

Solving the Optimal Experiment Design Problem with Mixed-Integer Convex Methods

Deborah Hendrych, Mathieu Besançon, Sebastian Pokutta

hendrych@zib.de

Zuse Institute Berlin

Research Seminar NASPDE Group · July 11, 2024



Berlin Mathematics Research Center



Outline

1. The Optimal Experiment Design Problem (OEDP)

2. Solution Methods

Branch-and-Bound with Frank-Wolfe Methods: Boscia.jl

Custom BnB for the OEDP under Matrix Means

Epigraph-based Outer Approximation: SCIP

Conic Outer Approximation: Direct and Second-Order

3. Computational Experiments

Results

4. Summary and Outlook



Table of Contents

The Optimal Experiment Design Problem (OEDP)

Solution Methods

Branch-and-Bound with Frank-Wolfe Methods: Boscia.jl

Custom BnB for the OEDP under Matrix Means

Epigraph-based Outer Approximation: SCIP

Conic Outer Approximation: Direct and Second-Order

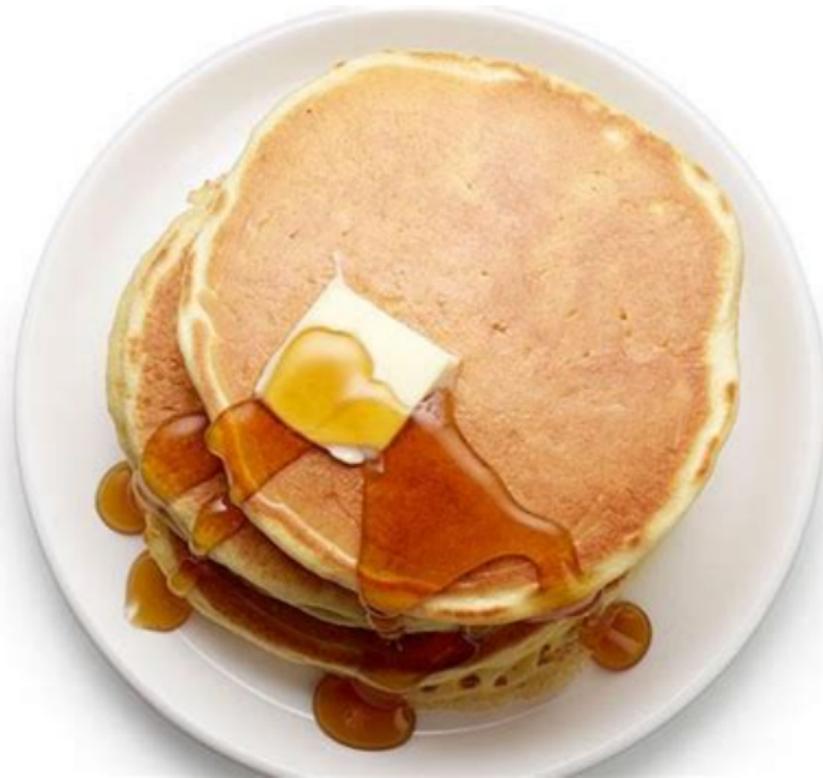
Computational Experiments

Results

Summary and Outlook



Think Pancakes



Set-up

- The ultimate goal is fitting a regression model.

$$\min_{\theta \in \mathbb{R}^n} \|A\theta - \mathbf{y}\|_2^2$$

- $A = [-\mathbf{a}_i -]_{i=1}^m$ with $\mathbf{a}_i \in \mathbb{R}^n$ is the *Experiment Matrix*. Assumed to have full column rank.
- θ is the unknown set of parameters to be found.
- \mathbf{y} is the not-yet measured response.
- Problem:** Running all m experiments too costly and/or too time intensive.

Information Matrix

The *Information Matrix* is a linear map $X : \mathbb{R}^m \rightarrow \mathbb{R}^{n \times n}$.

$$X(\mathbf{x}) = \sum_{i=1}^m x_i \mathbf{a}_i \mathbf{a}_i^\top = A^\top \operatorname{diag}(\mathbf{x}) A$$

- Variable $x_i \in \mathbb{Z}_{\geq 0}$ denotes how often experiment i is to be run. \mathbf{x} is a *design*.

Information Matrix

The *Information Matrix* is a linear map $X : \mathbb{R}^m \rightarrow \mathbb{R}^{n \times n}$.

$$X(\mathbf{x}) = \sum_{i=1}^m x_i \mathbf{a}_i \mathbf{a}_i^\top = A^\top \operatorname{diag}(\mathbf{x}) A$$

- Variable $x_i \in \mathbb{Z}_{\geq 0}$ denotes how often experiment i is to be run. \mathbf{x} is a *design*.
- $A^\top A$ is positive definite (PD).
- $X(\mathbf{x})$ is positive semi definite (PSD).

Information Matrix

The *Information Matrix* is a linear map $X : \mathbb{R}^m \rightarrow \mathbb{R}^{n \times n}$.

$$X(\mathbf{x}) = \sum_{i=1}^m x_i \mathbf{a}_i \mathbf{a}_i^\top = A^\top \operatorname{diag}(\mathbf{x}) A$$

- Variable $x_i \in \mathbb{Z}_{\geq 0}$ denotes how often experiment i is to be run. \mathbf{x} is a *design*.
- $A^\top A$ is positive definite (PD).
- $X(\mathbf{x})$ is positive semi definite (PSD).
- The inverse of $X(\mathbf{x})$ (if existent) is called the *Dispersion Matrix* $D(\mathbf{x})$ and is a measure of the variance of the parameters
Ahipaşaoğlu 2021.
- A design \mathbf{x} is "useful" if $X(\mathbf{x})$ is regular.

Information measure

- Function $\phi : \mathbb{S}_{++}^n \rightarrow \mathbb{R}$ where \mathbb{S}_{++}^n is the cone of $n \times n$ positive definite matrix.
- Necessary properties Pukelsheim 2006:

Information measure

- Function $\phi : \mathbb{S}_{++}^n \rightarrow \mathbb{R}$ where \mathbb{S}_{++}^n is the cone of $n \times n$ positive definite matrix.
- Necessary properties Pukelsheim 2006:
 - Respect Loewner Ordering: Let $D, B \in \mathbb{S}_+^n$. Then $D \succcurlyeq B$ if and only $D - B \in \mathbb{S}_+^n$. If $D \succcurlyeq B$, we require $\phi(D) \geq \phi(B)$.

Information measure

- Function $\phi : \mathbb{S}_{++}^n \rightarrow \mathbb{R}$ where \mathbb{S}_{++}^n is the cone of $n \times n$ positive definite matrix.
- Necessary properties Pukelsheim 2006:
 - Respect Loewner Ordering: Let $D, B \in \mathbb{S}_+^n$. Then $D \succcurlyeq B$ if and only $D - B \in \mathbb{S}_+^n$. If $D \succcurlyeq B$, we require $\phi(D) \geq \phi(B)$.
 - Concave: Prohibit linear interpolation.

Information measure

- Function $\phi : \mathbb{S}_{++}^n \rightarrow \mathbb{R}$ where \mathbb{S}_{++}^n is the cone of $n \times n$ positive definite matrix.
- Necessary properties Pukelsheim 2006:
 - Respect Loewner Ordering: Let $D, B \in \mathbb{S}_+^n$. Then $D \succcurlyeq B$ if and only $D - B \in \mathbb{S}_+^n$. If $D \succcurlyeq B$, we require $\phi(D) \geq \phi(B)$.
 - Concave: Prohibit linear interpolation.
 - Positively homogeneous: $\phi(\lambda X) = \lambda \phi(X)$ for all $\lambda \geq 0$.

Information measure

- Function $\phi : \mathbb{S}_{++}^n \rightarrow \mathbb{R}$ where \mathbb{S}_{++}^n is the cone of $n \times n$ positive definite matrix.
- Necessary properties Pukelsheim 2006:
 - Respect Loewner Ordering: Let $D, B \in \mathbb{S}_+^n$. Then $D \succcurlyeq B$ if and only $D - B \in \mathbb{S}_+^n$. If $D \succcurlyeq B$, we require $\phi(D) \geq \phi(B)$.
 - Concave: Prohibit linear interpolation.
 - Positively homogeneous: $\phi(\lambda X) = \lambda \phi(X)$ for all $\lambda \geq 0$.
 - Non-negative and non-constant: Convention.

Information measure

- Function $\phi : \mathbb{S}_{++}^n \rightarrow \mathbb{R}$ where \mathbb{S}_{++}^n is the cone of $n \times n$ positive definite matrix.
- Necessary properties Pukelsheim 2006:
 - Respect Loewner Ordering: Let $D, B \in \mathbb{S}_+^n$. Then $D \succcurlyeq B$ if and only $D - B \in \mathbb{S}_+^n$. If $D \succcurlyeq B$, we require $\phi(D) \geq \phi(B)$.
 - Concave: Prohibit linear interpolation.
 - Positively homogeneous: $\phi(\lambda X) = \lambda \phi(X)$ for all $\lambda \geq 0$.
 - Non-negative and non-constant: Convention.
 - Upper semi-continuous: The upper level sets $\{M \in \mathbb{S}_{++}^n \mid \phi(M) \geq \lambda\}$ are closed for all $\lambda \in \mathbb{R}$.
-

The Optimal Experiment Design Problem (OEDP)

$$\max_{\mathbf{x}} \log(\phi(X(\mathbf{x})))$$

$$\text{s.t. } \sum_{i=1}^m x_i = N \tag{OEDP}$$

$$\mathbf{l} \leq \mathbf{x} \leq \mathbf{u}$$

$$\mathbf{x} \in \mathbb{Z}_{\geq 0}^m,$$

- Let $m \gg N \geq n$ be the *number of allowed experiments*.
- \mathbf{l} and \mathbf{u} are lower and upper bounds, respectively. Usually, $\mathbf{l} = \mathbf{0}$.
- Denote $\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^m \mid \langle \mathbf{1}, \mathbf{x} \rangle = N, \mathbf{l} \leq \mathbf{x} \leq \mathbf{u}\}$.

The Optimal Experiment Design Problem (OEDP)

$$\max_{\mathbf{x}} \log(\phi(X(\mathbf{x})))$$

$$\text{s.t. } \sum_{i=1}^m x_i = N \tag{OEDP}$$

$$\mathbf{l} \leq \mathbf{x} \leq \mathbf{u}$$

$$\mathbf{x} \in \mathbb{Z}_{\geq 0}^m,$$

- Let $m \gg N \geq n$ be the *number of allowed experiments*.
- \mathbf{l} and \mathbf{u} are lower and upper bounds, respectively. Usually, $\mathbf{l} = \mathbf{0}$.
- Denote $\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^m \mid \langle \mathbf{1}, \mathbf{x} \rangle = N, \mathbf{l} \leq \mathbf{x} \leq \mathbf{u}\}$.
- If $C = A^\top \text{diag}(\mathbf{l}) A$ is positive definite, we have the *Fusion Problem*.

Matrix Means

Definition (Matrix Mean)

Let $B \in \mathbb{S}_+^n$ and let $\lambda(B)$ denote its eigenvalues. The *matrix mean* ϕ_p of B is defined as

$$\phi_p(B) = \begin{cases} \lambda_{\max}(B), & \text{for } p = \infty, \\ \left(\frac{1}{n} \operatorname{Tr}(B^p)\right)^{\frac{1}{p}}, & \text{for } p \neq 0, \pm\infty, \\ \det(B)^{\frac{1}{n}}, & \text{for } p = 0, \\ \lambda_{\min}(B), & \text{for } p = -\infty, \\ 0 & \text{for } p = [-\infty, 0] \text{ and } B \text{ singular.} \end{cases} \quad (1)$$

For $p \leq 1$, ϕ_p is a valid information measure (Pukelsheim 2006).

The A-Optimality and D-Optimality Criteria

The D-Optimal Problem ($p = 0$) The A-Optimal Problem ($p = -1$)

$$\begin{aligned} \min_{\mathbf{x}} & -\log \det(X(\mathbf{x})) \\ \text{s.t. } & \mathbf{x} \in \mathcal{P} \cap \mathbb{Z}_{\geq 0}^m \end{aligned} \quad (\text{DO})$$

minimal volume of standard ellipsoidal confidence region of θ

Ponte, Fampa, and Lee 2023; Ahipaşaoğlu 2021; Li et al. 2022

$$\begin{aligned} \min_{\mathbf{x}} & \text{Tr}\left((X(\mathbf{x}))^{-1}\right) \\ \text{s.t. } & \mathbf{x} \in \mathcal{P} \cap \mathbb{Z}_{\geq 0}^m \end{aligned} \quad (\text{AO})$$

minimize the diagonal of the bounding box of the confidence ellipsoid

Ahipaşaoğlu 2015; Sagnol and Pauwels 2019; Nikolov, Singh, and Tantipongpipat 2022

Both are \mathcal{NP} -hard. (Welch 1982; Nikolov, Singh, and Tantipongpipat 2022)

The D-Fusion Problem is also \mathcal{NP} -hard (Li et al. 2022). For the A-Fusion it is not known.



Table of Contents

The Optimal Experiment Design Problem (OEDP)

Solution Methods

Branch-and-Bound with Frank-Wolfe Methods: `Boscia.jl`

Custom BnB for the OEDP under Matrix Means

Epigraph-based Outer Approximation: `SCIP`

Conic Outer Approximation: Direct and Second-Order

Computational Experiments

Results

Summary and Outlook



Previous Approach

- Due to the hardness of the MINLP formulation, one often solves a continuous version.
- The *Limit Problem* is created by letting $N \rightarrow \infty$.
- \mathbf{x} can then be found of as a probability distribution.
- Different rounding schemes and exchange heuristics can then be employed to generate integer solution.
- Problem 1: Sparsity of the probability solution is often required.
- Problem 2: No guarantee of the optimality.

Boscia - Problem Setting

Mixed-Integer Convex Problems

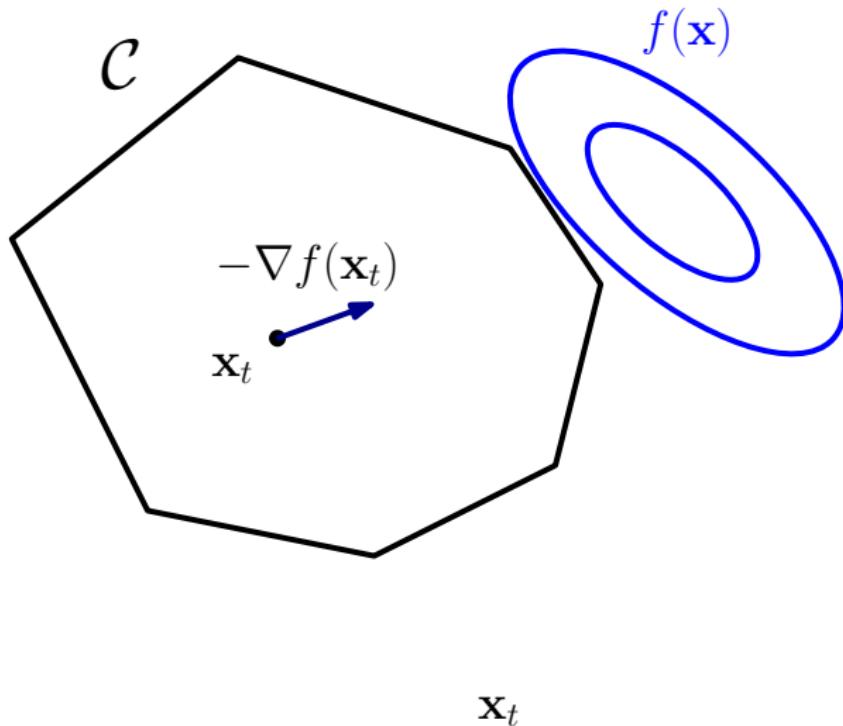
We want to solve MINLPs of the form

$$\begin{aligned} & \min_{\mathbf{x}} f(\mathbf{x}) \\ & \text{s.t. } \mathbf{x} \in \mathcal{X} \\ & \quad x_j \in \mathbb{Z} \quad \forall j \in J \end{aligned}$$

- f is a differentiable, non-linear, convex, L -smooth function.
- \mathcal{X} is polyhedral with combinatorial and integrality constraints.

The Frank-Wolfe algorithm I

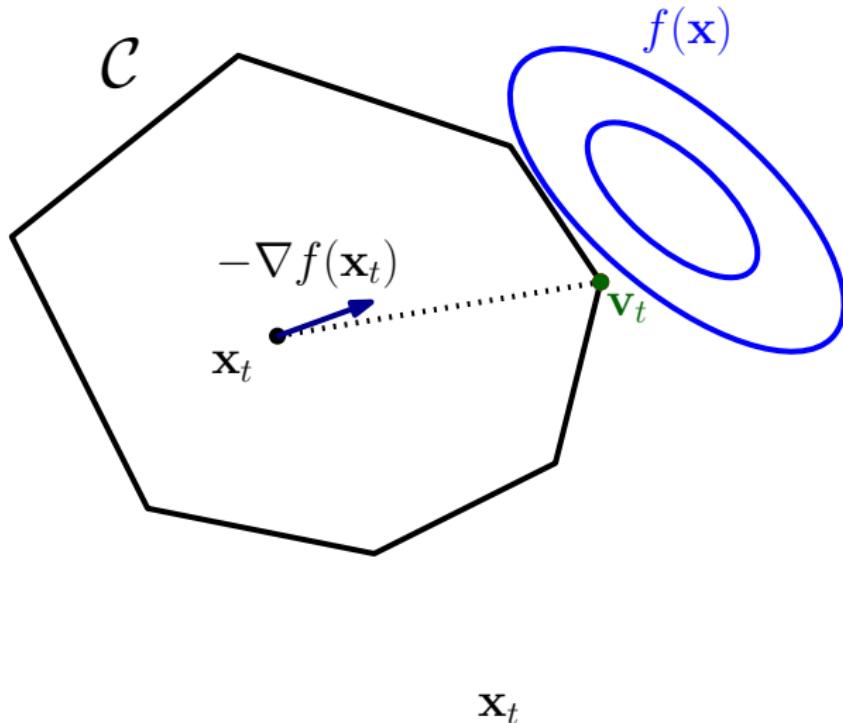
Frank and Wolfe 1956; Levitin and Polyak 1966



\mathbf{x}_t

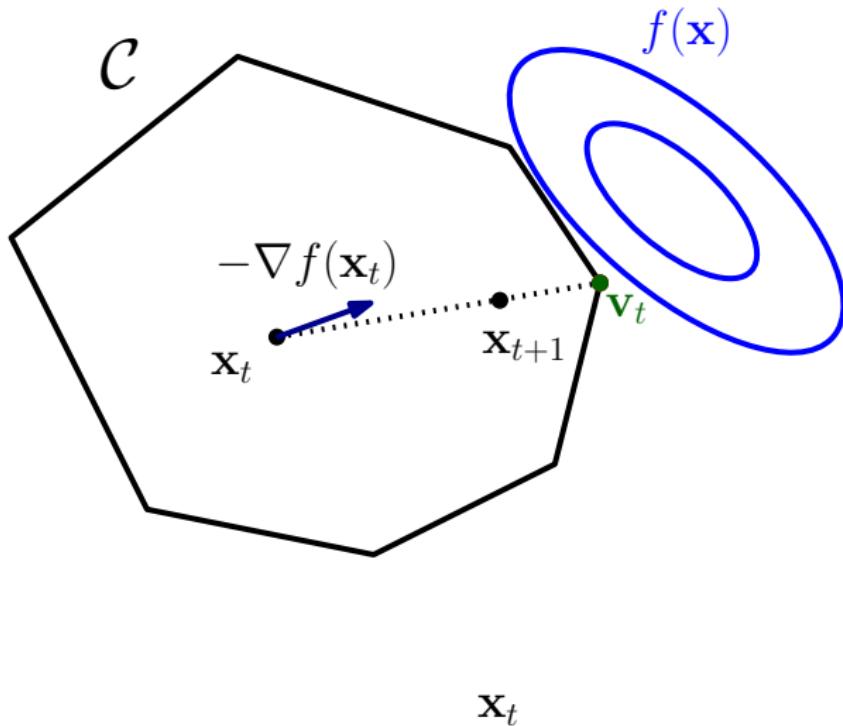
The Frank-Wolfe algorithm II

Frank and Wolfe 1956; Levitin and Polyak 1966



The Frank-Wolfe algorithm III

Frank and Wolfe 1956; Levitin and Polyak 1966



The Frank-Wolfe algorithm IV

Frank and Wolfe 1956; Levitin and Polyak 1966

$$\begin{aligned} \min_{\mathbf{x}} f(\mathbf{x}) \\ \text{s.t. } \mathbf{x} \in C \end{aligned}$$

- C is a compact convex set.
- f continuously differentiable, convex function.
- Requires f to be L -smooth, i.e. gradient ∇f Lipschitz-continuous.
- Assumes that the Linear Minimization Oracle (LMO) $\min_{\mathbf{x} \in C} \langle \mathbf{d}, \mathbf{x} \rangle$ is easy to solve.

The Frank-Wolfe Gap or Dual Gap

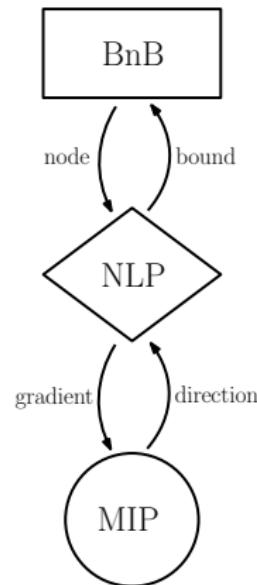
$$f(\mathbf{x}) - f(\mathbf{x}^*) \leq \langle \nabla f(\mathbf{x}), \mathbf{x} - \mathbf{x}^* \rangle \leq \min_{\mathbf{v} \in C} \langle \nabla f(\mathbf{x}), \mathbf{x} - \mathbf{v} \rangle$$



Branch-and-Bound with Frank-Wolfe: Boscia.jl

Hendrych et al. 2023

- Frank-Wolfe variants as node solver.
- Frank-Wolfe implemented in `FrankWolfe.jl` Besançon, Carderera, and Pokutta 2022.
- Exploits Frank-Wolfe's error-adaptiveness.
- Frank-Wolfe enables warm-starting.
- Linear Minimization Oracle is a *Bounded (Mixed-Integer) Linear Minimization Oracle (BLMO)*, usually a MIP solver but can also be a combinatorial solver.
- Integer feasible solutions from the root node.



Solution via Boscia.jl

- We have L -smoothness for the Fusion Problems, not for the Optimal Problems.



Solution via Boscia.jl

- We have L -smoothness for the Fusion Problems, not for the Optimal Problems.
- By Carderera, Besançon, and Pokutta 2021, we also have convergence if the objective is generalized self-concordant.
- Known for the $-\log \det$ as the barrier of the PSD cone.



Solution via Boscia.jl

- We have L -smoothness for the Fusion Problems, not for the Optimal Problems.
- By Carderera, Besançon, and Pokutta 2021, we also have convergence if the objective is generalized self-concordant.
- Known for the $-\log \det$ as the barrier of the PSD cone.

Theorem

The function $g(X) = \text{Tr}(X^{-p})$, with $p > 0$, is $\left(3, \frac{(p+2)\sqrt[4]{a^{2p}n}}{\sqrt{p(p+1)}}\right)$ -generalized self-concordant on $\text{dom}(g) = \{X \in \mathbb{S}_{++}^n \mid 0 < X \leq aI\}$ where $a \in \mathbb{R}_{>0}$ bounds the maximum eigenvalue of X .

A Custom Branch-and-Bound for OEDP under Matrix Means (Co-BnB)

Ahipaşaoğlu 2021

- Coordinate-Descent-like algorithm for the nodes.
- Developed for $m \gg n$.

$$\begin{aligned} & \max_{\mathbf{w}} \log(\phi(X(\mathbf{w}))) \\ \text{s.t. } & \sum_{i=1}^m w_i = 1 \\ & \mathbf{w} \in [0, 1]^m \\ & N\mathbf{w} \in \mathbb{Z}_{\geq 0}^m \end{aligned} \tag{M-OEDP}$$

- \mathbf{w} can be interpreted as a probability distribution.
- Exploit connection to the Minimum Volume Enclosing Ellipsoid Problem (MVEP) for the termination criteria.



Epigraph-based Outer Approximation: SCIP+OA

Bestuzheva et al. 2021; Kronqvist et al. 2019

- Requires Epigraph Formulation

$$\min_{t, \mathbf{x}} t$$

$$\text{s.t. } t \geq \log(\phi(X(\mathbf{x})))$$

$$\sum_{i=1}^m x_i = N$$

$$\mathbf{l} \leq \mathbf{x} \leq \mathbf{u}$$

$$\mathbf{x} \in \mathbb{Z}_{\geq 0}^m.$$

- Approximate feasible region by linear cuts using the gradient of the objective.
- Would require additional domain cuts, i.e. adding cutting planes to separate domain infeasible points. Hence, it's only used on the Fusion Problems.

Direct Conic Formulation

Coey, Kapelevich, and Vielma 2022a; Coey, Kapelevich, and Vielma 2022b; Coey, Lubin, and Vielma 2020

- Pajarito.jl is a mixed-integer convex solver with conic certificates.
- Hypatia.jl is an interior point solver for conic optimization problems.
- D-Criterion:

$$\mathcal{K}_{\log \det} := \text{cl} \left\{ (u, v, W) \in \mathbb{R} \times \mathbb{R}_{>0} \times \mathbb{S}_{++}^n \mid u \leq v \log \det(W/v) \right\}$$

- A-Criterion: Dual of

$$\mathcal{K}_{\text{sepspec}} := \text{cl} \left\{ (u, v, w) \in \mathbb{R} \times \mathbb{R}_{>0} \times \text{int}(Q) \mid u \geq v\varphi(w/v) \right\}$$

- Q is the PSD cone and φ is the negative square root.
- The convex conjugate of the negative square root is the trace inverse.

Second-Order Conic Formulation (SOCP)

Sagnol 2011; Sagnol and Harman 2015

- In Sagnol 2011 the SOCP formulation for the continuous problem was introduced.
- The SOCP formulation of the exact formulation of OEDP was shown in Sagnol and Harman 2015.
- In theory, a nice result, but in practice, the problem size becomes much larger.
- For the A-Optimal Problem, we have $2m(n + 1)$ variables and $2(n + 1) + m$ constraints.
- For the D-Optimal Problem, we have $2m(1 + n) + n^2 + 1$ variables and $n(m + 1) + 3m + 4$ constraints.

Table of Contents

The Optimal Experiment Design Problem (OEDP)

Solution Methods

Branch-and-Bound with Frank-Wolfe Methods: Boscia.jl

Custom BnB for the OEDP under Matrix Means

Epigraph-based Outer Approximation: SCIP

Conic Outer Approximation: Direct and Second-Order

Computational Experiments

Results

Summary and Outlook



Experimental Results I

Set up

- m between 50 and 120
- $n = \lfloor m/4 \rfloor$ and $n = \lfloor m/10 \rfloor$
- #allowed experiments $N = \lfloor 1.5n \rfloor$ in case of the Optimal Problems.
- #allowed experiments $N \in [m/20, m/3]$ in case of the Fusion Problem.
- Independent and correlated data for the experiment matrix A .
- Five random seeds, leading to 50 instances per problem and data.
- Carried out in Julia with a time limit of 1 hour.

Experimental Results II

Type	Corr.	Boscia		Co-BnB		Direct Conic		SOCP		SCIP+OA	
		% solved	Time (s)	% solved	Time (s)	% solved	Time (s)	% solved	Time (s)	% solved	Time (s)
A	no	58 %	208.53	42 %	640.68	14 %	1901.7	20 %	1499.12		
A	yes	82 %	98.5	50 %	541.22	20 %	1591.74	20 %	1844.74		
AF	no	80 %	54.82	78 %	82.96	12 %	2006.81	26 %	1591.48	38 %	464.82
AF	yes	26 %	1359.35	50 %	370.57	20 %	1132.66	20 %	2002.94	14 %	1471.59
D	no	74 %	81.07	58 %	442.28	24 %	732.57	22 %	2192.52		
D	yes	100 %	1.26	68 %	223.34	10 %	755.88	8 %	2623.3		
DF	no	94 %	3.32	86 %	38.28	14 %	1576.5	12 %	2748.23	50 %	333.25
DF	yes	60 %	50.68	54 %	185.07	14 %	1761.18	6 %	2970.18	28 %	753.6

Experimental Results III

Termination over time for the A-Criterion

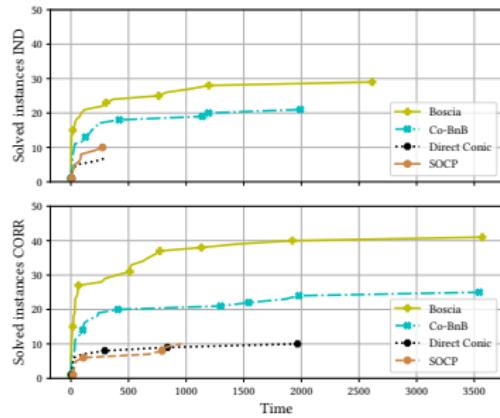


Figure: A-Optimal Problem

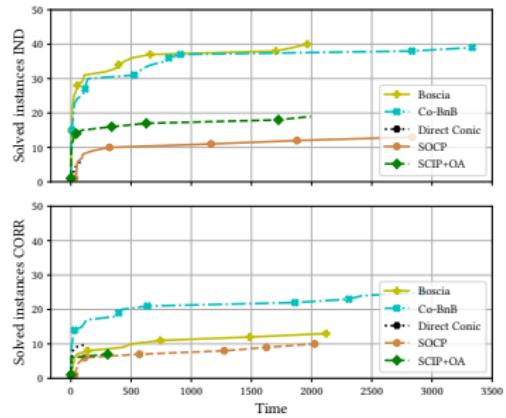


Figure: A-Fusion Problem

Experimental Results IV

Termination over time for the D-Criterion

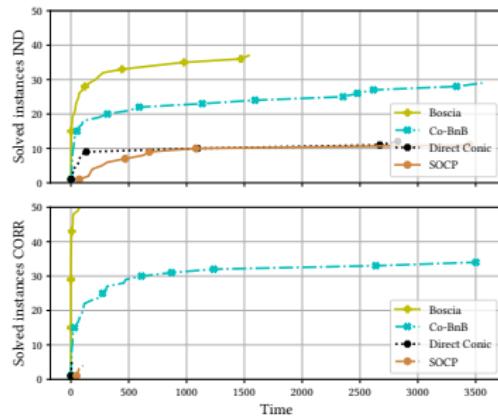


Figure: D-Optimal Problem

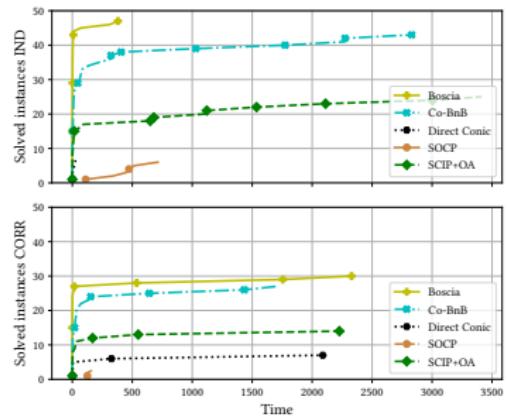


Figure: D-Fusion Problem

Table of Contents

The Optimal Experiment Design Problem (OEDP)

Solution Methods

Branch-and-Bound with Frank-Wolfe Methods: Boscia.jl

Custom BnB for the OEDP under Matrix Means

Epigraph-based Outer Approximation: SCIP

Conic Outer Approximation: Direct and Second-Order

Computational Experiments

Results

Summary and Outlook



Summary

- The Outer Approximation approaches are fast for small instances.
- Co-BnB is fast for instances where n is very small compared to m , i.e. $n = m/10$.
- Boscia.jl outperforms the other solvers, especially for medium to large scale instances.
- It also keeps the problem structure intact.
- Can easily handle additional constraints.

Outlook

On the Optimal Design side

- The E-Optimal Criterion and smoothing techniques.
- Knowledge of experiment matrix $A \rightarrow$ Robustness.
- Non-Linear regression and nearly convex functions.

On the general Boscia.jl side

- Exploiting Sharpness.
- Improvement of the Branch-and-Bound strategies.
- Preprocessing.
- Heuristics.
- Handling of non-linear constraints?

Thank you for your attention!



References I

- Ahipaşaoğlu, Selin Damla (2015). "A first-order algorithm for the A-optimal experimental design problem: a mathematical programming approach". In: *Statistics and Computing* 25.6, pp. 1113–1127.
- (2021). "A branch-and-bound algorithm for the exact optimal experimental design problem". In: *Statistics and Computing* 31.5, p. 65.
- Besançon, Mathieu, Alejandro Carderera, and Sebastian Pokutta (2022). "FrankWolfe.jl: A High-Performance and Flexible Toolbox for Frank-Wolfe Algorithms and Conditional Gradients". In: *INFORMS Journal on Computing*.
- Bestuzheva, Ksenia et al. (2021). *The SCIP Optimization Suite 8.0*. arXiv: 2112.08872 [math.OC].
- Carderera, Alejandro, Mathieu Besançon, and Sebastian Pokutta (2021). "Simple steps are all you need: Frank-Wolfe and generalized self-concordant functions". In: *Advances in Neural Information Processing Systems* 34, pp. 5390–5401.
- Coey, Chris, Lea Kapelevich, and Juan Pablo Vielma (2022a). "Conic optimization with spectral functions on Euclidean Jordan algebras". In: *Mathematics of Operations Research*.
- (2022b). "Performance enhancements for a generic conic interior point algorithm". In: *Mathematical Programming Computation*. DOI: <https://doi.org/10.1007/s12532-022-00226-0>.
- Coey, Chris, Miles Lubin, and Juan Pablo Vielma (2020). "Outer approximation with conic certificates for mixed-integer convex problems". In: *Mathematical Programming Computation* 12.2, pp. 249–293.

References II

- Frank, Marguerite and Philip Wolfe (1956). "An algorithm for quadratic programming". In: *Naval research logistics quarterly* 3.1-2, pp. 95–110.
- Hendrych, Deborah et al. (2023). Convex mixed-integer optimization with Frank-Wolfe methods. arXiv: 2208.11010 [math.OC].
- Kronqvist, Jan et al. (2019). "A review and comparison of solvers for convex MINLP". In: *Optimization and Engineering* 20, pp. 397–455.
- Levitin, Evgeny S and Boris T Polyak (1966). "Constrained minimization methods". In: *USSR Computational mathematics and mathematical physics* 6.5, pp. 1–50.
- Li, Yongchun et al. (2022). "D-optimal Data Fusion: Exact and Approximation Algorithms". In: *arXiv preprint arXiv:2208.03589*.
- Nikolov, Aleksandar, Mohit Singh, and Uthaipon Tantipongpipat (2022). "Proportional volume sampling and approximation algorithms for A-optimal design". In: *Mathematics of Operations Research* 47.2, pp. 847–877.
- Ponte, Gabriel, Marcia Fampa, and Jon Lee (2023). "Branch-and-bound for D-Optimality with fast local search and variable-bound tightening". In: *arXiv preprint arXiv:2302.07386*.
- Pukelsheim, Friedrich (2006). *Optimal design of experiments*. SIAM.
- Sagnol, Guillaume (2011). "Computing optimal designs of multiresponse experiments reduces to second-order cone programming". In: *Journal of Statistical Planning and Inference* 141.5, pp. 1684–1708.
- Sagnol, Guillaume and Radoslav Harman (2015). "Computing exact D-optimal designs by mixed integer second-order cone programming". In:

References III

- Sagnol, Guillaume and Edouard Pauwels (2019). "An unexpected connection between Bayes A-optimal designs and the group lasso". In: *Statistical Papers* 60.2, pp. 565–584.
- Welch, William J (1982). "Branch-and-bound search for experimental designs based on D optimality and other criteria". In: *Technometrics* 24.1, pp. 41–48.



SOCP A-Criterion

$$\begin{aligned} & \min_{\substack{\mu \in \mathbb{R}^m \\ \mathbf{x} \in \mathbb{Z}^m}} \quad \sum_{i=1}^m t_i \\ & \forall i \in [m] \quad Y_i \in \mathbb{R}^{1 \times n} \\ & \text{s.t.} \quad \sum_{i=1}^m A_i Z_i = I \\ & \quad \forall i \in [m] \quad \|Z_i\|_F^2 \leq t_i x_i \\ & \quad \sum_{i=1}^m x_i = N \end{aligned} \tag{A-SOCP}$$

SOCP D-Criterion

$$\begin{aligned} & \min_{\substack{\mathbf{w} \in \mathbb{R}_{\geq 0}^m, \mathbf{x} \in \mathbb{Z}^m \\ J \in \mathbb{R}^{nxn} \\ \forall i \in [m] Z_i \in \mathbb{R}^{1 \times n} \\ \forall i \in [m], j \in [n] t_{ij} \in \mathbb{R}_{\geq 0}}} \prod_{i=1}^m J_{ii}^{1/m} \\ \text{s.t. } & \sum_{i=1}^m A_i Z_i = J \\ & J \text{ lower triangle matrix} \tag{D-SOCP} \\ & \forall i \in [m], j \in [n] \|Z_i e_j\|_F^2 \leq t_{ij} w_i \\ & \forall j \in [n] \sum_{i=1}^m t_{ij} \leq J_{jj} \\ & \mathbf{l} \leq \mathbf{x} \leq \mathbf{u} \text{ and } \sum_{i=1}^m w_i = 1 \end{aligned}$$