

# MONOIDAL CATEGORIES FOR QUANTUM THEORY

*Students: H. Chan, K. Cherukuri, J. Go, D. Kareem, S. Karuturi,  
A. Khairari, A. Lala, T. Lucas, S. Narchetty, L. Patel   Instructor: Dheeran E. Wiggins*



## AXIOMS OF QUANTUM INFORMATION

Quantum information theory follows four major postulates or axioms.

**Axiom (State Space).** Any physical quantum system  $Q$  can be represented by a Hilbert space  $\mathcal{H}_Q$ .

**Axiom (Unitary Evolution).** Closed evolution over time of a quantum system  $Q$  can be represented by a unitary ( $U^\dagger U = UU^\dagger = \text{id}_{\mathcal{H}_Q}$ ) operator on  $\mathcal{H}_Q$ .

**Axiom (Multiple Systems).** Two quantum systems  $Q_1$  and  $Q_2$  can be considered as a joint system  $Q_1 Q_2$ . The associated Hilbert space should the tensor product:

$$\mathcal{H}_{Q_1 Q_2} = \mathcal{H}_{Q_1} \otimes \mathcal{H}_{Q_2}.$$

**Axiom (Measurement).** Measurement of a quantum system  $Q$  corresponds to orthonormal bases of the Hilbert space  $\mathcal{H}_Q$ . The corresponding probabilities of measurement should follow Born's rule.

A category is a choice of mathematical "setting" defined with objects and morphisms/processes.

**Example.** The category  $\text{Hilb}$  has objects that are Hilbert spaces and morphisms that are (bounded) linear transformations (a linear transformation is a function  $T : \mathcal{H} \rightarrow \mathcal{K}$  such that  $T(v+w) = T(v) + T(w)$  and  $cT(v) = T(cv)$  for any scalar  $c \in \mathbb{C}$ ).

**Example.** The category  $\text{Quant}$  should have physical systems as objects and physical processes as morphisms. We can choose many different models of  $\text{Quant}$ !

**Definition (Functor).** A functor is a structure-preserving map  $f : \mathcal{C} \rightarrow \mathcal{D}$  between two categories. This inherently means that there are two associated properties with the data of a functor: (1) an object  $f(x) \in \mathcal{D}$  for every object  $x \in \mathcal{C}$ , (2) a morphism  $f(x) \xrightarrow{f(\alpha)} f(y)$  in  $\mathcal{D}$  for every morphism  $x \xrightarrow{\alpha} y$  in  $\mathcal{C}$ . Firstly, a functor respects the composition property ( $f(\alpha\beta) = f(\alpha)f(\beta)$ ). Secondly, it also preserves the identity between categories ( $\text{id}_{f(x)} = f(\text{id}_x)$ ).

Since a functor "changes settings," somehow our axioms should be encoded in a functor from  $\text{Quant}$  to  $\text{Hilb}$ . But until we add more structure, we only have the State Space axioms working!

## (SYMMETRIC) MONOIDAL CATEGORIES

To model the Multiple Systems axiom using a functor, we need a good notion of tensor product.

**Definition (Monoidal Category).** A monoidal category includes a category  $\mathcal{C}$ , a functor  $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ , and a unit object  $\mathbf{1} \in \mathcal{C}$  such that  $\otimes$  is associative and unital with  $\mathbf{1}$  up to coherent natural isomorphisms.

For example,  $\text{Quant}$  is monoidal by considering joint systems (the unit is the empty system) and  $\text{Hilb}$  is monoidal via the usual tensor product  $\otimes$  (the unit is  $\mathbb{C}$ ).

**Definition (State).** A state on an object  $a \in \mathcal{C}$  is a morphism  $\mathbf{1} \rightarrow a$ .

This is the unit object morphing into another object in the category, so in the case of Hilbert spaces, it "picks" out a vector.

**Example.** States on  $\mathcal{H}$  in  $\text{Hilb}$  are the linear maps  $\mathbb{C} \rightarrow \mathcal{H}$ , i.e., vectors in  $\mathcal{H}$ .

**Example.** States on a quantum system  $Q$  in  $\text{Quant}$  correspond to the creation operators  $\emptyset \rightarrow Q$ .

**Definition (Effect).** An effect on an object  $a \in \mathcal{C}$  is a morphism  $a \rightarrow \mathbf{1}$ .

**Definition (Braiding, Symmetry).** A braiding on a monoidal category is a natural isomorphism that twists an object  $a \otimes b$  into  $b \otimes a$ , satisfying a coherence diagram. A braiding is called a symmetry if twisting twice gets us back to  $a \otimes b$ .

Both  $\text{Quant}$  and  $\text{Hilb}$  have symmetries  $\gamma$  and  $\sigma$ , so they are symmetric monoidal categories. We call a functor that preserves the structure of such categories a symmetric monoidal functor. Thus, a quantum theory satisfying both the State Space axiom and the Multiple Systems axiom is a symmetric monoidal functor

$$\mathcal{Z} : (\text{Quant}, \otimes, \mathbf{1}, \gamma) \rightarrow (\text{Quant}, \otimes, \mathbf{1}, \sigma).$$

Since  $\mathcal{Z}$  takes states to states, this means states of a quantum system  $Q$  are modeled by vectors in the Hilbert space  $\mathcal{Z}(Q)$ .

**Example.** If we choose  $\text{Quant} = \text{Bord}_2$ , then the result is called a 2D topological quantum field theory.

## DAGGER CATEGORIES

A dagger  $\dagger : \mathcal{C} \rightarrow \mathcal{C}$  is special "functor" that meets the following properties.

1. Given any morphism  $f : A \rightarrow B$ ,  $A, B \in \mathcal{C}$ , the dagger of  $f$ , or  $f^\dagger$ , is a map from  $B$  to  $A$ . In addition, daggers also reverse the order of composition:  $(g \circ h)^\dagger = h^\dagger \circ g^\dagger$ . This property is called *contravariance*

2. Given any morphism  $f$  in  $\mathcal{C}$ ,  $f^{\dagger\dagger} = f$ . This property is called *involutivity*

3. The dagger is the identity morphism on objects.

**Definition (Dagger Category).** A category equipped with a dagger  $(\mathcal{C}, \dagger)$  is called a dagger category.

We define two types of morphisms that behave nicely with daggers.

1. **Unitary morphisms.** A unitary morphism  $f : A \rightarrow B$  has the property that  $f \circ f^\dagger = \text{id}_B$  and  $f^\dagger \circ f = \text{id}_A$ .

Note. Unitary functions are useful because they allow us to *reverse* any transformation we apply to our objects. From a quantum perspective, it means we can reverse our evolutions!

2. **Isometry.** An isometry  $f$  only has the property that  $f^\dagger f = \text{id}_A$  (like half a unitary).

Note. These types of morphisms will be used when incorporating axiom 4.

**Definition (Dagger Monoidal Category).** A dagger monoidal category  $(\mathcal{C}, \otimes, \mathbf{1}, \alpha, \rho, \lambda, \dagger)$  is a monoidal category with a dagger operation where the coherence natural isomorphisms  $\alpha, \rho$ , and  $\lambda$  are unitary, and  $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$  ( $f, g$  are morphisms).

Note. If we also add a braiding/symmetry  $\gamma$  to our category, the only extra condition is for  $\gamma$  to be unitary! In this case, we add the adjective braided/symmetric.

**Example.** Both  $\text{Hilb}$  and  $\text{Quant}$  are dagger symmetric monoidal categories. In  $\text{Hilb}$ , we use the adjoint as the  $\dagger$ . In  $\text{Quant}$ , we reverse our closed evolution to take the  $\dagger$ .

Dagger functors are functors that preserve the dagger. This notion can be combined with that of a symmetric monoidal functor. A quantum theory satisfying the State Space axiom, the Unitary Evolution axiom, and the Multiple Systems axiom is then a dagger symmetric monoidal functor  $\text{Quant} \rightarrow \text{Hilb}$ .

## ENRICHMENT AND BIPRODUCTS

**Definition (Zero Object).** A zero object in a category  $\mathcal{C}$  is an object  $0 \in \mathcal{C}$  such that  $\forall a \in \mathcal{C}$ ,  $\exists!(a \rightarrow 0)$  and  $\exists!(0 \rightarrow a)$ .

**Definition (Zero Morphism).** Let  $(\mathcal{C}, 0)$  be a category with a zero object. Then a composite  $A \rightarrow 0 \rightarrow B$  in  $\mathcal{C}$  exists and is uniquely determined. We call it the zero morphism  $0_{A,B}$ .

We say a category with a zero object  $(\mathcal{C}, 0)$  is enriched in commutative monoids (CMon) if its morphisms can be added (in a way that is compatible with composition) associatively so that the zero morphisms act as units.

**Definition (Biproduct).** Let  $(\mathcal{C}, 0)$  be a category with a zero object that is enriched in CMon. Then, given  $a, b \in \mathcal{C}$ , their biproduct is an object  $a \oplus b \in \mathcal{C}$  with morphisms:

- $i_a : a \rightarrow a \oplus b$  and  $p_a : a \oplus b \rightarrow a$ .
- $i_b : b \rightarrow a \oplus b$  and  $p_b : a \oplus b \rightarrow b$ .

Such that:

- $a \xrightarrow{i_a} a \oplus b \xrightarrow{p_a} a = \text{id}_a$  and  $b \xrightarrow{i_b} a \oplus b \xrightarrow{p_b} b = \text{id}_b$ .
- $a \xrightarrow{i_a} a \oplus b \xrightarrow{p_b} 0_{a,b}$  and  $b \xrightarrow{i_b} a \oplus b \xrightarrow{p_a} 0_{b,a}$ .
- $i_1 p_1 + i_2 p_2 = \text{id}_{a \oplus b}$ .

Biproducts give us superposition. Note that a dagger biproduct is a biproduct in a dagger category where the  $i_a^\dagger = p_a$  and  $i_b^\dagger = p_b$ .

**Definition (Probability).** Given a state  $s$  and an effect  $e$  in a dagger monoidal category  $(\mathcal{C}, \otimes, \mathbf{1}, \dagger)$ , the probability  $\text{Prob}(s \text{ in } e)$  is given by  $s^\dagger \circ e^\dagger \circ e \circ s$  or

$$1 \xrightarrow{s} a \xrightarrow{e} \mathbf{1} \xrightarrow{e^\dagger} a \xrightarrow{s^\dagger} \mathbf{1}.$$

**Definition (Complete Set).** A set of effects  $\{a \xrightarrow{e_\lambda} \mathbf{1}\}_{\lambda \in \Lambda}$  is complete if for any non-zero morphism  $b \xrightarrow{f} a$  (ie  $f \neq 0_{b,a}$ ) there is an effect  $e_{\lambda'}$ ,  $\lambda' \in \Lambda$  such that  $e_{\lambda'} \circ f \neq 0_{b,\mathbf{1}}$ .

Using these two notions, we can now state Born's Rule.

**Theorem (Born's Rule).** If a set of effects  $\{e_1, e_2, \dots, e_n\}$ ,  $n \in \mathbb{N}$ , is complete in a dagger monoidal category with a zero object and dagger biproducts  $(\mathcal{C}, \otimes, \mathbf{1}, \dagger, 0, \oplus)$ , then for any isometry  $x : \mathbf{1} \rightarrow a$ , we have

$$\sum_{i=1}^n \text{Prob}(x \text{ in } e_i) = 1,$$

where  $\mathbf{1}$  is the identity morphism on  $\mathbf{1}$ .

We can quickly prove this theorem (assuming one other fact).

**Proof.** We can first expand and factor this sum via rules of biproduct and  $\text{Prob}(s \text{ in } e)$ :

$$\sum_{i=1}^n \text{Prob}(x \text{ in } e_i) = \sum_{i=1}^n x^\dagger e_i^\dagger e_i x = x^\dagger \left( \sum_{i=1}^n e_i^\dagger e_i \right) x.$$

From here, we assume the fact  $\sum_{i=1}^n e_i^\dagger e_i = \text{id}_a$  (proved using biproducts) to finish the proof:

$$x^\dagger \left( \sum_{i=1}^n e_i^\dagger e_i \right) x = x^\dagger \text{id}_a x = x^\dagger x = \mathbf{1} = 1.$$

Thus, a quantum theory satisfying all four axioms is a dagger symmetric monoidal functor from  $\text{Quant}$  to  $\text{Hilb}$  that fully preserves the biproduct structure!

## REFERENCES

- [1] Michael F. Atiyah. Topological quantum field theory. *Publications Mathématiques de l'IHÉS*, 68:175–186, 1988.
- [2] John Baez. Quantum quandaries: A category-theoretic perspective. In *The Structural Foundations of Quantum Gravity*. Oxford University Press, 2006.
- [3] Chris Heunen and Jamie Vicary. *Categories for Quantum Theory: An Introduction*. Oxford University Press, 2019.