General Topology

A collection of notes on major definitions and results, proofs, and commentary based on the corresponding course at Illinois, as instructed by Mineyev

Lecture Notes By

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Disclaimer
The lecture notes in this document were based on General Topology [535], as instructed by Igor Mineyev [Department of Mathematics] in the Fall semester of 2024 [FA24] at the University of Illinois Urbana-Champaign. All non-textbook approaches, exercises, and comments are adapted from Mineyev's lectures.

Textbook
Many of the exercises and presentations were selected from *Topology, Second Edition*, by James R. Munkres.

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All mathematics begins with the empty set.

– Igor Mineyev

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1.1 Notations

For the purpose of this course, the set of real numbers $\mathbb{R}$ will look like the real line, not paying as close attention to the precise logical model. Similarly, $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$, $\mathbb{R}^3 := \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, and so forth.

Definition 1.1.1 (2-Sphere) *We define the set*

$$\mathbb{S}^2 := \{x \in \mathbb{R}^3 : d(0, x) = 1\},$$

taking the standard Euclidean metric.

Following suit, we can define the n-sphere.

Definition 1.1.2 (n-Sphere)

$$\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : d(0.x) = 1\}.$$

Now, you will notice that our n-sphere lives in $n + 1$-space. In what sense is $\mathbb{S}^2$ two-dimensional? Well, after doing some groundwork, we can show that *locally*, the 2-sphere is akin to a plane.[1]

1: More precisely, open neighborhoods of the 2-sphere map homeomorphically onto $\mathbb{R}^2$.

Definition 1.1.3 (Family) *A family of subsets of X is any subset $\mathcal{F} \subseteq \mathcal{P}(X)$.*

Definition 1.1.4 (Family Intersection) *The intersection of a family $\mathcal{F} \subseteq \mathcal{P}(X)$ is*[2]

$$\bigcap \mathcal{F} := \bigcap_{S \in \mathcal{F}} S := \{x : \textit{ for all } S \in \mathcal{F}, x \in S\}.$$

2: The union is defined similarly.

Definition 1.1.5 (Open Ball) *An open ball $B(x, r)$ of radius $r > 0$ around $x \in E$ is defined as*[3]

$$B(x, r) := \{y \in E : d(x, y) < r\}.$$

3: We take a metric space (E, d).

1.2 Building a Plane Topology

Returning to the plane $\mathbb{R}^2$, let us look at the families

$$\mathcal{B} := \left\{B(x, r) : x \in \mathbb{R}^2, r \in (0, \infty)\right\} \subseteq \mathcal{P}(\mathbb{R}^2)$$
$$\mathcal{T}(\mathcal{B}) := \left\{\bigcup \mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\right\} \subseteq \mathcal{P}(\mathbb{R}^2).$$

Clearly $\mathcal{B} \subseteq \mathcal{T}$, because for all $B(x, r)$, $\mathcal{T} \ni \bigcup\{B(x, r)\} = \{B(x, r)\}$. Now, consider $(0, 1) \times (0, 1) \subset \mathbb{R}^2$.[4] Now, $B(x, r_x) \subseteq (0, 1) \times (0, 1)$, so

4: This is simply the open unit square. We can fill the open square with open balls via $B(x, r_x)$, where a formula for r_x comes from the infimum of the distance between x and a point on the boundary.

$$(0,1)\times(0,1) = \bigcup_{x\in(0,1)\times(0,1)} B(x,r_x) \in \mathcal{T}.$$

In general, we can find such open balls for any "similar" shape. We will develop the notion of open sets by identifying them with elements of $\mathcal{T}$.

Proposition 1.2.1

(i) For all $U_1, U_2 \in \mathcal{T}$, $U_1 \cup U_2 \in \mathcal{T}$.
(ii) For any subfamily $\mathcal{F} \subseteq \mathcal{T}$, $\bigcup \mathcal{F} \in \mathcal{T}$.[5]
(iii) For all $U_1, U_2 \in \mathcal{T}$, $U_1 \cap U_2 \in \mathcal{T}$.
(iv) For any finite subfamily of $\mathcal{F} \subseteq \mathcal{T}$, $\bigcap \mathcal{F} \in \mathcal{T}$.

5: That is, $\mathcal{T}$ is closed under arbitrary unions.

(i) *Proof.* We know $U_1 = \bigcup \mathcal{B}_1$ and $U_2 = \bigcup \mathcal{B}_2$ for some families $\mathcal{B}_1, \mathcal{B}_2$. Then,

$$U_1 \cup U_2 = \left(\bigcup \mathcal{B}_1\right) \cup \left(\bigcup \mathcal{B}_2\right) = \bigcup (\mathcal{B}_1 \cup \mathcal{B}_2) \in \mathcal{T}.$$

□

(ii) *Proof.* Take any $\mathcal{F} \subseteq \mathcal{T}$. Now, $\bigcup \mathcal{F} = \bigcup_{U\in\mathcal{F}} U$. We know U is a union of balls in $\mathbb{R}^2$, so $U = \bigcup \mathcal{B}_U$.[6] Then,

$$\bigcup \mathcal{F} = \bigcup_{U\in\mathcal{F}} U = \bigcup_{U\in\mathcal{F}} \left(\bigcup \mathcal{B}_U\right) = \bigcup_{B\in\bigcup_{U\in\mathcal{F}}\mathcal{B}_U} B = \bigcup \mathcal{F}' \in \mathcal{T}.$$

□

6: Take $\mathcal{B}_U \subseteq \mathcal{B}$.

(iii) *Proof.* We have $U_1 = \bigcup \mathcal{B}_1$ and $U_2 = \bigcup \mathcal{B}_2$, as before, so

$$U_1 \cap U_2 = \left(\bigcup \mathcal{B}_1\right) \cap \left(\bigcup \mathcal{B}_2\right) = \bigcup_{\substack{B_1\in\mathcal{B}_1\\ B_2\in\mathcal{B}_2}} (B_1 \cap B_2).$$

For each pair B_1, B_2, choose a family $\mathcal{F}_{B_1,B_2}$ of open balls such that $\bigcup \mathcal{F}_{B_1,B_2} = B_1 \cap B_2$. To construct $\mathcal{F}_{B_1,B_2}$, for all $x \in B_1 \cap B_2$, choose a sufficiently small radius r_x such that $B(x, r_x) \subseteq B_1 \cap B_2$.[7] Then, $B_1 \cap B_2 = \bigcup_{x\in B_1\cap B_2} B(x, r_x)$, so $U_1 \cap U_2 \in \mathcal{T}$, given our knowledge about unions.

(iv) Use induction. □

7: Is this possible? Well, we can pick the radius

$$r_x := \min \begin{Bmatrix} r_1 - d(x_1, x) \\ r_2 - d(x_2, x) \end{Bmatrix}$$

where $B_1 = B(x_1, r_1)$ and $B_2 = B(x_2, r_2)$. Proving that $B(x, r_x) \subseteq B_1 \cap B_2$ is just an exercise in the triangle inequality.

Remark 1.2.1 We cannot prove that for *any* subfamily, their intersection lies in $\mathcal{T}$. For instance, consider $B_\varepsilon := B(0, 1+\varepsilon)$ and $\mathcal{F} := \{B_\varepsilon : \varepsilon > 0\}$. Well, $\bigcap \mathcal{F} = \overline{B}(0,1) \notin \mathcal{T}$.[8]

8: In this case, $\overline{B}(0,1)$ is the closed ball, not the closure of the open ball.

We also trivially have that $\varnothing \in \mathcal{T}$ and $\mathbb{R}^2 \in \mathcal{T}$, taking unions as needed.[9]

9: These properties will provide motivation for the definition of a *topology* $\mathcal{T}$.

1.3 Spaces and Bases

Definition 1.3.1 (Metric Space) *Recall that a metric space is an ordered pair (X, d), where X is a set and $d : X \times X \to [0, \infty)$ such that*[10]

(i) $d(x, y) = 0$ if and only if $x = y$.

10: The function d is called the *metric* or *distance* function. The third property is the *triangle inequality*.

(ii) $d(x, y) = d(y, x)$.
(iii) $d(x, y) + d(y, z) \geq d(x, z)$.

Topologies can loosen our conditions so that we no longer need a metric function d on the set. Topologies are, thus, *weaker* structures than metrics.

Definition 1.3.2 (Topology) *A topology on a set X is any family $\mathcal{T}$ of subsets of X such that,*

(i) $\varnothing, X \in \mathcal{T}$
(ii) $\mathcal{T}$ *is closed under arbitrary unions.*
(iii) $\mathcal{T}$ *is closed under finite intersections.*

Remark 1.3.1 Every metric induces a topology, following the form of the proof from the discussion of the plane topology.[11]

Definition 1.3.3 (Topological Space) *The ordered pair $(X, \mathcal{T})$ is called a topological space.*[12]

Remark 1.3.2 As is usual with ordered tuple structures, we will prefer to write a space $(X, \mathcal{T})$ as simply X.[13]

Definition 1.3.4 (Open Set) *A subset $U \subseteq X$ is called an open set if $U \in \mathcal{T}$.*

Definition 1.3.5 (Closed Set) *A subset $C \subseteq X$ is called closed if $X \setminus C \in \mathcal{T}$.*[14]

Definition 1.3.6 (Basis of Topology) *Let X be a set. A basis $\mathcal{B}$ of a topology on X is a family of subsets*[15] *such that*

(i) *for all $x \in X$, there exists $B \in \mathcal{B}$, such that $x \in B$.*[16]
(ii) *for all $B_1, B_2 \in \mathcal{B}$, for any $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.*[17]

Based on our proof in $\mathbb{R}^2$, it is clear to see that the metric gives a basis, which provides us with a topology. In general, we simply need the basis to generate a topology.

Definition 1.3.7 (Topology Generated by a Basis) *Let X be a set and $\mathcal{B}$ be a basis of a topology. The following are equivalent definitions of the topology generated by the basis $\mathcal{T}(\mathcal{B})$:*[18]

(i)

$$\mathcal{T}(\mathcal{B}) := \{U \subseteq X : \text{ for all } x \in U \text{ there is } B \in \mathcal{B} \text{ s.t. } x \in B \subseteq U\}.$$

(ii)

$$\mathcal{T}(\mathcal{B}) := \left\{\bigcup \mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\right\}.$$

11: We form our topology by unioning the subfamilies of all possible balls.

12: We refer to the set X as the *underlying* set of the topological space.

13: This is only valid when the topology is clear for context, or clarified at the beginning of a statement.

14: That is, a set is closed if its complement is open.

15: Remember, this means $\mathcal{B} \subseteq \mathcal{P}(X)$.

16: Equivalently, $\bigcup \mathcal{B} = X$.

17: That is, $\mathcal{B}$ should be "dense enough." Equivalently, for any $B_1, B_2 \in \mathcal{B}$, there exists a subfamily $\mathcal{B}' \subseteq \mathcal{B}$ such that $\bigcup \mathcal{B}' = B_1 \cap B_2$.

18: To prove both topologies are equal, simply prove both directions of inclusion.

Proposition 1.3.1 *The generated topology $\mathcal{T}(\mathcal{B})$ is a topology.*

Proof. Follow the recipe shown for the plane. □

1.4 Comparing Topologies

Example 1.4.1 Let $X := [0, 1] \subset \mathbb{R}$. We could define a topology

$$\mathcal{T}_1 := \{\varnothing, [0, 1]\} \subset \mathcal{P}([0, 1]).$$

This is the *trivial* topology on $[0, 1]$.[19] We could also define another topology $\mathcal{T}_2 := \mathcal{P}([0, 1])$. This is called the *discrete* topology on $[0, 1]$.

19: In general, the trivial topology on any set X is just $\mathcal{T} := \{\varnothing, X\}$. It is the smallest, via cardinality, possible topology. Similarly, the discrete topology on X is $\mathcal{T} := \mathcal{P}(\mathcal{X})$

Now that we have multiple topologies, it is natural to wish to compare them. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X.

Definition 1.4.1 (Finer/Stronger Topology) *If we have $\mathcal{T}_1 \subseteq \mathcal{T}_2$, then we say $\mathcal{T}_1$ is smaller/courser/weaker and $\mathcal{T}_2$ is larger/finer/stronger.*

Remark 1.4.1 An equivalent way to characterize the discrete topology is to take a basis defined by

$$\mathcal{B}_{\text{disc}} := \{\{x\} : x \in X\},$$

the set of singletons.[20] Then, $\mathcal{T}(\mathcal{B}_{\text{disc}})$ is the discrete topology on X.

20: Sometimes you may see this called the *discrete* basis.

Lemma 1.4.1 (Basis Criterion) *Let $\mathcal{T}$ be a topology on X and let $\mathcal{C} \subseteq \mathcal{T}$. Then, the following are equivalent:*

(i) For any $U \in \mathcal{T}$ and for all $x \in U$, there exists $B \in \mathcal{C}$ such that $x \in B \subseteq U$.
(ii) $\mathcal{C}$ is a basis, and $\mathcal{T}(\mathcal{C}) = \mathcal{T}$.

Proof. For (i) $\Rightarrow$ (ii), take any $x \in X$. We have that $X \in \mathcal{T}$, so there exists a $B \in \mathcal{C}$ so that $x \in B \subseteq X$. Thus, $\bigcup \mathcal{C} = X$. Take any $B_1, B_2 \in \mathcal{C}$. Suppose $x \in B_1 \cap B_2$. Since $\mathcal{T}$ is a topology, $B_1 \cap B_2 \in \mathcal{T}$. Thus, there exists $B_3 \in \mathcal{C}$ such that $x \in B_3 \subseteq U$, so $\mathcal{C}$ is a basis. It remains to be shown that $\mathcal{T}(\mathcal{C}) = \mathcal{T}$, but this is easy. We leave (ii) $\Rightarrow$ (i) as an exercise. □

1.5 New Topologies From Old

Definition 1.5.1 (Subspace Topology) *Let $(X, \mathcal{T})$ be a topological space. Suppose $A \subseteq X$. We define the subspace topology*

$$\mathcal{T}_A := \{A \cap U : U \in \mathcal{T}\} \subseteq \mathcal{P}(A).$$

Then, $(A, \mathcal{T}_A)$ is the induced subspace of the ambient space $(X, \mathcal{T})$.

Proposition 1.5.1 *$\mathcal{T}_A$, as above, is a topology.*[21]

21: This is a very powerful construction, giving us a way to form as many topological subspaces as we like.

Proof. We have that $\varnothing \in \mathcal{T}_A$, as $\varnothing = A \cap \varnothing$. Similarly, $A \in \mathcal{T}_A$, as $A = A \cap X$. Pick any subfamily of $\mathcal{T}_A$. It must be of the form $\{A \cap U : U \in \mathcal{F}\}$, for some $\mathcal{F} \subseteq \mathcal{T}$. Well,

$$\bigcup_{U \in \mathcal{F}} (A \cap U) = A \cap \bigcup_{U \in \mathcal{F}} U = A \cap \bigcup \mathcal{F} \in \mathcal{T}_A.$$

Finally, taking $\mathcal{F}$ to be finite this time,

$$\bigcap_{U \in \mathcal{F}} (A \cap U) = A \cap \bigcap_{U \in \mathcal{F}} U = A \cap \bigcap \mathcal{F} \in \mathcal{T}_A.$$

□

Lemma 1.5.2 *Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. If $\mathcal{T} = \mathcal{T}(\mathcal{B})$ for some basis $\mathcal{B}$ in X, then consider $\mathcal{B}_A := \{A \cap B : B \in \mathcal{B}\}$. Then, $\mathcal{B}_A$ is a basis for $\mathcal{T}_A$.*[22]

22: Remember, this is *two* statements. We have that $\mathcal{B}_A$ is a basis of a topology, and that $\mathcal{T}(\mathcal{B}_A) = \mathcal{T}_A$.

Proof. The proof is left as an exercise. □

We now consider a nice canonical construction.

Definition 1.5.2 (Set Disjoint Union) *Given two sets X, Y, the disjoint union $X \coprod Y$ is a the union of a bijective copy of X and bijective copy of Y, such that the copies are disjoint.*[23]

23: However, you will also see the disjoint union used for when $X \cap Y = \varnothing$.

One way of creating these copies is to take $\{0, 1\} \times X \cup Y$, where we only keep X in the 0 slice and Y in the 1 slice.

Lemma 1.5.3 *The family $\mathcal{T}_X \cup \mathcal{T}_Y$ is a basis of a topology on $X \coprod Y$.*

Definition 1.5.3 (Disjoint Union Topology) *Let $\{(X_i, \mathcal{T}_i)\}_{i \in \Lambda}$ be an arbitrary family of topological spaces. Then, the topology on*

$$X := \coprod_{i \in \Lambda} X_i$$

is precisely the topology $\mathcal{T}$ generated by the union:

$$\mathcal{T} := \mathcal{T}\left(\bigcup_{i \in \Lambda} \mathcal{T}_i\right).$$

Definition 1.5.4 (Subbasis) *A subbasis on a set X is a family $\mathcal{S} \subseteq \mathcal{P}(X)$ such that $\bigcup \mathcal{S} = X$.*[24]

24: That is, for all $x \in X$, there is an $S \in \mathcal{S}$ such that $x \in S$.

Remark 1.5.1 Any basis is a subbasis.

Definition 1.5.5 (Basis From Subbasis) *Given a subbasis $\mathcal{S}$ on X, we can produce another family $\mathcal{B}(\mathcal{S})$ defined by*[25]

$$\mathcal{B}(\mathcal{S}) := \left\{ \bigcap \mathcal{S}' : \mathcal{S}' \subseteq \mathcal{S} \text{ and } \mathcal{S}' \text{ is finite} \right\}.$$

25: We also need $\mathcal{S}'$ to be nonempty.

We can now form a topology generated by the subbasis $\mathcal{T}(\mathcal{B}(\mathcal{S}))$.

Definition 1.5.6 (Topology Induced by Function) *Let $(X, \mathcal{T}_X)$ be a topological space, and let $f : X \to Y$, where Y is just a set.*[26] *Define an induced topology*

$$\mathcal{T}_Y := \left\{ V \subseteq Y : f^{-1}(V) \in \mathcal{T}_X \right\}.$$

26: Say f is surjective.

Example 1.5.1 Put any equivalence relation $\sim$ on $[0, 1]$. Identify $0 \sim 1$. Then, $[0, 1]/\sim$ can be visualized as a circle. We can construct a map $q : [0, 1] \to [0, 1]/\sim$ such that $x \mapsto [x]$. Clearly, q is surjective.[27]

27: Convince yourself that there is a homeomorphism between the circle and $[0, 1]/\sim$.

Definition 1.5.7 (Quotient Topology) *Given a space $(X, \mathcal{T}_X)$ and an equivalence relation on $\sim$ on X. Then, the quotient topology on $X/\sim$ is the topology induced by the canonical quotient map. That is,*

$$\mathcal{T}_{X/\sim} := \{V \subseteq X/\sim \, : q^{-1}(V) \in \mathcal{T}_X\}.$$

We now give an informal definition of a *cell complex*, for the purposes of this course.

Definition 1.5.8 (Disk) *We define the closed n-disk to be*

$$\mathbb{D}^n := \{x \in \mathbb{R}^n : d(0, x) \leq 1\} = \overline{B}(0, 1).$$

Per this definition, $\mathbb{D}^0 = \{\text{pt}\}$. Similarly, $\mathbb{D}^1 = [-1, 1]$. Now, in each of $n = 0, 1, 2, \ldots$, take several copies of $\mathbb{D}^0, \mathbb{D}^1, \mathbb{D}^2, \ldots$. In general, the boundary of $\mathbb{D}^n$ is $\mathbb{S}^{n-1}$. We now form a process of "gluing" each of our copies to one another. In particular, we identify the boundary of a copy of $\mathbb{D}^n$ with a copy of $\mathbb{D}^{n-1}$. This is how we form the equivalence relation on the disjoint union of these discs. We call the quotient of the disjoint union, in this form, a *cell complex*.[28]

28: Take 525 for a better definition.

Example 1.5.2 Going back to $[0, 1]$, if we take one copy of $\mathbb{D}^0$ and one copy of $\mathbb{D}^1$, we glue the boundary of $\mathbb{D}^1$; i.e., $\{0, 1\}$, to the single point in $\mathbb{D}^0$. Then, the rest of the equivalence classes are singletons.

Definition 1.5.9 (Saturated Set) *Let X be a set and $\sim$ an equivalence relation. A subset $U \subseteq X$ is called saturated with respect to $\sim$ if for all $x, x' \in X$, then if $x \in U$ and $x' \sim x$, then $x' \in U$.*

Using the definition of saturated set, we can reformulate our quotient topology by saying[29]

$$\mathcal{T}_{X/\sim} = \{q(U) : U \text{ is saturated and } U \in \mathcal{T}_X\}.$$

29: Check this equivalence.

1.6 Homeomorphisms and Continuity

We now introduce the notion of a *homeomorphism*, which is an isomorphism in the category Top of topological spaces.[30]

30: The morphisms in Top are precisely continuous maps.

Definition 1.6.1 (Homeomorphism) *Given topological spaces s$(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, a homeomorphism is a bijection $f : X \to Y$ such that*[31]

(i) for all $U \in \mathcal{T}_X$, $f(U) \in \mathcal{T}_Y$.
(ii) for any $V \in \mathcal{T}_Y$, $f^{-1}(V) \in \mathcal{T}_X$.

31: That is, $U \in \mathcal{T}_X$ if and only if $f(U) \in \mathcal{T}_Y$.

Remark 1.6.1 If there exists a homeomorphism $\varphi : X \to Y$ between spaces, then we say X is "homeomorphic" to Y, and write $X \simeq Y$.

How does one show that two topological spaces X and Y are *not homeomorphic*? Consider $\mathbb{S}^1$ and $\mathbb{R}$. We have that $\mathbb{S}^1 \not\simeq \mathbb{R}$, but how can we say that no such homeomorphism exists? Well, we need to build some topological *invariants* to distinguish our spaces.

Definition 1.6.2 (Projective Plane I) *Start with $\mathbb{S}^2$. Describe $\sim$ on $\mathbb{S}^2$ by: for all $x \in S^2$, $x \sim -x$. Then, the equivalence classes are $[x] = \{x, -x\}$, so the quotient space is*

$$\mathbb{S}^2/\sim = \{[x] : x \in \mathbb{S}^2\}.$$

We denote $\mathbb{R}\mathbf{P}^2 := \mathbb{S}^2/\sim$.

Definition 1.6.3 (Projective Plane II) *Start with a copy of $\mathbb{D}^2$ and a copy of $\mathbb{S}^1$. Well, $\partial\mathbb{D}^2 = \mathbb{S}^1$. Then, we can map $\partial\mathbb{D}^2 \to \mathbb{S}^1$, where half of $\partial\mathbb{D}^2$ gets sent to the complete circle $\mathbb{S}^1$. This gives us our equivalence relation $\sim'$.*[32] *Then, $\mathbb{R}\mathbf{P}^2 := (\mathbb{D}^2 \coprod \mathbb{S}^1)/\sim'$.*

32: That is, antipodal points in $\partial\mathbb{D}^2$ are in the same equivalence class as a single point in $\mathbb{S}^1$. Every point on the interior of $\mathbb{D}^2$ has singleton equivalence class.

Definition 1.6.4 (Projective Plane II') *Start with a copy of $\mathbb{D}^0$, $\mathbb{D}^1$, and $\mathbb{D}^2$. Identify the endpoints of $\mathbb{D}^1$ to the single point in $\mathbb{D}^0$. Then, glue a pair of antipodal points in $\mathbb{D}^2$ to the same equivalence class. As before, half of $\mathbb{D}^2$ traverses the interval, and then the other half traverses the same interval in the same direction. This cell complex is $\mathbb{R}\mathbf{P}^2$.*[33]

33: This is very similar to our second defintion. Constructing a homeomorphism between the first and second definition is a bit harder.

Definition 1.6.5 (Continuous) *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A function $f : X \to Y$ is continuous if for all $V \in \mathcal{T}_Y$, the preimage $f^{-1}(V) \in \mathcal{T}_X$.*[34]

34: As an exercise, give five different defintions of continuous.

Remark 1.6.2 That is, a homeomorphism is a continuous bijection whose inverse is *also continuous.*

Lemma 1.6.1 *Given $\mathcal{T}, \mathcal{T}'$ on X, the following are equivalent:*

(i) $\mathcal{T} \subseteq \mathcal{T}'$.
(ii) for all $U \in \mathcal{T}$ and $x \in U$, there exists $V \in \mathcal{T}'$ such that $x \in V \subseteq U$.[35]

35: Any open set containing the given point is called a *neighborhood*.

Proof. For (i) $\Rightarrow$ (ii), suppose $\mathcal{T}' \supseteq \mathcal{T}$. Let $V := U$. Now, for (ii) $\Rightarrow$ (i), let $U \in \mathcal{T}$. Then, by (ii), for any $x \in U$, there exists $V_x \in \mathcal{T}'$ such that $x \in V_x \subseteq U$. Then, $U = \bigcup_{x \in U} V_x \in \mathcal{T}'$. □

Lemma 1.6.2 *Suppose $\mathcal{T}, \mathcal{T}'$ are topologies on X, and they are generated by bases $\mathcal{B}, \mathcal{B}'$, respectively. Then, the following are equivalent:*

(i) $\mathcal{T} \subseteq \mathcal{T}'$.
(ii) for all $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

Proof. The proof is left as an exercise. □

We can state something similar with continuous functions.

Lemma 1.6.3 *Let X, Y be topological spaces, and $f : X \to Y$ be some function. Then, the following are equivalent:*

(i) f is continuous.[36]
(ii) for all $x \in X$, and for any $V \in \mathcal{T}_Y$, if $f(x) \in V$, then there exists $U \in \mathcal{T}_X$ such that $x \in U$ and $f(U) \subseteq V$.[37]

36: This is with respect to the standard definition, via preimages.

37: The condition (ii), for a single $x \in X$, is called *continuity at the point x*.

Sketch of Proof. Let us prove (ii) $\Rightarrow$ (i). We want to show f is continuous, so take $V \in \mathcal{T}_Y$. Then, consider $f^{-1}(V) \subseteq X$. Then, for all $x \in f^{-1}(V)$, there exists $U_x \in \mathcal{T}_X$ such that $x \in U_x \subseteq f^{-1}(V)$, so $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$, so $f^{-1}(V) \in \mathcal{T}_X$. Thus, f is continuous. □

Let us do one more of these.

Lemma 1.6.4 *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, $\mathcal{B}_X, \mathcal{B}_Y$ be bases generating the respective topologies. Let $f : X \to Y$ be a function. Then, the following are equivalent:*

(i) f is continuous.
(ii) for all $x \in X$, for all $B' \in \mathcal{B}_Y$, if $f(x) \in B'$, then there exists $B \in \mathcal{B}_X$ such that $x \in B$ and $f(B) \subseteq B'$.[38]

38: Once again, (ii) for a single $x \in X$, is precisely *continuity at the point* in terms of bases. Ranging over all x, we get an equivalent definition of continuity. When looking at a metric space, this is just ε, δ.

We could also state the definition of continuity in terms of closed sets: if $C \subseteq Y$ is closed, then $f^{-1}(C)$ is closed in X.

1.7 Product Topology

We now want to define a topology on the (cartesian) product of some topological spaces. If $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ are topological spaces, then define $X := X_1 \times X_2$. We can describe a set

$$\mathcal{B} := \{U_1 \times U_2 : U_1 \in \mathcal{T}_1, U_2 \in \mathcal{T}_2\}.$$

We want to know if we can squeeze something in the intersection of $U_1 \times U_2$ and $U_1' \times U_2'$. Well, if $x \in (U_1 \times U_2) \cap (U_1' \times U_2')$, then

$$x \in (U_1 \cap U_1') \times (U_2 \cap U_2') \subseteq (U_1 \times U_2) \cap (U_1' \times U_2').$$

Thus, $\mathcal{B}$ is a basis, so put the topology $\mathcal{T}(\mathcal{B})$ on X.[39]

Let $\{(X_i, \mathcal{T}_i)\}_{i \in I}$ be a family (finite, countable, or uncountable) of topological spaces.

39: For a product of n spaces, we have a natural way to place a topology on the product. But, what happens if we have an infinite family of spaces?

Definition 1.7.1 (Arbitrary Cartesian Product) *We define the cartesian product*

$$X := \prod_{i \in I} X_i := \left\{ \underline{x} : I \to \bigcup_{i \in I} X_i : \textit{ for all } j \in I, \underline{x}(j) \in X_j \right\}.$$

Before we define a topology on this product, suppose we have a function

$$\begin{aligned} Y &\xrightarrow{\ f\ } \prod_{i \in I} X_i \\ y &\longmapsto \left(\underline{x} : I \to \bigcup_{i \in I} X_i \right), \end{aligned}$$

where Y is a topological space. We could define $f_j : y \mapsto f(y)(j) \in X_j$, where we picked some $j \in I$.[40] Each f leads to a family of functions $\{f_j : Y \to X_j\}_{j \in I}$.

40: This is *akin* to a restriction to the jth coordinate.

Definition 1.7.2 (Coordinate Function) *The function f_j, as above, is called the jth coordinate function.*

Conversely, each indexed family $\{f_j : Y \to X_j\}_{j \in I}$ yields a function $f : Y \to \prod X_i$, where $f(y)(j) := f_j(y)$.[41] In other words, f is entirely determined by its coordinate functions, and vice-versa. As such, we want a way to relate the continuity of f with the continuity of its coordinate functions by picking the needed topology on the product.

41: This new function f, is precisely the f we had originally.

Consider the *canonical* projections

$$\begin{aligned} \prod_{i \in I} X_i &\xrightarrow{\ \pi_j\ } X_j \\ \underline{x} &\longmapsto \underline{x}(j). \end{aligned}$$

Using this projection, we an define f_j as the composition given by the diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "f_j := \pi_j \circ f"'] & \prod_{i \in I} X_i \arrow[d, "\pi_j"] \\ & X_j \end{tikzcd}$$

Figure 1.1: The existence of a unique f such that the diagram commutes is precisely the universal property of the product in Set. It turns out that via our construction of the product topology, we also get the universal property in Top.

Let $U_j \subseteq X_j$ and $U_j \in \mathcal{T}_j$. Then, the canonical preimage of U_j is

$$\pi_j^{-1}(U_j) = \left\{ \underline{x} : I \to \bigcup_{i \in I} X_i : \begin{array}{c} \text{for all } j \in I, \underline{x}(j) \in X_j \\ \text{and } \underline{x}(j) \in U_j \end{array} \right\}.$$

Consider the family[42]

$$\mathcal{S} := \{\pi_j^{-1}(U_j) : j \in I, U_j \in \mathcal{T}_j\}.$$

42: Why is this not a basis. Well consider $\pi_j^{-1}(U_j)$ and $\pi_{j'}^{-1}(U'_{j'})$. Then, we should be able to squeeze in such a family which contains an $\underline{x}$ in the intersection. This is not, generally possible.

Proposition 1.7.1 *The family $\mathcal{S}$, as above, is a subbasis.*[43]

43: Take any $\underline{x} \in \prod_{i \in I} X_i$. Then, $\underline{x} \in \pi_j^{-1}(X_j)$, by definition.

Definition 1.7.3 (Product Topology) *The product topology on $\prod X_i$ is $\mathcal{T}(\mathcal{B}(\mathcal{S}))$, the topology generated by the basis generated by the subbasis*

Remark 1.7.1 You will also hear the product topology referred to as the *Tychonoff* topology. It is the "most natural" one. One "proof" of this is via the *universal property*.

Remark 1.7.2 To assume that the arbitrary cartesian product of nonempty sets is nonempty is equivalent the axiom of choice.

It is worthwhile to mention that whereas $\mathcal{T}_{\text{prod}}$, as we have constructed it, picks finitely many indices in the basis. There is another topology on $\prod_i X_i$ known as the *box topology*.

Definition 1.7.4 (Box Topology) *The box topology on $\prod_i X_i$ is the topology $\mathcal{T}_{box} := \mathcal{T}(\mathcal{B}_{box})$ generated by the basis*[44]

$$\mathcal{B}_{box} := \left\{ \prod_{i \in I} U_i : \textit{for all } i \in I, U_i \in \mathcal{T}_i \right\}.$$

44: The box topology in finer than the Tychonoff topology. Check this as an exercise.

1.8 Closure, Interior, and Boundary

Definition 1.8.1 (Closure of Subset) *Let $(X, \mathcal{T}_X)$ be a topological space and*

$A \subseteq X$. Then, the closure $\overline{A}$ w.r.t. $(X, \mathcal{T}_X)$ is defined as the intersection[45]

$$\overline{A}^X := \bigcap_{A \subseteq C} C.$$

45: The $C \subseteq X$ are subsets which are closed with respect to $\mathcal{T}_X$ and contain A.

Lemma 1.8.1 *If $C \subseteq X$, then the following are equivalent:*

(i) C is closed in X.
(ii) $\overline{C}^X = C$.

Proof. For (i) $\Rightarrow$ (ii), we have that

$$\overline{C}^X = \bigcap_{C \subseteq C'} C' = C,$$

as $C \subseteq C$ is closed. Conversely, if $C = \overline{C}$, then

$$C^C = \left(\bigcap_{C \subseteq C'} C' \right)^C = \bigcup_{C \subseteq C'} (C')^C,$$

but the C' are closed, meaning $(C')^C \in \mathcal{T}_X$, so C^C is a union of open sets; i.e., its complement is open.[46] □

46: As such, we can characterize closedness via the closure.

Proposition 1.8.2 *A point $x \in X$ does* not *belong to $\overline{A}^X$ if and only if there exists an open neighborhood U of x such that $U \cap A = \varnothing$.*

Proof. Begin with ($\Leftarrow$). We have that U^C is closed, and $A \subseteq U^C$, so $\overline{A} \subseteq U^C \not\ni x$. Now, for ($\Rightarrow$), suppose $x \notin \overline{A} = \bigcap_{A \subseteq C} C$. Pick any C like this[47] such that $x \in C^C$. □

47: We know the intersection is nonempty, as it does not contain x.

Remark 1.8.1 (Alternative Definition of Closure) Via the proposition, we have that $x \in \overline{A}$ if and only if for all open neighborhoods U of x, $U \cap A \neq \varnothing$. Thus,

$$\overline{A} := \{x \in X : \text{ for all neighborhoods } U \text{ of } x, U \cap A \neq \varnothing\}.$$

Remark 1.8.2 The closure $\overline{A}$ is always closed.[48]

48: The lemma above gives the characterization, trivially.

Now, given $A_1 \subseteq A_2 \subseteq X$, what can we say about $\overline{A_1}$ and $\overline{A_2}$? Well, $\overline{A_1} \subseteq \overline{A_2}$. This is known as the *monotonicity of the closure*.[49]

49: We have that the LHS is an intersection of a *bigger* family than the RHS, so it is smaller.

Lemma 1.8.3 *Let $(X, \mathcal{T}_x), (Y, \mathcal{T}_Y)$ be topological spaces and $f : X \to Y$ a function. Then, the following are equivalent:*

(i) f is continuous.
(ii) for any $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$.

Proof. Using the closed set characterization of continuity, suppose f is continuous.[50] Take any set $A \subseteq X$. Let $x \in \overline{A}$. We want to show that $f(x) \in \overline{f(A)}$. Take any neighborhood V of $f(x)$. Since $f^{-1}(V)$ is open in X, and $x \in f^{-1}(V)$, we have that $A \cap f^{-1}(V) \neq \varnothing$, so $f(A) \cap f(f^{-1}(V)) \neq \varnothing$. As such, $f(A) \cap V \neq \varnothing$. Conversely, assume that for all $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$. Take any closed set $B \subseteq Y$. It will suffice to show that $f^{-1}(B)$ is closed in X; i.e., that $f^{-1}(B) = \overline{f^{-1}(B)}$. Well, we already know that $f^{-1}(B) \subseteq \overline{f^{-1}(B)}$. By assumption,

$$f(\overline{f^{-1}(B)}) \subseteq \overline{f(f^{-1}(B))} \subseteq \overline{B} = B,$$

using the monotonicity of the closure and the closedness of B. Taking the preimage, we get

$$\overline{f^{-1}(B)} \subseteq f^{-1}(B),$$

as desired. □

50: That is, $f^{-1}(C) \subseteq X$ is closed, for any closed $C \subseteq Y$.

Definition 1.8.2 (Hausdorff) *A topological space $(X, \mathcal{T})$ is Hausdorff if for all $x, y \in X$, there exists $U, V \in \mathcal{T}$ such that $x \in U$ and $y \in V$, and $U \cap V = \varnothing$.*[51]

51: Being Hausdorff gives us a notion of "separating" any two points. Morally, this means we would need a *fine enough* topology.

Exercise 1.8.1 Let X, Y be Hausdorff. Show that $X \times Y$ equipped with $\mathcal{T}_{\text{prod}}$ is also Hausdorff. Conclude that $X \times Y$ equipped with $\mathcal{T}_{\text{box}}$ is Hausdorff.

Definition 1.8.3 (Interior) *Let $(X, \mathcal{T})$ and $A \subseteq X$. Then, we define the interior of A in X w.r.t. $\mathcal{T}$ by*

$$\operatorname{int} A = \bigcup_{U \subseteq A} U,$$

where $U \in \mathcal{T}$.

Remark 1.8.3 It is immediate from the definition that $\operatorname{int} A \subseteq A \subseteq \overline{A}$.

Definition 1.8.4 (Boundary) *We define the boundary of $A \subseteq X$ w.r.t $\mathcal{T}$ by*

$$\partial A := \overline{A} \setminus \operatorname{int} A.$$

Definition 1.8.5 (Limit Point) *Given $(X, \mathcal{T})$ and $A \subseteq X$, a limit point of A in X is a point $x \in X$ such that for any neighborhood U of x in X,*[52]

$$(A \setminus \{x\}) \cap U \neq \varnothing.$$

52: The condition of being a limit point is stronger than being in the closure.

Remark 1.8.4 For any limit point $x \in X$ of A, $x \in \overline{A}$.

Proof. The proof is trivial. □

Example 1.8.1 Let $A := \{a_0, a_1\} \subseteq \mathbb{R}^2$. Then, a_0 is not a limit point, because there exists an open neighborhood U of a_0 such that $a_1 \notin U$, so

$$(\{a_0, a_1\} \setminus \{a_0\}) \cap U = \varnothing.$$

Yet, $a_0 \in A \subseteq \overline{A}$, so it is in the closure.

It turns out, this is *precisely* the general phenomenon. Define A' to be the *set of all limit points* of A in X w.r.t. $\mathcal{T}$.

Lemma 1.8.4 $\overline{A} = A \cup A'$.[53]

53: We now have another equivalent definition of the closure, and in turn, we could restate our definition of continuity using the RHS.

Proof. We have that $A \cup A' \subseteq \overline{A}$, trivially. Conversely, let $x \in \overline{A}$.

- If $x \in A$, we are done.
- Suppose $x \notin A$. Then, $A \setminus \{x\} = A$, and since $x \in \overline{A}$, for any neighborhood $U \in \mathcal{T}$ of x, $A \cap U \neq \varnothing$, completing the proof.

□

1.9 Aside on Norms and Sequences of Functions

Recall that the *Euclidean* metric on $\mathbb{R}^n$ is[54]

54: The "2" coming from the squaring and $1/2$ power.

$$d_2(x, y) := \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{\frac{1}{2}}.$$

Similarly,

$$d_1(x, y) := \sum_{i=1}^{n} |x_i - y_i|.$$

In general, our pth metric is[55]

55: We leave checking that the d_p are metrics if $p \in [1, \infty)$.

$$d_p(x, y) := \sqrt[p]{\sum_{i=1}^{n} |x_i - y_i|^p}.$$

Remark 1.9.1 (Infinity Metric) We define[56]

$$d_\infty := \max_{1 \le i \le n} \{|x_i - y_i|\}.$$

56: The geometrical interpretation of the unit ball in $\mathbb{R}^2$ for each of the d metrics is useful. In some sense, d_∞ acts like a limiting case of the other d_p.

Definition 1.9.1 (Norm) *A norm on a vector space* $(\mathcal{V}, \mathbb{F})$*, such as* $\mathbb{R}^n$*, is a function* $\|\cdot\| : \mathcal{V} \to [0, \infty)$ *such that*

(0) *for all* $x \in \mathcal{V}$, $\|x\| \geq 0$.
(i) *for all* $x \in \mathcal{V}$, $\|x\| = 0$ *if and only if* $x = 0$.
(ii) *for all* $x \in \mathcal{V}$ *and* $c \in \mathbb{F}$, $\|cx\| = |c| \cdot \|x\|$.
(iii) *for all* $x, y \in \mathcal{V}$, $\|x + y\| \leq \|x\| + \|y\|$.

Now, given a norm $\|\cdot\| : \mathscr{V} \to [0, \infty)$ on $(\mathscr{V}, \mathbb{F})$, we can define an *induced distance function*[57]

$$\begin{aligned} \mathscr{V} \times \mathscr{V} &\xrightarrow{\ d\ } [0, \infty) \\ (x, y) &\longmapsto \|x - y\|. \end{aligned}$$

57: It is trivial to check that the properties of the norm imply the properties of the induced metric.

Definition 1.9.2 (ℓ^2-Norm) *We define the ℓ^2-norm on $\mathbb{R}^n$ by*

$$\begin{aligned} \mathbb{R}^n &\xrightarrow{\ \|\cdot\|_2\ } [0, \infty) \\ x &\longmapsto \sqrt{\sum_{i=1}^{n} x_i^2}. \end{aligned}$$

Following this method, we can define the ℓ^p-norm on $\mathbb{R}^n$.[58]

58: We are simply tracing our steps backward through the metrics.

Definition 1.9.3 (ℓ^p-Norm) *Generally, the ℓ^p-norm on $\mathbb{R}^n$ is defined by*

$$\begin{aligned} \mathbb{R}^n &\xrightarrow{\ \|\cdot\|_p\ } [0, \infty) \\ x &\longmapsto \sqrt[p]{\sum_{i=1}^{n} |x_i|^p}. \end{aligned}$$

Definition 1.9.4 (Real Sequence Space) *The real sequence space is*

$$\mathbb{R}^\omega = \mathbb{R}^{\mathbb{Z}_+} := \{f : \mathbb{Z}_+ \to \mathbb{R}\}.$$

Definition 1.9.5 ($\ell^2(\mathbb{Z}_+, \mathbb{R})$ Sequence Space) *Define*[59]

$$\ell^2(\mathbb{Z}_+, \mathbb{R}) := \left\{ f : \mathbb{Z}_+ \to \mathbb{R} : \sum_{i \in \mathbb{Z}_+} |f(i)|^2 < \infty \right\} \subseteq \mathbb{R}^\omega.$$

59: On this set, $\|\cdot\|_2$ is a well-defined norm.

Definition 1.9.6 ($\ell^p(\mathbb{Z}_+, \mathbb{R})$ Sequence Space) *Following suit, define*

$$\ell^p(\mathbb{Z}_+, \mathbb{R}) := \left\{ f : \mathbb{Z}_+ \to \mathbb{R} : \sum_{i \in \mathbb{Z}_+} |f(i)|^p < \infty \right\}.$$

Here, the ℓ^p norm is precisely

$$\|f\|_p := \sqrt[p]{\sum_{i \in \mathbb{Z}_+} |f(i)|^p}.$$

As is standard in the literature, we will denote $\ell^p := \ell^p(\mathbb{Z}_+, \mathbb{R})$. Then, we have an inclusion $\iota : \ell^1 \hookrightarrow \ell^2$.[60]

60: The opposite inclusion certainly fails. Consider the harmonic sequence

$$f(i) = \frac{1}{i}.$$

We have that $f \notin \ell^1$, but $f \in \ell^2$.

Note that we can also define the ℓ^∞ norm via

$$\|f\|_\infty := \sup_{i\in\mathbb{Z}_+} \{|f(i)|\} \in [0,\infty] \subset \mathbb{R} \cup \{\infty\}.$$

In this sense,

$$\ell^\infty(\mathbb{Z}_+, \mathbb{R}) := \{f : \mathbb{Z}_+ \to \mathbb{R} : \|f\|_\infty < \infty\}.$$

Of course, $|\cdot|$ is a norm on $\mathbb{R}$, so $\|\cdot\|_\infty$ is a norm on ℓ^∞.[61]

61: Check this as an exercise.

Definition 1.9.7 (Lipchitz Equivalent Metrics) *Two metrics d, d' are Lipchitz equivalent on X if there exist $c_1, c_2 > 0$ such that*

$$d \le c_1 d' \quad \text{and} \quad d' \le c_2 d.$$

Definition 1.9.8 (Lipchitz Function) *A function $f : (X, d_X) \to (Y, d_Y)$ is called Lipchitz if there exists a constant $L > 0$ such that for all $x, x' \in X$,*

$$d_Y(f(x), f(x')) \le L d_X(x, x').$$

Definition 1.9.9 (Sequence Convergence) *A sequence $\{x_i\} \subseteq X$ on a topological space $(X, \mathcal{T})$ converges to $x \in X$ if for all neighborhoods U of x, there exists an $N \in \mathbb{Z}_+$ such that for all $n \ge N$, $x_n \in U$.*

Definition 1.9.10 (Sequentially Closed) *A subset $A \subseteq X$ is sequentially closed if for any sequence $\{x_i\} \subseteq A$ which converges to $x \in X$, then $x \in A$.*[62]

62: Prove that our standard defintion of being closed implies sequentially closed. However, the converse does not hold, generally.

Lemma 1.9.1 *Let X be a topological space and $A \subseteq X$, then $\overline{A}^{seq} \subseteq \overline{A}$. That is to say, for any sequence $\{x_n\}$ in A with $x_n \to x \in X$, then $x \in \overline{A}$.*

Proof. Take any sequence $\{x_n\}$ in A such that $x_n \to x \in X$. Then, $x \in \overline{A}^{\mathrm{seq}}$. Take any neighborhood U of x in X. Since $x_n \to x$, there exists $N \in \mathbb{Z}_+$ such that for all $n \ge N$, $x_n \in U$, but $x_n \in A$, so $U \cap A \ne \varnothing$. □

What about the converse? In general, it is not true. However, we can make our space *nice enough* for it to hold.

Definition 1.9.11 (Metrizable) *A topological space $(X, \mathcal{T})$ is metrizable if there exists a metric d on X such that $\mathcal{T} = \mathcal{T}_d$.*

Lemma 1.9.2 *Let X be a metrizable topological space. Then, $\overline{A} \subseteq \overline{A}^{seq}$.*

Proof. Let $x \in \overline{A}$. Consider the ball $B(x, 1)$. Well, $B(x,1) \cap A \ne \varnothing$. That is, there exists x_1 in the intersection. Similarly, consider the ball $B(x, 1/2) \cap A \ne \varnothing$, so we get an x_2. In general, we can define a sequence $\{x_n\}$ in A via $B(x, 1/n)$.[63] To show that $x_n \to x$, take any U of x. Then, there exists a ball B centered at x such that $B \subseteq U$. Let r be the radius of B. Let $N = 1/r + 1$. Then every x_n for $n \ge N$ is contained in B, so they are contained in U. □

63: We use the axiom of choice here.

Definition 1.9.12 (Basis at Point) *A basis at $x \in X$ is a family of open neighborhoods $\{U_i\}_{i\in\Lambda}$ of x such that for any neighborhood U of x, there exists an $i \in \Lambda$ such that $x \in U_i \subseteq U$.*

Definition 1.9.13 (First-Countable Space) *A topological space $(X, \mathcal{T})$ is called first-countable if for any $x \in X$, there exists a countable basis at x.*[64]

64: All metrizable spaces are first-countable, as we can take a basis of open balls of radius $1/n$.

If we have a countable basis $U_1, U_2, \ldots$, then we can define $U_1' := U_1$, $U_2' := U_1 \cap U_2$, $U_3' := U_1 \cap U_2 \cap U_3$, and so forth.[65]

65: Note that a *second-countable* space is one with a countable generating basis.

Proposition 1.9.3 *Let X be a first-countable topological space. Then, $\overline{A} \subseteq \overline{A}^{seq}$.*

Theorem 1.9.4 (Sequentially Continuous) *A function $f : X \to Y$ is sequentially continuous if for any sequence $\{x_n\}$ in X and any x in X, if $x_n \to x$, then $f(x_n) \to f(x)$.*[66]

66: That is, f maps convergent sequences (and in particular, the limit) to convergent sequences (image of the limit).

We can now ask, what is the relationship between sequential continuity and our standard continuity.

Lemma 1.9.5 *Let $f : X \to Y$ be a function between topological spaces. If X is continuous, then it is sequentially continuous. If we additionally have that X is first-countable (notably, metrizable), then the converse holds.*

Proof. Start with ($\Rightarrow$). Suppose $f : X \to Y$ is continuous. Take any sequence $\{x_n\} \subseteq X$ such that $x_n \to x \in X$. Take any open neighborhood V of $f(x)$. Then, $f^{-1}(V) \in \mathcal{T}_X$. Then, there exists an $N \in \mathbb{Z}_+$ such that $x_n \in f^{-1}(V)$ for all $n \geq N$, so $f(x_n) \in V$, meaning $f(x_n) \to f(x)$. Conversely, assume X is first-countable. Then, for any $A \subseteq X$, $\overline{A} = \overline{A}^{\text{seq}}$. Take a closed $B \subseteq Y$. We have that $\overline{f^{-1}(B)} = \overline{f^{-1}(B)}^{\text{seq}}$. Let $x \in \overline{f^{-1}(B)}^{\text{seq}}$. Then, there exists a sequence $\{x_n\} \subseteq f^{-1}(B)$ such that $x_n \to x$. Then, $f(x_n) \to f(x) \in \overline{B} = B$. Thus, $x \in f^{-1}(B)$, so $\overline{f^{-1}(B)} = f^{-1}(B)$, meaning $f^{-1}(B)$ is closed. $\square$

Now, note that

$$\text{second-countable} \Rightarrow \text{first-countable} \Leftarrow \text{metrizable}.$$

Take countably many copies of $[0, 1]$, take their disjoint union, and form the cell complex gluing all the 0s together. We will see this space later on, but it is worth noting that this graph is *not first-countable* with the quotient topology.[67] We will prove the result later on.

67: Obviously, if you take the topology induced by the plane, it *is* first-countable.

Remark 1.9.2 In general, given a cell complex which is not *locally finite*, we can say that it is not first-countable or metrizable via the quotient topology.

Consider the set of all possible functions $\mathfrak{F}(\mathbb{R}, \mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R}\}$. How can we put a topology on this set? Let us generalize a bit. Let $\{f_n\}$ be a sequence of functions in $\mathfrak{F}(X, Y)$, where X, Y are fixed topological spaces, Let $f : X \to Y$ be another function.

Definition 1.9.14 (Pointwise Convergence) *We say $f_n \to f$ pointwise if for all $x \in X$, $f_n(x) \to f(x)$.*[68]

68: We are using the well-defined notion of convergence in Y.

Suppose additionally that (Y, d_Y) is a metric space. Then, $f_n \to f$ pointwise is equivalent to: for all $x \in X$, for any $\varepsilon > 0$, there exists $N \in \mathbb{Z}_+$ so that for $n \geq N$, $f_n(x) \subseteq B(f(x), \varepsilon)$. What if we make this *uniform*? For all $\varepsilon > 0$, there exists $N \in \mathbb{Z}_+$ such that for all $x \in X$ and $n \geq N$, $f_n(x) \in B(f(x), \varepsilon)$. The uniform converge in a metric space codomain is precisely taking

$$\sup_x \{d(f_n(x), f(x))\} \leq \varepsilon.$$

Definition 1.9.15 (Supremum Metric) *Let $f, g : X \to Y$. Then,*[69]

$$d_{sup}(f, g) := \sup_x \{d_Y(g(x), f(x))\}.$$

69: Check that this is a distance function on the set of all bounded functions.

Definition 1.9.16 (Uniform Convergence) *Let $f, f_n : X \to Y$, where X is a set and Y is a metric space. We say $f_n \to f$ uniformly if for all $\varepsilon > 0$, there exists an $N \in \mathbb{Z}_+$ such that for all $n \geq N$,*

$$d_{sup}(f_n, f) \leq \varepsilon.$$

Theorem 1.9.6 (Uniform Limit Theorem) *Let X be a topological space and Y a metric space, and $f_n : X \to Y$ is a sequence of continuous functions converging uniformly to a function $f : X \to Y$. Then, f is continuous.*

Sketch of Proof. Let V be an open set $V \subseteq Y$. We wish to show that $f^{-1}(V)$ is open. Take any point $x \in f^{-1}(V)$. Consider $f(x)$. Pick a small enough $\varepsilon > 0$ so that $B(f(x_0), \varepsilon) \subseteq V$. Choose N large enough so that for any $x \in X$, the distance $d(f_n(x), f(x)) < \varepsilon/3$. Then, choose a neighborhood U of x_0 in X such that $f_N(U) \subseteq B(f_n(x_0), \varepsilon/3)$.[70] $\square$

70: We can do this step and the previous one via uniform convergence and continuity, respectively.

2 Connectedness and Compactness

We now consider some standard point-set topological invariants and characteristics. You likely have seen these in real analysis.

2.1 (Path-)Connectedness

Before we discuss connectedness, what would it mean for a space to *not* be connected? Well, $[0, 1] \cup [3, 4]$ is certainly not connected, intuitively.

Definition 2.1.1 (Separation) *A separation of a topological space X is a pair of open, nonempty subsets (U, V) in X such that $U \cap V = \varnothing$ and $X = U \cup V$.*[1]

Definition 2.1.2 (Connected) *A topological space X is connected if it does not admit a separation (U, V) on X.*[2]

Definition 2.1.3 (Path-Connected) *A space X is path-connected if for any $x, y \in X$, there exists a continuous function*

$$\gamma : [0, 1] \to X$$

such that $\gamma(0) = x$ and $\gamma(1) = y$.[3]

Lemma 2.1.1 *Any path-connected space is connected.*

Proof. Suppose X is not connected. Then, there exists a separation (U, V) of X such that $X = U \cup V$ and $U \cap V = \varnothing$.[4] Pick $x \in U$ and $y \in V$. Then, there exists a path γ from x to y. Then, $(\gamma^{-1}(U), \gamma^{-1}(V))$ is a separation of $[0, 1]$, a contradiction. $\square$

Define an equivalence relation $\sim_p$ on a topological space X by $x \sim_p y$ if and only if there exists a path $\gamma : [0, 1] \to X$ so that $\gamma(0) = x$ and $\gamma(1) = y$.

Lemma 2.1.2 *The relation $\sim_p$ is an equivalence.*[5]

Definition 2.1.4 (Path Component) *A path component is an equivalence class of $\sim_p$ in X.*

Define another equivalence relation $\sim$ on X by $x \sim y$ if and only if there exists a connected subset $A \subseteq X$ such that $x, y \in A$. Take the subspace topology on A.

1: The intuitive example admits a separation, by construction.

2: Read the Munkres proof of connectedness of intervals. It is not too difficult.

3: We call any such map γ a *path* in X.

4: Recall that $U, V \neq \varnothing$.

5: Use the constant map for reflexivity. For symmetry, pre-compose γ with $f(t) = 1 - t$ to get $\overline{\gamma}$ in the form we desire. For transitivity, concatenate the paths (though, proving continuity is a bit technical).

Lemma 2.1.3 *The relation $\sim$ is an equivalence.*[6]

6: Reflexivity comes from $A = \{x\}$. Symmetry is trivial. Transitivity is done via the union and the following lemma.

Lemma 2.1.4 *Let X be a topological space and $\{A_i\}_{i \in I}$ a collection of connected subsets $A_i \subseteq X$. Suppose there exists a point $x_0 \in X$ such that $x_0 \in \bigcap A_i$. Then, $\bigcup A_i$ is connected.*

Proof. Suppose $A := \bigcup A_i$ is not connected. Then, there exists a separation (U, V) of A.[7] Well, $x_0 \in U \coprod V$, so without loss of generality we can assume $x_0 \in U$, as U, V are disjoint. For each index $i \in I$, we know $x_0 \in A_i \subseteq U \coprod V$. Yet, A_i is connected, so $A_i \subseteq U$ or $A_i \subseteq V$, and since $x_0 \in U$, $A_i \subseteq U$. This works for every $i \in I$, so $V = \varnothing$, a contradiction. □

7: Recall that U, V are open in A via the subspace topology.

Definition 2.1.5 (Connected Component) *A connected component is an equivalence class of $\sim$ in X.*

Definition 2.1.6 (Locally Path-Connected) *A space X is locally path-connected if for all $x \in X$ and for all open neighborhoods U of x, there exists another neighborhood U' of x such that $x \in U' \subseteq U$ and U' is path-connected.*

Definition 2.1.7 (Locally Connected) *A space X is locally connected if for all $x \in X$ and for all open neighborhoods U of x, there exists another neighborhood U' of x such that $x \in U' \subseteq U$ and U' is connected.*

Remark 2.1.1 Use the subspace topology for U, for the local definitions.

Lemma 2.1.5 *If $x \sim_p y$ then $x \sim y$.*

Proof. Let γ be the path for $x \sim_p y$. Consider $\gamma([0,1]) \subseteq X$. Suppose $\gamma([0,1])$ admits a separation (U, V). Then, pre-images give us a contradiction, so $\gamma([0,1])$ is connected, meaning $x \sim y$.[8] □

8: The pre-images are nonempty, as $\gamma(0) = x$ and $\gamma(1) = y$.

Corollary 2.1.6 *The continuous image of any connected space is connected.*[9]

9: Use the same argument as above.

Proposition 2.1.7 *The continuous image of any path-connected space is path-connected.*[10]

10: If X is path-connected via γ and $f : X \to Y$ is continuous, then $f \circ \gamma$ is a path for Y.

Theorem 2.1.8 *If X is any topological space, $A \subseteq X$ is connected, and $f : X \to Y$ is continuous, then $f(A)$ is connected.*

From our discussion above, it is clear that for all $x \in X$, $[x]_{\sim_p} \subseteq [x]_{\sim}$. That is, the partition of X generated by $\sim_p$ is finer than that by $\sim$.

Lemma 2.1.9 *A space X is connected if and only if it has exactly one connected component.*

Proof. Suppose X is connected. Let $x \in X$. For each $x' \in X$, $x, x' \in X$ and X is connected. Since x' is arbitrary, $x \sim x'$, so $X \subseteq [x]_\sim$. Thus, $X = [x]_\sim$, since there is nothing larger. Conversely, assume there exists exactly one connected component: $\bigcup [x]_\sim$ is connected, but this is all of X. $\square$

Theorem 2.1.10 *If a topological space X is connected and locally path-connected, then it is path-connected.*

Proof. We want to show that X has precisely *one* path component. Since X is connected, there exists exactly one connected component: X. The path components give a partition of X. Let $x \in X$ and consider the path component $[x]_{\sim_p}$. Since the space is locally path-connected, there exists a path-connected neighborhood U'_x of x in X, so $U'_x \subseteq [x]_{\sim_p}$. Do this for all $x' \in [x]_{\sim_p}$. Thus, $\bigcup U'_{x'} = [x]_{\sim_p}$, so each path component of X is open. If there exists at least 2 path components, then we have a separation of X, which is a contradiction.[11] $\square$

11: We use that we can union the path components to form a nonempty, open, disjoint partition.

Remark 2.1.2 On the way to proving the theorem, we showed that if a space is locally path-connected, then the path components are open. The same implication holds for connectedness.

Lemma 2.1.11 *For any connected subset $A \subseteq X$, if $A \subseteq B \subseteq \overline{A}$, then B is connected.*[12]

12: In particular, $\overline{A}$ is closed

Now, let $C \subseteq X$ be a connected component. Then, C is indeed connected, so $\overline{C}$ is connected. Thus, $\overline{C} = C$, so C is closed.

Theorem 2.1.12 (Intermediate Value Theorem) *If X is a connected topological space and $f : X \to \mathbb{R}$ is continuous in the standard topology, then for any $x_0, x_1 \in X$ and $c \in \mathbb{R}$ between $f(x_0)$ and $f(x_1)$, inclusive, then there exists $x \in X$ so that $f(x) = c$.*

Proof. Suppose not. Then, $c \notin f(X)$. Then, the image $f(X) \subseteq (-\infty, c) \cup (c, \infty)$. Both $f(X) \cap (-\infty, c)$ and $f(X) \cap (c, \infty)$ are open by the subspace topology, so the intersections form a separation of $f(X)$. Take the pullback. Then, $f^{-1}(-\infty, c)$ and $f^{-1}(c, \infty)$ form a separation of X.[13] $\square$

13: This is a contradiction. The standard application is for intervals in $\mathbb{R}$.

Remark 2.1.3 (Topologist's Sine Curve) Union the graph of $\sin(1/x)$ in $\mathbb{R}^2$ to the interval on the y-axis. This is connected, but you get a nice contradiction to the continuity of any path from the interval to $\sin(1/x)$.

Remark 2.1.4 Note that if a space is locally path-connected, then the path components are open, so we could form a separation of a surrounding connected component. Hence, path components and connected components coincide exactly when the whole space is locally path-connected.[14]

14: This is another way to find the $x \sim y \Rightarrow x \sim_p y$ implication.

Theorem 2.1.13 $\mathbb{R} \not\simeq \mathbb{S}^1$.

Proof. Let $\varphi : \mathbb{S}^1 \xrightarrow{\sim} \mathbb{R}$ be a homeomorphism. Let $a \in \mathbb{S}^1$, and suppose $f(a) = b \in \mathbb{R}$. Consider $\mathbb{S}^1 \setminus \{a\}$ and $\mathbb{R} \setminus \{b\}$. Then, $\mathbb{S}^1 \setminus \{a\}$ is path-connected, but there is an obvious separation of $\mathbb{R} \setminus \{b\}$.[15] □

15: Do this rigorously with a restriction.

2.2 Compactness

We now want to establish a sense of "smallness" for arbitrary topological spaces, lacking a notion of boundedness.

Definition 2.2.1 (Cover) *A cover of a set X is a family $\{U_i \subseteq X\}_{i \in I}$ so that the union $\bigcup U_i = X$.*

If $(X, \mathcal{T})$ is a topological space, then an *open cover* of X is a cover $\{U_i\}_{i \in I}$ such that $U_i \in \mathcal{T}$ for all $i \in I$.

Definition 2.2.2 (Compact) *A topological space X is called compact if for any open cover $\{U_i\}_{i \in I}$ of X, there exists a finite subcover of X.*

Example 2.2.1 Consider $(0, 1) \simeq \mathbb{R}$. Consider the family $\{(i, i + 2)\}_{i \in \mathbb{Z}}$. Then, $\bigcup (i, i + 2) = \mathbb{R}$. Any finite subfamily would have a smallest and largest i. Then, $(i_{\min}, i_{\max} + 2) \neq \mathbb{R}$, so $\mathbb{R}$ is not compact, meaning $(0, 1)$ is not either.

Example 2.2.2 (Empty Set) $\varnothing$ is compact, vaccuously.

Example 2.2.3 (Singleton) Singletons $\{x\} \subset \mathbb{R}$ are compact.

Example 2.2.4 (Closed Interval) Consider $[0, 1] \subset \mathbb{R}$. We claim it is compact.

We might call the original definition of compactness the *intrinsic* definition. What would a *relative* or *extrinsic* definition be?

Definition 2.2.3 (Subset Compactness) *Let $(X, \mathcal{T}_X)$ be a topological space and $C \subseteq X$. We call C compact if for any family $\{U_i \in \mathcal{T}_X\}_{i \in I}$ so that $C \subseteq \bigcup_I U_i$, then there is a finite subfamily $\{U_i \in \mathcal{T}_X\}_{i \in I'}$ so that $I' \subseteq I$ with $|I'| < \infty$ and $C \subseteq \bigcup_{I'} U'$.*

Theorem 2.2.1 *If $C \subseteq X$, the two definitions of compactness are equivalent.*[16]

16: Chasing definitions makes this an easy proof.

Lemma 2.2.2 *If $(X, \mathcal{T}_X)$ is a compact topological space, then any closed set $C \subseteq X$ is compact.*

Proof. Pick a "generalized" open cover $\mathcal{F} := \{U_i \in \mathcal{T}_X\}_{i\in I}$ so that $C \subseteq \bigcup_I U_i$. Now, $X \setminus C$ is open, so $\mathcal{F}' := \{U_i\}_{i\in I} \cup (X \setminus C)$ is an open cover of X. Since X is compact, there exists a finite subfamily $\{U_{i_1}, \dots, U_{i_n}\}$ of $\mathcal{F}'$ that covers X. Thus, C is compact.[17] □

17: We use the second definition.

Lemma 2.2.3 *Let $(X, \mathcal{T}_X)$ be Hausdorff and $C \subseteq X$ a compact subspace. Then, C is closed in X.*

Proof. We want to show $X \setminus C$ is open. Fix a point $y \in X \setminus C$. Since X is Hausdorff, for any $x \in C$, we can find neighborhoods U_x of x and V_x of y such that $U_x \cap V_x = \varnothing$. Do this for each choice of $x \in C$. Consider the family $\{U_x\}_{x\in C}$. Well, $C \subseteq \bigcup_C U_x$.[18] Since C is compact, there exists a finite subcover $\{U_{x_1}, \dots, U_{x_n}\}$ of C. Consider the corresponding $\{V_{x_1}, \dots, V_{x_n}\}$. Then,

$$\underbrace{\left(\bigcap_{i=1}^{n} V_{x_i}\right)}_{V} \cap \left(\bigcup_{i=1}^{n} U_{x_i}\right) = \varnothing,$$

so $C \cap V = \varnothing$ and $y \in V$. This can be done for every choice of $y \in X \setminus C$. Hence, $X \setminus C$ is open, meaning C is closed.[19] □

18: That is, $\{U_x\}$ is a generalized open cover of C.

19: This is a nifty result. No matter how you homeomorphically embed a compact space in a Hausdorff space, it will always be closed, despite the fact that closedness is relative to ambient topology.

Lemma 2.2.4 *Let $f : X \to Y$ be a continuous function between topological spaces and $C \subseteq X$ be a compact subspace. Then, $f(C) \subseteq Y$ is compact.*

Proof. Let $\{U_i \in \mathcal{T}_Y\}_{i\in I}$ be an open cover of $f(C)$. Clearly $f^{-1}(U_i)$ is open for all $i \in I$. Since the U_i cover $f(C)$, the $f^{-1}(U_i)$ cover C. There is a finite subset $I' \subseteq I$ such that $\{f^{-1}(U_i)\}_{i\in I'}$ is a cover of C. Hence, $\{U_i\}_{i\in I'}$ is an open cover of $f(C)$. □

For any product $X \times Y$, we can pick any $x \in X$. Then, we can consider the *slice*

$$Y_x := \{(x, y) : y \in Y\} = \{x\} \times Y.$$

Define a map[20]

$$\begin{aligned} Y &\xrightarrow{\varphi_x} Y_x \\ y &\longmapsto (x, y). \end{aligned}$$

20: We claim that φ_x is a bijection.

In fact, the function φ_x is a homeomorphism with the subspace topology on $Y_x \subseteq X \times Y$, which can be seen using open rectangles and projections.

Lemma 2.2.5 *If X and Y are compact topological spaces, then $X \times Y$ is compact with respect to the product topology.*

Proof. Fix $x \in X$. Consider the corresponding slice Y_x. Since Y is compact, and $Y \simeq Y_x$, we know Y_x is compact with respect to the subspace topology from the product topology. Let $\mathcal{A} := \{A_i\}_{i\in I}$ be an open cover of $X \times Y$.[21] For any $y \in Y$, there exists an $i \in I$ such that $(x, y) \in A_i$. For each y, pick an open $U_{x,y} \subseteq X$ and open $V_{x,y} \subseteq Y$ such that $(x, y) \in U_{x,y} \times V_{x,y} \subseteq X \times Y$.

21: Our goal is to extract a finite cover.

The family $\{U_{x,y} \times V_{x,y}\}_{y \in Y}$ is an open cover of $\{x\} \times Y = Y_x$. Then, there exists a finite subcover $\{U_{x,y_1} \times V_{x,y_1}, \dots, U_{x,y_n} \times V_{x,y_n}\}$. Now,

$$\{x\} \times Y \subseteq \underbrace{\left(\bigcap_{i=1}^{n} U_{x,y_i} \right)}_{W_x \in \mathcal{T}_X} \times Y.$$

The set $\{W_x : x \in X\}$ is an open cover of X. Since X is compact, we can take finitely many W_{x_k} for $k \in [m]$. Then, $\{W_{x_k} \times Y\}_{k \in [m]}$ is a finite open cover of $X \times Y$. Tracing back through our steps, each tube is contained in one of the open rectangles, and each of the open rectangles is contained in one of the $A_i \in \mathcal{A}$. Since we have made finitely many choices of such A_i to cover $X \times Y$, we are done. □

Remark 2.2.1 There is a natural isomorphism in Set between $X \times Y \times Z \simeq (X \times Y) \times Z$, and taking the corresponding product topologies, we get that $X \times Y \times Z$ is compact.[22]

22: This works for all finite products, as you might expect.

Corollary 2.2.6 *All finite products of compact sets are compact.*

Theorem 2.2.7 (Tychonoff Theorem) *If $\{X_i\}_{i \in I}$ is a family of connected topological spaces, then $\prod X_i$ is compact.*[23]

23: We omit the proof for now. We will come back to this, though. It really is a neat proof.

2.3 Local and Sequential Compactness

Theorem 2.3.1 (Extreme Value Theorem) *Given a continuous function $f : X \to \mathbb{R}$, where X is a compact topological space, there exist $x_{min}, x_{max} \in X$ such that for all $x \in X$,*

$$f(x_{min}) \leq f(x) \leq f(x_{max}).$$

Proof. Suppose that for all $x' \in X$, there exists $x \in X$ so that $f(x') < f(x)$. We have $f(X) \subseteq \mathbb{R}$. Now, consider the set

$$\{(-\infty, a) : a \in f(X)\}.$$

Then, $f(X) \subseteq \bigcup_{a \in f(X)} (-\infty, a)$. In particular, this set is an open cover of $f(X)$. Now, X is compact and f is continuous, so $f(X)$ is compact, which means there exists a finite subcover $\{(-\infty, a_i) : i \in [n]\}$ of $f(X)$. Thus, $f(X) \subseteq \bigcup (-\infty, a_i)$. Pick $a := \max\{a_1, \dots, a_n\} \in f(X)$. Then, $f(X) \subseteq (-\infty, a)$, and $a \notin (-\infty, a)$, so $a \notin f(X)$. This is a contradiction.[24] □

24: We assume all the a are in $f(X)$.

Remark 2.3.1 The same process works for $x_{\min}$.

We will now define local compactness, which does *not* follow our usual pattern for locality.

Definition 2.3.1 (Locally Compact) *A space X is called locally compact if for any $x \in X$ there exists a neighborhood U of x which is contained in a compact subset $C \subseteq X$.*[25]

25:
$$x \in U \subseteq C \subseteq X$$

Proposition 2.3.2 *Compactness trivially implies local compactness. The converse, however, is not true.*

Example 2.3.1 We know $\mathbb{R}$ is not compact. Yet, $\mathbb{R}$ is locally compact.[26]

26: Take a closed interval C and a smaller open interval U.

Definition 2.3.2 (Subsequence) *A subsequence of $\{x_n\}$ given by $f : \mathbb{Z}_+ \to X$ is any composition*

$$\begin{array}{ccccc} \mathbb{Z}_+ & \hookrightarrow & \mathbb{Z}_+ & \xrightarrow{x} & X \\ k & \longmapsto & n_k & & \end{array}$$

such that $n_k < n_{k'}$ if $k < k'$.

Definition 2.3.3 (Sequentially Compact) *A topological space X is sequentially compact if for any sequence $\{x_n\} \subseteq X$, there exists a subsequence $\{x_{n_k}\} \subseteq X$ such that $x_{n_k} \to L$ for some $L \in X$.*

Theorem 2.3.3 *Let X be metrizable (or first-countable). Then, the following are equivalent:*

(i) X is compact.
(ii) X is sequentially compact.

Proof. We will just show (i) $\Rightarrow$ (ii).[27] Take any sequence $\{x_n\} \subseteq X$. Define $A := \{x_n : n \in \mathbb{Z}_+\} \subseteq X$. We defined that $y \in A'$ if and only if for any neighborhood U of y, $(A \setminus \{y\}) \cap U \neq \varnothing$. Start with the case $A' = \varnothing$. Then, $\overline{A} = A \cup A' = A$, so A is closed. Then, $X \setminus A$ is open. For any $a \in A$, the negation says there exists a neighborhood U_a of a such that $A \setminus a \cap U_a = \varnothing$, so $A \cap U_a = \{a\}$. Consider the family $\{U_a\}_{a \in A} \cup (X \setminus A)$. This is an open cover of X. Compactness tells us that there exists a finite subcover $\{U_{a_i}\}_{i=1}^m \cup \{X \setminus A\}$. That is, A is finite! From here you deduce that you can extract a convergent constant subsequence. Case two is when $A' \neq \varnothing$. Take $x \in A'$. Take an open ball U around x. We know $(A \setminus \{x\}) \cap U \neq \varnothing$. Successively, we take $B(x, 1), B(x, r_1)$, where $r_1 < 1$. Continue, taking smaller and smaller balls. Each time, we pick $y_i \in A$ with a larger index. We can check that the system of radii converges to zero, meaning we have a local basis at x. Hence, $y_i \to x$ in the sense of topology.[28] □

27: Read Munkres for the proof of the converse.

28: Take any neighborhood of x, then at some point, one of the balls in our local basis will fit in.

Definition 2.3.4 (Bounded) *A metric space (X, d) is bounded if the diameter*

$$\mathrm{Diam}(X) := \sup\{d(x_1, x_2) : x_1, x_2 \in X\} < \infty$$

Remark 2.3.2 In $\mathbb{R}^n$, boundedness of C amounts to picking a ball $B(x, r)$ so that $C \subset B(x, r)$, where the diameter $\mathrm{Diam}(X)$ is bounded by $2r$.[29]

29: The bound comes from the triangle inequality.

Theorem 2.3.4 (Heine-Borel) *For any subset $C \subseteq \mathbb{R}^n$, C is compact if and only if C is closed and bounded.*

Proof. Suppose C is compact. Since $\mathbb{R}^n$ is Hausdorff, C is closed. Consider the set $\{B(x, 1) : x \in C\}$. This forms an open cover of C. We have a finite subcover $\{B(x_i, 1) : 1 \leq i \leq n\}$ of C. Let $\mathscr{D} := \max\{d(x_1, x_i) : 1 \leq i \leq n\}$. Then, $C \subseteq B(x, \mathscr{D} + 2)$, so C is bounded. Conversely, suppose $C \subseteq \mathbb{R}^n$ is closed and bounded. There exists a cube $Q = [a, b]^n$ so that $C \subseteq Q$. Well, $\mathbb{R}^n \setminus C$ is open. Thus, $(\mathbb{R}^n \setminus C) \cap Q$ is open in Q. Well, $C = Q \setminus (Q \setminus C)$, so C is closed inside of Q. □

2.4 Lebesgue's Number Lemma

Let $\mathscr{A}$ be an open cover of some metric space (X, d) so that $\mathscr{A} \subseteq \mathscr{P}(X)$.[30] Let $x \in X$. Our goal is to determine if we can cover X in some small balls with equal radius so that each ball is only an $A \in \mathscr{A}$.

30: That is, we are taking a standard cover, not a generalized one.

Definition 2.4.1 (Lebesgue Number) *The Lebesgue number of the cover $\mathscr{A}$ is any number $\delta > 0$ such that any subset $S \subseteq X$ of $\mathrm{Diam}(S) < \delta$ is contained in A for some $A \in \mathscr{A}$.*

Remark 2.4.1 Looking at the open interval $(0, 1) \subseteq \mathbb{R}$, it is clear that the Lebesgue number does not exist, in general. However, it turns out, compactness gives us exactly what we need.

Definition 2.4.2 (Distance to Set) *For any $C \subseteq X$, define*

$$d(x, C) := \inf_{y \in C}\{d(x, y)\}.$$

Note that there is also the Hausdorff distance which uses supremum.

Theorem 2.4.1 (Lebesgue's Lemma) *If (X, d) is compact and $\mathscr{A} := \{A_i : i \in I\}$ is an open cover of X, then a Lebesgue number exists.*

Proof. There exists a finite subcover $\mathscr{A}' := \{A'_k : 1 \leq k \leq n\}$.[31] Denote $C_k := X \setminus A'_k$ for any $k \in \{1, \dots, n\}$. Note that if $A'_k = X$, then let $\delta := 1$, or whichever number you like. This is certainly a Lebesgue number for $\mathscr{A}'$. Assume $C_k \neq \varnothing$ for any k. Consider the function[32]

31: $\mathscr{A}'$ is a subfamily of $\mathscr{A}$. Note that if we can find a Lebesgue number for the finite cover, then we get one for $\mathscr{A}$ for free.

32: Since we excluded the empty case, our $d(x, C_k) \neq \infty$.

$$\begin{aligned} X &\xrightarrow{f} [0, \infty) \subseteq \mathbb{R} \\ x &\longmapsto \frac{1}{n}\sum_{k=1}^{n} d(x, C_k). \end{aligned}$$

From the triangle inequality of $d : X^2 \to [0, \infty)$,

$$d(x, y) - d(x', y) \le d(x, x')$$

and

$$d(x', y) - d(x, y) \le d(x, x'),$$

so $|d(x, y) - d(x', y)| \le d(x, x')$.[33] Similarly, we hope to have

$$\left|d(x, C) - d(x', C)\right| \le d(x, x').$$

33: Note that this bound does not depend on y at all. This means $d(\cdot, y) : X \to [0, \infty)$ is 1-Lipchitz.

We have that, for $c \in C$,

$$d(x, C) \le d(x, c) \le d(x, y) + d(y, c)$$

Hence,

$$d(x, C) - d(x, y) \le d(y, c).$$

The LHS is independent of c, so

$$d(x, C) - d(x, y) \le \inf_{c \in C} \{d(y, c)\} = d(y, C).$$

Interchanging the roles of x, y we get the opposite statement, so together the absolute value statement holds. Hence, the map

$$d(\cdot, C) : X \to [0, \infty)$$

is 1-Lipchitz.[34] Hence, $f : X \to [0, \infty)$ is continuous (the sum of finitely many continuous functions to $\mathbb{R}$ is continuous, as proven on the homework), plus $f(x) > 0$, as each A_k is open. Thus, there exists an $x_{\min} \in X$ such that for all $x \in X$, $0 < f(x_{\min}) \le f(x)$. Denote $\delta := f(x_{\min}) > 0$. Then, for any x, $f(x) \ge \delta$. Let C be any subset of X of diameter less than δ. Let $x_0 \in C$. Then, $C \subseteq B(x, \delta)$. We have that

34: Note that Lipchitz implies uniform continuity with $\delta := \varepsilon / L$.

$$0 < \delta \le f(x_0) = \frac{1}{n} \sum_{i=1}^{n} d(x_0, C_i) \le \max_{1 \le i \le n} \{d(x_0), C_i\}.$$

There exists a number $m \in \{1, \ldots, n\}$ so that

$$\max_{1 \le i \le n} \{d(x_0, C_i)\} = d(x_0, C_m),$$

meaning $0 < \delta \le d(x_0, C_m)$. Thus,

$$B(x_0, \delta) \cap C_m = \varnothing \Leftrightarrow B(x_0, \delta) \subseteq A'_m \in \mathscr{A},$$

as desired. Thus, δ is a Lebesgue number for $\mathscr{A}$. □

Definition 2.4.3 (Uniformly Continuous) *Let $f : X \to Y$ be a function between metric spaces. We call f uniformly continuous if for all $\varepsilon > 0$ there exists a $\delta > 0$ so that for all $x, x' \in X$,*

$$d_X(x, x') < \delta \Longrightarrow d_Y(f(x), f(x')) < \varepsilon.$$

It is pretty clear that

$$\text{Uniform Continuity} \Longrightarrow \text{Continuity}.$$

Certainly, the converse does not generally hold.

Example 2.4.1 The exponential function exp on $\mathbb{R}$ is *not* uniformly continuous. It goes wrong, because $\mathbb{R}$ is not compact!

Theorem 2.4.2 *If X is a compact metric space and $f : X \to Y$ is a continuous function between metric spaces, then f is uniformly continuous.*

Proof. To say that $d_X(x, x') < \delta$, we can say

$$\mathrm{Diam}\{x, x'\} < \delta.$$

Cover Y with open balls of radius $\varepsilon/2 > 0$. By pulling back on f, the set

$$\{f^{-1}(B(y, \varepsilon/2)) : y \in Y\}$$

is an open cover of X. Then, there exists a Lebesgue number δ[35] for this cover. Fill in the details as an exercise. □

35: This is where Lebesgue's lemma is useful.

2.5 Compactifications and Zorn's Lemma

Definition 2.5.1 (Compactification) *A compactification of a topological space X is any topological space Y so that $X \subseteq Y$ (as a subspace) so that Y is compact, Hausdorff, and $\overline{X}^Y = Y$.*

Definition 2.5.2 (One-Point Compactification) *If $Y \setminus X$, as above, is a singleton, then Y is a one-point compactification.*

Remark 2.5.1 We can compactify an open interval by adding two points and getting the closed interval, or by adding one and getting an object homeomorphic to the circle.

We briefly give the general outline of proving sequential compactness implies compactness for metrizable (first-countable) spaces.[36]

36: This proof was omitted from lecture for brevity.

- Prove that sequential compactness implies the existence of a Lebesgue number for any open cover. To do so, proceed by contradiction.
- Prove that for all $\varepsilon > 0$, there exists a finite cover of X consisting of ε-balls. Again, proceed by contradiction.
- Combine the previous parts.

Lemma 2.5.1 *Let X be compact and Y be Hausdorff. Define $f : X \hookrightarrow Y$ to be a continuous injection. Restrict the codomain to extract $f' : X \xrightarrow{\sim}_{\mathsf{Set}} f(X)$, a bijection in* **Set**. *Then, f' is a homeomorphism.*[37]

37: That is, we can "promote" an isomorphism in Set to an isomorphism in Top. This is not a rigorous notion, and the actual realizable functor is quite complex.

Proof. Take any closed $C \subseteq X$. Then, $f'(C)$ is compact in $f'(X) \subseteq Y$. Since Y is Hausdorff, then $f'(C)$ is closed in $f(X)$. Thus, $g' = f^{-1}$ is continuous. □

We can now give another description of compactness.

Proposition 2.5.2 *The following are equivalent:*

(i) *X is compact; i.e., any open cover of X admits a finite subcover.*
(ii) *For any family $\mathscr{C} := \{C_i \subseteq X\}_{i \in I}$ of closed sets so that $\bigcap_I C_i = \varnothing$, there exists a finite subfamily $\{C_{i_1}, \ldots, C_{i_n}\}$ so that $\bigcap_n C_{i_j} = \varnothing$.*[38]
(iii) *Let $\mathscr{C}$ be a family of closed subsets in X. If every finite subfamily of $\mathscr{C}$ has nonempty intersection,*[39] *then $\bigcap \mathscr{C} \neq \varnothing$.*

38: Duality of complements gives this formulation of compactness clearly.

39: We denote this condition by FIP (finite intersection property). Note that (iii) follows by taking the contrapositive of (ii).

Definition 2.5.3 (Non-Strict Partially Ordered Set) *A (non-strict) partially ordered set (poset) is a pair $(S, \leq)$ so that*

(i) *$\leq$ is reflexive.*
(ii) *$\leq$ is antisymmetric.*[40]
(iii) *$\leq$ is transitive.*

40: If $x \leq y$ and $y \leq x$, then $x = y$.

We call $\leq$, as above, a non-strict partial order, to be pedantic.[41]

41: We will prefer the non-strict convention.

Definition 2.5.4 (Strict Partial Order) *A relation $<$ is a strict partial order if*

(i) *for all $x \in S$, $x \nless x$.*
(ii) *for all $x, y \in S$, if $x < y$ then $y \nless x$.*
(iii) *for all $x, y, z \in S$, if $x < y$ and $y < z$ then $x < z$.*

Remark 2.5.2 The important thing to notice here is that we can interchange between $\leq$ and $<$ freely.

Example 2.5.1 (Posets)

(i) The class of all topologies $\mathscr{T}$ on a set S, where inclusion $\subseteq$ is the partial order, is a poset.
(ii) Another standard example is the subgroup lattice, taking all subgroups H of a set G, with a partial order $\leq$.

Definition 2.5.5 (Total Order) *A total (or simple/linear) order on a set S is a partial order $\leq$ on S if for all $a, b \in S$, $a \leq b$ or $b \leq a$.*

Definition 2.5.6 (Totally Ordered Set) *A totally ordered set (or chain) is a set with a total order $(S, \leq)$.*

Definition 2.5.7 (Upper Bound) *Given a subset $S' \subseteq S$ where S is a poset, an upper bound on S' in S is an element $u \in S$ so that for all $s' \in S'$, $s' \leq u$.*[42]

42: Similarly, you could talk about lower bounds.

Definition 2.5.8 (Maximal) *An element $m \in S$ is maximal if for all $s \in S$ so that $m \leq s$, we have $m = s$.*[43]

43: In totally ordered sets, this is the same as the "largest" element. However, in general they do not coincide. For instance, in $\mathbb{R}$, every closed subset has a maximal element which is also its largest. Yet, we could have a three-element set $\{a, b, c\}$ so that $a > b$, $c > b$, but a, c are not comparable. Then, both a, c are maximal, but there is no largest.

Lemma 2.5.3 (Zorn's) *Let $(S, \leq)$ be a nonempty poset such that any chain $\mathscr{C} \subseteq S$ has an upper bound in S. Then, S has a maximal element.*

Proof. The lemma is equivalent to the axiom of choice. We will simply assume ZFC hereafter, taking Zorn's for granted. $\square$

Remark 2.5.3 Let us take a brief aside on notation. Let C be a set. Then, $c \in C$ is an element. Consider now a family $\mathscr{C} \subseteq \mathscr{P}(C)$. In particular, $\mathscr{C} \in \mathscr{P}(\mathscr{P}(C))$. Then, we can form classes/supersets $\mathbf{C} \subseteq \mathscr{P}(\mathscr{P}(C))$, so $\mathbf{C} \in \mathscr{P}(\mathscr{P}(\mathscr{P}(C)))$. This will be useful in our proof of Tychonoff's theorem.

Theorem 2.5.4 (Tychonoff's Theorem) *Let $\{(X_i, \mathscr{T}_i) : i \in I\}$ be a family of compact spaces. Then, the product $X := \prod_I X_i$ with $\mathscr{T}_{prod}$ is compact.*

Proof. We will use the FIP characterization of compactness. Let $\mathscr{A}$ be a family of closed subsets of X satisfying the FIP. Since every $A \in \mathscr{A}$ is closed, we can write

$$\bigcap_{D \in \mathscr{D}} \overline{D} \subseteq \bigcap_{A \in \mathscr{A}} \overline{A} = \bigcap_{A \in \mathscr{A}} A = \bigcap \mathscr{A},$$

where we hope to construct such a $\mathscr{D}$, as proving $\bigcap_{\mathscr{D}} D \neq \varnothing$ will show $\bigcap \mathscr{A}$ is nonempty. Let

$$\mathbf{A} := \{\mathscr{A}' \subseteq \mathscr{P}(X) : \mathscr{A} \subseteq \mathscr{A}' \text{ and } \mathscr{A}' \text{ has FIP }\} \subseteq \mathscr{P}(\mathscr{P}(X)).$$

We can place a natural partial order $\leq$ on $\mathbf{A}$ by set theoretic inclusion. We will write $\mathscr{A}' \leq \mathscr{A}''$ if and only $\mathscr{A}' \subseteq \mathscr{A}''$. If $\mathscr{A} = \varnothing$, then $\bigcap \mathscr{A} = X \neq \varnothing$, using the axiom of choice.[44] Now, we must check that for any chain $\mathbf{B}$ in $\mathbf{A}$, $\mathbf{B}$ has an upper bound $\mathscr{C} \in \mathbf{A}$. Take any chain $\mathbf{B} \subseteq \mathbf{A}$. Define

$$\mathscr{C} := \bigcup_{\mathscr{B} \in \mathbf{B}} \mathscr{B}.$$

Clearly, for all $\mathscr{B} \in \mathbf{B}$, $\mathscr{B} \subseteq \mathscr{C}$, so $\mathscr{C}$ is an upper bound. We know $\mathscr{A} \subseteq \mathscr{C}$, as $\mathscr{A} \subseteq \mathscr{B}$ for all $\mathscr{B} \in \mathbf{B}$. Take any $C_1, \ldots, C_n \in \mathscr{C}$. We want to show

$$C_1 \cap \cdots \cap C_n \neq \varnothing.$$

For all $k \in [n]$, there exists a family $\mathscr{B}_k \in \mathbf{B}$ such that $C_k \in \mathscr{B}_k$. Since $\mathbf{B}$ is a chain with respect to $\subseteq$, there exists a largest family, say $\mathscr{B}_{k'}$.[45] Thus, $C_1, \ldots, C_n \in \mathscr{B}_{k'} \in \mathbf{B} \subseteq \mathbf{A}$. Well, $\mathscr{B}_{k'}$ satisfies the FIP, so $\mathscr{C} \in \mathbf{A}$. By Zorn's lemma, there exists a maximal family $\mathscr{D} \in \mathbf{A}$. In particular, $\mathscr{A} \subseteq \mathscr{D}$, and $\mathscr{D}$ satisfies the FIP. We also have that $\mathscr{D}$ is maximal in

$$\{A' \subseteq \mathscr{P}(X) : \mathscr{A}' \text{ satisfies FIP }\}.$$

We say that $\mathscr{D}$ is maximal with respect to the FIP. All that remains to be shown is that

$$\bigcap_{D \in \mathscr{D}} \overline{D} \neq \varnothing.$$

44: We select an element in X_i for each index X_i to form $\underline{x} : I \to \bigcup_I X_i$.

45: That is, for all $k \in [n]$, $\mathscr{B}_k \subseteq \mathscr{B}_{k'}$.

Define the family, for $j \in I$,

$$\{\overline{\pi_j(D)} : D \in \mathscr{D}\} \subseteq \mathscr{P}(X_j).$$

We claim that this family satisfies the FIP. Since X_j is compact,

$$\bigcap_{D \in \mathscr{D}} \overline{\pi_j(D)} \neq \varnothing.$$

Thus, there exists $x_j \in \bigcap_{\mathscr{D}} \overline{\pi_j(D)} \subseteq X_j$. Define $\underline{x}(j) := x_j$.[46] This gives us a point $\underline{x} \in \prod_I X_i = X$. Pick any $D \in \mathscr{D}$. We want to show $\underline{x} \in \overline{D}$. Take any open neighborhood U_j of x_j. Then, we have the subbasic set $\pi_j^{-1}(U_j)$. We have that $x_j \in U_j \cap \pi_j(D) \neq \varnothing$. Taking preimages, $\pi_j^{-1}(U_j) \cap D \neq \varnothing$. Via maximality and the lemma below, $\pi_j^{-1}(U_j) \in \mathscr{D}$, for any neighborhood U_j of x_j in X_j. Also by the lemma, every basic neighborhood containing $\underline{x}$ in X in $\mathscr{T}_{\text{prod}}$ is in $\mathscr{D}$. Take any such basic neighborhood U of $\underline{x}$ in X. For all $D \in \mathscr{D}$, we claim that $U \cap \mathscr{D} \neq \varnothing$.[47] Thus, $\underline{x} \in \overline{D}$ for each $D \in \mathscr{D}$, meaning $\underline{x} \in \bigcap_{\mathscr{D}} \overline{D}$, as desired. ☐

46: We need choice to turn nonemptiness into a function.

47: Use the FIP for $\mathscr{D}$.

Lemma 2.5.5 *If $\mathscr{D} \subseteq \mathscr{P}(X)$ is maximal with respect to the FIP, then*

(i) $\mathscr{D}$ is closed under finite intersections.
(ii) if $A \subseteq X$ is so that $A \cap D \neq \varnothing$ for all $D \in \mathscr{D}$, then $A \in \mathscr{D}$.

(i) *Proof.* Pick finitely many $D_1, \ldots, D_n \in \mathscr{D}$. We want to show that $D_1 \cap \cdots \cap D_n \in \mathscr{D}$. Consider

$$\mathscr{D}' := \mathscr{D} \cup \{D_1 \cap \cdots \cap D_n\}.$$

If we can show that $\mathscr{D}' \in \mathbf{A}$, then we win. Certainly $\mathscr{A} \subseteq \mathscr{D}'$. Pick any finite subfamily $D_1', \ldots, D_m' \in \mathscr{D}'$. If $D_i' \in \mathscr{D}$ for all i, then FIP implies $\bigcap_m D_i' \neq \varnothing$. If one of the $D_m' = D_1 \cap \cdots \cap D_n$, then

$$D_1' \cap \cdots \cap (D_1 \cap \cdots \cap D_n) \neq \varnothing,$$

also by FIP.[48] Thus, $\mathscr{D}' \in \mathbf{A}$. Yet, $\mathscr{D} \subsetneq \mathscr{D}'$, a contradiction to maximality. ☐

(ii) *Proof.* Proceed, again, by contradiction to maximality. ☐

48: Taking intersections is associative.

On Embeddings, Function Spaces, and Algebraic Topology

3 Separation and Embeddings

We have previously discussed this notion of being Hausdorff:

Definition 3.0.1 (Axiom T_2: Hausorff) *A topological space X is Hausdorff (or T_2) if for all $x \neq y \in X$, there exist neighborhoods U of x and V of y such that $U \cap V = \varnothing$.*

Naturally, we want to both weaken and strengthen this axiom.

3.1 Introducing the T_i Axioms

Definition 3.1.1 (Axiom T_1) *A topological space X satisfies T_1 if for all $x \neq y \in X$, there exists a neighborhood U of x such that $y \notin U$ and there exists a neighborhood V of y such that $x \notin V$.*

Remark 3.1.1 It is clear that $T_2 \Rightarrow T_1$.

Definition 3.1.2 (Axiom T_3: Regular) *A topological space X is regular (or T_3) if each singleton subset of X is closed, and if for all $x \in X$ and for all closed $C \subseteq X$, if $x \notin C$, then there exists a neighborhood U of x and there exists a neighborhood V of C such that $U \cap V = \varnothing$.*[1]

1: That is, you can separate any closed set from any point.

Remark 3.1.2 Since singletons are closed, we have $T_3 \Rightarrow T_2$.

Definition 3.1.3 (Axiom T_4: Normal) *A topological space X is normal (or T_4) if each singleton subset of X is closed, and for all closed subsets $C_1, C_2 \subseteq X$, if $C_1 \cap C_2 = \varnothing$, then there exist neighborhoods U of C_1 and V of C_2 such that $U \cap V = \varnothing$.*

Remark 3.1.3 Once again, $T_4 \Rightarrow T_3$.

Thus, we have the chain of implications

$$T_4 \Longrightarrow T_3 \Longrightarrow T_2 \Longrightarrow T_1.$$

Definition 3.1.4 (Axiom $T_{3\frac{1}{2}}$: Completely Regular) *A topological space X is completely regular if singletons are closed, and if for any point $x \in X$ and closed subset $C \subseteq X$ such that $x \notin C$, there exists a continuous function $f : X \to [0,1]$ such that $f : x \mapsto 0$ and $C \mapsto \{1\}$.*[2]

2: In some sense, we are separating the point and the set by a function.

We are using that $[0, 1]$ is Hausdorff. Take open intervals around 0 and 1 called V, V'. Then, $f^{-1}(V) \cap f^{-1}(V') = \varnothing$ is an open separation of x and the closed set C. As desired, this tells us

$$T_{3\frac{1}{2}} = \text{Completely Regular} \implies T_3.$$

It turns out, that $T_4 \Rightarrow T_{3\frac{1}{2}}$, which follows from what is called Urysohn's lemma. We will skip the proof of Urysohn's, though you can find the proof in any standard text.[3]

3: For instance, see Munkres.

3.2 Embeddings and Stone-Čech

We now want a way to construct maps which behave like inclusions, with the hope of using these functions to "embed" spaces into Hausdorff compact spaces. As you should remember, if we can force some subspace of our larger space to equal the closure of the embedded space, then we can form a compactification.

Definition 3.2.1 (Embedding) *An embedding of X into a topological space Y is a continuous injection $X \hookrightarrow Y$ such that $f' : X \xrightarrow{\sim}_{Top} f(X) \subseteq Y$ is a homeomorphism.*[4]

4: Let $f(X)$ have the subspace topology inherited from Y.

Theorem 3.2.1 (Embedding Theorem) *Let X be a topological space and $\{f_i : X \to [0,1]\}_{i\in I}$ be a family of continuous functions such that for all $x \in X$ and neighborhoods U of x, there exists an $i \in I$ so that $f_i(x) > 0$ and $f_i(X \setminus U) = \{0\}$. Then, the function*[5] *$F : X \to [0,1]^I$ defined by $F(x)(i) = f_i(x)$ is an embedding.*

5: By definition,

$$[0,1]^I := \prod_{i\in I} [0,1].$$

Proof. The proof is not particularly illuminating, so check Munkres. □

Lemma 3.2.2 (Existence of Stone-Čech) *Let X be completely regular. Then, a compactification exists.*

Proof. Let X be a $T_{3\frac{1}{2}}$ topological space. Let $\{f_i : i \in I\}$ be the set of all continuous *bounded* functions $f_i : X \to \mathbb{R}$. Applying the embedding theorem tells us that we have an embedding

$$F : X \to \prod_{i\in I} [\inf f_i(x), \sup f_i(x)].$$

Now, by the definition of an embedding $X \simeq F(X) \subseteq \overline{F(X)}^{\prod}$. Then, $\overline{F(X)}$ is Hausdorff and compact, so it is a compactification of X. □

Definition 3.2.2 (Stone-Čech Compactification) *We call $\beta(X) := \overline{F(X)}$, as above, the Stone-Čech compactification of X.*

Theorem 3.2.3 (Universal Property of Stone-Čech Compactification) *Let X be completely regular. For all bounded continuous functions $f : X \to \mathbb{R}$, there exists a continuous function $g : \beta(X) \to \mathbb{R}$ which extends f via the* *inclusion* $X \hookrightarrow \beta(X)$.

$$
\begin{tikzcd}
X \arrow[r, hook] \arrow[d, "\text{bounded}"', "\forall f"] & \beta(X) \arrow[dl, dashed, "\exists!"', "g"] \\
\mathbb{R} &
\end{tikzcd}
$$

Figure 3.1: Universal property of Stone-Čech compactification

Proof. We do not prove the universal property, but Munkres essentially constructs $\beta(X)$ via this property. □

We claim that if C is compact and Hausdorff, then the Stone-Čech compactification satisfies the diagram

$$
\begin{tikzcd}
X \arrow[r, hook] \arrow[d, "\text{bounded}"', "\forall f"] & \beta(X) \arrow[dl, dashed, "\exists!"', "g"] \\
C &
\end{tikzcd}
$$

Figure 3.2: Generalized universal property

Again, see Munkres for the proof. Using the universal property as a definition, you can move to arbitrary compact Hausdorff spaces C.[6]

6: We are going to use this for uniqueness, up to homeomorphism.

Theorem 3.2.4 (Uniqueness of Stone-Čech) *Let* $\beta_1(X)$ *and* $\beta_2(X)$ *be Stone-Čech compactifications of a* $T_{3\frac{1}{2}}$ *topological space. Then,* $\beta_1 \simeq \beta_2$.

Proof. Via the generalized universal property, we have the diagrams

$$
\begin{tikzcd}
X \arrow[r, hook] \arrow[d, tail] & \beta_1(X) \arrow[dl, dashed, "\exists!"', "g_1"] \\
\beta_2(X) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
X \arrow[r, hook] \arrow[d, tail] & \beta_2(X) \arrow[dl, dashed, "\exists!"', "g_2"] \\
\beta_1(X) &
\end{tikzcd}
$$

Gluing the diagrams together, we get that the diagram

$$\begin{tikzcd} & X \arrow[dl] \arrow[d] \arrow[dr] & \\ \beta_1(X) \arrow[r, "g_1"] & \beta_2(X) \arrow[r, "g_2"] & \beta_1(X) \end{tikzcd}$$

has $g_2 \circ g_1$ is continuous and equial to id_X. By symmetry, $g_1 \circ g_2$ is continuous and $g_1 \circ g_2 = \mathrm{id}_X$. We want to deduce that $g_2 \circ g_1 = \mathrm{id}_{\beta_1(X)}$ and $g_1 \circ g_2$ is $\mathrm{id}_{\beta_2(X)}$. We will be done via the following lemma, as $g_1^{-1} = g_2$, so g_1^{-1} is continuous. Thus, g_1 is a homeomorphism.[7] □

7: An isomorphism in any category is just an invertible. Recall that in the case of Top, we call these homeomorphisms. This is a homeomorphism extends identity on X.

Lemma 3.2.5 (Uniqueness of Extensions) *Let $A \subseteq X$ and $f : A \to Y$, where Y is Hausdorff, and $g, g' : \overline{A} \to Y$ are continuous extensions of f. Then, $g = g'$.*

Proof. Suppose not; i.e., $g \neq g'$. Then, there exists $x \in \overline{A}$ such that $g(x) \neq g'(x)$. Since Y is Hausdorff, we can separate the values by neighborhoods V of $g(x)$ and V' of $g'(x)$ so that $V \cap V' = \varnothing$. By continuity, $U := g^{-1}(V) \cap g'^{-1}(V') =: U'$ is a neighborhood of X. Then, $(U \cap U') \cap A \neq \varnothing$. Pick a point $y \in (U \cap U') \cap A$, then $g(y) \in V$ and $g'(y) \in V'$. Since $y \in A$, $g(y) = f(y) = g'(y)$. Thus, $V \cap V' \neq \varnothing$, a contradiction. □

4 Completeness and Function Spaces

For this discussion, let (X, d) be a metric space with a corresponding distance function $d : X^2 \to [0, \infty)$.

4.1 Cauchy Sequences and Completions

Recall a standard definition from analysis.

Definition 4.1.1 (Cauchy Sequence) *A Cauchy sequence in X is a sequence $\{x_n\}$ in X such that for all $\varepsilon > 0$, there exists an $N \in \mathbb{Z}_+$ such that for all $m, n \in \mathbb{Z}_+$, if $n, m \geq N$, then $d(x_n, x_m) < \varepsilon$.*

Proposition 4.1.1 *Any convergent sequence $\{x_n\} \subseteq X$ is Cauchy.*

Example 4.1.1 Consider a sequence in $\mathbb{Q}$ "converging" to $\sqrt{2}$ in $\mathbb{R}$. This is clearly Cauchy, but not convergent in $(\mathbb{Q}, d)$.

From this example, it is clear that the converse of the proposition above is not true.

Definition 4.1.2 (Complete) *A metric space (X, d) is complete if any Cauchy sequence in X is convergent in X.*

Our goal is now to construct a larger, complete space to embed a non-complete space within.

Definition 4.1.3 (Completion) *Given a metric space (X, d), a completion of X is a complete metric space $(\hat{X}, \hat{d})$ such that $X \subseteq \hat{X}$, $\hat{d} : \hat{X}^2 \to [0, \infty)$ is a metric extending d along the inclusion $\iota : X \hookrightarrow \hat{X}$, and $\overline{X}^{\mathcal{T}_{\hat{d}}} = \hat{X}$.*[1]

1: This forces a sort of minimality, like we did with completions.

Lemma 4.1.2 *Let (X, d) be any complete metric space. Let Y be a closed subspace $Y \subseteq X$. Then, Y is complete with respect to $d_Y := d|_Y$.*

Proof. Since the metrics agree, we have that $\{y_n\} \subseteq Y$ has $y_n \to x \in X$. To show $x \in Y$, it suffices to show that $x \in \overline{Y}$. Well, we showed that $\overline{Y}^{\text{seq}} = \overline{Y}$ for metric spaces, so we are done. □

Lemma 4.1.3 *In the setting above, the converse also holds.*

Proof. Pick $\overline{y} \in \overline{Y}$. Then, pick a sequence $\{y_n\}$ such that $y_n \to \overline{y} \in \overline{Y}$ and $y_n \to y' \in Y$. Since Y is Hausdorff, this forces $\overline{y} = y' \in Y$. □

Lemma 4.1.4 *If $\{x_n\}$ is a Cauchy sequence in a metric space (X, d), then the following are equivalent:*

(i) x_n converges in X.
(ii) x_n has a subsequence x_{n_k} which converges in X.

Proof. The implication (i) $\Rightarrow$ (ii) is trivial. Let $\{x_{n_k}\} \subseteq \{x_n\}$ be a subsequence which converges to x in X. Take any ε. Then, there exists $N \in \mathbb{Z}_+$ such that for all $m, n \geq N$, $d(x_m, x_n) < \varepsilon/2$. Choose $i \in \mathbb{Z}_+$ so that $n_i \geq N$ and $d(x_{n_i}, x) < \varepsilon/2$. Combining these via the triangle inequality, we get that $x_n \to x$, as desired. □

4.2 Bounded Space

Let X, Y be metric spaces.

Definition 4.2.1 (Bounded) *A function $f : X \to Y$ is bounded if $f(X)$ is bounded in Y. We can use that the diameter of $f(X)$ is finite, or that we can enclose it in a finite-radius ball in Y.*

Definition 4.2.2 ($\mathbb{B}(X, Y)$) *We define*

$$\mathbb{B}(X, Y) := \{f : X \to Y : f \text{ is bounded}\}.$$

Let d be the metric on Y.

Definition 4.2.3 (Uniform Metric) *Given $f, g \in \mathbb{B}(X, Y)$, let*

$$d_\infty(f, g) := \sup_{x \in X}\{d(f(x), g(x))\} < \infty.$$

Remark 4.2.1 Pretty easily, d_∞ is a metric on $\mathbb{B}(X, Y)$.[2]

2: It is finite, as we take all functions in $\mathbb{B}(X, Y)$ to be bounded.

Our goal is to construct an injective function $\Phi : X \to \mathbb{B}(X, \mathbb{R})$, sending $a \mapsto (\Phi_a : X \to \mathbb{R})$. Fix $x_0 \in X$. For all $a \in X$, define $\Phi_a : x \mapsto d(a, x) - d(x_0, x)$. Check that $\Phi_a \in \mathbb{B}(X, \mathbb{R})$. We have that

$$d(a, x) - d(x_0, x) \leq d(a, x_0)$$

and

$$d(x_0, x) - d(a, x) \leq d(a, x_0),$$

so $|\Phi_a(x)| \leq d(a, x_0)$. Thus, $\Phi_a \in \mathbb{B}(X, \mathbb{R})$. Suppose $a \neq b$. Then, $\Phi_a \neq \Phi_b$, we get $\Phi_a(a) \neq \Phi_b(a)$.

Remark 4.2.2 The function $\Phi : a \mapsto \Phi_a$ is an isometric embedding.[3]

3: That is, it is an injection, plus the metric is preserved:

$$d_\infty(\Phi_a, \Phi_b) = d(a, b).$$

Now, we will work towards proving the following lemma.

Lemma 4.2.1 *Let X be any set and (Y, d) a complete metric space. Then, $(\mathbb{B}(X, Y), d_\infty)$ is complete.*

Let $f : X \to Y$ be a function. We want to realize f as an element of some Cartesian product. The easiest way to do so, is to consider Y^X. That is,

$$Y^X := \prod_{x \in X} Y \equiv \mathfrak{F}(X, Y).$$

Clearly, $\mathbb{B}(X, Y) \subseteq Y^X$. If we can show completeness of the latter, and then show closedness of the former, we would be done with our lemma. In particular, we will need a metric for our function space.

Definition 4.2.4 (Forced-Boundedness Metric) *Given (Y, d), define $\overline{d}(y_1, y_2) := \min\{d(y_1, y_2), 1\} \leq 1$.*[4]

4: As an exercise, check that $\overline{d}$ is a metric.

Proposition 4.2.2 *If $a, b, c \in [0, \infty)$. Suppose $a \leq b + c$. Then, $\min\{a, 1\} \leq \min\{b, 1\} + \min\{c, 1\}$.*

Corollary 4.2.3 *The new metric $\overline{d}$ is a bounded metric.*[5]

5: It is trivially bounded by 1.

If $\varepsilon \leq 1$, then

$$\overline{d}(y_1, y_2) \leq \varepsilon \Leftrightarrow d(y_1, y_2) \leq \varepsilon.$$

Corollary 4.2.4 *If Y is complete with respect to a metric d, then it is complete with respect to $\overline{d}$.*

$$\begin{tikzcd}
 & \overline{d} \arrow[r, squiggly, "\infty"] & (\overline{d})_\infty \arrow[dd, equal, "\overline{d}_\infty"] \\
d \arrow[ur, squiggly, "\min"] \arrow[dr, squiggly, "\infty"'] & & \\
 & d_\infty \arrow[r, squiggly, "\min"'] & \overline{(d_\infty)}
\end{tikzcd}$$

Figure 4.1: We have two ways of moving from a metric space (Y, d) to a metric space of functions. We can either take the sup or the forced boundedness. It turns out the results are equal. Prove this as an exercise

Definition 4.2.5 (Isometric Embedding) *An isometric embedding is a function $f : X \to Y$ between metric spaces such that for all $x, x' \in X$,*

$$d_Y(f(x), f(x')) = d_X(x, x').$$

This implies that X is injective.

Now, we have a method of constructing a completion via

$$\begin{aligned} X &\hookrightarrow \mathbb{B}(X, \mathbb{R}) \\ a &\longmapsto \Phi_a. \end{aligned}$$

Lemma 4.2.5 *If (Y, d) is a complete metric space, then Y^X is complete with respect to $\overline{d}_\infty$.*

Proof. Let $\{f_n\}$ be any Cauchy sequence in $(Y^X, \overline{d}_\infty)$. For all $\varepsilon > 0$, there exists $N \in \mathbb{Z}_+$ so that for all $m, n \geq N$,

$$\overline{d}_\infty(f_m, f_n) < \varepsilon.$$

Then, for all $x \in X$,

$$\overline{d}(f_m(x), f_n(x)) \leq \overline{d}_\infty(f_m, f_n) < \varepsilon.$$

Thus, for each $x \in X$, the sequence $\{f_n(x)\} \subseteq Y$ is Cauchy. Hence, $\{f_n(x)\}$ converges in Y[6] to a point y_x. Define the function $f : X \to Y$ be defined by $x \mapsto y_x$. We need to check that $f_n \to f$ with respect to $\overline{d}_\infty$. Take any $\varepsilon > 0$. There exists $N \in \mathbb{Z}_+$ such that for all $m, n \geq N$,

6: Since d and $\overline{d}$ agree for small values, completeness in (Y, d) implies completeness in $(Y, \overline{d})$.

$$\overline{d}_\infty(f_m, f_n) < \frac{\varepsilon}{2}.$$

Again, for all $x \in X$,

$$\overline{d}(f_m(x), f_n(x)) \leq \overline{d}_\infty(f_m, f_n) < \frac{\varepsilon}{2},$$

and taking $m \to \infty$, we get

$$\overline{d}(f_m(x), f_n(x)) \xrightarrow{m\to\infty} \overline{d}(y_x, f_n(x)) = \overline{d}(f(x), f_n(x)),$$

as $\overline{d} : Y^2 \to [0, \infty)$ is continuous in the first variable. Then,

$$\overline{d}(f(x), f_n(x)) < \frac{\varepsilon}{2}$$

for all $x \in X$. Thus,[7]

$$\overline{d}_\infty(f, f_n) \leq \frac{\varepsilon}{2} < \varepsilon.$$

□

7: Taking the supremum gives us the non-strict inequality, and then we get our strict inequality trivially.

Definition 4.2.6 (Continous Space) *Let X be a topological space and Y be a topological space. Then, $\mathscr{C}(X, Y)$ is the set of continuous functions $f : X \to Y$.*

Theorem 4.2.6 *Let X be a topological space and Y be a metric space. Then, both $\mathscr{C}(X, Y)$ and $\mathbb{B}(X, Y)$ are closed in $(Y^X, \overline{d}_\infty)$.*[8]

8: We also want to show that $\mathbb{B}(X, Y)$ is bounded with respect to d_∞.

Proof. We first show that $\mathscr{C}(X, Y)$ is sequentially closed in Y^X. Suppose $\{f_n\} \subseteq \mathscr{C}(X, Y)$ is a sequence that converges to $f \in Y^X$. We want to use the uniform limit theorem. For all $\varepsilon > 0$, there exists an $n \in \mathbb{Z}_+$ such that for all $n \geq N$,

$$\overline{d}_\infty(f_n, f) < \varepsilon.$$

Again, for all $x \in X$,

$$\overline{d}(f_n(x), f(x)) \leq \overline{d}_\infty(f_n, f) < \varepsilon,$$

so $f_n \to f$ uniformly. We have shown, $\mathscr{C}(X, Y)$ is sequentially closed, and thus, is closed. Now, let $\{f_n\} \subseteq \mathbb{B}(X, Y)$ be a sequence such that $f_n \to f \in Y^X$. We want to show $f \in \mathbb{B}(X, Y)$. Let $\varepsilon = 1$. Then, there exists $N \in \mathbb{Z}_+$ such that for all $m, n \geq N$, $d_\infty(f_m, f_n) < 1$. Then, as before,

$$d(f_m(x), f_n(x)) < 1,$$

so take the limit $m \to \infty$ and get

$$d(f(x), f_n(x)) \leq 1.$$

We know f_n is bounded. Thus, f is bounded. □

Remark 4.2.3 Inspired by the low-dimensional topology *Coble Lectures* in Altgeld Hall, we will briefly mention n-manifolds.[9]

9: The lectures are presented by Peter Krohnheimer. He will focus on 3- and 4-manifolds.

Definition 4.2.7 (n-Manifold) *An n-manifold is a topological space M such that*

(i) M is locally Euclidean.[10]
(ii) M is Hausdorff (T_2) and second-countable.

10: That is, for all $x \in M$, there exists a neighborhood U of x in M such that U is homeomorphic to $\mathbb{B}^n \simeq \mathbb{R}^n$.

Example 4.2.1 Consider the image of the embedding $\mathbb{S}^1 \hookrightarrow K$, the right-trefoil knot. Then, $\mathbb{R}^3 \setminus K$ is a 3-manifold.

Proposition 4.2.7 *Every compact metric space is complete.*

Proof. Take any Cauchy sequence. Since the space is compact and metrizable, it is sequentially compact, so the sequence has a convergent subsequence. We showed this is equivalent to the sequence converging, so the space is complete. □

4.3 More on Function Spaces

If X is a set and Y is a topological space, denote for $x \in X$ and an open set $U \subseteq Y$,

$$S(x, U) := \{f : X \to Y : f(x) \in U\} \subseteq Y^X.$$

Well, for any $f \in Y^X$, if we take $x \in X$, then $f(x) \in U := Y$, so $S(x, U)$ is a subbasis.

Definition 4.3.1 (Topology of Pointwise Convergence) *Define a subbasis*

$$\mathcal{S}_{point} := \{S(x, U) : x \in X, U \in \mathcal{T}_Y\}.$$

Then, take all finite intersections to get $\mathcal{B}_{point}$. Finally, take all unions to get

$\mathcal{T}_{point}$, the topological of pointwise convergence.

Now, let X be a set and Y a metric space. Let $C \subseteq X$

Definition 4.3.2 (Topology of Compact Convergence) *Define*

$$B_C(f, \varepsilon) = \{g : X \to Y : \sup\{d(f(x), g(x)) : x \in C\} < \varepsilon\}.$$

Then, let $\mathcal{B}_{comp\ conv}$ be the basis of all such $B_C(f, \varepsilon)$. Then, $\mathcal{T}(\mathcal{B}_{comp\ conv}) =: \mathcal{T}_{comp\ conv}$ is the topology of compact convergence.

We have $\mathcal{T}_{\text{unif}}$ when our basis elements are $B_X(f, \varepsilon)$, ranging over the whole space.

Theorem 4.3.1 *Let X be a space. Let (Y, d) be a metric space. For the function space Y^X, one has the following inclusion of topologies:*[11]

$$\mathcal{T}_{unif} \supseteq \mathcal{T}_{comp\ conv} \supseteq \mathcal{T}_{point}.$$

11: This was proven on a day when I slept-in through lecture at 3:00 PM. Check Munkres for his proof.

Remark 4.3.1 If X is compact, then $\mathcal{T}_{\text{unif}} = \mathcal{T}_{\text{comp conv}}$. If X is discrete, then $\mathcal{T}_{\text{comp conv}} = \mathcal{T}_{\text{point}}$.[12]

12: Thus, we have a relation between all three of our new topologies on Y^X. We shall now construct a fourth.

Definition 4.3.3 (Compact-Open Topology) *Define*

$$S(C, U) := \{f \in \mathcal{C}(X, Y) : f(C) \subseteq U\}.$$

Let $\mathcal{S}_{comp\ open}$ be the subbasis of all such elements, where C is compact in X and U is open in Y.[13] *As usual, take finite intersections and arbitrary unions to get $\mathcal{T}_{comp\ open}$, either on Y^X or $\mathcal{C}(X, Y)$.*

13: We could just as well look at all of Y^X. Instead we mostly follow Munkres.

Consider $\mathcal{T}_{\text{comp open}}$ on $\mathcal{C}(X, Y)$, where X is a topological space and Y is a metric space. Our goal is to show that $\mathcal{T}_{\text{comp conv}}$ on Y^X, restricted to the subspace $\mathcal{C}(X, Y)$, coincides with $\mathcal{T}_{\text{comp open}}$.

Theorem 4.3.2 *If X is a topological space and (Y, d) is a metric space, then $\mathcal{T}_{comp\ conv}$ and $\mathcal{T}_{comp\ open}$ agree on $\mathcal{C}(X, Y) \subseteq Y^X$.*

Proof. We are looking at the subbasic elements $S(C, U)$ and basic elements $B_C(f, \varepsilon) \cap \mathcal{C}(X, Y)$. We want to show that $\mathcal{T}_{\text{comp conv}} \subseteq \mathcal{T}_{\text{comp open}}$. Take any $B_C(f', \varepsilon')$ for some choice of C, f', ε'. Let $f \in B_C(f', \varepsilon')$. There exists a $\varepsilon > 0$ such that $f \in B_C(f, \varepsilon) \subseteq B_C(f', \varepsilon')$.[14] We want to choose $C_1, \ldots, C_n$ and $U_1, \ldots, U_n$ so that

14: Namely, take
$$\varepsilon := \varepsilon' - \sup_{x \in C}\{d(f(x), f'(x))\}.$$

$$f \in S(C_1, U_1) \cap \cdots \cap S(C_n, U_n) \subseteq B_C(f, \varepsilon).$$

For each $x \in C$, choose a neighborhood V_x of x such that $\text{Diam}(f(\overline{V_x})) < \varepsilon$. To find this, first consider $U_x := B(f(x), \varepsilon/4) \subseteq Y$. Then, there exists a neighborhood V_x such that $f(V_x) \subseteq U_x$. Then, $f(\overline{V_x}) \subseteq \overline{B(f(x), \varepsilon/4)}$, by continuity. Well, $\overline{B(f(x), \varepsilon/4)} \subseteq B(f(x), \varepsilon/3)$. Thus,

$$\mathrm{Diam}(f(\overline{V_x})) \leq \mathrm{Diam}(B(f(x), \varepsilon/3)) \leq \frac{2\varepsilon}{3} < \varepsilon.$$

Since C is compact, there exists a finite subcover $\{V_{x_1}, \ldots, V_{x_n}\}$ of C. Denote $C_i := C \cap \overline{V_{x_i}}$. where $i \in \{1, \ldots, n\}$. Then, C_i is compact. We have that $C = \bigcup_n C_i$. Also, $\mathrm{Diam}(f(C_i)) \leq \mathrm{Diam}(f(\overline{V_{x_i}})) < \varepsilon$. Let $U_i := U_{x_i} = B(f(x_i), \varepsilon/4)$. We claim that

$$f \in \bigcap_{i=1}^{n} S(C_i, U_i) \subseteq B_C(f, \varepsilon).$$

The fact that f is in all the $S(C_i, U_i)$ is true by construction.[15] Take any $g \in \bigcap_n S(C_i, U_i)$. We want to show $g \in B_C(f, \varepsilon)$.[16] We have that $g(x), f(x) \in U_{x_i}$, so $d(g(x), f(x)) < \varepsilon$. Since the C_is cover C, for any $x \in C$, $d(f(x), g(x)) < \varepsilon$, meaning $g \in B_C(f, \varepsilon)$. We now proceed for the converse inclusion. Take any $S(C, U)$, where C is compact in X and U is open in Y. Also, take any $f \in S(C, U)$. Our goal is to find $B_C(f, \varepsilon)$. In particular, we will take f and C to be the same, only searching for ε. Consider $d(f(x), Y \setminus U) := \inf\{d(f(x), y) : y \in Y \setminus C\}$. Furthermore, we claim that $d(y', Y \setminus C)$ is continuous in y', as d is 1-Lipchitz. Then, $d(\cdot, Y \setminus U) \circ f$ is continuous, and is a function on C. Thus, there exists a minimum; i.e., there exists $x_{\min} \in C$ such that for all $x \in C$,[17]

$$0 < d(f(x_{\min}), Y \setminus U) \leq d(f(x), Y \setminus U).$$

Define $\varepsilon := d(f(x_{\min}), Y \setminus U)$. Then,

$$f \in B_C(f, \varepsilon) \cap \mathscr{C}(X, Y) \subseteq S(C, U),$$

meaning $\mathscr{T}_{\text{comp open}} = \mathscr{T}_{\text{comp conv}}$ on $\mathscr{C}(X, Y)$. □

15: We have $f : V_x \mapsto U_x$ via inclusion, so $f : C_i \mapsto U_i$ via inclusion.

16: We will check that for all $x \in C_i$, $d(g(x), f(x)) < \varepsilon$.

17: It strictly bounded by zero, using sequential closedness.

Corollary 4.3.3 *If X and Y are spaces, and $d : X^2 \to [0, \infty)$ is any metric inducing the topology on Y, then $\mathscr{T}_{comp\ conv}$ on $\mathscr{C}(X, Y)$. is independent of d.*

Corollary 4.3.4 *If X is compact, then $\mathscr{T}_{unif}$ on $\mathscr{C}(X, Y)$ is the same as $\mathscr{T}_{comp\ open}$.*

Consider sets X, Y, Z. Construct $Y^{X \times Z}$ of functions $f : X \times Z \to Y$. On the other hand, consider $(Y^X)^Z$, which is all functions $F : Z \to Y^X$. As sets, we have a bijection

$$\begin{aligned} Y^{X\times Z} &\xrightarrow[\text{bijection}]{\Phi} (Y^X)^Z \\ f &\longmapsto F(z)(x) := f(x, z), \end{aligned}$$

so $\Phi(f) := F$. Now, let us move to Top:

$$\mathscr{C}(X \times Z, Y) \xrightarrow{\quad\Phi\quad} \mathscr{C}(Z, \mathscr{C}(X, Y)).$$

Put $\mathscr{T}_{\text{comp open}}$ on $\mathscr{C}(X, Y)$. We want to show that $\Phi : f \mapsto F$ is bijective if X is locally compact and Hausdorff.

Definition 4.3.4 (Homotopy) *Any element $h \in \mathscr{C}(X \times [0,1], Y)$ is called a homotopy of $f, g \in \mathscr{C}(X, Y)$ if $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$.*[18]

18: That is, we can continuously "deform" f into g via h.

Using the above bijection

$$\mathscr{C}(X \times [0,1], Y) \longrightarrow \mathscr{C}([0,1], \mathscr{C}(X, Y)),$$

any homotopy can be viewed as a family of functions parameterized by the unit interval $[0, 1]$.

4.4 Arzelà-Ascoli Theorem

We now state a neat theorem which can be applied nicely to the theory of manifolds. Let X be a topological space and Y be a metric space.

Definition 4.4.1 (Surface) *A surface is a 2-manifold.*

Definition 4.4.2 (Equicontinuous) *We say that $\mathcal{F} \subseteq \mathscr{C}(X, Y)$ is equicontinuous at $x_0 \in X$ if for all $\varepsilon > 0$ there exists a neighborhood U of x_0 such that for all $f \in \mathcal{F}$,*[19]

$$f(U) \subseteq B(f(x_0), \varepsilon).$$

19: For $\mathcal{F}$ to be equicontinuous everywhere, the statement holds for all $x \in X$,

Definition 4.4.3 (Relatively Compact) *If $A \subseteq Y$, we call A relatively compact if $\overline{A}$ is compact in Y.*

Put the topology of compact convergence on $\mathscr{C}(X, Y)$. Remember, this is the same as the compact-open topology. Note that in $\mathbb{R}^n$, boundedness is very close to being relative compactness, so the Arzelà-Ascoli theorem is stated using this notion for $\mathscr{C}(X, \mathbb{R}^n)$. Now, let $\mathcal{F} \subseteq \mathscr{C}(X, Y)$. Pick $x_0 \in X$. Then, consider the set

$$S := \{f(x_0) : f \in \mathcal{F}\} \subseteq Y.$$

We say that $\mathcal{F}$ is *pointwise relatively compact* if S is relatively compact for any choice of x_0.

Theorem 4.4.1 (Arzelà-Ascoli I) *If $\mathcal{F} \subseteq \mathscr{C}(X, Y)$ is equicontinuous and pointwise relatively compact, then it is relatively compact.*[20]

20: The converse holds if X is locally compact Hausdorff.

Theorem 4.4.2 (Arzelà-Ascoli II) *Let X be Hausdorff and σ-compact.*[21] *Let f_n be a sequence in $\mathscr{C}(X, Y)$ which is equicontinuous and pointwise relatively compact. Then, there is a subsequence of f_n in $\mathscr{C}(X, Y)$.*

21: We say X is σ-comapct if it is the countable union of compact spaces. Alternatively, per Munkres, X is Hausdorff σ-compact if there exists a sequence of open subsets U_n and a sequence of compact subspaces C_n such that $U_n \subseteq C_n$. Plus, $\bigcup U_n = X$.

We have that I $\Rightarrow$ II.

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Consider the square $[0,1]^2 \subseteq \mathbb{R}^2$ with the subspace topology. Label opposite sides of the square by a, b respectively, orienting the b-sides up and the a-sides to the right. Then, define an equivalence relation $\sim$ where b-sides are equivalent, a-sides are equivalent, the corners are equivalent, and no points on the interior are equivalent. Taking the quotient space $[0,1]^2/\sim$ with the standard quotient topology, we get what is called the *2-torus* $\mathbb{T}^2$.

On the other hand, we could consider the set $\mathbb{S}^1 \times \mathbb{S}^1$ with each half of the product inheriting their respective $\mathbb{R}$-subspace topologies. Then, putting the product topology on $\mathbb{S}^1 \times \mathbb{S}^1$, this is *also* $\mathbb{T}^2$.

Consider a group action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ by

$$\begin{aligned} \mathbb{Z}^2 \times \mathbb{R}^2 &\xrightarrow{\text{action } +} \mathbb{R}^2 \\ (c,d)\cdot(x,y) &\longmapsto (x+c, y+d). \end{aligned}$$

The integer lattice of $[(x,y)]$ is precisely the *orbit* of the additive action. Then, take the quotient $\mathbb{R}^2/\mathbb{Z}^2$, with cosets given by the equivalence relation of the orbit of the action.[1] This is $\mathbb{T}^2$.

1: Recall that orbits are just the equivalence classes of the action.

Finally, consider the rotation of the circle $\mathbb{S}^1 \subseteq \mathbb{R}^3$ around the z-axis. This is also the same 2-torus $\mathbb{T}^2$.

Theorem 5.0.1 *The four definitions of* $\mathbb{T}^2$, $[0,1]^2/\sim$, $\mathbb{S}^1 \times \mathbb{S}^1$, $\mathbb{R}^2/\mathbb{Z}^2$, *and as a solid of revolution are homeomorphic.*

The equivalence of multiple definitions of $\mathbb{T}^2$ motivates our discussion of algebraic topology. Broadly speaking, algebraic topology precisely studies what its name suggests: how to take algebraic structures like groups, rings, modules, fields, and associate topological spaces with them. We begin this study with the *fundamental group*.

Remark 5.0.1 (Cell Complex on $\mathbb{T}^2$) Another way to construct $\mathbb{T}^2$ is as a cell complex. Take one copy of $\mathbb{D}^0$, two copies of $\mathbb{D}^1$, and one copy of $\mathbb{D}^2$. Glue $\partial\mathbb{D}^1$ of both copies to the point, yielding a shape resembling ∞. Then, glue the a-sides and the b-sides of $\mathbb{D}^2$ to half of each side of ∞.[2]

2: This is essentially the same as $[0,1]^2/\sim$, but we wanted emphasize the cell structure.

Similarly, we could form the Δ-*complex* structure on $\mathbb{T}^2$, taking one vertex, three 2-simplices, and two 3-simplices.[3] The issue with Δ-complexes is that we lose our information about our 2-simplices. The nice thing about *simplicial-complexes* is that we keep all information about the respective vertices, mapping the simplices injectively (any two points in the simplex should not be identified as one). One way to form the simplicial complex of $\mathbb{T}^2$ is to start with 4 vertices, 12 edges, and 8 triangles. Prescribe all the gluings as you would expect.

3: If a 3-simplex is a triangle, then a 4-simplex is a tetrahedron (the convex hull of four points in $\mathbb{R}^3$).

Definition 5.0.1 (Euler Characteristic) *Assigning alternating parity to the n-simplices of the complexes, starting with $+$ on the vertices, the Euler characteristic χ takes the number of each and sums via the parity.*

Theorem 5.0.2 *The Euler characteristic of $\mathbb{T}^2$ is $\chi = 0$.*[4]

4: It takes some fair amount of work to prove this generally.

It takes some effort, but proving the Euler characteristic is an invariant shows that $\mathbb{T}^2 \not\simeq \mathbb{S}^2$.

5.1 Fundamental Group

Let X be a topological space space with a *base point* $x_0 \in X$.

Definition 5.1.1 (Loop) *A loop at $x_0 \in X$ is a path $f : [0,1] \to X$ such that $f(0) = x_0$ and $f(1) = x_0$.*

We consider the set $\mathrm{Loops}(X, x_0) \subseteq \mathrm{Paths}(X)$.

Definition 5.1.2 (Concatenation Operation) *Let $f, g \in \mathrm{Paths}(X)$ or in $\mathrm{Loops}(X, x_0)$.*[5] *Suppose $f(1) = g(0)$. Define a new path $f * g$ by: divide $[0,1]$ into $[0,1/2]$ and $(1/2,1]$. Then consider the linear maps $[0,1/2] \to [0,1]$ prescribed by $t \mapsto 2t$ and $(1/2,1] \to [0,1]$ prescribed by $t \mapsto 2t-1$. Then, composing, our formula is*[6]

$$(f * g)(t) := \begin{cases} f(2t), & t \in [0,1/2] \\ g(2t-1), & t \in [1/2,1]. \end{cases}$$

5: For the former, we need the starting point of g to be the end point of f.

6: This is well-defined, since we took that $f(2(1/2)) = f(1) = g(0) = g(2(1/2)-1)$.

We claim that $(\mathrm{Loops}(X, x_0), *)$, where $*$ is the concatenation binary operation, is *not* a group.[7] It looks like we need to be a bit more careful. Given continuous maps $\varphi, \psi : X \to Y$, recall that a homotopy between φ and ψ is the continuous map

$$H : X \times [0,1] \to Y$$

such that $H(x,0) = \varphi(x)$ and $H(x,1) = \psi(x)$ for all $x \in X$.

7: It breaks even at associativity.

Definition 5.1.3 (Path Homotopy) *When X happens to also be an interval $X := [0,1]$, we can consider two paths $f, g : [0,1] \to Y$.*[8] *A path homotopy from f to g is a continuous function*

$$H : [0,1] \times [0,1] \to Y$$

such that for all $s \in [0,1]$, $H(s,0) = f(s)$ and $H(s,1) = g(s)$. Additionally, we require that for any $t \in [0,1]$, $H(0,t) = f(0)$ and $H(1,t) = f(1)$.[9]

8: We use s for the parameter of the path and t for the parameter of the homotopy.

9: That is, path homotopies preserve the end points.

Definition 5.1.4 (Homotopic) *Two paths $f, g : [0,1] \to Y$ are (path-) homotopic if there exists a path homotopy from f to g. We will write $f \sim g$.*

Example 5.1.1 (Trivial Loop) Consider $(\mathbb{S}^1, x_0)$. One loop is $c_0 : [0, 1] \to \mathbb{S}^1$, the constant map to x_0.

Theorem 5.1.1 *Being homotopic is an equivalence relation on* $\text{Loops}(X, x_0)$.[10]

10: Or, take $\text{Paths}(X)$.

Proof.

(i) *Reflexive:* Take $f \in \text{Paths}(X)$. We have the path homotopy $H(s,t) := f(s)$. This is continuous, because it is the composition of the projection from the square to the interval and f.

(ii) *Symmetry:* Suppose $f \sim g$. Then, there exists a path homotopy H from f to g. Let H' be defined by $H'(s,t) := H(s, 1-t)$.[11]

(iii) *Transitive:* Suppose $f \sim g$ and $g \sim h$. There exists a path homotopy H_1 from f to g and H_2 from g to h. Define

$$H := \begin{cases} H_1(s, 2t), & t \in [0, 1/2] \\ H_2(s, 2t-1), & t \in [1/2, 1], \end{cases}$$

noting that at $1/2$, $H(s, 1/2) = H_1(s, 1) = H_2(s, 0) = g(s)$.

□

11: Our H' is a composition $(s,t) \mapsto (s, 1-t) \mapsto H(s, 1-t)$. The end point mapping property is trivially true.

Definition 5.1.5 (Fundamental Group) *Define the fundamental group* $\pi_1(X, x_0) := \text{Loops}(X, x_0)/\sim$, *where* $\sim$ *is homotopy equivalence.*[12] *The operation* $\cdot : \pi_1(X, x_0)^2 \to \pi_1(X, x_0)$ *is defined by* $[f] \cdot [g] = [f * g]$.

12: That is, an element $[f]$ in $\pi_1(X, x_0)$ is a path homotopy equivalence class of loops.

Proposition 5.1.2 *The operation* $\cdot$ *on* $\pi_1(X, x_0)$ *is well-defined.*

Proof. Suppose $f' \sim f$ and $g' \sim g$. Then, there exists a path homotopy H_1 between f' and f, and likewise with H_2 and g, g'. Define H by crushing together the H_i horizontally in s, rather than in t:

$$H(s,t) := \begin{cases} H_1(2s, t), & s \in [0, 1/2] \\ H_2(2s-1, t), & s \in [1/2, 1]. \end{cases}$$

Thus, $[f * g] = [f' * g']$, meaning the operation is well-defined. □

Lemma 5.1.3 *The pair* $(\pi_1(X, x_0), \cdot)$ *is a group.*

Proof.

(i) *Associative:* We need to check that

$$([f] \cdot [g]) \cdot [h] = [f] \cdot ([g] \cdot [h]).$$

Well, $([f] \cdot [g]) \cdot [h] = [(f * g) * h]$. Similarly, $[f * (g * h)]$.[13]

13: Again, it will suffice to show that $f * (g * h)$ is path homotopic to $(f * g) * h$. We omit the proof. Define a path homotopy for the intervals $s \in [0, (t+1)/4]$, $s \in [(t+1)/4, (t+2)/4]$, and $s \in [(t+2)/4, 1]$.

(ii) *Identity:* Our candidate is $c_0 : [0,1] \to X$ defined by $[0,1] \mapsto \{x_0\}$. After concatenation, this is not *exactly* the same, but our goal is to show via path homotopy that

$$[f] \cdot [c_0] = [f * c_0] = [f],$$

meaning $f * c_0 \sim f$. Define the homotopy

$$H(s,t) := \begin{cases} f(\text{a linear map to } [0,1]), & s \in [0,(t+1)/2] \\ x_0, & s \in [(t+1)/2, 1]. \end{cases}$$

This is a path homotopy, so we have the identity $[c_0] \in \pi_1(X, x_0)$.

(iii) *Inverses:* Let $[f] \in \pi_1(X, x_0)$. Let $\overline{f} : [0,1] \to X$ be a loop at x_0 defined by $\overline{f}(s) := f(1-s)$.[14] Again, you can write the formula for $H(s,t)$, defined on the intervals $s \in [0,(1-t)/2]$, $s \in [(1-t)/2,(t+1)/2]$, and $s \in [(t+1)/2, 1]$. Thus, $[f] \cdot [\overline{f}] = [c_0]$. Then, just interchange the roles of $f, \overline{f}$, getting $[\overline{f}] \cdot [f] = [c_0]$.

Thus, $\pi_1(X, x_0)$ is a group under $\cdot$. □

14: This is just us flipping the interval, so we can trace backwards along the path.

Remark 5.1.1 Note that our fundamental group requires a base point. Thus, $\pi_1 : \mathsf{Top}_* \to \mathsf{Grp}$ is a functor, rather than from Top.

Theorem 5.1.4 *$\pi_1(\mathbb{R}^2, x_0) \simeq \{e\}$, the trivial group.*

Proof. Let $[f] \in \pi_1(\mathbb{R}^2, x_0)$. Define

$$\ell(t) := (1-t)f(s) + tx_0.$$

Then, we get a path homotopy[15]

$$H(s,t) := (1-t)f(s) + tx_0.$$

□

15: It should be clear that this is continuous.

Theorem 5.1.5 $\pi_1(\mathbb{T}^2, x_0) \simeq C_\infty \times C_\infty$.

Proof. We will do this informally. Recall the two paths a, b on the square forming $\mathbb{T}^2$. These paths satisfy $[a,b] = aba^{-1}b^{-1} = 1$, so we get[16]

$$\pi_1(\mathbb{T}^2, x_0) \simeq \langle a, b | [a,b] \sim c_0 \rangle \simeq C_\infty \times C_\infty \simeq \mathbb{Z}^2.$$

□

16: The two path generators a, b individually generate the free group, but we have to quotient by the relation of the commutator.

Example 5.1.2 A few more isomorphisms of fundamental groups:

(i) $\pi_1(\mathbb{T}^1, x_0) \simeq C_\infty$.
(ii) $\pi_1(\mathbb{S}^2, x_0) \simeq \{e\}$.
(iii) $\pi_1(R, x_0) \simeq F(2)$, the free group on two generators, where R is the two-rose.

5.2 Covering Spaces

What is $\pi_1(\mathbb{S}^1, x_0)$. We claim it is $C_\infty \simeq \mathbb{Z}$. The identity, per usual, is the constant loop $[c_0]$. Then, t would be one traversal around $\mathbb{S}^1$ in one direction. The group is generated by t, with negatives coming from traversing in the opposite direction. How do we do show this?[17]

17: We will sketch out the basic idea.

Well, consider $\mathbb{R} \to \mathbb{S}^1 \subseteq \mathbb{C}$. Then, we write $f : [0,1] \to \mathbb{S}^1 \subseteq \mathbb{C}$, sending $s \mapsto \cos(2\pi s), \sin(2\pi s)$. Then, $f^5 = f * \cdots * f$ is given by $s \mapsto (\cos 10\pi s, \sin 10\pi s)$. The question is why if we take any loop, then it will be path homotopic to one of this flavor. For any small open interval on $\mathbb{R}$, you can see that we would have a corresponding small ball on $\mathbb{S}^1$. Then, slowly traversing our path, we can lift each coordinate up into $\mathbb{R}$. Using this lifting (lift x_0 to 0) and compactness, the other endpoint will be at $-n$, where we go around n times. That is, $\mathbb{Z} \subseteq \mathbb{R}$ is the preimage of x_0. The map from the line to the circle is an important example of a covering space, lifting paths piece by piece.

Informally, a *covering space* of X is a function $p : \widetilde{X} \to X$ which is a local homeomorphism.[18]

18: This is *not* equivalent to the actual definition. We are just looking for intuition.

Remark 5.2.1 For a particular type of covering spaces, called regular covers, the $\pi_1(X, x_0) =: G$ acts on the covering space, in the way $\mathbb{Z}$ acts on $\mathbb{R}$ by translation.

Definition 5.2.1 (Covering Space) *A covering $p : \widetilde{X} \to X$ of X is a continuous map such that for all $x \in X$, there exists a neighborhood U_x of x and a discrete topological space D_x so that the preimage $p^{-1}(U_x) = \coprod_{D_x} V_k$, where $p|_{V_k} : V_k \to U_x$ is a homeomorphism for all k.*

Consider a circle which has $1/2$ the speed of traversal as $\mathbb{S}^1$. This is *also* a covering space of the circle $\mathbb{S}^1$, under the projection map $\times 2$. That is, antipodal points correspond to a single point on $\mathbb{S}^1$, precisely like our definition of $\mathbb{R}\mathbf{P}^2$. We could continue this process for all $\times n$ maps for $n \in \mathbb{Z}$. Then, we get corresponding subgroups associated with $n\mathbb{Z}$.

Definition 5.2.2 (Simply Connected) *A space X is simply connected if X is path connected and $\pi_1(X) = 1$.*

Among all covering spaces for $\mathbb{S}^1$, the only simply connected one is $\mathbb{R}$. In this case, we call $\mathbb{R}$ the *universal covering space* of $\mathbb{S}^1$. What is a universal covering space of $\mathbb{R}\mathbf{P}^2$?[19] It turns out, the answer is precisely $\mathbb{S}^2$, as one would expect.

19: In fact, universal covering spaces are unique up to homeomorphism.

What is the covering space of the 2-rose? We know the fundamental group is $F(2)$, the free group on two generators. If we trace up the fibers to a covering space, we could form a space which has an infinite amount of 4-branches. Then, we could either wrap around or make a *tree*. Recall that any subgroup of the free group is another free group.[20]

20: That is, subgroups of the fundamental group correspond to covering spaces.