Hidden Subgroups and Quantum Computation

Lecture 04

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Summer 2025
Illinois Mathematics and Science Academy

July 11, 2025

Overview

1. Setting the Stage
2. Quantum Circuit Model
3. Hidden Subgroups
4. Outlook

Today we will describe what a quantum circuit is, using the mathematical language we have built thus far.

Then, we can discuss some preliminaries on the HSP.

Suppose our quantum processing unit (QPU) has n qubits. Recall that by the multiple systems axiom, this means our state space $\mathcal{H}$ for the QPU is isomorphic to the n-fold tensor product

$$\mathcal{H} \simeq \mathbb{C}^2 \underbrace{\otimes \cdots \otimes}_{n \text{ times}} \mathbb{C}^2 = (\mathbb{C}^2)^{\otimes n}$$

of dimension 2^n.

Then, the states of our system should be given by the density operators on $\mathscr{H}$.

That is, we care about the operators

$$\mathscr{D}(\mathscr{H}) = \{\rho \in \mathbb{M}_{2^n}(\mathbb{C}) : \operatorname{tr}(\rho) = 1 \text{ and } \rho \geq 0\},$$

where $\rho \geq 0$ means $\langle\varphi|\rho|\varphi\rangle \geq 0$ for all $|\varphi\rangle \in \mathscr{H}$.

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Furthermore, the system evolution axiom tells us that if $\rho_t \in \mathcal{D}(\mathcal{H})$ is the state of our system at time t, then closed evolution in the times $t \in [t_1, t_2]$, the state of our system follows

$$\rho_{t_2} = U\rho_{t_1}U^\dagger.$$

Remember, we require $U \in \mathbb{M}_{2^n}(\mathbb{C})$ to be unitary:

$$UU^\dagger = U^\dagger U = I_{2^n}.$$

Note that one sort of density operator is a pure state, i.e., one of the form

$$\rho = |\varphi\rangle\langle\varphi| \,, \quad |\varphi\rangle \in \mathscr{H} \text{ of unit length.}$$

It is thus often easier to consider states as just normalized vectors $|\varphi\rangle$ in $\mathscr{H}$, where now closed evolution is of the form

$$|\varphi_{t_2}\rangle = U\,|\varphi_{t_1}\rangle\,,$$

since taking outer products gives back our original $U(-)U^\dagger$ form for pure states:

$$|\varphi_{t_2}\rangle\langle\varphi_{t_2}| = U\,|\varphi_{t_1}\rangle\,(U\,|\varphi_{t_1}\rangle)^\dagger = U\,|\varphi_{t_1}\rangle\langle\varphi_{t_1}|\,U^\dagger.$$

Observe that

$$U^{\dagger}\rho_{t_2}U = U^{\dagger}U\rho_{t_1}U^{\dagger}U = \rho_{t_1},$$

so quantum computation is reversible.

$$
\begin{tikzcd}
\rho_{t_1} \arrow[rr, bend left, "U(-)U^\dagger"] & & \rho_{t_2} \arrow[ll, dashed, bend left, "U^\dagger(-)U"]
\end{tikzcd}
$$

t

We call a unitary U acting on $s \leq n$ qubits $q_1, \ldots, q_s$ in the space $(\mathbb{C}^2)^{\otimes n}$ a quantum gate.

Quantum gates are the analogues of classical gates (e.g., AND, XOR, NOT) in the classical circuit diagrams you have seen.

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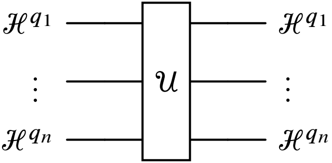
A quantum circuit should be something which looks like



We can interpret each wire as a part of the tensor product, where $\mathcal{H}^{q_j} \simeq \mathbb{C}^2$ for all $1 \leq j \leq n$ and $\mathcal{H} \simeq \bigotimes_{j=1}^{n} \mathcal{H}^{q_j}$.

Moving to the right along a wire represents moving forward in time, though we usually work under the assumption that our quantum gates U in $\mathcal{U}$ work instantaneously.

Assigned reading: §2.2 of Lomont's review for a slightly more rigorous treatment of quantum circuits.

https://arxiv.org/pdf/quant-ph/0411037.

Then, take a look at §3.1.

We now take a look at the statement of the HSP.

For a group element $j \in G$ and subgroup $H \leq G$, the (left) coset jH is the subset

$$jH = \{jh : h \in H\} \subseteq G.$$

When H is normal in G, these cosets were precisely the elements of our quotient group G/H.

Let $(G, \cdot)$ be a group, S be an arbitrary set, and $H \leq G$ be a subgroup. Then, a function

$$f : G \to S$$

separates H-cosets if for all elements $j, k \in G$,

$$f(j) = f(k) \quad \text{if and only if} \quad jH = kH.$$

The hidden subgroup problem (HSP) is as follows:

Let $(G, \cdot)$ be a group; let $S = \{s_1, \ldots, s_n\}$ be a finite set. If $f : G \to S$ separates H-cosets for some subgroup $H \leq G$, then use the value of f on some elements of G to determine a set $X \subseteq G$ such that $\langle X \rangle = H$.

Next time we will discuss

(i) the quantum fourier transform F_n on $\mathbb{Z}/n$.

(ii) the cyclic HSP algorithm.

Then, we will move on to the more general abelian HSP algorithm and applications to Simon's and Shor's algorithms.