

TWO-DIMENSIONAL DAGGER TQFTS ARE PRETTY SIMPLE

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It is well-known that two-dimensional topological quantum field theories are commutative Frobenius algebras. In 1994, Durhuus and Jonsson, showed that when Atiyah’s axioms are made \dagger -compatible, we can extract some positive real numbers λ_i that uniquely characterize such a theory. It is clear that this classification corresponds to the idea that “unitary two-dimensional field theories are semisimple commutative Frobenius algebras,” but I could not find this story told in the way I wanted anywhere, so I wrote this note.

1. DAGGER THINGS

Let $(\mathcal{B}ord_2, \amalg, \emptyset, \tau)$ and $(\mathcal{F}dHilb, \otimes, \mathbb{C}, \gamma)$ denote the bordism class category in dimension 2 and the category of finite-dimensional Hilbert spaces, respectively, equipped with their usual symmetric monoidal structures. The braidings are the standard bordism twist

$$\tau_{M,N} = \text{crossing} : M \amalg N \xrightarrow{\sim} N \amalg M$$

and the swap operator $\gamma_{A,B} : \mathcal{H}^A \otimes \mathcal{H}^B \xrightarrow{\sim} \mathcal{H}^B \otimes \mathcal{H}^A$ sending $a \otimes b \mapsto b \otimes a$.

The symmetric monoidal functor category therebetween $\mathcal{F}un^{\otimes}(\mathcal{B}ord_2, \mathcal{F}dHilb)$ is the category of two-dimensional topological quantum field theories. The famous classification result is that there is an equivalence

$$\Omega : \mathcal{F}un^{\otimes}(\mathcal{B}ord_2, \mathcal{F}dHilb) \xrightarrow{\sim} \mathcal{C}Frob$$

given by assigning to each theory \mathcal{Z} a commutative Frobenius algebra $\mathfrak{A} = \mathcal{Z}(\mathbb{S}^1)$ with multiplication $m = \mathcal{Z}(\text{cap})$, comultiplication $\Delta = \mathcal{Z}(\text{cup})$, unit $\eta = \mathcal{Z}(\text{point})$, and counit $\varepsilon = \mathcal{Z}(\text{point})$. However, this picture ignores a nifty piece of structure that can be tacked on to both bordisms and Hilbert spaces: a dagger. For our purposes, a dagger \dagger will mean an involutive contravariant endofunctor that does nothing on objects. Seeing as our categories $\mathcal{B}ord_2$ and $\mathcal{F}dHilb$ already have a symmetric monoidal structure, any dagger structure we add must be compatible with this, i.e., $(f \otimes g)^{\dagger} = f^{\dagger} \otimes g^{\dagger}$ for any maps f and g , and all associators, unitors, and braidings must be unitary.¹

It can be easily checked that on $\mathcal{B}ord_2$, the endofunctor \dagger given by reversing bordisms yields a symmetric monoidal dagger category. Likewise, the usual \dagger on $\mathcal{F}dHilb$ given by taking adjoints is a dagger structure compatible with \otimes .

Note that if we were to look at the dualities exhibited in $\mathcal{B}ord_2$, orientation-reversal would give the dual on objects, so the dual maps come from the “transposed” bordisms. In a sense, this analogy tells us that reversing the orientations corresponds

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James Pascaleff exposed me to semisimple topological field theories and [DJ94], so this note is just a slight rephrasing of his lectures. Charles Rezk made some helpful comments while I thought about this.

¹When working in a dagger category, we tend to borrow terminology from Hilbert spaces, so unitary means a morphism’s inverse is given by its dagger.

to taking the complex conjugate, since the dagger of a map is just a complex conjugate away from being the transpose. I find this to be helpful intuition, but dualities will not play any role for us.

Definition 1.1 (Dagger Functor). A dagger functor $f : (\mathcal{C}, \dagger) \rightarrow (\mathcal{D}, \dagger)$ between dagger categories is an ordinary functor so that the square

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{D} \\ \dagger \downarrow & & \downarrow \dagger \\ \mathcal{C} & \xrightarrow{f} & \mathcal{D} \end{array}$$

commutes. Of course, daggers are the identity on objects, so being a dagger functor only requires a condition on morphisms.

Write $\mathcal{F}\text{un}_{\dagger}^{\otimes}(\mathcal{B}\text{ord}_2, \mathcal{F}\text{dHilb})$ for the category of two-dimensional dagger (or unitary) field theories, i.e., the symmetric monoidal functor category that respects the \dagger in bordisms and Hilbert spaces in the above sense.²

Definition 1.2 (Dagger Frobenius Structure). A Frobenius structure $(\mathfrak{A}, m, \Delta, \eta, \varepsilon)$ in $\mathcal{F}\text{dHilb}$ is a dagger (or unitary) Frobenius algebra if $m^{\dagger} = \Delta$ and $\eta^{\dagger} = \varepsilon$.

Evidently, if \mathcal{Z} is a two-dimensional dagger field theory, then

$$\mathcal{Z}(\langle \mathbb{I} \rangle)^{\dagger} = \mathcal{Z}(\langle \mathbb{I}^{\dagger} \rangle) = \mathcal{Z}(\mathbb{I})$$

and

$$\mathcal{Z}(\mathfrak{m})^{\dagger} = \mathcal{Z}(\mathfrak{m}^{\dagger}) = \mathcal{Z}(\mathfrak{m}),$$

so we get a map

$$\begin{array}{ccc} \mathcal{F}\text{un}^{\otimes}(\mathcal{B}\text{ord}_2, \mathcal{F}\text{dHilb}) & \xrightarrow{\Omega} & \mathcal{C}\text{Frob} \\ \cup & & \cup \\ \mathcal{F}\text{un}_{\dagger}^{\otimes}(\mathcal{B}\text{ord}_2, \mathcal{F}\text{dHilb}) & \xrightarrow{\mathfrak{U}} & \mathcal{C}\text{Frob}^{\dagger} \end{array}$$

which is an equivalence. Thus, two-dimensional dagger field theories are exactly commutative dagger Frobenius algebras. But, what are dagger Frobenius algebras?

2. ON H^* -ALGEBRAS

To continue with our classification, we now diverge from field theories for a bit, turning our attention to a certain type of $*$ -algebra that crops up in quantum foundations. In 1945, Ambrose originally defined H^* -algebras as sort of Banach algebra, but following Heunen and Vicary, we omit the norm requirement.

Definition 2.1 (H^* -Algebra). An H^* -algebra is a \mathbb{C} -algebra \mathfrak{A} such that $(\mathfrak{A}, \langle - | - \rangle_{\mathfrak{A}})$ is simultaneously a Hilbert space equipped with an antilinear involution $\dagger : \mathfrak{A} \rightarrow \mathfrak{A}$ so that for all $a, b, c \in \mathfrak{A}$,

$$\langle ab | c \rangle_{\mathfrak{A}} = \langle b | a^{\dagger} c \rangle_{\mathfrak{A}} = \langle a | c b^{\dagger} \rangle_{\mathfrak{A}}.$$

²I am suppressing two adjectives here.

Per usual when working with $*$ -algebras, the quintessential ones are matrix algebras, or subalgebras of matrix algebras. Given a finite-dimensional Hilbert space \mathcal{H} and some positive real number λ , define an H^* -algebra $\mathfrak{B}(\mathcal{H}, \lambda)$ whose underlying algebra is the set of linear operators $\mathfrak{B}(\mathcal{H})$, whose involution is given by the adjoint, and whose inner product $\langle - | - \rangle : \mathfrak{B}(\mathcal{H}, \lambda)^2 \rightarrow \mathbb{C}$ is $\langle a | b \rangle = \lambda \operatorname{tr}(a^\dagger b)$. Ambrose uses these algebras $\mathfrak{B}(\mathcal{H}, \lambda)$ to give a Wedderburn-Artin classification result for finite-dimensional H^* -algebras.

Theorem 2.2 ([Amb45]). *Let \mathfrak{A} be a finite-dimensional H^* -algebra. Then, there is an orthogonal direct sum decomposition $\mathfrak{A} \simeq \mathfrak{B}(\mathcal{H}_1, \lambda_1) \oplus \mathfrak{B}(\mathcal{H}_2, \lambda_2) \oplus \cdots \oplus \mathfrak{B}(\mathcal{H}_n, \lambda_n)$ where n is a natural number, $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_n$ are finite-dimensional Hilbert spaces, and $\lambda_1, \lambda_2, \dots, \lambda_n$ are positive real numbers.*

The following can then be obtained rather easily.

Lemma 2.3 ([HV19]). *A monoid (\mathfrak{A}, m, η) internal to $\mathcal{F}\mathbf{dHilb}$ is a symmetric dagger Frobenius monoid if and only if it is a finite-dimensional H^* -algebra, where the involution is defined by sending a vector $a \in \mathfrak{A}$ to*

$$\mathbb{C} \xrightarrow{\eta} \mathfrak{A} \xrightarrow{m^\dagger} \mathfrak{A} \otimes \mathfrak{A} \xrightarrow{\operatorname{id}_{\mathfrak{A}} \otimes a} \mathfrak{A} \otimes \mathbb{C} \rightarrow \mathfrak{A}.$$

Thus, symmetric dagger Frobenius monoids in $\mathcal{F}\mathbf{dHilb}$ are just a kind of finite-dimensional H^* -algebra. The adjective “symmetric” here corresponds to the equation

which is obviously satisfied when things are commutative.

3. CHARACTERIZATION

Let \mathcal{Z} be a two-dimensional dagger field theory. We already know how to obtain a commutative dagger Frobenius algebra $\mathcal{Z}(\mathbb{S}^1) = \mathcal{Z}(\mathbb{S}^1)$ by passing through the usual equivalence. Using the language of H^* -algebras, we obtain the following.

Theorem 3.1 ([DJ94]). *Two-dimensional dagger field theories \mathcal{Z} give rise to and are classified by $n = \dim(\mathcal{Z}(\mathbb{S}^1))$ many positive real numbers $\lambda_1, \lambda_2, \dots, \lambda_n$.*

Proof. By Lemma 2.3, the Frobenius algebra $\mathcal{Z}(\mathbb{S}^1)$ is a finite-dimensional H^* -algebra. Then, Theorem 2.2 gives us a decomposition

$$\mathcal{Z}(\mathbb{S}^1) \simeq \mathfrak{B}(\mathcal{H}_1, \lambda_1) \oplus \mathfrak{B}(\mathcal{H}_2, \lambda_2) \oplus \cdots \oplus \mathfrak{B}(\mathcal{H}_n, \lambda_n).$$

Since $\mathcal{Z}(\mathbb{S}^1)$ is commutative with respect to $m = \mathcal{Z}(\mathbb{S})$, each of the summands $\mathfrak{B}(\mathcal{H}_i, \lambda_i)$ must be one-dimensional, so $n = \dim(\mathcal{Z}(\mathbb{S}^1))$. \square

Evidently, we have ended up with semisimplicity, since

$$\mathcal{Z}(\mathbb{S}^1) \simeq \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \cdots \oplus \mathbb{C}e_n$$

with pointwise operations, where the e_1, e_2, \dots, e_n form a basis of orthogonal idempotents. Then, the numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ are the images of the orthogonal idempotents under the trace map $\varepsilon = \mathcal{Z}(\mathbb{D})$.

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