EXERCISES FOR INTRODUCTION TO MATHEMATICAL QUANTUM ERROR CORRECTION

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1. Monday

Exercise 1.1 (Complex Arithmetic)**.** Simplify the following expressions in $\mathbb{C}$:

(i) $i^3 + i^2 + i + 1$
(ii) $(-3 + 2i)(6 - 8i)$
(iii) $(9i)(-i)$

Exercise 1.2 (Pauli Matrix Multiplication)**.** Let $I, X, Y, Z \in \mathbb{M}_2(\mathbb{C})$ be defined by

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; Y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(i) Compute the matrix products I^2, IX, IY, and IZ. What can you guess is true about multiplying matrices in $\mathbb{M}_2(\mathbb{C})$ by I, in general?
(ii) Compute the matrix products XY and YX. Simplify both products into the form cZ, where $c \in \mathbb{C}$. Is c different in each case?

The matrix I is called the identity. The other three matrices X, Y, Z are called the *Pauli matrices*. Together, these generate the *Pauli group*, a multiplicative group of order 16.

Exercise 1.3 (Recalling Definitions)**.**

(i) Let A, B be sets. What is an *injective* function $f : A \rightarrowtail B$? What is a *surjective* function $f : A \twoheadrightarrow B$? What do we call a function $f : A \xrightarrow{\sim} B$ which is *both* injective and surjective?
(ii) Write the definition of a *group* $(G, \cdot)$. What is the general difference between just a set and a group?[1]
(iii) What is the difference between a group and an *abelian* group?
(iv) Let $S \subseteq G$ be a subset of a group. Write the definition of the *normalizer* $\mathcal{N}_G(S)$ and *centralizer* $C_G(S)$ of S in G.

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[1] You can be informal about this.

2. Tuesday

Exercise 2.1 (Conjugates and Adjoints)**.** Compute the following, where $*$ denotes the complex *conjugate* $(a+bi)^* = a - bi$ in $\mathbb{C}$ and $\dagger$ denotes the *adjoint* in $\mathbb{C}^n$:

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}^\dagger = \begin{pmatrix} a_1^* & \cdots & a_n^* \end{pmatrix}.$$

(i) $i(i^*)$
(ii) $(7+i)^* + (1+3i) + (1-4i)^*$
(iii)

$$\begin{pmatrix} i \\ 1+i \\ 0 \\ 2+2i \end{pmatrix}^\dagger$$

Exercise 2.2 (Basics of $\mathbb{C}^2$)**.** As mentioned in lecture, $\mathbb{C}^2$ is an example of a vector space over $\mathbb{C}$. Notably, the elements of $\mathbb{C}^2$ are ordered pairs (z, w) where z, w are complex numbers. The functions (linear transformations) in this case are realizable as 2×2 complex matrices $\mathbb{M}_2(\mathbb{C})$.

(i) What is the *dimension* of $\mathbb{C}^2$? In turn, what is the size of a basis for $\mathbb{C}^2$?[2]
(ii) Write down a *basis* β for $\mathbb{C}^2$. That is, find a subset of $\mathbb{C}^2$ such that $\operatorname{span}\beta$ gives you every possible ordered pair/vector $(z, w) \in \mathbb{C}^2$.
(iii) Generalize your answers from (i) and (ii) to $\mathbb{C}^n$: determine its dimension and find a basis. These solutions should be analogous (look very similar) to your answers for $\mathbb{C}^2$.

Exercise 2.3 (Recalling Definitions)**.**

(i) Write down the definition of a *vector space* from lecture. How does a vector space differ from an *inner product* space?
(ii) What is a *Hilbert space* in finite dimensions?
(iii) How do we define the *direct sum/coproduct* $\bigoplus \mathcal{W}_i$ of an indexed family of vector spaces $\{\mathcal{W}_i\}_{i \in I}$? How is this different from a vector space *product*? Do the definitions differ when $I = \{1, \ldots, n\}$?

Exercise 2.4 (Challenge Problems)**.** These problems are more abstract and require quite a bit more thinking (and probably, more background) than I expect from just two days of watching lecture. Still, if you are interested, here are two relevant challenge problems. [These questions were adapted from Chris Dodd at Illinois.]

(i) Let $\mathcal{V}$ be a vector space of dimension n. A flag $\mathfrak{F}$ of length r in $\mathcal{V}$ is a collection of subspaces $\{0\} = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_r = \mathcal{V}$, where $\mathcal{V}_i \neq \mathcal{V}_{i+1}$ for all i. Show that $r \leq n$. If $r = n$, we call the flag $\mathfrak{F}$ *complete*. Show that $\mathfrak{F}$ is complete *if and only if* $\dim(\mathcal{V}_i/\mathcal{V}_{i+1}) = 1$ for all i.[3]
(ii) Let $T : \mathcal{V} \to \mathcal{W}$ and $S : \mathcal{W} \to \mathcal{U}$ be linear maps of vector spaces. Let T^* and S^* denote the dual maps $T^* : \mathcal{W}^* \to \mathcal{V}^*$ and $S^* : \mathcal{U}^* \to \mathcal{W}^*$, defined in the natural way $\varphi \mapsto \varphi \circ T$ or S. Show that $(S \circ T)^* = T^* \circ S^*$. Use this to show that if A, B are matrices so that AB exists, then $(AB)^t = B^t A^t$.

[2] Remember, this is the same as asking for the number of "copies" of $\mathbb{C}$ in $\mathbb{C}^2$.

[3] Here, $\mathcal{V}_i/\mathcal{V}_{i+1}$ is the *quotient space*. That is, the elements of the underlying group are additive cosets $v + \mathcal{V}_{i+1}$, where $v \in \mathcal{V}_i$, and the operations work via $(v + \mathcal{V}_{i+1}) + (k + \mathcal{V}_{i+1}) = (v + k) + \mathcal{V}_{i+1}$ and $c(v + \mathcal{V}_{i+1}) = cv + \mathcal{V}_{i+1}$, for a scalar $c \in \mathbb{C}$.

3. Wednesday

Exercise 3.1 (Poster)**.** If I assigned you a topic/result today, get it done during our work time today, or tonight, and send me the email at `dwiggins@imsa.edu`.

Exercise 3.2 (Computing Dimensions)**.** Recall that $\dim(\mathbb{C}^n \oplus \mathbb{C}^m) = n+m$ and $\dim(\mathbb{C}^n \otimes \mathbb{C}^m) = n \cdot m$, where $\oplus$ is the *direct sum* of Hilbert spaces and $\otimes$ is the *tensor product*, as described during lecture today. Compute the following dimensions, leaving your answers un-simplified (i.e., I want to see the products of the numbers, not a final simplified dimension).[4]

(i) $(\mathbb{C}^2 \otimes \mathbb{C}^4) \oplus \mathbb{C}^6$
(ii) $(\mathcal{H}^A \otimes \mathcal{H}^B) \oplus \mathcal{K}$, where $\mathcal{H}^A \simeq \mathbb{C}^4$, $\mathcal{H}^B \simeq \mathbb{C}^7$, and $\mathcal{K} \simeq \mathbb{C}^{32}$
(iii) $(\mathcal{H}^A \otimes \mathcal{H}^B) \oplus (\mathcal{H}^C \otimes \mathcal{H}^D) \oplus (\mathcal{H}^E \otimes \mathcal{H}^F)$, where $\mathcal{H}^A \simeq \mathbb{C}$, $\mathcal{H}^B \simeq \mathbb{C}^2$, $\mathcal{H}^C \simeq \mathbb{C}^3$, $\mathcal{H}^D \simeq \mathbb{C}^4$, $\mathcal{H}^E \simeq \mathbb{C}^5$, and $\mathcal{H}^F \simeq \mathbb{C}^6$.

Exercise 3.3 (Recalling Defintions)**.**

(i) What is a *superoperator*?
(ii) What is a *quantum channel*? Why are quantum channels the functions we care about, physically?
(iii) State the *Kraus Representation Theorem*, also known as the Operator-Sum Representation Theorem.
(iv) What is a *correctable error*?

Exercise 3.4 (Challenge Problems)**.** These problems are more abstract and require quite a bit more thinking (and probably, more background) than I expect from just three days of watching lecture. Still, if you are interested, here are two relevant challenge problems. [These questions were adapted from Eric Chitambar at Illinois.]

(i) Define the three-dimensional "dephasing" quantum channel Δ to be the superoperator

$$\mathbb{B}(\mathbb{C}^3) \xrightarrow{\Delta} \mathbb{B}(\mathbb{C}^3)$$

$$\begin{pmatrix} a & b & c \\ b^* & d & e \\ c^* & e^* & f \end{pmatrix} \longmapsto \begin{pmatrix} a & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & f \end{pmatrix}.$$

Determine the Choi matrix J_Δ for the superoperator Δ.

(ii) Remember, we say $\Phi : \mathbb{B}(\mathbb{C}^d) \to \mathbb{B}(\mathbb{C}^d)$ is a *unital* superoperator if $\Phi : I_d \mapsto I_d$. Prove that the following are equivalent (i.e., prove the "if and only if" relationship between the statements).
  (i) Φ is unital.
  (ii) $\mathrm{tr}_A(J_\Phi) = I^B$

This was one of our four results relating superoperators to their Choi matrices.

[4] You may use that dimension is an "isomorphism invariant." That is, if the dimensions of $\mathbb{C}^n$ is n, then the dimension of $\mathcal{H} \simeq \mathbb{C}^n$ is also n.

4. Thursday

Exercise 4.1 (Poster)**.** If I assigned you a topic/result today, get it done during our work time today and send me the email at `dwiggins@imsa.edu`.

Exercise 4.2 (Computing Commutation Relations)**.** During lecture, I listed the standard Pauli matrix commutation relations on the board. On Monday, you were asked to solve for one of them. Now, solve for the rest. That is, compute the following:[5]

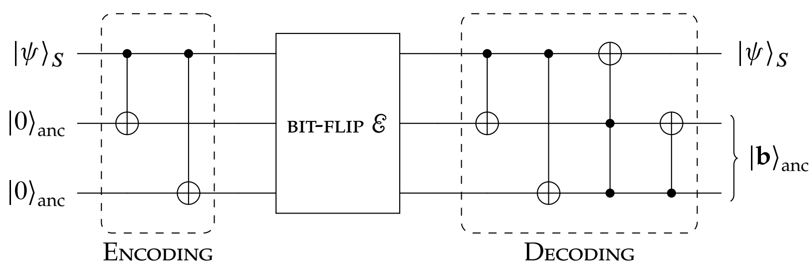
(i) YZ and ZY.
(ii) ZX and XZ.

You may choose whether or not to include i as a factor in your answers, though you should make clear the sign/parity of each product.

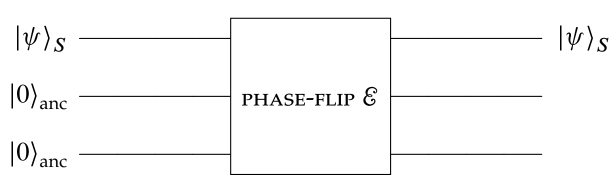
Exercise 4.3 (Recalling Definitions)**.**

(i) Define the *n-qubit Pauli group*.
(ii) State the main theorem of the *stabilizer formalism* of quantum error correction.
(iii) We defined the notion of an *isomorphism* for sets, groups, and vector spaces to be a function/homomorphism/linear transformation which admits an inverse. If you are given two general "mathematical objects" of some sort, say Ω and Ξ, plus a function between them $f : \Omega \to \Xi$ which preserves the so-called "structure" of the objects, how would you define an isomorphism $f : \Omega \xrightarrow{\sim} \Xi$ so that we could say the objects are *isomorphic* $\Omega \simeq \Xi$?

Exercise 4.4 (Challenge Problem)**.** Today I showed the three qubit bit-flip code:



Come up with an analogous three qubit "phase-flip" code which corrects errors $\mathcal{E}$ of the form $Z \otimes I \otimes I$, $I \otimes Z \otimes I$, and so forth:



Unlike with the bit-flip code, you do *not* need to output a $|\mathbf{b}\rangle_{\text{anc}}$ message for determining the error syndrome, you just need to recover $|\psi\rangle_S$. You will need the *Hadamard gate* H, which is defined in the form of a 2×2 matrix as[6]

$$H := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

[5] For your own reference, the Pauli matrices X, Y, Z are defined in the section for Monday.
[6] Hint: One way to do this would be with four CNOTs, six Hadamards, and one "Toffoli" (a CNOT with two controls).

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