PROBLEM SET 03: TENSOR PRODUCTS AND THE QUANTUM AXIOMS

Exercise 0.1 (Recalling Defintions)**.** Define the

(i) tensor product $\mathscr{H} \otimes \mathscr{K}$ of two finite dimensional spaces.
(ii) Kronecker product of two arbitrary sized matrices $\rho \in \mathsf{M}_{n\times m}(\mathbb{C})$ and $\sigma \in \mathsf{M}_{s\times t}(\mathbb{C})$.
(iii) multiple system axiom.
(iv) system evolution axiom.

Exercise 0.2 (Tensoring with $\mathbb{C}$)**.** Suppose $\mathscr{H}$ is a finite dimensional (complex) Hilbert space. Prove that $\mathscr{H} \otimes \mathbb{C}$ is isomorphic to $\mathscr{H}$.[1]

Exercise 0.3. If $\mathscr{H}, \mathscr{K} \in \mathsf{FdHilb}_{\mathbb{C}}$ have bases $\beta = \{b_1, b_2, \ldots, b_n\}$ and $\gamma = \{g_1, g_2, \ldots, g_m\}$, respectively, show that the set

$$\delta = \{b_i \otimes g_j : 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}$$

is a basis for $\mathscr{H} \otimes \mathscr{K}$.[2]

Exercise 0.4 (Direct and Tensor Products Commute)**.** Let $\{\mathscr{H}_1, \mathscr{H}_2, \ldots, \mathscr{H}_n\} \subseteq \mathsf{FdHilb}_{\mathbb{C}}$ be a collection of finite dimensional Hilbert spaces. Let $\mathscr{K}$ be another finite dimensional Hilbert space. Denote by $\mathscr{H}$ the direct product $\mathscr{H}_1 \times \mathscr{H}_2 \times \cdots \mathscr{H}_n$. Prove that

$$\mathscr{H} \otimes \mathscr{K} \simeq \prod_{i=1}^{n} (\mathscr{H}_i \otimes \mathscr{K})$$

by showing that the homomorphism $\varphi : \mathscr{H} \otimes \mathscr{K} \to \prod_{i=1}^{n} (\mathscr{H}_i \otimes \mathscr{K})$ given by

$$\varphi\big((h_1, h_2 \ldots, h_n) \otimes k\big) = (h_1 \otimes k, h_2 \otimes k, \ldots, h_n \otimes k)$$

is a bijection.

Exercise 0.5. In what sense does the tensor product bifunctor $(-)\otimes(-) : \mathsf{FdHilb}_{\mathbb{C}} \times \mathsf{FdHilb}_{\mathbb{C}} \to \mathsf{FdHilb}_{\mathbb{C}}$ provide a way to join finite quantum systems?

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Date: July 7, 2025.

[1] You do *not* need to explicitly construct an isomorphism.

[2] *Hint:* Construct a bilinear function $\mathscr{H} \times \mathscr{K} \to \mathbb{C}$ and then use the universal property of the tensor product bifunctor.