ABSTRACT ALGEBRA II

A COLLECTION OF NOTES ON MAJOR DEFINITIONS AND RESULTS, PROOFS, AND COMMENTARY BASED ON THE CORRESPONDING COURSE AT ILLINOIS, AS INSTRUCTED BY MINEYEV

LECTURE NOTES BY

DHEERAN E. WIGGINS

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ILLINOIS URBANA-CHAMPAIGN

Disclaimer
The lecture notes in this document were based on Abstract Algebra II [501], as instructed by Igor Mineyev [Department of Mathematics] in the Spring semester of 2025 [SP25] at the University of Illinois Urbana-Champaign. All non-textbook approaches, exercises, and comments are adapted from Mineyev's lectures.

Textbook
Many of the exercises and presentations were selected from *Advanced Modern Algebra* and *An Introduction to Homological Algebra*, by Joseph J. Rotman.

Author Information
Dheeran E. Wiggins is, at the time of writing [Spring, 2025], a second-year student at Illinois studying mathematics. All typesetting and verbiage are his own.

`dheeran2@illinois.edu`

Math is *one*.

– Igor Mineyev

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Looking at history, math originated, in some sense, via the so-called natural numbers $\mathbb{N} := \{1, 2, 3, 4, \dots\}$. Similarly, we can form the integers $\mathbb{Z}$, the rationals $\mathbb{Q}$, the reals $\mathbb{R}$, and the complex numbers $\mathbb{C}$. It is worthwhile, then, to consider counting multiple *copies* of the same object. Thus, we have the vague sense of mathematical *counting*.

Then, algebraic concepts originate from the visual pieces of mathematics *geometry* and *topology*.[1] Starting from geometric objects, we may form algebraic objects like *groups* or *modules*, which in turn, lead to (co)homology. Homological algebra is a branch of algebra, studying how we can compute (co)homology.

We can divide mathematics further, gleaning *algebraic geometry*: the study of solutions to polynomial equations. Let p be a polynomial in $R[x]$. Then, it leads to the set $V := \{x : p(x) = 0\}$. If $R = \mathbb{C}$, then $V \subseteq \mathbb{C}$. This set can visualized in the complex plane, and thus, is "geometric." More generally, we could take a polynomial of several variables $p \in R[x_1, \dots, x_n]$. Then, the corresponding zero-set

$$V = \{(x_1, \dots, x_n) : p(x_1, \dots, x_n) = 0\} \subseteq \mathbb{C}^n$$

is realizable geometrically, too.

Another instance of geometry inspiring algebra is in *geometric group theory*. Starting from a group, in the traditional sense, we can develop pictures (like a Cayley graph), which are geometric,[2] to study the group.

Category theory goes one step further, generalizing our notions of algebra, geometry, topology, and more. The language of categories and functoriality can be immensely worthwhile while translating between different classes of objects. Still, for understanding modules or groups, precisely, the general language of categories can be less than useful.

1.1 Review of Basic Structures

A *group* is a pair $(G, \cdot)$, where G is a set and $\cdot$ is an operation satisfying associativity, identity, and inverse. In turn, a *ring* is a triple $(R, +, \cdot)$, where R is a set, $+$ makes R an abelian group, and $\cdot$ is associative with unity. Finally, a *field* is a triple $(\mathbb{F}, +, \cdot)$ which is a ring such that $\cdot$ is commutative and has inverses for nonzero elements of $\mathbb{F}$.

Definition 1.1.1 (Vector Space) *A vector space $\mathcal{V}$ over a field $\mathbb{F}$ consists of two operations:*

(i) $+ : \mathcal{V}^2 \to \mathcal{V}$
(ii) $\cdot : \mathbb{F} \times \mathcal{V} \to \mathcal{V}$,

where $+$ makes $\mathcal{V}$ an abelian group and $\cdot$ is an action by the field of scalars, in the traditional sense.[3]

[1] If counting consists of numbers, geometry and topology deal with objects which are "visualizable."

[2] In some sense, geometry is perhaps more fundamental, as our intuitions for visualization can rederive any ideas in algebra.

[3] That is, $1_{\mathbb{F}} v = v$ for all $v \in \mathcal{V}$. Also, $c(u + v) = cu + cv$. Finally, $(c_1 + c_2)v = c_1 v + c_2 v$ and $c_1(c_2 v) = (c_1 c_2)v$.

Definition 1.1.2 (Left R-Module) *A (left) R-module M is a triple $(M, R, +, \cdot)$, defined the same way as a vector space, except R is an arbitrary ring.*[4]

Certainly, every ring can be thought of as a module over itself.

Definition 1.1.3 (Finite Direct Sum) *Let A, B be R-modules. The direct sum $A \oplus B$ of A and B is the module*

$$A \oplus B := \{(a, b) : a \in A, b \in B\} = A \times B,$$

where the operations are defined componentwise.[5]

Definition 1.1.4 (Module Homomorphism) *A homomorphism $f : A \to A'$ between R-modules is a group homomorphism with respect to $+$ and $f(ca) = cf(a)$ for all $c \in R$.*

That is, f must satisfy the commutative square

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\cdot r"'] & A' \arrow[d, "\cdot r"] \\
A \arrow[r, "f"'] & A'
\end{tikzcd}$$

Define the set of all R-module homomorphisms as the set

$$\mathrm{Hom}_R(A, B) := \left\{ \begin{matrix} \text{all } R\text{-module} \\ \text{homomorphisms } A \to B \end{matrix} \right\}.$$

What structure could we put on this set? Well, we let $f, g : A \rightrightarrows B$. Then, define $f + g : A \to B$ by $(f + g)(a) := f(a) + g(a)$. Furthermore, we cold define a scalar multiplication on $\mathrm{Hom}_R(A, B)$ by $r \cdot f : A \to B$ by $(r \cdot f)(a) = rf(a)$. We certainly have that this set forms an abelian group under the addition. However, the scalar multiplication admits

$$(rr'f)(a) = (r'r)f(a)$$

which violates our (left) R-module axioms, as long as R is *not* commutative.[6] Note that we have the commutative diagram

$$\begin{tikzcd}
 & A \arrow[dl, hook', "\alpha"'] \arrow[dr, "f"] & \\
A \oplus B \arrow[rr, dashed, "\theta", "\exists!"'] & & N \\
 & B \arrow[ul, hook, "\beta"] \arrow[ur, "g"'] &
\end{tikzcd}$$

The existence of θ comes from defining θ by $\theta(a, b) := f(a) + g(b)$. Uniqueness comes via the standard method for such a proof, using commutativity and the formula for θ, θ is the only choice.

[4] If the ring action comes from the right side, we call it a right R-module.

[5] In particular, $r(a, b) = (ra, rb)$, which we call a *diagonal* action.

[6] That is, the collection of pairwise R-module homomorphisms is an R-module if R is commutative.

This says that the module $A \oplus B$ together α and β satisfies the so-called universal property. In this case, we may say that the *coproduct* of modules A and B is the module $A \coprod B := A \oplus B$ with the inclusions α, β.

Using the standard trick of duality, reverse the arrows of the coproduct diagram to glean the *product* diagram:

$$\begin{tikzcd}
& A & \\
A \times B \arrow[ur, two heads, "p"] \arrow[dr, two heads, "q"'] & & N \arrow[ul, "f"'] \arrow[ll, dashed, "\theta", "\exists!"'] \arrow[dl, "g"] \\
& B &
\end{tikzcd}$$

We call $A \times B$ the product of the modules. This product exists in our category LMod_R of R-modules[7] Namely, the module has the underlying set $A \times B$. The function θ is given by

$$\theta(n) := (f(n), g(n)).$$

Remark 1.1.1 In this finite case, the coproduct and product module coincide.

It is worthwhile to note that the coproduct allows us to construct maps *out of* our module $A \oplus B$, whereas the product allows us to construct maps *into* our module $A \times B$.

We now wish to construct infinite versions of the coproduct and product of R-modules. Let $\{A_i : i \in I\}$ be a family of R-modules. Then, the coproduct $\coprod_I A_i$ of the A_i is given by the following universal property diagram:

$$\begin{tikzcd}
& A_i \arrow[dl, hook', "\alpha"'] \arrow[dr, "f"] & \\
\coprod\limits_{i \in I} A_i \arrow[rr, dashed, "\theta", "\exists!"'] & & N
\end{tikzcd}$$

which must commute for all $i \in I$.

Does it exist? Thankfully, the answer is yes. The (general) *direct sum* of the A_is satisfies this universal property.

Definition 1.1.5 (Direct Sum) *We define the direct sum of the A_i to be*

$$\bigoplus_{i \in I} A_i := \left\{ a : I \to \bigcup_{i \in I} A_i : \begin{array}{c} a(i) \in A_i \text{ and} \\ |\{i \in I : a(i) \neq 0\}| < \infty \end{array} \right\}.$$

Define addition pointwise, and likewise for scalar multiplication.

Check that $\bigoplus_I A_i$ satisfies the universal property of the coproduct $\coprod_I A_i$.

The product of a family $\{A_i : i \in I\}$ satisfies the universal property given by the following commutative diagram:
which must commute for all $i \in I$.

[7] We use the language, though we have not formally defined a category yet.

$$\begin{tikzcd}
& A_i & \\
\prod_{i\in I} A_i \arrow[ur, two heads, "p"] & & N \arrow[ul, "f"'] \arrow[ll, dashed, "\theta"', "\exists!"]
\end{tikzcd}$$

As you may expect, the "categorical" product exists in LMod_R in the form of the *Cartesian* product of the R-modules.

Definition 1.1.6 (Cartesian Product) *The Cartesian product of the A_i is the module*

$$\prod_{i\in I} A_i = \left\{ a : I \to \bigcup_{i\in I} A_I : a(i) \in A_i \right\},$$

where operations are given in the pointwise way you would expect.

Again, the projections $p_i(a)$ are just given by $a(i)$. For checking the universal property, we may define θ by

$$\theta(n)(i) := f_i(n).$$

1.2 Categorical Constructions

We now develop a convenient language for discussing structures.

Definition 1.2.1 (Category) *A category C consists of the following data:*

(i) a collection $\operatorname{ob}\mathsf{C}$ *of objects.*[8]
(ii) for any two objects $X, Y \in \operatorname{ob}\mathsf{C}$, a set $\operatorname{Hom}_{\mathsf{C}}(X, Y)$ of morphisms, written $\varphi : X \to Y$ or $X \overset{\varphi}{\to} Y$.
(iii) for any objects $X, Y, Z \in \operatorname{ob}\mathsf{C}$, a composition operation

$$\circ : \operatorname{Hom}_{\mathsf{C}}(Y, Z) \times \operatorname{Hom}_{\mathsf{C}}(X, Y) \to \operatorname{Hom}_{\mathsf{C}}(X, Z)$$

which is associative.
(iv) for any object $X \in \operatorname{ob}\mathsf{C}$, an identity $1_X \in \operatorname{Hom}_{\mathsf{C}}(X, X)$ which satisfies $1_X\varphi = \varphi = \psi 1_X$ for all $\varphi \in \operatorname{Hom}_{\mathsf{C}}(Y, X)$ and $\psi \in \operatorname{Hom}_{\mathsf{C}}(X, Y)$.

Example 1.2.1 Some standard examples of categories include

(i) Set of sets and functions.
(ii) Grp of groups and homomorphisms.
(iii) Ab of abelian groups and homomorphisms.
(iv) Ring of rings and homomorphisms.
(v) Set_G of G-sets and action-preserving functions.
(vi) LMod_R of R-modules and homomorphisms.
(vii) Top of (topological) spaces and continuous maps.
(viii) hTop of spaces and homotopy classes of continuous maps.
(ix) Δ of totally ordered sets and order-preserving functions.
(x) sSet of simplicial sets (Set-valued presheaves on Δ) and natural transformations.

[8] Oftentimes this collection is *not* a set. We will not pay too much attention to the foundations here, but NBG would be a reasonable system to make of use proper classes, here.

Alternative notations for the direct sum is to write $\bigoplus_I A_i$ as the set of sums $\sum_{i\in I} a_i$ with compact support. We could also write $a^1 + a^2 + \cdots + a^k$, where we have some $a^j \in A_{i_{(j)}}$.

Definition 1.2.2 (Free R-Module) *A free R-module M is an R-module isomorphic to a direct sum of the form*

$$M \simeq \bigoplus_{i\in I} R.$$

A basis for M over R, in the standard sense, is just a subset of M such that any element of M can be expressed *uniquely* as an R-linear combination of b_i with coefficients in R.

Proposition 1.2.1 *Every free R-module admits a standard basis.*[9]

Corollary 1.2.2 *Every vector space admits a basis.*

Example 1.2.2 Consider $\mathbb{Z}/3 := \mathbb{Z}/3\mathbb{Z}$ as a $\mathbb{Z}$-module. Then, the singleton $\{1\} \subseteq \mathbb{Z}/3$ can be used to express $2 = 5\cdot 1 = 8 \cdot 1 = 2$, so $\{1\}$ is not a basis. Thus, $\mathbb{Z}/3$ is not free.

Exercise 1.2.1 An R-module is free if and only if it has a basis.

Let I be any set and R a ring. Then, define

$$R[I] := \bigoplus_{i\in I} R.$$

Every element in $R[I]$ is a finite linear combination

$$a^1b^1 + a^2b^2 + \cdots + a^kb^k = \sum_{i\in I} a_ib_i.$$

Theorem 1.2.3 *Any R-module is isomorphic to a quotient of a free module.*

Proof. Let M be an R-module. Let $S \subseteq M$ be any subset that generates M.[10] We have a set inclusion $S \hookrightarrow M$. We claim that this extends *uniquely* to a surjective R-homomorphism $R[S] \twoheadrightarrow M$:

$$\begin{tikzcd}
\bigoplus_{s\in S} R \simeq R[S] \arrow[dr, dashed, "g", "\exists!"'] & \\
S \arrow[u, hook, "s\mapsto b_s"] \arrow[r, hook, "f"'] & M
\end{tikzcd}$$

Define the homomorphism $g : R[S] \twoheadrightarrow M$ explicitly by[11]

[9] We mean $\{b_i\}$, where $b_i = 1_R$ in the ith copy of R.

[10] This is generation in the traditional sense of a substructure.

[11] That is, define it by extending it R-linearly.

$$g\left(\sum_{s\in S} a_s \cdot s\right) := \sum_{s\in S} a_s f(s).$$

The proof of uniqueness follows in the standard way. By the first isomorphism theorem we get an isomorphism $R[S]/\ker g \xrightarrow{\sim} M$. ☐

Note that we could have taken f to be any function, rather than the inclusion. This gives us the universal property of free modules:[12]

$$\begin{tikzcd}
F \arrow[dr, dashed, "g", "\exists!"'] & \\
B \arrow[u, hook, "\iota"] \arrow[r, "f"'] & M
\end{tikzcd}$$

Figure 1.1: The triple $B \overset{\iota}{\to} F$ satisfies the universal property if for any R-module M and function $f : B \to M$, there is a unique g making the diagram commute.

Metamathematically, we have made the transition

$$\begin{tikzcd}
\text{set theory} \arrow[r, squiggly] & \text{categories}
\end{tikzcd}$$

elements and sets   objects and morphisms

Just to emphasize that not every category is a collection of enriched sets and functions, consider the category [1] given by the picture

$$\begin{tikzcd}
\bullet_1 \arrow[loop above, "1_0"] \arrow[r, "f"] & \bullet_1 \arrow[loop above, "1_0"]
\end{tikzcd}$$

In any category formed of enriched sets, we can form the *free object* F which comes with a set function $\iota : B \rightarrowtail F$ so that the following diagram commutes for any function $f \in \mathrm{Hom}_{\mathsf{Set}}(B, M)$.

$$\begin{tikzcd}
F \arrow[dr, dashed, "g", "\exists!"'] & \\
B \arrow[u, hook, "\iota"] \arrow[r, "f"'] & M
\end{tikzcd}$$

Proposition 1.2.4 *The free object in* Grp *of a set* B *is the free group* $F(B)$.

Proof. Use the standard construction of the free group as reduced words. ☐

[12] Here, we call F the free module and B the basis.

Proposition 1.2.5 *If $\iota : B \to F$ is not assumed to be injective, the universal property implies it.*

Definition 1.2.3 (Isomorphism) *Let C be a category. A morphism $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ is an isomorphism if there exists $g \in \mathrm{Hom}_{\mathsf{C}}(B, A)$ so that $fg = 1_B$ and $gf = 1_A$.*[13]

Definition 1.2.4 (Initial) *An initial object in a category C is an object $I \in \mathrm{ob}\,\mathsf{C}$ such that for any $A \in \mathrm{ob}\,\mathsf{C}$, there exists a unique morphism $f : I \to A$.*[14]

Just looking at the definition, we have no guarantee that such an object exists. In fact, they often will not.

Lemma 1.2.6 *If an initial object exists, it is unique.*

Proof. Suppose I and I' are initial in C. Then, there exists a unique morphism $f : I \to I'$. Similarly, there is a unique morphism $g : I' \to I$. Then, $gf : I \to I$ must be 1_I. Likewise, $fg = 1_{I'}$. Thus, $I \simeq I'$. ☐

Definition 1.2.5 (Terminal) *A terminal object in a category C is an object $T \in \mathrm{ob}\,\mathsf{C}$ such that for any $A \in \mathrm{ob}\,\mathsf{C}$, there exists a unique morphism $f : A \to T$.*[15]

Lemma 1.2.7 *Terminal objects are up to isomorphism.*

Proof. Proceed via duality. ☐

Fix $A, B \in \mathrm{ob}\,\mathsf{C}$. Define a new category $\mathsf{A}(A, B)$, where objects are diagrams $A \to M \leftarrow B$ and morphisms are commutative squares

$$\begin{tikzcd}
& A \arrow[dl] \arrow[dr] & \\
M \arrow[rr, "\varphi"] & & M' \\
& B \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Then, the existence of the coproduct $A \coprod B$ exists in C if and only if $\mathsf{A}(A, B)$ admits an initial object.[16]

This is the general format for any universal property.

Consider a sequence of inclusions in, say, Set or LMod_R:

$$A_1 \overset{\varphi_2^1}{\hookrightarrow} A_2 \overset{\varphi_3^2}{\hookrightarrow} A_3 \hookrightarrow \cdots \hookrightarrow A_n \hookrightarrow \cdots$$

[13] Write $A \simeq B$.

[14] That is,

$$\mathrm{Hom}_{\mathsf{C}}(I, A) = \{f\}.$$

[15] Again, this means

$$\mathrm{Hom}_{\mathsf{C}}(A, T) = \{f\}.$$

[16] We cuold do this for a family of objects.

We could take the coproduct of all A_n and then quotient via the canonical injection identification. This would give us an analogy for the union of a finite chain of this sort. The construction of this limiting module is a special case of the *direct limit*.

What should the *inverse limit* be? Well, suppose we have a chain

$$A_1 \xleftarrow{\psi_1^2} A_2 \xleftarrow{\psi_2^3} A_3 \leftarrow \cdots \leftarrow A_n \leftarrow \cdots$$

In some sense, we should consider all possible "compatible" sequences, with respect to the ψ_i^j morphisms.

Our goal will be to define the direct and inverse limit in the case of any indexing set.

Definition 1.2.6 (Direct System) *A direct system in* C *consists of a poset* $(I, \leq)$*, a family* $\{A_i : i \in I\}$ *of* C*-objects, and morphisms* $\varphi_j^i \in \mathrm{Hom}_{\mathsf{C}}(A_i, A_j)$ *for* $i \leq j$ *in* I.

Definition 1.2.7 (Direct Limit) *The direct limit of a direct system is defined by the following universal property.*

$$\begin{tikzcd}
& M_i \arrow[d, "\varphi_j^i"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
& M_j \arrow[dl] \arrow[dr] & \\
\varinjlim M_i \arrow[rr, dashed, "\exists!", "\theta"'] & & X
\end{tikzcd}$$

Figure 1.2: Universal property of the inverse limit

The direct limit is the same thing as a *colimit* of a diagram/indexing functor $F : \mathsf{C}I \to \mathsf{D}$, where I is our indexing set of the direct system.

Remark 1.2.1 Let I be a set with the discrete order relation: that is, $i \leq i$ for all $i \in I$ and $i \not\leq j$ for any distinct $i \neq j \in I$. Then, the colimit of the direct system is precisely the coproduct.

Definition 1.2.8 (Inverse Limit) *The inverse limit of a direct system is defined by the given universal property, flipping all arrows in the definition of the colimit.*

Again, the inverse limit is the same thing as a *limit* of the diagram $F : \mathsf{C}I \to \mathsf{D}$, where I is our indexing set.

Remark 1.2.2 Mirroring the dual case before, the discrete case yields the universal property of the product.

$$\begin{tikzcd}
 & M_i & \\
 & M_j \arrow[u, "\psi_i^j"'] & \\
\varprojlim M_i \arrow[ur] \arrow[uur, bend left] & & X \arrow[ul] \arrow[uul, bend right] \arrow[ll, dashed, "\exists!", "\theta"']
\end{tikzcd}$$

Figure 1.3: Universal property of the direct limit

Does this limit exist in LMod_R?

Theorem 1.2.8 *The limit exists for any inverse system of R-modules*

$$M_1 \xleftarrow{\psi_1^2} M_2 \xleftarrow{\psi_2^3} A_3 \leftarrow \cdots \leftarrow M_n \leftarrow \cdots .$$

Proof. Pick a sequence $(m_i)_{i\in I}$. This is just a map

$$m : I \to \bigcup_{j\in I} A_j,$$

which is an element of the Cartesian product. Define[17]

$$L := \{(m_i)_{i\in I} : \psi_i^j(m_j) = m_i \text{ for all } i \leq j\} \subseteq \prod_{i\in I} M_i.$$

Define $\alpha_i : L \to M_i$ for each $i \in I$ by $\alpha_i((m_k)_{k\in I}) := m_i$.[18] We wish to check that for all $i, j \in I$, for all $x \in L$, $(\psi_i^j \circ \alpha_j)(x) = \alpha_i(x)$. Write $x = (m_k)_{k\in I}$:

$$\left(\psi_i^j \circ \alpha_j\right)(m_k) = \psi_i^j(m_j) = m_i = \alpha_i(m_k).$$

as $x \in L$. Suppose $f_i : X \to M_i$ and f_j are chosen so that for all $i \leq j$, $\psi_i^j \circ f_j = f_i$. Define $\theta : X \to L$ by

$$\theta(x) := (f_k(x))_{k\in I}.$$

We wish to check $\alpha_i \circ \theta = f_i$:

$$(\alpha_i \circ \theta)(x) = \alpha_i((f_k(x))_{k\in I}) = f_i(x),$$

for all $x \in X$. Suppose $\theta' : X \to L$ is another morphism making satisfying the same universal property. That is, $\alpha_i \circ \theta' = f_i$. Look at the pieces $\theta(x) \in L$, which is of the form $(m_k)_{k\in I}$, and $f_i(x) = m_i$. Similarly, $\theta'(x) = (m_k')_{k\in I}$, and $f_i(x) = m_i' = m_i$. Thus, θ and θ' agree in terms of assignment for all $i \in I$, so $\theta = \theta'$. Thus, $\varprojlim M_i = L$ is the desired inverse limit. □

[17] This is our candidate for

$$\varprojlim M_i.$$

[18] This is essentially the projection restricted to L, keeping compatability.

Theorem 1.2.9 *The colimit exists for any direct system of R-modules*

$$A_1 \xrightarrow{\varphi_2^1} A_2 \xrightarrow{\varphi_3^2} A_3 \to \cdots \to A_n \to \cdots .$$

Left as an Exercise. The construction follows by taking the coproduct of the modules, then taking the quotient by where the modules agree. ☐

1.3 Functors and Exactness

Recall that the *kernel* of a homomorphism $\varphi : A \to B$ is the fiber over zero, written $\ker\varphi$. We now define the dual notion.[19]

Definition 1.3.1 (Cokernel) *Given a homomorphism $\varphi : A \to B$, the cokernel is given by*

$$\operatorname{coker}\varphi := B/\varphi(A).$$

Recall that $\operatorname{Hom}_R(A, B) = \operatorname{Hom}_{\mathsf{LMod}_R}(A, B)$ admits that structure of a $\mathbb{Z}$-module, or abelian group. We could view this process as "functoral:"

$$B \mapsto \operatorname{Hom}_R(A, B) \quad \text{or} \quad A \mapsto \operatorname{Hom}_R(A, B).$$

Definition 1.3.2 (Functor) *A (covariant) functor is a rule $F : \mathsf{C} \to \mathsf{D}$ with the following data:*

(i) *for all $c \in \mathsf{C}$, an object $F(c) \in \mathsf{D}$.*
(ii) *for all $f : c \to c' \in \mathsf{C}$, a morphism $F(f) =: f_* : F(c) \to F(c') \in \mathsf{D}$.*
(iii) *if g and f are composable in C, then $F(gf) = F(g)F(f)$.*[20]
(iv) *for all $c \in \mathsf{C}$, $F(\mathrm{id}_c) = \mathrm{id}_{F(c)} \in \operatorname{Hom}_{\mathsf{D}}(F(c), F(c))$.*

Definition 1.3.3 (Contravariant Functor) *A contravariant functor $G : \mathsf{C} \to \mathsf{D}$ has the same data as a covariant functor, except for any $f : \mathsf{C} \to \mathsf{C}'$, we have*

$$G(f) =: f^* : G(c') \to G(c),$$

so composition admits $G(gf) = G(f)G(g)$.[21]

We define a *covariant* functor:

$$\mathsf{LMod}_R \xrightarrow{\operatorname{Hom}_R(X,-)} \mathsf{LMod}_{\mathbb{Z}} = \mathsf{Ab}$$

$$\begin{tikzcd}
A \arrow[r, mapsto] \arrow[dd, "f"'] & \operatorname{Hom}_R(X, A) \arrow[dd, "f_*:\alpha\mapsto f\circ\alpha"] \\
{} \arrow[r, mapsto] & {} \\
B \arrow[r, mapsto] & \operatorname{Hom}_R(X, B)
\end{tikzcd}$$

[19] In categorical language, the kernel and cokernel can be defined up to isomorphism via a universal property. Though, the category needs a zero object (both initial and terminal).

[20] In the star notation, we have

$$(g \circ f)_* = g_* \circ f_*.$$

[21] In the star notation, this becomes

$$(g \circ f)^* = f^* \circ g^*.$$

That is, $f_*(\alpha)$ is given by post-composition with f. Let $f : A \to B$ and $g : B \to C$. We want to show that $(g \circ f)_* = g_* \circ f_*$:

$$\begin{aligned}(g \circ f)_*(\alpha : X \to A) &= (g \circ f) \circ \alpha \\ &= g \circ (f \circ \alpha) \\ &= g \circ (f_*(\alpha)) \\ &= g_*(f_*(\alpha)) \\ &= (g_* \circ f_*)(\alpha).\end{aligned}$$

Checking the other axioms is quick. Thus, $\mathrm{Hom}_R(X, -)$ is a (covariant) functor. As you might expect, we now define a *contravariant* functor:

$$\begin{tikzcd}
\mathsf{LMod}_R \arrow[rr, "\mathrm{Hom}_R(-,Y)"] & & \mathsf{LMod}_{\mathbb{Z}} = \mathsf{Ab} \\
A \arrow[dd, "f"'] & \longmapsto & \mathrm{Hom}_R(A, Y) \\
 & \longmapsto & \\
B & \longmapsto & \mathrm{Hom}_R(B, Y) \arrow[uu, "f^*:\beta \mapsto \beta \circ f"']
\end{tikzcd}$$

That is, $f^*(\beta)$ is given by pre-composition with f.[22] Again, check the functoriality axioms.

Definition 1.3.4 (Exact Sequence) *A sequence of modules* $A_i \in \mathsf{LMod}_R$ *with morphisms between given by*[23]

$$\cdots \to A_1 \xrightarrow{\partial_1} A_2 \xrightarrow{\partial_2} A_3 \to \cdots$$

is called exact if for all i, $\ker \partial_i = \operatorname{im} \partial_{i-1}$.

Definition 1.3.5 (Short Exact Sequence) *In* LMod_R, *a short exact sequence is one of the form*

$$0 \to A \to B \to C \to 0.$$

Consider the covariant Hom functor F. Run a sequence of modules through:

$$\cdots \to F(A_1) \xrightarrow{\partial_{1*}} F(A_2) \xrightarrow{\partial_{2*}} F(A_3) \to \cdots .$$

Naturally, we may ask the following: if our sequence of modules is exact, is the new sequence of abelian groups exact. The answer is *no*.[24] Instead, try applying the functor to a short exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0,$$

yielding

$$0 \to \mathrm{Hom}_R(X, A) \xrightarrow{i_*} \mathrm{Hom}_R(X, B) \xrightarrow{p_*} \mathrm{Hom}_R(X, C) \to 0$$

[22] We call these new covariant and contravariant functors Hom functors, or the *representable* functors.

[23] More generally, a bi-infinite sequence where kernels are contained in images, rather than equality, is called a *chain complex*.

[24] This is a *very* good thing, which yields nontrivial homology.

Certainly, our setup ensures that i and p are injective and surjective, respectively. Excluding the final term of the sequence $\to 0$, the new sequence *will be* exact. We call this sort of condition, excluding the right hand zero, *left exactness*.

Lemma 1.3.1 *The functor* $\mathrm{Hom}_R(X, -)$ *is left exact.*[25]

Proof. We want to check that i_* is injective. Let $f : X \to A$ be a homomorphism of R-modules be so that $i_*(f) = i \circ f = 0$. That is, for all $x \in X$, $(i \circ f)(x) = 0_B$. Since i is an injection, this forces $f(x) = 0$, so $f = 0 \in \mathrm{Hom}_R(X, A)$, meaning $\ker i_* = 0$, as desired.

Now, we want $i_*(\mathrm{Hom}(X, A)) \subseteq \ker p_*$. Given $i(A) \subseteq \ker B$, we can equivalently write $p \circ i = 0 \in \mathrm{Hom}_R(A, C)$. Then, our original statement is equivalent to

$$p_* \circ i_* = 0 : \mathrm{Hom}_{\mathsf{Ab}}\left(\mathrm{Hom}_R(X, A), \mathrm{Hom}_R(X, C)\right)$$

Well, $p \circ i = 0$ implies that $p_* \circ i_* = 0_* = 0$.

Finally, we want $\ker p_* \subseteq i_*(\mathrm{Hom}(X, A))$. Let $g \in \ker p_*$. Then, $g : X \to B$ is such that $p_*(g) = 0$, so $p \circ g = 0$. For all $x \in X$, $(p \circ g)(x) = 0$, so $p(g(x)) = 0$, meaning $g(x) \in \ker p \subseteq i(A)$. Yet, $i : A \rightarrowtail B$ is an injection. We claim there exists a lift

$$\begin{tikzcd}
A \arrow[r, tail, "i"] & B \arrow[r, "p"] & C \\
& X \arrow[ul, dashed, "\exists", "h"'] \arrow[u, "g"] \arrow[ur, dashed, "0"'] &
\end{tikzcd}$$

defined by $h(x) := a$, where a is the unique element in A such that $g(x) = i(a)$.[26] Certainly, $i(h(x)) = g(x)$. Thus, $g = i \circ h = i_*(h) \in i_*(\mathrm{Hom}(X, A))$, the image. □

Remark 1.3.1 Every functor $F : \mathsf{C} \to \mathsf{D}$, in particular, gives rise to a function

$$\mathrm{Hom}_{\mathsf{C}}(c, c') \to \mathrm{Hom}_{\mathsf{D}}(F(c), F(c')).$$

Some of the properties of the Hom functors are actually true of any so-called *additive functor*.

Definition 1.3.6 (Additive Category) *An additive category is a category* C *in which each* $\mathrm{Hom}_{\mathsf{C}}(c, c')$ *has a structure of an abelian group. Also, composition must be a bilinear operation on* C*, and* C *must admit all finitary biproducts.*[27]

Example 1.3.1 It should be clear that LMod_R is an additive category, from what we have done so far.

Definition 1.3.7 (Additive Functor) *A functor* $F : \mathsf{C} \to \mathsf{D}$ *between additive*

[25] That is, given a short exact sequence, the resulting sequence after applying the functor, will be left exact.

[26] Uniqueness comes from the fact that i is injective.

[27] It is a consequence of the first two axioms that the finitary products and coproducts agree.

categories C *and* D *if each function*

$$\mathrm{Hom}_{\mathsf{C}}(c, c') \to \mathrm{Hom}_{\mathsf{D}}(F(c), F(c'))$$

is a group homomorphism.

Example 1.3.2 The reason we care about noncommutative R is as follows. Let $G := \pi_1(X, x_0)$, the fundamental group of a space. Certainly, G could be nonabelian. In this case, the group ring $\mathbb{Z}G$ with integral coefficients is noncommutative.

Definition 1.3.8 (Contravariant Left Exact) *A contravariant additive functor* $G : \mathsf{C} \to \mathsf{D}$ *is left exact if the truncated exact sequence*[28]

$$(0 \to)A \to B \to C \to 0 \in \mathsf{C}$$

yields an exact sequence[29]

$$0 \to \mathrm{Hom}(C, Y) \to \mathrm{Hom}(B, Y) \to \mathrm{Hom}(A, Y) \in \mathsf{D}.$$

Lemma 1.3.2 *The functor* $\mathrm{Hom}_R(-, Y)$ *is left exact.*

Proof. Complete as an exercise. □

1.4 Tensor Products

We will define the notion of a tensor product of two modules, written $A \otimes_R B$. Note that if A is a *right* R-module, we will write A_R. Based on the notion of the tensor product, we can guess that $A \in \mathsf{RMod}_R$ and $B \in \mathsf{LMod}_R$.

Given right and left R-modules A and B, consider the following universal property.

$$\begin{tikzcd}
A \times B \arrow[r, "(-)\otimes(-)"] \arrow[d, "f"'] & A \otimes_R B \arrow[dl, dashed, bend left, "f'", "\exists!"'] \\
G &
\end{tikzcd}$$

Figure 1.4: Preview of tensor product universal property

Before we continue, it is worth noting that if we have the triangle

$$\begin{tikzcd}
A \arrow[r, two heads, "\alpha"] \arrow[d, "\beta"'] & B \\
C \arrow[ur, dashed, "\exists", "\gamma"'] &
\end{tikzcd}$$

---

[28] The implied $\mathbf{0} \to$ can be removed–it is equivalent.

[29] Note that if we realize a contravariant functor as a functor $\mathsf{C}^{\mathrm{op}} \to D$, then the notions of left exactness agree.

then it is *equivalent* to saying $\ker\alpha \subseteq \ker\beta$. We say β "factors through" α.

Remark 1.4.1 (Motivation) The goal and intuition behind our construction of $A \otimes_R B$ to be an abelian group together with formal elements $a \otimes b$. Furthermore, we want these to generate any other element of the abelian group. Plus, we need the following properties:[30]

(i)
$$ar \otimes b = a \otimes rb.$$

(ii)
$$(a + a') \otimes b = a \otimes b + a' \otimes b.$$

(iii)
$$a \otimes (b + b') = a \otimes b + a \otimes b'.$$

In particular, if $R \subseteq R'$ as a subring, then

$$R \otimes_R R' \simeq_{\mathsf{Ab}} R'.$$

The terminal goal of this, in turn, is to be able to *change coefficients*.

Definition 1.4.1 (Universal Property of Tensor Product) *The tensor product of $A \in \mathsf{RMod}_R$ and $B \in \mathsf{LMod}_R$ is an abelian group $C = A \otimes_R B$ together with an R-biadditive function*

$$(-) \otimes (-) : A \times B \to A \otimes_R B$$

satisfying the universal property that for any R-biadditive map $f : A \times B \to G$ to $G \in \mathsf{Ab}$, there exists a unique homomorphism $f' : A \otimes_R B \to G$ such tha the corresponding diagram commutes.

Definition 1.4.2 (R-Biadditive) *A function $f : A \times B \to G$ between two modules and a group G is called R-biadditive if:*[31]

(i) $f(ar, b) = f(a, rb)$.
(ii) $f(a + a', b) = f(a, b) + f(a', b)$.
(iii) $f(a, b + b') = f(a, b) + f(a, b')$.

Theorem 1.4.1 (Existence of Tensor Product) *The tensor product $A \otimes_R B$ exists in* LMod_R.

Proof. First, let $F := \mathbb{Z}[A \times B]$. Then, define a map $A \times B \hookrightarrow \mathbb{Z}[A \times B]$ by $(a, b) \mapsto 1 \cdot (a, b)$. Define the subgroup $K \le \mathbb{Z}[A \times B]$ generated by

$$K := \left\langle \begin{matrix} (ar, b) - (a, rb), \\ (a + a', b) - (a, b) - (a', b), \\ (a, b + b') - (a, b) - (a, b') \end{matrix} \right\rangle.$$

Define the quotient map $\mathbb{Z}[A \times B] \twoheadrightarrow \mathbb{Z}[A \times B]/K$. We now have a map $(-) \otimes (-)$ from $A \times B$ to $\mathbb{Z}[A \times B]/K$ via the composition. Then, certainly our construction forces $\otimes$ to be biadditive. Now, our goal is to satisfy the universal property. Consider the following diagram.

---

[30] Also, while taking the direct sum yields additive dimensions for vector spaces, we want the tensor product to yield *multiplicative* dimensions.

[31] We avoid saying *R-bilinear*, as we take $G \in \mathsf{Ab}$, not LMod_R.

$$\begin{tikzcd}
A \times B \arrow[rr, "(-)\otimes(-)"] \arrow[dr, hook, "i"] \arrow[ddr, bend right, "f"', "\forall"] & & A \otimes_R B \arrow[ddl, dashed, bend left, "f'", "\exists!"'] \\
& \mathbb{Z}[A \times B] \arrow[ur, two heads, "\pi"'] \arrow[d, dashed, "\varphi"] & \\
& G &
\end{tikzcd}$$

We know φ exists by extending f linearly against the inclusion $i : A \times B \hookrightarrow \mathbb{Z}[A \times B]$. Then, $K = \ker \pi \subseteq \ker \varphi$, as f is an R-biadditive map. Thus, f' exists, per our extension question from before. All that remains is uniqueness, which follows by our generation of $A \otimes_R B$ via the $a \otimes b$. ☐

Corollary 1.4.2 *From the construction,*

$$\{a \otimes b : a \in A, b \in B\} \subseteq A \otimes_R B$$

generates $A \otimes_R B$.[32]

Lemma 1.4.3 *Given homomorphisms* $f : A \to A' \in \mathsf{RMod}_R$ *and* $g : B \to B' \in \mathsf{LMod}_R$, *they induce a homomorphism of abelian groups*

$$f \otimes g : A \otimes_R B \to A' \otimes_R B'.$$

Proof. See the diagram[33]

$$\begin{tikzcd}
A \times B \arrow[rr, "(-)\otimes(-)"] \arrow[dr, "\varphi"'] & & A \otimes_R B \arrow[dl, dashed, "\varphi'"] \\
& A' \otimes_R B' &
\end{tikzcd}$$

Call $f \otimes g := \varphi'$. ☐

We define a *covariant* functor

$$\begin{tikzcd}
\mathsf{LMod}_R \arrow[rr, "A \otimes_R (-)"] & & \mathsf{LMod}_{\mathbb{Z}} = \mathsf{Ab} \\
B \arrow[d, "g"'] & \longmapsto & A \otimes_R B \arrow[d, "\mathrm{id}_A \otimes g"] \\
B' & \longmapsto & A \otimes_R B'
\end{tikzcd}$$

(with a further $\longmapsto$ printed between the vertical arrows)

Likewise, define $(-) \otimes_R B : \mathsf{RMod}_R \to \mathsf{Ab}$ in the intuitive way.

---

[32] As before, we can consider functoral maps $A \mapsto A \otimes_R B$ and $B \mapsto A \otimes_R B$.

[33] Define $\varphi(a, b) = f(a) \otimes g(b)$. We get φ' via the universal property:

$$\varphi'(a \otimes b) := \varphi(a, b)$$

on generators.

Proposition 1.4.4 (Tensor Distribution) *We have an isomorphism*

$$\left(\bigoplus_{i\in I} A_i\right) \otimes_R B \simeq \bigoplus_{i\in I} (A_i \otimes_R B).$$

Proof. The generators of the left-hand side could be mapped to $(a_i) \otimes b \mapsto (a_i \otimes b)$. First, define

$$\begin{tikzcd} \left(\bigoplus_{i\in I} A_i\right) \times B \arrow[r, "\varphi"] & \bigoplus_{i\in I} (A_i \otimes_R B) \\ ((a_i), b) \arrow[r, mapsto] & (a_i \otimes b), \end{tikzcd}$$

which is biadditive. We have a φ' by the universal property. Moreover, $\varphi' : (a_i) \otimes b \mapsto (a_i \otimes b)$ on simple tensors. Fix j. We want to define $\psi : a_j \otimes b \mapsto (\alpha_j(a_j)) \otimes b$, where $\alpha_j : A_j \hookrightarrow \bigoplus A_i$.[34] □

[34] Check that φ' and ψ' are inverses.

Remark 1.4.2 That is, $\otimes$ commutes with $\oplus$.

Proposition 1.4.5 *We have* $R \otimes_R R \simeq R$.

For free modules, we have

$$R[X] \otimes_R R[Y] \simeq \bigoplus_{x\in X} R \otimes_R \bigoplus_{y\in Y} R \simeq \bigoplus_{x\in X} \bigoplus_{y\in Y} R \simeq R[X \times Y].$$

Thus, if $R = \mathbb{F}$, a field, then $\dim(\mathcal{V} \otimes_{\mathbb{F}} \mathcal{W}) = \dim(\mathcal{V}) \dim(\mathcal{W})$.

Proposition 1.4.6 *We have*

$$\mathrm{Hom}_R\left(\bigoplus_{i\in I} A_i, B\right) \simeq \prod_{i\in I} \mathrm{Hom}_R(A, B).$$

Proof. The forward map is $f \mapsto (f_i)_{i\in I}$. □

Proposition 1.4.7 *We have*

$$\mathrm{Hom}_R\left(A, \prod_{i\in I} B_i\right) \simeq \prod_{i\in I} \mathrm{Hom}_R(A, B_i).$$

Proof. The forward map is $f \mapsto \pi_i f$. □

1.5 Tensor-Hom Adjunction and Naturality

We want a way to define the hom-set from a tensor product $A \otimes_R B$ to an object C. We need the structures to agree.

Definition 1.5.1 ((R, S)-Bimodule) *Let R and S be rings. An (R, S)-bimodule is an M in LMod_R and RMod_S such that*

$$(rm)s = r(ms).$$

Let C be a right S-module. Consider the symbol

$$\mathrm{Hom}_{-S}(A \otimes_R B, C).$$

Then, we would have to force B to be an (R, S)-bimodule, so that $A \otimes_R B$ is a right S module. In particular, we can define on the basis elements[35]

$$(a \otimes b)s := a \otimes (bs).$$

We may now state the *adjoint isomorphism* lemma.

Lemma 1.5.1 *There is an isomorphism*

$$\tau_{ABC} : \mathrm{Hom}_{-S}(A \otimes_R B, C) \xrightarrow{\sim} \mathrm{Hom}_{-R}(A, \mathrm{Hom}_{-S}(B, C)) \in \mathsf{Ab}.$$

Sketch of Proof. Begin by noting that B induces a right R-module structure on $\mathrm{Hom}_{-S}(B, C)$:

$$(gr)(b) := g(rb).$$

Take any $f \in \mathrm{Hom}_{-S}(A \otimes_R B, C)$. We will send it to an $f' \in \mathrm{Hom}_{-R}(A, \mathrm{Hom}_{-S}(B, C))$ given by

$$f'(a)(b) := f(a \otimes b) \in C.$$

The following must be checked:

(i) Show that f' is a right R module homomorphism.
(ii) Define $\tau_{ABC}(f) := f'$. Show that τ_{ABC} is a homomorphism in Ab.
(iii) Show that τ_{ABC} is a bijection.

□

Corollary 1.5.2 *If $\mathbb{F}$ is a field and $\mathscr{A}, \mathscr{B}$ are in $\mathsf{Vect}_{\mathbb{F}}$, then*

$$\mathscr{L}(\mathscr{A} \otimes \mathscr{B}, \mathbb{F}) \simeq \mathscr{L}(\mathscr{A}, \mathscr{B}^*).$$

Definition 1.5.2 (Natural Transformation) *Given two functors $F, G : \mathsf{C} \rightrightarrows \mathsf{D}$, a natural transformation $\tau : F \Rightarrow G$ is a class of morphisms*

$$\{\tau_c : F(c) \to G(c) \in \mathsf{D} : c \in \mathsf{C}\}$$

such that for any morphism $f : c \to c' \in \mathsf{C}$, we have $F(f)\tau_{c'} = G(f)\tau_c$.

Definition 1.5.3 (Natural Isomorphism) *A natural isomorphism is a natural transformation $\tau : F \Rightarrow G$ such that for all $c \in \mathsf{C}$, $\tau_c : F(c) \to G(c)$ is an isomorphism.*[36]

---

[35] Verify that this is well-defined using the coset construction of the tensor product.

[36] You will also hear this referred to as an *equivalence*.

$$\begin{tikzcd}
F(c) \arrow[r, "Ff"] \arrow[d, "\tau_c"'] & F(c') \arrow[d, "\tau_{c'}"] \\
G(c) \arrow[r, "Gf"] & G(c')
\end{tikzcd}$$

Figure 1.5: The commutative diagram of a natural transformation $\tau : F \Rightarrow G$

Proposition 1.5.3 *The* τ_{ABC} *can be viewed as a natural isomorphism in* A *and in* C*. That is,*

(i) if we fix B, C*, then*

$$\mathrm{Hom}_{-S}(- \otimes_R B, C) \xRightarrow{\tau_{-BC}} \mathrm{Hom}_{-R}(-, \mathrm{Hom}_{-S}(B, C))$$

is a natural isomorphism.

(ii) if we fix A, B*, then*

$$\mathrm{Hom}_{-S}(A \otimes_R B, -) \xRightarrow{\tau_{AB-}} \mathrm{Hom}_{-R}(A, \mathrm{Hom}_{-S}(B, -))$$

is a natural isomorphism.

Proof. Complete the proof as an exercise. □

Definition 1.5.4 (Adjoint) *A pair of functors* $F : \mathsf{C} \to \mathsf{D}$ *and* $G : \mathsf{D} \to \mathsf{C}$ *are adjoint if there exists a natural isomorphism*[37]

$$\mathrm{Hom}_{\mathsf{D}}(F(c), d) \simeq \mathrm{Hom}_{\mathsf{C}}(c, G(d)).$$

Remark 1.5.1 (Adjoint Pair) Then, we say G is left adjoint to F, and F is right adjoint to G. An adjoint pair is often written (G, F).

Proposition 1.5.4 *Consider functors*

$$F := (-) \otimes_R B : \mathsf{RMod}_R \to \mathsf{RMod}_S$$

and

$$G := \mathrm{Hom}_{-S}(B, -) : \mathsf{RMod}_S \to \mathsf{RMod}_R.$$

Then, (G, F) *is an adjoint pair.*[38]

Example 1.5.1 Note that if we consider R as an (R, R)-bimodule, then $R \otimes_R R$ is an (R, R)-bimodule, which is isomorphic to R. In turn, we may conclude that $R \simeq R \otimes_R R$ in Ring.

Lemma 1.5.5 *The functor* $A \otimes_R (-) : \mathsf{RMod}_R \to \mathsf{Ab}$ *is right exact.*

That is, for any exact $B' \to B \to B'' \to 0$ the induced sequence $A \otimes_R B' \to A \otimes_R B \to A \otimes_R B'' \to 0$ is exact. This can be shown directly, or indirectly via the tensor-Hom adjunction.

---

[37] We then say that there is an *adjunction* between C and D.

[38] This follows formally after the adjoint isomorphism lemma and the proposition of naturality are shown.

Homological Algebra

2

In topology, we study the related geometric notions of *simplicial complexes*, *cell complexes*, and *topological spaces*. These geometric constructions can provide useful motivation for our algebraic tools.

2.1 (Co)Homology of (Co)Chain Complexes

Definition 2.1.1 (n-Simplex) *The standard n-simplex Δ^n is the convex hull of $n + 1$ points in Euclidean n-space.*[1]

Suppose we are working with a simplicial complex with 11 edges. We could define the boundary homomorphisms,

$$0 \to \mathbb{Z}^4 \xrightarrow{\partial_2} \mathbb{Z}^{11} \xrightarrow{\partial_1} \mathbb{Z}^7 \to 0,$$

where each ∂_i has a domain of i-dimensional simplices. Since the $\mathbb{Z}^i \simeq \mathbb{Z}^{\oplus i}$ is free, it suffices to define on the basis $\partial_1(e_1) := v_2 - v_1$, where e_1 connects the vertices $v_1 \rightsquigarrow v_2$. Then, define $\partial_2(f)$ to be the sum of the edge. For cell complexes, we could generalize to disks and define boundary homomorphisms, and likewise for topological spaces, we could work with the so-called *singular chains*.

Definition 2.1.2 (Chain Complex) *A chain complex is a sequence $(C_\bullet, \partial_\bullet)$ of R-modules and R-module homomorphisms*

$$\cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \xrightarrow{\partial_{-1}} C_{-2} \to \cdots$$

such that for all $i \in \mathbb{Z}$, $\partial_i \circ \partial_{i+1} = 0$.[2]

Remark 2.1.1 Often, as seen in the definition above, we suppress the indices and write $C_\bullet$ to mean C_i, or $\bigoplus_{i \in \mathbb{Z}} C_i$.

Definition 2.1.3 (Homology) *The homology of a chain complex $C_\bullet$ is a sequence of R-modules*

$$H_i(C_\bullet) := \ker \partial_i \big/ \operatorname{Im} \partial_{i+1}.$$

Then, if two objects in Simp, the category of simplicial complexes, are isomorphic, then they yield identical chain complexes. The same holds going from Top to chain complexes. Thus, chain complexes are a topological invariant.[3]

Proposition 2.1.1 *If $C_\bullet$ is exact, then $H_i(C_\bullet) = 0$ for all i.*

---

[1] That is, the **0**-simplex is a point, whereas the **3**-simplex is the standard tetrahedron.

[2] That is, $\operatorname{Im} \partial_{i+1} \subseteq \ker \partial_i$.

[3] In turn, so is homology.

Remark 2.1.2 We will suppress notation further, writing $H_\bullet(C_\bullet)$.[4]

Definition 2.1.4 (Cochain Complex) *Built out of duality, a cochain complex of R-modules is a sequence $(C^\bullet, \delta^\bullet)$*

$$\cdots \leftarrow C^2 \xleftarrow{\delta^1} C^1 \xleftarrow{\delta^0} C^0 \xleftarrow{\delta^{-1}} C^{-1} \leftarrow \cdots$$

such that for all i, $\operatorname{Im}\delta^{i-1} \subseteq \ker \delta^i$.

Definition 2.1.5 (Cohomology) *The cohomology of a cochain complex is the sequence $H^\bullet(C^\bullet)$ of R-modules*

$$H^i(C^\bullet) := \ker \delta^i \big/ \operatorname{Im}\delta^{i-1}.$$

Formally, there is not much difference between homology of chain complexes and cohomology of cochain complexes.

Definition 2.1.6 (Exact) *An additive functor $F : \mathsf{LMod}_R \to \mathsf{Ab}$ is exact if for any short exact sequence in LMod_R, the induced sequence is short exact.*[5]

Example 2.1.1 The non-exactness of the tensor functor happens in the "left part" of the sequence. Thus, the tensor product does not preserve the injectivity. Note that since $(-) \otimes_R B$ is right exact, so is $A \otimes_R (-)$. On the other hand, we saw that both Hom functors are left exact.[6]

Informally, the Hom functors changes coefficients. Our goal is to use the non-exactness of the tensor and Hom functors to produce nontrivial homology. Note that the contravariant Hom functor $\operatorname{Hom}_R(-, B)$ takes chain complexes to cochain complexes.

2.2 Resolutions

A common construction is to take a R-module M, and then form

$$\cdots \to C_2 \to C_1 \to C_0 \to M \to 0 \to \cdots$$

which is exact at every index. If M is a module, can one construct an exact sequence as above such that each C_i is free?

Definition 2.2.1 (Free Resolution) *A free resolution of $M \in \mathsf{LMod}_R$ is an exact sequence $F_\bullet$ of the form*

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0 \to \cdots,$$

where the $F_\bullet$ are all free.

Lemma 2.2.1 *For any module M, there is a free resolution.*

[4] The • means two different things in either case.

[5] That is, it is one which is left and right exact.

[6] Again, this means Hom does not preserve surjectivity.

Proof. Proceed by induction on n, for F_n. Construct F_0 and a surjective $\alpha_0 : F_0 \twoheadrightarrow M$, using the universal property. We wish to build

$$\begin{tikzcd}
F_1 \arrow[r, dashed, "\alpha_1"] \arrow[dr, "\alpha_1'"'] & F_0 \arrow[r, "\alpha_0"] & M \arrow[r] & 0 \\
& \ker \alpha_0 \arrow[ur, hook, "\iota_0"'] & &
\end{tikzcd}$$

Repeat the process.[7] ☐

Let F be a free module and

$$B \xrightarrow{p} C \to 0$$

be exact. Further, suppose $f : F \to C$ is an arbitrary homomorphism. Then, there is a g so that the following diagram commutes:

$$\begin{tikzcd}
& F \arrow[dl, dashed, "g"', "\exists"] \arrow[d, "f"] & \\
B \arrow[r, two heads, "p"'] & C \arrow[r] & 0
\end{tikzcd}$$

Figure 2.1: Lifting property for R-modules

Theorem 2.2.2 *The lifting property holds for free modules.*[8]

Proof. We have $F \simeq R[X]$ for some X. Define g on the basis X:

$$g(x) := b \in p^{-1}(f(x)),$$

where we pick b arbitrarily, using that p is a surjection. Extend g by linearity. ☐

Definition 2.2.2 (Projective Module) *A module $P \in \mathsf{LMod}_R$ is projective if it satisfies the above lifting property.*

We saw that free implies projective. Let P be projective and let

$$0 \to A \to B \to C (\to 0)$$

be a short exact sequence. Consider the functor $\mathrm{Hom}_R(P, -)$, yielding an exact sequence

$$0 \to \mathrm{Hom}_R(P, A) \to \mathrm{Hom}_R(P, B) \to \mathrm{Hom}_R(P, C).$$

Well, the f of the lifting property is in $\mathrm{Hom}_R(P, C)$. Since P is projective, there is a $g \in \mathrm{Hom}_R(P, B)$ so that $p_* : g \mapsto f = p \circ g$. Thus, the existence of lifting for $0 \to A \to B \to C$ is equivalent to the surjectivity of p_*.

[7] Picking our F_n is not unique, so free resolutions are certainly not, either.

[8] We abbreviate the statement using the diagram above.

Lemma 2.2.3 *A left R-module P is projective if and only if the functor* $\operatorname{Hom}_R(P, -)$ *is exact.*

Definition 2.2.3 (Split Exact Sequence) *A short exact sequence*

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

is split[9] *if there exists a homomorphism $s : C \to B$ such that $ps = \mathrm{id}_C$.*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, tail, "i"] & B \arrow[r, two heads, bend left, "p"] & C \arrow[l, dashed, bend left, "s"] \arrow[r] & 0
\end{tikzcd}$$

Figure 2.2: Diagram of a split exact sequence. We call s a *section*.

Proposition 2.2.4 *If a short exact sequence, as above, splits, then $B \simeq A \oplus C$.*

Lemma 2.2.5 *Let P be a left R-module. Then, the following are equivalent.*

(i) *P is projective.*
(ii) *Any short exact sequence of the form*

$$0 \to A \to B \to P \to 0$$

splits.
(iii) *P is isomorphic to a direct summand of some free module.*

Proof. The direction (i) ⇒ (ii) is easy via

$$\begin{tikzcd}
 & & & P \arrow[dl, dashed, "g"'] \arrow[d, equal, "\mathrm{id}_P"] & \\
0 \arrow[r] & A \arrow[r, tail, "i"] & B \arrow[r, two heads, "p"] & P \arrow[r] & 0
\end{tikzcd}$$

We know there is a $g : P \to B$ such that $pg = \mathrm{id}_P$.[10] Now, for (ii) ⇒ (iii), let P be given, and let F be a free module such that there exists a surjective homomorphism $p : F \twoheadrightarrow P$.[11] We can turn this into

$$0 \to \ker p \hookrightarrow F \xrightarrow{p} P \to 0.$$

Then, (ii) tells us that p admits a section $s : P \rightarrowtail F$ such that $ps = \mathrm{id}_P$. Then, we get a decomposition $F \simeq P \oplus \ker p$.[12] Finally for (iii) ⇒ (i), we know P is a direct summand of some free module $F = P \oplus Q$. Then,

$$\begin{tikzcd}
P \oplus Q \simeq P \times Q \arrow[r, leftrightarrow, "\simeq"] & F \arrow[r, two heads, shift left, "q"] \arrow[d, dashed, "g"'] & P \arrow[l, hook', shift left, "j"] \arrow[d, "f"] \arrow[dl, dashed, bend right, "h"] & \\
 & B \arrow[r, two heads, "p"'] & C \arrow[r] & 0
\end{tikzcd}$$

Figure 2.3: By the lifting property for F, there is a g so that $pg = fq$. Define $h := gj$. Check that h solves the lifting problem.

□

[9] Or, there *exists a splitting*.

[10] This is exactly (ii).

[11] Again, we are guaranteed that one such module and homomorphism exists.

[12] That is, P is a direct summand of F.

Example 2.2.1 Not all projective modules are free. Consider $\mathbb{Z}/6$ in the form of equivalence classes, as a $\mathbb{Z}/6$-module. We could consider $\langle[2]\rangle$ and $\langle[3]\rangle$, the former of which is isomorphic to $\mathbb{Z}/3$, and the latter of which is isomorphic to $\mathbb{Z}/2$, in $\mathsf{LMod}_{\mathbb{Z}/6}$. We certainly have an isomorphism

$$\mathbb{Z}/6 \simeq \mathbb{Z}/3 \oplus \mathbb{Z}/2,$$

so $\mathbb{Z}/3$ is projective.[13] Yet, 3 is prime, so it is not free over $\mathbb{Z}/6$.

Definition 2.2.4 (Projective Resolution) *A projective resolution of an R-module A is an exact sequence*

$$\cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0=\varepsilon} A \to 0 \to \cdots,$$

such that $P_\bullet$ is projective.

Definition 2.2.5 (Truncated Resolution) *A deleted (or truncated) resolution $P_A = P_\bullet$ of an R-module A is a sequence*

$$\cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{0} 0.$$

2.3 Tor and Ext

We now wish to glean homology from these resolutions. Yet, these resolutions are exact, so we should apply a functor of some sort.[14] This naturally brings us to our first definition of Tor.

Definition 2.3.1 (Tor I) *Let A be a right R-module and B be a left R-module. Pick a projective resolution $P_A \twoheadrightarrow A$, then apply $(-) \otimes_R B$ to the truncated sequence P_A:*

$$\cdots \to P_1 \otimes_R B \xrightarrow{\partial_1 \otimes \mathrm{id}_B} P_0 \otimes_R B \to 0.$$

Then, we define

$$\mathrm{Tor}_n^R(A, B) := H_n(P_A \otimes_R B) = \ker(\partial_n \otimes \mathrm{id}_B) \big/ \mathrm{Im}(\partial_{n+1} \otimes \mathrm{id}_B).$$

Immediately, we give a second definition of Tor.[15]

Definition 2.3.2 (Tor II) *Pick a projective resolution $Q_B \twoheadrightarrow B$. Then, apply $A \otimes_R (-)$ to the truncated sequence Q_B. Define*

$$\mathrm{Tor}_n^R(A, B) := H_n(A \otimes_R Q_B) = \ker(\mathrm{id}_A \otimes \partial_n) \big/ \mathrm{Im}(\mathrm{id}_A \otimes \partial_{n+1}).$$

In the interest of dualizing, we will move on to the notion of injective modules and Ext.

[13] It is a summand.

[14] Certainly, an exact chain complex admits trivial homology.

[15] You should have two questions after seeing these defintions. Notably, is Tor resolution independent? And, do our definitions coincide.

Definition 2.3.3 (Injective Module) *A module I is called injective if it satisfies that for all exact sequences $B \leftarrow A \leftarrow 0$, and for any $f : A \to I$, there is a $g : B \to I$ so that I satisfies the extension property.*

$$\begin{tikzcd}
 & I & \\
B \arrow[ur, dashed, "g", "\exists"'] & A \arrow[l, tail, "i"] \arrow[u, "f"'] & 0 \arrow[l]
\end{tikzcd}$$

Figure 2.4: The extension property for the injective module I

Note that if we have a sequence $B \xrightarrow{p} C \to 0$, we can fill in

$$0 \to \ker p \to B \to C \to 0.$$

Likewise, if we have a sequence $0 \to A \xrightarrow{i} B$, we can fill in[16]

$$0 \to A \to B \to \operatorname{coker} i \to 0.$$

Definition 2.3.4 (Injective Resolution) *An injective resolution of a module B is an exact sequence of the form*

$$\cdots \xleftarrow{\delta^1} I^1 \xleftarrow{\delta^0} I^0 \leftarrow B \leftarrow 0,$$

where $I^\bullet$ is injective.

Theorem 2.3.1 *For any module B, there exists an injective resolution.*

Proof. See Rotman for the proof. □

We can now state our definitions of Ext.

Definition 2.3.5 (Ext I) *Let A be a left R-module and B be a left R-module. Pick a projective resolution $P_A \twoheadrightarrow A$, and then apply $\mathrm{Hom}_R(-, B)$ to P_A:*

$$\cdots \leftarrow \mathrm{Hom}_R(P_1, B) \xleftarrow{\partial_1^*} \mathrm{Hom}_R(P_0, B) \leftarrow 0.$$

Then, we define

$$\mathrm{Ext}_R^n(A, B) := H^n(\mathrm{Hom}_R(P_A, B)) = \ker(\partial_{n+1}^*) \big/ \mathrm{Im}(\partial_1^*).$$

Alternatively, we could use injective resolutions.

Definition 2.3.6 (Ext II) *Pick an injective resolution $B \rightarrowtail I^B$. Then, apply the $\mathrm{Hom}_R(A, -)$ functor, and define*

$$\mathrm{Ext}_R^n(A, B) := H^n(\mathrm{Hom}_R(A, I^B)) = \ker(\delta_*^n) \big/ \mathrm{Im}(\delta_*^{n-1}).$$

Note that there are functors

$$\mathrm{Tor}_\bullet^R(-, B) : \mathsf{RMod}_R \to \mathsf{ChMod}$$

---

[16] Recall that

$$\operatorname{coker} i := B/i(A)$$

and

$$\mathrm{Ext}^{\bullet}_{R}(-, B) : \mathsf{LMod}_R \to \mathsf{CoMod}.$$

2.4 (Co)Homology of Groups

Given a group G and the ring $\mathbb{Z}$, we can form the *group ring* $\mathbb{Z}G$ of G over $\mathbb{Z}$.[17] The underlying set is given by $\mathbb{Z}G := \mathbb{Z}[G]$.

Then, addition is defined

$$\left(\sum_{g\in G} a_g g\right) + \left(\sum_{g\in G} b_g g\right) = \sum_{g\in G}(a_g + b_g)g.$$

Multiplication, being careful, you must write[18]

$$\left(\sum_{g_1\in G} a_g g_1\right)\left(\sum_{g_2\in G} b_g g_2\right) = \sum_{g_1}\sum_{g_2}(a_{g_1} b_{g_2})(g_1 g_2).$$

Pick a left R-module B, where $R := \mathbb{Z}G$. Then, we define the *homology of G with coefficients in B* by

$$H_n(G, B) := \mathrm{Tor}_N^{\mathbb{Z}G}(\mathbb{Z}, B).$$

Explicitly, first pick a free (or projective) resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. That is, let $\mathbb{Z}$ be a right $\mathbb{Z}G$-module, taking the trivial right G-action. That is, $1 \cdot g := 1$.

We have a sequence

$$\cdots \to F_2 \to F_1 \to F_0 \to \mathbb{Z} \to 0$$

in $\mathsf{RMod}_{\mathbb{Z}G}$. Truncated, we have

$$\cdots \to F_2 \to F_2 \to F_0 \to 0.$$

We apply $(-) \otimes_{\mathbb{Z}G} B$. Then, take the homology.

Similarly, pick $B \in \mathsf{LMod}_{\mathbb{Z}G}$. Then, we define the *cohomology of G with coefficients in B* by

$$H^n(G, B) := \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, B).$$

Explicitly take a resolution as before, but consisting of $\mathbb{Z}G$-modules. Apply $\mathrm{Hom}_{\mathbb{Z}}(-, B)$, and then take cohomology.

Remark 2.4.1 (Alternative Convention for Topology) Sometimes, we wish to define $H_n(G, B)$ where $H_n(G, B) := \mathrm{Tor}_n^R(B, \mathbb{Z})$. This is useful when we want $G \simeq \pi_1(X, x_0)$, of some nice space $(X, x_0) \in \mathsf{Top}_*$.

This was, in sense, a topologically motivated construction of the (co)homology of groups. What about an algebraic approach?

Let G be a group. Denote

$$F_n := \mathbb{Z}[G^{n+1}] = \mathbb{Z}[\underbrace{G \times \cdots \times G}_{n+1}].$$

[17] If you like, we could take RG, for an arbitrary ring R.

[18] Reduce modulo when products $g_1 g_2 = g_1' g_2'$ agree.

An element of G^{n+1} is a tuple $(g_0, \dots, g_n)$, where $g_i \in G$. Define homomorphisms

$$\cdots \to F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \to \mathbb{Z}.$$

We may consider F_n as a free $\mathbb{Z}$-module.[19] On basis elements, define

$$\partial_n(g_0, g_1, \dots, g_n) := \sum_{i=0}^{n} (-i)^i (g_0, \dots, \hat{g_i}, \dots, g_n) : F_n \to F_{n-1}.$$

Check that $\partial_{n-1} \circ \partial_n = 0$. This is exact because there exists a contracting chain homotopy[20] in $\mathsf{LMod}_{\mathbb{Z}}$. Further, we can place a left $\mathbb{Z}G$-module structure on $F_n = \mathbb{Z}[G^{n+1}]$:

$$g \cdot (g_0, g_1, \dots, g_n) := (gg_0, gg_1, \dots, gg_n).$$

What must be checked is that ∂_n is a homomorphism of $\mathbb{Z}G$-modules.[21] Note that

$$\partial_0 = \varepsilon : F_0 \to \mathbb{Z}$$

is the *augmentation map*. We sent

$$\left(\sum_{g \in G} a_g g \right) \mapsto \sum_{g \in G} a_g.$$

The contracting chain homotopy here is given by

$$\begin{tikzcd} F_n \arrow[r, "h_n"] & F_{n+1} \\ (g_0, \dots, g_n) \arrow[r, mapsto] & (1, g_0, g_1, \dots, g_n). \end{tikzcd}$$

The standard (or homogeneous) bar resolution of a group is our resolution

$$\cdots \to F_2 \to F_1 \to F_0 \to \mathbb{Z}.$$

Let $F_n' := \mathbb{Z}G[G^n]$. As notation, we write elements as

$$[g_1|g_2|\cdots|g_n] \in G^n,$$

which is precisely the basis of F_n' over $\mathbb{Z}G$.[22]

Theorem 2.4.1 *We claim $F_n \simeq F_n'$ over $\mathbb{Z}G$.*

Sketch of Proof. The map $F_n' \xrightarrow{\sim} F_n$ is given by

$$[g_1|g_2|\cdots|g_n] \mapsto (1, g_1, \dots, g_n),$$

which is a basis of F_n over $\mathbb{Z}G$. □

We now have two isomorphic resolutions:

[19] We will move on to considering F_n as a $\mathbb{Z}G$-module.

[20] We will describe what the *chain homotopy equivalence* is later.

[21] It whould commute with the aciton.

[22] This is a basis as a free $\mathbb{Z}G$-module. Note that the bar notation is where the resolution gets its name.

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r, "\partial_2"] & F_1 \arrow[r, "\partial_1"] & F_0 \arrow[r, "\varepsilon"] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \arrow[d, equal] \\
\cdots \arrow[r] & F'_2 \arrow[r, "\partial'_2"'] \arrow[u, "\simeq"] & F'_1 \arrow[r, "\partial'_1"'] \arrow[u, "\simeq"] & F'_0 \arrow[r, "\partial'_0"'] \arrow[u, "\simeq"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Figure 2.5: The homogeneous bar resolution (top) and bar resolution (bottom) of a group G

Then, given the boundary map

$$\partial_n(g_0, g_2, \dots, g_n) := \sum_{i=0}^{n} (-i)^i (g_0, \dots, \hat{g_i}, g_n),$$

we can define $\partial'_n : \mathbb{Z}G[G^n] \to \mathbb{Z}G[G^{n-1}]$ by

$$\begin{aligned}\partial'_n[g_1|g_2|\cdots|g_n] := {} & x_1[x_2|\cdots|x_n] \\ & + \sum_{i=1}^{n-1}(-1)^i[x_1|\cdots|x_i x_{i+1}|\cdots|x_n] \\ & + (-1)^n[x_1|\cdots|x_{n-1}].\end{aligned}$$

2.5 Comparison Theorem and Chain Homotopies

Definition 2.5.1 (Chain Map) *Given a chain complex $C_\bullet$, a chain map $f_\bullet : C_\bullet \to C'_\bullet$ is a sequence of homomorphisms $f_n : C_n \to C'_n$ such that the natural diagram commutes.*

That is, we need a commutative lattice

$$\begin{tikzcd}
\cdots \arrow[r] & C_2 \arrow[r, "\partial_2"] \arrow[d, "f_2"'] & C_1 \arrow[r, "\partial_1"] \arrow[d, "f_1"'] & C_0 \arrow[r] \arrow[d, "f_0"'] & \cdots \\
\cdots \arrow[r] & C'_2 \arrow[r, "\partial'_2"'] & C'_1 \arrow[r, "\partial'_1"'] & C'_0 \arrow[r] & \cdots
\end{tikzcd}$$

Figure 2.6: The homogeneous bar resolution (top) and bar resolution (bottom) of a group G

Remark 2.5.1 (Notation) If we have a chain complex

$$C_{i+1} \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} C_{i-1},$$

then we define the *cycles* $Z_i := \ker \partial_i$ and the *boundaries* $B_i := \operatorname{Im} \partial_{i+1}$. Then, homology is given by $H_n(C_\bullet) := Z_n / B_n$. Likewise, we have *cocycles* $Z^i := \ker \delta^i$ and *coboundaries* $B^i := \operatorname{Im} \delta^{i-1}$ of a cochain complex, so cohomology is $H^n(C^\bullet) := Z^n / B^n$.

Lemma 2.5.1 *Each chain map $f_\bullet : C_\bullet \to C'_\bullet$ induces a homomorphism*

$f_{n*} : H_n(C_\bullet) \to H_n(C'_\bullet)$ given by

$$f_{n*}(x + \operatorname{Im} \partial_{n+1}) := f_n(x) + \operatorname{Im} \partial'_{n+1}$$

Proof. The diagram chase is trivial. ☐

Given two chain maps $f_\bullet, g_\bullet : C_\bullet \rightrightarrows C'_\bullet$, a *chain homotopy* between $f_\bullet$ and $g_\bullet$ is a sequence of homomorphisms $h_n : C_n \to C'_{n+1}$ such that for all n, $f_n - g_n = \partial'_{n+1} \circ h_n + h_{n-1} \circ \partial_n$:

$$\begin{tikzcd}
C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] \arrow[dl, dashed, "h_n"] & C_{n-1} \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] \arrow[dl, dashed, "h_{n-1}"] \\
C'_{n+1} \arrow[r, "\partial'_{n+1}"'] & C'_n \arrow[r, "\partial'_n"'] & C'_{n-1}
\end{tikzcd}$$

Figure 2.7: Chain homotopy $h_\bullet$

Lemma 2.5.2 *If there is a chain homotopy between $f_\bullet$ and $g_\bullet$, then the induced maps agree on homology.*

Proof. This is formal:

$$\begin{aligned}
(f_{n*} - g_{n*})(x + \operatorname{Im} \partial_{n+1}) &= (f_n - g_n)(x) + \operatorname{Im} \partial'_{n+1} \\
&= (\partial'_{n+1} \circ h_n + h_{n-1} \circ \partial_n)(x) + \operatorname{Im} \partial'_{n+1} \\
&= \operatorname{Im} \partial'_{n+1} \\
&= 0 \in H_n(C'_\bullet).
\end{aligned}$$

Thus, $f_{n*} - g_{n*} = 0 \in H_n(C'_\bullet)$. ☐

Theorem 2.5.3 (Comparison Theorem) *Suppose $A_\bullet \to M$ and $A'_\bullet \to M'$ are chain complexes over modules M and M':*

$$\cdots \to A_2 \to A_1 \to A_0 \to M \to 0$$

and

$$\cdots \to A'_2 \to A'_1 \to A'_0 \to M' \to 0,$$

where $A_\bullet$ is projective and $A'_\bullet$ is exact. Then, any homomorphism $f : M \to M'$ extends to a chain map $f_\bullet$ between the two chain complexes. Furthermore, any two such extensions $f_\bullet$ and $g_\bullet$ are chain homotopic.

Proof. Consider the given initial square.

$$\begin{tikzcd}
A_0 \arrow[r] \arrow[d, dashed, "f_0"'] & M \arrow[d, "f"] & \\
A'_0 \arrow[r] & M' \arrow[r] & 0
\end{tikzcd}$$

We get f_0 via the definition of projective module. Now, if we try the same for the next step, we see that

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, dashed, "?"'] & A_0 \arrow[d, "f_1"] & \\
A'_1 \arrow[r] & A'_0 \arrow[r, "?"] & 0
\end{tikzcd}$$

does not work.[23] Instead, we could take

$$\begin{tikzcd}
A_1 \arrow[dr, bend left, "f_1\partial_1"] & & \\
A'_1 \arrow[r] & \ker \partial'_0 \arrow[r] & 0
\end{tikzcd}$$

checking that $\mathrm{Im}(f_1\partial_1) \subseteq \ker \partial'_0$:

$$\partial'_0 f_1 \partial_1 = f \partial_0 \partial_1 = 0.$$

Thus, there is a lift

$$\begin{tikzcd}
A_1 \arrow[d, dashed, "f_2"'] \arrow[dr, bend left, "f_1\partial_1"] & & \\
A'_1 \arrow[r] & \ker \partial'_0 \arrow[r] & 0
\end{tikzcd}$$

Continue inductively, yielding a chain map $f_\bullet : C_\bullet \to C'_\bullet$. What about uniqueness, up to chain homotopy? Suppose $f_\bullet$ and $g_\bullet : C_\bullet \rightrightarrows C'_\bullet$ are two such extension chain maps of $f = g : M \to M'$. Let $h_{-1} : M \dashrightarrow A'_0$ be the zero map. Next, find h_0 so that $f_1 - g_1 = \partial_0 0 + \partial'_1 h_0$. This is equivalent to saying h_0 is a lift, so we want to solve

$$\begin{tikzcd}
 & A_0 \arrow[dl, dashed, bend right, "\exists"', "h_0"] \arrow[d, "f_1 - g_1"] & \\
A'_1 \arrow[r] & \ker \partial'_0 \arrow[r] & 0
\end{tikzcd}$$

Notably, we need to check that $f_1 - g_1$ takes place in the kernel. Well, $(\partial'_0(f_1 - g_1))(x) = (f\partial_0)(x) - (f\partial_0)(x) = 0$. Inductively, suppose we have h_n so that $f_n - g_n = \partial'_{n+1}h_n + h_{n-1}\partial_n$, we want to find $h_{n+1} : A_{n+1} \to A'_{n+2}$ so that

$$f_{n+1} - g_{n+1} = \partial'_{n+2}h_{n+1} + h_n\partial_{n+1}.$$

[23] Whereas in the first case we have exactness, this is not automatic in the next step.

That is, we equivalently want to find h_{n+1} so that

$$\partial'_{n+2}h_{n+1} = \underbrace{h_n\partial_{n+1} - f_{n+1} + g_{n+1}}_{R}\,.$$

Then, we want to have the diagram,

$$\begin{tikzcd}
 & A_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[dl, dashed, "h_{n+1}"'] & A_n \arrow[dl, "h_n"] \\
A'_{n+2} \arrow[r, "\partial'_{n+2}"'] & A'_{n+1} &
\end{tikzcd}$$

but to have exactness we instead seek to solve

$$\begin{tikzcd}
 & A_{n+1} \arrow[d, "R"] & \\
A'_{n+2} \arrow[r] & \ker \partial'_{n+1} \arrow[r] & 0
\end{tikzcd}$$

In order to be in a situation of this sort, we need to check that $\operatorname{Im} R \subseteq \ker \partial'_{n+1}$. We have

$$\begin{aligned}
\partial_{n+1}R &= \partial'_{n+1}(h_n\partial_{n+1} - f_{n+1} + g_{n+1}) \\
&= \partial'_{n+1}h_n\partial_{n+1} - \partial'_{n+1}(f_{n+1} - g_{n+1}) \\
&- (f_n - g_n - h_{n-1}\partial_n)\partial_{n+1} - \partial'_{n+1}(f_{n+1} = g_{n+1}) \\
&= 0,
\end{aligned}$$

using the commutativity of the diagram and that $A_\bullet$ is a chain complex. Therefore, there is a homotopy $h_{n+1} : A_{n+1} \dashrightarrow A'_{n+2}$. Completing our induction, $f_\bullet \simeq g_\bullet$ are chain homotopic.[24] ☐

Corollary 2.5.4 *If $P_\bullet \to M$ and $Q_\bullet \to M$ are two projective resolutions of a module M, then there is a chain map $f_\bullet : P_\bullet \to Q_\bullet$ that extends* $\mathrm{id} : M \to M$. *Also, there is a chain map $g_\bullet : Q_\bullet \to P_\bullet$ that extends* $\mathrm{id} : M \to M$.

Theorem 2.5.5 *Tor is resolution invariant.*

Proof. Apply the functor $(-) \otimes_R B$ to both resolutions, as above. Then, $f_\bullet \otimes \mathrm{id}_B : P_\bullet \otimes_R B \to Q_\bullet \otimes_R B$ is an induced chain map.[25] Let $f_{n*} : H_n(P_\bullet \otimes_R B) \to H_n(Q_\bullet \otimes_R B)$ be the induced map on homology. Do the same for g_{n*}. Then, $g_{n*}f_{n*} : H_n(P_\bullet \otimes_R B) \to H_n(P_\bullet \otimes_R B)$ is induced by the composite of the chain maps $g_\bullet f_\bullet : P_\bullet \to P_\bullet$. Let $\mathrm{id}_\bullet$ be the identity chain map on $P_\bullet$. Then, $\mathrm{id}_\bullet \simeq g_\bullet f_\bullet$ via chain homotopy, by the comparison theorem. Then, the induced maps $\mathrm{id}_{n*} = g_{n*}f_{n*}$, strictly. Well, $\mathrm{id}_{n*} = \mathrm{id}$ as a homomorphism on homology. Interchange the roles of f and g, and this tells us that $g_{n*}f_{n*} = \mathrm{id}_{n*} = f_{n*}g_{n*}$, so $f_{n*} = g_{n*}$, meaning f_{n*} is an isomorphism on homology.[26] ☐

[24] 24: That is, we can construct a chain homotopy $h_\bullet$ between them.

[25] 25: The same holds for $g_\bullet \otimes \mathrm{id}_B$.

[26] 26: Thus, Tor is independent of the choice of resolution.

We briefly describe some terminology you will hear.

Definition 2.5.2 (Derived Functors) *Let A be a module and F be an additive functor. Take a projective resolutions $P_A \to A$. Then, the left derived functor is a sequence $(L_\bullet F)(A)$ defined pointwise by*

$$(L_n F)(A) := H_n(F(P_A)).$$

Likewise, the right derived functor, given an injective resolution $A \to I^A$, is a sequence $(R^\bullet F)(A)$ defined pointwise by

$$(R^n F)(A) := H^n(F(I^A)).$$

Remark 2.5.2 Thus, Tor and Ext can be realized as the left derived functor of the tensor product and the right derived functor of the Hom functor, respectively.

Proposition 2.5.6 *By the comparison theorem, the left and right derived functors are, in fact, resolution invariant.*[27]

Analogous to the notion of homotopies and homotopy equivalences from topology, we now define chain homotopies between *chain complexes*.

Definition 2.5.3 (Chain Homotopy Equivalence) *A chain homotopy equivalence between chain complexes $C_\bullet$ and $C'_\bullet$ is a chain map $\varphi_\bullet : C_\bullet \to C'_\bullet$ such that there is a chain map $\psi_\bullet : C'_\bullet \to C_\bullet$ so that $\psi_\bullet\varphi_\bullet \simeq \mathrm{id}_\bullet$ and $\varphi_\bullet\psi_\bullet \simeq \mathrm{id}'_\bullet$.*

If there is a chain homotopy equivalence, as above, then the induced map

$$(\psi_\bullet\varphi_\bullet)_n = (\mathrm{id}_\bullet)_n : H_n(C_\bullet) \to H_n(C_\bullet)$$

on homology is the identity homomorphism. Well, this implies

$$(\psi_\bullet\varphi_\bullet)_n = \psi_{\bullet n}\varphi_{\bullet n} = \mathrm{id}_{H_n(C_\bullet)},$$

and likewise in the opposite direction.[28] Then, if $C_\bullet \simeq C'_\bullet$, then $H_n(C_\bullet) \simeq H_n(C'_\bullet) \in \mathsf{LMod}_R$. Consider the special case $\mathrm{id}_\bullet, 0_\bullet : C_\bullet \rightrightarrows C_\bullet$.

Definition 2.5.4 (Contracting Chain Homotopy) *A contracting chain homotopy is a chain homotopy between $\mathrm{id}_\bullet$ and $0_\bullet$.*

Let us break this down: $\mathrm{id}_\bullet - 0_\bullet = \partial_\bullet h_\bullet + h_\bullet \partial_\bullet$, suppressing indices. That is, for all $x \in C_n$,

$$x = \partial_{n+1} \circ h_n(x) + h_{n-1}\partial_n(x).$$

Then, if $\varphi_\bullet = 0 : 0_\bullet \hookrightarrow C_\bullet$, $\varphi_\bullet$ is a chain homotopy equivalence, so the induced maps gives us $H_n(C_\bullet) \xrightarrow{\sim} H_n(0_\bullet) = 0_n$, so $H_n(C_\bullet) \cong 0$. Thus, $\ker \partial_n = \operatorname{Im} \partial_{n+1}$, so $C_\bullet$ is exact.

[27] It is worthwhile to check that the derived functors *are* functors.

[28] That is,

$$\varphi_{\bullet n} : H_n(C_\bullet) \xrightarrow{\sim} H_n(C'_\bullet).$$

ON SPECTRAL SEQUENCES AND ALGEBRAS

The Standard Sequence

3

Given a simplicial complex, it may be a generally difficult problem to compute the homology of the whole complex. Instead, it may worthwhile to determine the homology of subcomplexes of the complex, forming an increasing sequence to study.

3.1 Filtrations

The notion of increasing sequences of modules leads us naturally into the definition of a *filtration*, which will eventually point us toward spectral sequences.

Definition 3.1.1 (Filtration) *A filtration of a module M is a sequence of submodules*

$$M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq \cdots \subseteq M$$

such that

$$\bigcup_i M_i = M.$$

A related characteristic, the *grading* of a filtration, yields

$$\mathrm{Gr}_n := M_n / M_{n-1}.$$

Remark 3.1.1 (Flag) In the case of finite dimensional vector spaces, consider a filtration, which is called a flag in this setting:

$$\mathscr{V}_0 \subseteq \mathscr{V}_1 \subseteq \mathscr{V}_2 \subseteq \cdots \subseteq \cdots \subseteq \mathscr{V}.$$

Then $\mathscr{V}$ is entirely determined by the grading.[1]

We write $F_p M$ for a filtration of M, instead of M_p.

Definition 3.1.2 (Chain Filtration) *Similarly, given a chain complex $C_\bullet$, a filtration $F_p C_\bullet$ is a sequence*[2]

$$F_0 C_\bullet \subseteq F_1 C_\bullet \subseteq F_2 C_\bullet \subseteq \cdots \subseteq \cdots \subseteq C_\bullet.$$

More generally, we could take a filtration of a chain complex $C_\bullet$ so that

$$\cdots \subseteq F_0 C_\bullet \subseteq F_1 C_\bullet \subseteq \cdots \subseteq C_\bullet$$

Note that the filtration is infinite on both sides. We require that $F_p C_\bullet$ is a chian complex for all p, along with the natural containment requirement. That is, $\partial_n(F_p C_n) \subseteq F_p C_{n-1}$; i.e., a sub-chain complex of $C_\bullet$.[3] We will, more often than not, be interested in filtrations which are "bounded."

[1] This is *not* true of modules, generally.

[2] That is, the inclusions hold generally for all n–every term of the chain complex.

[3] We need

$$\bigcup_{p \in \mathbb{Z}} F_p C_n = C_n$$

and

$$\bigcap_{p \in \mathbb{Z}} F_p C_n = 0.$$

In this context, we define again the grading

$$\mathrm{Gr}_p\, C_n := F_p C_n / F_{p-1} C_n.$$

3.2 Homological Type and Bigrading

We will concentrate on spectral sequence of a *homological type*, relating to chain complexes. As you would expect, there are spectral sequences of a *cohomological type*, relating to cochain complexes. These are dual in a natural sense, so we focus on the former.

Now, to visualize the notion of ascending chain complex filtrations, we place the filtration diagonally: the filtration of C_n is placed on the nth diagonal. That is, we let $q := n - p$:

[insert basic q-p diagonal]

We now change the grading notation:[4]

$$\mathrm{Gr}_p\, C_{p+q} := F_p C_{p+q} / F_{p-1} C_{p+q}.$$

The position of the term $F_p C_n$ is $(p, n-p)$.

[insert actual picture with them filled in]

Such a picture is called the "page" $\mathscr{D}^0 = (\mathscr{D}_{pq})_{p,q}$.

Definition 3.2.1 (Graded Module) *A graded module is a family of modules* $\{M_n\}_{n\in\mathbb{Z}} \subseteq \mathsf{LMod}_R$.

Definition 3.2.2 (Bigraded Module) *A bigraded module is a family of modules* $\{M_{pq} : p, q \in \mathbb{Z}^2\} \subseteq \mathsf{LMod}_R$.

If $(C_\bullet, \partial_\bullet)$ is a chain complex, then the boundary map $\partial_\bullet$ can be thought of as a differential of *degree* -1. Similarly, we can define maps $\partial_{\bullet\bullet} : M_{\bullet\bullet} \to M_{\bullet\bullet}$ of *bidegree* $(a, b) \in \mathbb{Z}^2$.

Remark 3.2.1 By "a page," we will precisely mean a "bigraded module."

Now, we could place the grading into our filtration into the same sort of page:

[insert grading page]

We call the above page $E^0 = (E^0_{pq})_{p,q}$ the 0th page of the spectral sequence, where

$$E^0_{pq} := \mathrm{Gr}_p\, C_{p+q} = F_p C_{p+q} / F_{p-1} C_{p+q}.$$

Note that there are "boundary maps" d^0 of bidegree $(0, -1)$.[5]

Definition 3.2.3 (Spectral Sequence) *A spectral sequence is a sequence of bigraded modules,*

$$E^r \equiv (E^r_{pq})_{(p,q)\in\mathbb{Z}^2},$$

---

[4] Keeping track of indices can be a hassle, and is often not discussed in introductory texts.

[5] The maps arise from the boundary maps associated to the numerator and denominator of the grading quotients. Well-definedness is a simple check.

called pages, together with a sequence

$$\partial^r \equiv (\partial^r_{pq})_{(p,q)\in\mathbb{Z}^2},$$

called differentials, such that[6]

$$\partial^r \circ \partial^r = 0,$$

and so that

$$E^{r+1}_{pq} \simeq H_{p+q}(E^r, \partial^r) \in \mathsf{LMod}_R.$$

Now, associated to a chain complex $C_\bullet$, we have cycles $Z_\bullet = \ker \partial_\bullet$ and boundaries $B_\bullet = \operatorname{Im} \partial_\bullet$.[7]

Definition 3.2.4 (Bounded) *A filtration is called bounded if for all n, there are $p', p'' \in \mathbb{Z}$ such that for all $p \leq p'$, we have $F_pC_n = 0$, and for all $p \geq p''$, $F_pC_n = C_n$.*

We will write $Z^r_{pq} := F_pC_{p+q} \cap \partial^{-1} F_{p-r}C_{p+q-1}$, where the position of the left half is (p,q) and the position of the right half is $(p-r, q+r-1)$. Explicitly, this is $\{x \in F_pC_{p+q} : \partial x \in F_{p-r}C_{p+q-1}\}$. After taking the preimage, we need to be in (p,q). Then, rewrite ∂ as a "morphism" $\partial^r_{pq} : F_pC_{p+q} \rightsquigarrow F_{p-r}C_{p+q-1}$.[8] Note that $\{Z^r_{pq}\}$ is a decreasing sequence in r. If r is very large, Z^r_{pq} stabilizes. Denote Z^∞_{pq} to be the module at which the sequence stabilizes. Explicitly, this is

$$Z^\infty_{pq} := F_pC_{p+q} \cap \partial^{-1}(0).$$

Consider the case when $r = 0$:

$$\begin{aligned} Z^0_{pq} &= F_pC_{p+q} \cap \partial^{-1} F_pC_{p+q-1} \\ &= F_pC_{p+q}. \end{aligned}$$

Thus, we have

$$Z^\infty_{pq} \subseteq \cdots \subseteq Z^2_{pq} \subseteq Z^1_{pq} \subseteq Z^0_{pq}.$$

Similarly, denote

$$B^r_{pq} := F_pC_{p+q} \cap \partial F_{p+r-1}C_{p+q+1}.$$

The left is at (p,q), the right is at $(p+r-1, q-r+2)$. Interpret $\partial : F_{p+r}C_{p+q+1} \rightsquigarrow F_pC_{p+q}$ of bidegree $(1-r, r-2)$. If we look at the term r, then this ∂^r corresponds to ∂^{r-1} from before. The sequence $\{B^r_{pq}\}$ increases with respect to r. For very large r, B^r_{pq} stabilizes, due to its boundedness, at the limiting term

$$B^\infty_{pq} := F_pC_{p+q} \cap \partial C_{p+q+1}$$

Thus, we have

$$B^0_{pq} \subseteq B^1_{pq} \subseteq B^2_{pq} \subseteq \cdots \subseteq B^\infty_{pq}.$$

In fact, we can combine

$$F_pC_{p+q} \cap \partial F_{p-1}C_{p+q+1} = B^0_{pq} \subseteq \cdots B^\infty_{pq} \subseteq Z^\infty_{pq} \subseteq \cdots \subseteq Z^0_{pq} = F_pC_{p+q}.$$

[6] The lower indices depend precisely on the bidgree.

[7] These are actual chain complexes, in their own right.

[8] This is a partial function. It is of bidegree $(-r, r-1)$ on the proper domain Z^r_{pq}.

Recall that a spectral sequence is a pair (E^r_{pq}, d^r_{pq}) of bigraded modules and bigraded differentials with bidegree $(-r, r-1)$.

Definition 3.2.5 (Partial Function) *A partial function* $C \rightsquigarrow C'$ *is a function* $\partial : X \to X'$, *where* $X \subseteq C'$ *and* $X' \subseteq C'$ *are subsets.*

Given a function $\partial : C \to C'$ and a some subsets $M \subseteq C$ and $M' \subseteq C'$, we can form a *canonical* partial function[9]

$$\partial^M_{M'} : C \rightsquigarrow C',$$

which is the full function[10]

$$\partial \downarrow^M_{M'} : (M \cap \partial^{-1} M') \to (\partial M \cap M').$$

In fact, $\partial \downarrow^M_{M'}$ is a surjection. Then, the approximate cycles and boundaries above take the form of the domain and range of this partial function. That is, we can reformulate our discussion above into the following:

(i) Z^r_{pq} is defined to be the domain of the canonical partial function ∂_n.
(ii) B^r_{pq} is defined to be the range of the canonical partial function ∂_n.

We denote

$$E^r_{pq} := Z^r_{pq} \Big/ B^r_{pq} + Z^{r-1}_{p-1,q+1}$$

We claim that this is the same as

$$Z^r_{pq} \Big/ \partial Z^{r-1}_{p+r-1,q-r+2} + Z^{r-1}_{p-1,q+1}.$$

Proposition 3.2.1 *That is, we want to show that*

$$B^r_{pq} = \partial Z^{r-1}_{p+r-1,q-r+2}.$$

Proof. We unravel the claimed inequality into the form

$$F_p C_{p+q} \cap \partial F_{p+r-1} C_{p+q+1} = \partial\big(F_{p+r-1} C_{p+q+1} \cap \partial^{-1} F_p C_{p+q}\big).$$

Since $\partial \downarrow^M_{M'}$ is epic,

$$M' \cap \partial M = \partial(M \cap \partial^{-1} M'),$$

so we are done, taking M to be $F_{p+r-1} C_{p+q+1}$. □

Via this reformulation, we may deduce that

$$E^0_{pq} = F_p C_{p+q} \Big/ F_{p-1} C_{p+q},$$

our original definition of the 0th page of the spectral sequence. This is a page (E^0, d^0) of bidegree $(0, -1)$. We want to check that taking the homology of E^0 yields E^1 in our new definition. In general, we take E^r to be defined by E^r_{pq} with bidegree-$(-r, r-1)$ morphisms $d^r_{pq} : E^r_{pq} \to E^r_{p-r,q+r-1}$. We prescribe these by taking the quotient map induced by ∂. That is, we have formed a sequence of pages $(E^r_{\bullet\bullet}, d^r_{\bullet\bullet}) = (E^r, d^r)$. For now, we omit the fact that we are taking successive homologies.

[9] 9: We will sometiems also use a dashed-arrow notation for partial functions.

[10] 10: This is the restriction of ∂.

3.3 Convergence of Spectral Sequences

What does it mean for a spectral sequence to "compute" the homology of a chain complex $(C_\bullet, \partial_\bullet)$? Notaionally, we denote such "convergence" of a spectral sequence by writing $E^r \Rightarrow H(C_\bullet)$.[11] Similarly, we could explicitly write the pages $E^r_{\bullet\bullet} \Rightarrow H_\bullet(C_\bullet)$.

Definition 3.3.1 (Convergence/Computation) *We say a spectral sequence converges $E^r \Rightarrow H(C_\bullet)$ (or computes the homology) if there exists a filtration of $H_\bullet(C_\bullet)$ called $\Phi_p(H_\bullet(C_\bullet)) \subseteq H_\bullet(C_\bullet)$.*[12] *We require that*

$$\Phi_p H_n(C_\bullet) \Big/ \Phi_{p-1} H_n(C_\bullet) \simeq E^\infty_{pq}.$$

Now, what is E^∞? Well, we are clearly, in some sense, looking at the limit of the E^r formula. First recall that we could write

$$E^r_{pq} = Z^r_{pq} \Big/ B^r_{pq} + Z^{r-1}_{p-1,q-1}.$$

Then, we glean

$$E^\infty_{pq} = Z^\infty \Big/ B^\infty_{pq} + Z^\infty_{p-1,q-1}.$$

This allows us to compute homology up to extensions in the filtration.

Lemma 3.3.1 *Let A, B, C be submodules of some ambient module. If $A \subseteq C$, then*

$$C \cap (A + B) = A + (C \cap B).$$

Theorem 3.3.2 *The spectral sequence obtained from a filtration $F_\bullet C_\bullet$ of a chain complex $(C_\bullet, \partial_\bullet)$ converges to the homology $E^r \Rightarrow H(C_\bullet)$.*

Proof. Define the filtration[13]

$$\Phi_p H_n(C_\bullet) := \mathrm{Im}(H_{p+q}(F_p C_\bullet) \to H_{p+q}(C_\bullet)),$$

where the map is the induced map of the inclusion on homology. We can write

$$\begin{aligned} \Phi_p H_{p+q}(C_\bullet) &= F_p C_{p+q} \cap Z_{p+q} + B_{p+q} \Big/ B_{p+q} \\ &\simeq Z^\infty_{pq} + B_{p+q} \Big/ B_{p+q}. \end{aligned}$$

Then,

$$\mathrm{Gr}_p\, H_{p+q}(C_\bullet) = \left(Z^\infty_{pq} + B_{p+q}/B_{p+q}\right) \Big/ \left(Z^\infty_{p-1,q-1} + B_{p+q}/B_{p+q}\right),$$

which "cancels" to

$$\begin{aligned} & Z^\infty_{pq} + B_{p+q} \Big/ Z^\infty_{p-1,q+1} + B_{p+q} \\ \simeq\ & Z^\infty_{pq} + \left(Z^\infty_{p-1,q-1} + B_{p+q}\right) \Big/ Z^\infty_{p-1,q+1} + B_{p+q}. \end{aligned}$$

[11] More often, the literature will write $E^0 \Rightarrow H(C_\bullet)$ or $E^1 \Rightarrow H(C_\bullet)$.

[12] More precisely,

$$\Phi_p(H_n(C_\bullet)) \subseteq H_n(C_\bullet).$$

[13] Per usual, $n = p + q$.

This is precisely

$$Z^\infty_{pq} \Big/ Z^\infty_{pq} \cap \left(Z^\infty_{p-1,q+1} + B_{p+q}\right),$$

which, by the lemma, is

$$Z^\infty_{pq} \Big/ Z^\infty_{p-1,q+1} + Z^\infty_{pq} \cap B_{pq}.$$

Finally, we can write

$$Z^\infty_{pq} \Big/ Z^\infty_{p-1,q+1} + F_p C_{p+q} \cap B_{p+q} = E^\infty_{pq}.$$

□

3.4 Bicomplexes and Tor's Equivalence

We consider some uses of spectral sequences. Notably, we pursue the special case of a *bicomplex*.

Definition 3.4.1 (Bicomplex) *A bicomplex is a triple* $(M_{\bullet\bullet}, d', d'')$, *where* $M_{\bullet\bullet} = (M_{pq})_{p,q\in\mathbb{Z}}$ *is a bigraded module,* d' *is a bigraded homomorphism* $M_{\bullet\bullet} \to M_{\bullet\bullet}$ *of bidegree* $(-1, 0)$, *and* d'' *is a bigraded homomorphism* $M_{\bullet\bullet} \to M_{\bullet\bullet}$ *of bidegree* $(0, -1)$. *Further,* $d' \circ d' = 0$, $d'' \circ d'' = 0$, *and the following square anticommutes:* $d'' \circ d' + d' \circ d'' = 0$.

$$\begin{tikzcd}
\bullet & \bullet \arrow[l, "d'"'] \arrow[d, "d''"] & \bullet^{(p,q)} \arrow[l, "d'"'] \arrow[d, "d''"] \\
 & \bullet & \bullet \arrow[l, "d'"] \arrow[d, "d''"] \\
 & & \bullet
\end{tikzcd}$$

Figure 3.1: Diagram of bicomplex with the two boundary maps

The *total complex* corresponding to a bicomplex $M_{\bullet\bullet}$ is defined as

$$\mathrm{Tot}_n(M_{\bullet\bullet}) := \bigoplus_{p,q\in\mathbb{Z}} M_{pq}, \quad n = p + q.$$

Then, there is an induced map

$$d := d' + d'' : \mathrm{Tot}_n(M_{\bullet\bullet}) \to \mathrm{Tot}_{n-1}(M_{\bullet\bullet}).$$

Check that $d \circ d = 0$, so $(\mathrm{Tot}_\bullet, d_\bullet)$ is a chain complex. Fix p_0 on the p-axis of a pq-plot of the total complex, where n is the diagonal. Define

$$F_p \, \mathrm{Tot}_n(M_{\bullet\bullet}) := \bigoplus_{i\le p_0} M_{i,n-i} \subseteq \mathrm{Tot}_n(M_{\bullet\bullet}).$$

It should be clear that this is a filtration for the total complex. Consider $\mathscr{D}^0$. At (p, q) for $p + q = n$, we have exactly $F_p \, \mathrm{Tot}_n(M_{\bullet\bullet})$. From there, construct the E^0 page by taking the grading:

$$E^0_{pq} = F_p \operatorname{Tot}_{p+q}(M_{\bullet\bullet}) \Big/ F_{p-1} \operatorname{Tot}_{p+q}(M_{\bullet\bullet}) \simeq M_{pq}.$$

Moreover, we have the classic $d^0 = d''$ maps induced on the quotients.[14] Why do we care about bicomplexes? Well, given to chain complex $(P_\bullet, \partial'_\bullet)$ and $(Q_\bullet, \partial''_\bullet)$, we may consider the bicomplex given by

$$\begin{tikzcd}
P_{p-1} \otimes Q_q & P_p \otimes Q_q \arrow[l, "d' := \partial' \otimes \mathrm{id}_q"'] \arrow[d, "d'' := \mathrm{id}_p \otimes \partial''"] \\
& P_p \otimes Q_{q-1}
\end{tikzcd}$$

Then, define $P_\bullet \otimes Q_\bullet$ to be the total complex of the tensored bicomplex above.

Remark 3.4.1 Consider two resolutions $P_A \twoheadrightarrow A$ and $Q_B \twoheadrightarrow B$. Truncating, we can look at $P_A \otimes Q_B$.[15] In particular, this is a chain complex. Moreover, this arises as a bicomplex of the two resolutions. Considering the filtration and given spectral sequence construction for total complexes, we can reconstruct E^0. Using homology with respect to the d^0, we could get the E^1 page with d^1. Taking the homology of the leftward d^1, we could get the next page. It can then be shown that the homology

$$H_n(P_A \otimes Q_B) = H_n(A \otimes Q_B).$$

It is worth noting, that the tensor bicomplex structure constructed above is *not* a bicomplex, as it fails the anticommutativity. However, multiplying $d'' \cdot (-1)^p$ gives the desired behavior.

Definition 3.4.2 (Flat) *A module $M \in \mathsf{LMod}_R$ is called flat if $(-) \otimes_R M : \mathsf{RMod}_R \to \mathsf{Ab}$ is exact.*

Remember, for a functor to be exact, it was must preserve exactness of short exact sequences.[16]

Example 3.4.1 (Base Ring) Consider $R \in \mathsf{LMod}_R$. This is flat:

$$A \otimes_R R \simeq A \in \mathsf{RMod}_R.$$

Example 3.4.2 (Free Module) Any free module $F \in \mathsf{LMod}_R$ is flat. Our functor is

$$(-) \otimes \bigoplus_{i \in I} R : \mathsf{RMod}_R \to \mathsf{Ab}.$$

Since tensors commute with direct sums, we get

$$A \otimes_R \bigoplus_{i \in I} R \simeq \bigoplus_{i \in I} (A \otimes_R R) \simeq \bigoplus_{i \in I} A \in \mathsf{RMod}_R.$$

Thus, since direct sums preserve kernels and images, we get exactness.

[14] That is, we have returned to our bicomplex, except with only d''.

[15] Here, we are talking about the total complex defined above.

[16] We know the tensor functor is already right exact. This definition just gives us the rest of the exactness.

We now classify, more generally, when direct sums of modules are flat.

Proposition 3.4.1 (Flat Sum) *The coproduct of modules* $\{M_j : j \in J\}$

$$\bigoplus_{j\in J} M_j \quad \textit{is flat}$$

if and only if each of the M_j *are flat.*

Proof. We are looking at two functors:

$$(-) \otimes_R \bigoplus_{j\in J} M_j \quad \text{and} \quad (-) \otimes_R M_j .$$

Consider a morphism $\varphi : A \to B$. Then, we get an induced square:[17]

$$\begin{tikzcd}
A \otimes_R \bigoplus_{j\in J} M_j \arrow[r, "\varphi\otimes\mathrm{id}_\oplus"] \arrow[d, "\simeq"'] & B \otimes_R \bigoplus_{j\in J} M_j \arrow[d, "\simeq"] \\
\bigoplus_{j\in J}(A \otimes_R M_j) \arrow[r, "\oplus(\varphi\otimes\mathrm{id}_{M_j})"] & \bigoplus_{j\in J}(B \otimes_R M_j)
\end{tikzcd}$$

□

Corollary 3.4.2 (Projective Module) *Every projective module is flat.*

Proof. Let P be projective. Using the equivalent condition, that means there is exists a free F and some module Q so that $F \simeq P \oplus Q$. □

Let $P_A \twoheadrightarrow A \to 0$ and $Q_B \twoheadrightarrow B \to 0$ be resolutions. We can then form the chain (total) complex

$$(P_A \otimes Q_B)_n := \mathrm{Tot}_n(P_p \otimes_R Q_q)_{(p,q)\in\mathbb{Z}^2} = \bigoplus_{p\in\mathbb{Z}} P_p \otimes Q_{n-p}.$$

The associated boundary map $d' + d''$ forms the chain $((P_A \otimes Q_B)_\bullet, d_\bullet)$.

Theorem 3.4.3 *The following series of isomorphisms exists:*

$$H_n(P_A \otimes_R B) \simeq H_n(P_A \otimes Q_B) \simeq H_n(A \otimes_R Q_B)$$

meaning the two definitions of $\mathrm{Tor}_n^R(A, B)$ *are equivalent.*

Proof. We will only show the first isomorphism, as the other argument is symmetric. Consider the chain complex $C_\bullet := P_A \otimes Q_B$. We need a filtration. Define

$$F_p C_n \equiv F_p(P_A \otimes Q_B)_n := \bigoplus_{i\le p}(P_i \otimes Q_{n-i}).$$

[17] Check that the square commutes.

We claim that $F_pC_\bullet$ is a subcomplex of $C_\bullet$. Thus, $F_\bullet C_\bullet$ is a filtration of $C_\bullet$. Our $\mathscr{D}^0$ page arises by putting F_pC_n on the $n = p + q$ diagonal. Take the quotients/gradings to yield the quotient

$$F_pC_{p+q} \Big/ F_{p-1}C_{p+q} \simeq P_p \otimes Q_q$$

on the nth diagonal of the E^0 page. Well, E^0 is the same as the bicomplex $P_p \otimes Q_q$ for all (p, q).[18] We wish to compute

$$E^1_{pq} := \ker d^0_{pq} / \operatorname{Im} d^0_{p,q+1}.$$

We have the chain complex

$$\begin{tikzcd}
P_p \otimes Q_q \arrow[d] & & & Q_q \arrow[d] \\
\vdots \arrow[d] & & P_p \otimes (-) & \vdots \arrow[d] \\
P_p \otimes Q_1 \arrow[d] & & & Q_1 \arrow[d] \\
P_p \otimes Q_0 \arrow[d] & {} \arrow[ur, Rightarrow, bend right] & & Q_0 \arrow[d] \\
0 & & & 0
\end{tikzcd}$$

which is $P_p \otimes Q_B$. If we instead included B on the left chain, we get $P_p \otimes (Q_q \to \cdots \to B \to 0)$, and since P_p is flat, the homology of the latter sequence is trivial.[19] That is,

$$H_n(P_p \otimes Q_B) = \begin{cases} P_p \otimes B, & n = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Our d^1 then must be induced by d'.[20] Taking the "horizontal" homology of this E^1 page (which only has a single row), we should get E^2. Yet, this is exactly

$$H_n(P_n \otimes B) \simeq E^2_{n0},$$

where $E^2_{pq} = 0$ if $q \neq 0$. On the page E^2, we need a differential d^2 of bidegree $(-2, 1)$. The only nonzero terms in E^2 are in the 0th row. Thus, $d^2 = 0$. Trivially, we have that $E^2 \simeq E^3 \simeq E^4 \simeq \cdots \simeq E^\infty$. In the E^∞ page, since only the bottom row is filled, and since our filtration comes on the diagonals, the only possible nonzero quotient yields

$$\operatorname{Tor}^R_n(A, B) := H_n(P_A \otimes B) \simeq E^2 \simeq E^\infty_{n0} \simeq H_n(P_A \otimes Q_B).$$

□

There are two ways for the other isomorphism for the equivalence of the definitions. One way is to interchange the two coordinates $(p, q) \mapsto (q, p)$ in the page $\mathscr{D}^0$ for the other result, and we are done. Alternatively, change the *first filtration*, which we have been using, to the *second filtration*. Forming this amounts to considering $i \leq q$ diagonal sums in the filtration, rather than $i \leq p$ sums. Certainly, this is equivalent to switching coordinates.

[18] Essentially, the downward map in E^0 is solely produced by d'', as the d' all cancel out.

[19] The only difference in H_n is for $n = 0$.

[20] This requires some checking. We leave it as an exercise.

3.5 A Remark on Cohomological Type

Thus far, we have looked at spectral sequences of homological type. How could we "dualize" this tool so that it applies to cohomology, rather than homology?

Definition 3.5.1 (Cohomological Type) *A spectral sequence of cohomological type is a sequence of pages (E_r, d_r), where $E_r = (E_r^{pq})_{(p,q)\in\mathbb{Z}^2}$ is a bigraded module and d_r is a bigraded homomorphism of bidegree $(r, 1-r)^{21}$ so that $d_r \circ d_r = 0$. Further, we require*

$$E_{r+1} \simeq H^\bullet(E_r, d_r).$$

Now, the question is, does a spectral sequence of cohomological type exist? By observation of the pattern of the bigraded homomorphisms, you can get the cohomological morphisms from the homological morphisms by flipping the arrows. Thus, any spectral sequence of cohomological type can be equivalently interpreted as a spectral sequence of homological type by setting $E_{pq}^r := E_r^{-p,-q}$, or vice-versa.

A natural question is if whether we can use spectral sequences of cohomological type to prove the equivalence of our two definitions of Ext. To this end, let $P_A \twoheadrightarrow A$ be projective and $I^B \leftarrowtail B$ be injective.

We can now consider $\mathrm{Hom}_R(P_p, I^q)$, using the contravariant and covariant aspects of our Hom functor, a bi(co)complex with the induced maps d'' and d'. Then, define a cochain complex structure

$$\left(\mathrm{Hom}(P_A, I^B)\right)^n := \bigoplus_{p+q=n} \mathrm{Hom}_R(P_p, I^q),$$

with the induced map is $d := d' + d''$.

Define the first (descending) filtration of the cochain complex $C^\bullet$ in the same way as before, this time going to the right on the $\mathscr{D}^0$ page:

$$F^p C^\bullet := \bigoplus_{i \geq p} \mathrm{Hom}_R(P_i, I^{n-i}).$$

Then, proceed as before.

Remark 3.5.1 Following our earlier remark of how to work with spectral sequences of cohomological type, we could flip the coordinates $(p, q) \mapsto (-p, -q)$, so our bicomplex of cohomological type, focused in the first quadrant, would yield a bicomplex of homologial type, focused in the third quadrant.

3.6 Exact Couples

Spectral sequences can be produced from *exact couples*, rather than via the aforementioned technique.

[21] Since we are dualizing, multiply the original bidegree by -1.

Definition 3.6.1 (Exact Couple) *An exact couple is a tuple* $(\mathscr{D}, E, \alpha, \beta, \gamma)$ *such that the* $\mathscr{D}$ *and* E *are bigraded modules and* α, β, γ *are bigraded homomorphisms which form the diagram*

$$\begin{tikzcd}
\mathscr{D} \arrow[rr, "\alpha"] & & \mathscr{D} \arrow[dl, "\beta"] \\
& E \arrow[ul, "\gamma"] &
\end{tikzcd}$$

and such that this diagram is exact at every term.

Suppose[22]

$$0 \to C'_\bullet \xrightarrow{\iota_\bullet} C_\bullet \xrightarrow{p_\bullet} C''_\bullet \to 0$$

is a short exact sequence of chain complexes. That is, we have a grid of the pictured form.

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
{} \arrow[r] & C'_{n+1} \arrow[r] \arrow[d, "\iota_{n+1}"] & C'_n \arrow[r, "\partial'_n"] \arrow[d] & C'_{n-1} \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & C_{n+1} \arrow[r] \arrow[d, "p_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & C''_{n+1} \arrow[r] \arrow[d] & C''_n \arrow[r, "\partial''_n"] \arrow[d] & C''_{n-1} \arrow[r] \arrow[d] & {} \\
& 0 & 0 & 0 &
\end{tikzcd}$$

Figure 3.2: Short exact sequence in ChMod_R

Then, such a short exact sequence of chain complexes leads to a long exact sequence

$$\begin{tikzcd}
H_n(C'_\bullet) \arrow[r, "\iota_{*n}"] & H_n(C_\bullet) \arrow[r, "p_{*n}"] & H_n(C''_\bullet) \arrow[dll, "\text{connecting morphism induced by } \partial_n" description] \\
H_{n-1}(C'_\bullet) \arrow[r, "\iota_{*n-1}"] & H_{n-1}(C_\bullet) \arrow[r, "p_{*n-1}"] & H_{n-1}(C''_\bullet)
\end{tikzcd}$$

Figure 3.3: Long exact sequence, induced on homology

Theorem 3.6.1 *Any filtration of a chain complex leads to an exact couple.*

Sketch of Proof. Suppose $F_\bullet C_\bullet$ is a filtration of a chain complex $C_\bullet$. Then,

[22] Each arrow is a morphism in ChMod_R. That is, they are chain maps.

there is a natural short exact sequence

$$0 \to F_{p-1}C_n \hookrightarrow F_pC_n \twoheadrightarrow F_pC_n \Big/ F_{p-1}C_\bullet \to 0.$$

Take the sum over all p.[23] By notation, take

$$0 \to F_\bullet C_\bullet \to F_\bullet \to C_\bullet \twoheadrightarrow \text{ sum over } p \to 0.$$

For each fixed n, we get short exact sequences of chain complexes. Then, using the above construction, we get a long exact sequence of homology. Then, form $\mathscr{D}^1$ by looking at the long exact sequence, and using the homology of the grading, we get the exact couple with E^1. ☐

Theorem 3.6.2 *Any exact couple*

$$\begin{tikzcd}
\mathscr{D}^1 \arrow[rr, "\alpha^1"] & & \mathscr{D}^1 \arrow[dl, "\beta^1"] \\
& E^1 \arrow[ul, "\gamma^1"] &
\end{tikzcd}$$

gives rise to another one

$$\begin{tikzcd}
\mathscr{D}^2 \arrow[rr, "\alpha^2"] & & \mathscr{D}^2 \arrow[dl, "\beta^2"] \\
& E^2 \arrow[ul, "\gamma^2"] &
\end{tikzcd}$$

by a particular algorithm.

Proof. See Rotman's book.[24] ☐

We are not going to explicitly use exact couples, but forming a sequence of exact couples from the $E^0 \to \mathscr{D}^0 \to \mathscr{D}^0 \to E^0$ couple *does* follow from taking homology.[25] However, we are not working with $\beta^0 \circ \beta^0$, as this is trivial by exactness. Define $d^0 := \beta \circ \gamma : E^0 \to E^0$, in the "wrong order." Certainly, $d^0 \circ d^0 = 0$. Then, define the homology $E^1 := H(E^0, d^0)$.

[23] This is the same as taking the sum over all $p - 1$.

[24] The proof is both perfetly precise and entirely unrevealing.

[25] At least, we have the $E^\bullet$.

Multilinear Algebra

4

We now diverge from our discussion of homology and ways to compute homology, moving on to some more classically algebraic notions. We will begin by describing how tensor products can be used to generate some neat algebras.

4.1 Promoting Modules to Algebras

Informally speaking, an *algebra* is an algebraic-structural combination of an R-module and a ring.

Definition 4.1.1 (Algebra) *An algebra over a commutative R is a tuple $(R \curvearrowright \mathfrak{A}, +, \cdot)$ so that $(R \curvearrowright \mathfrak{A}, +)$ is an R-module, $(\mathfrak{A}, +, \cdot)$ is a ring, and the ring-action $R \curvearrowright \mathfrak{A}$ is compatible with the multiplication of the algebra.*[1]

Example 4.1.1 Let R be a commutative ring.

(a) The ring of polynomials $R[x_1, \ldots, x_n]$ is an algebra, where the scalar action comes from multiplication by R-elements.

(b) A group ring RG, as a set, is $R[G]$. That is,

$$RG = \left\{ \sum_{g \in G} a_g \cdot g : a_g \in R \text{ and } \left|\{g : a_g \neq 0\}\right| < \infty \right\}.$$

The addition is, for $g \in G$,

$$\sum a_g g + \sum b_g g = \sum (a_g + b_g) g.$$

The multiplication is, for $g, h \in G$,

$$\left(\sum_{g \in G} a_g g\right)\left(\sum_{h \in G} b_h h\right) = \sum_{g \in G}\sum_{h \in G} (a_g b_h)(gh),$$

collecting like terms to yield[2]

$$\sum_{k \in G}\left(\sum_{g} (a_g b_{g^{-1}k})\right) k.$$

Further, add a scalar action by the usual action on the freely generated module $R[G] \simeq \bigoplus_G R$.

Remark 4.1.1 (Unity) For our purposes, we will tend to assume that algebras are *unital*. However, major examples in practice, like Banach algebras, or more specifically, C^*-algebras, are often not unital.

---

[1] That is, for all $r \in R$ and $a, b \in \mathfrak{A}$,

$$r(a \cdot b) = (ra) \cdot b = a \cdot (rb).$$

Equivalently, $r(-) : \mathfrak{A} \to \mathfrak{A}$ and $a \cdot (-) : \mathfrak{A} \to \mathfrak{A}$ are commutative, as are $r(-)$ and $(-) \cdot b : \mathfrak{A} \to \mathfrak{A}$.

[2] Let $k = gh$.

Given an R-module $M \in \mathsf{Mod}_R$, how could we *promote* M to an algebra? Let $m_1, m_2 \in M$. It can be done, but we will need to extend our space. The rough idea will be to start with M, form $M \otimes_R M$, in some reasonable sense, and then continue to get $M^{\otimes 3}$, and so on. However, our previous construction of the tensor product pushed $M \otimes_R M \in \mathsf{Ab}$, but on the way to get $M \otimes_R M \in \mathsf{Alg}_R$, we need to stop in Mod_R.

Remark 4.1.2 Recall that if R is commutative, then any R-module A has an (R, R)-bimodule structure on it.[3] Further, if A has an (S, R)-bimodule structure and B is a left R-module, then $A \otimes_R B$ can be given a left S-module structure.

Proof. Fix $s \in S$. Then, $s \cdot (-) \otimes \mathrm{id}_B : A \otimes_R B \to A \otimes_R B$ is a well-defined map which does what we want. Ranging over all of s, we get an action $S \times A \otimes_R B \to A \otimes_R B$.[4] ☐

If we have $A, B \in \mathsf{Mod}_R$, then both can viewed as (R, R)-bimodules, so we can form $A \otimes_R B \in \mathsf{Mod}_R$.

Definition 4.1.2 (Bilinear) *A map $h : A \times B \to C \in \mathsf{Mod}_R$ is R-bilinear if it is R-biadditive and the scalar factors pull out of the product.*

We can then define the universal property of the Mod_R tensor product:[5]

$$\begin{tikzcd}
A \times B \arrow[rr, "(-)\otimes(-)"] \arrow[dr, "\forall"', "f"] & & A \otimes'_R B \arrow[dl, dashed, "\exists!"', "\widetilde{f}"] \\
& C &
\end{tikzcd}$$

Figure 4.1: Here, $\otimes$ is the associated bilinear map,

Theorem 4.1.1 *The tensor product in Mod_R exists.*

Proof. Let $F := R[A \times B]$. Let $S' \subseteq F$ be the R-submodule generated by

$$\begin{pmatrix} (a + a', b) - (a, b) - (a', b) \\ (a, b + b') - (a, b) - (a, b') \\ (ra, b) - r \cdot (a, b) \\ (a, rb) - r \cdot (a, b) \end{pmatrix}.$$

Then, fill in the diagram, as we did for the abelian group case. ☐

Proposition 4.1.2 *The tensor products $A \otimes_R B \simeq A \otimes'_R B$.*

Proof. See the lemma.[6] ☐

Lemma 4.1.3 *The structure $A \otimes_R B$ is also an R-bilinear product.*

[3] This is done in the obvious way.

[4] Check the axioms.

[5] Rotman calls this object the R-bilinear product of A and B.

[6] We are using the uniquness clause of the universal property to make the leap.

Proof. Explicitly, $a \otimes b := (a, b) + S$, where $S \subseteq F$ is the biadditive relations-generated subgroup. Then, $(ra) \otimes b = r(a \otimes b)$, by the definition of the R-module structure on $A \otimes_R B$. Thus, our tensor map $(-) \otimes (-) : A \times B \to A \otimes_R B$ is, in fact, R-bilinear.[7] Let $f : A \times B \to C \in \mathsf{Mod}_R$ be a bilinear R-homomorphism. Considering the underlying biadditive maps and groups, there exists a group homomorphism $\widetilde{f} : A \otimes_R B \to C \in \mathsf{Ab}$ so that $f = \otimes \circ \widetilde{f}$. It will suffice to check on the elementary tensors. Let $r \in R$ and $a \otimes b \in A \otimes_R B$. Then,

$$\widetilde{f}(r(a \otimes b)) = \widetilde{f}(ra \otimes b) = f(ra, b) = rf(a, b) = r\widetilde{f}(a \otimes b),$$

so $\widetilde{f} : A \otimes_R B \to C \in \mathsf{Mod}_R$, as desired. □

Thus, $A \otimes_R B \simeq A \otimes'_R B$, so we will just write $A \otimes_R B$. Now, we want to promote $A \otimes_R B \in \mathsf{Mod}_R$ to an algebra $A \otimes_R B \in \mathsf{Alg}_R$. Let R be a commutative ring and let M be an R-module. We saw that our desire to force a product on M is to work with

$$R \oplus M \oplus (M \otimes M) \oplus (M \otimes M \otimes M) \oplus \cdots .$$

If $p \geq 1$, then some classical notations are $\bigotimes^p M = M^{\otimes p}$. If $p = 0$, then

$$\bigotimes^0 M = M^{\otimes 0} := R.$$

Remark 4.1.3 We should note that there are two ways to interpret the object $M \otimes \cdots \otimes M$. One way is to proceed by induction on $(M \otimes_R M) \otimes_R M$, and so forth.[8] Alternatively, you could proceed by the same construction as before, but with the free module

$$R\left[\prod_{i=1}^{p} M_i\right] \twoheadrightarrow M^{\otimes p},$$

which, after quotienting, satisfies the universal property

$$\begin{tikzcd}
M_1 \times \cdots \times M_p \arrow[rr, "(-)\otimes\cdots\otimes(-)"] \arrow[dr, "f"'] & & M_1 \otimes \cdots \otimes M_p \arrow[dl, dashed, "\widetilde{f}"] \\
& C &
\end{tikzcd}$$

Use R-multilinear maps, here.

Denote by $\mathfrak{T}(M)$ the direct sums of such tensor products:

$$T(M) := \bigoplus_{p \geq 0} M^{\otimes p}.$$

Lemma 4.1.4 *The module $\mathfrak{T}(M)$ has a natural R-algebra structure.*

Proof. The R-module structure is given by[9]

$$r \cdot (m_1 \otimes \cdots \otimes m_p) := (rm_1) \otimes m_2 \otimes \cdots \otimes m_p.$$

[7] This is the map composing the inclusion into the free abelian group on the product with the surjection onto the tensor quotient.

[8] In order for this to be canonical, in some sense, we would have to prove the associativity (up to isomorphism) of tensor products.

[9] Check that this is, in fact, well-defined.

The product is given by

$$(y_1 \otimes \cdots \otimes y_p) \cdot (y_1 \otimes \cdots \otimes y_q) := x_1 \otimes \cdots \otimes x_p \otimes y_1 \otimes \cdots \otimes y_q .$$

Again, check the multilinearity so that this definition is well-defined. Further, check that multiplying by $r \in R$ commutes with the algebra product. □

4.2 The Tensor Algebra

We now discuss some theory regarding $\mathfrak{T}(M)$.

Definition 4.2.1 (Graded Algebra) *A graded R-algebra (for commutative R) is an algebra $\mathfrak{A}$ such that there exist R-subalgebras $A^p \subseteq \mathfrak{A}$, one for each $p \geq 0$, such that, tautologically,*

$$\mathfrak{A} = \bigoplus_{p \geq 0} A^p .$$

Further, $A^p \cdot A^q \subseteq A^{p+q}$.

In particular, $\mathfrak{T}(M)$ is a graded R-algebra. We could form a category GrAlg_R by letting the objects be graded modules. The morphisms are given by graded homomorphisms.

Definition 4.2.2 (Graded Homomorphism) *A graded homomorphism is a homomorphism $\varphi : M \to N$ which satisfies $\varphi(M^p) \subseteq N^p$.*[10]

Importantly, when $R = \Bbbk \in \mathsf{Field}$, every $\Bbbk$-module is free. Everything reduces to a discussion about bases. Let $M = \mathcal{V} \in \mathsf{Mod}_R$ be free with basis X. Then, $\mathfrak{T}(\mathcal{V})$ has a nice, more explicit description. Recall

$$\mathfrak{T}(\mathcal{V}) := \bigoplus_{p \geq 0} \mathcal{V}^{\otimes p} = R \oplus \mathcal{V} \oplus (\mathcal{V} \otimes \mathcal{V}) \oplus (\mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V}) \oplus \cdots .$$

Well, $\mathcal{V} \simeq \bigoplus_{x \in X} R$. Substituting, and using the associativity and distributivity of tensors and coproducts we previously deduced, we get

$$\bigoplus_{x \in X} R \otimes \bigoplus_{y \in X} R \simeq \bigoplus_{x \in X} \bigoplus_{y \in X} (R \otimes R) \simeq \bigoplus_{x \in X} \bigoplus_{y \in X} R .$$

In this case, each kth tensor's dimension corresponds to a kth power of the dimension. Recall that free modules $F \in \mathsf{Mod}_R$ have the property

$$\begin{tikzcd}
F \arrow[dr, dashed, "\widetilde{\varphi}", "\exists!"'] & \\
X \arrow[u, "\iota"] \arrow[r, "\forall", "\varphi"'] & M
\end{tikzcd}$$

We could give the same diagram for Grp, or for Alg_R.[11]

---

[10] We write $M = \bigoplus_p M^p$ and $N = \bigoplus_p N^p$.

[11] Certainly, free groups exist. Note that the morphisms in Alg_R are R-module homomorphisms which are also R-homomorphisms (as a ring).

Definition 4.2.3 (Free Algebra) *A free R-algebra $\mathfrak{F} \in \mathsf{Alg}_R$ with basis X is one which satisfies the universal property, as above.*

The question is, per usual, does such a free algebra $\mathfrak{F}$ exist?[12]

Lemma 4.2.1 *For each $X \in \mathsf{Set}$, let $\mathscr{V} := \bigoplus_{x \in X} R$. The algebra $\mathfrak{T}(\mathscr{V}) \in \mathsf{Alg}_R$ satisfies the universal property for a free R-algebra.*

Proof. Complete as an exercise. ☐

Remark 4.2.1 The tensor algebra $\mathfrak{T}(\mathscr{V})$ of a free module of rank k corresponds to the algebra of polynomials over k noncommuting variables in X.

Remark 4.2.2 Let Σ be a Riemannian manifold. We could form the tangent space $\mathscr{V}$ at each point of Σ, and then collect these into the tangent bundle. The bundle can be promoted to one consisting of $\mathfrak{T}(\mathscr{V})$.

4.3 Quotients in Alg_R, GrAlg_R, and CAlg_R

If our goal is to build new algebras out of $\mathfrak{T}(\mathscr{V})$, then we must first deduce what quotients mean for R-algebras.[13] Let $\mathfrak{A}$ be an R-algebra. Let I be an ideal in $\mathfrak{A}$. We have the injection $R \rightarrowtail \mathfrak{A}$ given by $r \mapsto r \cdot 1$. By our definition of algebra, for all $r \in R$ and $a \in \mathfrak{A}$, we have $r \cdot a = (r \cdot 1) \cdot a$. The same holds for when we restrict attention to $r \in I$, so $r \cdot a = (r \cdot 1) \cdot a \in \mathfrak{A} I \subseteq I$. Thus, any ideal $I \subseteq \mathfrak{A}$ is a submodule.

Thus, we can always form a quotient $\mathfrak{A}/I \in \mathsf{Alg}_R$, using the inherited quotient ring and module structures.

Let $\mathscr{V}$ be a free R-module. What do ideals in $\mathfrak{T}(\mathscr{V})$ look like? Let I be the ideal in $\mathfrak{T}(\mathscr{V})$ generated by

$$\underbrace{v \otimes v' - v' \otimes v}_{\text{in } \mathscr{V} \otimes \mathscr{V}}, \quad v, v' \in \mathscr{V}.$$

What does this explicitly mean? Let $S \subseteq \mathfrak{A}$. Recall that the submodule generated by S is the set of all finite, formal R-linear combinations of $s \in S$.[14] Also, recall that a *cyclic* module is one with a singleton generating subset. The ideal generated by $S \subseteq \mathfrak{A}$ is given by

$$\langle S \rangle = \bigcap_{I \supseteq S} I = \mathfrak{A} S \mathfrak{A},$$

the same as an ideal generated in a ring.[15]

Definition 4.3.1 (Graded Ideal) *A graded ideal in a graded R-algebra*

[12] As we would hope, it does.

[13] In Grp, we use normality, in Mod_R, we use submodules, in Ring, we use ideals.

[14] Equivalently, it is just the smallest submodule of $\mathfrak{A}$ containing S.

[15] We could instad consider CAlg_R, the category of commutative algebras.

$\mathfrak{A} \in \mathsf{GrAlg}_R$ is an ideal $I \subseteq \mathfrak{A}$ such that

$$I = \bigoplus_{p\geq 0} I^P,$$

where $I^p := I \cap A^p$.

Lemma 4.3.1 *An ideal I in a graded R-algebra $\mathfrak{A}$ is graded if and only if it is generated by homogeneous elements.*

Definition 4.3.2 (Homogeneous Element) *An element $a \in \mathfrak{A} \in \mathsf{GrAlg}_R$ is called homogeneous if there exists $p \geq 0$ such that $a \in A^p$.*

We now return to our desire to enforce commutativity on a quotient of the tensor algebra $\mathfrak{T}(M)$, for some $M \in \mathsf{Mod}_R$. Our plan was to quotient by the ideal I defined by

$$\langle v \otimes m' - m' \otimes m : m, m' \in M \rangle \subseteq \mathfrak{T}(M).$$

What kind of structure does $\mathfrak{T}(M)/I$ have? Certainly, it lies in Alg_R. Observe that $I \subseteq \mathfrak{T}(M)$ is *homogeneous*.[16] By our lemma, proven on the homework, this is equivalent to I being graded.

Proposition 4.3.2 *The category GrAlg_R is closed under taking quotients by graded ideals.*

Proof. The proof is also left as an exercise for the homework. The multiplication on

$$\mathfrak{A}/I = \bigoplus_{p\geq 0} A^p \Big/ \bigoplus_{p\geq 0} I^p$$

is read from the quotient $\mathfrak{A}/I$:

$$(a + I)(b + I) = ab + I.$$

The natural guess for the grading on $\mathfrak{A}/I$ is

$$(\mathfrak{A}/I)^p := (A^P + I)\Big/ I.$$

Check that

$$\mathfrak{A}/I = \overset{\text{internal}}{\bigoplus_{p\geq 0}} (A^p + I)\Big/ I.$$

Alternatively, check that[17]

$$\mathfrak{A}/I \simeq \bigoplus_{p\geq 0} A^p/I^p.$$

□

[16] The generators live in $M^{\otimes 2}$, so they are homogeneous for $p = 2$.

[17] Rotman uses the second isomorphism theorem in Mod_R to deduce this. This is not illuminating.

Definition 4.3.3 (Symmetric Algebra) *Let* $I \subseteq \mathfrak{T}(M)$ *be an ideal. Denote*

$$\mathfrak{S}(M) := \mathfrak{T}(M)\Big/ I$$

for the symmetric algebra on $M \in \mathsf{Mod}_R$.

Example 4.3.1 Consider the term $M^{\otimes 2}/I^2 = M \otimes M/I^2$ in $\mathfrak{S}(M)$. In the quotient, $m \otimes m' = m' \otimes m$.[18]

Remark 4.3.1 (Dimension of Symmetric Algebra) Note that, for example, in the free case of degree 2 $\mathcal{V}^{\otimes 2}$, in the quotient $\mathfrak{S}(\mathcal{V})$ we have that the dimension follows

$$\frac{|X|(|X|+1)}{2},$$

where $X \times X$ is the basis of $\mathcal{V} \otimes \mathcal{V}$.

Exercise 4.3.1 When $\mathcal{V} \in \mathsf{Mod}_R$ is free with basis X, then $\mathfrak{S}(\mathcal{V})$ satisfies the university property

$$\begin{tikzcd}
\mathfrak{S}(\mathcal{V}) \arrow[dr, dashed, "\exists!"] & \\
X \arrow[u, hook] \arrow[r] & \mathfrak{A}
\end{tikzcd}$$

for free objects in CAlg_R, the category of commutative R-algebras.

Recall that the ring of polynomials $R[X] \in \mathsf{CAlg}_R$ satisfies an identical universal property.

Corollary 4.3.3 *We have that* $\mathfrak{S}(\mathcal{V}) \simeq R[X] \in \mathsf{CAlg}_R$.[19]

Another ideal in $\mathfrak{T}(M)$ is J, the ideal generated by $m \otimes m$ for all $m \in M$. This J is graded, as $m \otimes m \in M^{\otimes 2}$ are homogeneous. We will denote by

$$\bigwedge M := \mathfrak{T}(M)\Big/ J \in \mathsf{GrAlg}_R$$

the *exterior algebra on* M.[20]

Exercise 4.3.2 Check that $J^2 = (M \otimes M) \cap J = \langle m \otimes m : m \in M \rangle$ as an R-submodule of $M^{\otimes 2}$.

Using our grading notation, we write

$$\bigwedge M = \bigoplus_{p \geq 0} \bigwedge^p M.$$

According to the exercise, $\bigwedge^2 M \simeq M^{\otimes 2}/J^2$. The elements of the exterior

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[18] Hence the naming.

[19] In words, the symmetric algebra is the same as the ring of polynomials.

[20] This is the classic terminology.

algebra are denoted by

$$m \otimes m' + J := m \wedge m'.$$

Lemma 4.3.4 *We have that* $m \wedge m' = -m' \wedge m \in \bigwedge M$.

Proof. This is just a computation:

$$\cancelto{0}{(m + m') \wedge (m + m')} = \cancelto{0}{m \wedge m} + m \wedge m' + m' \wedge m + \cancelto{0}{m' \wedge m'}.$$

□

It is because of this behavior that you will also hear $\bigwedge M$ called the *alternating algebra on* M.

Remark 4.3.2 (Notation) We have been mixing some notational choices. Let $X \in \mathsf{Set}$, let $R \in \mathsf{Ring}$, and let

$$\mathcal{V} := R[X] = \bigoplus_{x \in X} R.$$

We will write

$$R\langle X \rangle := \mathfrak{T}(\mathcal{V}),$$

the tensor algebra on the set X. Rotman chooses to write

$$R[X] := \mathfrak{S}(\mathcal{V}),$$

the symmetric algebra on X.[21]

4.4 A Remark on Forms and de Rham Cohomology

For arbitrary $u \in \bigwedge^p \mathcal{V}$ and $v \in \bigwedge^p \mathcal{V}$, can we relate $u \wedge v$ with $v \wedge u$? First, write this in the case of elementary wedges:

$$\underbrace{u_1 \wedge u_2 \wedge \cdots \wedge u_p}_{u \in \bigwedge^p \mathcal{V}} \wedge \underbrace{v_1 \wedge v_2 \wedge \cdots \wedge v_q}_{v \in \bigwedge^q \mathcal{V}}.$$

Then, via the associativity of the tensor product, we can rearrange as we wish:

$$u_1 \wedge \cdots \wedge (u_p \wedge v_1) \wedge v_2 \wedge \cdots \wedge v_q,$$

interchanging with the -1 term. This yields

$$(-1)^p v_1 \wedge (u_1 \wedge u_2 \wedge \cdots \wedge u_p) \wedge v_2 \wedge \cdots \wedge v_q.$$

Doing the same for the v_j terms, we get

$$(-1)^{pq} v \wedge u.$$

[21] To distinguish, we will write $R[X]_{\text{alg}}$ for $\mathfrak{S}(\mathcal{V})$.

Thus, for basic wedges of degree p and q, we get (anti)commutation up to a pqth power of -1. Using bilinearity, write the us and vs in terms of some $x_{i_j} \in X$, the basic elements:[22]

$$x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_p}, \quad x_{i_j} \in X,$$

the basis in $\mathcal{V}^{\otimes p}$, so the dimension (say, if R is a field), is n^p. Consider the element

$$x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_p}$$

in the quotient. The question becomes combinatorial: count the subset of all tuples $(i_1, i_2, \ldots, i_p)$ such that in the wedge, $\bigwedge_{j=1}^{p} x_{i_j} \in \bigwedge^p \mathcal{V}$ forms a basis.

Lemma 4.4.1 (Binomial Theorem) *The set*

$$\left\{x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_p} : i_j \in [n] \text{ and } i_1 < i_2 < \cdots < i_p \subseteq \bigwedge^p \mathcal{V}\right\}$$

is an R-basis for $\bigwedge^p \mathcal{V}$.

Proof. The proof is left as an exercise. □

Corollary 4.4.2 *If $\mathcal{V} \in \mathsf{Mod}_R = \mathsf{Vect}_{\Bbbk}$, then*

$$\dim_R \bigwedge^p \mathcal{V} = \dim_{\Bbbk} \bigwedge^p \mathcal{V} = \binom{n}{p}.$$

Example 4.4.1 (Degree 2) We have

$$\dim_R \bigwedge^2 \mathcal{V} = \binom{n}{2} = \frac{n(n-1)}{2}.$$

Example 4.4.2 (Degree n) We have

$$\dim_R \bigwedge^n \mathcal{V} = \binom{n}{n} = 1.$$

Notably, the basis of $\bigwedge^n \mathcal{V}$ is

$$\{x_1 \wedge x_2 \wedge \cdots \wedge x_n\},$$

so $\bigwedge^n \mathcal{V} \simeq R$. Given $\dim \mathcal{V} = n$, an element $v_1 \wedge \cdots \wedge v_n$ for $v_i \in \mathcal{V}$ can be written as an R-multiple of $x_1 \wedge \cdots \wedge x_n$. Then, the coefficient of the basic element should be called the *determinant* of the list of vectors.[23]

Exercise 4.4.1 Check that the usual notion of determinant coincides with this description.

[22] For this discussion, suppose X is of size n.

[23] We are identifying the list of vectors with the matrix

$$\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}.$$

Remark 4.4.1 Let Σ be a Riemannian manifold.[24] The tangent space $T_x\Sigma$ for $x \in \Sigma$ we call $\mathscr{V} \in \mathsf{Vect}_{\mathbb{R}}$. The collection of all these spaces is called the tangent bundle. Replace each $\mathscr{V}$ with $\mathfrak{T}(\mathscr{V})$, which has $\dim \mathfrak{T}(\mathscr{V}) = \infty$ over $\mathbb{R}$. Instead, we could take the quotient and get $\bigwedge \mathscr{V}$. All of these operations are functorial on the fibers of the bundle.

Let $U \subseteq \mathbb{R}^n$ be open.[25] Consider $\mathscr{C}^\infty(U) \subseteq \mathrm{Map}(U, \mathbb{R})$, the collection of smooth functions $U \to \mathbb{R}$. Certainly, $\mathscr{C}^\infty(U) \in \mathsf{Alg}_{\mathbb{R}}$ under the natural operations. Formally, consider

$$\mathfrak{A}(U)^n \equiv \mathscr{C}^\infty(U)^n := \bigoplus_{i=1}^{n} \mathscr{C}^\infty(U) \in \mathsf{Mod}_{\mathscr{C}^\infty(U)}.$$

Then, $\mathfrak{A}(U)^n$ has a basis which we may denote $\mathrm{d}x_1, \dots, \mathrm{d}x_n$. We could consider the tensor algebra $\mathfrak{T}(\mathfrak{A}(U)^n)$, and then we may form $\bigwedge \mathfrak{A}(U)^n$ as the quotient.

Definition 4.4.1 (p-Form) *A p-form on U is an element of $\bigwedge^p \mathfrak{A}(U)^n$.*

The basis of $\bigwedge^p \mathfrak{A}(U)^n$ over $\mathscr{C}^\infty(U)$ is given by

$$\left\{\mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \wedge \cdots \wedge \mathrm{d}x_{i_p} : i_j \in [n] \text{ and } i_1 < i_2 < \cdots < i_p \subseteq \bigwedge^p \mathfrak{A}(U)^n\right\}.$$

Then, every such element is of the form

$$\sum f_{i_1,\dots,i_p} \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p} = \sum \left(f_{i_1,\dots,i_p} \mathrm{d}x_{i_1}\right) \wedge \mathrm{d}x_{i_2} \wedge \cdots \wedge \mathrm{d}x_{i_p} .$$

We write $\Omega^p(U)$ for the set of all p-forms on U. There exists a differential map $\Omega^p(U) \xrightarrow{d^p} \Omega^{p+1}(U)$. We get a cochain complex $(\Omega^\bullet(U), d^\bullet)$ so that $d^p d^{p-1} = 0$. In turn, the cohomology $H^\bullet(\Omega^\bullet(U), d^\bullet)$ is called the *de Rham cohomology* of U.

This description can be reformulated in terms of vector bundles, as heuristically described before.

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[24] For intuition, consider $\Sigma_1 \hookrightarrow \mathbb{R}^3$, taking the tangent plane at any point $x \in \Sigma_1$ on the torus.

[25] Here, U can be replaced with a smooth Riemannian manifold with a bit of extra work.,

Noetherian Rings and Representations

5

To close, we will now discuss some topics related to modules and representations which are often omitted in other courses, despite being rather useful. This may, at least in part, be review of familiar material.

5.1 Dimension, Rank, and Invariant Basis Number

Assume the axiom of choice. Then, every vector space $\mathcal{V}$ over $\Bbbk$ has a basis. Generally, there are two cases:

(i) If $\mathcal{V}$ has a finite basis β, then take any two bases $\beta, \beta' \subseteq \mathcal{V}$. Using reduction of the change-of-basis transformation to REF, we get that $|\beta| = |\beta'| < \infty$.[1]

(ii) Let $\beta, \beta' \subseteq \mathcal{V}$ be infinite bases. We want to show that $|\beta| = |\beta'|$. Each $b' \in \beta'$ can be written as a finite linear combination

$$b' = \sum_{b \in \beta}^{\text{finite}} r_b b, \quad b \in \beta.$$

That is, we have a function

$$\alpha : \beta' \to \mathrm{Fin}(\beta),$$

the set of finite subsets in β. Each $b' \in \mathrm{span}(\mathrm{supp}(b'))$. By linear independence, for all $C \in \mathrm{Fin}(\beta)$, there are finitely many $b' \in \beta'$ such that $b \in \mathrm{span}(C)$. Via some logic, $|\mathrm{Fin}(\beta)| = |\beta|$. Then, α is finite-to-1. Thus, the domain of α has cardinality no greater than $|\mathrm{Fin}(\beta)| = |\beta|$. Do the same in the opposite direction to yield $|\beta| \leq |\beta'|$. Via the Schröder-Bernstein theorem, $|\beta| = |\beta'|$, so we are done.

Thus, for $\mathsf{Vect}_{\Bbbk}$, the dimension of a vector space $\mathcal{V}$ can be defined as the cardinality of any basis of $\mathcal{V}$.

Let $R \in \mathsf{CRing}$. Consider a free $M \in \mathsf{Mod}_R$. Then, $M \simeq \bigoplus_{x \in X} R$. Suppose further that $M \simeq \bigoplus_{y \in Y} R$. Assume $R \neq 0$, by independence. Let $I \subseteq R$ be a maximal ideal in R. By Zorn's lemma, this is unique and nonempty. Then, IM is a submodule of M. Considering M/IM, we get an R/I-module, which is a vector space. Then, the argument reduces to the vector space case above.[2]

As such, every free module over a commutative ring has a well-defined notion of dimension.

Remark 5.1.1 (Rank of Free Group) Let F_n be the free group on n generators. Likewise, consider F_m. Suppose $F_n \simeq F_m$. Then, it can be shown in various ways that $n = m$. That is, rank is an isomorphism invariant for free groups.[3]

[1] This follows from the kernel being trivial.

[2] There is a slight difficult with checking that the coproduct structure is preserved.

[3] One sanity check is that the wedge of n circles is not homotopy equivalent to the wedge of m circles, unless $n = m$. Though, this fact is usually defended using that rank is isomorphism invariant.

Our next goal is to define a reasonable notion of "rank" for R-modules.

Theorem 5.1.1 *Let $M \in \mathsf{LMod}_R$. The following are equivalent:*

- *(i) M satisfies the ascending chain condition (ACC).*
- *(ii) For any nonempty family of submodules $\mathcal{F} := \{S_i \subseteq M : i \in I\}$, there exists a maximal element of $\mathcal{F}$ with respect to inclusion.*
- *(iii) Any submodule of M is finitely generated.*

Proof. We begin with (i) ⇒ (ii). Suppose there is a nonempty family $\mathcal{F}$ without a maximal element. Take any $S_0 \in \mathcal{F}$. Then, S_0 is not maximal in $\mathcal{F}$, so there exists $S_1 \in \mathcal{F}$ such that $S_0 \subsetneq S_1$. Similarly, construct $S_1 \subsetneq S_2$. We get an ascending chain

$$S_0 \subsetneq S_1 \subsetneq S_2 \subsetneq \cdots,$$

which is a contradiction to (i). Now, we work on (ii) ⇒ (iii). Let S be any submodule of M. Let $\mathcal{F}$ be the family of finitely generated submodules of S. Generally, we have no reason to believe $S \in \mathcal{F}$. By (ii) there exists a maximal submodule $S_{\max} \in \mathcal{F}$. We claim that $S_{\max} = S$. Suppose there exists an $x \in S \setminus S_{\max}$.[4] Consider

$$\langle S_{\max}, x\rangle = S_{\max} + Rx \neq S_{\max}.$$

Then, $\langle S_{\max}, x\rangle \in \mathcal{F}$, but $S_{\max} \subsetneq \langle S_{\max}, x\rangle$, a contradiction to the maximality of $S_{\max}$, so $S = S_{\max}$, as desired. Finally, we show (iii) ⇒ (i). Take any ascending chain

$$S_0 \subseteq S_1 \subseteq S_2 \subseteq \cdots \text{ in } M.$$

Let

$$S := \bigcup_{k=0}^{\infty} S_k \subseteq M.$$

We know S is finitely generated. Thus, we can write

$$S = \langle x_1, x_2, \ldots, x_n\rangle \subseteq M.$$

Then, by definition $\{x_1, x_2, \ldots, x_n\} \subseteq S$, so $x_1 \in S_i$ for some $i \in \mathbb{N}$, and likewise for all x_k. There exists some index $k_0 \in \mathbb{N}$ such that $x_1, \ldots, x_n \in S_{k_0}$. Thus,

$$S \subseteq S_{k_0} \subseteq S_{k_0+1} \subseteq S_{k_0+2} \subseteq \cdots \subseteq S,$$

so $S_{k_0} = S_{k_0+1} = \cdots$, giving us the desired stabilization.[5] □

The same properties are equivalent for a ring $R \in \mathsf{Ring}$. Just consider the module $R \in \mathsf{LMod}_R$.

Corollary 5.1.2 *The following are equivalent:*

- *(i) R satisfies the ACC for left ideals.*
- *(ii) For any family of left ideals $\mathcal{F}$ in R, there is a maximal left ideal in $\mathcal{F}$.*
- *(iii) Any left ideal in R is finitely generated.*

---

[4] We argue by contradiction.

[5] That is, M satisfies the ACC.

Definition 5.1.1 (Left Noetherian Ring) *If $R \in \mathsf{Ring}$ satisfies any of the above properties, it is called left noetherian.*

Theorem 5.1.3 *Let R be left noetherian. Then, any submodule of a finitely generated $M \in \mathsf{LMod}_R$ is finitely generated.*

Proof. Let $M \in \mathsf{LMod}_R$ be generated by

$$\{x_1, x_2, \ldots, x_n\}.$$

We proceed by induction on n. If $n = 1$, then $M = Rx_1$ is cyclic. Take any submodule $S \subseteq Rx_1$. There is the standard characterization

$$R \Big/ \operatorname{Ann}(x_1) \simeq M = Rx_1.$$

We have that $J := (r \cdot (-))^{-1}(S)$ is a left ideal in R. Clearly, $\operatorname{Ann}(x_1) \subseteq J$. Since R is left noetherian, J is finitely generated. Call $\alpha := r \cdot (-)$. Then, the restriction $\alpha : J \twoheadrightarrow S$ is a surjection. Thus, S is finitely generated.[6]

Let $S \subseteq M$ be a general submodule, and let us argue $n \mapsto n+1$, i.e., write $M = \langle x_1, \ldots, x_{n+1} \rangle_M$ all distinct. Let $M' := \langle x_1, \ldots, x_n \rangle_M \subseteq M$. Form the quotient $M'' := M/M'$ so that the following short sequence is exact:

$$0 \to M' \hookrightarrow M \twoheadrightarrow M'' \to 0.$$

Further, we can explicitly write $M'' \simeq Rx_{n+1}$. We have that M' is generated by at most n elements and M'' is generated by at most 1 element. Consider $S \cap M' \hookrightarrow S$. We can extend this to a short exact sequence

$$0 \to S \cap M' \hookrightarrow S \twoheadrightarrow S/(S \cap M') \to 0.$$

By induction, $S \cap M'$ is finitely generated by, say, $\{\alpha_i\}_i$. Further, we can write

$$S/(S \cap M') \simeq (S + M')/M' \subseteq M/M' = M'',$$

so $S/(S \cap M')$ is finitely generated by, say, $\{\beta_j\}$. Then, if $\pi : S \twoheadrightarrow S/(S \cap M')$ is the quotient map, we have

$$S \simeq \langle \alpha_i, \pi^{-1}(\beta_j) \rangle_M ,$$

taking any choice in the preimage of the generators. ☐

Corollary 5.1.4 *If R is left noetherian and $M \in \mathsf{LMod}_R$ is finitely generated, then M satisfies the ACC.*[7]

We now define the invariant basis number (IBN) for rings.

Definition 5.1.2 (Invariant Basis Number) *A ring R satisfies the IBN if for all $m, n \geq 0$,*

$$(R^m \simeq R^n) \Longrightarrow (m = n).$$

[6] This completes the base.

[7] This just runs through the equivalence of being left noetherian for modules.

Theorem 5.1.5 *If $R \neq 0$ is left noetherian, then R satisfies the* IBN.

Before we prove theorem, let us consider some natural examples.

Example 5.1.1 Let R be a principal ideal domain. Trivially, R is (left) noetherian,[8] so it satisfies the IBN.

Lemma 5.1.6 *Let R be a principal ideal domain and $M \in \mathsf{Mod}_R$ is finitely generated by at most n elements, then any submodule of M can be generated by at most n elements.*

Proof. Let $M = \langle x_1, \ldots, x_n \rangle_M$. In the base, $M = Rx_1$, consider $\alpha : R \to M$. It follows that $J := \alpha^{-1}(S)$ is an ideal, so J is generated by at most one element. Recycle the inductive argument from before. Then, $S \cap M'$ is generated by at most n elements, but $S/(S \cap M')$ is generated by at most 1 element, so S is generated by at most $n + 1$ elements. □

Proof of Theorem. Assume $\mathscr{A} := R^m \simeq R^n$. Without loss of generality, assume that $m \geq n$. Let φ be the composite endomorphism

$$\mathscr{A} \xrightarrow{\simeq} R^m \twoheadrightarrow R^n \xrightarrow{\simeq} \mathscr{A},$$

where $R^m \twoheadrightarrow R^n$ is the standard projection.[9] The kernel $\ker \varphi \simeq R^{m-n}$. We simply need to show that φ is an injection, forcing $m = n$. Thus, we are done, modulo the following lemma. □

Lemma 5.1.7 *Let R be left noetherian and let $\varphi : \mathscr{A} \twoheadrightarrow \mathscr{A}$ be a surjective homomorphism, where $\mathscr{A} \in \mathsf{LMod}_R$ is finitely generated. Then, $\varphi \in \mathrm{Aut}_{\mathsf{LMod}_R}(\mathscr{A})$.*

Proof. Consider the sequence $\varphi, \varphi^2, \varphi^3, \ldots$. Denote $K_i := \ker \varphi^i$. Certainly, $K_i \subseteq K_{i+1}$ for all i. Thus, we have a chain

$$K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots \subseteq K_i \subseteq \cdots .$$

Then, $\{K_i\}_{i \in \mathbb{N}}$ stabilizes.[10] Thus, there is an $i_0 \in \mathbb{N}$ such that $K_{i_0} = K_{i_0+j}$ for all $j \in \mathbb{N} \cup \{0\}$. We want to show that for sufficiently large i, $K_i = K_{i+1}$ implies $K_{i-1} = K_i$. We always have $K_{i-1} \subseteq K_i$. Take any $x \in K_i = K_{i+1}$. This means $\varphi^i(x) = 0$, but we need $\varphi^{i-1}(x) = 0$. Since φ is epic, there is $y \in \mathscr{A}$ such that $\varphi(y) = x$. Then,

$$\varphi^i(x) = \varphi^{i+1}(y) = 0,$$

so $y \in K_{i+1} = K_i$. Then, since $y \in K_i$, we get that $\varphi^i(y) = 0$. Splitting again, this means

$$\varphi^{i-1} \circ \varphi(y) = \varphi^{i-1}(x) = 0,$$

so $x \in K_{i_1}$, giving the desired inclusion. Keep applying this process inductively, starting from i_0. Then, we obtain

$$K_1 = \cdots = K_{i_0} = K_{i+1} = \cdots .$$

[8] We can suppress the "left," since domains are commutative.

[9] Realize R^n as a submodule of R^m along the standard inclusion.

[10] Use the corollary from earlier to deduce that $\mathscr{A}$ satisfies the ACC.

Extend the sequence on the left by K_0, where $K_0 = \ker \varphi^0 = \ker \mathrm{id} = 0$. Applying the argument one more term, we get that the whole sequence stabilizes at K_0, thus completing the argument. □

That is, we can talk about invariant basis number for nontrivial, left noetherian rings.

Example 5.1.2 (Fields) Of course, since principal ideal domains are left noetherian, so are fields.

Example 5.1.3 (Matrix Algebras) Let $\Bbbk$ be a field and let $R := \mathbb{M}_n(\Bbbk)$. Then, R is a $\Bbbk$-algebra in the usual way. We claim that R is left noetherian.[11] Of course, in particular, $R \in \mathsf{Vect}_{\Bbbk}$. Moreover, $\dim R < \infty$. Then, any flag $\mathfrak{F}$ in R must stabilize, due to the finiteness, so R satisfies the ACC.

We now give a classical, rich source of examples of left noetherian rings. Though, we will likely omit the proof.

Theorem 5.1.8 (Hilbert Basis Theorem) *If R is a left noetherian ring, then the ring of polynomials $R[x_1, \dots, x_n]$ is also left noetherian.*

5.2 Group Representations

Let $\Bbbk \in \mathsf{Field}$ be a field and let $\mathcal{V} \in \mathsf{Vect}_{\Bbbk}$ be a $\Bbbk$-module. Denote by $\mathrm{GL}(\mathcal{V})$ be the set of all linear maps with an inverse. That is, we simply defined the *general linear group* $\mathrm{GL}(\mathcal{V}) = \mathrm{Aut}_{\mathsf{Vect}_{\Bbbk}}(\mathcal{V})$.[12]

Definition 5.2.1 (Representation) *A G-representation of a group $G \in \mathsf{Grp}$ is any homomorphism*

$$\sigma : G \to \mathrm{GL}(\mathcal{V}),$$

for some $\mathcal{V} \in \mathsf{Vect}_{\Bbbk}$ over a field $\Bbbk$.

Remark 5.2.1 Equivalently, if $\mathsf{B}G$ denotes the delooping groupoid associated to G, then a G-representation is a functor

$$\Sigma : \mathsf{B}G \to \mathsf{Vect}_{\Bbbk}.$$

This form of the definition generalizes other, similar notions like G-sets $\mathsf{B}G \to \mathsf{Set}$ and G-spaces $\mathsf{B}G \to \mathsf{Top}$.

There is a nice relation between such representations and the theory of modules. For any group G and field $\Bbbk$, let $R := \Bbbk G$ denote the group algebra over $\Bbbk$.

Lemma 5.2.1 *Each G-representation $\sigma : G \to \mathrm{GL}(\mathcal{V})$ gives rise to a $\Bbbk G$-module $\mathcal{V}^\sigma$. Conversely, any $\Bbbk G$-module leads to a G-representation.*

[11] We will use the ACC for left ideals.

[12] The operation is composition.

Proof. Given $\sigma \in \mathrm{Hom}_{\mathsf{Grp}}(G, \mathrm{GL}(\mathcal{V}))$, our natural candidate for $\mathcal{V}^\sigma$ is $\mathcal{V}$ with a $\Bbbk G$-module structure:

$$\left(\sum_{g\in G}^{\text{finite}} a_g g\right) v := \sum_{g\in G}^{\text{finite}} a_g(\sigma(g)(v)), \quad v \in \mathcal{V}^\sigma.$$

Check that the defined map $\Bbbk G \times \mathcal{V} \to \mathcal{V}$ satisfies the axioms for a $\Bbbk G$-module.[13] Conversely, let $\mathcal{V}$ be a $\Bbbk G$-module. Define

$$\begin{aligned} G &\xrightarrow{\sigma} \mathrm{GL}(\mathcal{V}) \\ g &\longmapsto (v \mapsto (1g)v). \end{aligned}$$

Again, we should check that σ is a homomorphism, which is trivial by the definition of a $\Bbbk G$-module. ☐

Remark 5.2.2 The correspondence above can be encoded into an equivalence

$$\mathsf{Rep}_{\Bbbk}^{G} \simeq \mathsf{LMod}_{\Bbbk G}$$

between the category of representations of G over $\Bbbk$ and the category of $\Bbbk$-modules.

5.3 (Semi)Simplicity

Definition 5.3.1 (Simple Module) *A module $M \in \mathsf{LMod}_R$ is simple if it has no "proper" submodules.*[14]

Definition 5.3.2 (Simple Ring) *A ring $R \in \mathsf{Ring}$ if it has no "proper" ideals.*

Remark 5.3.1 A submodule N of M is simple precisely when N is minimal.

Definition 5.3.3 (Semisimple Module) *A module $M \in \mathsf{LMod}_R$ is called semisimple if it is a (possibly infinite) direct sum of its simple submodules.*

Definition 5.3.4 (Semisimple Ring) *A ring $R \in \mathsf{Ring}$ is called left semisimple if it is a (possibly infinite) direct sum of its minimal left ideals.*

Remark 5.3.2 A simple module is a semisimple module, trivially.

Remark 5.3.3 A simple ring does not have to be left semisimple.

We now state, without any proof, a nice characteristic of left semisimplicity.[15]

[13] This bit is left as an exercise.

[14] Here, "proper" means nonzero and not all of $\boldsymbol{M}$.

[15] If I remember correctly, my notes from Rezk's 500 course have these sorts of results proven.

Proposition 5.3.1 *For a module $M \in \mathsf{LMod}_R$, M is semisimple if and only if every submodule of M is a direct summand.*

Proposition 5.3.2 *For a ring $R \in \mathsf{Ring}$, R is left semisimple if and only if every left ideal of R is a direct summand.*[16]

5.4 Maschke's Theorem

We now state the classical result which witnesses a relation between representations of finite groups and left semisimple rings.

Theorem 5.4.1 (Mashcke's) *If $G \in \mathsf{Grp}$ is a finite group and $\Bbbk \in \mathsf{Field}$ is a field such that $\operatorname{char} \Bbbk \nmid |G|$,*[17] *then the group algebra $\Bbbk G \in \mathsf{Alg}_{\Bbbk G}$ is left semisimple.*

Sketch of Proof. It suffices to check that any left ideal $I \subseteq \Bbbk G$ is a direct summand, per the above proposition. First, observe that $\Bbbk G$ is a $\Bbbk$-linear space. Then, I is a $\Bbbk$-linear subspace. Pick a basis β_I in I, extend it to a basis β_F for all of $\Bbbk G$, resulting in a decomposition

$$I \oplus \mathscr{V} = \Bbbk G, \quad \mathscr{V} := \operatorname{span}\{\beta_F \setminus \beta_I\}.$$

Yet, we do not know that $\mathscr{V}$ is a left ideal of $\Bbbk G$. Still, we have a $\Bbbk$-linear projection $d : \Bbbk G \twoheadrightarrow I$. Moreover, d is a retraction:

$$(\forall u \in I)(d(u) = u).$$

That is, $I \hookrightarrow \Bbbk G \twoheadrightarrow I$ is the identity.[18] Using the fact that G is finite, we can average over G to get the desired result. Define $\mathscr{D} : \Bbbk G \to I$ by

$$\mathscr{D}(u) := \frac{1}{|G|} \sum_{x \in G} x \cdot d(x^{-1}u), \quad u \in \Bbbk G.$$

This is a $\Bbbk G$-retraction $\Bbbk G \to I$, as desired.[19] ☐

5.5 Classification via Wedderburn-Artin

We now "complete" our investigation of left semisimple rings.

Lemma 5.5.1 *A ring $R \in \mathsf{Ring}$ is left semisimple if and only if R is the direct sum of finitely many minimal left ideals.*

Lemma 5.5.2 (Schur's) *Let $M, M' \in \mathsf{LMod}_R$ be simple. Then,*

(i) any R-homomorphism $M \to M'$ is either 0 or an isomorphism.[20]
(ii) the endomorphism ring $\operatorname{End}_R(M)$ is a division ring.

[16] Here, this means direct summand as an R-module.

[17] For instance, let $\Bbbk \simeq \mathbb{Q}$.

[18] Recall that this is equivalent to being semisimple.

[19] Just use $\ker \mathscr{D}$ as the direct complement of the ideal.

[20] That is,

$$\operatorname{Hom}_R(M, M') \simeq \operatorname{Iso}(M, M') \cup \{*\}.$$

Lemma 5.5.3 *Submodules of semisimple modules are semisimple.*

Proof. Suppose $S \subseteq M$ is a submodule in a semisimple module. Then, let $A \subseteq S \subseteq M$ be a submodule. Then, A is a direct summand of M. Thus, there is a retraction $p : M \twoheadrightarrow A$, and restricting $p|_S$, we get another retraction of S onto A; A is a direct summand, so S is semisimple. ☐

Remark 5.5.1 The same result as above holds for quotients. That is, given $S \subseteq M$, we have that M/S is semisimple.

Proof. Let $S \subseteq M$. We get $M \simeq S \oplus S'$. Yet, $M/S \simeq S'$, so we are done. ☐

Exercise 5.5.1 For a ring $R \in \mathsf{Ring}$, the following are equivalent:[21]

(i) The ring R is left semisimple.
(ii) For all modules $M \in \mathsf{LMod}_R$, M is semisimple.
(iii) For all modules $M \in \mathsf{LMod}_R$, M is injective.
(iv) Every short exact sequence in LMod_R splits.
(v) For all modules $M \in \mathsf{LMod}_R$, M is projective.

[21] This result is, in some sense, the homologist's dream.

Recall that a module is noetherian if it satisfies the ACC. But why restrict ourselves to ascending chains?

Definition 5.5.1 (Artinian) *A module $M \in \mathsf{LMod}_R$ is called artinian if it satisfies the descending chain condition for modules (DCC).*

Lemma 5.5.4 *If R is left semisimple, then it is both noetherian and artinian.*

Theorem 5.5.5 (Wedderburn-Artin) *A ring $R \in \mathsf{Ring}$ is left semisimple if and only if R is a direct product of matricial rings over division rings Δ_α:*

$$R \simeq \prod_{\alpha \in \mathcal{A}} \mathbb{M}_{n_\alpha}(\Delta_\alpha).$$

This decomposition, in turn, is true if and only if

$$R \simeq \mathbb{M}_{n_1}(\Delta_1) \times \mathbb{M}_{n_2}(\Delta_2) \times \cdots \times \mathbb{M}_{n_m}(\Delta_m),$$

where Δ_k is a division ring for all $1 \leq k \leq m$.[22]

[22] That is, we can reduce the decomposition statement to a finite one without losing anything.

Corollary 5.5.6 *A ring R is left semisimple if and only if it is right semisimple.*

Idea of Proof. Use R^{op}:

$$R^{\mathrm{op}} \simeq \prod_{i=1}^{m} \mathbb{M}_{n_i}(\Delta_i^{\mathrm{op}}).$$

Then, R^{op} is left semisimple, so R is right semisimple. ☐