CALCULUS II, DISCUSSION ADP, EXAM I

DHEERAN E. WIGGINS

Abstract. I hope these notes prove to be useful while you study for the first exam of this course. I have structured the notes to mirror the order of topics as we saw them during our discussion sessions. Throughout the notes, I use a **Goal–Solution** format, which I hope helps you all internalize when to use each technique.

Contents

Date: Spring, 2026.

As a reminder, this is *not* official Math 231 review material. It may contain different problem-solving techniques than were presented in your lecture packets or discussion worksheets. It should serve as a supplement to those, not a replacement. I believe these notes closely reflect how I think about this stuff (and thus, how I presented it to you all). Per usual, I recommend going over *all* of the problems on your discussion sheets and homework. My favorite online reference for practice problems and alternative presentations of this material is Paul's Online Math Notes, Calculus II. Good luck!

1. The fundamental theorem

The fundamental theorem of calculus relates derivatives and integrals in two (main) ways.

1.1. **Definite integrals.**

Goal 1.1. We want to evaluate a definite integral of the form

$$\int_a^b f(x)\,\mathrm{d}x\,.$$

Solution 1.2. As long as f is continuous on $[a,b]$, then we can just compute

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a),$$

where the derivative of $F(x)$ is $f(x)$.

Thus, as long as we can find $F(x)$ from $f(x)$, we know how to compute the define integral of continuous functions. However, computing $F(x)$ can be hard, which is exactly why we must learn all the techniques in the following sections.

Example 1.3. Say we want to integrate

$$\int_2^5 x^2 + 1\,\mathrm{d}x\,.$$

Well, we know that $f(x) = x^2 + 1$ is continuous on $[2,5]$, since it is just a polynomial, so we can compute

$$\int_2^5 x^2 + 1\,\mathrm{d}x = \left[\frac{x^3}{3} + x\right]_2^5 = \left(\frac{5^3}{3} + 5\right) - \left(\frac{2^3}{3} + 2\right).$$

Here, our $F(x)$ was the function $x^3/3 + x$ we found by using the power rule for integration.

1.2. **Derivative of integral.**

Goal 1.4. We want to take the derivative of an integral

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t\,.$$

Note that we are differentiating with respect to the upper bound.

Solution 1.5. If f is continuous on $[a,b]$, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x).$$

That is, taking the derivative of an integral with respect to its upper bound returns the original function $f(x)$.

Example 1.6. Suppose we want to compute

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{-39}^x \ln\big(t^3 + t + 40\big)\,\mathrm{d}t\,.$$

The integrand $f(t) = \ln\big(t^3 + t + 40\big)$ is pretty messy. But using the fundamental theorem of calculus, we can just easily compute

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{-39}^x \ln\big(t^3 + t + 40\big)\,\mathrm{d}t = \ln\big(x^3 + x + 40\big).$$

2. Substitution

Goal 2.1. We want to simplify integrals by noticing that a function and its derivative are present in the integrand.

Solution 2.2. Given an integral of the form

$$\int f(g(x))g'(x)\,\mathrm{d}x\,,$$

we substitute $u = g(x)$, find the differential $\mathrm{d}u = g'(x)\,\mathrm{d}x$, and can rewrite the integral as

$$\int f(u)\,\mathrm{d}u\,.$$

Remark 2.3. If we are given a definite integral, remember to change the bounds after you substitute!

Example 2.4. We want to compute the definite integral

$$\int_0^1 x^2(x^3+1)^4\,\mathrm{d}x\,.$$

We can see both the function $g(x) = x^3 + 1$ and some of its derivative (the x^2) in the integrand, so we probably want to use u-substitution. Let $u = x^3 + 1$. Then,

$$\mathrm{d}u = 3x^2\,\mathrm{d}x\,, \quad \text{or} \quad \mathrm{d}x = \frac{\mathrm{d}u}{3x^2}.$$

It is time to change our bounds to be in terms of u. The lower x-bound is $x = 0$, so we compute

$$u = 0^3 + 1 = 1,$$

meaning the lower u-bound will be 1. The upper x-bound is $x = 1$, so we compute

$$u = 1^3 + 1 = 2,$$

meaning the upper u-bound will be 2. We can now substitute in our us:

$$\int_0^1 x^2(x^3+1)^4\,\mathrm{d}x = \frac{1}{3}\int_1^2 u^4\,\mathrm{d}u\,,$$

where the x^2 cancels out with the x^2 in our $\mathrm{d}x$, and the $1/3$ also comes from $\mathrm{d}x$. We can just use power rule and the fundamental theorem of calculus to finish the computation:

$$\frac{1}{3}\int_1^2 u^4 = \frac{1}{3}\left[\frac{u^5}{5}\right]_1^2 = \frac{1}{3}\left(\frac{2^5}{5} - \frac{1^5}{5}\right).$$

Remark 2.5. Note that because we changed our bounds to be in terms of us during the substitution process, we *do not* have to go back and turn everything back in terms of xs. Keep this in mind for when we do trigonometric things. We could have alternatively left off the bonds, just computed the indefinite integral

$$\int x^2(x^3+1)^4\,\mathrm{d}x = \frac{1}{3}\left(\frac{u^5}{5}\right) + C$$

using u-substitution, converted our solution into xs

$$\frac{1}{3}\left(\frac{(x^3+1)^5}{5}\right) + C,$$

and *then* substituted in the bounds and computed the definite integral

$$\int_0^1 x^2(x^3+1)^4\,\mathrm{d}x = \frac{1}{3}\left[\frac{(x^3+1)^5}{5}\right]_0^1 = \frac{1}{3}\left(\frac{(1^3+1)}{5} - \frac{(0^3+1)}{5}\right).$$

Either method is fine, but be consistent. Do not mix and match your xs with your u-bounds or vice-versa.

3. Integration by parts

Goal 3.1. We want to integrate things like

$$\int f(x)g(x)\,\mathrm{d}x\,,$$

where $f(x)$ and $g(x)$ are functions.

Solution 3.2. Find a $u = f(x)$ in the integrand and compute $\mathrm{d}u$ by taking the derivative. Find a $\mathrm{d}v = g(x)\,\mathrm{d}x$ in the integrand and compute v by taking the integral. Then, we have the formula

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u\,.$$

Remark 3.3. Picking the u and the $\mathrm{d}v$ are the hardest step in doing integration by parts. A helpful acronym I learned this semester from CAs Carter and Shreyansh is

L	ogarithms,	like $\ln(x)$
I	nverse trigonometric,	like $\arctan(x)$
A	lgebraic,	like x^3+2
T	rigonometric,	like $\sin(x)$
E	xponentials,	like e^x.

The point is that the functions near the top, like logarithms and inverse trigonometric functions, become easier to integrate once we take a derivative of them. This means we want the functions closer to the top of the acronym to be our u in the integration by parts.

Example 3.4. We want to integrate

$$\int xe^{5x}\,\mathrm{d}x\,.$$

We see a product of functions in the integrand, so our first thought is to use integration by parts. Since x is algebraic, which is above exponential in the acronym LIATE, we will set $u = x$ and $\mathrm{d}v = e^{5x}\,\mathrm{d}x$. Then, $\mathrm{d}u = 1\,\mathrm{d}x$ and

$$v = \frac{1}{5}e^{5x}.$$

We can then compute

$$\int xe^{5x}\,\mathrm{d}x = uv - \int v\,\mathrm{d}u = x\frac{1}{5}e^{5x} - \int \frac{1}{5}e^{5x}\,\mathrm{d}x = x\frac{1}{5}e^{5x} - \frac{1}{25}e^{5x} + C,$$

since u-substitution tells us

$$\int \frac{1}{5}e^{5x}\,\mathrm{d}x = \frac{1}{5}\int e^{5x}\,\mathrm{d}x \overset{u=5x}{=} \frac{1}{5}\int \frac{1}{5}e^{u}\,\mathrm{d}u = \frac{1}{25}e^{u} + C = \frac{1}{25}e^{5x} + C.$$

4. Trigonometric integrals

Goal 4.1. We want to integrate things that look like

$$\int \sin^n(x)\cos^m(x)\,\mathrm{d}x \quad \text{or} \quad \int \tan^n(x)\sec^m(x)\,\mathrm{d}x\,.$$

Solution 4.2. We can simplify these products of trigonometric functions using the trigonometric identities *we all remember*. Then, we can use u-substitution or integration by parts to compute the integral.

I will focus on the case of sines and cosines, but both are similar.

Example 4.3. We are asked to integrate

$$\int \sin^{41}(x)\cos^{10}(x)\,\mathrm{d}x\,.$$

The first thing I notice is that *one of the powers is odd*. This means I can split off one of my sines:

$$\int \sin(x)\sin^{40}(x)\cos^{10}(x)\,\mathrm{d}x\,.$$

Then, I can rewrite $\sin^{40}(x) = (\sin^2(x))^{20}$, since $2(20) = 40$:

$$\int \sin(x)(\sin^2(x))^{20}\cos^{10}(x)\,\mathrm{d}x\,.$$

Now, remember the identity $\sin^2(x) + \cos^2(x) = 1$. This means $\sin^2(x) = 1 - \cos^2(x)$, so my integral becomes

$$\int \sin(x)(1-\cos^2(x))^{20}\cos^{10}(x)\,\mathrm{d}x\,.$$

I now have an integral fully in terms of cosines, except for *one* leftover sine. (This is why we wanted an odd power.) The leftover sine can get canceled out in a u-substitution! Let $u = \cos(x)$ and $\mathrm{d}u = -\sin(x)\,\mathrm{d}x$. Then, the integral becomes

$$-\int (1-u^2)^{20}u^{10}\,\mathrm{d}u\,,$$

where the $-$ and the lack of $\sin(x)$ come from plugging in the differential $\mathrm{d}u$. From here, this is just a normal integral we can distribute and use power rule on (though it is very messy, so let's not bother).

Remark 4.4. What happens if both of the powers are even? Well, as you should know by now, this makes things a bit more challenging. Suddenly, our trick of using the Pythagorean identity followed by a u-substitution does not work. Instead, the usual method is to use is a double-angle and/or half-angle identity, turning the even powers into odd powers. It is absolutely worth memorizing these basic identities for the first exam. You should also make an effort to know the derivatives of $\tan(x)$ and $\sec(x)$.

Remark 4.5. See the solutions to Worksheet 03 on Canvas. Problem 4 discusses the strategies of trigonometric integrals thoroughly. I think it is worth it to keep all the cases in your mind. The type I worked out above is probably the easiest, so it is not too hard to get the hang of after, say, one or two practice problems.

5. Trigonometric substitution

Goal 5.1. Sometimes our integrals look like

$$\int \frac{1}{\sqrt{a^2+b^2x^2}}\,\mathrm{d}x\,,$$

where a and b are some numbers. We want to integrate this, but our immediate guess of doing u-substitution with $u = a^2 + x^2$ will not work, since there is no x in the numerator.

Solution 5.2. We recognize that $\sqrt{a^2+x^2}$ looks like the formula for the hypotenuse of a right triangle which has legs of length a and x. This means we can substitute x for some trigonometric function. Specifically, we make the following substitutions based on what sort of expression we see in the integrand:

$$\sqrt{a^2-b^2x^2} \qquad \text{substitute } x = \frac{a}{b}\sin(\theta), \qquad \text{where } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$\sqrt{a^2+b^2x^2} \qquad \text{substitute } x = \frac{a}{b}\tan(\theta), \qquad \text{where } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\sqrt{b^2x^2-a^2} \qquad \text{substitute } x = \frac{a}{b}\sec(\theta), \quad \text{where } 0 \le \theta < \frac{\pi}{2}, \pi \le \theta < \frac{3\pi}{2}.$$

Example 5.3. Let us compute

$$\int \frac{1}{x^4\sqrt{16-x^2}}\,\mathrm{d}x\,.$$

First, we see a factor in the integrand that looks like

$$\sqrt{a^2-b^2x^2}, \quad \text{where } a = 4 \text{ and } b = 1.$$

This tells us we should use the substitution

$$x = \frac{a}{b}\sin(\theta) = 4\sin(\theta), \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$$

We take a derivative to compute the differential

$$\mathrm{d}x = 4\cos(\theta)\,\mathrm{d}\theta\,.$$

Plugging these in, our integral becomes

$$\int \frac{1}{x^4\sqrt{16-x^2}}\,\mathrm{d}x = \int \frac{1}{(4\sin(\theta))^4\sqrt{16-(4\sin(\theta))^2}} 4\cos(\theta)\,\mathrm{d}\theta\,.$$

Simplifying gives

$$\int \frac{4}{4^4}\frac{\cos(\theta)}{\sin^4(\theta)4\sqrt{1-\sin^2(\theta)}}\,\mathrm{d}\theta = \int \frac{1}{4^4}\frac{\cos(\theta)}{\sin^4(\theta)\sqrt{\cos^2(\theta)}}\,\mathrm{d}\theta = \int \frac{1}{4^4}\frac{\cos(\theta)}{\sin^4(\theta)|\cos(\theta)|}\,\mathrm{d}\theta\,,$$

using the identity $1-\sin^2(\theta) = \cos^2(\theta)$. Now, since θ goes from $-\pi/2$ to $\pi/2$, we are working on the righthand side of the unit circle, so $\cos(\theta)$ is always positive, so $|\cos(\theta)| = \cos(\theta)$. This makes our integral

$$\frac{1}{4^4}\int \frac{\cos(\theta)}{\sin^4(\theta)\cos(\theta)}\,\mathrm{d}\theta = \frac{1}{4^4}\int \frac{1}{\sin^4(\theta)}\,\mathrm{d}\theta\,.$$

From here, this is just a (bit tricky) trigonometric integral.

6. Partial fractions

Goal 6.1. We want to evaluate things that look like

$$\int \frac{p(x)}{q(x)}\,\mathrm{d}x\,, \quad \text{where } \deg(p(x)) < \deg(q(x)).$$

Solution 6.2. We can use a partial fraction decomposition, based on what $q(x)$ looks like:

$$(ax+b)^k \qquad \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$$

$$(ax^2+bx+c)^k \qquad \frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k},$$

where a, b, and c are some numbers, k is some power, and the A_i and B_i are the coefficients you have to solve for.

Example 6.3. Say we want to evaluate the integral[1]

$$\int \frac{x^2-29x+5}{(x-4)^2(x^2+3)}\,\mathrm{d}x\,.$$

Well, we spot $(x-4)^2$ and (x^2+3) in the denominator. The former means we have the first case, so we get terms that look like

$$\frac{A_1}{x-4} + \frac{A_2}{(x-4)^2}.$$

The latter means we also have the second case, so we get a term that looks like

$$\frac{A_3x+A_4}{x^2+3}.$$

Now, we set the integrand equal to the terms we found:

$$\frac{x^2-29x+5}{(x-4)^2(x^2+3)} = \frac{A_1}{x-4} + \frac{A_2}{(x-4)^2} + \frac{A_3x+A_4}{x^2+3}.$$

We can find a common denominator on the righthand side:

$$\frac{A_1(x-4)(x^2+3) + A_2(x^2+3) + (A_3x+A_4)(x-4)^2}{(x-4)^2(x^2+3)}.$$

This denominator matches the one on the lefthand side, so we can set the numerators to be equal:

$$x^2-29x+5 = A_1(x-4)(x^2+3) + A_2(x^2+3) + (A_3x+A_4)(x-4)^2.$$

Now, we just have to multiply everything on the righthand side out to figure out what A_1, A_2, A_3, and A_4 are:

$$(A_1+A_3)x^3 + (-4A_1+A_2-8A_3+A_4)x^2 + (3A_1+16A_3-8A_4)x - 12A_1 + 3A_2 + 16A_4.$$

There are no x^3 terms on the left, so $A_1+A_3=0$. The coefficient of x^2 on the left is 1, so $-4A_1+A_2-8A_3+A_4=1$. The coefficient on x is -29 on the left, so $3A_1+16A_3-8A_4=-29$. Finally, we have a constant term 5 on the left, so $-12A_1+3A_2+16A_4=5$. Solving these gives $A_1=1$, $B=-5$, $C=-1$, and $D=2$, but this takes some time to see. Practice solving such systems for the exam, but do not waste your time doing these difficult ones. We can now rewrite our integral as

$$\int \frac{1}{x-4} - \frac{5}{(x-4)^2} + \frac{-x+2}{x^2+3}\,\mathrm{d}x\,,$$

[1] I took this problem from Paul's Online Math Notes.

using the partial fractions decomposition we just found. The first term can be integrated using a logarithm, the second can be integrated using a u-substitution and power rule, the third can be done using a u-substitituion and a logarithm, and the fourth can be done using arctan.

7. Improper integrals

Goal 7.1. We want to integrate things that look like

(i)
$$\int_a^\infty f(x)\,\mathrm{d}x\,.$$

(ii)
$$\int_{-\infty}^b f(x)\,\mathrm{d}x\,.$$

(iii)
$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x$$

(iv)
$$\int_a^b f(x)\,\mathrm{d}x\,,$$

where $f(x)$ has some problems somewhere between a and b.

Or, we want to just tell if something that looks like the case (i)

(v)
$$\int_a^\infty f(x)\,\mathrm{d}x$$

converges to a finite number, where $f(x)$ is greater than or equal to zero for all $x \geq a$.

Solution 7.2 (i)**.** We can evaluate

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{t\to\infty} \int_a^t f(x)\,\mathrm{d}x\,,$$

taking the limit at the end after applying the fundamental theorem of calculus.

Solution 7.3 (ii)**.** Similarly, we can evaluate

$$\int_\infty^b f(x)\,\mathrm{d}x = \lim_{t\to\infty} \int_{-t}^b f(x)\,\mathrm{d}x\,.$$

Solution 7.4 (iii)**.** Combining (i) and (ii), we can evaluate

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \int_{-\infty}^b f(x)\,\mathrm{d}x + \int_b^\infty f(x)\,\mathrm{d}x\,,$$

doing each improper integral separately (using the limit solution seen in (i) and (ii)). Here, we just pick some point b in the middle to split the bounds up. A good choice is usually $b = 0$.

Solution 7.5 ((iv))**.** If $f(x)$ has a problem at $a \leq c \leq b$, we can split the integral up into

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^a f(x)\,\mathrm{d}x\,.$$

Now, since the problem is at c, we need to take the limit, approaching from the left from a and from the right from b, respectively:

$$\lim_{t \to c^-} \int_a^t f(x)\,\mathrm{d}x + \lim_{t \to c^+} \int_t^b f(x)\,\mathrm{d}x\,.$$

Solution 7.6 ((v))**.** As long as $f(x) \geq 0$, let us guess whether

$$\int_a^\infty f(x)\,\mathrm{d}x$$

converges. If we think it does, then we should find some function $g(x)$ such that $g(x) \geq f(x)$ (is bigger) and such that

$$\int_a^\infty g(x)\,\mathrm{d}x$$

converges. This will usually be some kind of p-integral, since we know

$$\int_a^\infty \frac{1}{x^p}\,\mathrm{d}x$$

converges for all $p > 1$ and any positive lower bound a. This implies that our original integral converges, since it is less than something finite. On the other hand, if we think our integral diverges, then we pick a nonnegative function $h(x) \leq f(x)$ such that

$$\int_a^\infty h(x)\,\mathrm{d}x$$

diverges. For example, this is something like $1/x$, or any p-integral with $p \leq 1$. Then, this implies that our original integral diverges, since it is bigger than something infinite.

8. Approximate integration

Goal 8.1. We want to approximate the definite integral

$$\int_a^b f(x)\,\mathrm{d}x$$

by dividing the interval $[a, b]$ into n even parts. (Note that we can drop the condition that n must be even for the trapezoid rule, but it is necessary for Simpson's rule.)

Solution 8.2 (Trapezoid Rule)**.** Define the quantity

$$\Delta x = \frac{b-a}{n},$$

where a and b are the bounds on the integral and n is the number of parts. Then,

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{\Delta x}{2}(f(a) + 2f(a+\Delta x) + 2f(a+2\Delta x) + \cdots + 2f(a+(n-1)\Delta x) + f(b)).$$

Alternatively, if we write $a = x_0$ and $b = x_n$, then we could write

$$x_i = a + (i)\Delta x, \quad \text{where } 0 \leq i \leq n.$$

Solution 8.3 (Simpson's Rule)**.** Again, define the quantity

$$\Delta x = \frac{b-a}{n}.$$

Then, the definite integral is approximately

$$\frac{\Delta x}{3}(f(a) + 4f(a+\Delta x) + 2f(a+2\Delta x) + \cdots + 2f(a+(n-2)\Delta x) + 4f(a+(n-1)\Delta x) + f(b)).$$

Remark 8.4. In practice, all you have to do is compute the quantity Δx and evaluate the given function $f(x)$ at the n points $x_0, x_1, \dots, x_n$. Then, you plug this information into the above formula.

Remark 8.5. Memorize that

(i) for trapezoid rule, the coefficient pattern is that there are 2s on every function evaluation, except for the first and last. That is,

$$1, 2, 2, 2, \dots, 2, 2, 2, 1.$$

Also, you multiply the whole thing by $\Delta x/2$.

(ii) for Simpson's rule, the coefficient pattern is that you alternate between 4s and 2, starting with 4, except for the first and last. That is,

$$1, 4, 2, 4, \dots, 4, 2, 4, 1.$$

Because n is even, your second to last coefficient after alternating will always be a 4.

University of Illinois Urbana-Champaign, Illinois, 61801
Email address: dheeran2@illinois.edu