MATH 231 EXAM II REVIEW

DHEERAN E. WIGGINS

ABSTRACT. I hope these notes prove to be useful while you study for the second exam of this course. I have structured the notes to mirror the order of topics as we saw them during our discussion sessions.

CONTENTS

1.	Arc Length and Surface Area	2
2.	Moments and Center of Mass	3
3.	Sequence Notation	5
4.	Series Notation	5
5.	(Most) Series Tests	7
6.	Direct Comparison Test	8
7.	Limit Comparison Test	8

Date: Fall 2025.

As a reminder, this is *not* official Math 231 review material. It may contain different problem-solving techniques than were presented in your lecture packets or discussion worksheets. It should serve as a supplement to those, not a replacement. I believe these notes closely reflect how I think about this stuff (and thus, how I presented it to you all). Per usual, I recommend going over *all* of the problems on your discussion sheets and homework. My favorite online reference for practice problems and alternative presentations of this material is Paul's Online Math Notes, Calculus II. Good luck!

1. Arc Length and Surface Area

Goal 1.1 (Arc Length)**.** Given a curve f on an interval $[a, b]$, we want to be able to describe the length of f.

Solution 1.2 (Arc Length)**.** Define what a small part of the curve is ($\mathrm{d}s$), and then add that up along f by integrating.

Setup 1.3 (Arc Length)**.** Let $y = f(x)$ be a function on the interval $[a, b]$.

(i) If we want to find the arc length of f by integrating with respect to x, we need to take the derivative of y with respect to x and plug it into the arc length differential formula.

(ii) If we want to find the arc length of f by integrating with respect to y, we first need to find the inverse of $f(x)$, so that we can write $x = g(y)$ on the interval $[f(a), f(b)]$. To do this, solve $y = f(x)$ for x (e.g., if $y = x^3$, then we can solve for x and get $x = \sqrt[3]{y}$). Then, we can take the derivative of x with respect to y and plug it into the arc length differential formula.

Name	Integrate with respect to	Formula
Arc length differential $\mathrm{d}s$	x	$\mathrm{d}s = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$
Arc length differential $\mathrm{d}s$	y	$\mathrm{d}s = \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y$
Arc length L	x	$L = \int_a^b \mathrm{d}s$
Arc length L	y	$L = \int_{f(a)}^{f(b)} \mathrm{d}s$

Goal 1.4 (Surface Area)**.** Given a curve f on an interval $[a, b]$, we want to compute the surface area of the surface S given by rotating f around some axis.

Solution 1.5 (Surface Area)**.** Use the arc length differential $\mathrm{d}s$, multiplied by the formula for circumference $2\pi r$, and then add it up along f by integrating.

Setup 1.6 (Surface Area)**.** Let $y = f(x)$ be a function on the interval $[a, b]$.

(i) If we want to find the surface area of the surface sketched out by $f(x)$ when it is rotated about the x-axis, then we compute $\mathrm{d}s$ as before and plug it into the surface area formula, multiplying $\mathrm{d}s$ by $2\pi y$ and integrating. If we want to integrate with respect to x, then use the corresponding formula for $\mathrm{d}s$ and use $y = f(x)$. If we want to integrate with respect to y, then use the corresponding formula for $\mathrm{d}s$ and use $y = y$.

(ii) If we want to find the surface area of the surface sketched out by $f(x)$ when it is rotated about the y-axis, then we compute $\mathrm{d}s$ as before and plug it into the surface area formula, multiplying $\mathrm{d}s$ by $2\pi x$ and integrating. If we want to integrate with respect to x, then use the corresponding formula for $\mathrm{d}s$ and use $x = x$. If we want to integrate with respect to y, then use the corresponding formula for $\mathrm{d}s$ and use $x = g(y)$, like we did with arc length.

Name	Rotate about the	Integrate with respect to	Formula
Surface area A	x-axis	x	$A = \int_a^b 2\pi f(x)\,\mathrm{d}s$
Surface area A	x-axis	y	$A = \int_{f(a)}^{f(b)} 2\pi y\,\mathrm{d}s$
Surface area A	y-axis	x	$A = \int_a^b 2\pi x\,\mathrm{d}s$
Surface area A	y-axis	y	$A = \int_{f(a)}^{f(b)} 2\pi g(y)\,\mathrm{d}s$

2. Moments and Center of Mass

There are generally three types of problems here. Since the point of this section is to test your ability to apply integration, *focus on the 2D continuous* case!

Goal 2.1 (1D Discrete)**.** Given some points on a line that have some masses, we want to compute their center of mass.

Solution 2.2 (1D Discrete)**.** Divide the formulas for the moment and mass to get the center of mass.

Setup 2.3 (1D Discrete)**.** Let $x_1, x_2, \ldots, x_n$ be n points on a line and $m_1, m_2, \ldots, m_n$ be their respective masses. We can add up all the masses to get the total mass of the system. Likewise, we can take the weighted sum of the points and the masses to find the moment. Dividing the moment by the mass gives the center of mass.

Goal 2.4 (2D Discrete)**.** Given some points in the plane that have some masses, we want to compute their center of mass.

Solution 2.5 (2D Discrete)**.** Divide the formulas for the moments and mass to get the center of mass coordinates.

Setup 2.6 (2D Discrete)**.** Let $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be n points in the plane with masses $m_1, m_2, \ldots, m_n$, respectively. We can add up all the masses to get the total mass of the system. We can take the weighted sum of the x_i to get the moment about y and the weighted sum of the y_i to get the moment about x.

Goal 2.7 (2D Continuous)**.** Given some positive function $f(x)$ on an interval $[a, b]$ we want to find the center of mass of the uniform-density "lamina" (region under the curve) sketched out between the x-axis and $f(x)$ from a to b.

Solution 2.8 (2D Continuous)**.** Find the mass of the lamina. Then, use the moment formulas and divide them by the mass of the lamina to get the centroid/center of mass.

Setup 2.9 (2D Continuous)**.** Let $f(x)$ be positive on the interval $[a, b]$. Compute the mass by multiplying ρA, where ρ is the density given in the problem and A is the area under the curve (the integral of $f(x)$ from a to b). Using the formulas for the moments about x and y, divide by the mass to get the coordinates of the centroid/center of mass.

Remark 2.10 (Intuition for Moments). You should think of the moment M_x about x as the "tendency of the system to rotate over the x-axis" and the moment M_y about y as the "tendency of the system to rotate over the y-axis." Thus, if the system is symmetric about the x-axis, then it won't rotate over the x-axis, so M_x should be 0. Likewise, if it is symmetric about the y-axis, then it won't rotate over the y-axis, and so M_y should be 0.

Name	Case	Formula
Mass m	1D Discrete	$m = m_1 + m_2 + \cdots + m_n$
Moment M	1D Discrete	$M = m_1 x_1 + m_2 x_2 + \cdots + m_n x_n$
Center of Mass $\overline{x}$	1D Discrete	$\overline{x} = \dfrac{M}{m}$
Mass m	2D Discrete	$m = m_1 + m_2 + \cdots + m_n$
Moment M_x about x	2D Discrete	$M_x = m_1 y_1 + m_2 y_2 + \cdots + m_n y_n$
Moment M_y about y	2D Discrete	$M_y = m_1 x_1 + m_2 x_2 + \cdots + m_n x_n$
Center of Mass $(\overline{x}, \overline{y})$	2D Discrete	$(\overline{x}, \overline{y}) = \left(\dfrac{M_y}{m}, \dfrac{M_x}{m} \right)$
Mass m	2D Continuous	$m = \rho A = \rho \displaystyle\int_a^b f(x)\,\mathrm{d}x$
Moment M_x about x	2D Continuous	$M_x = \rho \displaystyle\int_a^b \frac{1}{2} f(x)^2\,\mathrm{d}x$
Moment M_y about y	2D Continuous	$M_y = \rho \displaystyle\int_a^b x f(x)\,\mathrm{d}x$
Centroid/Center of Mass $(\overline{x}, \overline{y})$	2D Continuous	$(\overline{x}, \overline{y}) = \left(\dfrac{M_y}{m}, \dfrac{M_x}{m} \right)$

Remark 2.11. In your packets, there is a formula for the center of mass in the 2D continuous case:

$$\overline{x} = \frac{1}{A} \int_a^b x f(x)\,\mathrm{d}x \quad \text{and} \quad \overline{y} = \frac{1}{A} \int_a^b \frac{1}{2} f(x)^2\,\mathrm{d}x\,.$$

This is the same thing as what is in the table, which you can see by simplifying M_y/m and M_x/m. The point is that while we care about the density ρ when computing the moments about x and y, when you compute the centroid/center of mass, it ends up canceling out since ρ is in the mass formula.

3. Sequence Notation

Definition 3.1 (Sequence)**.** A sequence a_n is an infinite list of numbers

$$a_1, a_2, a_3, a_4, \ldots$$

Definition 3.2 (Convergent Sequence)**.** A sequence a_n converges to a number L (or is convergent) if

$$\lim_{n\to\infty} a_n = L.$$

Intuitively, this means that eventually the list becomes/gets closer and closer to a certain number L.

Definition 3.3 (Divergent Sequence)**.** A sequence a_n diverges (or is divergent) if it is not convergent. That is, if

$$\lim_{n\to\infty} a_n$$

does not exist, is $-\infty$, or is ∞, then the sequence a_n diverges.

When given a sequence, compute its limit by doing some algebra inside the limit and getting it into a form you can evaluate.

Remark 3.4 (Sequence in Function). Keep in mind that if $f(x)$ is continuous, then $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$. For example,

$$\lim_{n\to\infty} e^{\frac{1}{n}} = e^{\lim\limits_{n\to\infty}\frac{1}{n}} = e^0 = 1, \quad \text{since } \lim_{n\to\infty} \frac{1}{n} = 0.$$

4. Series Notation

Goal 4.1 (Series)**.** Given a sequence a_n, we want to define what it means to add up all the numbers in the sequence. We want something like "$a_1 + a_2 + a_3 + \cdots$."

Solution 4.2 (Series)**.** We can define certain finite sums, called the partial sums, and then take their limit. This is what we call a series.

Definition 4.3 (Partial Sum)**.** Let a_n be a sequence. The Nth partial sum S_N is the sum

$$S_N = a_1 + a_2 + \cdots + a_N.$$

Alternatively, we could write this in Sigma-notation as

$$S_N = \sum_{n=1}^{N} a_n.$$

Remark 4.4 (Sequence From Partial Sums). Say you're given a formula for the Nth partial sum S_N of some sequence. How can you figure out the sequence a_n? Well,

$$S_N = a_1 + a_2 + \cdots + a_N,$$

so subtracting a bunch of terms from the right-hand-side, we get

$$a_N = S_N - (a_1 + a_2 + \cdots + a_{N-1}).$$

But, $(a_1 + a_2 + \cdots + a_{N-1})$ is the $(N-1)$th partial sum S_{N-1}, per our definition above. Thus,

$$a_n = S_n - S_{n-1}$$

for all n.

Definition 4.5 (Series)**.** Let a_n be a sequence. Let S_N be the Nth partial sum of the a_n. Then, the series of the a_n is defined to be the limit

$$\lim_{N\to\infty} S_N.$$

We write this limit as an "infinite sum"

$$\sum_{n=1}^{\infty} a_n,$$

since $S_N = \sum_{n=1}^{N} a_n$.

Definition 4.6 (Convergent Series)**.** We say that the series $\sum_{n=1}^{\infty} a_n$ is convergent if the limit

$$\lim_{N\to\infty} S_N$$

is convergent, where $S_N = \sum_{n=1}^{N} a_n$ is the Nth partial sum.

Definition 4.7 (Divergent Series)**.** We say that the series $\sum_{n=1}^{\infty} a_n$ is divergent if it is not convergent. That is, if the limit

$$\lim_{N\to\infty} S_N$$

does not exist, is $-\infty$, or is ∞, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Definition 4.8 (Absolutely Convergent Series)**.** We say that the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent. That is, a series is absolutely convergent if it is convergent after "forgetting about any negatives inside."

Definition 4.9 (Conditionally Convergent Series)**.** If a series $\sum_{n=1}^{\infty} a_n$ is convergent, but $\sum_{n=1}^{\infty} |a_n|$ is divergent (that is, it is *not* absolutely convergent), then we say that $\sum_{n=1}^{\infty} a_n$ is conditionally convergent. The idea is that convergence of $\sum_{n=1}^{\infty} a_n$ is *conditional* on some of the negative signs inside the series.

Remark 4.10 (Indexing)*.* Sometimes our series start from a number k that is different from 1. That is, we can be given a series

$$\sum_{n=k}^{\infty} a_n,$$

where k is some number greater than or equal to 0. This series is still just a limit of the partial sums, except now we do not have the first $k-1$ terms $a_1 + a_2 + \cdots + a_{k-1}$ in the series (if $k \geq 1$) or we have an extra a_0 term in the series (if $k = 0$).

Reindexing $n = k$ to start from 0 or 1 can occasionally be helpful. Here is how we do this. Say we want to start from 0. Then, let $m = n - k$ be our new indexing variable (the name does not matter). Since n starts at k, our new variable m starts at $k - k = 0$, which is what we wanted. This tells us that $n = m + k$, so substituting this in, we get

$$\sum_{n=k}^{\infty} a_n = \sum_{m=0}^{\infty} a_{m+k}.$$

Reindexing to start at 1 is basically the same:

$$\sum_{n=k}^{\infty} a_n = \sum_{m=1}^{\infty} a_{m+k-1}$$

Do not memorize these–just remember that if you want to move the starting point $n = k$ in the Sigma-notation *down*, all indices in the sum should move *up* by the same amount, and vice-versa.

5. (Most) Series Tests

Test	Series looks like	If	then the series
Divergence	anything $\sum_{n=k}^{\infty} a_n$	$\lim_{n\to\infty} a_n \neq 0$,	diverges
Geometric	$\sum_{n=0}^{\infty} ar^n$ or $\sum_{n=1}^{\infty} ar^{n-1}$	$\|r\| < 1$, $\|r\| \geq 1$,	converges to $\frac{a}{1-r}$ diverges
Telescoping	$\sum_{n=1}^{\infty}(b_n - b_{n+1})$ (e.g., after partial fractions)	$\lim_{n\to\infty} b_{n+1} = B$ finite,	converges to $b_1 - B$
Integral	$\sum_{n=k}^{\infty} f(n)$, where $f(x)$ is continuous, $f(x)$ is positive, and $f(x)$ is decreasing ($f'(x)$ is negative)	$\int_k^{\infty} f(x)\,\mathrm{d}x$ converges, $\int_k^{\infty} f(x)\,\mathrm{d}x$ diverges,	converges diverges
Sum of Two	$\sum_{n=k}^{\infty}(b_n + c_n)$	$\sum_{n=k}^{\infty} b_n$ & $\sum_{n=k}^{\infty} c_n$ converge,	converges
p-Series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$, $p \leq 1$,	converges diverges
Alternating	$\sum_{n=k}^{\infty}(-1)^n b_n$ or $\sum_{n=k}^{\infty}(-1)^{n+1} b_n$	$b_n \geq 0$, $\lim_{n\to\infty} b_n = 0$, and $b_{n+1} \leq b_n$,	converges
Absolute	anything $\sum_{n=k}^{\infty} a_n$	it absolutely converges,	converges
Ratio	anything $\sum_{n=k}^{\infty} a_n$	$\underbrace{\lim_{n\to\infty} \left\| \frac{a_{n+1}}{a_n} \right\|}_{L}$ is $L < 1$, $L > 1$, $L = 1$,	absolutely converges diverges [no information]
Root	anything $\sum_{n=k}^{\infty} a_n$	$\underbrace{\lim_{n\to\infty} \|a_n\|^{\frac{1}{n}}}_{L}$ is $L < 1$, $L > 1$, $L = 1$,	absolutely converges diverges [no information]

6. Direct Comparison Test

Let $\sum_{n=k}^{\infty} a_n$ be a series with $a_n > 0$.

(i) *Guess* based on intuition whether it converges or diverges by simplifying a_n to look like one of our standard series, like a geometric series or a p-series.

(ii) *If we guessed convergent*, then find a sequence $b_n > 0$ such that $a_n \leq b_n$ and

$$\sum_{n=k}^{\infty} b_n \quad \text{converges.}$$

This tells us that our guess was correct, and $\sum_{n=k}^{\infty} a_n$ converges too.

(iii) *If we guessed divergent*, then find a sequence $b_n > 0$ such that $b_n \leq a_n$ and

$$\sum_{n=k}^{\infty} b_n \quad \text{diverges.}$$

This tells us that our guess was correct, and $\sum_{n=k}^{\infty} a_n$ diverges too.

(iv) *If we cannot find a b_n* to match our guess, it may be that we guessed incorrectly. In this case, just guess the opposite and try to find a b_n for the new guess.

Remark 6.1. Use an easy b_n, like a p-series or a geometric series, that looks like a simpler version of a_n, to do the direct comparison test.

7. Limit Comparison Test

Let $\sum_{n=k}^{\infty} a_n$ be a series with $a_n > 0$.

(i) *Guess* based on intuition whether it converges or diverges by simplifying a_n to look like one of our standard series like a geometric series or a p-series.

(ii) *If we guessed convergent*, then find a sequence $b_n > 0$ such that

$$\sum_{n=k}^{\infty} b_n \quad \text{converges.}$$

If the limit

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c, \quad \text{where } c \neq 0 \text{ is a finite number,}$$

then our guess was correct, and $\sum_{n=k}^{\infty} a_n$ converges too.

(iii) *If we guessed divergent*, then find a sequence $b_n > 0$ such that

$$\sum_{n=k}^{\infty} b_n \quad \text{diverges.}$$

If the limit

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c, \quad \text{where } c \neq 0 \text{ is a finite number,}$$

then our guess was correct, and $\sum_{n=k}^{\infty} a_n$ diverges too.

Remark 7.1. To find b_n, simplify a_n by removing constants and irrelevant coefficients. This simplified sequence b_n helps you make your guess *and* it can function as the b_n to do the limit comparison test.

University of Illinois Urbana-Champaign, Illinois, 61801
Email address: `dheeran2@illinois.edu`