AN INTRODUCTION TO SHEAF COHOMOLOGY

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1. Motivations

Many mathematical objects, such as functions, sections of bundles, or solutions to differential equations, are naturally defined on small, local regions of a space. A natural question to ask is whether having a locally defined object everywhere gives rise to a global object; can we "glue" together local objects to provide a global object? Sheaf cohomology provides a powerful framework to quantify the obstructions to this gluing process, revealing whether local data can be assembled into a global whole and, if not, what prevents it.

For example, consider the topological space $X = S^1$, the circle, with two proper open sets U_1 and U_2, each covering half the circle plus some overlap (i.e., $U_1 \cup U_2 = S^1$, and $U_1 \cap U_2$ consists of two arcs). Suppose we define a continuous function $f_1 : U_1 \to \mathbb{R}$ and $f_2 : U_2 \to \mathbb{R}$. To obtain a global continuous function $f : S^1 \to \mathbb{R}$, we need f_1 and f_2 to agree on the overlap $U_1 \cap U_2$. If they do not, there is an obstruction to gluing them into a global function. The question now arise: what compatibility conditions need to be satisfied on triple, quadruple, and higher order intersections? Cohomology provides a tool in handling more complicated layers of consistency.

2. Abelian Categories

To study sheaf cohomology, we need a categorical framework to do homological algebra–in particular, we want categorical notions of kernels and cokernels that behave in the desired way. This will lead us to the setting of abelian categories.

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Definition 2.1 (Additive Category)**.** A category $\mathscr{C}$ is called *additive* if

(i) for all objects $X, Y \in \mathrm{ob}\,\mathscr{C}$, the hom-set $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ is an abelian group.

(ii) composition of morphisms is bilinear, i.e.,

$$g \circ (f_1 + f_2) = g \circ f_1 + g \circ f_2, \quad (f_1 + f_2) \circ h = f_1 \circ h + f_2 \circ h,$$

whenever the compositions are defined.

(iii) for every pair of objects $X, Y \in \mathrm{ob}\,\mathscr{C}$, their coproduct $X \oplus Y$ exists.

In other words, additive categories are Ab-categories with finite coproducts.

Proposition 2.2. *If $\mathscr{C}$ is additive,*

(i) *finite products exist and agree with coproducts* (*up to isomorphism*)*.*

(ii) *there is a unique* (*up to isomorphism*) *object* $0 \in \mathrm{ob}\,\mathscr{C}$ *such that* $\mathrm{Hom}_{\mathscr{C}}(0, x)$ *and* $\mathrm{Hom}_{\mathscr{C}}(x, 0)$ *are one element sets for all* $x \in \mathrm{ob}\,\mathscr{C}$*.*

Definition 2.3 (Monomorphism, Epimorphism)**.** Let $f : X \to Y$ be a morphism in $\mathscr{C}$.

(i) f is a *monomorphism* if for every $Z \in \mathrm{ob}\,\mathscr{C}$ and $g_1, g_2 : Z \rightrightarrows X$, $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$.

(ii) f is an *epimorphism* if for every $Z \in \mathrm{ob}\,\mathscr{C}$ and $g_1, g_2 : Y \rightrightarrows Z$, $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$.

Remark 2.4. Looking at monomorphisms and epimorphisms, we should think of them as "injective" morphisms and "surjective" morphisms, respectively. It is the case that every isomorphism is both a monomorphism and an epimorphism, but the converse is not true generally. We will focus on the case of abelian categories which, among other benefits, do not have this shortcoming.

Definition 2.5 (Kernel, Cokernel, Image)**.** Let $f : X \to Y$ be a morphism in an additive category $\mathscr{C}$.

(i) The *kernel* of f is the limit of

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & Y \end{tikzcd} .$$

(ii) The *cokernel* of f is the colimit of the same diagram.

(iii) The *image* of f, denoted $\mathrm{im}\, f$, is the kernel of the cokernel map $p : B \to \mathrm{coker}(f)$.

Definition 2.6 (Abelian Category)**.** An additive category $\mathscr{C}$ is called *abelian* if:

(i) every morphism has a kernel and a cokernel.

(ii) for $f : X \to Y$, the natural map $\mathrm{coker}((\mathrm{ker}((f)) \to \mathrm{ker}((\mathrm{coker}((f))$ is an isomorphism.

Theorem 2.7 (Properties of Abelian Categories)**.** *Let $f : X \to Y$ be a morphism in an abelian category $\mathscr{C}$.*

(i) *f can be decomposed into a pair $X \xrightarrow{p} \mathrm{im}(f) \xrightarrow{i} Y$ where p is an epimorphism and i a monomorphism.*

(ii) *f is a monomorphism if and only if* $\mathrm{ker}(f) = 0$ *if and only if* $\mathrm{im}(f) = Y$*.*

(iii) *f is an epimorphism if and only if* $\mathrm{coker}(f) = 0$ *if and only if* $\mathrm{im}(f) = X$*.*

(iv) *f is an isomorphism if and only if it is a monomorphism and an epimorphism.*

Example 2.8. The category of abelian groups $\mathsf{Ab} = {}_{\mathbb{Z}}\mathsf{Mod}$ is an example of an abelian category. More generally ${}_R\mathsf{Mod}$ is an abelian category for any ring R. By the next theorem we will see that many abelian categories act like the category of R-modules. It is, in some sense, the "model" for how we want abelian categories to act.

Theorem 2.9 (Freyd-Mitchell Embedding)**.** *If $\mathscr{C}$ is an abelian category that is small, that is the objects and hom-sets form sets, then there is a fully faithfull and exact functor $F : \mathscr{C} \to {}_R\mathsf{Mod}$ for some ring R.*

Theorem 2.10. *Exact functors preserve kernels and cokernels.*

3. Presheaves

Presheaves are the natural way in which to define "local data" on parts of a topological space. Additionally given "global data", presheaves will respect the restriction of that data to a subset. Sheaves in turn we be a special type of presheaf in which certain circumstances allow "local data" to be glued back together to "global data".

Definition 3.1 (Opposite Category)**.** Let $\mathscr{C}$ be a category. The *opposite category* $\mathscr{C}^{\text{op}}$ is defined as follows:

- (i) The objects of $\mathscr{C}^{\text{op}}$ are the same as the objects of $\mathscr{C}$.
- (ii) For any two objects X,Y, the set of morphisms from X to Y in $\mathscr{C}^{\text{op}}$ is defined by
$$\mathrm{Hom}_{\mathscr{C}^{\text{op}}}(X,Y) := \mathrm{Hom}_{\mathscr{C}}(Y,X).$$
- (iii) The composition of morphisms in $\mathscr{C}^{\text{op}}$ is given by
$$f^{\text{op}} \circ g^{\text{op}} := (g \circ f)^{\text{op}},$$
where $f^{\text{op}} : Y \to Z$ and $g^{\text{op}} : X \to Y$ are morphisms in $\mathscr{C}^{\text{op}}$, corresponding to $f : Z \to Y$ and $g : Y \to X$ in $\mathscr{C}$.

Definition 3.2 (Poset Category)**.** Every poset $(P, \leq)$ can be regarded as a category $\mathscr{P}$, often called a *poset category*, where

- (i) objects in $\mathscr{P}$ are the elements of P.
- (ii) there exists a unique morphism in $\mathrm{Hom}_{\mathscr{P}}(p,q)$ if $p \leq q$.

Definition 3.3 (Topological Space)**.** A topological space is a pair $(X, \mathscr{T})$, where X is a set and $\mathscr{T} \subseteq \mathscr{P}(X)$ is a subfamily of the power set of X consisting of *open sets* satisfying:

- (i) $\varnothing, X \in \mathscr{T}$.
- (ii) If $X_i \in \mathscr{T}$ for all $i \in I$ a set, then $(\bigcup_{i \in I} X_i) \in \mathscr{T}$.
- (iii) If $X_i \in \mathscr{T}$ for $i \in I$ a finite set, then $(\bigcap_{i \in I} X_i) \in \mathscr{T}$.

Example 3.4. Let $X = \mathbb{R}$. A set $U \subseteq X$ is open in the *standard topology* if for all $p \in U$ there is an open interval I such that $p \in I \subseteq U$.

Remark 3.5 (Topology as a Poset and Poset Category). Given a topological space $(X, \mathscr{T})$, the set of open subsets $\mathscr{T}$ forms a partially ordered set (poset) under inclusion: for $U, V \in \mathscr{T}$, we write $U \leq V$ if $U \subseteq V$. This relation is reflexive, antisymmetric, and transitive, making $(\mathscr{T}, \subseteq)$ a poset. In turn, we can define the poset category Open_X.

Definition 3.6 (Presheaf)**.** Let X be a topological space. A *presheaf of sets* $\mathscr{F}$ on X consists of:

- (i) A set, $\mathscr{F}(U)$, for each open set $U \subseteq X$.
- (ii) A restriction map,
$$\rho_{U,V} : \mathscr{F}(U) \to \mathscr{F}(V), \quad s \mapsto s|_V,$$
for each inclusion $V \subseteq U$, such that the following properties hold:
  - (a) (Identity) For every open set U, the restriction map $\rho_{U,U}$ is the identity.

(b) (Composition) For every chain of inclusions $W \subseteq V \subseteq U$, we have:

$$\begin{tikzcd} \mathscr{F}(U) \arrow[rr, "\rho_{U,V}"] \arrow[dr, "\rho_{U,W}"'] & & \mathscr{F}(V) \arrow[dl, "\rho_{V,W}"] \\ & \mathscr{F}(W) & \end{tikzcd}$$

Remark 3.7. In categorical terms, a presheaf of sets is a functor

$$\mathscr{F} : \mathrm{Open}_X^{\mathrm{op}} \to \mathsf{Set}.$$

In this way, one can define a presheaf with values in a category $\mathscr{C}$ to be a functor

$$\mathscr{F} : \mathrm{Open}_X^{\mathrm{op}} \to \mathscr{C}.$$

Example 3.8. Let $X = \mathbb{R}$, and define, for all $U \in \mathrm{Open}_{\mathbb{R}}$, $\mathscr{F}(U) := \mathrm{Map}(U, \mathbb{R})$, the set of real-valued continuous functions on U. For $V \subseteq U$, take the restriction map given by function restriction, i.e. $f|_V$ is the function sending $v \mapsto f(v)$ for $v \in V \subseteq U$. This choice defines a presheaf of abelian groups (or rings) on $\mathbb{R}$. Diagrammatically, the presheaf $\mathscr{F}$ does the following on the inclusion $\iota : V \hookrightarrow U$:

$$\mathrm{Open}_{\mathbb{R}}^{\mathrm{op}} \xrightarrow{\mathscr{F}} \mathsf{Ab}$$

$$\begin{tikzcd} U \arrow[r, mapsto] \arrow[d, "\iota^{\mathrm{op}}"'] & \mathrm{Map}(U, \mathbb{R}) \arrow[d, "\iota^*"] \\ V \arrow[r, mapsto] & \mathrm{Map}(V, \mathbb{R}) \end{tikzcd}$$

Here, $\iota^* : \mathrm{Map}(U, \mathbb{R}) \to \mathrm{Map}(V, \mathbb{R})$ denotes the pre-composition $f \mapsto f \circ \iota = f|_V$.

Example 3.9 (Constant Presheaf)**.** Let X be a topological space and A a set (or an abelian group). The *constant presheaf* on X with values in A, denoted $\mathscr{F}_{\mathrm{const}}$, is defined as:

- For every open set $U \subseteq X$, set
$$\mathscr{F}_{\mathrm{const}}(U) = A$$
- For every inclusion of open sets $V \subseteq U$, define the restriction map
$$\rho_{U,V} : \mathscr{F}_{\mathrm{const}}(U) \to \mathscr{F}_{\mathrm{const}}(V)$$
to be the identity map on A, that is,
$$\rho_{U,V}(a) = a, \quad \text{for all } a \in A$$

This presheaf does not generally satisfy the sheaf gluing condition, as discussed in the next section, and thus is not a sheaf unless the topological space X is very simple (e.g., discrete).

4. The Category of Sheaves

To do sheaf cohomology we will want to define not only (pre)sheaves but also maps between them. One should hope that (pre)sheaves will live in some abelian category and we will see that this is often the case.

Definition 4.1 (Vertical Composite)**.** Let $F, G, H : \mathscr{C} \to \mathscr{D}$ be functors with natural transformations $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$. There is a natural composition $\beta \circ \alpha : F \Rightarrow H$, called the *vertical composite*, defined objectwise $(\beta \circ \alpha)_X = \beta_X \circ \alpha_X$ for all $X \in \mathrm{ob}\mathscr{C}$ (Fig. 1).

$$\begin{tikzcd}[column sep=huge] \mathscr{C} \arrow[r, bend left=60, "F"{name=F}] \arrow[r, "G"{description, name=G}] \arrow[r, bend right=60, "H"'{name=H}] \arrow[from=F, to=G, phantom, "\Downarrow\alpha"] \arrow[from=G, to=H, phantom, "\Downarrow\beta"] & \mathscr{D} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd}[column sep=huge] \mathscr{C} \arrow[r, bend left=60, "F"{name=F}] \arrow[r, bend right=60, "H"'{name=H}] \arrow[from=F, to=H, phantom, "\Downarrow \beta\circ\alpha"] & \mathscr{D} \end{tikzcd}$$

Figure 1. Vertical composition of natural transformations

Exercise 4.2. The vertical composite is, in fact, a natural transformation.

Definition 4.3 (Functor Category)**.** Let $\mathscr{C}, \mathscr{D}$ be an categories. Then, define the functor category $\mathrm{Fun}(\mathscr{C}, \mathscr{D}) \equiv [\mathscr{C}, \mathscr{D}]$ to be the category which has functors $F : \mathscr{C} \to \mathscr{D}$ as objects and natural transformations $\alpha : F \Rightarrow G$ as morphisms between $F, G : \mathscr{C} \rightrightarrows \mathscr{D}$. Composition is given by vertical composition.

Theorem 4.4. *If $\mathscr{D}$ is an abelian category, then so is $[\mathscr{C}, \mathscr{D}]$*

Example 4.5 (Presheaf Category)**.** Let X be a topological space, $\mathscr{C} = \mathrm{Open}_X^{\mathrm{op}}$ and $\mathscr{D} = \mathsf{Ab}$. Then, the category $\mathrm{Psh}(X)$ of abelian group-valued presheaves on X is the functor category $[\mathscr{C}, \mathscr{D}] = [\mathrm{Open}_X^{\mathrm{op}}, \mathsf{Ab}]$. Observe that since $\mathsf{Ab} = {}_{\mathbb{Z}}\mathsf{Mod}$ is an abelian category, so is the functor category $\mathrm{Psh}(X)$.

Definition 4.6 (Sheaf)**.** A *sheaf* is a presheaf, $\mathscr{F}$, which additionally satisfies the so-called sheaf condition[1]:

(i) For every open cover $\{U_i\}_{i\in I}$ of an open set $U \subseteq X$, if there exists a family $\{f_i\}_{i\in I}$ with $f_i \in \mathscr{F}(U_i)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$, then there is a unique $f \in \mathscr{F}(U)$ such that $f|_{U_i} = f_i$ for all $i \in I$.
(ii) $\mathscr{F}(\varnothing)$ is terminal.

Example 4.7. The prior example of the presheaf on $X = \mathbb{R}$ defined by $\mathscr{F}(U) = \mathrm{Map}(U, \mathbb{R})$ is a sheaf. However, if we wanted to send U to the *bounded* continuous functions $\mathbb{B}(U, \mathbb{R})$, we would have a presheaf, but not a sheaf. In particular, the set of open intervals $\{(n, n+2)\}_{n\in\mathbb{Z}}$ cover $\mathbb{R}$ and the function $f(x) = x$ is bounded on each of these intervals. However, the function $f(x) = x$ fails to be bounded on $\mathbb{R}$.

Example 4.8. Let X be any topological space and $p \in X$ a point. Let S be any set. Then

$$\iota_{*,p}S(U) = \begin{cases} S & \text{if } p \in U \\ \{*\} & \text{if } p \notin U, \end{cases}$$

[1] There are many (equivalent) ways to define the sheaf condition–we give one. See [Vak24] for others.

with restriction maps id_S and $s \mapsto *$, defines a sheaf on X with values in Set.

Definition 4.9 (Full Subcategory)**.** We say a category $\mathscr{C}$ is a full subcategory of a category $\mathscr{D}$ if $\mathrm{ob}\,\mathscr{C} \subseteq \mathrm{ob}\,\mathscr{D}$, and for all $x, y \in \mathrm{ob}\,\mathscr{C}$, we have $\mathrm{Hom}_{\mathscr{C}}(x,y) = \mathrm{Hom}_{\mathscr{D}}(x,y)$.

Example 4.10 (Category of Sheaves)**.** We define the category of sheaves on X (with values in Ab), denoted $\mathrm{Sh}(X)$, to be the full subcategory of $\mathrm{Psh}(X)$ restricted to objects which are sheaves.

Remark 4.11*.* Just as for presheaves, the category of sheaves on X with values in Ab is an abelian category.

5. Sheaf Cohomology

Seeing as (pre)sheaves in abelian groups form an abelian category, we can define the cohomology of a sheaf. As is done in group cohomology, the cohomology is defined using derived functors.

Definition 5.1 (Injective Object)**.** An object E in an abelian category $\mathscr{A}$ is *injective* if, for every monomorphism $g : A \to B$ and every $f : A \to E$, there exists $h : B \to E$ such that $f = hg$.

Definition 5.2 (Injective Resolution)**.** An *injective resolution* of $A \in \mathrm{ob}\,\mathscr{C}$, where $\mathscr{C}$ is an abelian category, is an exact sequence

$$\mathbf{E} \quad := \quad 0 \to A \xrightarrow{\mu} E^0 \xrightarrow{d_0} E^1 \xrightarrow{d_1} E^2 \to \dots$$

in which each E^n is injective. If $\mathbf{E}$ is an injective resolution of A, then its *deleted injective resolution* is the complex

$$\mathbf{E}^A \quad := \quad 0 \to E^0 \xrightarrow{d_0} E^1 \xrightarrow{d_1} E^2 \to \dots$$

Theorem 5.3. *The category of sheaves on abelian groups has enough injectives, i.e. for every object it is possible to construct an injective resolution.*[2]

Definition 5.4 (Additive Functor)**.** If $\mathscr{C}$ and $\mathscr{D}$ are additive categories, a functor $T : \mathscr{C} \to \mathscr{D}$ (of either variance) is *additive* if, for all A, B and all $f, g \in \mathrm{Hom}_{\mathscr{C}}(A,B)$, we have

$$T(f+g) = Tf + Tg;$$

that is, the function $\mathrm{Hom}_{\mathscr{C}}(A,B) \to \mathrm{Hom}_{\mathscr{D}}(TA,TB)$, given by $f \mapsto Tf$, is a homomorphism of abelian groups.

Definition 5.5 (Right Derived Functor)**.** A *right derived functor* R^nT, where $T : \mathscr{A} \to \mathscr{C}$ is an additive covariant functor between abelian categories. Choose, once for all, an injective resolution

$$\mathbf{E} \quad := \quad 0 \to B \xrightarrow{\mu} E^0 \xrightarrow{d_0} E^1 \xrightarrow{d_1} E^2 \to \dots$$

of every object B, form the complex $T\mathbf{E}^B$, where $\mathbf{E}^B$ is the deleted injective resolution, and take homology:

$$(R^nT)B = H^n(T\mathbf{E}^B) = \frac{\ker Td^n}{\mathrm{im}\,Td^{n-1}}.$$

[2]See [Rot09] Proposition 5.97 for a proof.

Definition 5.6 (Global Sections Functor)**.** Let X be a topological space, and let $\mathrm{Sh}(X)$ be the category of sheaves of abelian groups on X. The *global sections functor* $\Gamma : \mathrm{Sh}(X) \to \mathsf{Ab}$ is defined by

$$\Gamma(\mathscr{F}) = \mathscr{F}(X),$$

where $\mathscr{F}(X)$ is the group of sections of $\mathscr{F}$ over the entire space X. For a morphism $\varphi : \mathscr{F} \to \mathscr{G}$ of sheaves, $\Gamma(\varphi) = \varphi_X : \mathscr{F}(X) \to \mathscr{G}(X)$.

Definition 5.7 (Sheaf Cohomology)**.** If X is a topological space, then *sheaf cohomology* is defined, for every sheaf $\mathscr{F}$ over X, by

$$H^q(\mathscr{F}) = R^q\Gamma(\mathscr{F}).$$

In short, take an injective resolution $\mathbf{E}$ of $\mathscr{F}$, delete $\mathscr{F}$ to obtain $\mathbf{E}^{\mathscr{F}}$, apply Γ, and take homology:

$$H^q(\mathscr{F}) = H^q(\Gamma\mathbf{E}^{\mathscr{F}}).$$

Remark 5.8. In practice, sheaf cohomology can be computed using finer resolutions, like Čech cohomology, which is more concrete for explicit calculations. We explore this in the examples below.

6. Examples

To better understand sheaf cohomology, we compute it explicitly for a simple case using Čech cohomology, which approximates the derived functor definition with combinatorial tools. We begin by briefly investigating conditions to use Čech cohomology, and then consider a few easy examples.

Definition 6.1 (Locally Finite)**.** A family of subsets $\{A_\alpha\}$ of a topological space X is called *locally finite* if for every point $x \in X$, there exists an open neighborhood U of x such that U intersects only finitely many of the A_αs.

Definition 6.2 (Paracompact)**.** A topological space X is called *paracompact* if every open cover $\{U_\alpha\}$ of X has an open locally finite refinement $\{V_\beta\}$ such that for each β, there exists α with $V_\beta \subseteq U_\alpha$.

Definition 6.3 (Hausdorff)**.** A topological space X is called *Hausdorff* (or a T_2 space) if for any two distinct points $x, y \in X$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.

Remark 6.4. Some commonly seen examples of paracompact Hausdorff spaces are: manifolds, metric spaces, CW complexes.

Theorem 6.5. *If X is a paracompact Hausdorff space and $\mathscr{F}$ is a sheaf of X, then $\check{H}^i(X, \mathscr{F}) \simeq H^i(X, \mathscr{F})$.*[3]

Definition 6.6 (Čech Cohomology, Informal)**.** For a presheaf $\mathscr{F} \in \mathrm{Sh}(X)$, an open cover $\mathscr{U} = \{U_i\}_{i\in I}$ of X, and $q \geq 0$, the *Čech cochain group* $\check{C}^q(\mathscr{U}, \mathscr{F})$ assigns to each set of $q+1$ open sets in $\mathscr{U}$ with nontrivial intersection a section over $U_{i_0} \cap \cdots \cap U_{i_q}$. The differential $d : \check{C}^q \to \check{C}^{q+1}$ is defined by alternating sums of restrictions. The *Čech cohomology* is

$$\check{H}^q(\mathscr{U}, \mathscr{F}) = H^q(\check{C}^\bullet(\mathscr{U}, \mathscr{F})).$$

[3]A detailed proof can be found in Theorem 5.10.1 from Godement's *Topologie algébrique et théorie des faisceaux*.

Definition 6.7 (Čech Cohomology, Explicit)**.** Let X be a topological space, $\mathscr{U} = \{U_i\}$ an open cover of X, and $\mathscr{F}$ a presheaf of abelian groups on X. The *Čech cochain complex* $\check{C}^\bullet(\mathscr{U}, \mathscr{F})$ is defined by

$$\check{C}^q(\mathscr{U}, \mathscr{F}) := \prod_{i_0 < \cdots < i_q} \mathscr{F}(U_{i_0} \cap \cdots \cap U_{i_q}).$$

The coboundary map $\delta^q : \check{C}^q \to \check{C}^{q+1}$ is given by

$$(\delta\alpha)_{i_0 \cdots i_{q+1}} = \sum_{j=0}^{q+1} (-1)^j \, \alpha_{i_0 \cdots \hat{i}_j \cdots i_{q+1}} \Big|_{U_{i_0} \cap \cdots \cap U_{i_{q+1}}}.$$

The *Čech cohomology* of $\mathscr{U}$ with coefficients in $\mathscr{F}$ is

$$\check{H}^q(\mathscr{U}, \mathscr{F}) := H^q\left(\check{C}^\bullet(\mathscr{U}, \mathscr{F})\right).$$

Remark 6.8*.* There exists connection between Čech cohomology and singular cohomology:

- *Singular cohomology* $H^n_{\text{sing}}(X, A)$ is defined by using the cochains on all continuous maps from standard simplices into X.
- *Čech cohomology* $\check{H}^n(X, A)$ is defined using open covers and examining the intersections of open sets and assembling data over them.
- If X is a nice space–specifically if it is homotopy equivalent to a CW complex–then
$$\check{H}^n(X, A) \simeq H^n_{\text{sing}}(X, A).$$

Definition 6.9 (Constant Sheaf)**.** For a locally connected space X, we define the *constant sheaf* with values in S as, for each open set $U \subseteq X$

$$S_X(U) := \prod_{CC(U)} S.$$

Where $CC(U)$ denotes the set of connected components of U. These can be thought of as functions taking a constant value in S on each connected component of U. The restriction maps thus correspond to function restriction.

Remark 6.10*.* The constant sheaf is a more general construction, in particular it is the image of the constant presheaf under a specific adjoint functor $\mathrm{Psh}(X) \to \mathrm{Sh}(X)$.

Example 6.11. Let $X = S^1$, and let $\mathbb{Z}_{S^1}$ be the constant sheaf. Choose a cover $\mathscr{U} = \{U_1, U_2\}$, where U_1 and U_2 are open arcs covering S^1, with $U_1 \cap U_2 = V_1 \sqcup V_2$, two disjoint open intervals. We compute

- $\check{C}^0(\mathscr{U}, \mathbb{Z}_{S^1}) = \mathbb{Z}_{S^1}(U_1) \times \mathbb{Z}_{S^1}(U_2) = \mathbb{Z} \times \mathbb{Z}$.
- $\check{C}^1(\mathscr{U}, \mathbb{Z}_{S^1}) = \mathbb{Z}_{S^1}(V_1) \times \mathbb{Z}_{S^1}(V_2) = \mathbb{Z} \times \mathbb{Z}$.
- Differential $d : \check{C}^0 \to \check{C}^1$, $d(n_1, n_2) = (n_2 - n_1, n_2 - n_1)$.
- Then, compute the cohomology:
  - $\check{H}^0 = \ker d = \{(n, n) : n \in \mathbb{Z}\} \simeq \mathbb{Z}$, the global sections.
  - $\check{H}^1 = \check{C}^1 / \operatorname{im} d = (\mathbb{Z} \times \mathbb{Z}) / \{(m, m) : m \in \mathbb{Z}\} \simeq \mathbb{Z}$.

Thus, $H^0(S^1, \mathbb{Z}_{S^1}) \simeq \mathbb{Z}$, $H^1(S^1, \mathbb{Z}_{S^1}) \simeq \mathbb{Z}$, reflecting the topology of S^1.

Example 6.12. Consider $X = \mathbb{R}$ with the constant sheaf $\mathbb{Z}_{\mathbb{R}}$.

Choose an open cover $\mathscr{U} = \{U_1, U_2\}$ where $U_1 = (-\infty, 1)$ and $U_2 = (0, \infty)$, with $U_1 \cap U_2 = (0, 1)$. Then, we hvae

- $\check{C}^0(\mathscr{U}, \mathbb{Z}_{\mathbb{R}}) = \mathbb{Z}_{\mathbb{R}}(U_1) \times \mathbb{Z}_{\mathbb{R}}(U_2) = \mathbb{Z} \times \mathbb{Z}$.
- $\check{C}^1(\mathscr{U}, \mathbb{Z}_{\mathbb{R}}) = \mathbb{Z}_{\mathbb{R}}(U_1 \cap U_2) = \mathbb{Z}$.
- Differential $d : \check{C}^0 \to \check{C}^1$, $d(n_1, n_2) = n_2 - n_1$.
- We compute the cohomology:
  - $\check{H}^0(\mathscr{U}, \mathbb{Z}_{\mathbb{R}}) = \ker d = \{(n,n) \in \mathbb{Z}^2\} = \mathbb{Z}$.
  - $\check{H}^1(\mathscr{U}, \mathbb{Z}_{\mathbb{R}}) = \mathbb{Z}/\operatorname{im} d = \mathbb{Z}/\{n_2 - n_1\} = \mathbb{Z}/\mathbb{Z} = 0$.

Thus, $\check{H}^0(\mathbb{R}, \mathbb{Z}_{\mathbb{R}}) \simeq \mathbb{Z}$ and $\check{H}^1(\mathbb{R}, \mathbb{Z}_{\mathbb{R}}) \simeq 0$. Note that the higher cohomology vanishes due to $\mathbb{R}$ being contractible.

7. Applications

There are useful applications of sheaf cohomology in various areas of mathematics. One such area is complex analysis, where it can be used to construct global functions from local data. An example of this is known as the first cousin problem. For this problem, the type of topological space we will be considering sheaves on is a complex manifold.

Definition 7.1 (Complex Manifold)**.** A *complex manifold* is a topological manifold that is locally homeomorphic to $\mathbb{C}^n$ with a holomorphic atlas of charts. That is, for any charts (U_i, φ) and (V_i, ψ) such that $U_i \cap V_i \neq \varnothing$,

$$\varphi \circ \psi^{-1}$$

is holomorphic.

On a complex manifold, a function is considered holomorphic or meromorphic if it is with respect to all charts. Let $\mathscr{O}$ and $\mathscr{M}$ denote the sheaves of holomorphic functions and meromorphic functions on a complex manifold, respectively.

Example 7.2 (The First Cousin Problem)**.** Let X be a complex manifold and let $\mathscr{U}$ be a cover of X by open sets U_i. For all U_i, let $h_i \in \mathscr{M}(U_i)$ be such that

$$h_i - h_j \in \mathscr{O}(U_i \cap U_j)$$

for all j. The problem, then, is to find an $h \in \mathscr{M}(X)$ such that

$$h|_{U_i} - h_i \in \mathscr{O}(U_i)$$

for all i. Now, consider the short exact sequence

$$0 \to \mathscr{O} \to \mathscr{M} \xrightarrow{\varphi} \mathscr{M}/\mathscr{O} \to 0,$$

from which we can derive the long exact sequence

$$0 \to H^0(X, \mathscr{O}) \to H^0(X, \mathscr{M}) \xrightarrow{\varphi_*} H^0(X, \mathscr{M}/\mathscr{O}) \xrightarrow{\delta} H^1(X, \mathscr{O}) \to \cdots.$$

We can then define a global section s on $\mathscr{M}/\mathscr{O}$ from the h_i. This means $s \in H^0(X, \mathscr{M}/\mathscr{O})$, and a specific first cousin problem has a solution if there is an $h \in H^0(X, \mathscr{M})$ such that

$$\varphi_*(h) = s$$

More generally, a first cousin problem is solvable for all open covers $\mathscr{U}$ and compatible sets of meromorphic functions if and only if the map φ_* is surjective, and this is the case if $H^1(X, \mathscr{O}) = 0$.

It is always the case that $H^1(X, \mathscr{O}) = 0$–and thus, that the first cousin problem is solvable–for a class of complex manifolds known as *Stein manifolds*. This fact is the result of a theorem in complex geometry known as Cartan's Theorem B.

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