MAC for emerging applications in ultra-dense multichannel wireless networks, this paper demonstrates the novelty of MFG analysis, which can be used to study other distributed MAC protocols in ultra-dense wireless networks.

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1 INTRODUCTION

The proliferation of smart wireless devices has brought revolutionary changes in many domains, such as smart homes, smart cities, autonomous cars, virtual-reality/argumented reality, the Internet of the Things (IoT). To accommodate the increasing demand of emerging wireless applications on spectrum, large amounts of spectrum bands that were previously unused or unavailable have recently been released for public use as unlicensed bands for large-scale access, which calls for spectrum access algorithms that are both distributed and efficient.

We consider a scenario in which a large number of smart wireless devices need to constantly communicate their recent status to a fusion center or to nearby peers. This setting includes applications such as sensing and monitoring in smart cities, factories or power stations, and safety messages in autonomous driving. In such applications, an old message can usually be discarded when a new message arrives, because the outdated information is no longer useful when new information is available. Managing wireless channel access for such an ultra-dense deployment of devices with a non-traditional traffic load is a challenge.

In this paper, our focus is on performance analysis of distributed Medium Access Control (MAC) protocols in such ultra-dense multichannel wireless networks. Given the sheer number and the heterogeneous nature of the ownership and applications of the devices, as well the large unlicensed bands that they operate over, it is difficult (if not impossible) to have a centralized scheduler to allocate channels (frequency bands) to devices. Therefore, distributed MAC protocols of simple plug-and-play type are essential. However, performance analysis of even simple distributed MAC in large-scale systems is challenging.

Under our model, each device generates update packets at some rate, and the device drops any previously created packet when a new one is generated, i.e., only the most recent packet at each device is a candidate for transmission. The devices employ a simple MAC protocol under which each device has a clock, and when the clock ticks randomly probes several spectrum bands, and randomly picks one that is free. However, since such probing incurs an energy cost and the number of bands is large, the device can neither probe at a high frequency, nor probe all bands at each clock tick. Thus, it must optimally determine both the frequency of its clock, as well as how many bands to probe when its clock ticks.

We seek to understand the performance of such a MAC protocol when the number of devices, M and available spectrum bands, N are related as $M = mN$, where m is a constant. The devices need to share access to the available spectrum bands, and desire to maximize their individual steady state throughputs while accounting for the energy that they expend in probing. Each device has an exponential clock, and can select any desired clock rate, k. When the clock ticks, the device probes d/k bands, where d is a parameter that

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it chooses[1]. Now, the optimal choice of parameters (k_i, d_i) for a device i depends on the probability that a randomly probed band is currently utilized, which in turn depends on the parameters selected by the other devices. Hence, the devices engage in a strategic game of observing channel utilization, and choosing the tuple (k_i, d_i) while trading off steady state throughput and probing cost.

While this system can be modeled as an M-player Baysian game where each device makes myopic probing/transmision decisions based on local observations, it is infeasible to compute the Nash equilibria for large M due to the curse of dimensionality. In this paper, we use a mean field game (MFG) approach to overcome this difficulty by studying the asymptotic performance of the system as the numbers of devices and spectrum bands both go to infinity. In this large-population regime, the distribution of channel states converges weakly to a point mass, which can be computed explicitly. Therefore, instead of interacting with $M-1$ other players, each device optimizes its strategy with respect to a fixed channel state distribution, which dramatically simplifies the problem.

Main Results

Our main results are detailed as follows.

MFG Formulation: We first introduce the model and the M-player Baysian game in Section 2. The problem is hard to analyze because it involves an M-dimensional Markov chain. To overcome this difficulty, we adopt the MFG approach, developed in [13]. We first show that fixed k and d for each device, in the mean-field limit, the fraction of busy channels, denoted by γ, converges weakly to a constant (the result is presented in **Theorem 1**). Therefore, in the mean-field limit, each device maximizes its utility (throughput minus the probing cost) with respect to a constant γ instead of the probability distribution of N channel states, which makes the analysis tractable. In Section 3, we also prove that probing one channel with rate d dominates probing d/k channels with rate k, which reduces the policy space of each device to a single parameter d. In the mean-field limit, the M-player Baysian game becomes an MFG. Specifically, given γ, the fraction of busy channels, each device chooses a myopic d to maximize its utility, which defines the mapping $T_2 : \gamma \longrightarrow d$. Given d, we can calculate the fraction of busy channels in the mean-field limit, which defines the mapping: $T_1 : d \longrightarrow \gamma$. The Mean Field Nash Equilibrium (MFNE) is a pair (d^*, γ^*) such that

$$d^* = T_2(T_1(d^*)).$$

Existence, Uniqueness and Convergence to MFNE: In Section 5, we present a comprehensive analysis of the existence and uniqueness of the MFNE. **Theorem 2** states that there exists a unique MFNE when the traffic load is high, that the MFNE results in $d^* = \infty$ (i.e. each device probes channels continuously without any waiting) when the traffic load is low, and that the system jumps between a finite probing rate and infinite probing rate when the load is in between. The precise meanings of "high" and "low" are defined in Theorem 2.

In Section 5, we examine convergence to the MFNE. We focus on the most interesting regime, namely, the high load regime, under which d^* is finite in the unique MFNE. **Proposition 3** shows that the composition of T_2 and T_1 is a contraction mapping, which implies the convergence to the unique MFNE from any initial condition following the Banach fixed point theorem.

Price of Anarchy: In Section 6, we compare the performance of the distributed MAC protocol with a solution that solves a centralized optimization problem and forces the resulting probing rate upon all the devices. The key difference between the two is that the central solution knows exactly how changing the probing rate of a device level will affect the fraction of busy channels in the network, i.e. it knows the function $\gamma = T_1(d)$; whereas in the distributed algorithm, each device optimizes its probing rate d assuming that γ is a constant. We show that the price of anarchy is upper bounded by 0.5, i.e., the loss of efficiency is at most half. Numerical studies show that the price of anarchy is close to zero when the load is light and approaches the upper bound 0.5 when the traffic load increases. Furthermore, comparing with the global optimal solution, each device probes channels with a higher rate at the MFNE, which results in higher throughput but also consumes more energy during probing.

Numerical Evaluation: Finally, we evaluate the algorithm with extensive simulations. In particular, we compare the performance of the distributed MAC in finite population systems with the MFG solution. We observe that the performance predicted using MFG is close to the performance of finite population systems even with moderate N, which confirms the effectiveness of the MFG approach. We also observe that the proposed algorithm significantly outperforms other simple distributed MAC protocols.

Related Work

The mean field approach is a method of identifying the steady-state behavior of an $M-$dimensional Markov chain, where M is the number of particles (devices in our case), whose states are modeled via the Markov chain. The goal is to characterize the steady-state distribution (time becomes asymptotically large) for a finite M, and then determine the limiting steady-state distribution as M becomes asymptotically large.

In order to do so, the mean field method proceeds to take the two limits (particles and time) in the reverse order. The main idea is to use the fact that under mild conditions, as the number of particles, M becomes asymptotically large, the state distribution of the limiting Markov chain can be accurately represented using an ordinary differential equation (ODE). Then the steady-state distribution of the limiting Markov chain is the same as the infinite time limiting state of the ODE (if it exists). Finally, if it can be shown that the order of taking the particle and time limits can be interchanged (yield the same limit) for the Markov chain, then the limiting state of the ODE provides the desired solution referred to as the Mean Field Equilibrium (MFE) (see [2] and references within). A recent approach based on Stein's method [7, 8, 17, 18] can directly establish the convergence of steady-state distributions to the MFE without the interchange of the limits argument and provide the rate of convergence.

When we do have convergence of the steady-state distribution to a deterministic limit of the ODE, we have a further property referred to as Propagation of Chaos ([9, 15]), under which the states of any finite set of particles are independent of each other given

[1]The form d/k is for notational convenience, and the optimal choice will turn out to be an integer.

the state distribution as a whole. Such an independence property is particularly useful in identifying the behavior of a given particle in the large M limit, and to determine the corresponding ODE of the system. In the context of wireless MAC protocols, such an independence assumption regarding the backoff processes of the devices using 802.11 MAC enabled the derivation of steady-state performance in the limiting case of a large number of devices that always have packets to transmit (called "saturated") [3].

This assumption was questioned in [2], in which it was shown that simply having a unique fixed point of the corresponding ODE is insufficient, and that all trajectories have to converge to that fixed point in order for the independence claim to hold. Later, it was shown that there exist natural parameter selections for 802.11 under which the sufficiency conditions of [2] are satisfied for the cases of infinite and finite backoff stages [4, 6]. More recently, the performance of 802.11 MAC in the unsaturated case was characterized using the mean field approach [5]. However, existing work considers the case of a single interference channel or an interference graph, unlike our setup of channel selection under a high bandwidth regime.

The mean field regime has also been studied under a game theoretic setting. Initial work in this space and many that followed consider a one shot game under which the mean field independence property is used to simplify decision making [13]. More recent work has considered repeated games under a variety of different application settings [1, 10, 14, 16]. Here, the MFG is considered as the extension of a Bayesian repeated game to infinite players, with the independence property being used to enable the identification of existence and structural properties of a Mean Field Nash Equilibrium (MFNE). However, no claim is typically made about the convergence of the steady-state distribution of the finite player system to the mean field in the limit as the number of players increases. This paper not only establishes the existence and convergence of MFNE in the limit but also shows the convergence of the steady-state distribution to the MFE under a given policy.

2 SYSTEM MODEL AND AN M-PLAYER GAME

We consider a multi-channel ultra-dense wireless networks with N channels and $M = mN$ devices. At each time instance, one and only one device can transmit over a given channel due to interference. As in many IoT applications, each device wants to continuously communicate their latest status to corresponding receivers, which could be an access point or another IoT device. The messages are called status messages in this paper. We note after a new status message is generated, the device does not need to transmit old, unsent status messages currently in the buffer, so the old status messages will be discarded. This communication model is an example where the system wants the most fresh information and wants to minimize the "age of information" [11].

We assume for each device, status messages are generated according to a Poisson process with rate λ. When the device is probing an idle channel to transmit, it only stores the latest status message. If the device is transmitting a status message when a new status message arrives, the device keeps the newest status message in the buffer and transmits it immediately after finishing sending the one in transmission. A channel being used to transmit a status message is in busy state, otherwise the channel is in idle state. We further assume that the time it takes to transmit a message is exponentially distributed with mean one.

When a device has a status message to transmit, it searches for an idle channel to transmit the message. A device cannot afford to continuously monitor all N frequency bands at all times, because channel probing costs energy and battery powered smart wireless devices are energy constrained. We assume each device maintains an internal exponential clock with rate k. When the exponential clock ticks, the device probes $\frac{d}{k}$ channels. If one of the $\frac{d}{k}$ channels is idle, the device occupies the channel and transmits the message in the buffer. A device has three possible states: *idle (0)*, *probing (1)* and *transmitting (2)*. Let $Q_i(t)$ denote the number of devices in state i at time t. Each device is associated with a *continuous-time* Markov chain with three states as shown in Figure 1 in principle. The Markov-chain includes three states and the transitions occur as follows:

- The state moves from idle to probing when a message arrives, which occurs with rate λ.
- Let d_l and k_l denote the probing parameters used by device l, and $\mathbf{d}$ and $\mathbf{k}$ denote M-dimensional vectors that represent the probing parameters of all M devices. Given $Q_2(t)$, the number of devices in the transmitting state, by probing $\frac{d_l}{k_l}$ channels, the probability of finding an idle channel is
$$1-\left(\frac{Q_2(t)}{N}\right)^{\frac{d_l}{k_l}}.$$
Therefore, the state of the Markov chain transits from probing to transmitting with rate
$$k_l\left(1-\left(\frac{Q_2(t)}{N}\right)^{\frac{d_l}{k_l}}\right).$$
- The state transits from transmitting to idle when (1) the status message is transmitted, which occurs with rate one, and (2) no new status message arrives during the transmission, which occurs with probability $\frac{1}{1+\lambda}$. To see this let T denote the transmission time of a message, which is an exponential random variable with mean one. Under the Poisson arrival, the probability of no arrival during a period of duration t is $e^{-\lambda t}$. Therefore, the probability that there is no new message arrival during the transmission is
$$\begin{aligned}&\Pr(\text{no arrival during transmission})\\ =&E\left[\Pr(\text{no arrival during duration } T|T)\right]\\ =&\int_{t=0}^{\infty} e^{-\lambda t}e^{-t}\,dt\\ =&\int_{t=0}^{\infty} e^{-(\lambda+1)t}\,dt\\ =&\frac{1}{1+\lambda}.\end{aligned}$$
Therefore, the transition rate is $\frac{1}{1+\lambda}$.

Suppose $Q_2(t)$ is a constant, then the stationary distribution of this three-state Markov chain, denoted by π, can be calculated using

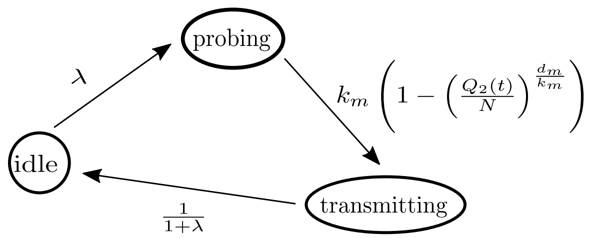

Figure 1: The Continuous-Time Markov Chain

the global balance equations:

$$\lambda\pi_0 = k_l\left(1-\left(\frac{Q_2}{N}\right)^{\frac{d_l}{k_l}}\right)\pi_1 = \frac{1}{1+\lambda}\pi_2,$$

from which, we have

$$\begin{aligned}\pi_0 &= \frac{1}{\lambda(1+\lambda)}\pi_2\\ \pi_1 &= \frac{1}{(1+\lambda)k_l\left(1-\left(\frac{Q_2}{N}\right)^{\frac{d_l}{k_l}}\right)}\pi_2\\ \pi_2 &= \frac{1}{1+\frac{1}{\lambda(1+\lambda)}+\frac{1}{(1+\lambda)k_l\left(1-\left(\frac{Q_2}{N}\right)^{\frac{d_l}{k_l}}\right)}}.\end{aligned}\tag{1}$$

However, $Q_2(t)$ is a random process whose stationary distribution is determined by $\mathbf{d}$ and $\mathbf{k}$ so is difficult to calculate. Now let $\pi^{(l)}(\mathbf{d},\mathbf{k})$ denote the stationary distribution of the Markov chain associated with device l. As mentioned earlier, calculation of $\pi^{(l)}$ is difficult even for fixed $\mathbf{k}$ and $\mathbf{d}$.

Making the problem even more difficult, each device needs to balance the energy consumed for probing and the amount of information transmitted. We consider the following cost function for each device:

$$\hat{J}(d_l,k_l) = -\pi_2^{(l)}(\mathbf{d},\mathbf{k}) + c\left(\pi_1^{(l)}(\mathbf{d},\mathbf{k})d_l\right)^2. \tag{2}$$

In the equation above, the first term $\pi_2^{(l)}(\mathbf{d},\mathbf{k})$ is the fraction of time the device is in the transmitting state, so can be viewed as the average throughput. In the second term, $\pi_1^{(l)}(\mathbf{d},\mathbf{k})$ is the fraction of time the device is in the probing state and d_l is the number of channels it probes per unit time when it is in the probing state, so $\pi_1^{(l)}(\mathbf{d},\mathbf{k})d_l$ is the average number of channels probed per unit time. c is a constant. The quadratic form is in keeping with the idea that energy usage for a given task is convex for most communication applications. Given other devices' probing parameters $\mathbf{d}_{-l}$ and $\mathbf{k}_{-l}$, device l aims at finding the optimal d_l^* and k_l^* such that

$$\begin{aligned}(d_l^*,k_l^*) &\in \arg\min_{d_l,k_l}\hat{J}(d_l,k_l)\\ &= \arg\min_{d_l,k_l} -\pi_2^{(l)}(\mathbf{d},\mathbf{k}) + c\left(\pi_1^{(l)}(\mathbf{d},\mathbf{k})d_l\right)^2.\end{aligned}\tag{3}$$

We note that this is an M-player game and the difficulty in solving the Nash equilibrium of this M-player game is in calculating $\pi^{(l)}(\mathbf{d},\mathbf{k})$ as discussed earlier.

3 MEAN-FIELD GAME FOR ULTRA-DENSE WIRELESS NETWORKS

Since solving the M-player game (3) is difficult, we use the MFG approach with $N,M\to\infty$. In the next section, we will show that assuming all devices use the same probing policy (d,k), then as $N,M\to\infty$, $Q_i(\infty)/M$ converges weakly to q_i^*, which is the equilibrium point of the following mean-field model:

$$\begin{aligned}\frac{dq_0}{dt} &= -\lambda q_0 + \frac{1}{1+\lambda}q_2\\ \frac{dq_1}{dt} &= \lambda q_0 - k(1-(mq_2)^{d/k})q_1\\ \frac{dq_2}{dt} &= k\left(1-(mq_2)^{d/k}\right)q_1 - \frac{1}{1+\lambda}q_2\end{aligned}\quad. \tag{4}$$

We defer the derivation of this mean-field model and the proof of convergence to the Technical report []. Intuitively, $q_i(t)$ is an approximation of $Q_i(t)/M$ and q_i^* is an approximation of $Q_i(\infty)/M$ at the mean-field limit.

Given q_2^*, the fraction of devices are in transmitting state, the fraction of busy channels is $\gamma^* = mq_2^*$. Now to introduce the MFG, we assume time-scale separation such that devices adapt their probing strategies in a slower time scale than the convergence of the mean-field model. Under this assumption, when it is the time for devices to adapt their probing policies, all devices can measure γ, which can be done accurately under the time-scale separation assumption. Then after measuring the fraction of busy channels is γ, each device can compute the stationary distribution of its three-state Markov chain according to (1) by substituting $\gamma = Q_2/N$, and also the corresponding cost $J(d,k)$. Each device optimizes its probing strategy (d^*,k^*) such that

$$(d^*,k^*) \in \arg\min_{d,k} J(d,k), \tag{5}$$

where

$$\begin{aligned}J(d,k) = &-\frac{1}{1+\frac{1}{\lambda(1+\lambda)}+\frac{1}{(1+\lambda)k\left(1-\gamma^{\frac{d}{k}}\right)}}\\ &+c\left(\frac{d}{(1+\lambda)k\left(1-\gamma^{\frac{d}{k}}\right)+\frac{k\left(1-\gamma^{\frac{d}{k}}\right)}{\lambda}+1}\right)^2.\end{aligned}\tag{6}$$

In other words, choosing a probing strategy to minimize its cost for given γ. Note that the cost function $J(d,k)$ is different from $\hat{J}(d,k)$ defined in (2) because γ is a constant in $J(d,k)$ but it is a function of (d,k) in $\hat{J}(d,k)$. We can view $\hat{J}(d,k)$ as the true cost function and $J(d,k)$ is an estimate of the true cost obtained by assuming γ does not change even when the device changes its probing strategy. We use different notations to emphasize the difference.

In summary, given (d,k), the mean-field model (4) maps (d,k) to the fraction of busy channels γ. Let T_1 denote this mapping, i.e.

$$T_1 : (d,k) \to \gamma.$$

Given the fraction of busy channels γ, each device minimizes the cost function J in (d,k), which maps γ to policy (d,k). Let T_2 denote this mapping, i.e.

$$T_2 : \gamma \to (d,k).$$

With the notation defined above, we formally define the MFG and Mean Field Nash Equilibrium (MFNE).

MFG for Distributed MAC:

- **Initialization:** All devices are initialized with a common probing policy (d, k).
- **System Adaptation:** The mean-field model (4) converges under policy (d, k) and the fraction of busy channels converges to a constant γ.
- **Policy Optimization:** All devices learn γ in the system adaptation step, and optimize their probing strategies by minimizing $J(d, k)$. Go to the system adaptation step. □

A policy (d^*, k^*) is called the MFNE if

$$(d^*, k^*) = T_2(T_1(d^*, k^*)).$$

At the MFNE where all devices use the policy (d^*, k^*), no device has incentive to unilaterally change the strategy in the mean-field limit. We also remark that the assumption that all devices use the same policy (d, k) at the beginning is not critical. Under the assumption all devices have the same cost function, the optimal probing strategy is determined only by γ. Therefore, even devices have different probing strategies at the beginning, after they measure γ in the policy optimization step, they will start to use the same probing policy.

In the next section, we prove the weak convergence of $Q_i(\infty)/M$ to q_i^*, which is the key assumption we have used to derive the MFG.

4 MEAN-FIELD LIMIT WITH FIXED (d, k)

Assume all devices have the same cost function. Then given the fraction of busy channels γ, the solution of the optimal policy (d^*, k^*) is the same for all devices. Therefore, without loss of generality, we assume all devices use the same policy (d, k) and consider the convergence of the fraction of busy channels to its mean-field limit in this homogeneous case. Before proving this result, we first present the following lemma.

Lemma 1. *The cost function $J(k, d)$ satisfies for any $k < d$,*

$$J(d, d) < J(d, k).$$

Proof. Given γ, k and d, the stationary distribution of the three-state Markov chain is given by (1) with $Q_2/N = \gamma$. The cost function $J(k, d)$, therefore, can be written in terms of γ, k, and d as

$$J(k, d) = -\frac{(1+\lambda)k(1-\gamma^{d/k})}{(1+k(1-\gamma^{d/k})(1+\lambda+\frac{1}{\lambda}))} + c\left(\frac{d}{(1+k(1-\gamma^{d/k})(1+\lambda+\frac{1}{\lambda}))}\right)^2.$$

The transition rate from the probing state to the transmitting state is $k(1-\gamma^{d/k})$. Note that $k(1-\gamma^{d/k})$ is increasing in k when $\frac{d}{k} \geq 1$ because

$$\frac{\partial}{\partial k}\left(k\left(1-\gamma^{\frac{d}{k}}\right)\right) = 1-\gamma^{\frac{d}{k}} + \gamma^{\frac{d}{k}}\frac{d}{k}\log\gamma.$$

Now define

$$f(y, \gamma) = 1-\gamma^y + \gamma^y y\log\gamma.$$

We next prove that $f(y) > 0$ for $y \geq 1$ and $0 < \gamma \leq 1$. Note that

$$\frac{\partial}{\partial y}f(y, \gamma) = -\gamma^y\log\gamma + \gamma^y\log\gamma + \gamma^y y(\log\gamma)^2 = \gamma^y y(\log\gamma)^2 > 0.$$

Now consider

$$f(1, \gamma) = 1-\gamma+\gamma\log\gamma.$$

We have

$$\frac{\partial}{\partial\gamma}f(1, \gamma) = \log\gamma < 0.$$

Therefore, we conclude that for $y \geq 1$ and $0 < \gamma \leq 1$, we have

$$f(y, \gamma) > f(1, \gamma) \geq f(1, 1) = 0,$$

i.e.

$$\frac{\partial}{\partial k}\left(k\left(1-\gamma^{\frac{d}{k}}\right)\right) = 1-\gamma^{\frac{d}{k}} + \gamma^{\frac{d}{k}}\frac{d}{k}\log\gamma > 0$$

Define $x = k(1-\gamma^{d/k})$. We obtain

$$J(x) = -\frac{(1+\lambda)}{\frac{1}{x}+1+\lambda+\frac{1}{\lambda}} + c\left(\frac{d}{1+x(1+\lambda+\frac{1}{\lambda})}\right)^2,$$

which is clearly a decreasing function of x. Therefore, for fixed d, $J(d, k)$ is a decreasing function of k. Therefore, we have $J(d, d) < J(d, k)$ when $d > k$. □

According to the lemma above, given γ, the optimal policy (d^*, k^*) satisfies $k^* = d^*$. In other words, given d, it is optimal to one channel at a time with rate d. Therefore, in the following discussion, we focus on probing policies such that $d = k$. Since $d = k$, we will now proceed assuming that each device wishes to optimize a cost function written in terms of d. This function can be written as:

$$J(d) = -\frac{(1+\lambda)d(1-\gamma)}{1+d(1-\gamma)(1+\lambda+\frac{1}{\lambda})} + c\left(\frac{d}{1+d(1-\gamma)(1+\lambda+\frac{1}{\lambda})}\right)^2. \tag{7}$$

and the dynamical system can be written as:

$$\begin{aligned}
\frac{dq_0}{dt} &= -\lambda q_0 + \frac{1}{1+\lambda}q_2 \\
\frac{dq_1}{dt} &= \lambda q_0 - d(1-mq_2)q_1 \\
\frac{dq_2}{dt} &= d(1-mq_2)q_1 - \frac{1}{1+\lambda}q_2
\end{aligned} \tag{8}$$

Theorem 1. *Assume that all devices use the same policy (d, d). Let $\gamma^{(N)}(\infty)$ denote the fraction of busy channels at the steady state in a system with N channels and mN devices. Then $\gamma^{(N)}(\infty)$ converges weakly to γ, which is the unique equilibrium of mean-field model* (4) *with $d = k$, and is the unique solution of the following equation:*

$$\gamma = \frac{m(1+\lambda)k(1-\gamma)}{1+d(1-\gamma)(1+\lambda+\frac{1}{\lambda})}. \tag{9}$$

Due to space constraints the proof can be found in the Technical report [], where we also briefly discuss the derivation of the mean-field model (4). Figure 2 shows the simulation results with $m = 5$, and $c = 10$, $\lambda = 0.7$, and $d = 0.065$. We varied N from 10, to 100 and then to 1,000. We can clearly see that γ converges to the mean-field limit as N increases, and when $N = 1,000$, γ concentrates to the mean-field limit.

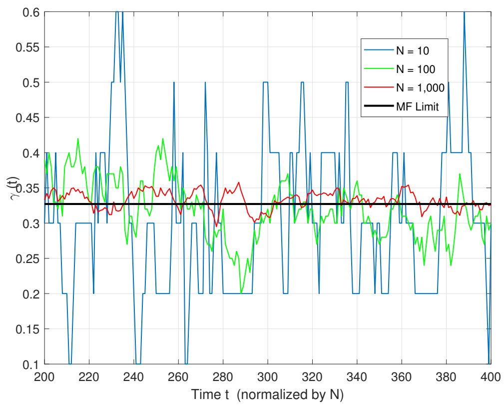

Figure 2: Convergence to the Mean Field Limit with Fixed d

5 UNIQUENESS AND CONVERGENCE OF MFNE

In the previous section, we have shown that given policy (d, d), the stationary distribution of the mN-device system converges to a unique mean-field limit, which defines mapping

$$T_1 : d \longrightarrow \gamma. \tag{10}$$

The mapping

$$T_2 : \gamma \longrightarrow d \tag{11}$$

is obtained by solving the optimization problem $\min_k J(d)$ for given γ.

The following lemma provides the closed-form expression of mapping T_2.

Lemma 2. *Given $0 < \gamma < 1$ and $d \geq 0$, $J(d)$ has a unique minimizer*

$$d = \frac{a}{\max\{2c - ab, 0\}},$$

where $a = (1-\gamma)(1+\lambda)$ and $b = (1-\gamma)\left(1+\lambda+\frac{1}{\lambda}\right)$.

Proof. Define $a = (1+\lambda)(1-\gamma)$ and $b = (1-\gamma)(1+\lambda+\frac{1}{\lambda})$. Then $J(d)$ can be written as

$$J(d) = -\frac{ad}{1+bd} + c\left(\frac{d}{1+bd}\right)^2,$$

and

$$\frac{\partial J(d)}{\partial d} == \frac{1}{(1+bd)^2}\left(-a + \frac{2cd}{1+bd}\right).$$

We now consider

$$h(d) = -a + \frac{2cd}{1+bd}.$$

Note that $h(d)$ is an increasing function for $d \geq 0$. Furthermore $h(0) = -a$ and

$$h(d) \leq \lim_{d\to\infty} h(d) = -a + \frac{2c}{b}.$$

Therefore, if $\frac{2c}{b} \leq a$, (i.e. $h(d) \leq 0$), then $J(d)$ is a strictly decreasing function and the minimum is achieved at $d = \infty$. Otherwise, the minimum is achieved when

$$d = \frac{a}{2c - ab}.$$

In summary, $J(d)$ is minimized at

$$d = \frac{a}{\max\{2c - ab, 0\}}.$$

□

Now given mapping T_1 characterized in Theorem 1 and mapping T_2 characterized in Lemma 2, the following theorem establishes the existence and uniqueness of the MFNE.

Theorem 2. *The existence of MFG equilibria depends on the traffic load λ and constant c. The results can be divided into three cases. For fixed c, the following three cases correspond to"low", "high" and "medium" traffic regimes.*

- **Case I (Low Traffic Regime)**: *If*

$$2c \leq \left(\max\left\{0, 1 - \frac{m(1+\lambda)}{1+\lambda+\frac{1}{\lambda}}\right\}\right)^2 (1+\lambda)\left(1+\lambda+\frac{1}{\lambda}\right), \tag{12}$$

then $d^* = \infty$ is the unique MGF equilibrium. In other words, in this case, a device should continuously probe idle channels (with no waiting) when there is a message to transmit.

- **Case II (High Traffic Regime)**: *If*

$$2c > \left(1-\gamma^*\right)^2 (1+\lambda)\left(1+\lambda+\frac{1}{\lambda}\right), \tag{13}$$

where

$$\gamma^* = 1 + \frac{c}{m(1+\lambda)^2} - \sqrt{\frac{c^2}{m^2(1+\lambda)^4} + \frac{2c}{m(1+\lambda)^2}},$$

then there exists a unique MGF equilibrium

$$d^* = \frac{(1-\gamma^*)(1+\lambda)}{2c - (1-\gamma^*)^2(1+\lambda)\left(1+\lambda+\frac{1}{\lambda}\right)}. \tag{14}$$

- **Case III (Medium Traffic Regime)**: *Otherwise, MFNE does not exist and devices switch probing strategy between $d = \infty$ and*

$$d = \frac{(1-\tilde{\gamma})(1+\lambda)}{2c - (1-\tilde{\gamma})^2(1+\lambda)\left(1+\lambda+\frac{1}{\lambda}\right)},$$

where

$$\tilde{\gamma} = \min\left\{1, \frac{m(1+\lambda)}{1+\lambda+\frac{1}{\lambda}}\right\}.$$

Proof. We first consider Case I such that

$$2c \leq \left(\max\left\{0, 1 - \frac{m(1+\lambda)}{1+\lambda+\frac{1}{\lambda}}\right\}\right)^2 (1+\lambda)\left(1+\lambda+\frac{1}{\lambda}\right). \tag{15}$$

Under this condition, we have

$$1 - \frac{m(1+\lambda)}{1+\lambda+\frac{1}{\lambda}} > 0. \tag{16}$$

Recall (q_0^*, q_1^*, q_2^*) denote the unique equilibrium point of mean field model (34) for a given d. For any $d \geq 0$, we have

$$q_2^* \leq \frac{1+\lambda}{1+\lambda+\frac{1}{\lambda}}.$$

This upper bound holds because the following equations holds for all $d > 0$:

$$\lambda q_0^* = \frac{1}{1+\lambda} q_2^* \tag{17}$$

$$\sum_i q_i^* = 1, \tag{18}$$

which implies

$$\frac{\frac{1}{\lambda}}{1+\lambda} q_2^* + q_1^* + q_2^* = 1$$

and

$$\left(1 + \frac{\frac{1}{\lambda}}{1+\lambda}\right) q_2^* \leq 1.$$

Recall that $\gamma^* = m q_2^*$, so

$$\gamma^* \leq \frac{m(1+\lambda)}{1+\lambda+\frac{1}{\lambda}}.$$

Substituting this inequality into (38), we have that the following inequality holds for any $d \geq 0$:

$$2c \leq \left(1-\gamma^*\right)^2 (1+\lambda)\left(1+\lambda+\frac{1}{\lambda}\right) = ab, \tag{19}$$

where a and b are defined in Lemma 2. Therefore, $2c \leq ab$, and $d^* = \infty$ according to Lemma 2. Furthermore, given $d^* = \infty$, we have

$$\gamma^* = \frac{m(1+\lambda)}{1+\lambda+\frac{1}{\lambda}} > 0$$

according to Theorem 1 by taking $d \to \infty$. Therefore, $d^* = \infty$ is the unique MFG equilibrium.

Now if $d^* < \infty$ is a MFG equilibrium, it satisfies the following two equations

$$d^* = \frac{(1-\gamma^*)(1+\lambda)}{2c - (1-\gamma^*)^2(1+\lambda)\left(1+\lambda+\frac{1}{\lambda}\right)} \tag{20}$$

$$\gamma^* = \frac{m d^*(1-\gamma^*)(1+\lambda)}{1+d^*(1-\gamma^*)(1+\lambda+\frac{1}{\lambda})}.$$

Substituting the first equation into the second one, we obtain

$$\begin{aligned}\gamma^* &= \frac{m(1-\gamma^*)(1+\lambda)\frac{(1-\gamma^*)(1+\lambda)}{2c-(1-\gamma^*)^2(1+\lambda)\left(1+\lambda+\frac{1}{\lambda}\right)}}{1+(1-\gamma^*)(1+\lambda+\frac{1}{\lambda})\frac{(1-\gamma^*)(1+\lambda)}{2c-(1-\gamma^*)^2(1+\lambda)\left(1+\lambda+\frac{1}{\lambda}\right)}} \\ &= \frac{m(1-\gamma^*)(1+\lambda)(1-\gamma^*)(1+\lambda)}{2c-(1-\gamma^*)^2(1+\lambda)\left(1+\lambda+\frac{1}{\lambda}\right)+(1-\gamma^*)^2(1+\lambda+\frac{1}{\lambda})(1+\lambda)} \\ &= \frac{m(1+\lambda)^2}{2c}(1-\gamma^*)^2.\end{aligned}$$

Note that $\gamma^* = \frac{m(1+\lambda)^2}{2c}(1-\gamma^*)^2$ has a unique solution $\gamma^* \in (0,1)$ since γ^* is an increasing function (increasing from 0 to 1) and $(1-\gamma^*)^2$ is a decreasing function (decreasing from 1 to 0). In particular, the unique solution is

$$\gamma^* = 1 + \frac{c}{m(1+\lambda)^2} - \sqrt{\frac{c^2}{m^2(1+\lambda)^4} + \frac{2c}{m(1+\lambda)^2}}. \tag{21}$$

Now to guarantee $d^* < \infty$, it requires

$$2c > (1-\gamma^*)^2(1+\lambda)\left(1+\lambda+\frac{1}{\lambda}\right)$$

according to (43), which concludes Case II.

Finally we consider Case III. When condition

$$2c > (1-\gamma^*)^2(1+\lambda)\left(1+\lambda+\frac{1}{\lambda}\right)$$

does not hold, after learning γ^* defined in (44), all devices choose strategy $d = \infty$. However, when

$$2c > \left(\max\left\{0, 1 - \frac{m(1+\lambda)}{1+\lambda+\frac{1}{\lambda}}\right\}\right)^2 (1+\lambda)\left(1+\lambda+\frac{1}{\lambda}\right), \tag{22}$$

$d = \infty$ is not an MFG equilibrium because

$$\tilde{\gamma} = T_1(\infty) = \min\left\{1, \frac{m(1+\lambda)}{1+\lambda+\frac{1}{\lambda}}\right\}$$

but

$$\tilde{d} = T_2(\tilde{\gamma}) < \infty$$

when

$$2c > ab = (1-\tilde{\gamma})^2(1+\lambda)\left(1+\lambda+\frac{1}{\lambda}\right).$$

Therefore, after all devices choosing $d = \infty$, the fraction of busy channels is $\tilde{\gamma}$ in the mean-field limit. After learning the fraction of busy channels is $\tilde{\gamma}$, all devices change their policy to $d = \tilde{d}$. It can be verified that $T_1(\tilde{d}) \leq \gamma^*$, so under policy $\tilde{d}$, the fraction of busy channels in the mean-field limit is at most γ^*. Then after learning the fraction of busy channels, all devices switch to policy $d = \infty$. Therefore no MFG equilibrium exists in this case. The system switches between $d = \infty$ and $d = \tilde{d}$. □

The theorem above presents the conditions under which an MFNE exits. Next, we study the convergence (i.e, stability) of the MFNE. For Case I, the convergence is immediate as indicated in the proof of Theorem 2, where we can see that all devices choose strategy $d^* = \infty$ after learning the fraction of busy channels and reach the MFNE. We now focus on Case II under which d^* is a finite value and have the following global convergence result. Since no MFNE exits in Case III, the question of convergence is irrelevant.

Theorem 3. *Consider Case II in Theorem 2. For any $c > c_{m,\lambda}$ where $c_{m,\lambda}$ is a positive constant such that*

$$m\frac{(1+\lambda)}{(1+\lambda+1/\lambda)}\frac{\frac{2c_{m,\lambda}}{(1+\lambda)\left(1+\lambda+\frac{1}{\lambda}\right)}+1}{\left(\frac{2c_{m,\lambda}}{(1+\lambda)\left(1+\lambda+\frac{1}{\lambda}\right)}-1\right)^2} = 1,$$

the system converges to the MFNE starting from any initial condition.

We remark that convergence to the mean-field limit (Theorem 1) and convergence to the MFNE (Theorem 3) are two fundamentally different concepts. Convergence to the mean-field limit shows that the stationary distributions of finite size systems converge weakly to the fixed point of the mean-field model for *fixed* (d, k), so no "game" is involved but the result does justify the MFG approach. On contrast, convergence to the MFNE does not involve finite-size stochastic systems, but investigates the dynamics of the MFG. The result shows that the iterative process, defined as the MFG for distributed MAC in Section 3, converges to the unique MFNE.

Proof. Recall mappings T_1 and T_2. Given policy (d, d), the stationary distribution of the mN-device system converges to a unique mean-field limit, which defines the following mapping

$$T_1 : d \to \gamma. \tag{23}$$

The mapping

$$T_2 : \gamma \to d \tag{24}$$

is obtained by solving the optimization problem $\min_d J(d)$ for given γ.

We begin by showing that, for fixed m, T_1 always has Lipschitz constant which is upper bounded by $m(1+\lambda)$. Based on (9), we first obtain

$$\frac{\partial \gamma}{\partial d} = -\frac{m(1+\lambda)k}{\left(1+d(1-\gamma)(1+\lambda+\frac{1}{\lambda})\right)^2}\frac{\partial \gamma}{\partial d} + \frac{m(1+\lambda)(1-\gamma)}{\left(1+k(1-\gamma)(1+\lambda+\frac{1}{\lambda})\right)^2}$$

which implies that

$$\begin{aligned}\left|\frac{\partial \gamma}{\partial d}\right| &= \frac{m(1+\lambda)(1-\gamma)}{m(1+\lambda)d + \left(1+d(1-\gamma)(1+\lambda+1/\lambda)\right)^2}\\ &< m(1+\lambda)(1-\gamma)\\ &< m(1+\lambda).\end{aligned}$$

Recall that T_2 is a map from γ to d which gives us the unique minimizer for the cost function $J(d)$, and that we consider Case II such that

$$2c > (1-\gamma)^2(1+\lambda)\left(1+\lambda+\frac{1}{1+\lambda}\right),$$

and

$$d = \frac{(1-\gamma)(1+\lambda)}{2c-(1-\gamma)^2(1+\lambda)(1+\lambda+1/\lambda)}.$$

Define $\alpha = \frac{2c}{(1+\lambda)(1+\lambda+\frac{1}{\lambda})}$, we further obtain

$$k = \frac{1}{(1+\lambda+1/\lambda)}\frac{1-\gamma}{\alpha-(1-\gamma)^2},$$

from which, we have

$$\begin{aligned}\left|\frac{\partial d}{\partial \gamma}\right| &= \frac{1}{1+\lambda+1/\lambda}\frac{\alpha+(1-\gamma)^2}{(\alpha-(1-\gamma)^2)^2}\\ &< \frac{1}{1+\lambda+1/\lambda}\frac{\alpha+1}{(\alpha-1)^2}.\end{aligned}$$

Define $T(d) = T_2(T_1(d))$. From the discussion above, we have

$$\frac{\partial T}{\partial d} = \left|\frac{\partial d}{\partial \gamma}\right|\left|\frac{\partial \gamma}{\partial d}\right| \le m\frac{(1+\lambda)}{(1+\lambda+1/\lambda)}\frac{\alpha+1}{(\alpha-1)^2}.$$

Note

$$\frac{\alpha+1}{(\alpha-1)^2}$$

is a decreasing function of α for $\alpha > 1$ because

$$\frac{d}{d\alpha}\left(\frac{\alpha+1}{(\alpha-1)^2}\right) = -\frac{\alpha+3}{(\alpha-1)^3} < 0,$$

so is a decreasing function of c according to the definition of α. Furthermore,

$$\lim_{\alpha\to\infty}\frac{\alpha+1}{(\alpha-1)^2} = 0.$$

Therefore, given m and λ, there exists $c_{m,\lambda} > 0$ such that

$$m\frac{(1+\lambda)}{(1+\lambda+1/\lambda)}\frac{\frac{2c_{m,\lambda}}{(1+\lambda)(1+\lambda+\frac{1}{\lambda})}+1}{\left(\frac{2c_{m,\lambda}}{(1+\lambda)(1+\lambda+\frac{1}{\lambda})}-1\right)^2} = 1.$$

For any $c > c_{m,\lambda}$, we have a contraction mapping and the system converges to the MFG equilibrium. □

6 PRICE OF ANARCHY

In this section, we analyse the performance of the distributed MAC with respect to a global optimal solution where a centralized controller chooses the optimal k for minimizing

$$\hat{J}(d) = -\frac{(1+\lambda)d(1-\gamma)}{1+d(1-\gamma)(1+\lambda+\frac{1}{\lambda})} + c\left(\frac{d}{1+d(1-\gamma)(1+\lambda+\frac{1}{\lambda})}\right)^2, \tag{25}$$

where

$$\gamma = \frac{m(1+\lambda)d(1-\gamma)}{1+d(1-\gamma)(1+\lambda+\frac{1}{\lambda})}. \tag{26}$$

Denote by $\hat{d}$ the optimal solution. All devices are forced to use probing rate $\hat{d}$. We will call the cost corresponding to this probing rate the global optimal cost and compare it with the cost at the MFNE.

Recall that for the MFNE, each device minimizes it cost function by assuming that γ is fixed. For the centralized case, the controller solves (25) by considering γ to be a function of d as defined in (26). This is the reason the global optimal solution differs from the cost at the MFNE. Let $\hat{\gamma}$ denote the fraction of busy channels that occurs as a result of the central controller picking an optimal sampling rate. Define

$$1 - \frac{|J(\gamma^*)|}{|\hat{J}(\hat{\gamma})|}$$

to be the price of anarchy. The following theorem shows that the price of anarchy is at most 0.5. Note that the cost at the MFNE and the global optimal cost are both negative because the policy that does not probe any channel and does not transmit any message has cost zero. Therefore, lower the cost, the larger its absolute value.

Theorem 4. *The price of anarchy,* $1 - |J(\gamma^*)|/|\hat{J}(\hat{\gamma})|$, *is at most* $1/2$. *In Case I, the low traffic regime defined in Theorem 2, the price of anarchy is zero.*

We note that in the low traffic regime (Case I in Theorem 2), both the distribution MAC and the centralized solution use probing strategy with $k^* = \infty$, so the price of anarchy is zero. We provide a proof for Case II defined in Theorem 2.

Proof. By substituting (26) into (25), we obtain

$$\hat{J}(\gamma) = -\frac{\gamma}{m} + c\left(\frac{\gamma}{m(1+\lambda)(1-\gamma)}\right)^2$$

The optimal solution to minimize $\hat{J}(\gamma)$ can be obtained by setting $\frac{\partial \hat{J}}{\partial \gamma}$ to be zero, which yields that the minimizer $\hat{\gamma}$ is the unique solution to the following equation

$$\hat{\gamma} = \frac{m(1+\lambda)^2(1-\hat{\gamma})^3}{2c}.$$

By simple substitution, we further obtain

$$\hat{J}(\hat{\gamma}) = -\frac{(1+\lambda)^2}{4c}(1-\hat{\gamma})^3(1+\hat{\gamma}) \tag{27}$$

It can be shown (and indeed we show this in the technical report) that γ^* is the unique solution of the following equation

$$\gamma^* = \frac{m(1+\lambda)^2(1-\gamma^*)^2}{2c}.$$

By substituting it into (25), we have

$$J(\gamma^*) = -\frac{(1+\lambda)^2}{4c}(1-\gamma^*)^2. \tag{28}$$

The ratio of the cost function at MFNE to the optimal cost function is given by:

$$\frac{|J(\gamma^*)|}{|\hat{J}(\hat{\gamma})|} = \frac{1}{(1+\hat{\gamma})}\frac{(1-\gamma^*)^2}{(1-\hat{\gamma})^3} = \frac{1}{(1+\hat{\gamma})}\frac{\gamma^*}{\hat{\gamma}}, \tag{29}$$

where the last equality holds because

$$\frac{\gamma^*}{\hat{\gamma}} = \frac{\frac{m(1+\lambda)^2(1-\gamma^*)^2}{2c}}{\frac{m(1+\lambda)^2(1-\hat{\gamma})^3}{2c}} = \frac{(1-\gamma^*)^2}{(1-\hat{\gamma})^3}.$$

Observe that $\hat{\gamma}$ is strictly smaller than γ^* because otherwise

$$\frac{\gamma^*}{\hat{\gamma}} < \frac{(1-\gamma^*)^2}{(1-\hat{\gamma})^3}.$$

Therefore, we conclude that

$$1 > \frac{|J(\gamma^*)|}{|\hat{J}(\hat{\gamma})|} > \frac{1}{(1+\hat{\gamma})} > \frac{1}{2}.$$

Which implies that:

$$0 < \text{Price of Anarchy} < \frac{1}{2}$$

In other words, the price of anarchy is upper bounded by 0.5. □

Focusing on Case II defined in Theorem 2, Figure 3 shows the price of anarchy with $c = 0.1$ and $m = 5$ with λ varying from 0.5 to 2. We can see that the price of anarchy increases as λ increases and approaches 0.5.

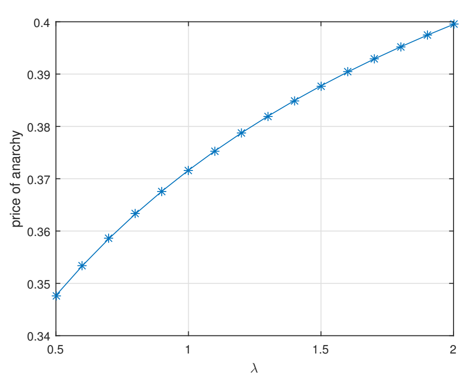

Figure 3: Price of Anarchy versus λ

7 SIMULATIONS

In this section, we use simulations to compare the distributed MAC policy, named DMAC-G for short with other similar light-weight distributed protocols. We simulated $N = 1,000$ devices with $m = 5$, and $c = 10$, and the average λ varying from 0.5 to 1. These choices of parameters guarantee the existence and convergence to the MFNE. We used uniformization to simulate the CTMC described in our system model in Section 2.

DMAC -G : We simulated two different scenarios for the DMAC -G protocol:homogeneous case where all devices have the same arrival rate and the same parameter, c and heterogeneous case where devices have different arrival rates and different values of parameter c. Since we ran the simulations on a laptop without parallelization, to speed up the simulations, the fraction of busy channels was measured as a common variable shared by all devices. In this way, we were able to simulate an M-device system efficiently using uniformization.

- **The homogeneous case** In the homogeneous case every device has the same arrival rate λ and energy parameter c. Hence, each device has the same utility function and so will choose the same sampling rate when given the common random variable for the fraction of busy channels.
- **The heterogeneous case** Each device follows the policy (d, d), however, the devices have different arrival rates and parameters c. The arrival rates were picked uniformly at random from $[0.75\lambda, 1.25\lambda]$. Similarly the values of the parameter c were chosen uniformly at random from $[0.75c, 1.25c]$ for some c.

Therefore, both cases have the same average arrival rates and cost parameters. Figures 4, 5 and 6 show that both scenarios yield very similar cost, fraction of busy channels and delay. We compare our algorithm under these two scenarios with a CSMA protocol with exponential back off.

E-CSMA : Each device maintains an exponential clock with initial rate $k = 1$. When the clock ticks, the device probes one of the N channels, chosen uniformly at random. If the probed channel is idle, the device starts to transmit the packet, if not the device halves its sampling rate and the clock restarts. We simulated this protocol under both homogeneous and heterogeneous scenarios.

We evaluated the performance of the protocols in terms of the cost and per-packet delay(for those successfully transmitted packets). We can observe from Figure 4 that DMAC-G yield a lower cost than E-CSMA and the gap increases as λ increases. Note, that the cost function is a linear combination of the probing cost minus the throughput. From Figure 6, we can also observe that our algorithm has much lower per-packet delay. The average delay is less than 2 for all the λ under DMAC-G, which reduces the probing rate when the traffic load increases, which reduces overall cost and per-packet delay(increases the freshness of the information).

These simulations confirm: (i) the analytical results in this paper, while derived for the homogeneous case, also match the performances of the heterogeneous case reasonably well; and (ii) our low-complexity, adaptive MAC protocol signficantly outperforms the exponential back-off MAC protocol (a commonly used MAC protocol).

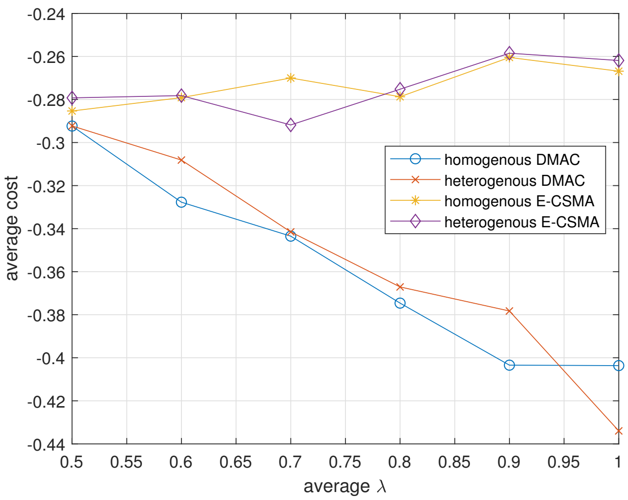

Figure 4: Average Costs under the Four Different Scenarios.

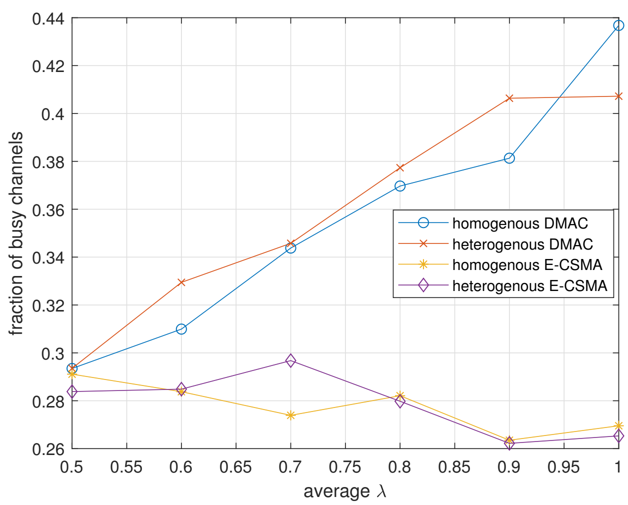

Figure 5: Average Fraction of Busy Channels under the Four Different Scenarios.

8 CONCLUSION

This paper formulated a multichannel ultra-dense wireless network with distributed MAC as a mean-field game, and provided a comprehensive analysis of the system including the existence and uniqueness of the MFNE, convergence to the MFNE and the price of anarchy compared with a global optimal solution. Numerical evaluations confirmed our theoretical results.

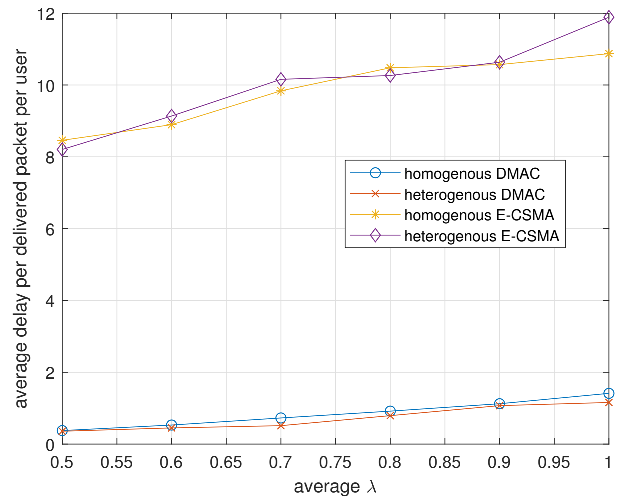

Figure 6: Average delay per delivered packet per user under the Four Different Scenarios.

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A PROOF OF THEOREM 1

To understand the mean-field model (4), consider $Q_1(t)$ and a sufficiently small time interval δ. According to a standard argument of continuous-time Markov chain, we have

$$\begin{aligned} & E[Q_1(t+\delta)-Q_1(t)|\mathbf{Q}(t)=\mathbf{Q}] \\ = \quad & \lambda Q_0\delta - k\left(1-\left(\frac{Q_2}{N}\right)^{d/k}\right)Q_1\delta + O(\delta^2), \end{aligned} \tag{30}$$

where $\lambda Q_0\delta$ is the probability that during $[t, t+\delta]$, one of the devices moves from idle to probing, and $k\left(1-\left(\frac{Q_2}{N}\right)^{d/k}\right)Q_1\delta$ is the probability that during $[t, t+\delta]$, one of the devices moves from probing to transmitting. Now dividing $M\delta$ on both sides, we obtain

$$\begin{aligned} & \frac{E[Q_1(t+\delta)-Q_1(t)|\mathbf{Q}(t)=\mathbf{Q}]}{M\delta} \\ = \quad & \lambda\frac{Q_0}{M} - k\left(1-\left(\frac{mQ_2}{M}\right)^{d/k}\right)\frac{Q_1}{M} + O(\frac{\delta}{M}), \end{aligned} \tag{31}$$

Now defining $q_i = \frac{Q_i}{M}$ and

$$\dot{q}_1 = \lim_{\delta\to 0}\frac{E[Q_1(t+\delta)-Q_1(t)|\mathbf{Q}(t)=\mathbf{Q}]}{M\delta},$$

we have

$$\dot{q}_1 = \lambda q_0 - k(1-(mq_2)^{d/k})q_1, \tag{32}$$

i.e. the mean-field model for q_1. The rest of the mean-field model can be similarly obtained. We can see that the mean-field model approximates the original stochastic system by using the expected drift (31) as the system dynamic.

In the following lemma, we first show that mean-field model (4) has a unique equilibrium.

Lemma 3. *Given $d > 0$ and $k > 0$, mean field model (4) has a unique equilibrium.*

Proof. The equilibrium point of mean field model (1) satisfies the following fix point equations:

$$\begin{aligned} q_0^* &= \frac{k(1-(mq_2^*)^{d/k})}{\lambda(1+k(1-(mq_2^*)^{d/k})(1+\lambda+\frac{1}{\lambda}))} \\ q_1^* &= \frac{1}{(1+k(1-(mq_2^*)^{d/k})(1+\lambda+\frac{1}{\lambda}))} \quad . \\ q_2^* &= \frac{(1+\lambda)k(1-(mq_2^*)^{d/k})}{(1+k(1-(mq_2^*)^{d/k})(1+\lambda+\frac{1}{\lambda}))} \end{aligned}$$

Note that q_0^* and q_1^* are uniquely determined by q_2^*. Therefore, we next show that q_2^* has a unique solution. Recall that $\gamma^* = mq_2^*$. Substituting it into the third equation above, we have

$$\gamma^* = m\frac{(1+\lambda)k(1-(\gamma^*)^{d/k})}{(1+k(1-(\gamma^*)^{d/k})(1+\lambda+\frac{1}{\lambda}))}. \tag{33}$$

Define function $\theta(\cdot)$ such that

$$\theta(\gamma^*) = \frac{m(1+\lambda)k(1-(\gamma^*)^{\frac{d}{k}})}{1+k(1-(\gamma^*)^{\frac{d}{k}})((1+\lambda)+\frac{1}{\lambda})}$$

Notice that $\theta(\gamma^*)$ is monotonically decreasing function in γ^* because

$$\begin{aligned} \frac{d\theta(\gamma^*)}{d\gamma^*} &= -m(1+\lambda)d(\gamma^*)^{d/k-1}\left(\frac{1}{\left(1+k(1-(\gamma^*)^{d/k})((1+\lambda)+\frac{1}{\lambda})\right)^2}\right) \\ &< 0. \end{aligned}$$

Further note that

$$\theta(0) = \frac{m(1+\lambda)k}{\left(1+k((1+\lambda)+\frac{1}{\lambda})\right)} > 0$$

and

$$\theta(1) = 0.$$

Since γ^* is strictly increasing in γ^* and $\theta(\gamma^*)$ is strictly decreasing in γ^*, we conclude that $\gamma^* = \theta(\gamma^*)$ has a unique solution, which concludes the lemma. □

The proof is an application of the main result in [17]. Recall each device uses policy (d, d). The mean field model under this policy is similar to (4) and is the following nonlinear system:

$$\begin{aligned} \frac{dq_0}{dt} &= -\lambda q_0 + \frac{1}{1+\lambda}q_2 \\ \frac{dq_1}{dt} &= \lambda q_0 - d(1-mq_2)q_1 \\ \frac{dq_2}{dt} &= d(1-mq_2)q_1 - \frac{1}{1+\lambda}q_2 \end{aligned} \tag{34}$$

Let (q_0^*, q_1^*, q_2^*) denote the unique equilibrium point of this dynamical system. The uniqueness of the equilibrium point is due to Lemma 3. Define $\epsilon_i(t)$ to be

$$\epsilon_i(t) = q_i(t) - q_i^*.$$

Then the dynamical system (34) can be equivalently represented by :

$$\begin{aligned} \frac{d\epsilon_0}{dt} &= -\lambda\epsilon_0 + \frac{1}{1+\lambda}\epsilon_2 \\ \frac{d\epsilon_1}{dt} &= \lambda\epsilon_0 - d(1-mq_2^*)\epsilon_1 + mdq_1\epsilon_2 \\ \frac{d\epsilon_2}{dt} &= d(1-mq_2^*)\epsilon_1 - mkq_1\epsilon_2 - \frac{1}{1+\lambda}\epsilon_2 \end{aligned} \tag{35}$$

It is clear from the definition that $\sum_{i\in 0,1,2}\epsilon_i = 0$ for any time t.

Lemma 4. *The dynamical system described by (35) is asymptotically stable for any valid ϵ_i and locally exponentially stable near the origin.*

Proof. We prove the first part of the lemma using the Lyapunov theorem [12]. Define Lyapunov function $V(\epsilon)$ such that

$$V(\epsilon) = |\epsilon_0| + |\epsilon_1| + |\epsilon_2|. \tag{36}$$

Note that $\sum_i \epsilon_i(t) = 0$ for all t, so at least one of the ϵ_i is negative and one is positive when $\epsilon \neq 0$.

We first analyze the cases where only one ϵ_i is strictly negative, which includes the following three cases.

Case I: $\epsilon_0 < 0$, $\epsilon_2 \geq 0$ and $\epsilon_1 \geq 0$. In this case, we have

$$V(\epsilon) = -\epsilon_0 + \epsilon_1 + \epsilon_2.$$

Therefore,

$$\begin{aligned}\frac{dV}{dt} &= -\frac{d\epsilon_0}{dt} + \frac{d\epsilon_1}{dt} + \frac{d\epsilon_2}{dt}\\ &= \lambda\epsilon_0 - \frac{1}{1+\lambda}\epsilon_2 + \lambda\epsilon_0 - k(1-mq_2^*)\epsilon_1\\ &\quad + md\epsilon_2 q_1 + d(1-mq_2^*)\epsilon_1 - md\epsilon_2 q_1 - \frac{1}{1+\lambda}\epsilon_2\\ &= -2\frac{d\epsilon_0}{dt}\\ &= 2\lambda\epsilon_0 - 2\frac{1}{1+\lambda}\epsilon_2 < 0.\end{aligned}$$

Case II: $\epsilon_1 < 0$, $\epsilon_0 > 0$, and $\epsilon_2 \geq 0$. In this case, we have

$$V(\epsilon) = \epsilon_0 - \epsilon_1 + \epsilon_2.$$

Therefore,

$$\begin{aligned}\frac{dV}{dt} &= \frac{d\epsilon_0}{dt} - \frac{d\epsilon_1}{dt} + \frac{d\epsilon_2}{dt} = -2\frac{d\epsilon_1}{dt}\\ &= -2\lambda\epsilon_0 + 2d(1-mq_2^*)\epsilon_1 - 2mdq_1\epsilon_2 < 0.\end{aligned}$$

Case III: $\epsilon_2 < 0$, $\epsilon_0 > 0$, and $\epsilon_1 \geq 0$. In this case, we have

$$V(\epsilon) = \epsilon_0 + \epsilon_1 - \epsilon_2.$$

Therefore,

$$\begin{aligned}\frac{dV}{dt} &= \frac{d\epsilon_0}{dt} + \frac{d\epsilon_1}{dt} - \frac{d\epsilon_2}{dt} = -2\frac{d\epsilon_2}{dt}\\ &= -2d(1-mq_2^*)\epsilon_1 + \frac{1}{1+\lambda}(2mdq_1 + 2)\epsilon_2 < 0.\end{aligned}$$

For the cases where one ϵ_i is strictly positive, we can similarly show $\frac{dV}{dt} < 0$. For example, when $\epsilon_0 > 0$, $\epsilon_2 \leq 0$ and $\epsilon_1 \leq 0$, following a similar analysis to Case I, we have

$$\frac{dV}{dt} = 2\frac{d\epsilon_0}{dt} = -2\lambda\epsilon_0 + 2\frac{1}{1+\lambda}\epsilon_2 < 0.$$

Therefore, based on the Lyapunov theorem, we conclude that the system is asymptotically stable.

To prove that the system is locally exponentially stable, we need to show that the linearized system matrix around its equilibrium is negative definite, i.e, has strictly negative eigenvalues. The linearized dynamical system is given by:

$$\begin{aligned}\frac{d\epsilon_0}{dt} &= -\lambda\epsilon_0 + \frac{1}{1+\lambda}\epsilon_2\\ \frac{d\epsilon_2}{dt} &= \left(-d(1-mq_2^*) - mdq_1^* - \frac{1}{1+\lambda}\right)\epsilon_2 - d(1-mq_2^*)\epsilon_0\end{aligned}, \tag{37}$$

where we used the fact $\epsilon_1 = -\epsilon_0 - \epsilon_2$ and eliminated one of the equations from the dynamical system.

The matrix corresponding to the linearized form can be written as:

$$A = \begin{bmatrix} -\lambda & \frac{1}{1+\lambda} \\ -d(1-mq_2^*) & -d(1-mq_2^*) - mdq_1^* - \frac{1}{1+\lambda} \end{bmatrix}$$

Let η be an eigenvalue of A, Then η must satisfy

$$(-\lambda-\eta)\left(-d(1-mq_2^*) - mdq_1^* - \frac{1}{1+\lambda} - \eta\right) + \frac{d}{1+\lambda}(1-mq_2^*) = 0.$$

If $\eta \geq 0$, then the first term is strictly positive and

$$(-\lambda-\eta)\left(-d(1-mq_2^*) - mdq_1^* - \frac{1}{1+\lambda} - \eta\right) + \frac{d}{1+\lambda}(1-mq_2^*) > 0.$$

Therefore, the eigenvalues of A are strictly negative, and the dynamical system is locally exponentially stable. □

The theorem holds by invoking Theorem 1 in [17].