TWO-DIMENSIONAL DAGGER TQFTS ARE PRETTY SIMPLE

DHEERAN E. WIGGINS

It is well-known that two-dimensional topological quantum field theories are commutative Frobenius algebras. In 1994, Durhuus and Jonsson, showed that when Atiyah's axioms are made †-compatible, we can extract some positive real numbers λ_i that uniquely characterize such a theory. It is clear that this classification corresponds to the idea that "unitary two-dimensional field theories are semisimple commutative Frobenius algebras," but I could not find this story told in the way I wanted anywhere, so I wrote this note.

1. Dagger Things

Let $(\mathfrak{Bord}_2, \amalg, \varnothing, \tau)$ and $(\mathfrak{FdHilb}, \otimes, \mathbb{C}, \gamma)$ denote the bordism class category in dimension 2 and the category of finite-dimensional Hilbert spaces, respectively, equipped with their usual symmetric monoidal structures. The braidings are the standard bordism twist

$$\tau_{M,N} = \quad : M \amalg N \xrightarrow{\simeq} N \amalg M$$

and the swap operator $\gamma_{\mathsf{A},\mathsf{B}} : \mathcal{H}^{\mathsf{A}} \otimes \mathcal{H}^{\mathsf{B}} \xrightarrow{\simeq} \mathcal{H}^{\mathsf{B}} \otimes \mathcal{H}^{\mathsf{A}}$ sending $a \otimes b \mapsto b \otimes a$.

The symmetric monoidal functor category therebetween $\mathfrak{Fun}^{\otimes}(\mathfrak{Bord}_2, \mathfrak{FdHilb})$ is the category of two-dimensional topological quantum field theories. The famous classification result is that there is an equivalence

$$\Omega : \mathfrak{Fun}^{\otimes}(\mathfrak{Bord}_2, \mathfrak{FdHilb}) \xrightarrow{\simeq} \mathfrak{CFrob}$$

given by assigning to each theory $\mathcal{Z}$ a commutative Frobenius algebra $\mathfrak{A} = \mathcal{Z}(\mathbb{S}^1)$ with multiplication $m = \mathcal{Z}(\;)$, comultiplication $\Delta = \mathcal{Z}(\;)$, unit $\eta = \mathcal{Z}(\;)$, and counit $\varepsilon = \mathcal{Z}(\;)$. However, this picture ignores a nifty piece of structure that can be tacked on to both bordisms and Hilbert spaces: a dagger. For our purposes, a dagger † will mean an involutive contravariant endofunctor that does nothing on objects. Seeing as our categories $\mathfrak{Bord}_2$ and $\mathfrak{FdHilb}$ already have a symmetric monoidal structure, any dagger structure we add must be compatible with this, i.e., $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$ for any maps f and g, and all associators, unitors, and braidings must be unitary.[1]

It can be easily checked that on $\mathfrak{Bord}_2$, the endofunctor † given by reversing bordisms yields a symmetric monoidal dagger category. Likewise, the usual † on $\mathfrak{FdHilb}$ given by taking adjoints is a dagger structure compatible with $\otimes$.

Note that if we were to look at the dualities exhibited in $\mathfrak{Bord}_2$, orientation-reversal would give the dual on objects, so the dual maps come from the "transposed" bordisms. In a sense, this analogy tells us that reversing the orientations corresponds

---

Date: December 2025.

James Pascaleff exposed me to semisimple topological field theories and [DJ94], so this note is just a slight rephrasing of his lectures. Charles Rezk made some helpful comments while I thought about this.

[1]When working in a dagger category, we tend to borrow terminology from Hilbert spaces, so unitary means a morphism's inverse is given by its dagger.

to taking the complex conjugate, since the dagger of a map is just a complex conjugate away from being the transpose. I find this to be helpful intuition, but dualities will not play any role for us.

Definition 1.1 (Dagger Functor)**.** A dagger functor $f : (\mathfrak{C}, \dagger) \to (\mathfrak{D}, \dagger)$ between dagger categories is an ordinary functor so that the square

$$\begin{array}{ccc} \mathfrak{C} & \xrightarrow{f} & \mathfrak{D} \\ {\scriptstyle \dagger}\downarrow & & \downarrow{\scriptstyle \dagger} \\ \mathfrak{C} & \xrightarrow{f} & \mathfrak{D} \end{array}$$

commutes. Of course, daggers are the identity on objects, so being a dagger functor only requires a condition on morphisms.

Write $\mathfrak{Fun}^{\otimes}_{\dagger}(\mathfrak{Bord}_2, \mathfrak{FdHilb})$ for the category of two-dimensional dagger (or unitary) field theories, i.e., the symmetric monoidal functor category that respects the $\dagger$ in bordisms and Hilbert spaces in the above sense.[2]

Definition 1.2 (Dagger Frobenius Structure)**.** A Frobenius structure $(\mathfrak{A}, m, \Delta, \eta, \varepsilon)$ in $\mathfrak{FdHilb}$ is a dagger (or unitary) Frobenius algebra if $m^\dagger = \Delta$ and $\eta^\dagger = \varepsilon$.

Evidently, if $\mathcal{Z}$ is a two-dimensional dagger field theory, then

$$m^\dagger = \mathcal{Z}(\ )^\dagger = \mathcal{Z}(\ ^\dagger) = \mathcal{Z}(\ ) = \Delta$$

and

$$\eta^\dagger = \mathcal{Z}(\ )^\dagger = \mathcal{Z}(\ ^\dagger) = \mathcal{Z}(\ ) = \varepsilon,$$

so we get a map

$$\begin{array}{ccc} \mathfrak{Fun}^{\otimes}(\mathfrak{Bord}_2, \mathfrak{FdHilb}) & \xrightarrow{\Omega} & \mathfrak{C}\mathrm{Frob} \\ \cup & & \cup \\ \mathfrak{Fun}^{\otimes}_{\dagger}(\mathfrak{Bord}_2, \mathfrak{FdHilb}) & \overset{\mho}{\dashrightarrow} & \mathfrak{C}\mathrm{Frob}^\dagger \end{array}$$

which is an equivalence. Thus, two-dimensional dagger field theories are exactly commutative dagger Frobenius algebras. But, what are dagger Frobenius algebras?

2. On H*-Algebras

To continue with our classification, we now diverge from field theories for a bit, turning our attention to a certain type of $*$-algebra that crops up in quantum foundations. In 1945, Ambrose originally defined H*-algebras as sort of Banach algebra, but following Heunen and Vicary, we omit the norm requirement.

Definition 2.1 (H*-Algebra)**.** An H*-algebra is a $\mathbb{C}$-algebra $\mathfrak{A}$ such that $(\mathfrak{A}, \langle - | - \rangle_{\mathfrak{A}})$ is simultaneously a Hilbert space equipped with an antilinear involution $\dagger : \mathfrak{A} \to \mathfrak{A}$ so that for all $a, b, c \in \mathfrak{A}$,

$$\langle ab | c \rangle_{\mathfrak{A}} = \langle b | a^\dagger c \rangle_{\mathfrak{A}} = \langle a | c b^\dagger \rangle_{\mathfrak{A}} .$$

[2]I am suppressing two adjectives here.

Per usual when working with $*$-algebras, the quintessential ones are matrix algebras, or subalgebras of matrix algebras. Given a finite-dimensional Hilbert space $\mathcal{H}$ and some positive real number λ, define an H*-algebra $\mathfrak{B}(\mathcal{H}, \lambda)$ whose underlying algebra is the set of linear operators $\mathfrak{B}(\mathcal{H})$, whose involution is given by the adjoint, and whose inner product $\langle - | - \rangle : \mathfrak{B}(\mathcal{H}, \lambda)^2 \to \mathbb{C}$ is $\langle a | b \rangle = \lambda \operatorname{tr}(a^\dagger b)$. Ambrose uses these algebras $\mathfrak{B}(\mathcal{H}, \lambda)$ to give a Wedderburn-Artin classification result for finite-dimensional H*-algebras.

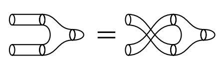
Theorem 2.2 ([Amb45])**.** *Let $\mathfrak{A}$ be a finite-dimensional H*-algebra. Then, there is an orthogonal direct sum decomposition $\mathfrak{A} \simeq \mathfrak{B}(\mathcal{H}_1, \lambda_1) \oplus \mathfrak{B}(\mathcal{H}_2, \lambda_2) \oplus \cdots \oplus \mathfrak{B}(\mathcal{H}_n, \lambda_n)$ where n is a natural number, $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n$ are finite-dimensional Hilbert spaces, and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are positive real numbers.*

The following can then be obtained rather easily.

Lemma 2.3 ([HV19])**.** *A monoid $(\mathfrak{A}, m, \eta)$ internal to $\mathfrak{F}$dHilb is a symmetric dagger Frobenius monoid if and only if it is a finite-dimensional H*-algebra, where the involution is defined by sending a vector $a \in \mathfrak{A}$ to*

$$\mathbb{C} \xrightarrow{\eta} \mathfrak{A} \xrightarrow{m^\dagger} \mathfrak{A} \otimes \mathfrak{A} \xrightarrow{\mathrm{id}_{\mathfrak{A}} \otimes a} \mathfrak{A} \otimes \mathbb{C} \to \mathfrak{A}.$$

Thus, symmetric dagger Frobenius monoids in $\mathfrak{F}$dHilb are just a kind of finite-dimensional H*-algebra. The adjective "symmetric" here corresponds to the equation



which is obviously satisfied when things are commutative.

3. Characterization

Let $\mathcal{Z}$ be a two-dimensional dagger field theory. We already know how to obtain a commutative dagger Frobenius algebra $\mho(\mathcal{Z}) = \mathcal{Z}(\mathbb{S}^1)$ by passing through the usual equivalence. Using the language of H*-algebras, we obtain the following.

Theorem 3.1 ([DJ94])**.** *Two-dimensional dagger field theories $\mathcal{Z}$ give rise to and are classified by $n = \dim(\mathcal{Z}(\mathbb{S}^1))$ many positive real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$.*

Proof. By Lemma 2.3, the Frobenius algebra $\mathcal{Z}(\mathbb{S}^1)$ is a finite-dimensional H*-algebra. Then, Theorem 2.2 gives us a decomposition

$$\mathcal{Z}(\mathbb{S}^1) \simeq \mathfrak{B}(\mathcal{H}_1, \lambda_1) \oplus \mathfrak{B}(\mathcal{H}_2, \lambda_2) \oplus \cdots \oplus \mathfrak{B}(\mathcal{H}_n, \lambda_n).$$

Since $\mathcal{Z}(\mathbb{S}^1)$ is commutative with respect to $m = \mathcal{Z}(\ )$, each of the summands $\mathfrak{B}(\mathcal{H}_i, \lambda_i)$ must be one-dimensional, so $n = \dim(\mathcal{Z}(\mathbb{S}^1))$. □

Evidently, we have ended up with semisimplicity, since

$$\mathcal{Z}(\mathbb{S}^1) \simeq \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \cdots \oplus \mathbb{C}e_n$$

with pointwise operations, where the $e_1, e_2, \ldots, e_n$ form a basis of orthogonal idempotents. Then, the numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the images of the orthogonal idempotents under the trace map $\varepsilon = \mathcal{Z}(\ )$.

References

[Amb45] Warren Ambrose. Structure theorems for a special class of Banach algebras. *Transactions of the American Mathematical Society*, 57(3), 1945.

[DJ94] Bergfinnur Durhuus and Thordur Jonsson. Classification and construction of unitary topological field theories in two dimensions. *Journal of Mathematical Physics*, 35(10), 1994.

[HV19] Chris Heunen and Jamie Vicary. *Categories for Quantum Theory: An Introduction*. Oxford University Press, 2019.

Department of Mathematics, University of Illinois Urbana, Illinois, 61801
Email address: `dheeran2@illinois.edu`