JOURNAL OF ALGORITHMS 6, 104–111 (1985)

# A "Retraction" Method for Planning the Motion of a Disc

COLM Ó'DÚNLAING AND CHEE K. YAP*

*Computer Science Department, Courant Institute of Mathematical Sciences, New York University, New York, New York 10012*

Received November 2, 1982; revised October 14, 1983

A new approach to certain motion-planning problems in robotics is introduced. This approach is based on the use of a generalized Voronoi diagram, and reduces the search for a collision-free continuous motion to a search for a connected path along the edges of such a diagram. This approach yields an $O(n \log n)$ algorithm for planning an obstacle-avoiding motion of a single circular disc amid polygonal obstacles. Later papers will show that extensions of the approach can solve other motion-planning problems, including those of moving a straight line segment or several coordinated discs in the plane amid polygonal obstacles. © 1985 Academic Press, Inc.

## 0. INTRODUCTION

The piano movers' problem has been the subject of a number of recent papers [1, 4, 9, 11]. The 2-dimensional version of the problem which will concern us here is (i) to determine whether there exists a continuous collision-free motion, between two specified placements, of a given rigid body $B$ free to move within a 2-dimensional region $\Omega$ (a "room") bounded by a set $S$ of finitely many polygonal obstacles, and (ii) to produce such a motion when one exists. The method described in the papers cited above is roughly as follows: for a given set $S$ of obstacles and a body $B$, the set $FP$ of all pairs [position, orientation] which represent placements of $B$, in which $B$ does not touch any obstacle in $S$, is an open manifold, and a continuous obstacle-avoiding path between two given placements of $B$ exists if and only if these placements belong to the same connected component of $FP$. Thus

*This author was supported in part by the ONR under Contract N00014-82-K-0381.



0196-6774/85 $3.00

the problem is reduced to that of finding the connected components of $FP$, which is solved by partitioning $FP$ into a collection of smaller cells having relatively simple structure, and by determining the adjacency relationships between these cells. These relationships yield a finite "connectivity graph," which can then be searched to determine the connected components of $FP$. This approach can be applied in general situations, and has been shown in [12] to yield polynomial-time algorithms for planning the motion of various bodies or groups of bodies $B$.

In this paper we suggest a different approach ("retraction") to the movers' problem and apply it to the simple case in which the body $B$ is a circular disc. A motion-planning algorithm of complexity $O(n \log n)$ is shown for this case. As suggested in [4] and used in [5], the idea of solving the motion-planning problem for a disc circumscribing a 2-dimensional body of more complex shape can be used as an initial heuristic in motion-planning for the body. Only if no continuous motion exists for this disc should more refined algorithms (taking more detail of the body shape into account) be brought to bear. This "retraction" approach is based on the observation that a continuous motion can be "retracted" onto a suitable subspace of the set of free positions: namely, the Voronoi diagram of the room $\Omega$. The use of the Voronoi diagram for motion-planning has been independently proposed in an earlier work [10], but the treatment in [10] is from the perspective of artificial intelligence, and uses digitized images.

The approach described in this note will be extended in [6–8] to give an efficient solution to the problem of moving a line segment (a "ladder") within the room $\Omega$. This requires a suitable definition of the Voronoi diagram; the mathematical and computational details are far more complex than those of this paper. The "retraction" approach is also used in [13] to give efficient algorithms for coordinating the motion of several discs. This last paper demonstrates that retraction algorithms need not always involve generalizations of Voronoi diagrams.

Throughout this paper we assume that the boundary $\partial\Omega$ of the "room" $\Omega$ is a finite union of polygons, specified as a set $S$ of points ("corners") and open line segments ("walls"). The size of the problem will be measured by the number $n$ of objects in $S$.

The basic idea is to construct the *Voronoi diagram* $\mathrm{Vor}(\Omega)$ (see [2]). This diagram is a planar network of straight and parabolic arcs, and is characterized as the set of points in $\Omega$ which are equidistant from at least two distinct objects in $S$; it can be computed in time $O(n \log n)$ by an algorithm due to Kirkpatrick [2] (a simpler $O(n \log^2 n)$ algorithm due to Lee and Drysdale [3] is also available). Let $B$ be a circular disc of radius $r$. The motion-planning algorithm is based on the fact (proved below) that there exists a continuous obstacle-avoiding motion $\pi$ of the disc $B$ between two specified positions $x_0$ and $x_1$ of its center $C$ if and only if there exists another continuous obstacle-avoiding motion $\pi'$ of $B$ from $x_0$ to $x_1$ during which, except for its initial and final phases (in which $C$ is moved to and from the Voronoi diagram), the center $C$ moves entirely along the Voronoi diagram Vor($\Omega$). This enables us to reduce the problem to searching the Voronoi diagram.

## 1. Motion-planning for a Disc

In the ensuing sections, $\Omega$ will be an open polygonal region, and $S$ will be the set of $n$ corners and walls forming the boundary $\partial\Omega$ of this region; $B$ will be a disc of radius $r$ free to move in $\Omega$. "Distance" will always mean Euclidean distance in the plane. An instance of the movers' problem will be characterized by specifying $S$, the radius $r$ of $B$, and two points $x_0$ and $x_1$ in $\Omega$. (Each wall in $S$ is an oriented line segment with the convention that the immediate interior of $\Omega$ lies to its left.) Given $S$, $r$, $x_0$, and $x_1$, the movers' problem is to move the center $C$ of the disc $B$ from $x_0$ to $x_1$ in a continuous motion in which $B$ remains within $\Omega$ (and thus never touches a wall or corner). For any point $x$ in the plane, let

$$\text{Clearance}(x)$$

denote the (Hausdorff) distance of $x$ from the complement $\Omega'$ of $\Omega$. One can show that Clearance($x$) is continuous in $x$. Note that $\Omega$ then consists of the set of points whose clearance is strictly positive. Any placement of the disc $B$ in the plane may be identified uniquely with the position of its center $C$. Under this identification, it is clear that the set $FP$ of *free placements* (in which the entire disc is within $\Omega$) corresponds to the set of points in the plane whose clearance exceeds $r$. For any point $x$ in the plane, define

$$\text{Near}(x)$$

as the set of points in $\partial\Omega$ closest to $x$. Since $\partial\Omega$ is closed, Near($x$) is always nonempty; if $x$ is in $\Omega$ then Near($x$) can be characterized as the set of points in which $\partial\Omega$ intersects the circle of radius Clearance($x$) around $x$. The *Voronoi diagram* Vor($\Omega$) of $\Omega$ is the set of points

$$\{x \text{ in } FP\text{: Near}(x) \text{ contains more than one point}\}.$$

One can define a map Im from $\Omega$ onto Vor($\Omega$) as follows: let $x$ be any point in $\Omega$. If $x$ is on Vor($\Omega$) then define Im($x$) = $x$; otherwise, Near($x$) = $\{p\}$ for some point $p$ in $\partial\Omega$. Let $L$ be the semi-infinite straight line from $p$ through $x$, and define Im($x$) to be the first point $y$ (if it exists), where $L$ intersects Vor($\Omega$). Intuitively, Im($x$) is obtained by "pushing" $x$ away from the closest wall (or corner) until it lies on the Voronoi diagram. If $\Omega$ is unbounded, then $\mathrm{Im}(x)$ may not be defined; $x$ could be "pushed" away to infinity. The following theorem will provide the theoretical basis for the motion-planning algorithm in this paper. (Recall that a *retraction* is a continuous map from a topological space $X$, onto a subspace $Y$ of $X$, which fixes every point in $Y$; $Y$ is called a *retract* of $X$.)

THEOREM 1 (The retraction theorem). *If $\Omega$ is bounded, then* (i) *the map* Im *is a continuous retraction of $\Omega$ onto* $\mathrm{Vor}(\Omega)$ (*so* $\mathrm{Vor}(\Omega)$ *is a retract of* $\Omega$), *and* (ii) *if* $\mathrm{Im}(x) \neq x$, *then the clearance is strictly increasing along the line-segment joining $x$ to* $\mathrm{Im}(x)$.

The proofs of Theorems 1 and 2 are given in the Appendix. Recall that *FP* may be identified with the set of points in $\Omega$ whose clearance exceeds the radius of the disc.

THEOREM 2. *If $\Omega$ is bounded, then given any two points $x_0$ and $x_1$ in FP, there exists a continuous path connecting $x_0$ to $x_1$ in FP if and only if there exists a continuous path from* $\mathrm{Im}(x_0)$ *to* $\mathrm{Im}(x_1)$ *in* $\mathrm{Vor}(\Omega) \cap FP$.

Theorem 2 fails if $\Omega$ is unbounded, since then Im is not everywhere defined. Figure 1 illustrates how Theorem 2 can fail: however, it is easy to adapt the algorithm to cope with unbounded $\Omega$. Accordingly, for the rest of the paper we shall assume that $\Omega$ is bounded.

Theorem 2 justifies the following motion-planning algorithm:

(1) (Preprocessing) Decompose the boundary $\partial\Omega$ into a set $S$ of $n$ corners and walls.

Construct the Voronoi diagram $\mathrm{Vor}(\Omega)$, using, for instance, Kirkpatrick's $O(n \log n)$ algorithm [2]. The diagram $\mathrm{Vor}(\Omega)$ is a union of $O(n)$ straight and parabolic arcs. Kirkpatrick's algorithm can specify it as a (combinatorial) graph $N$, in which each edge is labelled with its two endpoints and with an algebraic equation defining it as a curve in the plane, and each vertex is labelled with its coordinates. Finally, from the equation defining an edge $e$ and from the coordinates of its endpoints, one can compute its *width*—the minimum clearance along $e$; this information can also be linked to $e$. The overall cost of this extra processing is $O(n)$.

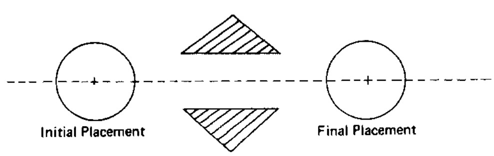


FIG. 1. A case where the disc must leave the Voronoi diagram.

(2) (Motion planning) Given two points $x_0$ and $x_1$ in the plane, it is required to move the center of the disc $B$ from $x_0$ to $x_1$ while not touching any obstacle. By considering in turn every object in $S$, one can compute the quantity Clearance($x_0$) and the set Near($x_0$). At this point, if Clearance($x_0$) $\leqslant r$, the initial placement is not free: stop with an error message. If not, one can then compute Im($x_0$), as follows: if Near($x_0$) is not a singleton then Im($x_0$) $= x_0$; else let $p$ be the unique element of Near($x_0$). By considering all edges in $N$, one can locate the point $y_0$, where the semi-infinite line $L$ from $p$ through $x_0$ first intersects Vor($\Omega$), and also determine the edge $e_0$ containing $y_0$. Then $y_0 =$ Im($x_0$). Similarly, one can determine the edge $e_1$ of $N$ containing $y_1 =$ Im($x_1$), and stop if $x_1$ is not free. All this can be done in time $O(n)$.

If $e_0 = e_1$, then one can determine explicitly whether the clearance exceeds $r$ between $y_0$ and $y_1$ along this edge. This takes constant time. If so, we are done; otherwise, it is still possible that a connecting path exist along some other edges. However, checking this possibility can be reduced to the case $e_0 \neq e_1$ by separating the edge at the point of minimal clearance. If $e_0 \neq e_1$, then one can determine which, if any, of the endpoints of $e_0$ can be reached from $y_0$ while maintaining sufficient clearance; similarly for $y_1$. Assuming that an endpoint $z_0$ of $e_0$ and $z_1$ of $e_1$ can be so reached, the motion-planning problem now reduces to finding a path from $z_0$ to $z_1$ in $N$ along edges of width greater than $r$. This can be done in time $O(n)$.

## 3. Final Remarks

It is instructive to compare the *Voronoi diagram* approach to that used in previous papers [4, 9, 11] which can be called the *critical region* approach. (We discuss only the case of a disc.)

When applied to a single moving disc, the critical region approach also gives rise to an $O(n \log n)$ algorithm. Indeed, this particularly simple path problem reduces to computing the "$r$-fringe" of the given set of obstacles $S$, where the $r$-fringe consists of those points in $\Omega$ which are at distance $r$ from the nearest obstacle. Since the $r$-fringe is the boundary of $FP$, we can clearly deduce the connected components of $FP$. (However, we do not know how to obtain the $r$-fringe in $O(n \log n)$ time without using an $O(n \log n)$ algorithm to compute the Voronoi diagram.)

In practical terms, the solution path obtained by the Voronoi diagram approach should be a particularly reliable way to avoid collisions, since it keeps the disc as far as possible from the obstacles. In fact, the algorithm can easily be modified to find a path of maximum clearance. Again, note that for both approaches the preprocessing time is $O(n \log n)$ and the path-finding time is $O(n)$. However, in the critical region approach the data structure established during preprocessing can only be used to plan motions for a disc of fixed size; a significant advantage of our approach is that the data structure established in the preprocessing phase can be used to solve motion-planning problems for discs of any size.

One drawback of the Voronoi diagram algorithm is that it will yield paths which may be arbitrarily longer than the shortest paths. For instance, if we wish to move the disc between two points which are close together and close to a wall in a large, bare, room, then the Voronoi diagram method will prescribe that the disc be first moved close to the center of the room. Since the shortest path will usually involve touching the walls at several points, one cannot maintain the requirement of high clearance, and the Voronoi diagram approach is in principle unsuitable for generating short paths. In practice, however, the Voronoi paths can be improved locally; it is a subject for further investigation.

## APPENDIX: Proofs of Theorems 1 and 2

LEMMA A. $\mathrm{Vor}(\Omega)$ *is closed with respect to* $\Omega$.

*Proof.* It is enough to show that its complement in $\Omega$ is open. Let $x$ be any point in $\Omega \setminus \mathrm{Vor}(\Omega)$. By assumption, $\mathrm{Near}(x) = \{p\}$ for some point $p$. Since $\partial\Omega$ is polygonal, it follows that we can choose a closed disc $D$ about $x$, of radius slightly greater than $\mathrm{Clearance}(x)$, such that the intersection $C$ of $D$ with $\partial\Omega$ has one of the following two forms: (i) a line segment containing $p$ (when $p$ is not a corner), or (ii) two line segments meeting at $p$ such that the angle they form with the line $xp$ is at least $\pi/2$. Since $\mathrm{Clearance}(\cdot)$ is continuous, there exists an open neighborhood $V$ of $x$ in $\Omega$ such that, for every $y$ in $V$, the closed disc $D_y$ of radius $\mathrm{Clearance}(y)$, about $y$, is contained in $D$; thus $D_y$ touches $\partial\Omega$ within the set $C$. It follows easily that $\mathrm{Near}(y)$ contains exactly one point, so $y$ does not lie in $\mathrm{Vor}(\Omega)$. Hence $V$ does not intersect $\mathrm{Vor}(\Omega)$ and $\Omega \setminus \mathrm{Vor}(\Omega)$ is open. Q.E.D.

LEMMA B. *Let $x$ be any point in $\Omega$ and suppose that $p \in \mathrm{Near}(x)$. Let $L$ be the semi-infinite line from $p$ through $x$. Then* (i) $\mathrm{Near}(y) = \{p\}$ *for every point $y$ strictly between $p$ and $x$ on $L$,* (ii) *if $x \in \mathrm{Vor}(\Omega)$ then $p \notin \mathrm{Near}(y)$ for any point $y$ beyond $x$ in $L$, and* (iii) *if* $\mathrm{Near}(x) = \{p\}$ *then there exists an open interval $I$ about $x$ in $L$ such that, for any $y$ in $I$,* $\mathrm{Near}(y) = \{p\}$.

*Proof.* For any point $y$ in $L$, let $D_y$ be the closed disc centered at $y$ and touching $p$. Given any two points $y$ and $y'$ in $L$, where $y'$ lies beyond $y$ (i.e., further from $p$), it is clear that $D_y$ is contained in $D_{y'}$ and any point $q \neq p$ on the circumference of $D_y$ is in the interior of $D_{y'}$.

Since $\mathrm{Near}(x)$ is contained in the circumference of $D_x$, and the interior of $D_x$ is contained in $\Omega$, it follows that for any $y$ between $p$ and $x$ the interior of $D_y$ is contained in $\Omega$ and $D_y$ intersects $\partial\Omega$ only at $p$; therefore $\text{Near}(y) = \{p\}$, as asserted in (i). Again, if $x$ is in $\text{Vor}(\Omega)$, then $D_x$ also touches $\partial\Omega$ at some point $q \neq p$; therefore, for any $y$ beyond $x$ in $L$, $q$ is interior to $D_y$ so $p \notin \text{Near}(y)$, proving (ii). Finally, to prove (iii), suppose that $\text{Near}(x) = \{p\}$. There exists a disc $D$ strictly containing $D_x$ such that $D$ intersects the complement of $\Omega$ in a set $C$ (containing $p$) as described in Lemma A. Let $I$ be an open interval around $x$ in $L$ such that for any $y$ in $I$, $D_y$ can only touch $\partial\Omega$ in $C$; clearly, $p$ is the point in $C$ closest to each $y$ in $I$, so $\text{Near}(y) = \{p\}$ as asserted. Q.E.D.

*Proof of Theorem* 1. Im *is defined everywhere in* $\Omega$. Let $x$ be any point in $\Omega$. Certainly, if $x$ is in $\text{Vor}(\Omega)$, then $\text{Im}(x)$ is defined. Otherwise, suppose that $\text{Near}(x) = \{p\}$, and let $L$ be the line from $p$ through $x$. If $\text{Near}(y) = \{p\}$ for every point $y$ on $L$, then $\text{Clearance}(y) = d(y, p)$ for every such point $y$, and $L$ would contain points of arbitrarily large clearance; this contradicts the assumption that $\Omega$ is bounded. Therefore the set

$$H = \{y \text{ in } L \colon \text{Near}(y) \neq \{p\}\}$$

is nonempty. Let $y_0$ be the point closest to $p$ in the closure of $H$. By Lemma B, $y_0$ lies beyond $x$ in $L$, and by continuity of the clearance function, $d(y_0, p) = \text{Clearance}(y_0)$ so $p \in \text{Near}(y_0)$ (and $y_0 \in \Omega$). By Lemma B (iii), $\text{Near}(y_0)$ contains some point $q \neq p$, so $y_0$ is in $\text{Vor}(\Omega)$.

*The clearance-increasing property of* Im. Suppose that $x$ is any point in $\Omega \setminus \text{Vor}(\Omega)$, so $\text{Near}(x) = \{p\}$ for some point $p$, $p \in \text{Near}(y)$, where $y = \text{Im}(x)$, and $p$, $x$, and $y$ are collinear. By Lemma B, $\text{Near}(y') = \{p\}$ for any point $y'$ between $x$ and $y$, so $\text{Clearance}(y') = d(y', p)$ is strictly increasing as $y'$ moves from $x$ to $y$ along the line joining them.

*Continuity of the map* Im. Let $x_n$ be a sequence converging to some point $x$ in $\Omega$, and suppose (by proceeding to a subsequence, if necessary) that $\text{Im}(x_n) = y_n$ converges to some point $y$ in the plane. Since $\text{Clearance}(y_n) \geqslant \text{Clearance}(x_n)$ for all $n$, $\text{Clearance}(y) \geqslant \text{Clearance}(x)$ so $y$ is in $\Omega$. Since each $y_n$ is in $\text{Vor}(\Omega)$ it follows from Lemma A that $y$ is in $\text{Vor}(\Omega)$.

If $x_n$ is in $\text{Vor}(\Omega)$ for infinitely many $n$, then by proceeding to a subsequence if necessary we can assume that all $x_n$ are in $\text{Vor}(\Omega)$, so $y_n = x_n$ for all $n$ and hence $y = x = \text{Im}(x)$. Therefore it remains to consider the case where $x_n \notin \text{Vor}(\Omega)$ for all but finitely many $n$; so by proceeding to a subsequence if necessary we can assume that for all $n$ $\text{Near}(x_n) = \{p_n\}$, where $p_n$ converges to some point $p$ in $\partial\Omega$. By continuity, $p \in \text{Near}(x)$, $p \in \text{Near}(y)$, and $y$ lies on the line $L$ from $p$ through $x$. Thus by Lemma B, $\text{Near}(y') = \{p\}$ for any point $y'$ between $p$ and $y$ on $L$, so $y = \text{Im}(x)$. Therefore Im is continuous. Q.E.D.

*Proof of Theorem* 2. Suppose that $x_0$ and $x_1$ are connected by a continuous path $\pi$ in $FP$. Such a path can be defined as a continuous map from $[0,1]$ into $FP$ such that $\pi(0) = x_0$ and $\pi(1) = x_1$. Then the composition $\mathrm{Im} \cdot \pi$ is a continuous path joining $\mathrm{Im}(x_0)$ to $\mathrm{Im}(x_1)$. Since Clearance $(\mathrm{Im}(\pi(t)) \geqslant$ Clearance $(\pi(t)) > r$ for every $t$ in $[0,1]$, this path joins $\mathrm{Im}(x_0)$ to $\mathrm{Im}(x_1)$ in $FP \cap \mathrm{Vor}(\Omega)$.

Conversely, if there is a continuous path from $\mathrm{Im}(x_0)$ to $\mathrm{Im}(x_1)$ in $FP \cap \mathrm{Vor}(\Omega)$, then Theorem 1(ii) shows that the path can be extended to a path from $x_0$ to $x_1$ in $FP$. Q.E.D.

## Acknowledgments

The encouragement and generous help which Jack Schwartz and Micha Sharir gave us are much appreciated. We are indebted to David Kirkpatrick for pointing out the work of P. Rowat [10].

## References

1. J. Hopcroft, D. Joseph, and S. Whitesides, "On the Movement of Robot Arms in 2-Dimensional Bounded Regions," Tech. Report TR 82-486, Comput. Sci. Dept., Cornell University, 1982; "Proc. 23rd IEEE Ann. Sympos. Found. Comput. Sci.," Chicago, 1982.
2. D. Kirkpatrick, Efficient computation of continuous skeletons, *in* "Proc. 20th IEEE Ann. Sympos. Found. Comput. Sci.," 1979, pp. 18–27.
3. D. T. Lee and S. Drysdale, Generalization of Voronoi diagrams in the plane, *SIAM J. Comput.* **10**, No. 1 (1981), 73–87.
4. T. Lozano-Perez and M. Wesley, An algorithm for planning collision-free paths among polyhedral obstacles, *Comm. ACM* **22** (1979), 560–570.
5. H. P. Moravec, "Robot Rover Visual Navigation," UMI Research, Ann Arbor, Mich., 1981.
6. C. Ó'Dúnlaing, M. Sharir, and C. Yap, "Generalized Voronoi Diagrams for Moving a Ladder I: Topological Considerations," technical report, Courant Institute of Mathematical Sciences, New York, 1983.
7. C. Ó'Dúnlaing, M. Sharir, and C. Yap, "Generalized Voronoi Diagrams for Moving a Ladder II: Efficient Construction of the Ladder," technical report, Courant Institute of Mathematical Sciences, New York, 1983.
8. C. Ó'Dúnlaing, M. Sharir, and C. Yap, Retraction: A New Approach to Motion-Planning, *in* "Proc. 15th ACM Sympos. Theory of Comput.," 1983, pp. 207–220.
9. J. Reif, Complexity of the mover's problem and generalizations, *in* "Proc. 20th IEEE Ann. Sympos. Found. Comput. Sci.," 1979, pp. 421–427.
10. P. F. Rowat, "Representing Spatial Experience and Solving Spatial Problems in a Simulated Robot Environment," Ph.D. thesis, University of British Columbia, 1979.
11. J. Schwartz and M. Sharir, On the piano-movers' problem: I. The case of a two-dimensional rigid body moving amidst polygonal barriers, *Comm. Pure Appl. Math.* **36** (1983), 345–398.
12. J. Schwartz and M. Sharir, On the piano-movers' problem: II. General techniques for computing topological properties of real algebraic manifolds, *Adv. Appl. Math.* **4** (1983), 298–351.
13. C. Yap, "Coordinating the Motion of Several Discs," technical report, Courant Institute of Mathematical Sciences, New York, 1983.