WHAT IS A TQFT?
A SYMMETRIC MONOIDAL FUNCTOR.

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Abstract. In this note I spell out how to realize topological field theories as functors on a suitable category of cobordisms. [Really, this ended up being an introduction to symmetric monoidal categories. At some point, I plan complete this note, i.e., type up sequel sections on conformal and topological conformal field theories.] This is intended for an audience who is familiar with Atiyah's axioms for a topological (quantum) field theory, but is maybe only familiar with the rudiments of ordinary category theory.

Contents

A *functorial quantum field theory* (FQFT) tends to refer to an assignment

$$\text{physical spaces and trajectories} \leftrightsquigarrow \text{state spaces and linear maps.}$$

At the very least, we like to ask that such an assignment respects the usual notions of sewing and multiplication/correlation in either setting. This tells us that an FQFT should be some symmetric monoidal functor from a category of cobordisms to a category of vector spaces. Of course, if we add structure to the domain and target categories, then the type of functor should be tweaked to respect that.

Three of our familiar field theories can be viewed as FQFTs.

(i) A $(d + 1)$-dimensional *topological quantum field theory* (TQFT) is a symmetric monoidal functor from a category of d-manifolds and $(d + 1)$-dimensional cobordisms to the category of complex vector spaces and $\mathbb{C}$-linear maps.
(ii) A *conformal field theory* (CFT) is a continuous symmetric monoidal functor from a category of 1-manifolds with parametrization and 2-dimensional conformal cobordisms to the category of complex vector spaces and $\mathbb{C}$-linear maps.
(iii) A *topological conformal field theory* (TCFT) is a chain complex-enriched homotopy-symmetric monoidal functor from a dg-category of 1-manifolds

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Date: Fall 2025.

These notes were compiled to supplement a talk given by myself for *MATH 595: Topological Field Theories in 2D*, as instructed by Prof. James Pascaleff in the fall semester of 2025 at the *University of Illinois*.

with parametrization and 2-dimensional conformal cobordisms to the dg-category of chain complexes of complex vector spaces and chain maps.

Motivated by these three examples, we build the necessary categorical machinery to define some FQFTs, though we will only have the time to make it to (i).

1. Categorical Preliminaries

Recall that a category $\mathcal{C}$ consists of a collection of objects $|\mathcal{C}|$, for every pair of objects $a, b \in \mathcal{C}$, a set $\mathcal{C}(a, b)$ of morphisms $a \to b$, and for every triple of objects $a, b, c \in \mathcal{C}$, a composition map

$$\circ_{a,b,c} : \mathcal{C}(b, c) \times \mathcal{C}(a, b) \to \mathcal{C}(a, c).$$

We require that composition is associative and that for all $a \in \mathcal{C}$ there is a morphism $\mathrm{id}_a : a \to a$ so that id_a is a two-sided identity for $\mathcal{C}(a, a)$ with respect to $\circ_{a,a,a}$.

Example 1.1. We usually populate our categories with algebraic objects, giving rise to the category

(i) Vect, of finite-dimensional complex vector spaces and $\mathbb{C}$-linear maps.
(ii) coCh, of cochain complexes of complex vector spaces and chain maps.
(iii) Rep_G, of G-representations $(\mathcal{V}, \varphi)$ and G-equivariant maps.
(iv) FdHilb of finite-dimensional Hilbert spaces and bounded $\mathbb{C}$-linear maps.
(v) CFrob, of finite-dimensional complex commutative Frobenius algebras and Frobenius algebra maps.

Example 1.2. We often care about categories of spaces with continuous maps, like Top, of all topological spaces, or the nicer category CGWH, of compactly generated weak Hausdorff spaces.

Example 1.3. Adding a bit more structure, we could consider the category SMan of smooth manifolds and smooth maps, or more generally, the category $\mathsf{D}^p\mathsf{Man}$ of C^p manifolds and C^p maps for any $p \in 0 \cup \mathbb{N} \cup \infty$.

Yet, it is certainly not the case that a category must have objects that are sets with extra structure and morphisms that are structure-preserving functions. Notably, if our goal is for trajectories or cobordisms to correspond to linear maps, we should have a category with morphisms given by cobordisms. Since two diffeomorphically equivalent cobordisms should eventually give rise to the same map, the morphisms in such a category should be defined as equivalence classes.

Definition 1.4 (Bordism Category)**.** The category Bord_{d+1} has objects that are closed and oriented d-manifolds. Given $M_0, M_1 \in \mathsf{Bord}_{d+1}$, the morphisms $M_0 \to M_1$ are given by equivalence classes of cobordisms from M_0 to M_1, where a pair of cobordisms $W, W' : M_0 \rightrightarrows M_1$ are equivalent if there is a diffeomorphism $\varphi : W \to W'$ so that

$$\begin{tikzcd} W \arrow[rr, "\varphi", "\simeq"'] & & W' \\ & \overline{M_0} \amalg M_1 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

commutes. The composition law is given by the gluing of cobordisms. That is, we have a map

$$\circ_{M_0,M_1,M_2} : \mathsf{Bord}_{d+1}(M_1, M_2) \times \mathsf{Bord}_{d+1}(M_0, M_1) \to \mathsf{Bord}_{d+1}(M_0, M_2)$$

prescribed by

$$W_{12} \circ_{M_0,M_1,M_2} W_{01} = W_{01} \coprod_{M_1} W_{12} = W_{02}.$$

For each d-manifold $M \in \mathsf{Bord}_{d+1}$, we use the natural choice of identity morphism $\mathrm{id}_M \in \mathsf{Bord}_{d+1}(M, M)$ given by the cylinder

$$\mathrm{id}_M = M \times [0,1].$$

1.1. **Functoriality.** Recall that a functor $f : \mathcal{C} \to \mathfrak{D}$ between categories $\mathcal{C}$ and $\mathfrak{D}$ assigns an object $f(a) \in \mathfrak{D}$ for each object $a \in \mathcal{C}$ and a morphism $f(\alpha) : f(a) \to f(b) \in \mathfrak{D}$ for each morphism $\alpha : a \to b \in \mathcal{C}$. Functoriality then requires that functors respect the composition and unitality structure of the categories:

$$f(\beta\alpha) = f(\beta)f(\alpha), \quad \text{for all } \alpha \in \mathcal{C}(a,b) \text{ and } \beta \in \mathcal{C}(b,c)$$

and

$$f(\mathrm{id}_a) = \mathrm{id}_{f(a)}, \quad \text{for all } a \in \mathcal{C}.$$

Functors give us a natural notion of morphisms between categories, and so we can assemble all categories into a single category Cat.

Likewise, given parallel functors $f, g : \mathcal{C} \rightrightarrows \mathfrak{D}$, a natural transformation $\eta : f \Rightarrow g$ is an assignment, for each object $a \in \mathcal{C}$, of a component morphism $\eta_a : f(a) \to g(a) \in \mathfrak{D}$ so that the naturality square

$$\begin{tikzcd} f(a) \arrow[r, "\eta_a"] \arrow[d, "f(\alpha)"'] & g(a) \arrow[d, "g(\alpha)"] \\ f(b) \arrow[r, "\eta_b"] & g(b) \end{tikzcd}$$

commutes for all morphisms $\alpha : a \to b \in \mathcal{C}$. Natural transformations are the natural notion of morphisms between functors, and so after fixing a pair of categories $\mathcal{C}$ and $\mathfrak{D}$, we can assemble a category of functors $\mathrm{Fun}(\mathcal{C}, \mathfrak{D})$ which has functors $\mathcal{C} \to \mathfrak{D}$ for objects, natural transformations as morphisms, and the (vertical) composition of natural transformations is done componentwise.

Now, an isomorphism $a \simeq b$ in an arbitrary category $\mathcal{C}$ is a morphism $\varphi : a \to b$ so that there is a two-sided inverse $\varphi^{-1} : b \to a$ with respect to the identity morphisms. Thus, an isomorphism of categories is such a functor in Cat. However, this turns out to be a bit too restrictive, and so we weaken this to the notion of equivalence.

Definition 1.5 (Natural Isomorphism)**.** We call a natural transformation $\eta : f \Rightarrow g$ a natural isomorphism if its component morphisms η_a are all isomorphisms. In this case, we call f and g naturally isomorphic, writing $f \approx g$.

Definition 1.6 (Equivalence)**.** A functor $f : \mathcal{C} \to \mathfrak{D}$ is an equivalence if there is a functor $g : \mathfrak{D} \to \mathcal{C}$ such that $\mathrm{id}_{\mathcal{C}} \approx gf$ and $fg \approx \mathrm{id}_{\mathfrak{D}}$.

1.2. **Monoidal Structure.**

Definition 1.7 (Monoidal Category)**.** A monoidal category consists of

(i) a category $\mathcal{C}$.
(ii) a *tensor product* bifunctor $(-) \otimes (-) : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.
(iii) a *unit* object $\mathbb{1} \in \mathcal{C}$.
(iv) an *associator* natural isomorphism

$$\alpha : ((-) \otimes (-)) \otimes (-) \Rightarrow (-) \otimes ((-) \otimes (-))$$

with components $\alpha_{a,b,c} : (a \otimes b) \otimes c \to a \otimes (b \otimes c)$, for all triples $a, b, c \in \mathcal{C}$.

(v) a *left unitor* natural isomorphism

$$\lambda : \mathbb{1} \otimes (-) \Rightarrow \mathrm{id}_{\mathcal{C}}$$

with components $\lambda_a : \mathbb{1} \otimes a \to a$, for all $a \in \mathcal{C}$.

(vi) a *right unitor* natural isomorphism

$$\rho : (-) \otimes \mathbb{1} \Rightarrow \mathrm{id}_{\mathcal{C}}$$

with components $\rho_a : a \otimes \mathbb{1} \to a$, for all $a \in \mathcal{C}$.

These data must satisfy commutativity of the triangle (Fig. 1) and pentagon (Fig. 2) diagrams for all objects $a, b, c, d \in \mathcal{C}$.

$$\begin{tikzcd}
(a \otimes \mathbb{1}) \otimes b \arrow[rr, "\alpha_{a,\mathbb{1},b}"] \arrow[dr, "\rho_a \otimes \mathrm{id}_b"'] & & a \otimes (\mathbb{1} \otimes b) \arrow[dl, "\mathrm{id}_a \otimes \lambda_b"] \\
& a \otimes b &
\end{tikzcd}$$

FIGURE 1. Triangle diagram

$$\begin{tikzcd}
(a \otimes (b \otimes c)) \otimes d \arrow[rr, "\alpha_{a,b\otimes c,d}"] & & a \otimes ((b \otimes c) \otimes d) \arrow[d, "\mathrm{id}_a \otimes \alpha_{b,c,d}"] \\
((a \otimes b) \otimes c) \otimes d \arrow[u, "\alpha_{a,b,c} \otimes \mathrm{id}_d"] \arrow[dr, "\alpha_{a\otimes b,c,d}"'] & & a \otimes (b \otimes (c \otimes d)) \\
& (a \otimes b) \otimes (c \otimes d) \arrow[ur, "\alpha_{a,b,c\otimes d}"'] &
\end{tikzcd}$$

FIGURE 2. Pentagon diagram

We will abbreviate the data of such a monoidal category by writing $(\mathcal{C}, \otimes, \mathbb{1})$.

Example 1.8. The category Set, of sets and functions between them, is monoidal with tensor product given by the Cartesian product, unit object given by a chosen singleton $*$, associator given by

$$((p, q), r) \mapsto (p, (q, r)),$$

and unitors given by $(*, q) \mapsto q$ and $(p, *) \mapsto p$, where p, q, and r live in some sets.

Example 1.9. The category Vect is monoidal with tensor product given by the usual tensor product $\otimes = \otimes_{\mathbb{C}}$, unit object given by $\mathbb{C}$, associator sending

$$(p \otimes q) \otimes r \mapsto p \otimes (q \otimes r),$$

and unitors sending $1 \otimes q \mapsto q$ and $p \otimes 1 \mapsto p$, where p, q, and r live in some vector spaces.

Example 1.10. The category Bord_{d+1} is monoidal with tensor product given by $\amalg$, unit object given by $\varnothing$. The associator and unitors are again the natural ones.

To formalize the notion of swapping systems, we introduce the additional structure of *braiding* on a monoidal category.

Definition 1.11 (Braided Monoidal Category)**.** A braided monoidal category consists of a monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$ and a *braiding* natural isomorphism

$$\gamma : (-) \otimes (-) \Rightarrow (-) \otimes (-)$$

with components $\gamma_{a,b} : a \otimes b \to b \otimes a$, for all $a, b \in \mathcal{C}$. These data must satisfy commutativity of the hexagon diagrams (Fig. 3) for all objects $a, b, c \in \mathcal{C}$.

$$\begin{tikzcd}
& a \otimes (b \otimes c) \arrow[dl, "\alpha^{-1}_{a,b,c}"'] \arrow[r, "\gamma_{a,b\otimes c}"] & (b \otimes c) \otimes a & \\
(a \otimes b) \otimes c \arrow[dr, "\gamma_{a,b} \otimes \mathrm{id}_c"'] & & & b \otimes (c \otimes a) \arrow[ul, "\alpha^{-1}_{b,c,a}"'] \\
& (b \otimes a) \otimes c \arrow[r, "\alpha_{b,a,c}"'] & b \otimes (a \otimes c) \arrow[ur, "\mathrm{id}_b \otimes \gamma_{a,c}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& (a \otimes b) \otimes c \arrow[dl, "\alpha_{a,b,c}"'] \arrow[r, "\gamma_{a\otimes b,c}"] & c \otimes (a \otimes b) & \\
a \otimes (b \otimes c) \arrow[dr, "\mathrm{id}_a \otimes \gamma_{b,c}"'] & & & (c \otimes a) \otimes b \arrow[ul, "\alpha_{c,a,b}"'] \\
& a \otimes (c \otimes b) \arrow[r, "\alpha^{-1}_{a,c,b}"'] & (a \otimes c) \otimes b \arrow[ur, "\gamma_{a,c} \otimes \mathrm{id}_b"'] &
\end{tikzcd}$$

Figure 3. Hexagon diagrams

We abbreviate the above data by writing $(\mathcal{C}, \otimes, \mathbb{1}, \gamma)$.

Example 1.12. For any pair of vector spaces $\mathcal{V}, \mathcal{U}$, there is a unique linear map

$$\gamma_{\mathcal{V},\mathcal{U}} : \mathcal{V} \otimes \mathcal{U} \to \mathcal{U} \otimes \mathcal{V}$$

given by $v \otimes u \mapsto u \otimes v$ on the basic tensors. Moreover, this map is natural and an isomorphism, so the components assemble into a braiding γ on Vect.

Example 1.13. The twist cobordism τ gives a braiding on Bord_{d+1}.

Definition 1.14 (Symmetric Monoidal Category)**.** A symmetric monoidal category is a braided monoidal category $(\mathcal{C}, \otimes, \mathbb{1}, \gamma)$ where $\gamma_{b,a}\gamma_{a,b} = \mathrm{id}_{a\otimes b}$ for all $a, b \in \mathcal{C}$.

Proposition 1.15. *The braiding τ on Bord_d and the braiding γ on Vect are symmetric.*

Proof. Let $\mathcal{V}$ and $\mathcal{U}$ be $\mathbb{C}$-linear spaces with $v \in \mathcal{V}$ and $u \in \mathcal{U}$. Then,

$$\gamma_{\mathcal{U},\mathcal{V}}(\gamma(\mathcal{V}, \mathcal{U})(v \otimes u)) = \gamma_{\mathcal{U},\mathcal{V}}(u \otimes v) = v \otimes u = \mathrm{id}_{\mathcal{V}\otimes\mathcal{U}}(v \otimes u).$$

Now, let M_0 and M_1 be closed and oriented d-manifolds. Then,

$$\tau_{1,0} \circ \tau_{01} : M_0 \amalg M_1 \to M_0 \amalg M_1$$

is a cobordism diffeomorphic to the parallel cylinders $(M_0 \times [0,1]) \amalg (M_1 \times [0,1])$. Moreover, this diffeomorphism can be taken relative on the boundary M_0 and M_1. Finally, we have another diffeomorphism equivalence of these parallel cylinders to

$$(M_0 \amalg M_1) \times [0,1] = \mathrm{id}_{M_0 \amalg M_1}\,.$$

□

1.3. **Symmetric Monoidal Maps.** Now that our categories have the added structure of a tensor product, we should add structure to the definition of functor. This leads us to the definition of a monoidal functor, i.e., a morphism of monoidal categories.

Definition 1.16 (Monoidal Functor)**.** Let $(\mathfrak{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ and $(\mathfrak{D}, \boxtimes, \mathbf{1}, \beta, \mathfrak{l}, \mathfrak{r})$ be monoidal categories. Then, a (strong) monoidal functor $f : \mathfrak{C} \to \mathfrak{D}$ consists of an ordinary functor $f : \mathfrak{C} \to \mathfrak{D}$, a natural isomorphism

$$\varphi : f(-) \boxtimes f(-) \Rightarrow f(- \otimes -)$$

with components $\varphi_{a,b} : f(a) \boxtimes f(b) \to f(a \otimes b)$, for all $a, b \in \mathfrak{C}$, and an isomorphism

$$\underline{f} : \mathbf{1} \to f(\mathbb{1}).$$

These data must fulfill the intuitive compatibility requirements with the associator (Fig. 1.3) and unitors (Fig. 1.3) for all objects $a, b, c \in \mathfrak{C}$.

$$\begin{tikzcd}
(f(a) \boxtimes f(b)) \boxtimes f(c) \arrow[r, "\beta_{f(a),f(b),f(c)}"] \arrow[d, "\varphi_{a,b} \boxtimes \mathrm{id}_{f(c)}"'] & f(a) \boxtimes (f(b) \boxtimes f(c)) \arrow[d, "\mathrm{id}_{f(a)} \boxtimes \varphi_{b,c}"] \\
f(a \otimes b) \boxtimes f(c) \arrow[d, "\varphi_{a \otimes b,c}"'] & f(a) \boxtimes f(b \otimes c) \arrow[d, "\varphi_{a,b \otimes c}"] \\
f((a \otimes b) \otimes c) \arrow[r, "f(\alpha_{a,b,c,})"'] & f(a \otimes (b \otimes c))
\end{tikzcd}$$

FIGURE 4. Associator compatibility for a monoidal functor

$$\begin{tikzcd}
\mathbf{1} \boxtimes f(a) \arrow[r, "\mathfrak{l}_{f(a)}"] \arrow[d, "\underline{f} \boxtimes \mathrm{id}_{f(a)}"'] & f(a) \arrow[d, "f(\lambda_a^{-1})"] \\
f(\mathbb{1}) \arrow[r, "\varphi_{\mathbb{1},a}"'] & f(\mathbb{1} \otimes a)
\end{tikzcd}$$

FIGURE 5. Left unitor compatibility for a monoidal functor. The right unitor compatibility is completely analogous.

Once we have a monoidal functor $f : \mathfrak{C} \to \mathfrak{D}$, if we have the additional structure of a braiding on both the domain and target categories, it is reasonable to ask for f to be compatible with this as well.

Definition 1.17 (Braided Monoidal Functor)**.** Let $\mathfrak{C}$ and $\mathfrak{D}$ be monoidal categories with braidings γ and σ, respectively. Then, a braided monoidal functor is a monoidal functor $f : \mathcal{C} \to \mathfrak{D}$ such that the square

$$\begin{tikzcd} f(a) \boxtimes f(b) \arrow[r, "\sigma_{f(a),f(b)}"] \arrow[d, "\varphi_{a,b}"'] & f(b) \boxtimes f(a) \arrow[d, "\varphi_{b,a}"] \\ f(a \otimes b) \arrow[r, "f(\gamma_{a,b})"'] & f(b \otimes a) \end{tikzcd}$$

commutes for all objects $a, b \in \mathcal{C}$.

If we further have that the braidings on our domain and target categories are actually symmetries, then a braided monoidal functor between them is called a *symmetric monoidal functor*. Note that we do not have to enforce the symmetry.

Finally, we should ask that natural transformations between monoidal functors are compatible with the added structure.

Definition 1.18 (Monoidal Natural Transformation)**.** Let $f, g : \mathcal{C} \rightrightarrows \mathfrak{D}$ be a pair of parallel monoidal functors, using the same notation as above. Let ψ be the natural isomorphism for g. Then, a monoidal natural transformation consists of a natural transformation $\eta : f \Rightarrow g$ such that the naturality square

$$\begin{tikzcd} f(a) \boxtimes f(b) \arrow[r, "\varphi_{a,b}"] \arrow[d, "\eta_a \boxtimes \eta_b"'] & f(a \otimes b) \arrow[d, "\eta_{a\otimes b}"] \\ g(a) \boxtimes g(b) \arrow[r, "\psi_{a,b}"'] & g(a \otimes b) \end{tikzcd}$$

and the triangle

$$\begin{tikzcd} f(\mathbb{1}) \arrow[rr, "\eta_{\mathbb{1}}"] & & g(\mathbb{1}) \\ & \mathbf{1} \arrow[ul, bend left, "\underline{f}"] \arrow[ur, bend right, "\underline{g}"'] & \end{tikzcd}$$

commute for all $a, b \in \mathcal{C}$.

Definition 1.19 (Symmetric Monoidal Functor Category)**.** Let $\mathcal{C}$ and $\mathfrak{D}$ be symmetric monoidal categories. We can form a category $\mathrm{Fun}^s_\otimes(\mathcal{C}, \mathfrak{D})$ with objects given by the symmetric monoidal functors $\mathcal{C} \to \mathfrak{D}$ and morphisms given by the corresponding monoidal natural transformations. The maps are composed as in $\mathrm{Fun}(\mathcal{C}, \mathfrak{D})$.

Remark 1.20 (Strictness). A result due to Saunders Mac Lane says that every monoidal category is monoidally equivalent to a *strict* monoidal category, i.e., one where the associator and unitors are the identity natural isomorphism. Thus, suppressing them is less of an abuse than it seems at first glance. This can be extended to the symmetric monoidal case, in which case the resulting structure is called a permutative category.

2. Topological Quantum Field Theories

Our goal was to functorially assign complex vector spaces and linear maps to manifolds and cobordisms therebetween, all while respecting the (symmetric) multiplicative structure of both.

Definition 2.1 (TQFT)**.** A $(d + 1)$-dimensional topological quantum field theory is a (strong) symmetric monoidal functor

$$\mathcal{Z} : (\mathsf{Bord}_{d+1}, \amalg, \varnothing, \tau) \to (\mathsf{Vect}, \otimes, \mathbb{C}, \gamma).$$

We should probably further require that $\underline{\mathcal{Z}}$ is the identity isomorphism $\mathbb{C} \to \mathcal{Z}(\varnothing)$.

Looking back at our definition of an isomorphism of TQFTs, it is clear that this is encoded in a monoidal natural isomorphism. Thus, the sensible choice of morphism between two TQFTs is exactly a monoidal natural transformation.

Definition 2.2 (TQFT Category)**.** The $(d+1)$-dimensional TQFT category is the symmetric monoidal functor category

$$\mathsf{TQFT}_{d+1} = \mathrm{Fun}^s_{\otimes}(\mathsf{Bord}_{d+1}, \mathsf{Vect}).$$

Now, recall our construction Ω which took a two-dimensional topological quantum field theory $\mathcal{Z} \in \mathsf{TQFT}_2$ and outputted a Frobenius algebra. We set $\mathfrak{A} = \mathcal{Z}(\mathbb{S}^1)$, let the multiplication m be given by the pair of pants with unit cup, and let the comultiplication Δ be given by the reversed pair of pants and counit cap. Coupling this construction with the obvious action of Ω on morphisms, we conclude the following.

Theorem 2.3 (Classification of 2D TQFTs)**.** *The functor* $\Omega : \mathsf{TQFT}_2 \to \mathsf{CFrob}$ *is an equivalence of categories.*

Of Atiyah's axioms, our functorial picture recovers

(i) diffeomorphism invariance via
(ii)
(iii) unitality via the definition of strong monoidal functor.

Note that involutivity and duality behave a bit differently from each other, even though we previously treated them as a singular axiom. We will skip over duality, only

Definition 2.4 (Dagger Frobenius Algebra)**.** A dagger Frobenius algebra is a Frobenius algebra $(\mathfrak{A}, m, \Delta, \eta, \varepsilon)$ such that $m^\dagger = \Delta$ and $\eta^\dagger = \varepsilon$.

The commutative dagger Frobenius algebras assemble into a category $\mathsf{CFrob}^\dagger$.

Put the usual $\dagger$ structure on FdHilb. On Bord_2, we have an involution given by reversing cobordisms. Both are compatible with the previously described symmetric monoidal structures, so we can consider the category of all symmetric monoidal dagger functors $\mathsf{Bord}_2 \to \mathsf{FdHilb}$. Call the resulting category UTQFT_2 and call these objects unitary TQFTs.

Proposition 2.5 (Classification of Unitary 2D TQFTs)**.** *The equivalence* Ω *from before descends to an equivalence on the subcategories of* $\dagger$*-structures identified above*:

$$\begin{tikzcd}
\mathsf{TQFT}_2 \arrow[r, "\Omega", "\simeq"'] & \mathsf{CFrob} \\
\mathsf{UTQFT}_2 \arrow[u, hook] \arrow[r, dashed, "\simeq"] & \mathsf{CFrob}^\dagger \arrow[u, hook]
\end{tikzcd}$$

Proof. Follows from the definition of a symmetric monoidal $\dagger$-functor. □

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