



MONOIDAL CATEGORIES FOR QUANTUM THEORY



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AXIOMS OF QUANTUM INFORMATION

Quantum information theory follows four major postulates or axioms.

Axiom (State Space). Any physical quantum system Q can be represented by a Hilbert space \mathcal{H}_Q .

Axiom (Unitary Evolution). Closed evolution over time of a quantum system Q can be represented by a unitary ($U^\dagger U = UU^\dagger = \text{id}_{\mathcal{H}_Q}$) operator on \mathcal{H}_Q .

Axiom (Multiple Systems). Two quantum systems Q_1 and Q_2 can be considered as a joint system $Q_1 Q_2$. The associated Hilbert space should be the tensor product:

$$\mathcal{H}_{Q_1 Q_2} = \mathcal{H}_{Q_1} \otimes \mathcal{H}_{Q_2}.$$

Axiom (Measurement). Measurement of a quantum system Q corresponds to orthonormal bases of the Hilbert space \mathcal{H}_Q . The corresponding probabilities of measurement should follow Born's rule.

A category is a choice of mathematical “setting” defined with objects and morphisms/processes.

Example. The category **Hilb** has objects that are Hilbert spaces and morphisms that are (bounded) linear transformations (a linear transformation is a function $T : \mathcal{H} \rightarrow \mathcal{K}$ such that $T(v+w) = T(v) + T(w)$ and $cT(v) = T(cv)$ for any scalar $c \in \mathbb{C}$).

Example. The category **Quant** should have physical systems as objects and physical processes as morphisms. We can choose many different models of **Quant**!

Definition (Functor). A functor is a structure-preserving map $f : \mathcal{C} \rightarrow \mathcal{D}$ between two categories. This inherently means that there are two associated properties with the data of a functor: (1) an object $f(x) \in \mathcal{D}$ for every object $x \in \mathcal{C}$, (2) a morphism $f(x) \xrightarrow{f(\alpha)} f(y)$ in \mathcal{D} for every morphism $x \xrightarrow{f} y$ in \mathcal{C} . Firstly, a functor respects the composition property ($f(\alpha\beta) = f(\alpha)f(\beta)$). Secondly, it also preserves the identity between categories ($\text{id}_{f(x)} = f(\text{id}_x)$).

Since a functor “changes settings,” somehow our axioms should be encoded in a functor from **Quant** to **Hilb**. But until we add more structure, we only have the State Space axioms working!

(SYMMETRIC) MONOIDAL CATEGORIES

To model the Multiple Systems axiom using a functor, we need a good notion of tensor product.

Definition (Monoidal Category). A monoidal category includes a category \mathcal{C} , a functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, and a unit object $\mathbf{1} \in \mathcal{C}$ such that \otimes is associative and unital with $\mathbf{1}$ up to coherent natural isomorphisms.

For example, **Quant** is monoidal by considering joint systems (the unit is the empty system) and **Hilb** is monoidal via the usual tensor product \otimes (the unit is \mathbb{C}).

Definition (State). A state on an object $a \in \mathcal{C}$ is a morphism $\mathbf{1} \rightarrow a$.

This is the unit object morphing into another object in the category, so in the case of Hilbert spaces, it “picks” out a vector.

Example. States on \mathcal{H} in **Hilb** are the linear maps $\mathbb{C} \rightarrow \mathcal{H}$, i.e., vectors in \mathcal{H} .

Example. States on a quantum system Q in **Quant** correspond to the creation operators $\emptyset \rightarrow Q$.

Definition (Effect). An effect on an object $a \in \mathcal{C}$ is a morphism $a \rightarrow \mathbf{1}$.

Definition (Braiding, Symmetry). A braiding on a monoidal category is a natural isomorphism that twists an object $a \otimes b$ into $b \otimes a$, satisfying a coherence diagram. A braiding is called a symmetry if twisting twice gets us back to $a \otimes b$.

Both **Quant** and **Hilb** have symmetries γ and σ , so they are symmetric monoidal categories. We call a functor that preserves the structure of such categories a symmetric monoidal functor. Thus, a quantum theory satisfying both the State Space axiom and the Multiple Systems axiom is a symmetric monoidal functor

$$\mathcal{Z} : (\text{Quant}, \otimes, \mathbf{1}, \gamma) \rightarrow (\text{Quant}, \otimes, \mathbf{1}, \sigma).$$

Since \mathcal{Z} takes states to states, this means states of a quantum system Q are modeled by vectors in the Hilbert space $\mathcal{Z}(Q)$.

Example. If we choose **Quant** = **Bord**₂, then the result is called a 2D topological quantum field theory.

DAGGER CATEGORIES

A dagger $\dagger : \mathcal{C} \rightarrow \mathcal{C}$ is special “functor” that meets the following properties.

- Given any morphism $f : A \rightarrow B$, $A, B \in \mathcal{C}$, the dagger of f , or f^\dagger , is a map from B to A . In addition, daggers also reverse the order of composition: $(g \circ h)^\dagger = h^\dagger \circ g^\dagger$. This property is called *contravariance*.
- Given any morphism f in \mathcal{C} , $f^{\dagger\dagger} = f$. This property is called *involutivity*.
- The dagger is the identity morphism on objects.

Definition (Dagger Category). A category equipped with a dagger (\mathcal{C}, \dagger) is called a dagger category.

We define two types of morphisms that behave nicely with daggers.

- Unitary morphisms.** A unitary morphism $f : A \rightarrow B$ has the property that $f \circ f^\dagger = \text{id}_B$ and $f^\dagger \circ f = \text{id}_A$.
Note. Unitary functions are useful because they allow us to *reverse* any transformation we apply to our objects. From a quantum perspective, it means we can reverse our evolutions!
- Isometry.** An isometry f only has the property that $f^\dagger f = \text{id}_A$ (like half a unitary).
Note. These types of morphisms will be used when incorporating axiom 4.

Definition (Dagger Monoidal Category). A dagger monoidal category $(\mathcal{C}, \otimes, \mathbf{1}, \alpha, \rho, \lambda, \dagger)$ is a monoidal category with a dagger operation where the coherence natural isomorphisms α, ρ , and λ are unitary, and $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$ (f, g are morphisms).

Note. If we also add a braiding/symmetry γ to our category, the only extra condition is for γ to be unitary! In this case, we add the adjective *braided/symmetric*.

Example. Both **Hilb** and **Quant** are dagger symmetric monoidal categories. In **Hilb**, we use the adjoint as the \dagger . In **Quant**, we reverse our closed evolution to take the \dagger .

Dagger functors are functors that preserve the dagger. This notion can be combined with that of a symmetric monoidal functor. A quantum theory satisfying the State Space axiom, the Unitary Evolution axiom, and the Multiple Systems axiom is then a dagger symmetric monoidal functor $\text{Quant} \rightarrow \text{Hilb}$.

ENRICHMENT AND BIPRODUCTS

Definition (Zero Object). A zero object in a category \mathcal{C} is an object $0 \in \mathcal{C}$ such that $\forall a \in \mathcal{C}$, $\exists ! (a \rightarrow 0)$ and $\exists ! (0 \rightarrow a)$.

Definition (Zero Morphism). Let $(\mathcal{C}, 0)$ be a category with a zero object. Then a composite $A \rightarrow 0 \rightarrow B$ in \mathcal{C} exists and is uniquely determined. We call it the zero morphism $0_{A,B}$.

We say a category with a zero object $(\mathcal{C}, 0)$ is enriched in commutative monoids (**CMon**) if its morphisms can be added (in a way that is compatible with composition) associatively so that the zero morphisms act as units.

Definition (Biproduct). Let $(\mathcal{C}, 0)$ be a category with a zero object that is enriched in **CMon**. Then, given $a, b \in \mathcal{C}$, their biproduct is an object $a \oplus b \in \mathcal{C}$ with morphisms:

- $i_a : a \rightarrow a \oplus b$ and $p_a : a \oplus b \rightarrow a$.
- $i_b : b \rightarrow a \oplus b$ and $p_b : a \oplus b \rightarrow b$.

Such that:

- $a \xrightarrow{i_a} a \oplus b \xrightarrow{p_a} a = \text{id}_a$ and $b \xrightarrow{i_b} a \oplus b \xrightarrow{p_b} b = \text{id}_b$.
- $a \xrightarrow{i_a} a \oplus b \xrightarrow{p_b} 0_{a,b}$ and $b \xrightarrow{i_b} a \oplus b \xrightarrow{p_a} 0_{b,a}$.
- $i_1 p_1 + i_2 p_2 = \text{id}_{a \oplus b}$.

Biproducts give us superposition. Note that a dagger biproduct is a biproduct in a dagger category where the $i_a^\dagger = p_a$ and $i_b^\dagger = p_b$.

Definition (Probability). Given a state s and an effect e in a dagger monoidal category $(\mathcal{C}, \otimes, \mathbf{1}, \dagger)$, the probability $\text{Prob}(s \text{ in } e)$ is given by $s^\dagger \circ e^\dagger \circ e \circ s$ or

$$\mathbf{1} \xrightarrow{s} a \xrightarrow{e} \mathbf{1} \xrightarrow{e^\dagger} a \xrightarrow{s^\dagger} \mathbf{1}.$$

Definition (Complete Set). A set of effects $\{a \xrightarrow{e_\lambda} \mathbf{1}\}_{\lambda \in \Lambda}$ is complete if for any non-zero morphism $b \xrightarrow{f} a$ (ie $f \neq 0_{b,a}$) there is an effect $e_{\lambda'}$, $\lambda' \in \Lambda$ such that $e_{\lambda'} \circ f \neq 0_{b,\mathbf{1}}$.

Using these two notions, we can now state Born's Rule.

Theorem (Born's Rule). If a set of effects $\{e_1, e_2, \dots, e_n\}$, $n \in \mathbb{N}$, is complete in a dagger monoidal category with a zero object and dagger biproducts $(\mathcal{C}, \otimes, \mathbf{1}, \dagger, 0, \oplus)$, then for any isometry $x : \mathbf{1} \rightarrow a$, we have

$$\sum_{i=1}^n \text{Prob}(x \text{ in } e_i) = 1,$$

where $\mathbf{1}$ is the identity morphism on $\mathbf{1}$.

We can quickly prove this theorem (assuming one other fact).

Proof. We can first expand and factor this sum via rules of biproduct and $\text{Prob}(s \text{ in } e)$:

$$\sum_{i=1}^n \text{Prob}(x \text{ in } e_i) = \sum_{i=1}^n x^\dagger e_i^\dagger e_i x = x^\dagger \left(\sum_{i=1}^n e_i^\dagger e_i \right) x.$$

From here, we assume the fact $\sum_{i=1}^n e_i^\dagger e_i = \text{id}_a$ (proved using biproducts) to finish the proof:

$$\begin{aligned} x^\dagger \left(\sum_{i=1}^n e_i^\dagger e_i \right) x &= x^\dagger \text{id}_a x \\ &= x^\dagger x = \text{id}_{\mathbf{1}} = 1. \end{aligned}$$

□

Thus, a quantum theory satisfying all four axioms is a dagger symmetric monoidal functor from **Quant** to **Hilb** that fully preserves the biproduct structure!

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