MATRIX LIE GROUPS

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Contents

Heuristically, a *Lie group* is a group paired with a smooth structure. By differentiating, we obtain a *Lie algebra*, which is, in some sense, the infinitesimal analogue of a group. What about integration? How much information about the group structure can we recover from the Lie algebra?

1. Part I

Recall that we write $\mathrm{GL}_n(\mathbb{C})$ for the group of $n \times n$ invertible complex matrices. Well, $\mathrm{GL}_n(\mathbb{C}) \subseteq \mathbb{M}_n(\mathbb{C})$, which is isomorphic to $\mathbb{C}^{n^2}$. We can equip $\mathbb{C}^{n^2}$ with the usual topology, thus giving $\mathrm{GL}_n(\mathbb{C})$ the subspace topology.

Definition 1.1 (Matrix Lie Group)**.** A matrix Lie group G is a subgroup of $\mathrm{GL}_n(\mathbb{C})$ which is closed in the subspace topology.

Notably, since $\mathrm{GL}_n(\mathbb{C})$ is, in particular, closed, it is a matrix Lie group.

Example 1.2. The general linear group of $\mathbb{R}^n$, written $\mathrm{GL}_n(\mathbb{R})$, embeds into $\mathrm{GL}_n(\mathbb{C})$. That is, we follow the restriction $A \in \mathbb{M}_n(\mathbb{R}) \hookrightarrow \mathbb{M}_n(\mathbb{C})$. The subgroup $\mathrm{GL}_n(\mathbb{R})$ is a closed subset.

Example 1.3. The orthogonal group on $\mathbb{R}^n$, denoted $\mathrm{O}(n)$, is a matrix Lie group, restricting to $\mathbb{M}_n(\mathbb{R})$, as before, and requiring that $A^t A = I_n$.

Example 1.4. The special linear group of $\mathbb{R}^n$, denoted $\mathrm{SL}_n(\mathbb{R})$, is a matrix Lie group by restricting to real matrices with $\det(A) = 1$.

Example 1.5. The special orthogonal group $\mathrm{SO}(n)$ is a matrix Lie group, restricting to the real, orthogonal, and special operators.

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1.1. **Matrix Exponential Map.** Using the usual power series expansion of the exponential, we define a useful tool for working with matrix Lie groups.

Definition 1.6 (Matrix Exponential)**.** Let $X \in \mathbb{M}_n(\mathbb{C})$. The matrix exponential $\exp(X)$ is

$$\sum_{k=0}^{\infty} \frac{X^k}{k!}.$$

Exercise 1.7. Check that $\exp(-)$ is well-defined and continuous.

Proposition 1.8. *Let $X, Y \in \mathbb{M}_n(\mathbb{C})$ and $C \in \mathrm{GL}_n(\mathbb{C})$. Then,*

(*i*) $\exp\big(CXC^{-1}\big) = C\exp(X)C^{-1}$
(*ii*) $\exp(X) \in \mathrm{GL}_n(\mathbb{C})$ *and* $\exp(X)^{-1} = \exp(-X)$.
(*iii*) $\det\big(\exp(X)\big) = \exp(\mathrm{tr}(X))$.
(*iv*) *if* $XY = YX$, *then* $\exp(X + Y) = \exp(X)\exp(Y)$.
(*v*) $\exp(X)^m = \exp(mX)$.

Proof. Left as an exercise. □

Recall that

$$\log(x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-1)^k}{k}$$

converges absolutely for $|x - 1| < 1$.

Definition 1.9 (Matrix Logarithm)**.** For $\|A - I_n\| < 1$ with $A \in \mathrm{GL}_n(\mathbb{C})$, we have[1]

$$\log(A) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(A - I_n)^k}{k}.$$

Theorem 1.10. *Let $A \in \mathrm{GL}_n(\mathbb{C})$ and $X \in \mathbb{M}_n(\mathbb{C})$, if $\|A - I_n\| < 1$,*

$$\exp\big(\log(A)\big) = A,$$

and if $\|X\| < \log 2$, then

$$\log\big(\exp(X)\big) = X.$$

Proposition 1.11. *Let $X, Y \in \mathbb{M}_n(\mathbb{C})$. Then, we have*

$$(\exp(X/m)\exp(Y/m)) \xrightarrow[m\to\infty]{} \exp(X + Y).$$

Proof. Define

$$A_m = \exp(X/m)\exp(Y/m) = I + X/m + Y/m + O(1/m^2).$$

Since $X/m, Y/m \to 0$ as $m \to \infty$, we have that $A_m \to I_n$ as $m \to \infty$. Thus, for large enough m,

$$\|A_m - I_n\| < 1,$$

meaning we can apply our logarithm. Now,

$$\log(A_m) = X/m + Y/m + O(1/m^2).$$

Thus,

$$A_m = \exp\big(\log(A_m)\big) = \exp\big(X/m + Y/m + O(1/m^2)\big),$$

[1] We mean the Hilbert-Schmidt norm, but the operator norm could work, too.

so by (v),

$$A_m^m = \exp(X + Y + O(1/m)) \xrightarrow[m\to\infty]{} \exp(X + Y).$$

$\square$

1.2. **Lie Algebras.** We now define the so-called infinitesimal analogue of the Lie group.

Definition 1.12 (Lie algebra)**.** A Lie algebra over $\mathbb{R}$ consists of a $\mathbb{R}$-linear space $\mathfrak{g}$, along with a bilinear map

$$[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

such that $[-,-]$ satisfies

(i) *antisymmetry*: for all $X, Y \in \mathfrak{g}$, $[X, Y] = -[Y, X]$.
(ii) *the Jacobi identity*: for all $X, Y, Z \in \mathfrak{g}$,

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

Example 1.13. Let $\mathfrak{A}$ be an associative $\mathbb{R}$-algebra. Then, $\mathfrak{A}$ with the commutator bracket

$$[a, b] = ab - ba, \quad a, b \in \mathfrak{A},$$

is a Lie algebra.

Exercise 1.14. Check that $(\mathfrak{A}, [-,-])$, as above, is a Lie algebra.

Example 1.15. Let $\mathfrak{A} = \mathbb{M}_n(\mathbb{C})$. Then, the associated Lie algebra, using the commutator bracket, is denoted $\mathfrak{gl}_n(\mathbb{C})$.

Definition 1.16. Let $\mathfrak{g}$ be a Lie algebra. A Lie subalgebra is a $\mathbb{R}$-linear subspace $\mathfrak{h} \subseteq \mathfrak{g}$ so that $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$.

Exercise 1.17. A Lie subalgebra is a Lie algebra.

Example 1.18. We define a Lie subalgebra

$$\mathfrak{sl}_n(\mathbb{C}) = \{X \in \mathfrak{gl}_n(\mathbb{C}) : \mathrm{tr}(X) = 0\} \subseteq \mathfrak{gl}_n(\mathbb{C}).$$

Proof. Observe that, since the trace is cyclic,

$$\mathrm{tr}[X, Y] = \mathrm{tr}(XY) - \mathrm{tr}(YX) = 0,$$

so $\mathfrak{sl}_n(\mathbb{C})$ is a bracket-invariant $\mathbb{R}$-subspace. $\square$

Example 1.19. The following are Lie subalgebras:

(i) $\mathfrak{u}(n) = \{X \in \mathfrak{gl}_n(\mathbb{C}) : X^* = -X\}$.
(ii) $\mathfrak{o}(n) = \{X \in \mathfrak{gl}_n(\mathbb{R}) : X^t = -X\}$.
(iii) $\mathfrak{su}(n) = \mathfrak{sl}_n(\mathbb{C}) \cap \mathfrak{u}(n)$.
(iv) $\mathfrak{so}(n) = \mathfrak{sl}_n(\mathbb{R}) \cap \mathfrak{o}(n)$.

Definition 1.20 (Adjoint Map)**.** Let $\mathfrak{g}$ be a Lie algebra. Then, for every $X \in \mathfrak{g}$, define $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ by $Y \mapsto [X, Y]$. Then, we may define the adjoint map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ by $X \mapsto \mathrm{ad}_X$.

Remark 1.21. The Jacobi identity is equivalent to both of the following:

(i) $\mathrm{ad}_X[Y, Z] = [\mathrm{ad}_X(Y), Z] + [Y, \mathrm{ad}_X(Z)]$.
(ii) $\mathrm{ad}_{[X,Y]}(Z) = [\mathrm{ad}_X, \mathrm{ad}_Y](Z)$.[2]

[2] The second bracket here is the commutator bracket in $\mathrm{End}(\mathfrak{g})$.

1.3. **Lie Algebras of Lie Groups.** We want to associate to each matrix Lie group a Lie algebra which gives the directions inside $\mathrm{GL}_n(\mathbb{C})$ that stay in, or are tangent to, G.

Definition 1.22 (Associated Lie Algebra)**.** Let $G \subseteq \mathrm{GL}_n(\mathbb{C})$ be a matrix Lie group. Then, its associated Lie algebra $\mathfrak{g}$ is

$$\mathfrak{g} = \{X \in \mathfrak{gl}_n(\mathbb{C}) : \text{ for all } t \in \mathbb{R}, \exp(tX) \in G\}.$$

Proposition 1.23. *Let G be a matrix Lie group and $\mathfrak{g}$ be its Lie algebra. Then, for all $X, Y \in \mathfrak{g}$ and $C \in G$,*

(*i*) *for all $A \in G$, $AXA^{-1} \in \mathfrak{g}$.*
(*ii*) *for all $s \in \mathbb{R}$, $sX \in \mathfrak{g}$.*
(*iii*) *$X + Y \in \mathfrak{g}$.*
(*iv*) *$XY - YX \in \mathfrak{g}$.*

That is, $\mathfrak{g}$ is a $\mathbb{R}$-linear space that becomes a Lie algebra using the commutator bracket.

Proof. We have that for all $t \in \mathbb{R}$,

$$\exp\bigl(tAXA^{-1}\bigr) = A\exp(tX)A^{-1} \in G,$$

so $\mathfrak{g}$ is closed under G-conjugation. For all $s \in \mathbb{R}$,

$$\exp(t(sX)) = \exp((ts)X) \in G,$$

so $sX \in \mathfrak{g}$. Observe that

$$\exp(t(X+Y)) = \lim_{m\to\infty} (\exp(tX/m)\exp(tY/m))^m \in G,$$

and since G is closed, $\exp(t(X+Y)) \in \mathfrak{g}$, so $\mathfrak{g}$ is closed under addition. Finally, since we have that $\mathfrak{g}$ is a $\mathbb{R}$-linear space, observe that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(tX)Y\exp(-tX) = XY - YX,$$

and the first expression is in $\mathfrak{g}$, so $XY - YX \in \mathfrak{g}$ by closure. □

Proposition 1.24. *The Lie algebra of* $\mathrm{GL}_n(\mathbb{C})$ *is* $\mathfrak{gl}_n(\mathbb{C})$ *and* $\mathrm{SL}_n(\mathbb{C})$ *is* $\mathfrak{sl}_n(\mathbb{C})$*.*

Proof. The first is trivial. Recall that $A \in \mathrm{SL}_n(\mathbb{C})$ if it is invertible and $\det(A) = 1$. Well, for all $t \in \mathbb{R}$,

$$\det\bigl(\exp(tX)\bigr) = \exp(t\,\mathrm{tr}(X)),$$

so applying the derivative at $t = 0$, we get $\mathrm{tr}(X) = 0$. □

Definition 1.25 (Exponential)**.** The exponential map of a Lie group G is

$$\exp\big|_{\mathfrak{g}} = \exp : \mathfrak{g} \to G.$$

Theorem 1.26. *Let U_ε denote the ε-ball at* 0 *in $\mathfrak{g}$, using the standard metric. There exists* $0 < e < \log 2$ *such that the restriction of the exponential to $U_\varepsilon \to \exp(U_\varepsilon)$ is a homeomorphism.*

Proof. Let V_ε be the image $\exp(U_\varepsilon)$. Let D denote $\mathfrak{g}^\perp$ in $\mathfrak{gl}_n(\mathbb{C})$. Define a map

$$\Phi : \mathfrak{g} \oplus D \to \mathrm{GL}_n(\mathbb{C})$$

by $(X, Y) \mapsto \exp(X)\exp(Y)$. Note that the differential of Φ at the origin is

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi(tX, 0) = X,$$

and likewise for Y on the second component. Thus, the differential of Φ has $D_{(0,0)}\Phi = \mathrm{id}$. By the inverse function theorem, there is an $\varepsilon > 0$ such that the restriction of Φ to U_ε is a homeomorphism. Then, to show $\exp = \Phi|_{U_\varepsilon \oplus 0}$ is a local homeomorphism, we simply need to show that V_ε is open. Suppose V_ε is not open. Then, there is a sequence $\{A_m : m \in \mathbb{N}\}$ converging to I that is not in V_ε, so $\log(A_m) \notin \mathfrak{g}$. For large enough m, A_m lies in the image of the local homeomorphism Φ, so $A_m = \exp(X_m)(Y_m)$, where $\{X_m : m \in \mathbb{N}\} \subseteq \mathfrak{g}$ and $\{Y_m : m \in \mathbb{N}\} \subseteq D$. Then, $\exp(Y_m) \in G$. Observe that $Y_m/\|Y_m\|$ is on the unit sphere in D, which is compact. Thus, we may assume that $Y_m/\|Y_m\| \to Y$ on the unit sphere in D. Let us try and show that $Y \in \mathfrak{g}$.[3] Since $A_m \to I$, $Y_m \to 0$. Define k_m to be hte floor of $t/\|Y_m\|$. Then,

$$|k_m \|Y_m\| - t| \leq \|Y_m\| \to 0.$$

Thus,

$$\exp(tY) = \lim_{m\to\infty} \exp(k_m Y_m) = \lim_{m\to\infty} \exp(Y_m)^{k_m} \in G,$$

as G is closed. Yet, this would suggest $Y \in \mathfrak{g}$, but $\mathfrak{g} \cap D = 0$, so we have our contradiction. □

Proposition 1.27. *Let G be a connected matrix Lie group. Every element $A \in G$ may be written as*

$$A = \prod_{i=1}^{n} \exp(X_i),$$

where $X_1, \dots, X_n \in \mathfrak{g}$.

Definition 1.28 (Commutative Lie Algebra)**.** A Lie algebra $\mathfrak{g}$ is commutative if for all $X, Y \in \mathfrak{g}$, the bracket $[X, Y] = 0$.

Proposition 1.29. *Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. Suppose G is abelian. Then, so is $\mathfrak{g}$. If G is connected, then the converse also holds.*

Exercise 1.30. Let $\mathfrak{g}$ be a two-dimensional noncommutative Lie algbera. Show that there exists a basis $\{x, y\}$ for $\mathfrak{g}$ such that $[x, y] = x$.

Exercise 1.31. The Heisenberg group is defined as

$$\mathrm{Heis} = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_3(\mathbb{R}) : a, b, c \in \mathbb{R} \right\}.$$

Show that this is a group. Further, check that $\mathrm{Heis} \subseteq \mathrm{GL}_3(\mathbb{R})$ is closed. Then, show that

$$\mathfrak{heis} = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \in \mathrm{GL}_3(\mathbb{R}) : a, b, c \in \mathbb{R} \right\}$$

is the Lie algebra of $\mathfrak{heis}$. Finally, show that the exponential map $\exp : \mathfrak{heis} \to \mathrm{Heis}$ is epic. This group is called the Heisenberg group because the Lie bracket on its Lie algebra is $[X, Y] = Z$, for a suitable basis, which is reminiscent of the commutation relation between position and momentum.

Exercise 1.32. Show that the Lie algebra of each usual matrix Lie group is the correspondingly named Lie algebra defined in this section.

[3] That is, we want to show that for all $t \in \mathbb{R}$, the exponential map $\exp(tY) \in G$.

2. Part II

We now hope to answer the question of whether we can produce a (matrix) Lie group from a Lie algebra. We begin our discussion with the Baker-Campbell-Hausdorff (BCH) formula.

2.1. Baker-Campbell-Hausdorff.

Our goal is to write the product $\exp(X)\exp(Y)$ of two exponentials as $\exp(Z)$.[4] Recall that there exists $U_\varepsilon \subseteq \mathfrak{g}$ such that $\exp : U_\varepsilon \to V_\varepsilon$ is a local homeomorphism, so for small X, Y, we may write

$$Z = \log\bigl(\exp(X)\exp(Y)\bigr).$$

Remark 2.1. Let $B_1(1) \subseteq \mathbb{C}$ be the open ball of radius 1. Define

$$g(z) = \frac{\log z}{1 - z^{-1}} = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(z-1)^m}{m(m+1)}.$$

The theorem we hope to prove is as follows.

Theorem 2.2 (BCH, Integral Form)**.** *There exists $\varepsilon > 0$ such that for all $X, Y \in \mathfrak{gl}_n(\mathbb{C})$ so that $\|X\|, \|Y\| < \varepsilon$, we have*

$$\log\bigl(\exp(X)\exp(Y)\bigr) = X + \int_0^1 g(\exp(\mathrm{ad}_X)\exp(\mathrm{ad}_Y))\,\mathrm{d}t\,.$$

Observe that we have

$$\frac{1 - e^{-z}}{z} = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{(n+1)!},$$

so we can define

$$\frac{I_n - \exp(-A)}{A} = \sum_{n=0}^{\infty} (-1)^n \frac{A^n}{(n+1)!}.$$

Lemma 2.3. *Let $J \subseteq \mathbb{R}$ be an interval and $X : J \to \mathbb{M}_n(\mathbb{C})$ be smooth. Then,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\exp(X(t)) = \exp(X(t))\left(\frac{I_n - \exp\bigl(-\mathrm{ad}_{X(t)}\bigr)}{\mathrm{ad}_{X(t)}}\left(\frac{\mathrm{d}X}{\mathrm{d}t}\right)\right).$$

Proof. Left as an exercise. □

Lemma 2.4. *Let $A \in \mathrm{GL}_n(\mathbb{C})$ and define $\mathrm{Ad}_A : \mathfrak{gl}_n(\mathbb{C}) \to \mathfrak{gl}_n(\mathbb{C})$ by $X \mapsto AXA^{-1}$. Then,*

$$\mathrm{Ad}_{\exp(X)} = \exp(\mathrm{ad}_X).$$

Proof. Define $A(t) = \mathrm{Ad}_{\exp(tX)}$. Likewise, define $B(t) = \exp(t\,\mathrm{ad}_X)$. Then,

$$\begin{aligned}
A'(t)(Y) &= \frac{\mathrm{d}}{\mathrm{d}t}(\exp(tX)Y\exp(-tX)) \\
&= \exp(tX)XY\exp(-tX) - \exp(tX)YX\exp(-tX) \\
&= \exp(tX)[X, Y]\exp(-tX) \\
&= A(t)\,\mathrm{ad}_X(Y).
\end{aligned}$$

Then, $A(0) = \mathrm{Ad}_{I_n} = I_n$. Further,

$$B'(t)(Y) = B(t)\,\mathrm{ad}_X(Y),$$

[4] Of course, in general, this is not possible.

and $B(0) = \exp(0) = I_n$. Thus, both satisfy the same ODE, meaning $A(t) = B(t)$ for all $t \in \mathbb{R}$. Thus,

$$\mathrm{Ad}_{\exp(X)} = A(1) = B(1) = \exp(\mathrm{ad}_X).$$

□

Proof of BCH. Let $Z(t) = \log\bigl(\exp(X)\exp(tY)\bigr)$. Then, by the first lemma,

$$\exp(-Z(t))\frac{\mathrm{d}}{\mathrm{d}t}\exp(Z(t)) = \left(\frac{I - n - \exp\bigl(\mathrm{ad}_{Z(t)}\bigr)}{\mathrm{ad}_{Z(t)}}\right)\left(\frac{\mathrm{d}Z}{\mathrm{d}t}\right).$$

Yet,

$$\exp(-Z(t))\frac{\mathrm{d}}{\mathrm{d}t}\exp(Z(t)) = \exp(-tY)\exp(-X)\exp(X)\exp(tY)Y = Y,$$

so

$$\left(\frac{I_n - \exp\bigl(-\mathrm{ad}_{Z(t)}\bigr)}{\mathrm{ad}_{Z(t)}}\right)\left(\frac{\mathrm{d}Z}{\mathrm{d}t}\right) = Y.$$

Then,

$$D\left(\frac{I_n - \exp(-A)}{A}\right)h = \sum_{n=0}^{\infty}(-1)^n\frac{nA^{n-1}h}{(n+1)!}.$$

Now, for $A = 0$, this becomes $-h/2 \neq 0$. By the inverse function theorem, we can invert

$$\frac{I_n - \exp\bigl(-\mathrm{ad}_{Z(t)}\bigr)}{\mathrm{ad}_{Z(t)}}$$

for small $Z(t)$. Write

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = \left(\frac{I_n - \exp\bigl(-\mathrm{ad}_{Z(t)}\bigr)}{\mathrm{ad}_{Z(t)}}(Y).\right)$$

Now,

$$\mathrm{Ad}_{\exp(Z(t))} = \mathrm{Ad}_{\exp(X)\exp(ty)} = \mathrm{Ad}_{\exp(X)}\mathrm{Ad}_{\exp(tY)},$$

so by the second lemma,

$$\exp\bigl(\mathrm{ad}_{Z(t)}\bigr) = \exp(\mathrm{ad}_X)\exp(t\,\mathrm{ad}_Y).$$

We can thus use the logarithm to get

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = \left(\frac{I_n - (\exp(\mathrm{ad}_X)\exp(t\,\mathrm{ad}_Y))^{-1}}{\log\bigl(\exp(\mathrm{ad}_X)\exp(t\,\mathrm{ad}_Y)\bigr)}\right)(Y) = g(\exp(\mathrm{ad}_X)\exp(t\,\mathrm{ad}_Y))(Y).$$

By the fundamental theorem of calculus,

$$Z(1) = Z(0) + \int_0^1 \frac{\mathrm{d}Z}{\mathrm{d}t}\,\mathrm{d}t = X + \int_0^1 g(\exp(\mathrm{ad}_X)\exp(t\,\mathrm{ad}_Y))(Y)\,\mathrm{d}t\,,$$

as desired. □

Remark 2.5 (BCH, Series Form). We have that $g(z) = 1 + \frac{1}{2}(z-1) +$ higher order terms. Likewise,

$$\exp(\mathrm{ad}_X)\exp(t\,\mathrm{ad}_Y) - I = \mathrm{ad}_X + t\,\mathrm{ad}_Y + \text{higher order terms}.$$

Thus, by the BCH formula,

$$\log\bigl(\exp(X)\log(Y)\bigr) = X + Y + \frac{1}{2}[X,Y] + \text{ higher order terms}.$$

2.2. **Lie Subgroups and Subalgebras.** Recall that if $\mathfrak{g}$ is a Lie algebra, then a Lie subalgebra is a linear subspace $\mathfrak{h} \subseteq \mathfrak{g}$ such that for all $X, Y \in \mathfrak{h}$, $[X, Y] \in \mathfrak{h}$.

Definition 2.6 (Connected Lie Subgroup)**.** If G is a matrix Lie group with Lie algebra $\mathfrak{g}$, then a subset $H \subseteq G$ is a connected Lie subgroup if

(i) $H \leq G$.
(ii) the associated Lie algebra $\mathfrak{h}$ is a Lie subalgebra.
(iii) every element $A \in H$ can be written as

$$A = \prod_{i=1}^{m} \exp(X_i), \quad X_i \in \mathfrak{h}.$$

Theorem 2.7. *Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Lie subalgebra. Then, there exists a unique connected Lie subgroup $H \subseteq G$ with Lie algebra $\mathfrak{h}$.*

Example 2.8. Consider the 2-torus

$$\mathbb{T}^2 = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\varphi} \end{pmatrix} : \theta, \varphi \in \mathbb{R} \right\} \subseteq \mathrm{GL}_2(\mathbb{C}).$$

We have $\mathfrak{t} = \langle X, Y \rangle$ where $[X, Y] = 0$. Let $\alpha \in \mathbb{R}^\times$ and let $\mathfrak{h}_\alpha = \langle X + \alpha Y \rangle$. Then,

$$H_\alpha = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\alpha\theta} \end{pmatrix} : \theta \in \mathbb{R} \right\}$$

is a connected Lie subgroup with Lie algebra $\mathfrak{h}_\alpha$. If α is a 2π-scaled rational, then $H_\alpha \simeq \mathbb{S}^1$, so H_α would be a closed subgroup. However, if α is 2π-scaled irrational, then $H_\alpha \simeq \mathbb{R}$, and $\overline{H_\alpha} = \mathbb{T}^2$.

Given a Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, where $\mathfrak{g}$ is associated to $G \subseteq \mathrm{GL}_n(\mathbb{C})$, we can construct our corresponding connected Lie subgroup

$$H = \{\exp(X_1) \cdots \exp(X_m) : m \geq 0 \text{ and } X_i \in \mathfrak{h}\}.$$

We have that the third condition is satisfied, by definition. Since H is the subgroup generated by $\exp(\mathfrak{h})$, the subgroup condition is also satisfied. The tricky part is showing that $\mathrm{Lie}(H) = \mathfrak{h}$. Pick a complement $N \subseteq \mathfrak{g}$ of $\mathfrak{h}$ so that $\mathfrak{g} = \mathfrak{h} \oplus N$. Since the exponential map is a diffeomorphism around the origin, we have open neighborhoods $0 \in U \subseteq \mathfrak{h}$ and $0 \in V \subseteq N$ so that $f : U \times V \to f(U \times V)$ sending $(X, Y) \to \exp(X)\exp(Y)$ is a diffeomorphism. Note that $\mathfrak{h}$ is contained in the Lie algebra associated to H, so we want to show the opposite inclusion.

Suppose $Y \in \mathrm{Lie}(H)$. Then, $\exp(tY) \in H$ for all $t \in \mathbb{R}$. We must show that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(tY) \in \mathfrak{h}$$

Since $\exp(0Y) = \mathrm{id}$ in $f(U \times V)$, we find two maps $\mathcal{C}^1$ $X(t) \in \mathfrak{h}$ and $Z(t) \in N$ such that

$$\exp(X(t))\exp(Z(t)) = \exp(tY).$$

Then,

$$Y = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(tY) = X'(0) + Z'(0),$$

so we must simply show that $Z'(0)$ vanishes.

Lemma 2.9. *The subset defined by*

$$\{Z \in V : \exp(Z) \in H\} \subseteq N$$

is countable.

Definition 2.10 (Rational Element)**.** Let $\mathfrak{h}$ be a Lie algebra. Fix a basis β of $\mathfrak{h}$. A rational element of $\mathfrak{h}$ is an element with rational coefficients with respect to β.

Observe that if $\dim \mathfrak{h} = n$, so that $\mathfrak{h} \simeq \mathbb{R}^n$, then the rational elements of $\mathfrak{h}$ are in bijective correspondence with $\mathbb{Q}^n$. In particular, there are only countably many. Furthermore, the rational elements lie $\|-\|_{\mathbb{R}^n}$-dense in $\mathfrak{h}$.

Lemma 2.11. *Let $\mathfrak{h}$ be a Lie algebra with a fixed basis. Let $\delta > 0$ and take $A \in H$. Then, there are rational elements $R_1, \ldots, R_m \in \mathfrak{h}$ and an $X \in \mathfrak{h}$ such that*

$$A = \exp(R_1) \cdots \exp(R_m) \exp(X)$$

and $\|X\| < \delta$.

Proof. We prove this in two steps. Given $X \in \mathfrak{h}$, we have

$$\exp(X/h)^h = \exp(X),$$

so we can write

$$A = \exp(X_1) \cdots \exp(X_N)$$

with all having norm less than δ. The second step is by induction on N. Of course, for $N = 1$, we are done. By BCH, there is some $\varepsilon > 0$ such that for $\|X\|, \|Y\| < \varepsilon$, $C(X, Y)$ exists[5] and $\exp(X)\exp(Y) = \exp(C(X, Y))$.

Thus, C is continuous, so we may assume $\delta < \varepsilon$, and for $\|X\|, \|Y\| < \delta$, we have $\|C(X, Y)\| < \varepsilon$. Thus, we have rational $R_1, \ldots, R_m$ and X with $\|X\| < \delta$:

$$A = \exp(X_1) \cdots \exp(X_N) \exp(X_{N+1}) = \exp(R_1) \cdots \exp(R_m) \exp(X) \exp(X_N + 1).$$

This is $\exp(R_1) \cdots \exp(R_m) \exp(C(X, X_{N+1}))$, and $\|C(X, X_{N+1})\| < \varepsilon$. Since rational elements are dense, we may find a sequence $\{R^j : j \in \mathbb{N}\}$ which converges to $C(X, X_{N+1})$ as $j \to \infty$. Since $C(-Z, Z) = 0$, we have

$$C(R^j, C(X, X_{N+1})) \xrightarrow[j \to \infty]{} 0.$$

Therefore, there exists a rational R_{m+1} such that

$$\|C(-R_{m+1}, C(X, X_{n+1}))\| < \delta.$$

We can now write

$$A = \prod_{i=1}^{m} \exp(R_i) \cdot \exp(C(X, X_{N+1}))$$

as

$$\prod_{i=1}^{m+1} \exp(R_i) \cdot \exp(C(-R_{m+1}, C(X, X_{N+1}))).$$

$\square$

[5]Here, $C(X, Y)$ is the map from the BCH formula.

Proof of Countability. We have that $A \in f(U \times V)$, so we can write $A = \exp(X)\exp(Y)$ with $X \in U$ and $Y \in V$, uniquely. Let $\delta > 0$ be small enough such that for $\|X\|, \|Y\| < \delta$, we have $X \in U, Y \in V$, and $C(X,Y)$ exists and is in $U \times V$. Now, suppose

$$\exp(Z_j) = \prod_{i=1}^{m} \exp(R_i) \cdot \exp(X_j), \quad X_j \in \mathfrak{h}$$

in $\exp(V)$, for $j \in \{1,2\}$. Then,

$$\exp(-Z_1) = \exp(-X_1)\exp(X_2)\exp(-Z_2) = \exp(C(-X_1, X_2))\exp(-Z_2)$$

and $C(X_1, X_2) \in U$. Since f is a bijection, $Z_1 = Z_2$ and $C(-X_1, X_2) = 0$, so $X_1 = X_2$. Any $\exp(Z) \in H$ has a representation

$$\exp(Z) = \prod_{i=1}^{n} \exp(R_i) \cdot \exp(X),$$

where the R_i are rational and $X \in \mathfrak{h}$ with $\|X\| < \delta$. Thus, $E = V \cap \log^{-1}(H)$ is countable. $\square$

Proof of Theorem. For $Y \in \mathrm{Lie}(H)$, we have $\exp(tY) = \exp(X(t))\exp(Z(t))$, as before, and $\exp(tY), \exp(X(t)) \in H$, so $\exp(Z(t))$ is in H too. Thus, $Z(t)$ takes values in the intersection E, meaning $Z(t)$ is constant. That is, $Z'(0) = 0$ and $Y = X'(0) \in \mathfrak{h}$, as we had hoped. $\square$

2.3. **Lie's Third Theorem.** We can split our introductory question in two:

(i) Can we embed any Lie algebra into $\mathfrak{gl}_n(\mathbb{C})$?
(ii) Is every connected Lie subgroup a matrix Lie group?

Fortunately, both have positive answers. Unfortunately, both require some heavy machinery.

Theorem 2.12 (Ado)**.** *Every finite dimensional real Lie algebra $\mathfrak{g}$ can be identified with a real Lie subalgebra of $\mathfrak{gl}_n(\mathbb{C})$, for some sufficiently large n.*

That is, every finite dimensional real Lie algebra admits a faithful representation.

Theorem 2.13 (Goto)**.** *Every connected Lie subgroup of* $\mathrm{GL}_n(\mathbb{C})$ *is a matrix Lie subgroup.*

Theorem 2.14 (Lie's Third Theorem)**.** *Let $\mathfrak{g}$ be a finite dimensional real Lie algebra. Then, there exists a matrix Lie group G so that* $\mathrm{Lie}(G) = \mathfrak{g}$.

Proof. By Ado's theorem, we may view the given $\mathfrak{g}$ in $\mathfrak{gl}_n(\mathbb{C})$. Then, there is a $G \subseteq \mathrm{GL}_n(\mathbb{C})$ which is a connected Lie subgroup. By Goto's theorem, this is a matrix Lie group. $\square$

Exercise 2.15. Show that all connected Lie subgroups of the 2-torus $\mathbb{T}^2$ are $\{I\}, H_\alpha, \mathbb{T}^2$, where α takes its usual values *and* $0, \infty$.

Proof. Evidently we have a flag

$$\mathfrak{i} \hookrightarrow \mathfrak{h}_\alpha \hookrightarrow \mathfrak{t}$$

inside the Lie algbera of the torus. By the correspondence between connected Lie subgroups and Lie subalgebras, this means that each of the above uniquely corresponds to a connected Lie subgroup. By drawing a figure, it is clear that these are $\{I\}$, H_α, and $\mathbb{T}^2$ itself. $\square$

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