

# TWO-DIMENSIONAL DAGGER TQFTS ARE PRETTY SIMPLE

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It is well-known that two-dimensional topological quantum field theories are commutative Frobenius algebras. In 1994, Durhuus and Jonsson, showed that when Atiyah's axioms are made  $\dagger$ -compatible, we can extract some positive real numbers  $\lambda_i$  that uniquely characterize such a theory. It is clear that this classification corresponds to the idea that “unitary two-dimensional field theories are semisimple commutative Frobenius algebras,” but I could not find this story told in the way I wanted anywhere, so I wrote this note.

## 1. DAGGER THINGS

Let  $(\mathcal{B}\text{ord}_2, \amalg, \emptyset, \tau)$  and  $(\mathcal{F}\text{dHilb}, \otimes, \mathbb{C}, \gamma)$  denote the bordism class category in dimension 2 and the category of finite-dimensional Hilbert spaces, respectively, equipped with their usual symmetric monoidal structures. The braidings are the standard bordism twist

$$\tau_{M,N} = \otimes \otimes : M \amalg N \xrightarrow{\sim} N \amalg M$$

and the swap operator  $\gamma_{A,B} : \mathcal{H}^A \otimes \mathcal{H}^B \xrightarrow{\sim} \mathcal{H}^B \otimes \mathcal{H}^A$  sending  $a \otimes b \mapsto b \otimes a$ .

The symmetric monoidal functor category therebetween  $\mathcal{F}\text{un}^\otimes(\mathcal{B}\text{ord}_2, \mathcal{F}\text{dHilb})$  is the category of two-dimensional topological quantum field theories. The famous classification result is that there is an equivalence

$$\Omega : \mathcal{F}\text{un}^\otimes(\mathcal{B}\text{ord}_2, \mathcal{F}\text{dHilb}) \xrightarrow{\sim} \mathcal{C}\text{Frob}$$

given by assigning to each theory  $Z$  a commutative Frobenius algebra  $\mathfrak{A} = Z(\mathbb{S}^1)$  with multiplication  $m = Z(\mathbb{D})$ , comultiplication  $\Delta = Z(\mathbb{C})$ , unit  $\eta = Z(\emptyset)$ , and counit  $\varepsilon = Z(\mathbb{O})$ . However, this picture ignores a nifty piece of structure that can be tacked on to both bordisms and Hilbert spaces: a dagger. For our purposes, a dagger  $\dagger$  will mean an involutive contravariant endofunctor that does nothing on objects. Seeing as our categories  $\mathcal{B}\text{ord}_2$  and  $\mathcal{F}\text{dHilb}$  already have a symmetric monoidal structure, any dagger structure we add must be compatible with this, i.e.,  $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$  for any maps  $f$  and  $g$ , and all associators, unitors, and braidings must be unitary.<sup>1</sup>

It can be easily checked that on  $\mathcal{B}\text{ord}_2$ , the endofunctor  $\dagger$  given by reversing bordisms yields a symmetric monoidal dagger category. Likewise, the usual  $\dagger$  on  $\mathcal{F}\text{dHilb}$  given by taking adjoints is a dagger structure compatible with  $\otimes$ .

Note that if we were to look at the dualities exhibited in  $\mathcal{B}\text{ord}_2$ , orientation-reversal would give the dual on objects, so the dual maps come from the “transposed” bordisms. In a sense, this analogy tells us that reversing the orientations corresponds

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James Pascaleff exposed me to semisimple topological field theories and [DJ94]. Charles Rezk made helpful comments while I thought about these things.

<sup>1</sup>When working in a dagger category, we tend to borrow terminology from Hilbert spaces, so unitary means a morphism's inverse is given by its dagger.

to taking the complex conjugate, since the dagger of a map is just a complex conjugate away from being the transpose. I find this to be helpful intuition, but dualities will not play any role for us.

**Definition 1.1** (Dagger Functor). A dagger functor  $f : (\mathcal{C}, \dagger) \rightarrow (\mathcal{D}, \dagger)$  between dagger categories is an ordinary functor so that the square

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{D} \\ \dagger \downarrow & & \downarrow \dagger \\ \mathcal{C} & \xrightarrow{f} & \mathcal{D} \end{array}$$

commutes. Of course, daggers are the identity on objects, so being a dagger functor only requires a condition on morphisms.

Write  $\mathcal{F}\text{un}_\dagger^\otimes(\mathcal{B}\text{ord}_2, \mathcal{F}\text{dHilb})$  for the category of two-dimensional dagger (or unitary) field theories, i.e., the symmetric monoidal functor category that respects the  $\dagger$  in bordisms and Hilbert spaces in the above sense.<sup>2</sup>

**Definition 1.2** (Dagger Frobenius Structure). A Frobenius structure  $(\mathfrak{A}, m, \Delta, \eta, \varepsilon)$  in  $\mathcal{F}\text{dHilb}$  is a dagger (or unitary) Frobenius algebra if  $m^\dagger = \Delta$  and  $\eta^\dagger = \varepsilon$ .

Evidently, if  $\mathcal{Z}$  is a two-dimensional dagger field theory, then

$$m^\dagger = \mathcal{Z}(\triangleright)^\dagger = \mathcal{Z}(\triangleright^\dagger) = \mathcal{Z}(\triangleleft) = \Delta$$

and

$$\eta^\dagger = \mathcal{Z}(\odot)^\dagger = \mathcal{Z}(\odot^\dagger) = \mathcal{Z}(\oslash) = \varepsilon,$$

so we get a map

$$\begin{array}{ccc} \mathcal{F}\text{un}^\otimes(\mathcal{B}\text{ord}_2, \mathcal{F}\text{dHilb}) & \xrightarrow{\Omega} & \mathcal{C}\text{Frob} \\ \cup & & \cup \\ \mathcal{F}\text{un}_\dagger^\otimes(\mathcal{B}\text{ord}_2, \mathcal{F}\text{dHilb}) & \dashrightarrow^{\mathcal{V}} & \mathcal{C}\text{Frob}^\dagger \end{array}$$

which is an equivalence. Thus, two-dimensional dagger field theories are exactly commutative dagger Frobenius algebras. But, what are dagger Frobenius algebras?

## 2. ON H\*-ALGEBRAS

To continue with our classification, we now diverge from field theories for a bit, turning our attention to a certain type of  $*$ -algebra that crops up in quantum foundations. In 1945, Ambrose originally defined  $H^*$ -algebras as sort of Banach algebra, but following Heunen and Vicary, we omit the norm requirement.

**Definition 2.1** ( $H^*$ -Algebra). An  $H^*$ -algebra is a  $\mathbb{C}$ -algebra  $\mathfrak{A}$  such that  $(\mathfrak{A}, \langle -|-\rangle_{\mathfrak{A}})$  is simultaneously a Hilbert space equipped with an antilinear involution  $\dagger : \mathfrak{A} \rightarrow \mathfrak{A}$  so that for all  $a, b, c \in \mathfrak{A}$ ,

$$\langle ab|c\rangle_{\mathfrak{A}} = \langle b|a^\dagger c\rangle_{\mathfrak{A}} = \langle a|cb^\dagger\rangle_{\mathfrak{A}}.$$

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<sup>2</sup>I am suppressing two adjectives here.

Per usual when working with  $*$ -algebras, the quintessential ones are matrix algebras, or subalgebras of matrix algebras. Given a finite-dimensional Hilbert space  $\mathcal{H}$  and some positive real number  $\lambda$ , define an  $H^*$ -algebra  $\mathfrak{B}(\mathcal{H}, \lambda)$  whose underlying algebra is the set of linear operators  $\mathfrak{B}(\mathcal{H})$ , whose involution is given by the adjoint, and whose inner product  $\langle - | - \rangle : \mathfrak{B}(\mathcal{H}, \lambda)^2 \rightarrow \mathbb{C}$  is  $\langle a | b \rangle = \lambda \operatorname{tr}(a^\dagger b)$ . Ambrose uses these algebras  $\mathfrak{B}(\mathcal{H}, \lambda)$  to give a Wedderburn-Artin classification result for finite-dimensional  $H^*$ -algebras.

**Theorem 2.2** ([Amb45]). *Let  $\mathfrak{A}$  be a finite-dimensional  $H^*$ -algebra. Then, there is an orthogonal direct sum decomposition  $\mathfrak{A} \simeq \mathfrak{B}(\mathcal{H}_1, \lambda_1) \oplus \mathfrak{B}(\mathcal{H}_2, \lambda_2) \oplus \cdots \oplus \mathfrak{B}(\mathcal{H}_n, \lambda_n)$  where  $n$  is a natural number,  $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_n$  are finite-dimensional Hilbert spaces, and  $\lambda_1, \lambda_2, \dots, \lambda_n$  are positive real numbers.*

The following can then be obtained rather easily.

**Lemma 2.3** ([HV19]). *A monoid  $(\mathfrak{A}, m, \eta)$  internal to  $\mathfrak{FdHilb}$  is a symmetric dagger Frobenius monoid if and only if it is a finite-dimensional  $H^*$ -algebra, where the involution is defined by sending a vector  $a \in \mathfrak{A}$  to*

$$\mathbb{C} \xrightarrow{\eta} \mathfrak{A} \xrightarrow{m^\dagger} \mathfrak{A} \otimes \mathfrak{A} \xrightarrow{\operatorname{id}_{\mathfrak{A}} \otimes a} \mathfrak{A} \otimes \mathbb{C} \rightarrow \mathfrak{A}.$$

Thus, symmetric dagger Frobenius monoids in  $\mathfrak{FdHilb}$  are just a kind of finite-dimensional  $H^*$ -algebra. The adjective “symmetric” here corresponds to the equation

$$\text{Diagram: } \textcircled{1} \otimes \textcircled{2} = \textcircled{2} \otimes \textcircled{1},$$

which is obviously satisfied when things are commutative.

### 3. CHARACTERIZATION

Let  $\mathcal{Z}$  be a two-dimensional dagger field theory. We already know how to obtain a commutative dagger Frobenius algebra  $\mathcal{Z}(\mathcal{Z}) = \mathcal{Z}(\mathbb{S}^1)$  by passing through the usual equivalence. Using the language of  $H^*$ -algebras, we obtain the following.

**Theorem 3.1** ([DJ94]). *Two-dimensional dagger field theories  $\mathcal{Z}$  give rise to and are classified by  $n = \dim(\mathcal{Z}(\mathbb{S}^1))$  many positive real numbers  $\lambda_1, \lambda_2, \dots, \lambda_n$ .*

*Proof.* By Lemma 2.3, the Frobenius algebra  $\mathcal{Z}(\mathbb{S}^1)$  is a finite-dimensional  $H^*$ -algebra. Then, Theorem 2.2 gives us a decomposition

$$\mathcal{Z}(\mathbb{S}^1) \simeq \mathfrak{B}(\mathcal{H}_1, \lambda_1) \oplus \mathfrak{B}(\mathcal{H}_2, \lambda_2) \oplus \cdots \oplus \mathfrak{B}(\mathcal{H}_n, \lambda_n).$$

Since  $\mathcal{Z}(\mathbb{S}^1)$  is commutative with respect to  $m = \mathcal{Z}(\mathbb{D})$ , each of the summands  $\mathfrak{B}(\mathcal{H}_i, \lambda_i)$  must be one-dimensional, so  $n = \dim(\mathcal{Z}(\mathbb{S}^1))$ .  $\square$

Evidently, we have ended up with semisimplicity, since

$$\mathcal{Z}(\mathbb{S}^1) \simeq \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \cdots \oplus \mathbb{C}e_n$$

with pointwise operations, where the  $e_1, e_2, \dots, e_n$  form a basis of orthogonal idempotents. Then, the numbers  $\lambda_1, \lambda_2, \dots, \lambda_n$  are the images of the orthogonal idempotents under the trace map  $\varepsilon = \mathcal{Z}(\mathbb{O})$ .

## REFERENCES

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