

EXERCISES FOR INTRODUCTION TO MATHEMATICAL QUANTUM ERROR CORRECTION

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1. MONDAY

Exercise 1.1 (Complex Arithmetic). Simplify the following expressions in \mathbb{C} :

- (i) $i^3 + i^2 + i + 1$
- (ii) $(-3 + 2i)(6 - 8i)$
- (iii) $(9i)(-i)$

Exercise 1.2 (Pauli Matrix Multiplication). Let $I, X, Y, Z \in \mathbb{M}_2(\mathbb{C})$ be defined by

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- (i) Compute the matrix products I^2, IX, IY , and $. What can you guess is true about multiplying matrices in $\mathbb{M}_2(\mathbb{C})$ by I , in general?$
- (ii) Compute the matrix products XY and YX . Simplify both products into the form cZ , where $c \in \mathbb{C}$. Is c different in each case?

The matrix I is called the identity. The other three matrices X, Y, Z are called the *Pauli matrices*. Together, these generate the *Pauli group*, a multiplicative group of order 16.

Exercise 1.3 (Recalling Definitions).

- (i) Let A, B be sets. What is an *injective* function $f : A \rightarrow B$? What is a *surjective* function $f : A \rightarrow B$? What do we call a function $f : A \rightarrow B$ which is *both* injective and surjective?
- (ii) Write the definition of a *group* (G, \cdot) . What is the general difference between just a set and a group?¹
- (iii) What is the difference between a group and an *abelian* group?
- (iv) Let $S \subseteq G$ be a subset of a group. Write the definition of the *normalizer* $\mathcal{N}_G(S)$ and *centralizer* $C_G(S)$ of S in G .

Date: January 15, 2025.

¹You can be informal about this.

2. TUESDAY

Exercise 2.1 (Conjugates and Adjoint). Compute the following, where $*$ denotes the complex *conjugate* $(a + bi)^* = a - bi$ in \mathbb{C} and \dagger denotes the *adjoint* in \mathbb{C}^n :

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}^\dagger = (a_1^* \quad \cdots \quad a_n^*).$$

- (i) $i(i^*)$
- (ii) $(7 + i)^* + (1 + 3i) + (1 - 4i)^*$
- (iii)

$$\begin{pmatrix} i \\ 1 + i \\ 0 \\ 2 + 2i \end{pmatrix}^\dagger$$

Exercise 2.2 (Basics of \mathbb{C}^2). As mentioned in lecture, \mathbb{C}^2 is an example of a vector space over \mathbb{C} . Notably, the elements of \mathbb{C}^2 are ordered pairs (z, w) where z, w are complex numbers. The functions (linear transformations) in this case are realizable as 2×2 complex matrices $M_2(\mathbb{C})$.

- (i) What is the *dimension* of \mathbb{C}^2 ? In turn, what is the size of a basis for \mathbb{C}^2 ?
- (ii) Write down a *basis* β for \mathbb{C}^2 . That is, find a subset of \mathbb{C}^2 such that $\text{span } \beta$ gives you every possible ordered pair/vector $(z, w) \in \mathbb{C}^2$.
- (iii) Generalize your answers from (i) and (ii) to \mathbb{C}^n : determine its dimension and find a basis. These solutions should be analogous (look very similar) to your answers for \mathbb{C}^2 .

Exercise 2.3 (Recalling Definitions).

- (i) Write down the definition of a *vector space* from lecture. How does a vector space differ from an *inner product space*?
- (ii) What is a *Hilbert space* in finite dimensions?
- (iii) How do we define the *direct sum/coproduct* $\bigoplus \mathcal{W}_i$ of an indexed family of vector spaces $\{\mathcal{W}_i\}_{i \in I}$? How is this different from a vector space *product*? Do the definitions differ when $I = \{1, \dots, n\}$?

Exercise 2.4 (Challenge Problems). These problems are more abstract and require quite a bit more thinking (and probably, more background) than I expect from just two days of watching lecture. Still, if you are interested, here are two relevant challenge problems. [These questions were adapted from Chris Dodd at Illinois.]

- (i) Let \mathcal{V} be a vector space of dimension n . A flag \mathfrak{F} of length r in \mathcal{V} is a collection of subspaces $\{0\} = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_r = \mathcal{V}$, where $\mathcal{V}_i \neq \mathcal{V}_{i+1}$ for all i . Show that $r \leq n$. If $r = n$, we call the flag \mathfrak{F} *complete*. Show that \mathfrak{F} is complete *if and only if* $\dim(\mathcal{V}_i/\mathcal{V}_{i+1}) = 1$ for all i .³
- (ii) Let $T : \mathcal{V} \rightarrow \mathcal{W}$ and $S : \mathcal{W} \rightarrow \mathcal{U}$ be linear maps of vector spaces. Let T^* and S^* denote the dual maps $T^* : \mathcal{W}^* \rightarrow \mathcal{V}^*$ and $S^* : \mathcal{U}^* \rightarrow \mathcal{W}^*$, defined in the natural way $\varphi \mapsto \varphi \circ T$ or S . Show that $(S \circ T)^* = T^* \circ S^*$. Use this to show that if A, B are matrices so that AB exists, then $(AB)^t = B^t A^t$.

²Remember, this is the same as asking for the number of “copies” of \mathbb{C} in \mathbb{C}^2 .

³Here, $\mathcal{V}_i/\mathcal{V}_{i+1}$ is the *quotient space*. That is, the elements of the underlying group are additive cosets $v + \mathcal{V}_{i+1}$, where $v \in \mathcal{V}_i$, and the operations work via $(v + \mathcal{V}_{i+1}) + (k + \mathcal{V}_{i+1}) = (v + k) + \mathcal{V}_{i+1}$ and $c(v + \mathcal{V}_{i+1}) = cv + \mathcal{V}_{i+1}$, for a scalar $c \in \mathbb{C}$.

3. WEDNESDAY

Exercise 3.1 (Poster). If I assigned you a topic/result today, get it done during our work time today, or tonight, and send me the email at dwiggins@imsa.edu.

Exercise 3.2 (Computing Dimensions). Recall that $\dim(\mathbb{C}^n \oplus \mathbb{C}^m) = n + m$ and $\dim(\mathbb{C}^n \otimes \mathbb{C}^m) = n \cdot m$, where \oplus is the *direct sum* of Hilbert spaces and \otimes is the *tensor product*, as described during lecture today. Compute the following dimensions, leaving your answers un-simplified (i.e., I want to see the products of the numbers, not a final simplified dimension).⁴

- (i) $(\mathbb{C}^2 \otimes \mathbb{C}^4) \oplus \mathbb{C}^6$
- (ii) $(\mathcal{H}^A \otimes \mathcal{H}^B) \oplus \mathcal{K}$, where $\mathcal{H}^A \simeq \mathbb{C}^4$, $\mathcal{H}^B \simeq \mathbb{C}^7$, and $\mathcal{K} \simeq \mathbb{C}^{32}$
- (iii) $(\mathcal{H}^A \otimes \mathcal{H}^B) \oplus (\mathcal{H}^C \otimes \mathcal{H}^D) \oplus (\mathcal{H}^E \otimes \mathcal{H}^F)$, where $\mathcal{H}^A \simeq \mathbb{C}$, $\mathcal{H}^B \simeq \mathbb{C}^2$, $\mathcal{H}^C \simeq \mathbb{C}^3$, $\mathcal{H}^D \simeq \mathbb{C}^4$, $\mathcal{H}^E \simeq \mathbb{C}^5$, and $\mathcal{H}^F \simeq \mathbb{C}^6$.

Exercise 3.3 (Recalling Definitions).

- (i) What is a *superoperator*?
- (ii) What is a *quantum channel*? Why are quantum channels the functions we care about, physically?
- (iii) State the *Kraus Representation Theorem*, also known as the Operator-Sum Representation Theorem.
- (iv) What is a *correctable error*?

Exercise 3.4 (Challenge Problems). These problems are more abstract and require quite a bit more thinking (and probably, more background) than I expect from just three days of watching lecture. Still, if you are interested, here are two relevant challenge problems. [These questions were adapted from Eric Chitambar at Illinois.]

- (i) Define the three-dimensional “dephasing” quantum channel Δ to be the superoperator

$$\mathbb{B}(\mathbb{C}^3) \xrightarrow{\Delta} \mathbb{B}(\mathbb{C}^3)$$

$$\begin{pmatrix} a & b & c \\ b^* & d & e \\ c^* & e^* & f \end{pmatrix} \longmapsto \begin{pmatrix} a & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & f \end{pmatrix}.$$

Determine the Choi matrix J_Δ for the superoperator Δ .

- (ii) Remember, we say $\Phi : \mathbb{B}(\mathbb{C}^d) \rightarrow \mathbb{B}(\mathbb{C}^d)$ is a *unital* superoperator if $\Phi : I_d \mapsto I_d$. Prove that the following are equivalent (i.e., prove the “if and only if” relationship between the statements).
 - (i) Φ is unital.
 - (ii) $\text{tr}_A(J_\Phi) = I^B$

This was one of our four results relating superoperators to their Choi matrices.

⁴You may use that dimension is an “isomorphism invariant.” That is, if the dimensions of \mathbb{C}^n is n , then the dimension of $\mathcal{H} \simeq \mathbb{C}^n$ is also n .