PORTRYAGIN-THOM CONSTRUCTION

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Contents

We consider two, seemingly unrelated problems. On one hand, there is the classification of smooth manifolds, and on the other, there is homotopy theory and computing the homotopy groups of spheres. We hope to find a correspondence of sorts, so as to transfer tools between the seemingly disjoint topics. The correspondence comes in the form of Pontryagin-Thom construction.

$$\begin{tikzcd} \begin{array}{c}\text{classification of}\\ \text{smooth manifolds}\\ \text{up to cobordism}\end{array} \arrow[r, leftrightarrow, bend left, "\text{Pontryagin-Thom}"] & \begin{array}{c}\text{homotopy classes}\\ \text{of maps } \mathbb{S}^k \to \mathbb{S}^n\end{array} \end{tikzcd}$$

1. Smooth Manifolds

Smooth manifolds, in some sense, are where we can do analysis. Recall that if $m \geq 0$, then $M \subseteq \mathbb{R}^n$ is a smooth m-manifold if for all points $p \in M$, there is a neighborhood $U_p \subseteq M$ such that

(i) $p \in U_p$.
(ii) U_p is open in M.
(iii) there is a diffeomorphism $\varphi_p : U_p \xrightarrow{\sim} \mathbb{R}^m$ called a *chart*.

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Example 1.1. We have

(a) 0-dimensional manifolds: $*$.
(b) 1-dimensional manifolds: $\mathbb{R}$, $\mathbb{S}^1$.
(c) 2-dimensional manifolds: $\mathbb{R}^2$, $\mathbb{S}^2$, $\mathbb{T}^2$, and generally, any Σ_p.
(d) m-dimensional manifolds: $\mathbb{S}^m \subseteq \mathbb{R}^{m+1}$.

Definition 1.2 (Smooth)**.** Let M be an m-manifold and N be a k-manifold. Then, a map $f : M \to N$ is smooth at a point $p \in M$ if $\psi_p f \varphi_p^{-1}$ is smooth at $\varphi_p(p)$, where $\varphi_p M \xrightarrow{\sim} \mathbb{R}^m$ is a chart at $p \in M$ and $\psi_p : N \xrightarrow{\sim} \mathbb{R}^k$ is a chart at $f(p)$.

Let T_pM be the tangent space of M at p.[1]

Exercise 1.3. Check that T_pM is a $\mathbb{R}$-linear space of dimension $\dim(M)$.

Definition 1.4 (Differential)**.** The differential at $p \in M$ is the map

$$(\mathrm{d}f)_p : T_pM \to T_{f(p)}N$$

given by $\mathrm{d}(\psi_p f \varphi_p^{-1})$.

Recall that a smooth map $f : M \to N$ is a diffeomorphism if it has a smooth inverse.

1.1. **Regular Values.** It turns out that if we have a smooth map $M \to N$, then picking a so-called regular value q of f gives us a new manifold $f^{-1}(q)$.

Definition 1.5 (Regular Value)**.** Let $f : M \to N$ be a smooth map. Let $q \in N$. We say q is a regular value of f, writing $q \in \mathrm{Reg}(f)$, if for all $p \in f^{-1}(q)$, the differential map

$$T_pM \xrightarrow{(\mathrm{d}f)_p} T_qN$$

is a surjection.

Example 1.6.

(a) Given two manifolds, $M \times N$ is a manifold. Further, the projection $\pi : M \times N \to M$ is smooth and $\mathrm{Reg}(\pi) = M$, as we would hope.
(b) Map Σ_1 to the line via h, forming a 1-dimensional annulus A with boundary $\partial(A) = \{\pm 1, \pm 1/2\}$. Then, $\mathrm{Reg}(h) = \partial(A)$.

Let $f : M \to N$ be smooth.

Theorem 1.7 (Sard's, Weak)**.** *Then,* $\mathrm{Reg}(f)$ *is dense in* N*. Thus, if* N *is nonempty, then so is* $\mathrm{Reg}(f)$*.*

Theorem 1.8. *If q is a regular value of f, then $f^{-1}(q) \subseteq M$ is a manifold of dimension* $\dim(M) - \dim(N)$*.*

Corollary 1.9. *Let $f : M \to N$ be smooth with* $\dim(M) < \dim(N)$*. Then, f is not a surjection.*[2]

[1]For example, we could take it to be the image of the differential of the chart $\mathrm{d}\varphi_p$ at $p \in M$.
[2]That is, there are no space-filling smooth curves.

1.2. **Framed Submanifold.** Suppose p is a regular value of $f : M \to N$. Then, there is a submanifold $f^{-1}(p) \subseteq m$. In particular, it is actually a submanifold which admits a *framing*.

Definition 1.10 (Orthogonal Complement)**.** Let $K \subseteq M \hookrightarrow \mathbb{R}^n$ be a submanifold. Given a point $p \in K$, we set

$$(T_p K)^\perp = \{v \in T_p M : v \perp T_p K\},$$

using the inherited inner product in $\mathbb{R}^n$.

Definition 1.11 (Framing)**.** A framing on $K \subseteq M$, as above, is a function β from K to the normal bundle which assigns to each point $p \in K$ a basis of $(T_p K)^\perp$. Further, β depends continuously on the choice of point p.

Definition 1.12 (Framed Submanifold)**.** A framed submanifold of M is a pair (K, β), where $K \subseteq M$ and β is a framing.

Exercise 1.13. Heuristically, explain why $\mathbb{S}^1 \times I$ can be framed, whereas the Möbius strip M cannot.

Theorem 1.14. *Let $f : M \to N$ be smooth.*

(*i*) *If $q \in \mathrm{Reg}(f)$, then $f^{-1}(q)$ can be framed.*

(*ii*) *A submanifold $K \subseteq N$ can be framed if and only if there is an open neighborhood $U \subseteq N$ of K and*

$$U \cong K \times \mathbb{R}^{\dim(N)-\dim(K)}.$$

We call U, as above, a *tubular neighborhood*.

Exercise 1.15. Show that $\mathbb{S}^1 \vee \mathbb{S}^1$ is not a smooth manifold.

Exercise 1.16. Show $\mathbb{S}^2$ is a manifold by constructing charts.

Exercise 1.17. Show that any polynomial $f \in \mathbb{C}[z]$ is a smooth map $\mathbb{S}^2 \to \mathbb{S}^2$, where we identify $\mathbb{S}^2$ with $\mathbb{C} \cup \{\infty\}$ and define $f(\infty) = \infty$.

Exercise 1.18. Let $f : M \to N$ be a smooth map. Let M be a compact manifold. Show that if f is not constant, then f has no less than two critical values.

Exercise 1.19. A *knot* is an embedded submanifold $\mathbb{S}^1 \hookrightarrow \mathbb{R}^3$. Show that any knot can be framed. What does a tubular neighborhood of a knot look like.

2. Homotopy Groups

Recall that if $(X, x_0), (Y, y_0) \in \mathsf{Top}_*$, then a pointed homotopy h between a pair of pointed maps $f, g : (X, x_0) \rightrightarrows (Y, y_0)$ is a homotopy $h : f \simeq g$ so that $h(x_0, t) = y_0$ for all $t \in I$. We will write $[X, Y]_*$ for the set of pointed maps $X \to Y$ modulo pointed homotopies.

Definition 2.1 (nth Homotopy Group)**.** Let $(X, x_0) \in \mathsf{Top}_*$. Let $\mathbb{S}^n$ be pointed at $s_0 = e_1 \in \mathbb{R}^{n+1}$. Then, the nth homotopy group of (X, x_0) is

$$\pi_n X = \pi_n(X, x_0) = [\mathbb{S}^n, X]_*.$$

Of course, $\pi_1 X$ is just the usual fundamental group of X.

Remark 2.2. We must define the group operation on $\pi_n X$. Think of each map $f : \mathbb{S}^n \to X$ as a map

$$f : I^n \to X$$

such that $\partial[0,1]^n \mapsto x_0$. If we have maps $f, g : I^n \rightrightarrows X$, then define $f * g$ to be the concatenation map, which makes $\pi_n X$ into a group.[3]

[3] The proof is similar to the proof for $\pi_1 X$.

2.1. **Simple Results.**

Exercise 2.3. The group $\pi_n X$ is abelian for $n \geq 2$.

Example 2.4. Using the usual proof via covering spaces, $\pi_1 \mathbb{S}^1 \simeq \mathbb{Z}$.

Example 2.5. We have that $\pi_n \mathbb{S}^n \simeq \mathbb{Z}$ using the map $[f : \mathbb{S}^n \to \mathbb{S}^n] \mapsto \deg(f)$. Here, if f is a smooth map and $p \in \mathbb{S}^n$ is a regular value, then $f^{-1}(p)$ is a finite set, and we say $\deg(f)$ is the cardinality counted with orientation. Of course, f need not be smooth, but every map is homotopic to a smooth one. Alternatively, if ω is a volume form on $\mathbb{S}^n$, then

$$\deg(f) = \int_{\mathbb{S}^n} f^* \omega.$$

Example 2.6. Let $i < n$. Then, of course, $\pi_i \mathbb{S}^n$ is trivial.

Proof. We know $f : \mathbb{S}^i \to \mathbb{S}^n$ is homotopic to a smooth map g. Then, g is not surjective, per the corollary of Sard's theorem. We can then factor

$$\begin{tikzcd}
\mathbb{S}^1 \arrow[r, "g"] \arrow[d] & \mathbb{S}^n \\
\mathbb{S}^n \setminus * \arrow[r, "\simeq"] & \mathbb{R}^n \arrow[u]
\end{tikzcd}$$

Sicne $\mathbb{R}^n \simeq *$, we have that $g \simeq f \simeq c_{x_0}$. ∎

Example 2.7. What about $\pi_i \mathbb{S}^n$ for $i > n$? Let us focus on $\pi_i \mathbb{S}^1$.

Proof. Use the universal covering map $\exp : \mathbb{R} \to \mathbb{S}^1$. Then, we have

$$\begin{tikzcd}
 & \mathbb{R} \arrow[d, "\exp"] \\
\mathbb{S}^i \arrow[r, "f"'] & \mathbb{S}^1
\end{tikzcd}$$

Applying $\pi_i(-) : \mathsf{Top}_* \to \mathsf{Grp}$, we get

$$\begin{tikzcd}
 & 0 \arrow[d, "\exp_*"] \\
\mathbb{Z} \arrow[r, "f_*"'] & \mathbb{Z}
\end{tikzcd}$$

Certainly, $f_*(\pi_i \mathbb{S}^i) = 0$, so by monodromy, there is a (commuting) lift meaning f is nullhomotopic.[4]

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "\exp"] \\ \mathbb{S}^i \arrow[r, "f"'] \arrow[ur, dashed, "\exists!"] & \mathbb{S}^1 \end{tikzcd}$$

□

For $n > 1$, things become rather difficult to calculate, and in general, is an open problem. The earliest example is $\pi_3 \mathbb{S}^2$, due to Hopf. The proof is given by using the Hopf fibration $\eta : \mathbb{S}^3 \to \mathbb{S}^2$.

We can realize the Hopf fibration in the following way.

$$\begin{tikzcd} \mathbb{C}^\times \arrow[r, "\text{fibre}"] & \mathbb{C}^2 \setminus \{0\} \arrow[r, two heads, "\text{quotient}"] & \mathbb{CP}^1 \\ \mathbb{S}^1 \arrow[r] \arrow[u, hook] & \mathbb{S}^3 \arrow[r, "\eta"'] \arrow[u, hook] & \mathbb{S}^2 \arrow[u, equal] \end{tikzcd}$$

Proposition 2.8 (Hopf)**.** *We have $\pi_3 \mathbb{S}^2 \simeq \mathbb{Z}$, generated by η.*

2.2. **Stabilization.** Let X be a space. The *suspension* SX of X is the quotient of the cylinder $X \times I$ which identifies $S \times 0$ with a point and $S \times 1$ with another. For example, $S(\mathbb{S}^n) \simeq \mathbb{S}^{n+1}$. Given a map $f : \mathbb{S}^{k+n} \to \mathbb{S}^n$, then we get a new map $Sf : \mathbb{S}^{k+n+1} \to \mathbb{S}^{n+1}$.

Theorem 2.9 (Freudenthal Suspension)**.** *The function*

$$S : \pi_{k+n} \mathbb{S}^k \to \pi_{k+n+1} \mathbb{S}^{n+1}$$

is a group homomorphism that is

(*i*) *surjective if $k = n - 1$.*
(*ii*) *bijective if $k \leq n - 2$.*

As a consequence, the sequence

$$\pi_k \mathbb{S}^0 \xrightarrow{S} \pi_{k+1} X \xrightarrow{S} \pi_{k+2} X \xrightarrow{S} \pi_{k+3} \mathbb{S}^3 \xrightarrow{S} \cdots$$

stabilizes at $\mathbb{S}^{k+2}$. Further, the stable value is called the kth *stable homotopy group* of spheres. We will denote this by $\pi_k^{\text{st}} \mathbb{S}^0$.

Exercise 2.10. Show that the degree is multiplicative.

Exercise 2.11. We saw that any degree n polynomial $p \in \mathbb{C}[z]$ defines a self-map of $\mathbb{S}^2$. Show that this map has degree n.

Exercise 2.12. Show that the exponential map $\exp : \mathbb{R} \to \mathbb{S}^1$, or any covering space, for that matter, gives an isomorphism on homotopy groups π_n whenever $n \geq 2$.

[4] Again, we use that $\mathbb{R} \simeq *$.

Exercise 2.13. Show that any continuous function between smooth manifolds is homotopic to a smooth one.

3. Cobordisms

Let M be a manifold. An *orientation* on M is a function θ which continuously assigns to each point $p \in M$ an orientation on T_pM, as a vector space. Then, we call (M, θ) an orientable manifold.

Definition 3.1 (Manifold with Boundary)**.** An m-manifold with boundary $M \subseteq \mathbb{R}^n$ locally looks like $\mathbb{R}^m$ or $\mathbb{H}^m$, the upper half-space.

We define ∂M, for a manifold with boundary, to be the points $p \in M$ that look like $\mathbb{H}^m$.

Remark 3.2. The boundary ∂M of a manifold with boundary is a submanifold of dimension $\dim(M)-1$. Further, it can be framed.

Example 3.3. Some examples include $\mathbb{D}^n$ and $M \times I$.

Write M^n for a manifold M of dimension n.

Definition 3.4 (Cobordant)**.** Let M^n and N^m be compact with empty boundary. We say that M is cobordant to N if there exists a manifold with boundary W^{n+1} which is compact and so that $\partial W = M \coprod N$. We write $\sim_{\text{cob}}$ for the relation of being cobordant.

Example 3.5. We have that $\mathbb{S}^1 \sim_{\text{cob}} \mathbb{S}^1 \coprod \mathbb{S}^1$ via the pair of pants.

Definition 3.6. We define

$$\Omega_k(\mathbb{R}^n) = \left\{k\text{-dimensional submanifolds in } \mathbb{R}^n\right\} \Big/ \sim_{\text{cob}} .$$

Remark 3.7. The set $\Omega_k(\mathbb{R}^n)$ is an abelian group. We have the operation $[M]+[N] = [M \coprod N]$. The identity is $[\varnothing]$. Observe that $[M] = [\varnothing]$ if and only if M is ∂W for some compact W. Further, observe that

$$2[M] = [M] + [M] = [M \coprod M] = [\varnothing]$$

in $\Omega_k(\mathbb{R}^n)$, via the trivial cobordism. Thus, every element of $\Omega_k(\mathbb{R}^n)$ is 2-torsion.

3.1. **Framed Cobordism.**

Definition 3.8 (Framed Cobordism)**.** Let (K_0, β_0) and (K_1, β_1) be framed k-submanifolds of M. We say K_0 is framed cobordant to K_1 if there is a $W^{k+1} \subseteq M \times I$ that is compact with framing β such that

(i) $\partial W = K_0 \coprod K_1$.
(ii) β restricts to β_0 and β_1.

Definition 3.9. We define

$$\Omega_k^{\text{f}}(\mathbb{R}^n) = \left\{k\text{-submanifolds framed in } \mathbb{R}^n\right\} \Big/ \sim_{\text{fcob}} .$$

Remark 3.10. If we instead work with oriented cobordism, then we have a group $\Omega_k^{\text{or}}(\mathbb{R}^n) = \Omega_k^{\text{SO}}(\mathbb{R}^n)$.

Example 3.11. Certainly, $\Omega_0(\mathbb{R}^n) = \{[\varnothing], [*]\} \simeq \mathbb{Z}/2$.

Exercise 3.12. Compute $\Omega_0^{\text{or}}(\mathbb{R}^n)$.

Note that in $\Omega_k^{\text{or}}(\mathbb{R}^n)$, we get that $-[M] = [\overline{M}]$, where $\overline{M}$ is M with the opposite orientation.

Theorem 3.13 (Unstable)**.** *There is a group isomorphism*

$$\Omega_k^{\mathrm{f}}(\mathbb{R}^{n+k}) \xrightarrow{\mathrm{PT}} \pi_{n+k}\mathbb{S}^n,$$

where we call PT *the Pontryagin-Thom collapse map.*

Proof. We will call the inverse Φ. Think of $\mathbb{S}^n$ as the compactification of $\mathbb{R}^n$. Let $[f] \in \pi_{n+k}\mathbb{S}^n$ such that $f : \mathbb{S}^{n+k} \to \mathbb{S}^n$ is smooth, and so that $f(\infty) = \infty$ and $0 \in \mathbb{S}^n$ is a regular value. Consider the framed submanifold $f^{-1}(0) \hookrightarrow \mathbb{S}^{n+k}$. Define[5]

$$\Phi(f) = [f^{-1}(0)].$$

Now, let us show that Φ is surjective. Pick out $[M] \in \Omega_k^{\mathrm{f}}(\mathbb{R}^{n+k})$. Since M has a framing, there is a neighborhood U of M so that $U \cong M \times \mathbb{R}^n$. We can project and include to get

$$\begin{tikzcd} U \arrow[r, "\cong"] \arrow[rrr, bend right, "g"'] & M \times \mathbb{R}^n \arrow[r, two heads, "p"] & \mathbb{R}^n \arrow[r, hook] & \mathbb{S}^n \end{tikzcd}$$

Now, we define $\mathrm{PT}([M])$ as a map $\mathbb{S}^{n+k} \to \mathbb{S}^n$ by

$$x \mapsto \begin{cases} g(x), & x \in U \\ \infty, & x \notin U. \end{cases}$$

Check that $\Phi(\mathrm{PT}([M])) = [M]$. Checking that the map is monic is left as an exercise. $\square$

3.2. **Stable Version.** We have the commutative square

$$\begin{tikzcd} \Omega_k^{\mathrm{f}}(\mathbb{R}^{n+k}) \arrow[r, "\mathrm{PT}", "\simeq"'] \arrow[d, dashed, "E"'] & \pi_{n+k}\mathbb{S}^n \arrow[d, "S"] \\ \Omega_k^{\mathrm{f}}(\mathbb{R}^{n+k+1}) \arrow[r, "\mathrm{PT}", "\simeq"'] & \pi_{n+k+1}\mathbb{S}^{n+1} \end{tikzcd}$$

We saw that applying the suspension enough times yields an isomorphism, so we hope that the same is true of E.

Definition 3.14 (kth Framed Cobordism Group)**.** Let Ω_k^{f} be $\Omega_k^{\mathrm{f}}(\mathbb{R}^N)$ for $N \gg 0$.

Theorem 3.15 (Stable)**.** *We have an isomorphism* $\mathrm{PT} : \Omega_k^{\mathrm{f}} \to \pi_k^{\mathrm{st}}\mathbb{S}^0$.

Exercise 3.16. Recall that a compact manifold M is cobordant to $\varnothing$ if there exists a compact manifold with boundary W such that $\partial W \cong M$. SHow that the compactness of W is essential here, since every manifold M is the boundary of some W.

Exercise 3.17. Show that the boundary $\partial M \subseteq M$ of any manifold is a framed submanifold of dimension $\dim(M) - 1$.

[5]Show that the multitude of choices made here do not change the prescription.

Exercise 3.18. Compute the following groups of cobordisms:

(i) Ω_0.
(ii) Ω_0^{fr}.
(iii) Ω_1.
(iv) Ω_1^{or}.
(v) Ω_2.
(vi) Ω_2^{or}.

4. Low Dimensional Examples

Our hope is to now study $\pi_i \mathbb{S}^n$ for small $i \geq n$ using the Pontryagin-Thom construction. We will start with $\pi_n \mathbb{S}^n$, since we know what to expect. Well, $\pi_n \mathbb{S}^n$ is isomorphic to $\Omega_0^{\mathrm{f}}(\mathbb{R}^n)$. Each class is represented by a fixed set of points $x_1, \ldots, x_k \in \mathbb{R}^n$, and finding a framing of the normal bundle of $\mathbb{R}^n$ amounts to taking linear isomorphisms $\varphi_i : \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n$ for $1 \leq i \leq k$. Two such tuples represent some framed cobordism class. Since we are just taking signed sums, this means $\Omega_0^{\mathrm{f}}(\mathbb{R}^n) \simeq \mathbb{Z}$. Further, $\mathrm{PT} : \pi_n \mathbb{S}^n \to \Omega_0^{\mathrm{f}}(\mathbb{R}^n)$ is precisely the degree map.

Proposition 4.1 (Hopf)**.** *Recall the result that $\pi_3 \mathbb{S}^2 \simeq \mathbb{Z}$.*

Proof. The Hopf fibration $\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \xrightarrow{\eta} \mathbb{S}^2$ gives a piece of the long exact sequence so $\pi_3 \mathbb{S}^2 \simeq \mathbb{Z}$, as

$$\begin{tikzcd}
\pi_3 \mathbb{S}^1 \arrow[r] \arrow[d, equal] & \pi_3 \mathbb{S}^3 \arrow[r, "\simeq"] \arrow[d, "\simeq"] & \pi_3 \mathbb{S}^2 \arrow[r] \arrow[d, "\simeq"] & \pi_2 \mathbb{S}^1 \arrow[d, equal] \\
0 & \mathbb{Z} & \mathbb{Z} & 0
\end{tikzcd}$$

desired. □

By Freudenthal suspension, $S^n : \pi_3 \mathbb{S}^2 \to \pi_{n+3} \mathbb{S}^{n+2}$ is surjective.

Proposition 4.2. *We have $\pi_{n+1} \mathbb{S}^n \simeq \mathbb{Z}/2$ for $n \geq 3$.*

Proof. We will compute $\Omega_1^{\mathrm{f}}(\mathbb{R}^{n+1})$. Our strategy is to produce homomorphisms

$$\mathbb{Z}/2 \xrightarrow{J_n} \Omega_1^{\mathrm{f}}(\mathbb{R}^{n+1}) \xrightarrow{\omega} \mathbb{Z}/2,$$

where the composite is an isomorphism.[6] Now, elements of $\Omega_1^{\mathrm{f}}(\mathbb{R}^{n_1})$ are represented by framed links

$$\left(\coprod \mathbb{S}^1, \text{ framing} \right).$$

There is a framed cobordism $\mathbb{S}^1 \to \varnothing$, equipping $\mathbb{S}^1$ with the standard framing in $\mathbb{R}^3$, by using the disk $\mathbb{D}^2 \subseteq \mathbb{R}^3$ with a matching framing. We can obtain interesting elements by changing the framing. Any other framing φ of $\mathbb{S}^1$ in $\mathbb{R}^{n+1}$ gives

$$\mathbb{S}^1 \times \mathbb{R}^n \xrightarrow[\text{standard}]{\simeq} N\mathbb{S}^1 \xrightarrow[\varphi]{\simeq} \mathbb{S}^1 \times \mathbb{R}^n,$$

where $N\mathbb{S}^1$ is the normal bundle. Note that these isomorphisms can be distinct. For any $x \in \mathbb{S}^1$, we get a linear isomorphism $\varphi_x : \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n$. In other words,

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[6] If we know $\Omega_1^{\mathrm{f}}(\mathbb{R}^{n+1})$ is cyclic, then this forces is to be $\mathbb{Z}/2$

$$\begin{tikzcd} \mathbb{S}^1 \arrow[r] \arrow[dr] & \mathrm{GL}_n(\mathbb{R}) \\ & \mathrm{SO}(n) \arrow[u, "\text{up to homotopy}"'] \end{tikzcd}$$

The upshot is that there is a bijection

$$\left\{\text{framings of } \mathbb{S}^1 \subseteq \mathbb{R}^{n+1}\right\} \Big/ \simeq \quad \xrightarrow{\simeq} \quad [\mathbb{S}^1, \mathrm{SO}(n)] = \pi_1 \,\mathrm{SO}(n).$$

Now, $\pi_1 \,\mathrm{SO}(n)$ is $\mathbb{Z}$ for $N = 2$ and $\mathbb{Z}/2$ for $n \geq 3$. For fixed $\mathbb{S}^1 \subseteq \mathbb{R}^{n+1}$, define $\omega(\mathbb{S}^1) \in \mathbb{Z}/2$. For general framed links, add ω of the components, so we get our desired ω for $n \geq 3$. Also, note that we have the map

$$J_n : \pi_1 \,\mathrm{SO}(n) \to \Omega_1^{\mathrm{f}}(\mathbb{R}^{n+1})$$

given by $(\mathbb{S}^1 \xrightarrow{\alpha} \mathrm{SO}(n)) \mapsto \mathbb{S}^1 \subseteq \mathbb{R}^{n+1}$ with the framing corresponding to α.[7] □

Exercise 4.3. Show the Hopf fibration η in $\pi_{n+1}\mathbb{S}^n$ corresponds to the nontrivial framing in $\Omega_1^{\mathrm{f}}(\mathbb{R}^{n+1})$.

Proposition 4.4. *We have $\pi_{n+2}\mathbb{S}^n \simeq \mathbb{Z}/2$ for $n \geq 2$, generated by η^2.*[8]

To show this, a good discussion of framed surfaces inside $\mathbb{R}^{n+2}$ would be needed, which we simply do not have the time for.

Remark 4.5. Recall the Hopf map

$$S(\mathbb{C}^2) = \mathbb{S}^3 \xrightarrow{\eta} \mathbb{C}\mathrm{P}^1 \xrightarrow{\sim} \mathbb{S}^2.$$

Similarly, there is a Hopf map for quaternionic projective space

$$S(H^2) = \mathbb{S}^7 \xrightarrow{\nu} \mathbb{S}^4 \xrightarrow{\sim} H\mathrm{P}^1,$$

which is *not* nullhomotopic. This defines an element $\nu \in \pi_3^{\mathrm{st}}\mathbb{S}^0$.

Proposition 4.6. *We have $\pi_3^{\mathrm{st}}\mathbb{S}^0 \simeq \mathbb{Z}/24$, generated by ν.*

Sketch of Proof. The goal is, then, to describe the third framed cobordism group $\Omega_3^{\mathrm{f}} = \Omega_3^{\mathrm{f}}(\mathbb{R}^N)$, where $N \gg 0$. Well, let us look at $\mathbb{S}^3 \subseteq \mathbb{R}^N$. Certainly, $\mathbb{S}^3$ is a Lie group isomorphic to SU(2). Any Lie group G has a trivial tangent bundle. That is, $TG \simeq G \times \mathbb{R}^n$. In particular, $T\mathbb{S}^3$ is trivial, so the normal bundle $N\mathbb{S}^3$ will be trivial (stably, for $N \gg 0$). Thus, $\mathbb{S}^3$ with this framing defines an element of $\Omega_3^{\mathrm{f}}(\mathbb{R}^N)$. The claim is then that this element generates the group and is of order 24, thus completing the proof. □

Definition 4.7 (K_3 Surface)**.** A K_3 surface is a 2-dimensional complex manifold with trivial canonical bundle and trivial odd dimensional singular cohomology.

Example 4.8. For example, $x^4 + y^4 + z^4 + w^4 = 0$ in $\mathbb{C}\mathrm{P}^4$.

[7] This is a special case of the J-homomorphism $\pi_k \,\mathrm{SO}(n) \to \pi_{n+k}\mathbb{S}^n$.

[8] That is, $\mathbb{S}^{n+2} \xrightarrow{\eta} \mathbb{S}^{n+1} \xrightarrow{\eta} \mathbb{S}^n$, taking appropriate suspensions of the Hopf map.

Lemma 4.9. *Let X be a K_3 surface. Then, the Euler characteristic $\chi(X)$ is 24. Thus, there exists a vector field V on X with 24 zeros. Further, TX has the structure of $\mathbb{S}^3 = \mathrm{SU}(2) \subseteq \mathrm{SO}(4)$, so we have operators $\hat{i}, \hat{j}, \hat{k}$ acting on TX.*

Now, define $Y \subseteq X$ by adding small disks $\mathbb{D}^4$ around the zeros of C. The upshot is that Y has boundary

$$\partial Y = \coprod_{24} \mathbb{S}^3.$$

The vector field V is nowhere vanishing on Y, so we get four linearly independent vector fields $V, \hat{i}V, \hat{j}V, \hat{k}V$ on Y. Thus, $TY \simeq Y \times \mathbb{R}^4$, meaning Y is (stably) framed. What remains is to check that the framing on the copies of $\mathbb{S}^3$ is the standard Lie group framing.

Remark 4.10. Considering framed cobordism classes of stable normal manifolds, equipped with a map to a space X, forms a group $\Omega_n^U(X)$. In fact, this is a cohomology theory with coefficients in the homotopy groups of spheres. For further reading, see the works of Quillen, Connor-Floyd, and COCTALOS by Hopkins.

Exercise 4.11. Show that every Lie group G has trivial tangent bundle. Conclude that any Lie group can be given the structure of a stably framed manifold, and hence, under the Pontryagin-Thom isomorphism, defines an element of the stable homotopy groups of spheres.

Exercise 4.12. The Hopf map $\eta : \mathbb{S}^3 \to \mathbb{S}^2$ defines a generator for the group $\pi_3\mathbb{S}^2 \simeq \mathbb{Z}$. Under the Pontryagin-Thom isomorphism, it corresponds to an element of $\Omega_1^{\mathrm{fr}}(\mathbb{R}^3)$, i.e., to a cobordism class of framed 1-submanifolds of $\mathbb{R}^3$. Show that this submanifold can be taken to be the usual *unknot* $\mathbb{S}^1$ in $\mathbb{R}^3$. What is the correct framing? Try working out the Pontrytagin-Thom map for this submanifold explicitly and see the relation to the Hopf map directly.

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