Inverse kinematics of the KUKA LBR iiwa R800 (7 DOF)

With the help of inverse kinematics, the required joint angles can be calculated for a target position. The *Closed Form Solution* and the numerical solution are available for calculation. Because of the iterative nature of the numerical solution and the resulting slow calculation, the Closed Form Solution is preferred. The Closed Form Solution is based on analytical expressions or on the solution of a polynomial of maximum degree four. Therefore, when designing robot arms, care is taken to ensure that as many angles α_i 0 or $\pm 90°$ as possible are created by axis alignment, so that a Closed Form Solution also exists. The basis for the calculation of the inverse kinematics are the DH parameters.[4]

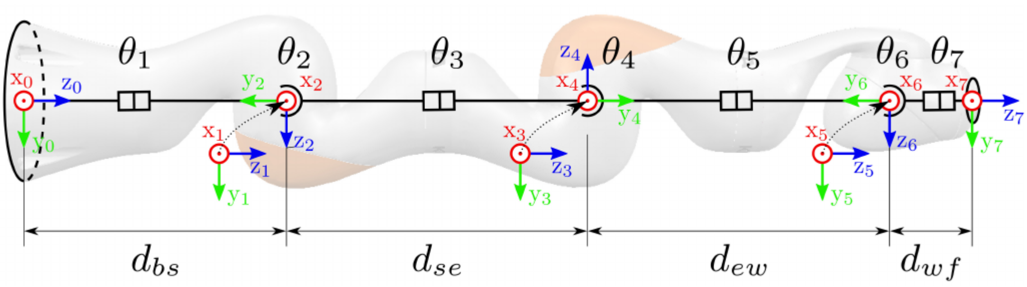

Figure 0.1: Illustration of the robot arm used (7-DOF) without gripper and the alignment of the coordinate systems as well as the dimensions. In my case the arm is vertically mounted.

i	α_i	d_i	a_i	θ_i
1	$-\frac{\pi}{2}$	d_{bs}	0	θ_1
2	$\frac{\pi}{2}$	0	0	θ_2
3	$\frac{\pi}{2}$	d_{se}	0	θ_3
4	$-\frac{\pi}{2}$	0	0	θ_4
5	$-\frac{\pi}{2}$	d_{ew}	0	θ_5
6	$\frac{\pi}{2}$	0	0	θ_6
7	0	d_{wf}	0	θ_7

with $d_{bs} = 340mm$, $d_{se} = 400mm$, $d_{ew} = 400mm$ and $d_{wf} = 126mm$

From the DH parameters and based on the homogeneous transformation matrix, which transforms the coordinate system $(i-1)$ into the coordinate system i,

$$T_i^{i-1} = \begin{bmatrix} cos(\theta_i) & -sin(\theta_i)cos(\alpha_i) & sin(\theta_i)sin(\alpha_i) & a_i cos(\theta_i) \\ sin(\theta_i) & cos(\theta_i)cos(\alpha_i) & -cos(\theta_i)sin(\alpha_i) & a_i sin(\theta_i) \\ 0 & sin(\alpha_i) & cos(\alpha_i) & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{0.1}$$

the following transformation matrices result:

$$T_1^0 = \begin{bmatrix} cos(\theta_1) & 0 & -sin(\theta_1) & 0 \\ sin(\theta_1) & 0 & cos(\theta_1) & 0 \\ 0 & -1 & 0 & d_{bs} \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_2^1 = \begin{bmatrix} cos(\theta_2) & 0 & sin(\theta_2) & 0 \\ sin(\theta_2) & 0 & -cos(\theta_2) & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{0.2}$$

$$T_3^2 = \begin{bmatrix} cos(\theta_3) & 0 & sin(\theta_3) & 0 \\ sin(\theta_3) & 0 & -cos(\theta_3) & 0 \\ 0 & 1 & 0 & d_{se} \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_4^3 = \begin{bmatrix} cos(\theta_4) & 0 & -sin(\theta_4) & 0 \\ sin(\theta_4) & 0 & cos(\theta_4) & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{0.3}$$

$$T_5^4 = \begin{bmatrix} cos(\theta_5) & 0 & -sin(\theta_5) & 0 \\ sin(\theta_5) & 0 & cos(\theta_5) & 0 \\ 0 & -1 & 0 & d_{ew} \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_6^5 = \begin{bmatrix} cos(\theta_6) & 0 & sin(\theta_6) & 0 \\ sin(\theta_6) & 0 & -cos(\theta_6) & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{0.4}$$

$$T_7^6 = \begin{bmatrix} cos(\theta_7) & -sin(\theta_7) & 0 & 0 \\ sin(\theta_7) & cos(\theta_7) & 0 & 0 \\ 0 & 0 & 1 & d_{wf} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{0.5}$$

It follows for the final transformation matrix:

$${}_7^0T =_1^0 T(\theta_1)_2^1T(\theta_2)_3^2T(\theta_3)_4^3T(\theta_4)_5^4T(\theta_5)_6^5T(\theta_6)_7^6T(\theta_7) = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{0.6}$$

For example, the position of the wrist is P_w, related to the base of the robot arm can be calculated as follows (forward kinematics):

$$P_w' =_7^0 T \cdot P_w =_1^0 T(\theta_1)_2^1T(\theta_2)_3^2T(\theta_3)_4^3T(\theta_4)_5^4T(\theta_5)_6^5T(\theta_6)_7^6T(\theta_7) \cdot [X, Y, Z, 1]_w^T \tag{0.7}$$

However, the following must be solved

$${}_7^0T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} =_1^0 T(\theta_1)_2^1T(\theta_2)_3^2T(\theta_3)_4^3T(\theta_4)_5^4T(\theta_5)_6^5T(\theta_6)_7^6T(\theta_7) \tag{0.8}$$

to θ_i to calculate the required angles (inverse kinematics). Thus seven nonlinear equations have to be solved. Four equations belong to the rotation matrix and three to the position vector. In order for a solution to be found, the target position must lie within the working range of the robot.

Calculation of the joint angles θ_i [3] [5] [6]

To calculate the angles θ_1 and θ_2, the elbow position (X,Y,Z) is required in relation to the base B:

$$P^B_{elbow} = A^B_3 P^3_{elbow} = A^B_1 A^1_2 A^2_3 P^3_{elbow} \tag{0.9}$$

with $P^3_{elbow} = [0, 0, 0, 1]^T$ (Origin of the coordinate system 3)
This results in the following nonlinear system of equations:

$$\begin{bmatrix} P^B_{elbow}.x \\ P^B_{elbow}.y \\ P^B_{elbow}.z \\ 1 \end{bmatrix} = \begin{bmatrix} d_{se}cos(\theta_1) \cdot sin(\theta_2) \\ d_{se}sin(\theta_1) \cdot sin(\theta_2) \\ d_{bs} + d_{se} \cdot cos(\theta_2) \\ 1 \end{bmatrix} \tag{0.10}$$

This results in θ_1,

$$\boxed{\theta_1 = atan2\left(P^B_{elbow}.y, P^B_{elbow}.x\right)} \tag{0.11}$$

and θ_2,

$$\boxed{\theta_2 = acos\left(\frac{P^B_{elbow}.z - d_{bs}}{d_{se}}\right)} \tag{0.12}$$

The angles θ_3 and θ_4 are calculated in the same way:

$$P^B_{elbow} = A^B_5 P^5_{wrist} = A^B_1 A^1_2 A^2_3 A^3_4 A^4_5 P^5_{wrist} \tag{0.13}$$

with $P^5_{wrist} = [0, 0, 0, 1]^T$ (Origin of the coordinate system 5)

$$(A^1_2)^{-1}(A^B_1)^{-1}P^B_{wrist} = A^2_3 A^3_4 A^4_5 P^5_{wrist} \tag{0.14}$$

From this follows

$$\begin{bmatrix} m \\ n \\ p \\ 1 \end{bmatrix} = \begin{bmatrix} -d_{ew}cos(\theta_3) \cdot sin(\theta_4) \\ -d_{ew}sin(\theta_3) \cdot sin(\theta_4) \\ d_{se} + d_{ew} \cdot cos(\theta_4) \\ 1 \end{bmatrix} \tag{0.15}$$

and thus

$$\boxed{\theta_3 = atan2(n, m)} \tag{0.16}$$

$$\boxed{\theta_4 = acos\left(\frac{p - d_{se}}{d_{ew}}\right)} \tag{0.17}$$

The angles θ_5 and θ_6 can be calculated as follows:

$$P^B_{target} = A^B_7 P^5_{target} = A^B_1 A^1_2 A^2_3 A^3_4 A^4_5 A^6_5 A^6_7 P^7_{target} \tag{0.18}$$

mit $P^7_{target} = [0, 0, 0, 1]^T$ (Origin of the coordinate system 7)

$$(A_4^3)^{-1}(A_3^2)^{-1}(A_2^1)^{-1}(A_1^B)^{-1}P_{target}^B = A_5^4 A_5^6 A_7^6 P_{target}^7 \tag{0.19}$$

$$\begin{bmatrix} m \\ n \\ p \\ 1 \end{bmatrix} = \begin{bmatrix} -d_{wf}cos(\theta_5) \cdot sin(\theta_6) \\ -d_{wf}sin(\theta_5) \cdot sin(\theta_6) \\ d_{ew} + d_{wf} \cdot cos(\theta_6) \\ 1 \end{bmatrix} \tag{0.20}$$

This results in:

$$\boxed{\theta_5 = atan2(n, m)} \tag{0.21}$$

$$\boxed{\theta_6 = acos\left(\frac{p - d_{ew}}{d_{wf}}\right)} \tag{0.22}$$

The angle θ_7 describes the rotation around the Z axis. If the *Target Center Axis* T_c now lies parallel to the Y-axis of the coordinate system 7, the projection of T_c onto the X-axis is zero.

$$T_c^7 = A_B^7 T_c^B = A_6^7 A_6^5 A_4^5 A_3^4 A_2^3 A_1^2 A_B^1 T_c^B \tag{0.23}$$

with $T_c^7 = [0, T_c^7.y, T_c^7.z, 0]^T$ and $T_c^B = [T_c^B.x, T_c^B.y, T_c^B.z, 0]^T$

$$(A_6^7)^{-1} T_c^7 = A_7^6 T_c^7 = A_5^6 A_4^5 A_3^4 A_2^3 A_1^2 A_B^1 T_c^B \tag{0.24}$$

$$\begin{bmatrix} T_c^7.y \cdot \sin(\theta_7) \\ T_c^7.y \cdot \cos(\theta_7) \\ T_c^7.z \\ 0 \end{bmatrix} = \begin{bmatrix} m \\ n \\ p \\ 0 \end{bmatrix} \tag{0.25}$$

From this follows for θ_7:

$$\boxed{\theta_7 = atan2(m, n)} \tag{0.26}$$

Redundancy circle

The LBR iiwa R800 robot system from KUKA is an anthropomorphic robot arm with 7 degrees of freedom. The combination of rotary and articulated joints corresponds to the ball and socket joint of a human shoulder. Since the arm has one more degree of freedom than necessary, there are several possibilities or solutions for gripping or hitting an object. If there are no obstacles in the working area, the solution that requires the least distance to be travelled is chosen. This additional degree of freedom, represented by the redundancy circle, makes it possible, for example, to avoid obstacles or even to perform a zero space movement with unchanged tool position and direction [2, p. 3]. This redundancy refers to the first four joints.
The analytical, inverse kinematics derived in the previous chapter requires the position of the shoulder and wrist, the elbow, and the targent center axis for calculation.
The position of the shoulder is always (the base points in vertical direction (Z axis)):

$$\vec{P}_{Shoulder} = [0, 0, d_{bs}]^T \tag{0.27}$$

For the position of the wrist several calculation steps are necessary. On the one hand, its position depends on the target position of the EEF, on the other hand, it also depends on the alignment (θ, φ) of the EEF. It follows:

$$\vec{P}_{wrist} = \vec{P}_{EEF} - Ry(\theta_{wrist}) \cdot Rz(\varphi_{wrist}) \cdot \vec{e_z} \cdot d_{wf} \tag{0.28}$$

The elbow position is evaluated via a redundancy circle. For an arm with 7-DOF follows:

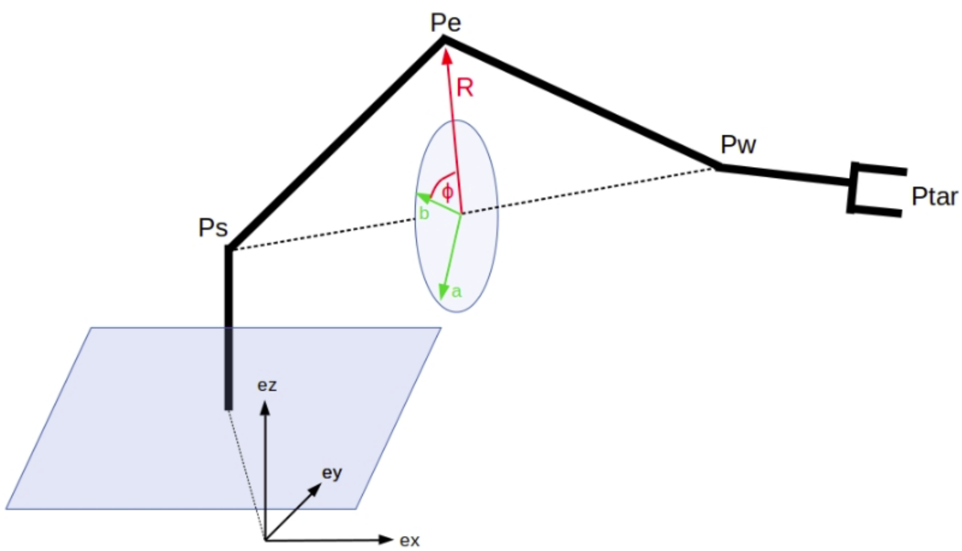

Figure 0.2: Redundancy circuit of the robot arm. A plane (unit circle) is formed in which the robot arm moves on a circular disk with radius R. The unit vectors $\vec{a}$ and $\vec{b}$ span the circular plane and are perpendicular to each other. The clamped circular plane is again perpendicular to the vector from the shoulder to the wrist. The arm angle ϕ describes the position on the circular disc.

For the unit vectors $\vec{a}$ and $\vec{b}$ it has to be defined which position on the redundancy circle corresponds to the arm angle $\phi = 0$. This fixed definition is achieved by the following formula [1, p. 12]:

$$\vec{b} = \frac{\vec{a} - (\vec{a} \cdot \vec{P}_c) \cdot \vec{P}_c}{\left\| \vec{a} - (\vec{a} \cdot \vec{P}_c) \cdot \vec{P}_c \right\|} \tag{0.29}$$

The unit vector $\vec{a}$ can be chosen arbitrarily. If, for example, the arm joint at $\phi = 0$ is to point in the X-axis direction, the unit vector $\vec{a} = (1, 0, 0)^T$ must be. Known are the shoulder and arm joint positions as well as the lengths of the upper and lower arm. First, the length of the vector $\vec{Pc}$ from the shoulder to the center of the circle is required.
Then the circle radius can be calculated:

$$\alpha = asin\left(\frac{Pc_{length}}{d_{se}}\right), \qquad R = cos(\alpha) \cdot d_{se} \tag{0.30}$$

Finally follows for the position of the elbow:

$$\vec{Pe} = R \cdot [cos(\phi) \cdot \vec{a} + sin(\phi) \cdot \vec{b}] + \vec{Pc} + \vec{P}_{Shoulder} \tag{0.31}$$

The position of the elbow on the circular disk is limited by the permissible joint angles of the robot system and constraints. As soon as obstacles touch the elbow, this part of the redundancy circle is avoided.[2, S. 3]
Finally, the Target Center Axis must be determined. The inverse kinematics was programmed so that the user specifies the position and orientation of the EEF. From this, the program calculates the Target Center Axis. A vector u is sought that is orthogonal to the vector $\vec{P}_{wrist}$. This is achieved by the following calculations:

$$\vec{u} = [0, 1, 0]^T, \qquad Tc = Rz(\varphi) \cdot Ry(\theta) \cdot \vec{u} \tag{0.32}$$

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